Lorentz Transformations; The Lie Algebra, Representations and Fields

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1 Introduction

It has been more than a year since we were exposed to representations of Lorentz lie algebra in some capacity. Every course since then has contributed to a more physical understanding. This article presents my thoughts on the subject as I attend the lectures on Quantum Field Theory II at NISER. The goal is to understand the essence of representation theory at the level of basic Quantum Field Theory and Particle Physics and note the different fields that one can study viz. quantize, all at the level of Physics. These notes are verified neither by my peers nor the instructors and are very rough sketches of thoughts. Much of the understanding comes from the courses taken by Dr. Dr Sayantani Bhattacharya (Special Theory Of Relativity), Dr. Chethan N. Gowdigere (Particle Physics), and Dr. Yogesh Srivastava (Quantum Field Theory I and II). This began out of excitement when Dr. Yogesh adopted a very structured treatment to the theory and things started fitting in slowly. The notes will undergo modifications, as I discuss them with my fellow QFT mates, Sajag and Deepak and the instructor Dr. Yogesh.

2 Questions that are currently not resolved or Thoughts that I want to highlight

Skip this section if this is your first time reading.

- The significance of $L^{\rho\sigma}$, the infinite dimensional representation of the Lorentz Lie Algebra. What does it say? Yes, it is the basic feature of any field, but...
- The groups generated by the representations of the Lorentz Lie Algebra need not all be the same!

3 The Lie Algebra

Under Lorentz Transformations we have

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \tag{3.1}$$

where,

$$\lambda^{\alpha}_{\mu}\eta_{\alpha\beta}\Lambda^{\beta}_{\nu} = \eta_{\mu\nu}\ (\Lambda^T\eta\Lambda = \eta) \tag{3.2}$$

We then have,

- $\Lambda^T\eta\Lambda = \eta \Rightarrow \det(\Lambda)^2 = 1$
  - $\det(\Lambda) = +1$ — Proper Lorentz Transformation (L.T)
  - $\det(\Lambda) = -1$ — Improper L.T
- $\eta_{00} = \eta_{\alpha\beta}\Lambda^{\alpha}{}_{0}\Lambda^{\beta}{}_{0} \Rightarrow 1 = (\Lambda^{0}{}_{0})^2 - (\Lambda^{i}{}_{0})^2$
  - $\Lambda^{0}{}_{0} \geq 1 \rightarrow$ Orthochronus L.T
  - $\Lambda^{0}{}_{0} \leq -1 \rightarrow$ Non-Orthochronus L.T

Figure 1: Classification of the Lorentz Transformation based on two features. Clearly, the Proper and Improper, similarly Orthochronus and Non-Orthochronus L.Ts are disconnected subsets of the total L.Ts.

Claim: Every Improper and/or Non-Orthochronus Lorentz Transformation can be expressed in terms of Proper and/or Orthochronus L.T together with a Space(P) or Time(T) Inversion Operators,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, T = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For a P.O.L.T,

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu \tag{3.3}$$

$$\omega^\mu_\nu \to 0 \implies \Lambda^\mu_\nu \to \delta^\mu_\nu$$

And indeed $det(\delta) = +1$ and $\delta^0_0 = 1)$ as it should be for a P.O.L.T

Using the defining relation of $Lambda$ - 3.1, we have

$$(\delta^\alpha_\mu + \omega^\alpha_\mu)(\eta_{\alpha\beta}(\delta^\beta_\nu + \omega^\beta_\nu) = \eta_{\mu\nu}) \tag{3.4}$$

Which gives.

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \tag{3.5}$$

That is, an anti symmetric tensor (rank 4 $\implies$ 6 independent elements). We thus write,

$$\omega^\mu_\nu = \frac{1}{2}\Omega_A(M^A)^\mu_\nu \equiv \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu_\nu; \tag{3.6}$$

Where, A labels the six basis elements to write a general anti symmetric matrix; $(M^A)^\mu_\nu; A = 1, ...6 \equiv (M^{\rho\sigma})^\mu_\nu$; Anti-symmetric in$\rho, \sigma$.

Such an $(M^{\rho\sigma})^{\mu\nu}$ anti-symmetric w.r.t $\rho \leftrightarrow \sigma, \mu \leftrightarrow \nu$ can be written as

$$(M^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\nu\rho}\eta^{\mu\sigma} \tag{3.7}$$

$$i.e, (M^{\rho\sigma})^\mu_\nu = \eta^\rho\mu\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu \tag{3.8}$$

Further we have,

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau}M^{\rho\nu} - \eta^{\rho\tau}M^{\sigma\nu} - (\tau \leftrightarrow \nu) \tag{3.9}$$

This is called the Lorentz Lie Algebra! $M^{\rho\sigma}$ are called the generators of the Lorentz Group and are said to satisfy the above lie algebra. Λ is called the fundamental representation of the Lorentz Lie Algebra and leads to classification like scalars, vectors, or general tensors under L.Ts.

Example: Consider a space-time vector T^μ, it transforms as

$$T'^\mu = \Lambda^\mu_\nu T^\nu \tag{3.10}$$

$$\tag{3.11}$$

$$
\Lambda^{\mu}_{\nu} = \begin{cases} \left(1 + \frac{1}{2}\Omega_{\alpha\beta}(M^{\alpha\beta})^{\mu}_{\nu}\right) & \textit{for infinitesimal transformations} \\ \\ e^{\frac{1}{2}\Omega_{\alpha\beta}(M^{\alpha\beta})^{\mu}_{\nu}} & \textit{for finite transformations} \end{cases} \tag{3.12}
$$

These objects are different from fields. How are they different? Can we say four-velocity which is a four-vector as a field? But it doesn't transform as a vector field, but as a four-vector. Is a vector field said to be transforming like a four-vector? That internal x transformation subtelty...

4 Fields mathematically manifest as representations

4.1 Representations

More representations $\iff$ more QFTs to study...

Definition 2.1: We say a set of matrices represents the Lorentz group if the operations between the matrices preserve the Lorentz group operation, i.e $\Lambda_1\Lambda_2 = \Lambda_3 \implies D(\Lambda_1)D(\Lambda_2) = D(\Lambda_3)$, where $D(\Lambda)$ is a representation of Λ that satisfies the Lorentz Lie Algebra.

Consider a scalar field; after the Canonical Quantization, we achieve the particle states with spin 0! Starting from the Vector Field, we have particle states with spin 1! Clearly, these are not the only spins that we observe in reality and we thus look for other kinds of fields, characterized by their behavior under the Lorentz Transformations. Thus the motivation to look for different representations of the Lorentz Lie Algebra comes from the fact that, in QFT we aim to study all those fields that are physically relevant. The physical relevance of these fields is captured by definition 2.1, the way they transform under the L.Ts obeying the Lorentz Lie Algebra.

Why is Lorentz Lie Algebra considered the fundamental physical relation that needs to be obeyed by any representation? One answer could be that such an algebra results in representations (thus fields) that give out all the physically observed spin particles upon quantization. (With great advantages come the great troubles of having mathematically infinitely many fields that might have not been observed yet in their quantized forms viz. particles, like say gravitino?).

As we saw in the Special Theory of Relativity Course of Fall 2022, there's a way to directly see all the finite dimensional representations of the Lorentz Lie Algebra labeled by their resultant quantum spins. See Chapter-33, Pg-211, QFT by Srednicki.

Let's study the fields we already know, and place them in the picture of representation theory.

4.2 The Fundamental Representation of Lorentz Group

4.3 Scalar Fields

Under $x'^{\mu} \to x^{\mu} = \Lambda^{\mu}_{\nu}x'^{\nu}$

$$
\phi'^{(x)=\phi(\Lambda^{-1}x)} \tag{4.1}
$$

$$
\tag{4.2}
$$

i.e. the scalar field essentially follows the same old field, hence called scalar.

$$\phi'(x^\mu) = \phi((\delta 6\mu_\nu - \omega^\mu_\nu)x^\nu) \tag{4.3}$$
$$= \phi(x^\mu - \omega^\mu_\nu) \tag{4.4}$$
$$= \phi(x^\mu) - \omega^\mu_\nu x^\partial_\mu \phi(x) \tag{4.5}$$
$$\delta\phi = -\omega^\rho_\nu x^\nu \partial_\rho \phi(x) \tag{4.6}$$

Using the generators, write,

$$\omega^\rho_\nu = \frac{1}{2}\Omega_{\alpha\beta}(M^{\alpha\beta})^\rho_\nu \tag{4.7}$$
$$\delta\phi = \frac{1}{2}\Omega_{\alpha\beta}(x^\beta\partial^\alpha - x^\alpha\partial^\beta)\phi \tag{4.8}$$
$$\tag{4.9}$$

Define,

$$L^{\alpha\beta} = x^\alpha\partial^\beta - x^\beta\partial^\alpha \tag{4.10}$$

Thus,

$$\phi'(x) = \left(1 + \frac{1}{2}\Omega_{\alpha\beta}L^{\alpha\beta}\right)\phi(x) \tag{4.11}$$

And $L^{\alpha\beta}$ satisfies the Lorentz Lie Algebra!

$$\phi'(x) = D(\Lambda)\phi(x) \tag{4.12}$$
$$\tag{4.13}$$

with,

$$D(\Lambda) = \begin{cases} \left(1 + \frac{1}{2}\Omega_{\alpha\beta}L^{\alpha\beta}\right) & \textit{for infinitesimal transformations} \\ \\ e^{\frac{1}{2}\Omega_{\alpha\beta}L^{\alpha\beta}} & \textit{for finite transformations} \end{cases} \tag{4.14}$$

We thus realize the scalar representation of the Lorentz Group (which is infinite dimensional) *on the space of fields* with the generators $L^{\alpha\beta}$ which rightly satisfies the Lorentz Lie Algebra.

4.4 Vector Fields

Under L.T, we have

$$A'^\mu(x) = \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x) \tag{4.15}$$

From 4.2 and 4.27 we have,

$$A'^{\mu}(x) = \left(\delta^{\mu}_{\nu} + \frac{1}{2}\Omega_{\alpha\beta}(M^{\alpha\beta})^{\mu}_{\nu}\right)\left(1 + \frac{1}{2}\Omega_{\alpha\beta}L^{\alpha\beta}\right)A^{\nu}(x) \tag{4.16}$$

$$= \delta^{\mu}_{\nu} + \frac{1}{2}\Omega_{\alpha\beta}\left((M^{\alpha\beta})^{\mu}_{\nu} + L^{\alpha\beta}\delta^{\mu}_{\nu}\right) + \mathcal{O}(\Omega^2_{\alpha\beta}) \tag{4.17}$$

$$\equiv \delta^{\mu}_{\nu} + \frac{1}{2}\Omega_{\alpha\beta}(J^{\alpha\beta})^{\mu}_{\nu} \tag{4.18}$$

where,

$$(J^{\alpha\beta})^{\mu}_{\nu} = (M^{\alpha\beta})^{\mu}_{\nu} + L^{\alpha\beta}\delta^{\mu}_{\nu} \tag{4.19}$$

We write,

$$A'^{\mu}(x) = D(\Lambda)^{\mu}_{\nu}A^{\nu} \tag{4.20}$$

with,

$$D(\Lambda) = \begin{cases} \left(1 + \frac{1}{2}\Omega_{\alpha\beta}(J^{\alpha\beta})\right) & \textit{for infinitesimal transformations} \\ \\ e^{\frac{1}{2}\Omega_{\alpha\beta}(J^{\alpha\beta})} & \textit{for finite transformations} \end{cases} \tag{4.21}$$

We thus realize the vector representation of the Lorentz Group *on the space of fields* with the generators $J^{\alpha\beta}$ which rightly satisfies the Lorentz Lie Algebra.

From the previous two sections we deduce a (one to one?) correspondence between the representations of the Lorentz Group and the physical fields that we want to study,

We thus hunt for more representations of the L.G. In such a hunt, Dirac makes use of the Clifford Algebra to find the so-called, *Spinor Representation*; thus giving rise to a new QFT study...(the history might be other way around you start with spinor QFT and try to fit into the game of Lorentz Group representation and the algebra consistency, I don't know...)

4.5 Spinor Fields

Introduce γ^{μ}, the γ-matrices such that,

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}I \tag{4.22}$$

$S^{\mu\nu} := \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$ *satifies the Lorentz Lie Algebra!*

We thus have another finite dimensional representation of the Lorentz Group. The subtleties of dimension of the matrices and many other properties of γ-matrices will not be discussed here, like the details of the fundamental and scalar representation of Lorentz Group which have been omitted. The goal is to show the different representations and realize the QFTs that one can study.

The *fields* that transform via the Spinor representation(also called the Dirac representations) are called spinors or spinor filds, written in four dimensions as,

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{4.23}$$

Thus, under L.T the spinors transform as,

$$\Psi'(x) = S[\Lambda]\Psi(\Lambda^{-1}x) \tag{4.24}$$

with,

$$S[\Lambda] = \begin{cases} \left(1 + \frac{1}{2}\Omega_{\alpha\beta}S^{\alpha\beta}\right) & \textit{for infinitesimal transformations} \\ \\ e^{\frac{1}{2}\Omega_{\alpha\beta}S^{\alpha\beta}} & \textit{for finite transformations} \end{cases} \tag{4.25}$$

We thus write,

$$\Psi'(x) = D(\Lambda)\Psi(x) \tag{4.26}$$

with,

$$D(\Lambda) = \begin{cases} \left(1 + \frac{1}{2}\Omega_{\alpha\beta}(\Xi^{\alpha\beta})\right) & \textit{for infinitesimal transformations} \\ \\ e^{\frac{1}{2}\Omega_{\alpha\beta}(\Xi^{\alpha\beta})} & \textit{for finite transformations} \end{cases} \tag{4.27}$$

where,

$$(\Xi^{\alpha\beta})^{\mu}_{\nu} = (S^{\alpha\beta})^{\mu}_{\nu} + L^{\alpha\beta}\delta^{\mu}_{\nu} \tag{4.28}$$

We thus realize the Spinor or Dirac representation of the Lorentz Group *on the space of fields* with the generators $\Xi^{\alpha\beta}$ which rightly satisfies the Lorentz Lie Algebra.

5 Summary and some subtleties

Λ Fundamental Representation (What are the objects called?)

$$\left\{\begin{array}{ll} \text{Scalars}: & a \\ \text{Vectors}: & A^\mu \\ \text{Tensors}: & T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} \end{array}\right\} \Rightarrow SO(1,3) \text{ generators } M^{\rho\sigma}$$

{ LORENTZ LIE ALGEBRA }

Λ, L Fundamental Representation on Fields (Objects are Fields)

$$\left\{\begin{array}{ll} \text{Scalar Fields}: & \Phi'(x') = \Phi(x) \\ \text{Vector Fields}: & A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x) \\ \text{Tensor Fields}: & \text{any physical example?} \end{array}\right\} \Rightarrow SO(1,3) \text{ generators } M^{\rho\sigma}.$$

$S(\Lambda), L$ Spinor Representation on Fields (Objects are called Spinor Fields...)

$$\left\{\begin{array}{ll} \text{Scalar Fields}: & \Phi'(x') = \Phi(x) \\ \text{Spinor Fields}: & \Psi'(x') = S(\Lambda)\Psi(x) \\ \text{Tensor Fields}: & \cdots \end{array}\right\} \Rightarrow SL(2,\mathbb{C}) \text{ generators } S^{\rho\sigma}$$

allowing a phase change

...

- The infinite dimensional representation $L^{\rho\sigma}$ of the LORENTZ LIE ALGEBRA.

Moving from objects that are constant in space-time to objects that depend on space-time viz. fields, we have consider a scalar field, the most simplest of all...

Under $x' \longrightarrow x \quad \Phi \longrightarrow \Phi'$

we have

$$\begin{aligned} \Phi'(x) &= \Phi(x') \\ &= \Phi(\Lambda^{-1}x) \\ &= \Phi(x^\mu - w^\mu{}_\nu x^\nu) \\ &= \left(1 + \tfrac{1}{2}\Omega_{\rho\sigma}L^{\rho\sigma}\right)\Phi(x) \end{aligned}$$

Where $L^{\rho\sigma}$ satisfies the Lorentz Lie Algebra...

WHAT IS THE SIGNIFICANCE of $L^{\rho\sigma}$ satisfying L.L.A?