FINE STRUCTURE

from P.A. Dirac to Darmen...

- $H_{DIRAC} = c\bar{\alpha}\cdot\bar{p} + \beta mc^2$
  - $\alpha_i^2 = \beta^2 = 1 \quad i = 1,2,3$
  - $\{\alpha_i, \beta\} = 0$
  - $\{\alpha_i, \alpha_j\} = 0 \quad (i \neq j)$

$\downarrow \; H\psi = E\psi \qquad \psi \equiv \begin{pmatrix} \chi \\ \eta \end{pmatrix}$

$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

$\chi \equiv \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \eta \equiv \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$

$$\boxed{\begin{aligned} &c(\bar{\sigma}\cdot\bar{p})\eta = (E - mc^2)\chi \\ &c(\bar{\sigma}\cdot\bar{p})\chi = (E + mc^2)\eta \end{aligned}}$$

$\Rightarrow \quad \chi = \frac{2c}{v}\eta \Rightarrow$ η small Compared to χ!

$\downarrow$ Solve for χ.

$$\ldots \quad E'\chi = \frac{(\bar{\sigma}\cdot\bar{p})(\bar{\sigma}\cdot\bar{p})}{2m}\chi$$

But let's add magnetic & Electric fields!

$$\bar{\sigma}\cdot(c\bar{p} - q\bar{A})\eta + (mc^2 + q\phi)\chi = E\chi \qquad \text{— 1(a)}$$

$$\bar{\sigma}\cdot(c\bar{p} - q\bar{A})\chi + (q\phi - mc^2)\eta = E\eta \qquad \text{— 1(b)}$$

$\downarrow$

$$\boxed{(E' - q\phi) = \frac{(\bar{\sigma}\cdot\bar{\pi})K(\bar{\sigma}\cdot\bar{\pi})}{2m}\chi} \qquad \text{— (2)}$$

$$K = \left\{\frac{2mc^2}{E' + 2mc^2 - q\phi}\right\}.$$

$$(\bar{\sigma}\cdot\bar{\pi})(\bar{\sigma}\cdot\bar{\pi}) = \pi^2 + i\bar{\sigma}\cdot\underline{(\bar{\pi}\times\bar{\pi})}$$

- $(\bar{\sigma}\cdot\bar{\pi})(\bar{\sigma}\cdot\bar{\pi}) = \left(\bar{p} - \frac{q\bar{A}}{c}\right)^2 - \frac{iq}{c}\bar{\sigma}\cdot(\underbrace{\bar{A}\times\bar{p} + \bar{p}\times\bar{A}})$

$(\bar{p}\times\bar{A}) = -i\hbar(\bar{\nabla}\times\bar{A}) - \bar{A}\times\bar{p}$

$$= \left(\bar{p} - \frac{q\bar{A}}{c}\right)^2 - \frac{q\hbar}{c}\bar{\sigma}\cdot(\bar{\nabla}\times\bar{A})$$

so for $K = 1$, $(E' - q\phi) = \frac{(\bar{\sigma}\cdot\bar{\pi})^2}{2m}\chi = \left\{\frac{\left(\bar{p} - \frac{q\bar{A}}{c}\right)^2}{2m} - \frac{q\hbar}{2mc}\bar{\sigma}\cdot(\bar{\nabla}\times\bar{A})\right\}\chi.$

— (3).

$$K = \left(\frac{1}{1 + \frac{E' - q\phi}{2mc^2}}\right) = 1 - \left(\frac{E' - q\phi}{2mc^2}\right) + \ldots \qquad \text{— (4)}$$

$$(E' - q\phi)\chi = \frac{1}{2m}\left((\bar{\sigma}\cdot\bar{\pi})^2 - (\bar{\sigma}\cdot\bar{\pi})\frac{(E'-q\phi)}{2mc^2}(\bar{\sigma}\cdot\bar{\pi})\right). \quad \text{———} (*)$$

But crap :/ doesn't explain Fine structure yet...

!! "Foldy - Wouthuysen Transformation" later...

$\psi = \Omega\chi$, $\chi = \Omega^{-1}\psi$. # η-eliminated ! :0 ?

$$\boxed{\Omega^{-1} = 1 \pm \frac{(\bar{\sigma}\cdot\bar{\pi})^2}{8m^2c^2}} \rightsquigarrow \text{order of } v^2/c^2 \ldots$$

now Do $\Omega^{-1}\{\text{Eqn }(*)\}\underbrace{\Omega^{-1}\psi}_{\chi}$ ...

$$\Omega^{-1}(E'-q\phi)\Omega^{-1}\psi = \frac{\Omega^{-1}}{2m}\left\{(\bar{\sigma}\cdot\bar{\pi})^2 - (\bar{\sigma}\cdot\bar{\pi})\frac{(E'-q\phi)}{2mc^2}(\bar{\sigma}\cdot\bar{\pi})\right\}\Omega^{-1}\psi$$

$$= (E'-q\phi)\psi - \frac{(\bar{\sigma}\cdot\bar{\pi})^2(E'-q\phi)\psi}{8m^2c^2} - \frac{(E'-q\phi)(\bar{\sigma}\cdot\bar{\pi})^2\psi}{8m^2c^2} + \ldots$$

$$\boxed{= \frac{(\bar{\sigma}\cdot\bar{\pi})^2}{2m} - \frac{(\bar{\sigma}\cdot\bar{\pi})^4}{8m^3c^2}} + \ldots$$

upto $O(v^2/c^2)$.

$$= \frac{(\bar{\sigma}\cdot\bar{\pi})(E'-q\phi)(\bar{\sigma}\cdot\bar{\pi})}{4m^2c^2}\psi + \ldots$$

Thus,

$$(E'-q\phi)\psi = \left\{\frac{(\bar{\sigma}\cdot\bar{\pi})^2}{2m} - \frac{(\bar{\sigma}\cdot\bar{\pi})^4}{8m^3c^2} + \frac{(\bar{\sigma}\cdot\bar{\pi})^2(E'-q\phi)}{8m^2c^2} + \frac{(E'-q\phi)(\bar{\sigma}\cdot\bar{\pi})^2}{8m^2c^2} - \frac{(\bar{\sigma}\cdot\bar{\pi})(E'-q\phi)(\bar{\sigma}\cdot\bar{\pi})}{4m^2c^2}\right\}\psi$$

• Simplifying further using:

• $(\bar{\sigma}\cdot\bar{\pi})(E'-q\phi) - (E'-q\phi)(\bar{\sigma}\cdot\bar{\pi}) = i\hbar q(\bar{\sigma}\cdot\nabla\phi)$

• $(\bar{\sigma}\cdot\bar{\pi})(\bar{\sigma}\cdot\bar{\nabla}\phi) - (\bar{\sigma}\cdot\bar{\nabla}\phi)(\bar{\sigma}\cdot\bar{\pi}) = -i\hbar\, \bar{\nabla}\cdot\bar{\nabla}\phi + 2i(\bar{\sigma}\cdot\bar{\pi})\times\nabla\phi$

• $(\bar{\sigma}\cdot\bar{\pi})^2(E'-q\phi) - (E'-q\phi)(\bar{\sigma}\cdot\bar{\pi})^2$
$= q\hbar^2(\bar{\nabla}\cdot\bar{\nabla}\phi) - 2\hbar(\bar{\sigma}\cdot\bar{\pi})\times\nabla\phi + 2(\bar{\sigma}\cdot\bar{\pi})(E'-q\phi)\cdot(\bar{\sigma}\cdot\bar{\pi})$

$$(E'-q\phi)\psi = \frac{1}{2m}\left\{(\bar{\sigma}\cdot\bar{\pi})^2 - \frac{(\bar{\sigma}\cdot\bar{\pi})^4}{4m^2c^2} + \frac{q\hbar^2(\bar{\nabla}\cdot\bar{\nabla}\phi)}{4m^2c^2} - \frac{q\hbar}{2mc^2}(\bar{\sigma}\cdot\bar{\pi})\times\bar{\nabla}\phi\right\}\psi .$$

$\boxed{A=0}$

$$\cancel{\boxed{\mathcal{H} = \frac{\hat{p}^2}{2m} + V - \frac{\hat{p}^4}{8m^3c^2} + \frac{e^2}{2m^2c^2}\frac{\bar{L}\cdot\bar{S}}{r^3} + \frac{\pi}{2}\frac{e^2\hbar^2}{m^2c^2}\delta(r)}}$$

$$\boxed{\mathcal{H} = \underbrace{\frac{\hat{p}^2}{2m} + V}_{\underline{H(0)}} - \underbrace{\frac{\hat{p}^4}{8m^3c^2}}_{\underline{\delta H_{rel}}} + \underbrace{\frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\bar{S}\cdot\bar{L}}_{\underline{\delta H_{SO}}} + \underbrace{\frac{\hbar^2}{8m^2c^2}\nabla^2 V}_{\underline{\delta H_{Darwin}}}}$$

$\downarrow$

$$\boxed{\mathcal{H} = \underbrace{\frac{\hat{p}^2}{2m} + V}_{\underline{H(0)}} - \underbrace{\frac{\hat{p}^4}{8m^3c^2}}_{\underline{\delta H_{rel}}} + \underbrace{\frac{e^2}{2m^2c^2}\frac{\bar{S}\cdot\bar{L}}{r^3}}_{\underline{\delta H_{\text{Spin-Orbital}}}} + \underbrace{\frac{\pi}{2}\frac{e^2\hbar^2}{m^2c^2}\delta(r)}_{\underline{\delta H_{Darwin}}} .}$$