TUTORIAL-04

Practice Problems on Master theorem:-

The master theorem applies to recurrences of the following form: $T(n) = aT(n/b) + f(n)$ where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically positive function. There are 3 cases:-

(1) if $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

(2) if $f(n) = \Theta(n^{\log_b a} \log^k n)$ with $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

(3) if $f(n) = \Omega(n^{\log_b a + \epsilon})$ with $\epsilon > 0$ and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$. Regularity condition: $af(n/b) \leq cf(n)$ for some constants $c < 1$ and all sufficiently large n.

Q(1) $T(n) = 3T(n/2) + n^2$

$a = 3$, $b = 2$, $f(n) = n^2$

$\because$ a and b are constant and $f(n)$ is a +ve function

$\because$ Master's theorem is applicable

$c = \log_b a = \log_2 3 = 1.58$

$\Rightarrow n^c = n^{1.58}$, which is $n^2 > n^{1.58}$

$\therefore$ case 3 is applied here $\Rightarrow$ $\boxed{T(n) = \Theta(n^2)}$

Q(2) $T(n) = 4T(n/2) + n^2$

$a = 4$, $b = 2$, $f(n) = n^2$

$\because$ a and b are constant & $f(n)$ is +ve func.

$\therefore$ Master's theorem is applicable

$c = \log_b a = \log_2 4 = \log_2 2^2 = 2\log_2 2 = 2$

$\therefore n^c = n^2$, $n^c = f(n)$

$\therefore$ case 2 is applied here.

$\Rightarrow \boxed{T(n) = O(n^2 \log n)}$

Q(3) :- $T(n) = T(n/2) + 2^n$

$a = 1$, $b = 2$, $f(n) = 2^n$

$\therefore$ a & b are constants & f(n) is +ve func.

$\therefore$ Master's theorem is applicable.

$c = \log_b a = \log_2 1 = 0$

$\therefore n^c = n^0 = 1$

$\therefore f(n) > n^c$

$\therefore$ case 3 is applied

$\Rightarrow \boxed{T(n) = O(2^n)}$

Q(4) :- $T(n) = 2^n T(n/2) + n^n$

$a = 2^n$, $b = 2$, $f(n) = n^n$

$\therefore$ a is not constant, value depend on 'n'.

$\therefore$ Master's theorem is not applicable

Q(5) :- $T(n) = 16\,T(n/4) + n$

$a = 16$, $b = 4$, $f(n) = n$

$\therefore$ a & b are constant & f(n) is +ve func.

$\therefore$ Master's theorem is applicable

$c = \log_b a = \log_4 16 = \log_4 4^2 = 2\log_4 4 = 2$

$\Rightarrow n^c = n^2$, $f(n) < n^c$

$\therefore$ case 1 is applied

$\Rightarrow \boxed{T(n) = O(n^2)}$

Q(6) :- $T(n) = 2T(n/2) + n\log n$

$a = 2$, $b = 2$, $f(n) = n \log n$

$\therefore$ a & b are constant and f(n) is a +ve func.

$\therefore$ Master's theorem is applicable.

$c = \log_b a = \log_2 2 = 1$

$\therefore n^c = n^1 = n$, $f(n) > n^c$

Case 3 is applied

$\Rightarrow \boxed{T(n) = \Theta(n \log n)}$

Q7:- $T(n) = 2T(n/2) + n/\log n$

$a = 2$, $b = 2$, $f(n) = n/\log n$

$\therefore$ a and b are constant & f(n) is +ve func.

$c = \log_b a = \log_2 2 = 1$

$\therefore n^c = n^1 = n$

f(n) is non-polynomial difference b/w f(n) & n^c

$\therefore$ Master's theorem is not applicable

Q8:- $T(n) = 2T(n/4) + n^{0.51}$

$a = 2$, $b = 4$, $f(n) = n^{0.51}$

$\therefore$ a and b are constant & f(n) is a +ve func.

$\therefore$ Master's theorem is applicable

$c = \log_b a = \log_4 2 = 0.50$

$n^c = n^{0.50}$

$\therefore f(n) > n^c$

$\therefore$ Case 3 is applicable

$\Rightarrow \boxed{T(n) = \Theta(n^{0.51})}$

Q9:- $T(n) = 0.5\, T(n/2) + 1/n$

$a = 0.5$, $b = 2$, $f(n) = 1/5$

$\therefore a < 1$

$\therefore$ Master's theorem is not applicable

Q10:- $T(n) = 16\, T(n/4) + n!$

$a = 16$, $b = 4$, $f(n) = n!$

$\therefore$ a & b are constant & f(n) is +ve func.

$\therefore$ Master's theorem is applicable.

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$c = \log_b a = \log_4 16 = \log_4 4^2 = 2\log_4 4 = 2$

$\therefore n^c = n^2$

$\therefore f(n) > n^c$

$\therefore$ case 3 is applied here

$\Rightarrow \boxed{T(n) = \Theta(n!)}$

Q(11):- $T(n) = 4T(n/2) + \log n$

$a = 4$, $b = 2$, $f(n) = \log n$

$\therefore$ a & b are constant & f(n) is +ve func.

$\therefore$ Master's theorem is applicable

$c = \log_b a = \log_2 4 = \log_2 2^2 = 2\log_2 2 = 2$

$\therefore n^c = n^2$

$\therefore f(n) < n^c$

$\therefore$ case 1 is applied

$\Rightarrow \boxed{T(n) = \Theta(n^2)}$

Q(12):- $T(n) = \sqrt{n}\, T(n/2) + \log n$

$a = \sqrt{n}$, $b = 2$, $f(n) = \log n$

$\therefore$ a is not constant

$\therefore$ Master's theorem is not applicable

Q(13):- $T(n) = 3T(n/2) + n$

$a = 3$, $b = 2$, $f(n) = n$

$\therefore$ a & b are constant & f(n) is +ve func.

$\therefore$ Master's theorem is applicable.

$c = \log_b a = \log_2 3 = 1.58$

$\therefore n^c = n^{1.58}$, $f(n) < n^c$

$\therefore$ case 1 is applied.

$\Rightarrow \boxed{T(n) = \Theta(n^{1.58})}$

Q(14):- $T(n) = 3T(n/3) + \sqrt{n}$

$a = 3$, $b = 3$, $f(n) = \sqrt{n}$

$\therefore$ a & b are constant & f(n) is +ve func.

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$\therefore$ Master's theorem is applicable.

$c = \log_b a = \log_3 3 = 1$

$\therefore$ $n^c = n^1 = n$, $f(n) < n^c$

$\therefore$ case 1 is applicable

$\Rightarrow$ $\boxed{T(n) = \Theta(n)}$

Q(15):- $T(n) = 4T(n/2) + c.n$

$a = 4$, $b = 2$, $f(n) = c.n$

$\therefore$ a & b are constant & f(n) is +ve func.

$\therefore$ Master's theorem is applicable

$c = \log_b a = \log_2 4 = \log_2 2^2 = 2\log_2 2 = 2$

$\therefore$ $n^c = n^2$ $\therefore$ $f(n) < n^c$

$\therefore$ case 1 is applied

$\Rightarrow$ $\boxed{T(n) = \Theta(n^2)}$

Q(16):- $T(n) = 3T(n/4) + n\log n$

$a = 3$, $b = 4$, $f(n) = n\log n$

$\therefore$ a & b are constant & f(n) is +ve func.

$\therefore$ Master's theorem is applicable

$c = \log_b a = \log_4 3 = 0.79$

$n^c = n^{0.79}$ $\therefore$ $f(n) > n^c$

$\therefore$ case 3 is applied

$\Rightarrow$ $\boxed{T(n) = \Theta(n\log n)}$

Q(17):- $T(n) = 3T(n/3) + n/2$

$a = 3$, $b = 3$, $f(n) = n/2$

a & b are constant & f(n) is +ve func.

$\therefore$ Master's theorem is applicable.

$c = \log_b a = \log_3 3 = 1$

$n^c = n$, $f(n) = n^c$

$\therefore$ case 2 is applied

$\Rightarrow$ $\boxed{T(n) = \Theta(n.\log n)}$

Q(18):- $T(n) = 6T(n/3) + n^2 \log n$

$a = 6$, $b = 3$, $f(n) = n^2 \log n$

$\therefore$ a & b are constant & f(n) is +ve func.

$\therefore$ Master's theorem is applicable

$c = \log_b a = \log_3 6 = 1.63$

$n^c = n^{1.63}$ $\therefore$ $f(n) > n^c$

$\therefore$ case 3 is applied

$\Rightarrow \boxed{T(n) = \Theta(n^2 \log n)}$

Q(19):- $T(n) = 4T(n/2) + n/\log n$

$a = 4$, $b = 2$, $f(n) = n/\log n$

$\therefore$ a and b are constant & f(n) is +ve func.

$\therefore$ Master's theorem is applicable

$c = \log_b a = \log_2 4 = \log_2 2^2 = 2\log_2 2 = 2$

$\therefore$ $n^c = n^2$, $\therefore$ $f(n) < n^c$

$\therefore$ case 1 is applied

$\Rightarrow \boxed{T(n) = \Theta(n^2)}$

Q(20):- $T(n) = 64T(n/8) - n^2 \log n$

$\therefore$ a and b are constant and f(n) is -ve func.

$\therefore$ Master's theorem is not applicable

Q(21):- $T(n) = 7T(n/3) + n^2$

$a = 7$, $b = 3$, $f(n) = n^2$

$\therefore$ a & b are constant & f(n) is +ve func.

$\therefore$ Master's theorem is applicable

$\therefore$ $c = \log_b a = \log_3 7 = 1.77$

$\therefore$ $n^c = n^{1.77}$ $\therefore$ $f(n) > n^c$

$\therefore$ case 3 is applied, $\Rightarrow \boxed{T(n) = \Theta(n^2)}$

Q(22):- $T(n) = T(n/2) + n(2 - \cos n)$

$\therefore$ f(n) is not regular function

$\therefore$ Master's theorem is not applicable.