Evaluate the iterated integral.

i) $$\int_0^4 \int_0^{\sqrt{y}} x y^2 \, dx \, dy$$

$$= \int_0^4 \left. \frac{x^2}{2} y^2 \right]_{x=0}^{\sqrt{y}} dy$$

$$= \int_0^4 \frac{y^3}{3} \, dy = \left. \frac{y^4}{4 \times 3} \right]_0^4 = \frac{4^4}{4 \times 3} = \frac{64}{3}$$

ii) $$\int_0^1 \int_{x^2}^{x} (1+2y) \, dy \, dx$$

$$= \int_0^1 \left(\int_{x^2}^{x} (1+2y) \, dy \, dx \right)$$

$$= \int_0^1 \left. y + y^2 \right]_{x^2}^{x} dx$$

$$= \int_0^1 \left(x + \cancel{x^2} - \cancel{x^2} - x^4 \right) dx$$

$$= \left. \frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$$

Evaluate the double integral.

i) $\iint_D y^2 \, dA, \quad D = \{(x,y) \mid -1 \le y \le 1, \ -y-2 \le x \le y\}$

Soln: $\iint_D y^2 \, dA = \int_{y=-1}^{1} \left(\int_{x=-y-2}^{y} y^2 \, dx \right) dy$

$$= \int_{y=-1}^{1} x y^2 \Big]_{x=-y-2}^{y} \, dy$$

$$= \int_{y=-1}^{1} \left(y^3 - (-y-2) y^2 \right) dy$$

$$= \int_{y=-1}^{1} \left(y^3 + y^3 + 2y^2 \right) dy$$

$$= \int_{y=-1}^{1} \left(2y^3 + 2y^2 \right) dy$$

$$= \frac{\cancel{2} y^4}{\cancel{4}\, 2} + \frac{2y^3}{3} \Bigg]_{y=-1}^{1} = \left(\frac{1}{2} + \frac{2}{3} \right) - \left(\frac{1}{2} - \frac{2}{3} \right)$$

$$= 4/3$$

ii) $\iint_D \frac{y}{x^5+1}\, dA, \quad D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x^2\}$

Soln:

$$\iint_D \frac{y}{x^5+1}\, dA = \int_{x=0}^{1} \left[\int_{y=0}^{x^2} \frac{y}{x^5+1}\, dy \right] dx$$

$$= \int_{x=0}^{1} \left. \frac{y^2}{2(x^5+1)} \right]_{y=0}^{x^2} dx$$

$$= \int_{x=0}^{1} \frac{x^4}{2(x^5+1)}\, dx$$

put $1+x^5 = t$ | as $x \to 0,\ t \to 1$

$\Rightarrow 5x^4\, dx = dt$ | as $x \to 1,\ t \to 2$

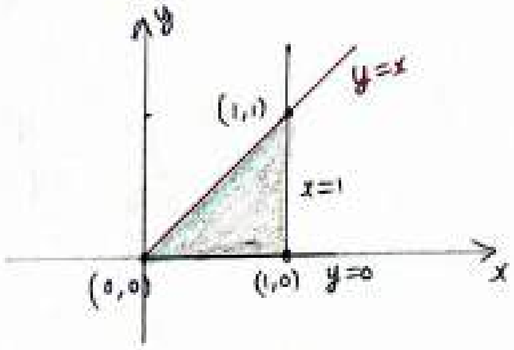
$$= \int_{t=1}^{2} \frac{dt}{10\, t}$$

$$= \frac{1}{10} \log t \Big]_{t=1}^{2}$$

$$= \frac{1}{10}(\log 2 - \log 1) = \frac{\log 2}{10}$$

Express D as a region of type I and also a region of type II. Then evaluate the double integral in two ways.

i) $\iint_D x\,dA$, D is enclosed by the lines $y=x$, $y=0$, $x=1$



Type I

$$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$$

$$\iint_D x\,dA = \int_{x=0}^{1}\left[\int_{y=0}^{x} x\,dy\right]dx$$

$$= \int_{x=0}^{1} xy\Big]_{y=0}^{x} dx = \int_{x=0}^{1} x^2\,dx = \frac{x^3}{3}\Big]_{x=0}^{1} = \frac{1}{3}$$

Type 2

$$D = \{(x,y) \mid 0 \le y \le 1,\ y \le x \le 1\}$$

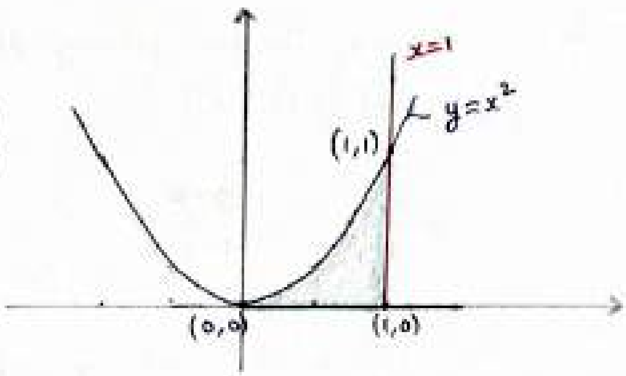
$$\iint_D x\,dA = \int_{y=0}^{1}\left[\int_{x=y}^{1} x\,dx\right]dy$$

$$= \int_{y=0}^{1} \frac{x^2}{2}\Big]_{x=y}^{1} dy = \int_{y=0}^{1}\left(\frac{1}{2}-\frac{y^2}{2}\right)dy = \frac{y}{2}-\frac{y^3}{6}\Big]_{y=0}^{1}$$

$$= \frac{1}{2}-\frac{1}{6} = \frac{1}{3}$$

Evaluate the double integral.

1) $\iint_D x \cos y \, dA$, D is bounded by $y=0$, $y=x^2$, $x=1$



Let

$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x^2\}$

$\iint_D x \cos y \, dA$

$= \int_{x=0}^{1} \int_{y=0}^{x^2} x \cos y \, dy \, dx$

$= \int_{x=0}^{1} x \sin y \Big]_{y=0}^{x^2} dx$

$= \int_{x=0}^{1} x \sin x^2 \, dx$ put $x^2 = t$ $\Rightarrow 2x\,dx = dt$ | as $x \to 0$, $t \to 0$; as $x \to 1$, $t \to 1$

$= \int_{t=0}^{1} \frac{1}{2} \sin t \, dt$

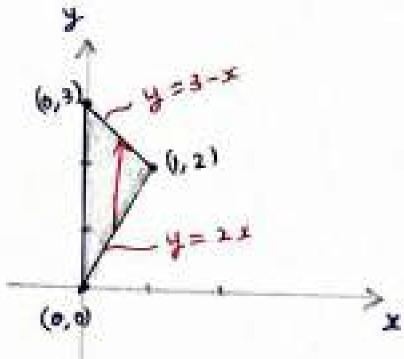
$= \frac{1}{2} \left(-\cos t\right)\Big|_{t=0}^{1} = -\frac{\cos 1}{2} + \frac{1}{2}$

ii) $\iint_D 2xy \, dA$, D is the Triangular region with vertices $(0,0)$, $(1,2)$, and $(0,3)$.

Soln: Eqn of the line joining the points $(0,0)$ and $(1,2)$ is

$$\frac{y-0}{x-0} = \frac{2-0}{1-0}$$

$$\Rightarrow \frac{y}{x} = 2 \quad \text{or} \quad y = 2x$$



Eqn of the line joining the points $(0,3)$ and $(1,2)$ is

$$\frac{y-3}{x-0} = \frac{2-3}{1-0}$$

$$\Rightarrow y-3 = -x$$

or $\quad y = 3-x$

Here $D = \{(x,y) \mid 0 \le x \le 1, \; 2x \le y \le 3-x\}$

$$\therefore \iint_D 2xy \, dA = \int_{x=0}^{1} \int_{y=2x}^{3-x} 2xy \, dy \, dx = \int_{x=0}^{1} xy^2 \Big]_{y=2x}^{3-x} dx$$

$$= \int_{x=0}^{1} \left[x(3-x)^2 - x(2x)^2\right] dx$$

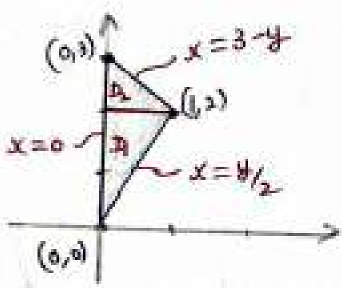
$$= \int_{x=0}^{1} \left(x\,(9+x^2-6x) - 4x^3\right) dx$$

$$= \int_{x=0}^{1} \left(9x - 6x^2 - 3x^3\right) dx$$

$$= 9\frac{x^2}{2} - \frac{6x^3}{3} - \frac{3x^4}{4}\Big]_{x=0}^{1} = \frac{9}{2} - 2 - \frac{3}{4} = \frac{7}{4}$$

Let us change the order and integrate.

To write D as Type 2 region, we have to divide D as follows.



$D = D_1 \cup D_2$, where

$$D_1 = \left\{ (x,y) \;\middle|\; 0 \le y \le 2,\; 0 \le x \le y/2 \right\}$$

$$D_2 = \left\{ (x,y) \;\middle|\; 2 \le y \le 3,\; 0 \le x \le 3-y \right\}$$

$$\therefore \iint_D 2xy \, dA = \iint_{D_1} 2xy\, dA + \iint_{D_2} 2xy\, dA$$

$$= \int_{y=0}^{2}\int_{x=0}^{y/2} 2xy\, dx\,dy + \int_{y=2}^{3}\int_{x=0}^{3-y} 2xy\, dx\, dy$$

$$= \int_{y=0}^{2} x^2 y\Big]_{x=0}^{y/2} + \int_{y=2}^{3} x^2 y\Big]_{x=0}^{3-y} dy$$

$$= \int_{y=0}^{2} \frac{y^3}{4}\, dy + \int_{y=2}^{3} (3-y)^2 y\, dy$$

$$= \frac{y^4}{4\times 4}\Bigg]_{y=0}^{2} + \int_{y=2}^{3} \left(9y + y^3 - 6y^2\right) dy$$

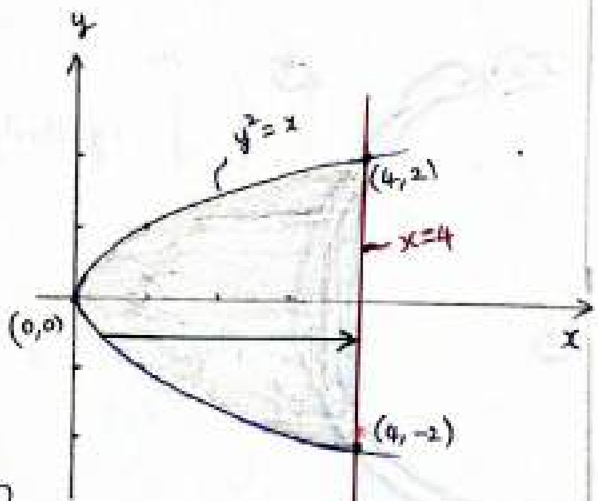
$$= 1 + \left[\frac{9y^2}{2} + \frac{y^4}{4} - \frac{6y^3}{3}\right]_{y=2}^{3}$$

$$= 1 + \frac{81}{2} + \frac{81}{4} - \frac{27\times 6}{3} - \frac{36}{2} - \frac{16}{4} + \frac{48}{3}$$

$$= 7/4$$

Find the volume of the solid under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$

Soln: Given $f(x,y) = 1 + x^2y^2$, D is the region enclosed by $x = y^2$ and $x = 4$.



At the point of intersection:

$$y^2 = 4$$

$$\Rightarrow y = \pm 2$$

$\therefore$ pts are $(4,2)$ and $(4,-2)$

$$D = \left\{ (x,y) \,\middle|\, -2 \le y \le 2,\ y^2 \le x \le 4 \right\}$$

The required volume

$$V = \iint_D f(x,y)\, dA$$

$$= \int_{y=-2}^{2} \int_{x=y^2}^{4} (1 + x^2y^2)\, dx\, dy$$

$$= \int_{y=-2}^{2} x + \frac{x^3}{3}y^2 \Big]_{x=y^2}^{4} dy$$

$$= \int_{y=-2}^{2} \left[4 + \frac{4^3y^2}{3} - \left(y^2 + \frac{y^8}{3}\right)\right] dy$$

$$= \int_{y=-2}^{2} \left[4 + \frac{61y^2}{3} - \frac{y^8}{3}\right] dy$$

$$= 4y + \frac{61}{9}y^3 - \frac{y^9}{27}\Big]_{y=-2}^{y=2}$$

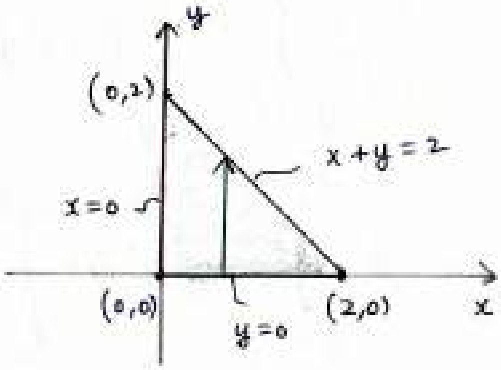
$$= 4(2) + \frac{61\times 8}{9} - \frac{512}{27} - \left(-4(2) - \frac{61\times 8}{9} + \frac{512}{27}\right)$$

$$= \frac{1168}{27} + \frac{1168}{27} = \frac{2336}{27}$$

Find the volume of the solid enclosed by the paraboloid $z = x^2+y^2+1$ and the planes $x=0$, $y=0$, $z=0$, and $x+y=2$.

Let D be the region bounded by the planes $x=0, y=0, z=0,$ and $x+y=2$.

$$D = \{ (x,y) \mid 0 \le x \le 2,\ 0 \le y \le 2-x \}$$



The required volume

$$V = \iint_D (x^2+y^2+1)\, dA$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} (x^2+y^2+1)\, dy\, dx$$

$$= \int_{x=0}^{2} \left(x^2 y + \frac{y^3}{3} + y \right) \Bigg|_{y=0}^{2-x} dx$$

$$= \int_{x=0}^{2} \left[x^2(2-x) + \frac{(2-x)^3}{3} + (2-x) \right] dx$$

$$= \int_{x=0}^{2} \left[2x^2 - x^3 + \frac{8}{3} - \frac{x^3}{3} - \frac{12x}{3} + \frac{6x^2}{3} + 2 - x \right] dx$$

$$= \int_{x=0}^{2} \left(4x^2 - \frac{4x^3}{3} - 5x + \frac{14}{3} \right) dx$$

$$= 4\frac{x^3}{3} - \frac{\cancel{4}x^4}{\cancel{4}\times 3} - 5\frac{x^2}{2} + \frac{14x}{3}\Big]_{x=0}^{2}$$

$$= \frac{4\times 8}{3} - \frac{16}{3} - \frac{20}{2} + \frac{28}{3}$$

$$= \frac{32-16+28}{3} - 10 = \frac{44}{3} - 10 = \frac{14}{3}$$

Evaluate the integral by reversing the order of integration

i) $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$

Soln: Given $\int_{y=0}^{1} \int_{x=3y}^{3} e^{x^2}\,dx\,dy = \iint_D e^{x^2}\,dA$

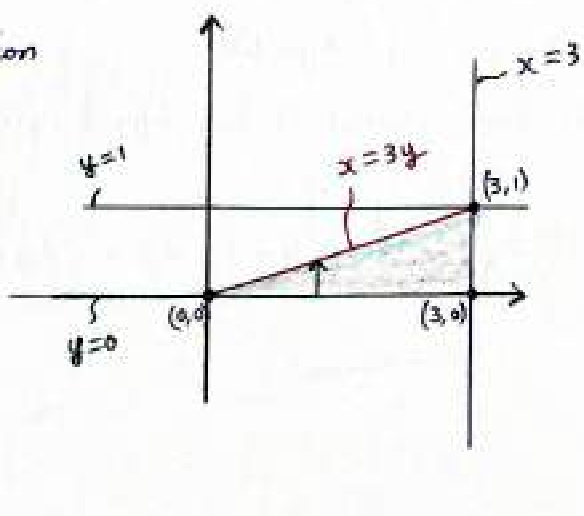
where

$$D = \{(x,y) \mid 0 \le y \le 1,\ 3y \le x \le 3\}$$

This is type II region.

Let us intepete D as type I region

$$D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le \tfrac{x}{3}\}$$



Therefore,

$$\iint_D e^{x^2}\,dA = \int_{x=0}^{3} \int_{y=0}^{x/3} e^{x^2}\,dy\,dx$$

$$= \int_{x=0}^{3} y e^{x^2} \Big]_{y=0}^{x/3} dx$$

$$= \int_{x=0}^{3} \frac{x e^{x^2}}{3} dx$$

put $x^2 = t$ $\Rightarrow 2x\,dx = dt$

as $x \to 0, t \to 0$
as $x \to 3, t \to 9$

$$= \int_{t=0}^{9} \frac{e^t}{3} \frac{dt}{2} = \frac{1}{6} e^t \Big]_{t=0}^{9} = \frac{e^9 - 1}{6}$$

ii) $\displaystyle\int_0^1 \int_{e^x}^{e} \frac{1}{\log y}\, dy\, dx$

Soln: Given

$$\int_{x=0}^{1} \int_{y=e^x}^{e} \frac{1}{\log y}\, dy\, dx = \iint_D \frac{1}{\log y}\, dA$$

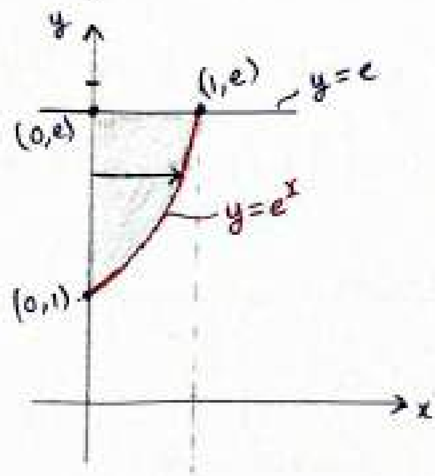
where

$$D = \{(x,y) \mid 0 \le x \le 1,\ e^x \le y \le e\}.$$

This is of type I.

We can interpret D as type II as follows:

$$D = \{(x,y) \mid 1 \le y \le e,\ 0 \le x \le \log y\}$$



Therefore,

$$\iint_D \frac{1}{\log y}\, dA = \int_{y=1}^{e} \int_{x=0}^{\log y} \frac{1}{\log y}\, dx\, dy$$

$$= \int_{y=1}^{e} \frac{x}{\log y} \Big|_{x=0}^{\log y} dy$$

$$= \int_{y=1}^{e} \left(\frac{\log y}{\log y} - 0 \right) dy$$

$$= \int_{y=1}^{e} dy = y \Big]_{y=1}^{e} = e - 1$$

Exercises

1) Evaluate the iterated integral.

i) $\displaystyle\int_1^5 \int_0^x (8x - 2y)\, dy\, dx$ $\left(\text{Ans } \frac{868}{3}\right)$

ii) $\displaystyle\int_0^{\pi/2} \int_0^x x \sin y\, dy\, dx$ $\left(\text{Ans. } \frac{\pi^2}{8} - \frac{\pi}{2} + 1\right)$

2) Evaluate the double integral.

i) $\displaystyle\iint_D \frac{y}{x^2+1}\, dA, \quad D = \left\{ (x,y) \,\middle|\, 0 \le x \le 4,\ 0 \le y \le \sqrt{x} \right\}$ $\left(\text{Ans } \frac{\log 17}{4}\right)$

ii) $\displaystyle\iint_D e^{-y^2}\, dA, \quad D = \left\{ (x,y) \,\middle|\, 0 \le y \le 3,\ 0 \le x \le y \right\}$ $\left(\text{Ans } \frac{1 - e^{-9}}{2}\right)$

3) Evaluate the double integral.

i) $\iint_D xy^2\,dA$, D is enclosed by $x=0$ and $x=\sqrt{1-y^2}$

(Ans $\frac{2}{15}$)

ii) $\iint_D y^2\,dA$, D is the Triangular region with vertices $(0,1)$, $(1,2)$, $(4,1)$.

(Ans $\frac{11}{3}$)

4) Find the volume of the given solid.

i) Under the plane $x-2y+z=1$ and above the region bounded by $x+y=1$ and $x^2+y=1$. (Ans $\frac{17}{60}$)

ii) Enclosed by the paraboloid $z=x^2+3y^2$ and the planes $x=0$, $y=1$, $y=x$, $z=0$. (Ans $\frac{5}{6}$)

5) Evaluate the integral by reversing the order of integration.

i) $\displaystyle\int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}} \cos(x^2)\,dx\,dy$ (Ans 0)

ii) $\displaystyle\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ (Ans. $\frac{3}{8}$)

iii) $\displaystyle\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$.

(Ans. $1-\frac{1}{\sqrt{2}}$)

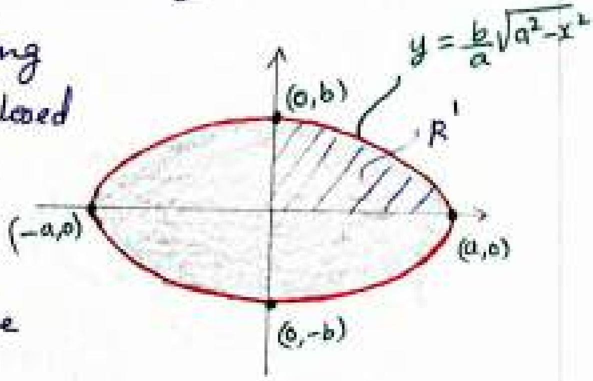
Area and double integrals

Let R be a bounded region in the plane and A be the area enclosed by R. Then

$$\boxed{A = \iint_R dA}$$

Ex 1: Find the area of a ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Soln: Since area is symmetric along coordinate axis, we find area enclosed in first quadrant and multiple it by 4.



Let A' be the area enclosed in the 1st quadrant. Then

$$A = 4A'$$

$$= 4\iint_{R'} dA = 4\int_{x=0}^{a}\int_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx = 4\int_{x=0}^{a} \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$\Rightarrow A = 4\int_{\theta=\pi/2}^{0} \frac{b}{a}\, a\sin\theta\,(-a\sin\theta)\,d\theta$$

$$= 4ab\int_{\theta=0}^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right) = 4ab\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\theta=0}^{\pi/2}$$

$$= \pi ab$$

put $x = a\cos\theta$

$dx = -a\sin\theta\, d\theta$

as $x \to 0$

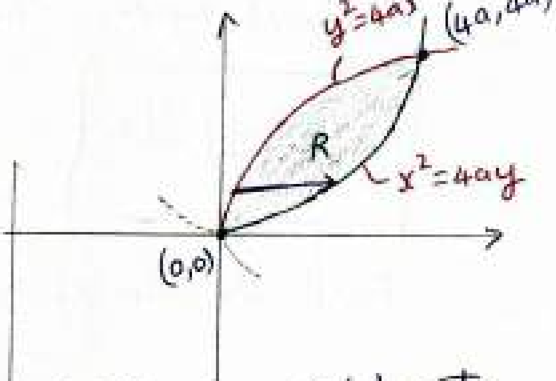
$\theta \to \pi/2$

as $x \to a$

$\theta \to 0$

Ex 2: Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16a^2}{3}$

Soln: Area enclosed



$$A = \iint_R dA$$

$$\int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{x=\sqrt{4ay}} dx\, dy$$

$$= \int_{y=0}^{4a} \left[\sqrt{4ay} - \frac{y^2}{4a}\right] dy$$

$$= \sqrt{4a}\, \frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\Big]_0^{4a}$$

$$= \sqrt{4a} \cdot \frac{2}{3}\, a^{3/2}\, 8 - \frac{4^3 a^3}{12a}$$

$$= \frac{16 \times 2}{3} a^2 - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}$$

At the pts of intersection

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\Rightarrow \frac{x^4}{16a^2} = 4ax$$

$$\Rightarrow x\left(\frac{x^3}{16a^2} - 4a\right) = 0$$

$$\Rightarrow x = 0 \text{ or } x^3 = 4^3 a^3$$

$$\Rightarrow x = 4a$$

Pts are

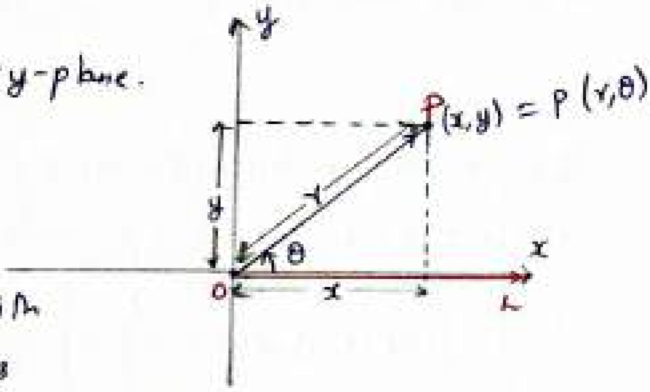
(0,0) and (4a, 4a)

Double integrals in Polar Coordinates

Let $P(x,y)$ be a point in the xy-plane.

Let O – pole

OL – initial line.



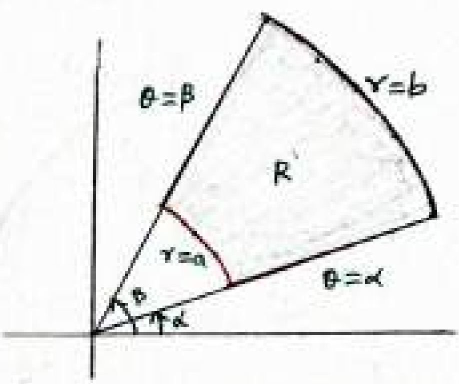
The location of the point P with reference to polar coordinates is (r,θ), where r is called radius vector, θ is called vectorial angle.

The transformation from cartesian coordinates to polar is given by:

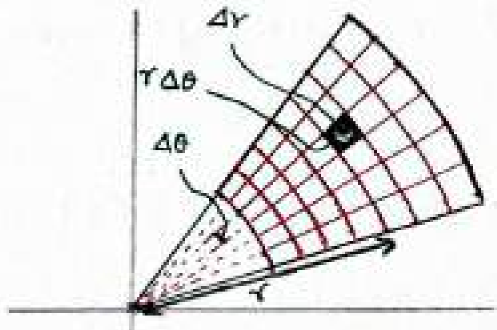
$x = r\cos\theta$ $y = r\sin\theta$	or	$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Consider the polar rectangle (as shown in below figure)

$$R = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$$



Polar rectangle



Dividing R into polar subrectangle.

If shaded is an area element ΔA,

then $\Delta A \simeq r\,\Delta r\,\Delta\theta$

As Δr and $\Delta\theta \to 0$, $dA = r\,dr\,d\theta$.

Thus,

If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.}$$

Ex 1: Evaluate the volume of the solid bounded by the plane $z=0$ and the paraboloid $z = 1-x^2-y^2$

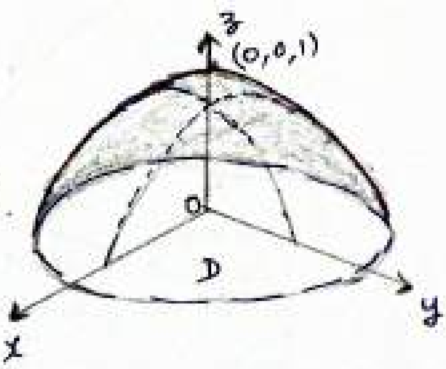
Soln: put $z=0$ in the Equation of the paraboloid $z = 1-x^2-y^2$;

we get $x^2+y^2=1$.

This means that the xy-plane intersect paraboloid in the circle $x^2+y^2=1$.

so the volume of the solid lies under the paraboloid and above the circular disk $D = \{(x,y) \mid x^2+y^2 \le 1\}$

put $x = r\cos\theta$, $y = r\sin\theta$.



$1-x^2-y^2 = 1-r^2$

and $D = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

The volume is

$$V = \iint_D (1-x^2-y^2)\,dA$$

$$= \int_0^{2\pi} \int_0^1 (1-r^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\, dr$$

$$= \theta\Big]_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1$$

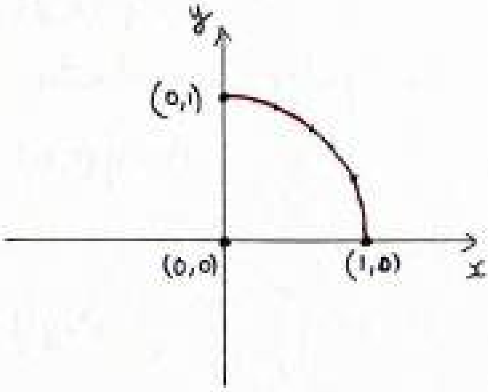
$$= 2\pi \left(\frac{1}{2} - \frac{1}{4}\right) = \pi/2$$

Ex 2: Evaluate

$$\int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} \sin(x^2+y^2)\, dx\, dy$$

Here the region

$$R = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le \sqrt{1-y^2}\}$$



Suitable transformation is

$x = r\cos\theta$

$y = r\sin\theta$,

Then

$$\sin(x^2+y^2) = \sin r^2$$

$$R = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le \pi/2\}.$$

Thus

$$\int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} \sin(x^2+y^2)\, dx\, dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \sin(r^2)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} -\frac{\cos(r^2)}{2}\Big]_{r=0}^{1} d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left(-\frac{\cos 1}{2} + \frac{1}{2}\right) d\theta$$

$$= \frac{1}{2}(1-\cos 1)\ \theta \Big]_0^{\pi/2}$$

$$= \frac{\pi}{4}(1-\cos 1)$$

Ex 3: Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ by changing to polar coordinates. Hence show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

Soln: The region of integration is 1st quadrant of the xy-plane, That is

$$R = \{(x,y) \mid 0 \le x < \infty,\ 0 \le y < \infty\}.$$

In polar coordinates,

$$R = \{(r,\theta) \mid 0 \le \theta \le \pi/2,\ 0 \le r < \infty\}$$

Therefore,

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{-e^{-r^2}}{2}\Big]_{r=0}^{\infty} d\theta$$

$$= \int_0^{\pi/2} (0 + 1/2)\, d\theta = \pi/4$$

Note that: If $f(x,y) = g(x).h(y)$, then

$$\int_c^d \int_a^b f(x,y)\, dx\, dy = \int_c^d h(y)\, dy \cdot \int_a^b g(x)\, dx.$$

Thus,

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = \int_0^\infty e^{-x^2}\, dx \cdot \int_0^\infty e^{-y^2}\, dy$$

$$= \left(\int_0^\infty e^{-x^2}\, dx \right)^2 \quad \left(\begin{array}{l} x \text{ and } y \text{ are} \\ \text{dummy variables} \end{array} \right.$$

$$= \pi/2^2$$

$$\Rightarrow \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

Exercise:

1) Use polar coordinates to find the volume of the solid under the cone $z = \sqrt{x^2+y^2}$ and above the disk $x^2+y^2 \leq 4$. $\left(\text{Ans. } \frac{16\pi}{3}\right)$

2) Evaluate the iterated integral by converting to polar coordinates

a) $\displaystyle\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\, dx\, dy$ $\left(\text{Ans. } \frac{2\sqrt{2}}{3}\right)$

b) $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx$ $\left(\text{Ans } \frac{\pi}{4}(1-e^{-4})\right)$

Change of variables in Double integrals

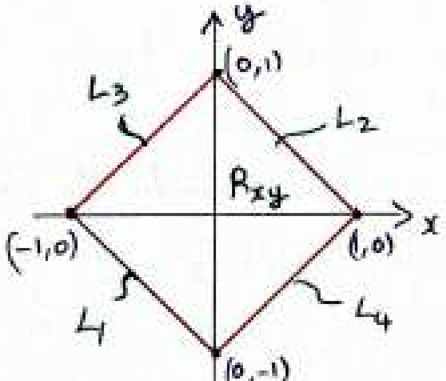
Suppose T is a Transformation from the xy-plane to uv-plane defined by

$$u = u(x,y), \quad v = v(x,y), \quad x = x(u,v), \quad y = y(u,v)$$

If it maps the region R_{xy} in the xy-plane to R_{uv} in the uv-plane and $J = \dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$, then

$$\boxed{\iint_{R_{xy}} f(x,y)\, dA = \iint_{R_{uv}} f\big(x(u,v), y(u,v)\big) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv}$$

Ex 1 Evaluate $\displaystyle\iint_R \left(\frac{x-y}{x+y-2}\right)^2 dx\, dy$ over the region R pictured



Soln: Integrand and Equation of lines suggest that integral will be simplified if we change the variables

$$u = x + y, \quad v = x - y$$

$$\Rightarrow x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}$$

Equation of the line

L_1: $x + y = -1$

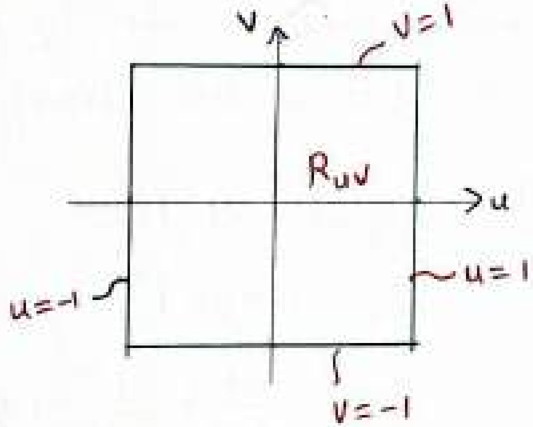
L_2: $x + y = 1$

L_3: $x - y = -1$

L_4: $x - y = 1$

Now,

$$R_{uv} = \{ (u,v) \mid -1 \le u \le 1, \quad -1 \le v \le 1 \}$$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix}$$

$$= -1/2$$

Thus

$$\iint_{R_{xy}} \left(\frac{x-y}{x+y-2}\right)^2 dx\,dy = \iint_{R_{uv}} \left(\frac{v}{u-2}\right)^2 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_{v=-1}^{1} \int_{u=-1}^{1} \left(\frac{v}{u-2}\right)^2 \tfrac{1}{2}\, du\,dv$$

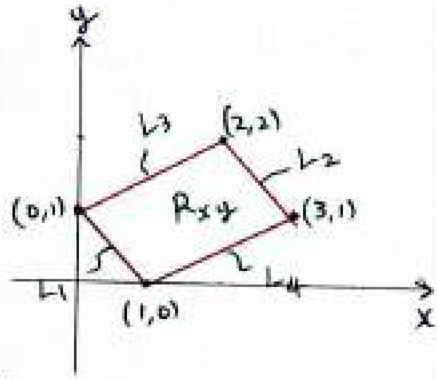
$$= \frac{1}{2} \int_{u=-1}^{1} \left(\frac{1}{u-2}\right)^2 du \cdot \int_{v=-1}^{1} v^2\,dv$$

$$= \frac{1}{2} \left(\frac{-1}{u-2}\right)\Bigg|_{u=-1}^{1} \cdot \frac{v^3}{3}\Bigg|_{v=-1}^{1}$$

$$= \frac{-1}{2}\left(-1 + \frac{1}{3}\right) \cdot \left(\frac{1}{3} + \frac{1}{3}\right)$$

$$= \frac{2}{9}$$

Ex 2 Evaluate $\iint_R (x+y)^2\,dx\,dy$, where R is the parallelogram in the xy-plane with vertices $(1,0)$, $(3,1)$, $(2,2)$, $(0,1)$. Use suitable transformation.



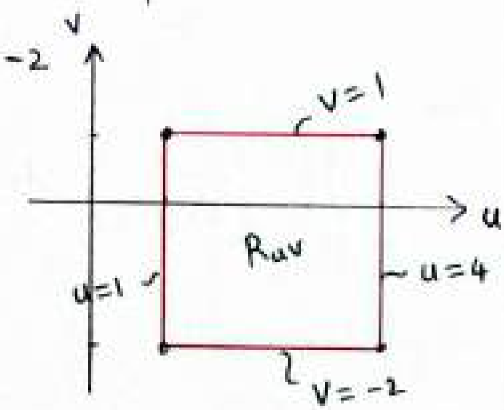
Soln: Equation of line

$L_1: \; x+y=1$

$L_2: \; \dfrac{y-1}{x-3} = \dfrac{1}{-1} \Rightarrow x+y=4$

$L_3: \; \dfrac{y-2}{x-2} = \dfrac{-1}{-2} \Rightarrow x-2y=-2$

$L_4: \; \dfrac{y-1}{x-3} = \dfrac{-1}{-2} \Rightarrow x-2y=1$



Suitable transformation is

$u = x+y, \quad v = x-2y$

Given the region in xy-plane is a parallelogram and the corresponding region in the uv-plane R_{uv} is a rectangle

$$R_{uv} = \{(u,v) \mid 1 \le u \le 4, \; -2 \le v \le 1\}.$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = -\tfrac{1}{3}$$

Thus,

$$\iint_{R_{xy}} (x+y)^2\, dx\, dy = \iint_{R_{uv}} u^2\, |J|\, du\, dv$$

$$= \int_{v=-2}^{1} \int_{u=1}^{4} u^2 \cdot \frac{1}{3}\, du\, dv$$

$$= \int_{v=-2}^{1} \left. \frac{u^3}{3} \cdot \frac{1}{3} \right]_{u=1}^{4} dv$$

$$= \left(\frac{64}{9} - \frac{1}{9}\right) \int_{v=-2}^{1} dv$$

$$= \frac{63}{9} \cdot 3 = 21$$

Ex 3 Evaluate $\iint_R \frac{x-2y}{3x-y}\, dA$, where R is the parallelogram enclosed by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$, and $3x-y=8$. Use appropriate change of variables.

Soln: put $u = x-2y$
$v = 3x-y$

Thus

$$\iint_{R_{xy}} \frac{x-2y}{3x-y}\, dA = \iint_{R_{uv}} \frac{u}{v}\, |J|\, dA'$$

where $R_{uv} = \{(u,v) \mid 0 \le u \le 4,\ 1 \le v \le 8\}$

and

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}} = \frac{1}{5}$$

Thus

$$\iint_{R_{xy}} \frac{x-2y}{3x-y}\, dA = \int_{v=1}^{8}\int_{u=0}^{4} \frac{u}{v}\cdot\frac{1}{5}\, du\, dv$$

$$= \frac{1}{5}\int_{v=1}^{8} \frac{1}{v}\, dv \cdot \int_{u=0}^{4} u\, du$$

$$= \frac{1}{5} \log v \Big|_{v=1}^{8} \cdot \frac{u^2}{2}\Big]_{u=0}^{4}$$

$$= \frac{1}{5}\left(\log 8 - \log 1\right)\cdot 8 = \frac{8}{5}\log 8$$

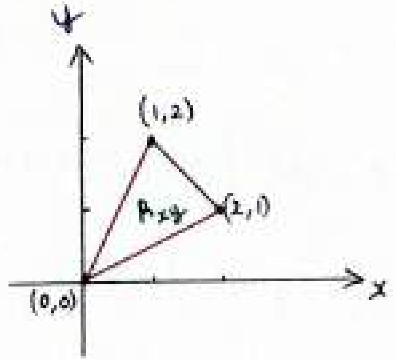
Ex 4: Evaluate $\iint_R (x-3y)\, dA$, where R is the triangular region with vertices $(0,0)$, $(2,1)$, and $(1,2)$. Use the transformation $x = 2u+v$, $y = u+2v$.

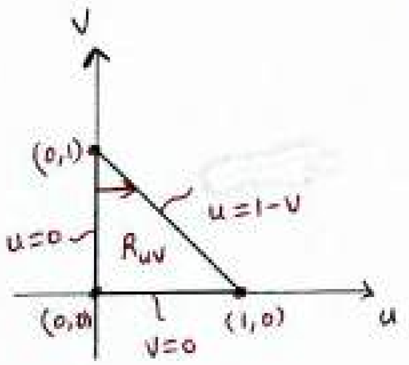
Soln: Given transformation $x = 2u+v$, $y = u+2v$ is linear, it transforms triangle to triangle.

Let us find corresponding three vertices in uv-plane

(x,y)	$(0,0)$	$(2,1)$	$(1,2)$
(u,v)	$(0,0)$	$(1,0)$	$(0,1)$

(put $x=2$, $y=1$ in the transformation $2 = 2u+v$
$1 = u+2v$
solving, $u=1$
$v=0$)





and $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

Thus,

$$\iint_{R_{xy}} (x-3y)\, dA = \iint_{R_{uv}} \left((2u+v) - 3(u+2v)\right) |J|\, dA'$$

$$= \int_{v=0}^{1} \int_{u=0}^{1-v} (-u-5v)\, 3\, du\, dv$$

$$= \int_{v=0}^{1} -\frac{u^2}{2} - 5vu \Big]_{u=0}^{1-v} 3\, dv$$

$$= \int_{v=0}^{1} \left(-\frac{(1-v)^2}{2} - 5v(1-v)\right) 3\, dv$$

$$= \int_{v=0}^{1} (9v^2 - 8v - 1)\frac{3}{2}\, dv$$

$$= \frac{3}{2}\left(3v^3 - 4v^2 - v\right)\Big|_{v=0}^{1} = -3$$

Exercise

1) Evaluate $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$, where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$ and $(0,1)$. Use appropriate change of variables. $\left(\text{Ans: } \frac{3}{2}\sin 1\right)$

2) Evaluate $\iint_R x^2\, dA$, where R is the region bounded by the ellipse $9x^2+4y^2=36$. By using the Transformation $x=2u$, $y=3v$. $(\text{Ans. } 6\pi)$

3) Evaluate $\iint_R xy\, dA$, where R is the region in the first quadrant bounded by the lines $y=x$ and $y=3x$ and the hyperbolas $xy=1$, $xy=3$. By using the transformation $x=u/v$, $y=v$ $(\text{Ans. } 2\log 3)$