Unit 1: Linear Algebra - Practice Problems

1. Find the row echelon and reduced row echelon form of the following matrices. Hence find the rank.

(a) $\begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 4 & 4 & 1 \\ 0 & 1 & -2 & 2 \\ 3 & 3 & 1 & 4 \\ 0 & 1 & -3 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ -2 & 3 & -2 & 3 \\ 0 & -3 & 3 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & -5 \\ 4 & 4 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 3 & -3 & 6 \\ 2 & 3 & -5 \\ 1 & -3 & 6 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 5 & 3 \end{bmatrix}$

2. Use Gauss-Jordan method to find the inverse for the following matrices:

(a) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & -5 \\ 4 & 4 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & -3 & 6 \\ 2 & 3 & -5 \\ 1 & -3 & 6 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & -2 \\ 8 & 4 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 3 & 2 & 4 \\ 4 & 1 & 5 \\ 6 & 5 & 7 \end{bmatrix}$

(g) $\begin{bmatrix} -2 & 1 & 1 & 2 \\ 3 & 0 & 2 & -2 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 3 & 5 \end{bmatrix}$ (h) $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$

Note: For more problems, See page 40, Higher Engineering Mathematics by BS Grewal

3. Test the following system of equations for consistency and solve by Gauss elimination method

   (a) $3x + 3y + 2z = 1$; $x + 2y = 4$; $10y + 3z = -2$; $2x - 3y - z = 5$

   (b) $3x_1 + 2x_2 + 4x_3 = 7$; $2x_1 + x_2 + x_3 = 4$; $x_1 + 3x_2 + 5x_3 = 2$

   (c) $x + y + z + u = 2$; $2x - y + 2z - u = -5$; $3x + 2y + 3z + 4u = 7$; $x - 2y - 3z + 2u = 5$

   (d) $x + y + z = 6$; $x - y + 2z = 5$; $3x + y + z = 8$

   (e) $2x_1 + x_2 + 4x_3 = 12$; $4x_1 + 11x_2 - x_3 = 33$; $8x_1 - 3x_2 + 2x_3 = 20$

4. Derive the solution set for each of the following systems using Gauss-Jordan method

   (a) $x+y+z=9$; $2x+y-z=0$; $2x+5y+7z=52$

   (b) $x+y+z=9$; $2x-3y+4z=13$; $3x+4y+5z=40$

   (c) $2x + 5y + 7z = 52$; $2x + y - z = 0$; $x + y + z = 9$

   (d) $20x + y - 2z = 1$; $3x + 20y - z = -18$; $2x - 3y + 20z = 25$

   (e) $x_1 - x_2 + x_3 - x_4 = 2$; $x_1 - x_2 + x_3 + x_4 = 0$; $4x_1 - 4x_2 + 4x_3 = 4$; $-2x_1 + 2x_2 - 2x_3 + x_4 = -3$

5. Solve each of the following homogeneous systems:

   (a) $2x_1 + 4x_2 - 5x_3 + 3x_4 = 0$; $3x_1 + 6x_2 - 7x_3 + 4x_4 = 0$; $5x_1 + 10x_2 - 11x_3 + 6x_4 = 0$

   (b) $x - 2y - 3z = 0$; $2x + y + 3z = 0$; $3x - 4y - 2z = 0.$

   (c) $x + 2y + 3z + t = 0$; $2x + 4y + 7z + 4t = 0$; $3x + 6y + 10z + 5t = 0$

   (d) $x - 2y - 3z = 0$; $2x + 5y + 2z = 0$; $3x - y - 4z = 0$

   (e) $x_1 + 3x_2 + 2x_3 - x_4 - x_5 = 0$; $2x_1 + 6x_2 + 5x_3 + x_4 - x_5 = 0$; $5x_1 + 15x_2 + 12x_3 + x_4 - 3x_5 = 0$

   (f) $2x_1 - 4x_2 + 3x_3 - x_4 + 2x_5 = 0$; $3x_1 - 6x_2 + 5x_3 - 2x_4 + 4x_5 = 0$; $5x_1 - 10x_2 + 7x_3 - 3x_4 + 18x_5 = 0$

6. Find the value of λ for which the system $x + 2y + 3z = 14;\ x + 4y + 7z = 30;\ x + y + z = \lambda$ is consistent. Find the solution for this value of λ.

7. Find the values of λ and μ such that the equations
$$2x + 3y + 5z = 9; 7x + 3y - 2z = 8;\ 2x + 3y + \lambda z = \mu$$
have (i) no solution (ii) unique solution (iii) an infinite number of solutions.

8. Find the values of a and b for the system of equations
$$2x - y + 2z = -1;\quad x - 3z + y = 2;\quad x + 2az = 3b$$
to have (i) unique solution (ii) infinite solutions (iii) no solution

9. Find the values of a and b for which the equations
$$x + ay + z = 3; x + 2y + 2z = b; x + 5y + 3z = 9$$
are consistent. When will these equations have a unique solution?

10. Find the values of λ and μ such that the system of equations
$$x + y + z = 6;\quad x + 2y + 3z = 10;\quad x + 2y + \lambda z = \mu,$$
may have (i) no solution (ii) unique solution (iii) an infinite number of solutions

Note: See Higher Engineering Mathematics by BS Grewal, pages 50-51 for more problems.

Gauss Seidel Method

1. Solve the following using Gauss – Seidel method. Perform 5 iterations.

a) $4x + 2y + z = 14$
$x + 5y - z = 10$
$x + y + 8z = 20$

b) $10x + 4y - 2z = 20$
$3x + 12y - z = 28$
$x + 4y + 7z = 2$

2. Solve the following using Gauss – Seidel method. Iterate until absolute relative error is less than 10^{-3}

a) $28x + 4y - z = 32$
$x + 3y + 10z = 24$
$2x + 17y + 4z = 35$

b) $x + y + 54z = 110$
$27x + 6y - z = 85$
$6x + 15y + 2z = 72$

Eigenvalues and Eigenvectors

1. Find the eigenvalues and the corresponding eigenvectors for each of the following matrices:

a) $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$

d) $\begin{bmatrix} 3 & -8 \\ 2 & 3 \end{bmatrix}$ e) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$ f) $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

2. Two eigenvalues of the matrix A are equal to 1 each. Find the eigenvalues of A^{-1}, where

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Note: See page 60, Higher Engineering Mathematics by BS Grewal for extra problems.