Double integral over rectangles

Intended Learning Outcomes

At the end of the lecture, student will be able to:

- Explain the concept of double integrals over rectangles
- Evaluate integral of functions of two variables over rectangles

Double integrals

- In this and later sessions, we extend the idea of a definite integral to double integrals of functions of two variables.
- These idea are then used to compute volumes, masses and centroids of more general regions.
- We also use double integrals to calculate probabilities when two random variables are involved.
- We will see that polar coordinates are useful in computing double integrals over some types of regions.

Double integrals

- Our attempt to solve the area problem led to the definition of a definite integral.

- We now seek to find the volume of a solid. In the process, we arrive at the definition of a double integral.

Volumes and Double integrals

Consider a function f of two variables defined on a closed rectangle

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$$

and we suppose that $f(x,y) \geq 0$.

- The graph of f is a surface with equation $z = f(x,y)$.
- Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq f(x,y),\ (x,y) \in R\}$$

- Our goal is to find the volume of S (see Figure 1).

Double integrals

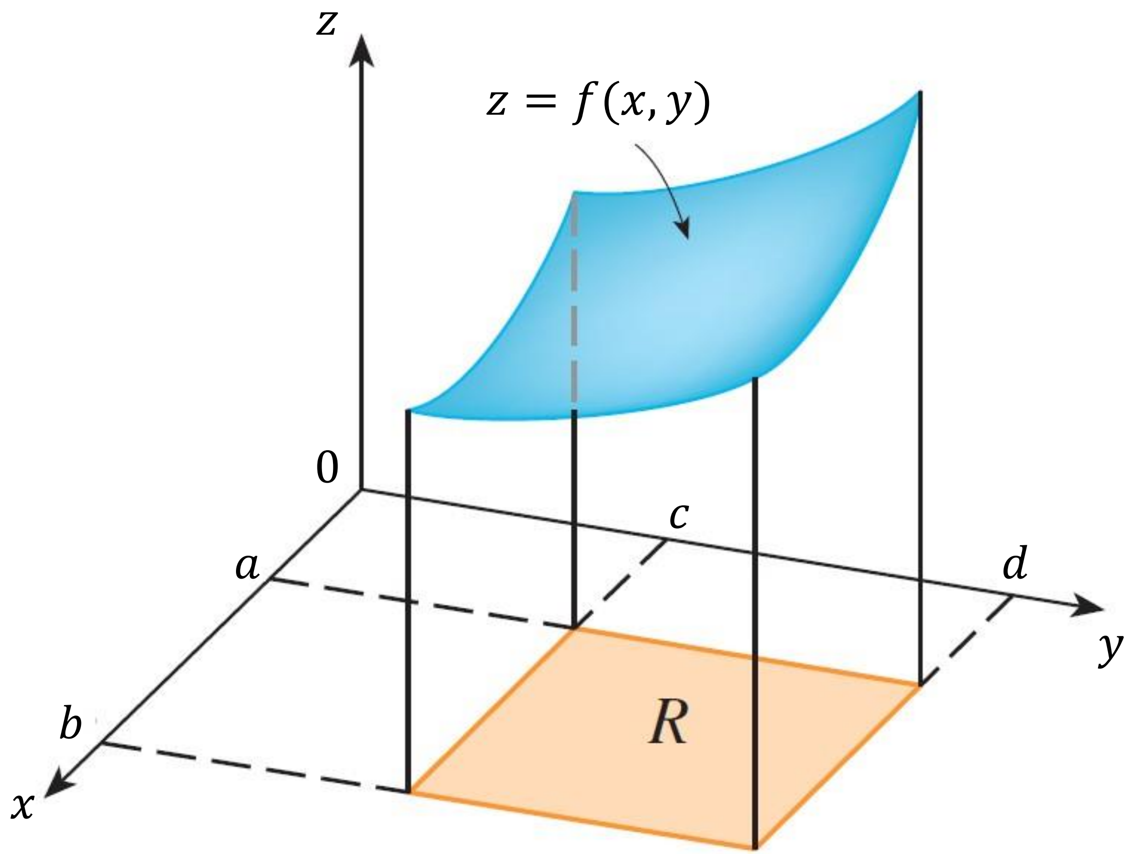

Figure 1

Volumes and Double integrals

- The first step is to divide the rectangle R into subrectangles.
- This is accomplished by dividing the intervals $[a, b]$ and $[c, d]$ into m and n equal subintervals, respectively, such that

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_m = b$$

$$c = y_0 < y_1 < \cdots < y_{j-1} < y_j < \cdots < y_n = d$$

- Thus, we form subrectangles

$$\begin{aligned} R_{ij} &= [x_{i-1}, x_i] \times [y_{j-1}, y_j] \\ &= \{(x, y) \mid x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\} \end{aligned}$$

from $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

- There are mn of these subrectangles, and they cover R.

Volumes and Double integrals

- Let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$. Then the area of R_{ij} is $\Delta A_{ij} = \Delta x_i \Delta y_i$

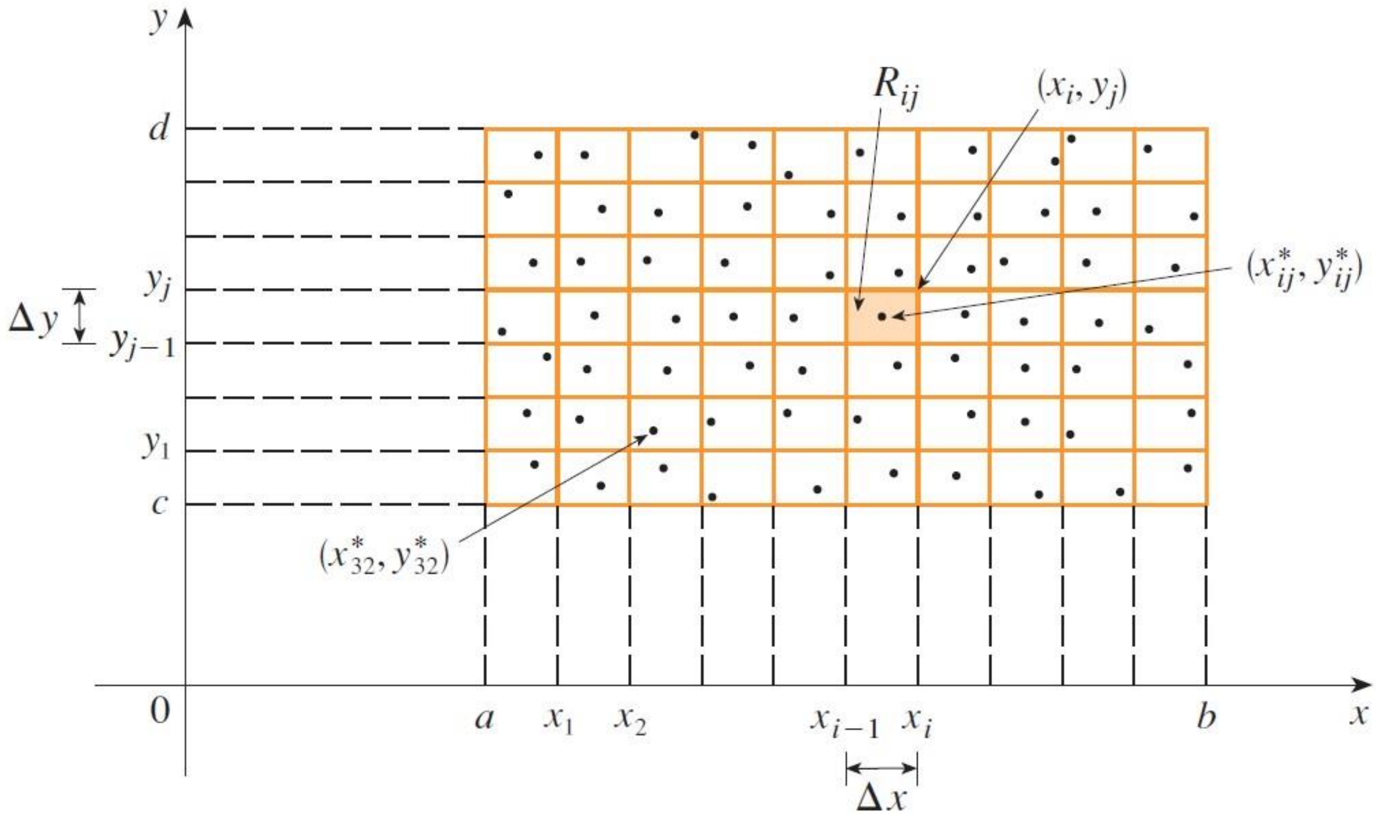

Figure 2

Volumes and Double integrals

- Let's choose a sample point $(x_{ij}^{*},\ y_{ij}^{*})$ in each R_{ij}.
- We can approximate the part of S that lies above each R_{ij} by a thin rectangle box with:
  - — Base R_{ij}
  - — Height $f(x_{ij}^{*},\ y_{ij}^{*})$
  - — See Figure 3.
- The volume of this box is the height of the box times the area of the rectangle:

$$f\left(x_{ij}^{*}, y_{ij}^{*}\right) \Delta A$$

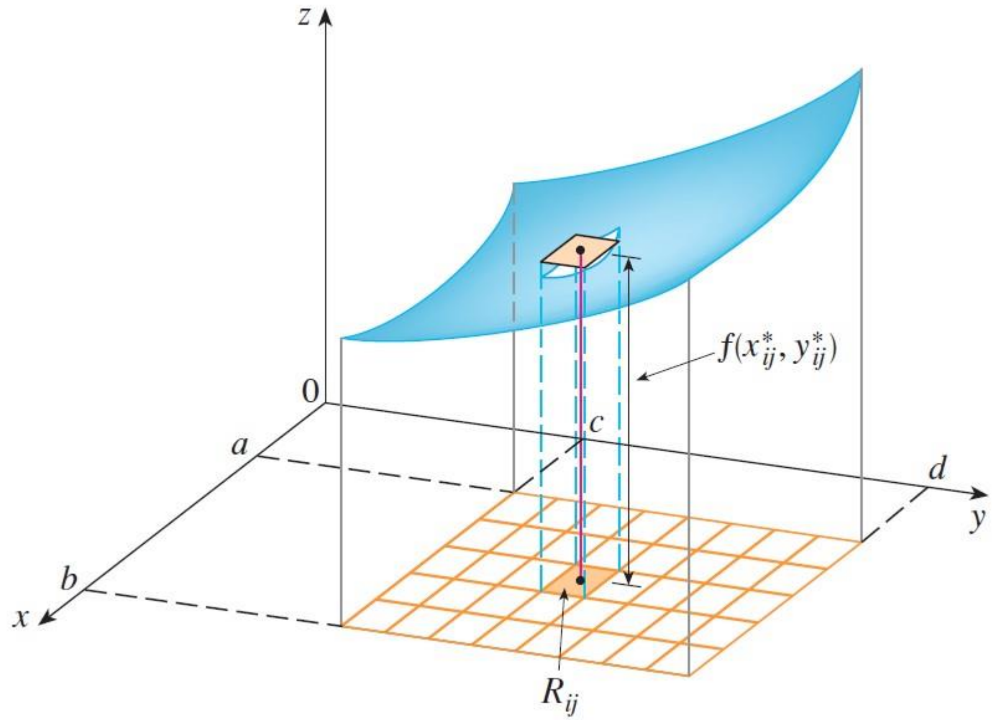

Figure 3

Volumes and Double integrals

- We follow this procedure for all the rectangles and add the volumes of the corresponding boxes.
- Thus, we get an approximation to the total volume of S:

$$V = \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^{*}, y_{ij}^{*}\right) \Delta A$$

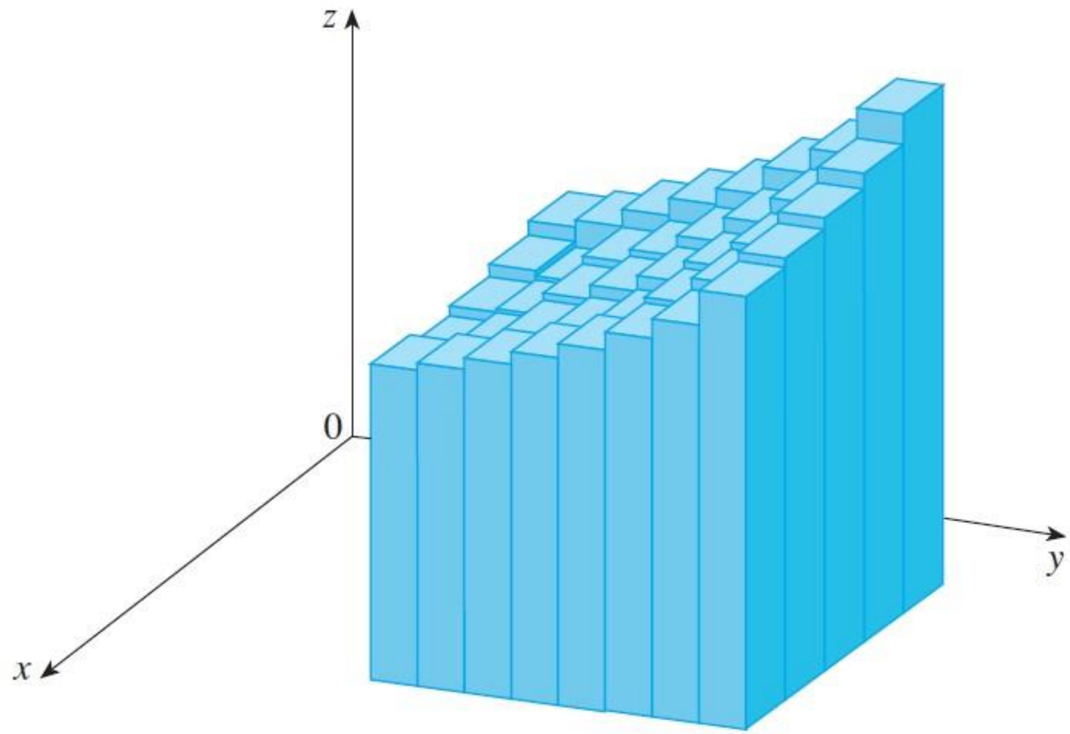

Figure 4

Volumes and Double integrals

- The approximation to the total volume of S becomes better as m and n become larger (length of subintervals become smaller), So,

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A.$$

- Above expression defines the volume of the solid S that lies between the surface $z = f(x, y)$ and the rectangle R.

- So we make the following definition.

Definition The double integral of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A.$$

If limit exists.

Iterated Integrals

- Evaluation of the double integral using first principle is difficult.
- Evaluation become simple by expressing double integral as an iterated integrals.
- Let $f(x, y)$ be a function defined on $R = [a, b] \times [c, d]$.
  - Integral $\int_c^d f(x, y)dy$ means that x is held fixed and f is integrated with respect to y from $y = c$ to $y = d$ (is called partial integration of f with respect to y).
  - Integral in the previous step gives the area $A(x)$ bounded by the curve, $\mathrm{z} = f(x, y)$, where x is held constant and $c \leq y \leq d$, that is,

$$A(x) = \int_c^d f(x, y)\, dy.$$

Iterated Integrals

- Integration of $A(x)$ with respect to x from $x = a$ to $x = b$ gives the volume V of the solid S that lies above R and under the surface $z = f(x, y)$, that is,

$$\iint_R f(x,y)dA = V = \int_a^b A(x)\,dx = \int_a^b \left(\int_c^d f(x,y)dy \right) dx.$$

- Extreme right side of last equation is called iterated integrals, usually brackets are omitted. Thus

$$\int_a^b \int_c^d f(x,y)dy\,dx = \int_a^b \left(\int_c^d f(x,y)dy \right) dx$$

means that we first integrate with respect to y from c to d and then with respect x from a to b.

- Similar argument shows that

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dx\,dy.$$

Iterated Integrals

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

Fubini's Theorem If f is continuous on the rectangle

$R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy\, dx = \int_c^d \int_a^b f(x, y) dx\, dy.$$

This is true if f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example

Example 2 Evaluate the double integral $\iint_R (x - 3y^2)\,dA$, where $R = \{(x, y) \mid 0 \le x \le 2, \quad 1 \le y \le 2\}$.

Solution 1 Fubini's Theorem gives

$$\iint_R (x - 3y^2)\,dA = \int_0^2 \int_1^2 (x - 3y^2)\,dy\,dx$$

$$= \int_0^2 \left[xy - y^3\right]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\,dx = \left[\frac{x^2}{2} - 7x\right]_0^2 = -12$$

Example…

Solution 2 Again applying Fubini's Theorem, but this time integrating with respect to x first, we have

$$\iint_R (x - 3y^2)\,dA = \int_1^2 \int_0^2 (x - 3y^2)\,dx\,dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy$$

$$= \int_1^2 (2 - 6y^2)\,dy$$

$$= 2y - 2y^3\Big]_1^2 = -12.$$

Example…

Example 3 Evaluate $\iint_R y\sin(xy)\;dA$, where $R=[1,2]\times[0,\pi]$.

Solution We have

$$\iint_R y\sin(xy)\,dA=\int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy$$

$$=\int_0^{\pi}[-\cos(xy)]_{x=1}^{x=2}\,dy$$

$$=\int_0^{\pi}(-\cos 2y+\cos y)\,dy$$

$$=\left[-\frac{1}{2}\sin 2y+\sin y\right]_0^{\pi}=0.$$

Summary

Let $f(x, y)$ be a function defined on $R = [a, b] \times [c, d]$.

1. The volume V of the solid that lies above R and under the graph of f is
$$V = \iint_R f(x, y) dA$$
2. Double integral in the right side of last expression can be evaluated as an iterated integrals, that is,
$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx\, dy.$$
3. Right side of last equation means that we first integrate with respect to x from a to b and then with respect y from c to d.
4. Moreover,
$$\int_c^d \int_a^b f(x, y) dx\, dy = \int_a^b \int_c^d f(x, y) dy\, dx\,.$$

Double integral over general regions

Intended Learning Outcomes

At the end of the lecture, student will be able to:

- Explain the concept of double integrals over general regions
- Evaluate integral of functions of two variables over general regions

Topics

- Double integrals over general regions
- Examples

Double Integrals

- For single integrals, the region over which we integrate is always an interval.
- For double integrals, we want to be able to integrate a function f not just over rectangles but also over regions D of more general shape.
  - One such shape is illustrated in Figure 1

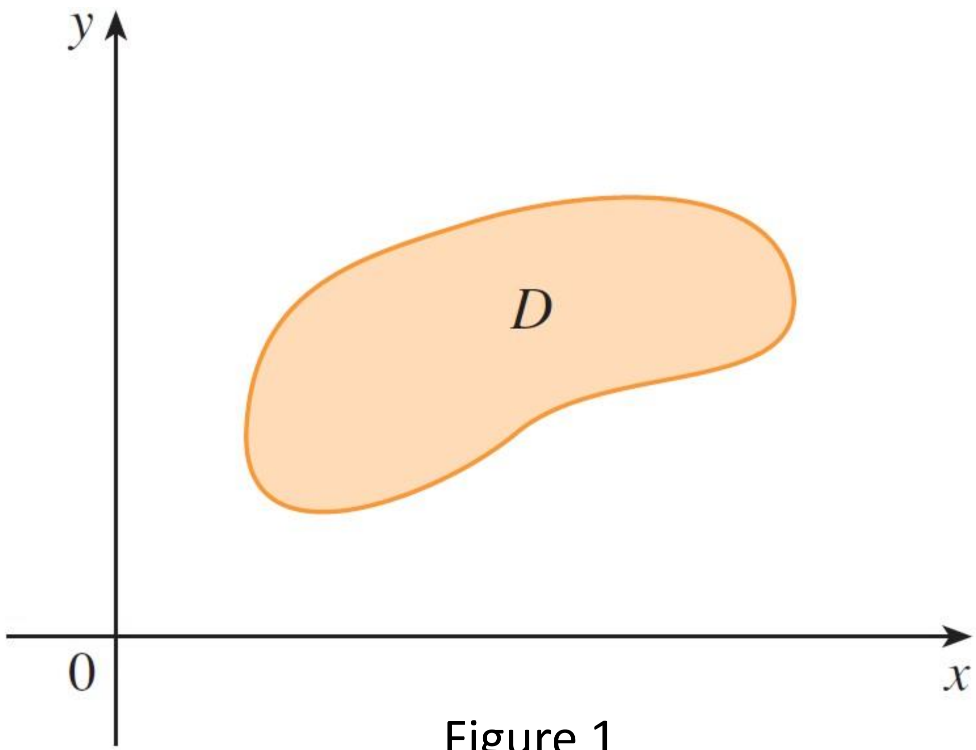

Figure 1

Double Integrals – Type 1 Region

- A plane region D is said to be of type 1, if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$.

- Some examples of type 1 regions are shown Figure 5.

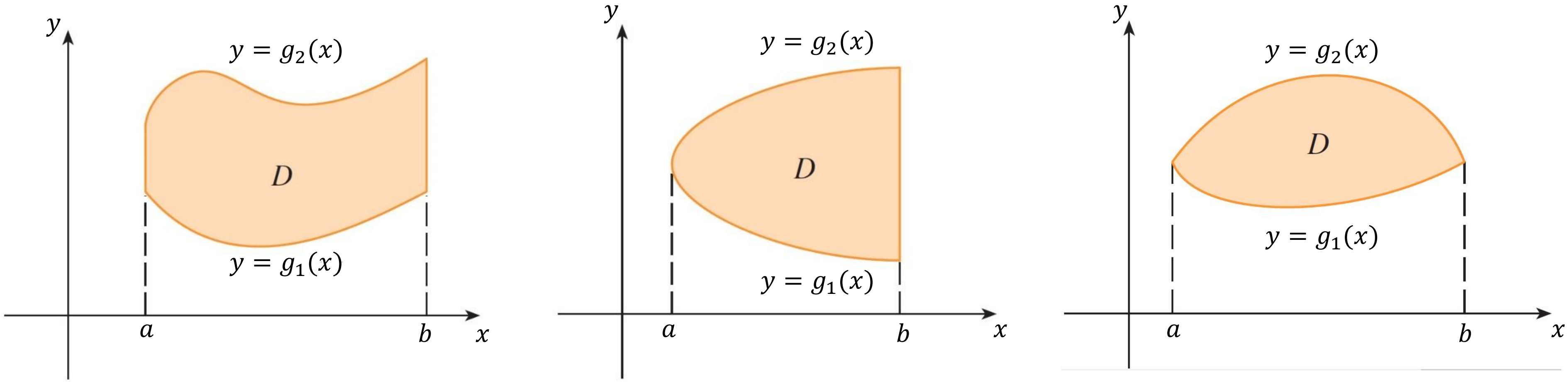

Figure 5

Double Integrals – Type 2 Region

- A plane region D is said to be of type 2, if it can be expressed as:

$$D = \{(x, y) \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\} \ldots\ldots (4)$$

where h_1 and h_2 are continuous.

- Two such regions are illustrated in Figure 7

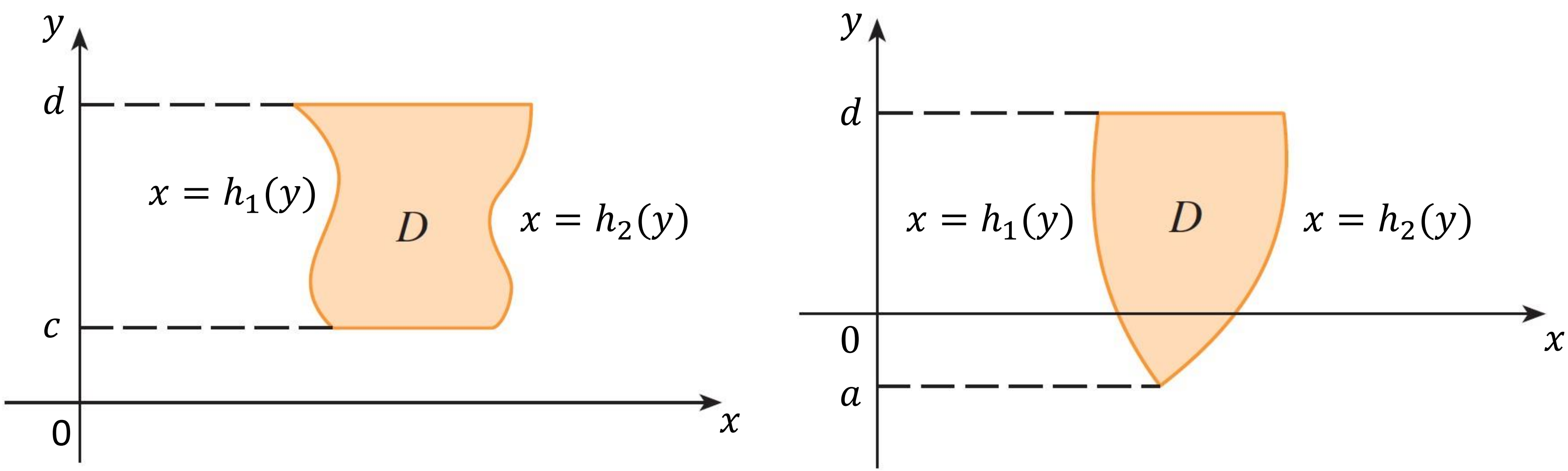

Figure 7

Double Integrals – Type 2 Region

- Using the same methods that were used in establishing Equation 3, we can show that:

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy \ldots\ldots (5)$$

where D is a type 2 region given by Equation (4).

Example

Example 1 Evaluate

$$\iint_D (x + 2y)\, dA$$

where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$. Thus,

$$x = \pm 1.$$

- We note that the region D is a type 1 region.
- So, we can write

$$D = \{(x, y) \mid -1 \leq x \leq 1,\ 2x^2 \leq y \leq 1 + x^2\}$$

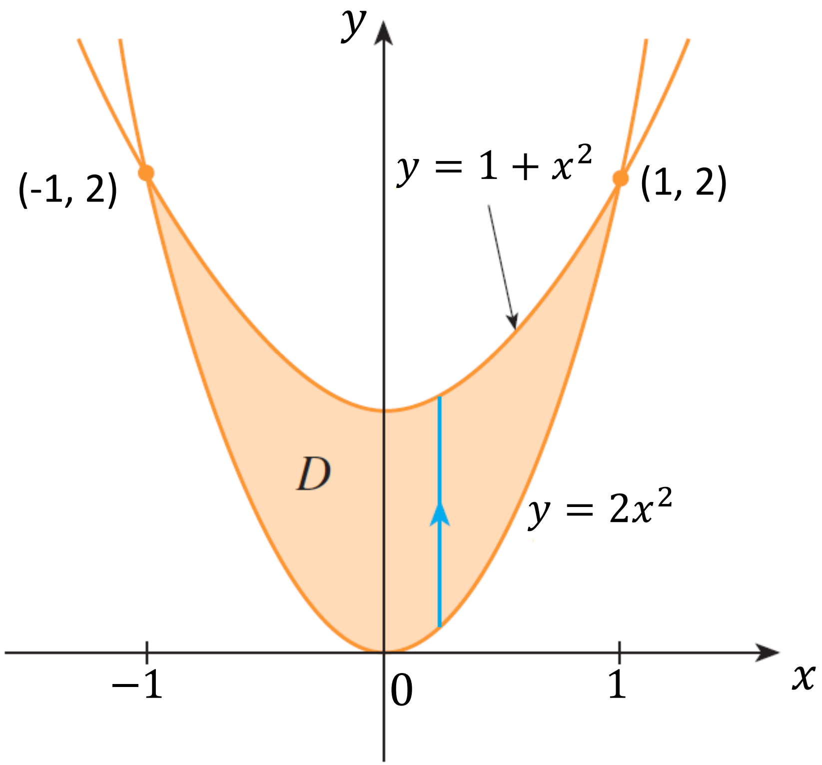

Figure 7

Example...

- The lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$.
  - So, from Equation 3 gives,

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} \left[x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2\right] dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1) dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \Big|_{-1}^{1} = \frac{32}{15}.$$

Example…

Note:

- When we set up a double integral as in Example 1, it is essential to draw a diagram as shown in Figure 7.
  - Often, it is helpful to draw a vertical arrow as shown.
- Then, the limits of integration for the inner integral can be read from the diagram:
  - The arrow starts at the lower boundary $y = g_1(x)$, which gives the lower limit in the integral.
  - The arrow ends at the upper boundary $y = g_2(x)$, which gives the upper limit of integration.
- For a type 2 region, the arrow is drawn horizontally from the left boundary to the right boundary.

Example…

Example 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy–plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution 1 From the Figure 8, we see that is a type 1 region and

$D = \{(x, y) \mid 0 \leq x \leq 2,\ 2x^2 \leq y \leq 2x\}$

- So, the volume under $z = x^2 + y^2$ and above D is calculated as follows.

$$V = \iint_D (x^2 + y^2)\, dA$$

$$= \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx$$

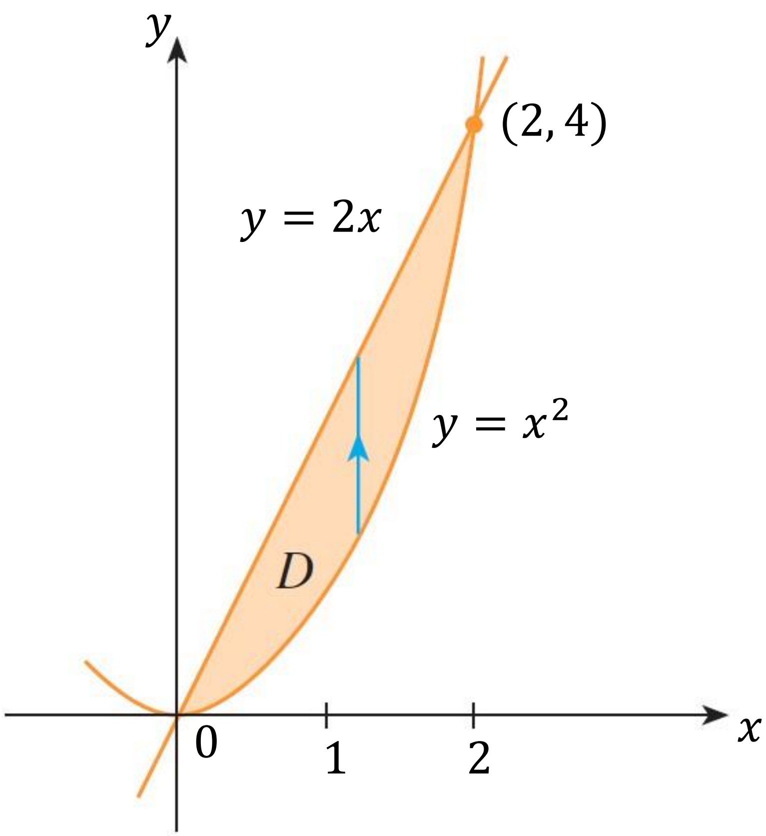

Figure 8

Example...

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{y=x^2}^{y=2x} dx$$

$$= \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2 x^2 - \frac{(x^2)^3}{3}\right] dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3}\right) dx$$

$$= -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6}\Bigg|_0^2$$

$$= \frac{216}{35}$$

Example...

Solution 2 From the Figure 9, we see that D can also be written as a type 2 region:

$$D = \{(x, y) \mid 0 \le y \le 4,\ 1/2y \le x \le \sqrt{y}\}$$

- So, another expression for V is as follows.

$$V = \iint_D (x^2 + y^2)\, dA$$

$$= \int_0^4 \int_{\frac{1}{2}}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy$$

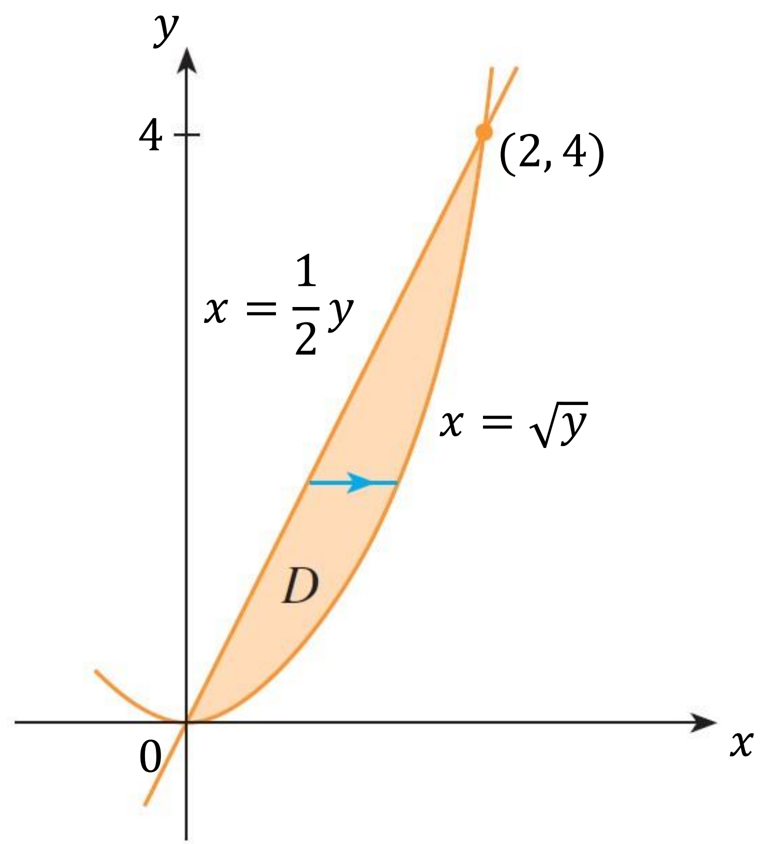

Figure 8

Example...

$$= \int_0^4 \left[\frac{x^3}{3} + y^2 x\right]_{x=\frac{1}{2}y}^{x=\sqrt{y}} dy$$

$$= \int_0^4 \left[\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2}\right] dy$$

$$= \frac{2}{15} y^{5/2} + \frac{2}{7} y^{7/2} - \frac{13}{96} y^4 \Big|_0^4$$

$$= \frac{216}{35}.$$

Summary

1. Let $f(x, y)$ be a function defined in a region D (type 1 region)

$$D = \{(x, y) \mid a \leq x \leq b,\, g_1(x) \leq y \leq g_2(x)\}.$$

Then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx\,.$$

2. Let $f(x, y)$ be a function defined in a region D (type 2 region)

$$D = \{(x, y) \mid c \leq y \leq d,\, h_1(y) \leq x \leq h_2(y)\}.$$

Then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy\,.$$