Hiwi documentation

Shubhankar Riswadkar

April 7, 2025

1 Frame assignment and DH parameters

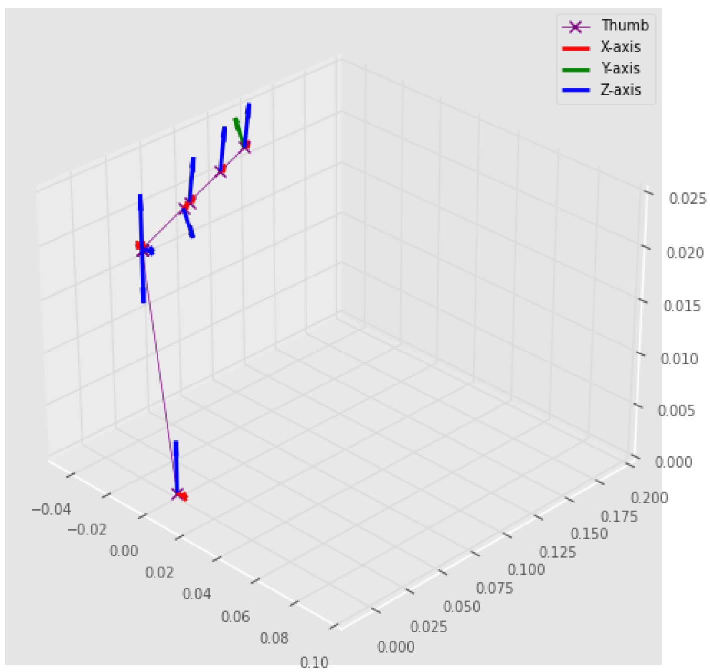

Figure 1: Thumb coordinate frame assignment.

Table 1: Denavit-Hartenberg Parameters

Link	**Link Length (a_i) [mm]**	**Link Twist (α_i) [deg]**	**Link Offset (d_i) [mm]**	**Joint Angle (θ_i) [deg]**
1	28	0	20	153.5
2	0	90	0	q_1
3	0	90	0	q_2
4	48.945	-60	0	16.5
5	7.5	-90	0	q_3
6	38.22	0	0	q_4
7	30.81	0	0	q_5

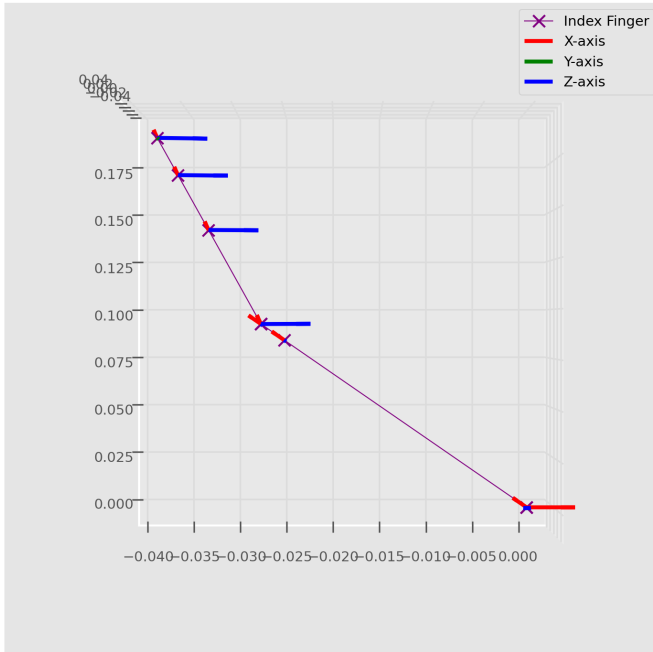

Figure 2: Index finger coordinate frame assignment.

Table 2: Denavit-Hartenberg Parameters

Link	**Link Length** (a_i) **[mm]**	**Link Twist** (λ_i) **[deg]**	**Link Offset** (d_i) **[mm]**	**Joint Angle** (θ_i) **[deg]**
1	0	0	0	106.5
2	88	0	0	0
3	7.5	0	0	q_1
4	0	90	0	-10
5	47.775	0	0	q_2
6	27.885	0	0	q_3
7	18.915	0	0	q_4

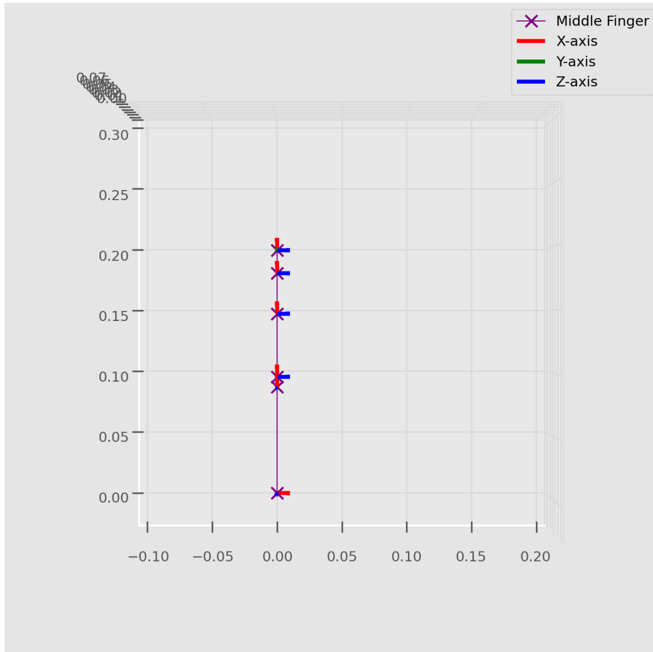

Figure 3: Middle finger coordinate frame assignment.

Table 3: Denavit-Hartenberg Parameters

Link	**Link Length (a_i) [mm]**	**Link Twist (α_i) [deg]**	**Link Offset (d_i) [mm]**	**Joint Angle (θ_i) [deg]**
1	87	0	0	90
2	8.5	90	0	q_1
3	51.87	0	0	q_2
4	33.15	0	0	q_3
5	18.915	0	0	q_4

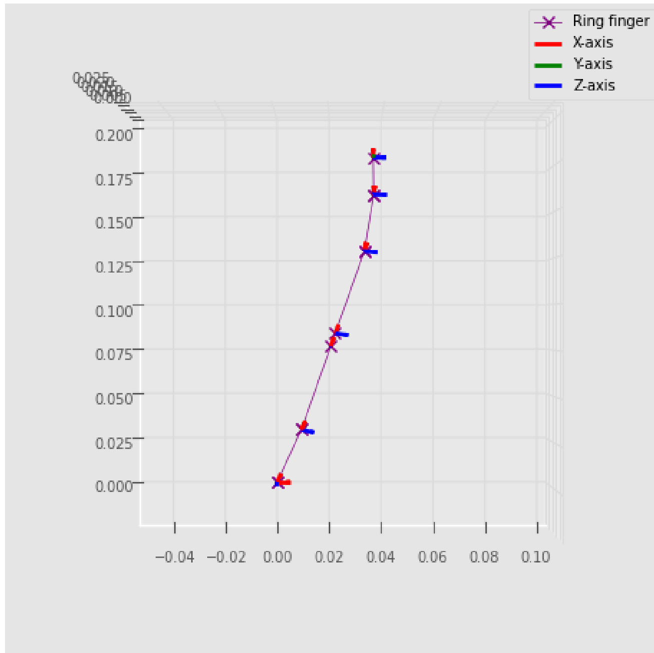

Figure 4: Ring finger coordinate frame assignment.

Table 4: Denavit-Hartenberg Parameters

Link	**Link Length (a_i) [mm]**	**Link Twist (α_i) [deg]**	**Link Offset (d_i) [mm]**	**Joint Angle (θ_i) [deg]**
1	0	0	0	72.5
2	31	90	0	0
3	48.5	90	0	q_1
4	8.5	-90	0	q_2
5	47.58	-90	0	q_3
6	0	90	0	7
7	32.175	-90	0	q_4
8	0	90	0	7
9	20.865	0	0	q_5

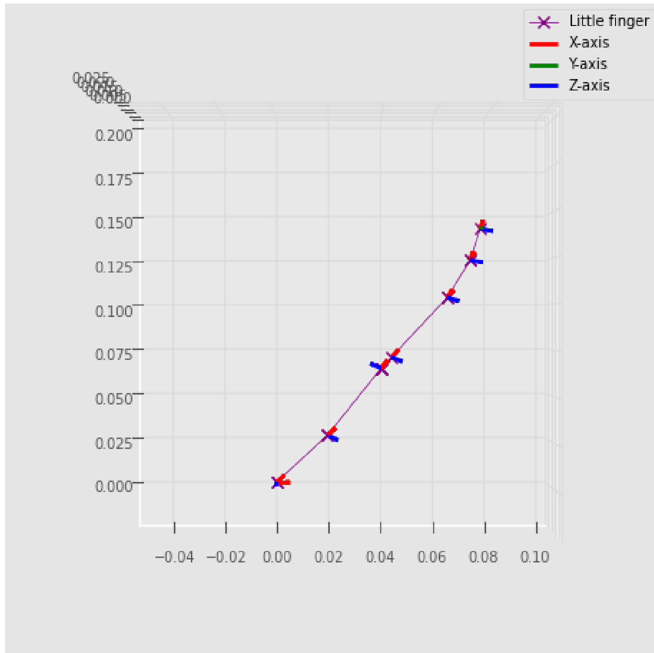

Figure 5: Little finger coordinate frame assignment.

Table 5: Denavit-Hartenberg Parameters

Link	**Link Length** (a_i) **[mm]**	**Link Twist** (α_i) **[deg]**	**Link Offset** (d_i) **[mm]**	**Joint Angle** (θ_i) **[deg]**
1	0	0	0	63.5
2	33	90	0	0
3	43	90	0	q_1
4	8.5	-90	0	q_2
5	39.78	-90	0	q_3
6	0	90	0	10
7	22.815	-90	0	q_4
8	0	90	0	10
9	18.135	0	0	q_5

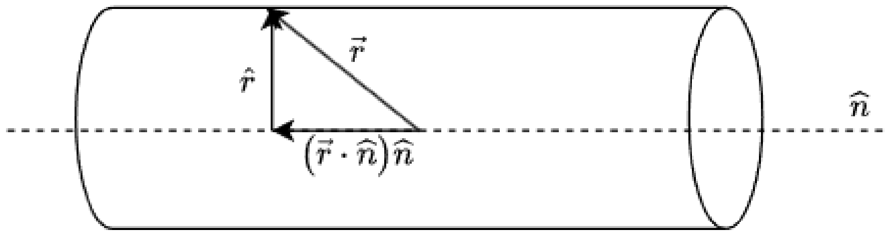

Figure 6: Force decompostion

$$
\begin{aligned}
{}^{0}T_{\text{end-effector}} &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
F_{\text{finger}} &= c \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}
\end{aligned} \tag{1}
$$

$$\hat{r} = \frac{\vec{r}}{\|\mathbf{r}\|} - (\vec{r} \cdot \hat{n})\hat{n} \tag{2}$$

$$\hat{t} = \hat{n} \times \hat{r} \tag{3}$$

$$F_{\text{axial}} = F_{\text{finger}} \cdot \hat{n} \tag{4}$$

$$F_{\text{radial}} = F_{\text{finger}} \cdot \hat{r} \tag{5}$$

$$F_{\text{tangential}} = F_{\text{finger}} \cdot \hat{t} \tag{6}$$

Polyhedral Approximation of Friction Cone

To approximate the friction cone with a polyhedron, the tangential directions $\mathbf{d}_k$ are defined as unit vectors along the edges of the polyhedron. These vectors are evenly distributed around the surface of the cone in the tangential plane.

Force Closure Condition

According to Salisbury [**?**], a grasp achieves
textbfforce closure if and only if:

1. The set of primitive wrenches from all contact points positively spans the entire wrench space ($\mathbb{R}^6$ in 3D space).

2. There exists a strictly positive linear combination of the primitive wrenches that sums to zero (i.e., an internal force loop).

To satisfy the force closure condition, the primitive wrenches $\mathbf{w}_i$ must satisfy the following:

$$\sum_{i=1}^{m\times n} \beta_i \mathbf{w}_i = \mathbf{0}, \quad \beta_i > 0 \quad \forall i. \tag{7}$$

Here:

- $\mathbf{w}_i$ are the primitive wrenches derived from the edges of the friction cone polyhedron for each contact point.
- β_i are positive coefficients that form a strictly positive linear combination of the wrenches.
- $m \times n$ is the total number of primitive wrenches from all contact points.

The existence of such β_i coefficients ensures that the origin lies within the interior of the convex hull formed by the primitive wrenches, satisfying the force closure condition.

1. **Positive Spanning Condition:** The primitive wrenches must span the entire wrench space, ensuring that any external force or torque can be counterbalanced by a combination of the contact forces.

2. **Strictly Positive Combination:** A strictly positive linear combination summing to zero ensures that the contact forces can form an internal loop, balancing each other without relying on external forces or moments.

Setup of Direction Unit Vectors $\mathbf{d}_k$

The direction unit vectors $\mathbf{d}_k$ can be expressed in terms of the tangential plane defined at each contact point. Let $\mathbf{n}_i$ be the normal vector at the ith contact point. The tangential plane is spanned by two orthogonal unit vectors $\mathbf{t}_i$ and $\mathbf{s}_i$ lying in the plane perpendicular to $\mathbf{n}_i$. The n unit vectors $\mathbf{d}_k$ are then given by:

$$\mathbf{d}_k = \cos(\theta_k)\mathbf{t}_i + \sin(\theta_k)\mathbf{s}_i, \quad \theta_k = \frac{2\pi k}{n}, \quad k = 1, 2, \ldots, n, \tag{8}$$

where:

- n is the number of edges of the polyhedron (e.g., $n = 4$ for a square pyramid or $n = 8$ for an octagonal approximation).
- θ_k is the angle defining the direction of each edge in the tangential plane.
- $\mathbf{t}_i, \mathbf{s}_i$ are two orthonormal vectors spanning the tangential plane.

Procedure to Compute $\mathbf{t}_i$ and $\mathbf{s}_i$

Given the normal vector $\mathbf{n}_i$, the tangential vectors $\mathbf{t}_i$ and $\mathbf{s}_i$ can be computed as follows:

1. Choose an arbitrary vector $\mathbf{a}$ that is not parallel to $\mathbf{n}_i$ (e.g., $\mathbf{a} = [1, 0, 0]^T$ or $[0, 1, 0]^T$).
2. Compute $\mathbf{t}_i$ as:
$$\mathbf{t}_i = \frac{\mathbf{a} \times \mathbf{n}_i}{\|\mathbf{a} \times \mathbf{n}_i\|}. \tag{9}$$
3. Compute $\mathbf{s}_i$ as:
$$\mathbf{s}_i = \mathbf{n}_i \times \mathbf{t}_i. \tag{10}$$
4. Ensure $\mathbf{t}_i$ and $\mathbf{s}_i$ are orthogonal and normalized.

Summary of Unit Vector Setup

The direction unit vectors $\mathbf{d}_k$ are calculated as:

$$\mathbf{d}_k = \cos\left(\frac{2\pi k}{n}\right)\mathbf{t}_i + \sin\left(\frac{2\pi k}{n}\right)\mathbf{s}_i, \quad k = 1, 2, \ldots, n. \tag{11}$$

These vectors evenly approximate the friction cone in the tangential plane at the ith contact point.

Optimization of Grasp Contact Forces

Objective: Minimize the total magnitude of the contact forces:

$$\min \sum_{i=1}^{m} \|\mathbf{f}_i\|^2, \tag{12}$$

where $\mathbf{f}_i \in \mathbb{R}^3$ represents the contact force at the ith contact point, and m is the number of contact points.

Variables:

- $\mathbf{f}_i \in \mathbb{R}^3$: Contact force at the ith contact point.
- $f_{i,n} \geq 0$: Normal component of the contact force at the ith contact point.
- $\alpha_{i,k} \geq 0$: Coefficients for the edges of the friction cone polyhedral approximation at the ith contact point.

Constraints:

1. **Friction Limit:** Friction cone constraint for a single contact point i:

$$\sqrt{f_{s,i}^2 + f_{t,i}^2} \leq \mu_i f_{n,i}$$

Friction cone constraint for all contact points $i = 1, 2, \ldots, n_c$:

$$\sqrt{f_{s,i}^2 + f_{t,i}^2} \leq \mu_i f_{n,i}, \quad \forall i = 1, 2, \ldots, n_c$$

2. **Equilibrium Constraint:**

$$G\mathbf{f} = \mathbf{w}_{\text{ext}}, \tag{13}$$

where G is the grasp matrix, $\mathbf{f}$ is the stacked vector of all contact forces, and $\mathbf{w}_{\text{ext}}$ is the external wrench (force and torque) acting on the object.

3. **Force Closure Constraint:**

$$\sum_{i=1}^{m \times n} \beta_i \mathbf{w}_i = \mathbf{0}, \quad \beta_i > 0, \quad \forall i, \tag{14}$$

where $\mathbf{w}_i$ are the primitive wrenches (edges of the friction cone approximation) across all contact points, and β_i are positive coefficients.

Full Optimization Problem:

Stage 1: Minimum Norm Force Optimization with Friction Cone and Equilibrium Constraints

$$\begin{aligned}
\text{Minimize:} \quad & \sum_{i=1}^{m} \frac{1}{2} \|\mathbf{f}_i\|^2 & (15)\\
\text{Subject to:} \quad & \sqrt{f_{s,i}^2 + f_{t,i}^2} \leq \mu_i f_{n,i}, \quad \forall i = 1, 2, \ldots, n_c & (16)\\
& G\mathbf{f} = -\mathbf{w}_{\text{ext}} & (17)\\
& \sum_{i=1}^{m \times n} \beta_i \mathbf{w}_i = 0, \quad \beta_i > 0, \quad \forall i & (18)
\end{aligned}$$

Stage 2: Linear Program for Force Closure Test

$$\begin{aligned}
\text{Minimize:} \quad & \sum_{i=1}^{m \times n} \beta_i & (19)\\
\text{Subject to:} \quad & \sum_{i=1}^{m \times n} \beta_i \mathbf{w}_i = 0 & (20)\\
& \beta_i > 0, \quad \forall i & (21)
\end{aligned}$$

Alternative Linear friction cone Problem Formulation

Contact Force Decomposition

At each contact point i, the contact force $\mathbf{f}_i \in \mathbb{R}^3$ can be decomposed into:

$$\mathbf{f}_i = f_{n,i}\mathbf{n}_i + \sum_{j=1}^{k_e} \alpha_{ij}\mathbf{d}_{ij} \tag{15}$$

where:

- $\mathbf{n}_i \in \mathbb{R}^3$: Unit normal vector at contact i.
- $\mathbf{d}_{ij} \in \mathbb{R}^3$: Unit direction vectors approximating the friction cone at contact i. These vectors lie in the tangent plane orthogonal to $\mathbf{n}_i$.
- $f_{n,i} \geq 0$: Normal force magnitude at contact i.
- $\alpha_{ij} \geq 0$: Positive scalar weights representing tangential force components in direction $\mathbf{d}_{ij}$.
- k_e: Number of linearized friction cone edges per contact.

Collecting all contact forces into a single vector $\mathbf{f} \in \mathbb{R}^{3n_c}$ for n_c contacts, we have:

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_{n_c} \end{bmatrix} \tag{16}$$

Wrench Equilibrium Constraint

The grasp matrix $\mathbf{G} \in \mathbb{R}^{6\times 3n_c}$ maps contact forces to the object wrench space. The equilibrium condition requires:

$$\mathbf{G}\mathbf{f} = -\mathbf{w}_{\text{ext}} \tag{17}$$

or explicitly:

$$\mathbf{G}_N\mathbf{f}_n + \mathbf{G}_D\alpha = -\mathbf{w}_{\text{ext}} \tag{18}$$

where:

- $\mathbf{f}_n = [f_{n,1}, \ldots, f_{n,n_c}]^\top \in \mathbb{R}^{n_c}$
- $\boldsymbol{\alpha} = [\alpha_{11}, \ldots, \alpha_{1k_e}, \ldots, \alpha_{n_c1}, \ldots, \alpha_{n_ck_e}]^\top \in \mathbb{R}^{n_ck_e}$
- $\mathbf{G}_N = \mathbf{G}\mathbf{N} \in \mathbb{R}^{6\times n_c}$ with $\mathbf{N} \in \mathbb{R}^{3n_c\times n_c}$: matrix of contact normals.
- $\mathbf{G}_D = \mathbf{G}\mathbf{D} \in \mathbb{R}^{6\times n_ck_e}$ with $\mathbf{D} \in \mathbb{R}^{3n_c\times n_ck_e}$: block diagonal matrix of tangential vectors.

Friction Cone Inequality Constraints

At each contact i, the linearized friction cone inequality is:

$$\mu_i f_{n,i} \geq \sum_{j=1}^{k_e} \alpha_{ij} \tag{19}$$

or equivalently for all contacts:

$$\boldsymbol{\mu} \odot \mathbf{f}_n - \mathbf{A}_{\text{ineq}} \boldsymbol{\alpha} \geq \mathbf{0} \tag{20}$$

where:

- $\boldsymbol{\mu} = [\mu_1, \dots, \mu_{n_c}]^\top$ are the friction coefficients.
- $\odot$ denotes element-wise multiplication.
- $\mathbf{A}_{\text{ineq}} \in \mathbb{R}^{n_c \times n_c k_e}$: block diagonal matrix where each block is a row of ones $[1 \dots 1]$.

Non-Negative Normal Force and Non-Negative Alpha

$$f_{n,i} \geq \epsilon, \quad \forall i = 1, \dots, n_c \tag{21}$$

$$\alpha_{ij} \geq 0, \quad \forall i = 1, \dots, n_c,\ j = 1, \dots, k_e \tag{22}$$

Objective Function

Minimize the squared norm of the contact forces (normal and tangential components):

$$\min_{\mathbf{f}_n, \boldsymbol{\alpha}} \frac{1}{2} \left(\|\mathbf{f}_n\|^2 + \|\boldsymbol{\alpha}\|^2 \right) \tag{23}$$

Full Optimization Problem

$$\min_{\mathbf{f}_n, \boldsymbol{\alpha}} \quad \frac{1}{2} \left(\|\mathbf{f}_n\|^2 + \|\boldsymbol{\alpha}\|^2 \right) \tag{24}$$

$$\text{subject to} \quad \mathbf{G}_N \mathbf{f}_n + \mathbf{G}_D \boldsymbol{\alpha} = -\mathbf{w}_{\text{ext}} \tag{25}$$

$$\boldsymbol{\mu} \odot \mathbf{f}_n - \mathbf{A}_{\text{ineq}} \boldsymbol{\alpha} \geq \mathbf{0} \tag{26}$$

$$\mathbf{f}_n \geq \epsilon \tag{27}$$

$$\boldsymbol{\alpha} \geq \mathbf{0} \tag{28}$$

Alternative LP force closure test

Let number of generators be ke = 3

$$
\begin{aligned}
f_i &= S_i \sigma_i \\
f_i &= \begin{bmatrix} 1 & 1 & 1 \\ \mu \cos \frac{2\pi}{3} & \mu \cos \frac{4\pi}{3} & \mu \cos 2\pi \\ \mu \sin \frac{2\pi}{3} & \mu \sin \frac{4\pi}{3} & \mu \sin 2\pi \end{bmatrix} \begin{bmatrix} \sigma_{i1} \\ \sigma_{i2} \\ \sigma_{i3} \end{bmatrix} \\
f_i &= \begin{bmatrix} \sigma_{i1} + \sigma_{i2} + \sigma_{i3} \\ \sigma_{i1} \mu \cos \frac{2\pi}{3} + \sigma_{i2} \mu \cos \frac{4\pi}{3} + \sigma_{i3} \mu \cos 2\pi \\ \sigma_{i1} \mu \sin \frac{2\pi}{3} + \sigma_{i2} \mu \sin \frac{4\pi}{3} + \sigma_{i3} \mu \sin 2\pi \end{bmatrix}
\end{aligned}
\tag{29}
$$

$$
f_i = f_{n,i} n_i + f_{s,i} s_i + f_{t,i} t_i \tag{30}
$$

Approximate Force Closure Tests

Any of the friction cones discussed can be approximated as the non-negative span of a finite number n_g of generators s_{ij} of the friction cone. Given this, one can represent the set of applicable contact wrenches at contact i as follows:

$$
\mathbf{G}_i \boldsymbol{\lambda}_i = \mathbf{S}_i \boldsymbol{\sigma}_i, \quad \boldsymbol{\sigma}_i \geq 0
$$

where $\mathbf{S}_i = \begin{bmatrix} s_{i1} \cdots s_{ing} \end{bmatrix}$ and $\boldsymbol{\sigma}_i$ is a vector of non-negative generator weights.

Generators of the Friction Cone

If contact i is frictionless,

$$
n_g = 1, \quad \mathbf{S}_i = \begin{bmatrix} \hat{\mathbf{n}}_i^\top & ((\mathbf{c}_i - \mathbf{p}) \times \hat{\mathbf{n}}_i)^\top \end{bmatrix}^\top
$$

If contact i is of type HF, we represent the friction cone by the non-negative sum of uniformly spaced contact force generators whose non-negative span approximates the Coulomb cone with an inscribed regular polyhedral cone.

This leads to the following definition of $\mathbf{S}_i$:

$$
\mathbf{S}_i = \begin{pmatrix} \cdots & 1 & \cdots \\ \cdots & \mu_i \cos\left(\frac{2k\pi}{n_g}\right) & \cdots \\ \cdots & \mu_i \sin\left(\frac{2k\pi}{n_g}\right) & \cdots \end{pmatrix}
$$

where k varies from 1 to n_g.

If one prefers to approximate the quadratic friction cone by a circumscribing polyhedral cone, one simply replaces μ_i in the above definition with $\mu_i / \cos(\pi/n_g)$.

Soft Finger Model (SF)

The adjustment needed for the SF model is:

$$
\mathbf{S}_i = \left(\begin{array}{ccc|cc} \cdots & 1 & \cdots & 1 & 1 \\ \cdots & \mu_i \cos\left(\frac{2k\pi}{n_g}\right) & \cdots & 0 & 0 \\ \cdots & \mu_i \sin\left(\frac{2k\pi}{n_g}\right) & \cdots & 0 & 0 \\ \hline \cdots & 0 & \cdots & b\nu_i & -b\nu_i \end{array}\right)
$$

where b is the characteristic length used to unify units.

Contact Wrench Set

The set of total contact wrenches that may be applied by the hand without violating the contact friction law at any contact can be written as:

$$
\mathbf{G}\boldsymbol{\lambda} = \mathbf{S}\boldsymbol{\sigma}, \quad \boldsymbol{\sigma} \geq 0
$$

where

$$
\mathbf{S} = \left[\mathbf{S}_1, \ldots, \mathbf{S}_{n_c}\right]
$$

and

$$
\boldsymbol{\sigma} = \left[\boldsymbol{\sigma}_1^\top \cdots \boldsymbol{\sigma}_{n_g}^\top\right]^\top
$$

Dual Formulation of the Friction Constraints

It is convenient to reformulate the friction constraints in a dual form:

$$
\mathbf{F}_i \boldsymbol{\lambda}_i \geq 0
$$

Each row of $\mathbf{F}_i$ is normal to a face formed by two adjacent generators of the approximate cone. For an HF contact, row i of $\mathbf{F}_i$ can be computed as the cross product of s_i and s_{i+1}. In the case of an SF contact, the generators are of dimension four, so simple cross products will not suffice. General methods exist to perform the conversion from the generator form to the face normal form.

The face normal constraints for all contacts can be combined into the following compact form:

$$
\mathbf{F}\boldsymbol{\lambda} \geq 0
$$

where

$$
\mathbf{F} = \text{Blockdiag}(\mathbf{F}_1, \ldots, \mathbf{F}_{n_c})
$$

LP2: Frictional Form Closure Test

Let $\mathbf{e}_i \in \mathbb{R}^{n_{\lambda_i}}$ be the first row of $\mathbf{H}_i$. Further let:

$$
\mathbf{e} = \left[\mathbf{e}_1, \ldots, \mathbf{e}_{n_c}\right] \in \mathbb{R}^{n_\lambda}
$$

and

$$\mathbf{E} = \text{Blockdiag}(\mathbf{e}_1, \ldots, \mathbf{e}_{n_c}) \in \mathbb{R}^{n_\lambda \times n_c}$$

The following linear program is a quantitative test for frictional form closure. The optimal objective function value d^* is a measure of the distance the contact forces are from the boundaries of their friction cones, and hence a crude measure of how far a grasp is from losing frictional form closure.

Linear Program LP2

$$\begin{aligned}
\text{maximize:} \quad & d \\
\text{subject to:} \quad & \mathbf{G}\boldsymbol{\lambda} = -\mathbf{w}_{ext} \\
& \mathbf{F}\boldsymbol{\lambda} - \mathbf{1}d \geq \mathbf{0} \\
& d \geq 0 \\
& \mathbf{e}^\top \boldsymbol{\lambda} \leq n_c
\end{aligned}$$

The last inequality in LP2 is simply the sum of the magnitudes of the normal components of the contact forces.

LP3: Force Closure Verification

If the grasp has frictional form closure, the last step to determine the existence of force closure is to verify the condition:

$$\mathcal{N}(\mathbf{G}) \cap \mathcal{N}(\mathbf{J}^\top) = \mathbf{0}$$

This condition is easy to verify with another linear program LP3:

Linear Program LP3

$$\begin{aligned}
\text{maximize:} \quad & d \\
\text{subject to:} \quad & \mathbf{G}\boldsymbol{\lambda} = \mathbf{0} \\
& \mathbf{J}^\top \boldsymbol{\lambda} = \mathbf{0} \\
& \mathbf{E}\boldsymbol{\lambda} - \mathbf{1}d \geq 0 \\
& d \geq 0 \\
& \mathbf{e}^\top \boldsymbol{\lambda} \leq n_c
\end{aligned}$$

Summary of Force Closure Testing

1. Compute rank($\mathbf{G}$):

   (a) If rank($\mathbf{G}$) $\neq n_v$, then force closure does not exist. Stop.

   (b) If $\text{rank}(\mathbf{G}) = n_v$, continue.

2. Solve LP2: Test *frictional form closure*.

   (a) If $d^* = 0$, then frictional form closure does not exist. Stop.

   (b) If $d^* > 0$, then frictional form closure exists and d^* is a crude measure of how far the grasp is from losing frictional form closure.

3. Solve LP3: Verify *force closure*.

Modifying LP2 for force closure for a given external wrench acting on the object

$$
\begin{aligned}
\text{maximize:} \quad & d \\
\text{subject to:} \quad & \mathbf{G}\boldsymbol{\lambda} = -\mathbf{w}_{ext} \\
& \mathbf{F}\boldsymbol{\lambda} - \mathbf{1}d \geq 0 \\
& d \geq 0 \\
& \mathbf{e}^\top \boldsymbol{\lambda} \leq n_c
\end{aligned}
\tag{31}
$$