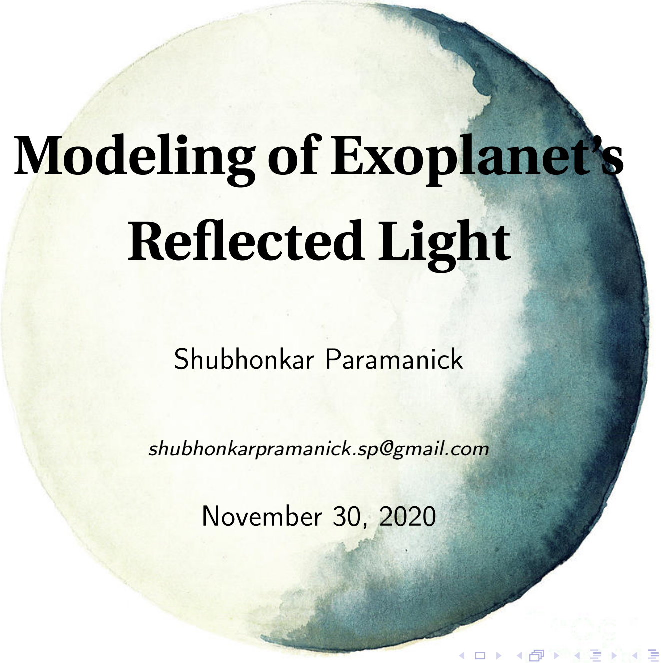

Modeling of Exoplanet's Reflected Light

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Overview

1. Description of the Geometry
2. Plane Parallel Rays
3. Obtaining the Phase Curve

General Problem

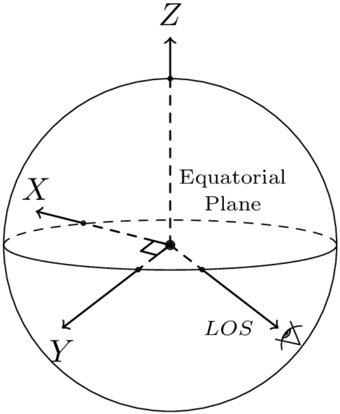
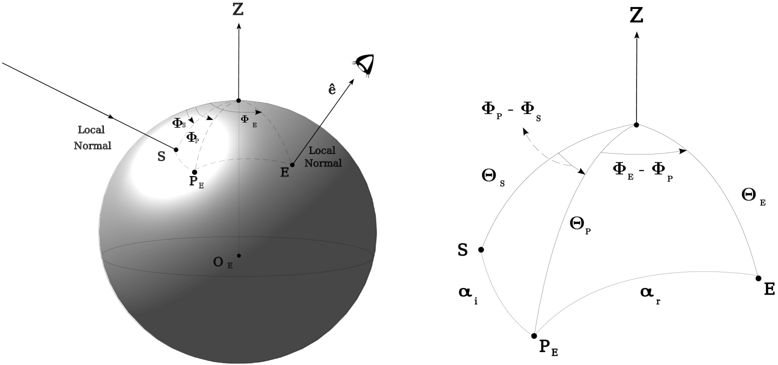


Figure: Planeto-centric Frame



(a) Sphere irradiated at S. (b) Arcs on the Geodesics.

$$\cos(\alpha_i) = \cos(\Theta_S)\cos(\Theta_P) + \sin(\Theta_S)\sin(\Theta_P)\cos(\Phi_P - \Phi_S)$$

$$\cos(\alpha_r) = \cos(\Theta_E)\cos(\Theta_P) + \sin(\Theta_E)\sin(\Theta_P)\cos(\Phi_E - \Phi_P)$$

Surface Flux

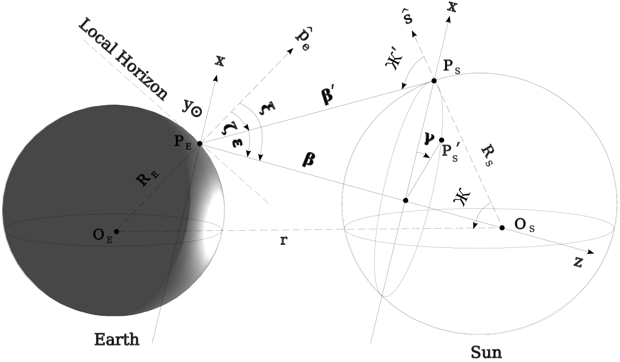


Figure: Geometry to determine the intensity of light received at some point on the planet (P_E) from a point P_S on the star.

$$\mathscr{F}_{E,Ref.}(R_E, \Theta_P, \Phi_P, \nu, t) = \int_{\Omega} \mathscr{I}_{E,Ref.}(R_E, \Theta_P, \Phi_P, \nu, t)\, \hat{e} \cdot \widehat{p_e}\; d\Omega_{E\,@\,Earth}$$

Continued...

If the reflected intensity distribution is isotropic (i.e. the surface of the planet is Lambertian),

$$
\begin{aligned}
\mathscr{F}_{\mathrm{E,Ref.}}(\mathrm{R_E}, \Theta_\mathrm{P}, \Phi_\mathrm{P}, \nu, \mathrm{t}) &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \mathscr{I}_{\mathrm{E,Ref.}}(\mathrm{R_E}, \Theta_\mathrm{P}, \Phi_\mathrm{P}, \nu, \mathrm{t}) \cos\alpha_\mathrm{r} \, \sin\alpha_\mathrm{r} \, \mathrm{d}\alpha_\mathrm{r} \, \mathrm{d}\omega_\mathrm{r} \\
&= \pi \times \mathscr{I}_{\mathrm{E,Ref.}}(\mathrm{R_E}, \Theta_\mathrm{P}, \Phi_\mathrm{P}, \nu, t)
\end{aligned}
$$

where $\mathscr{F}_{\mathrm{E,Ref.}}$ is the emergent flux and $\mathscr{I}_{\mathrm{E,Ref.}}$ is the outgoing intensity from the planet's surface.

$$
\mathscr{F}_{\mathrm{E,Ref.}}(\mathrm{R_E}, \Theta_\mathrm{P}, \Phi_\mathrm{P}, \nu, \mathrm{t}) = \mathscr{F}_{\mathrm{E,Ref.}}(\Theta_\mathrm{P}, \Phi_\mathrm{P}, \nu, \mathrm{t})
$$

Let $\mathrm{I}_{\mathrm{Emit.}}(\text{Ж})$ be the intensity of radiation emitted from the point P_S on the star, Then the reflected flux can be written as

Intensity Distribution of the Reflected Light

$$\mathscr{F}_{\mathrm{E,Ref.}}(\Theta_{\mathrm{P}}, \Phi_{\mathrm{P}}, \nu, \mathrm{t}) = \int\limits_{\substack{\Omega\,=\,\text{Area of the}\\ \text{Star visible @ E}}} \mathrm{I}_{\mathrm{Emit.}}(\text{Ӂ})\hat{s}\cdot\widehat{\beta'}\mathbb{R}(\alpha_r, \omega_r, \Theta_{\mathrm{P}}, \Phi_{\mathrm{P}})\, d\Omega_{\text{Star @ P}_{\mathrm{E}}}$$

$$= \frac{\mathbb{R}(\alpha_{\mathrm{r}}, \omega_{\mathrm{r}}, \Theta_{\mathrm{P}}, \Phi_{\mathrm{P}}) \times {R_S}^2}{\beta'^2} \int\limits_{\substack{\text{Area of the}\\ \text{Star visible @ E}}} \mathrm{I}_{\mathrm{Emit.}}(\text{Ӂ}) \cos \text{Ӂ}' \sin \text{Ӂ} \cos\zeta \, d\text{Ӂ}\, d\gamma$$

$$\mathscr{I}_{\mathrm{E,Ref.}}(\Theta_{\mathrm{P}}, \Phi_{\mathrm{P}}, \nu, \mathrm{t}) = \frac{\mathbb{R}(\alpha_r, \omega_r, \Theta_{\mathrm{P}}, \Phi_{\mathrm{P}}) \times \mathrm{R_S}^2}{\pi\beta'^2} \times \int\limits_{\substack{\text{Area of the}\\ \text{Star visible @ E}}} \mathrm{I}_{\mathrm{Emit.}}(\text{Ӂ}) \cos \text{Ӂ}' \sin \text{Ӂ} \cos\zeta \, d\text{Ӂ}\, d\gamma$$

Reflected Luminosity

The luminosity of the planet ($\mathscr{L}_{\mathrm{E,Ref.}}(\alpha_r, \nu, \mathrm{t})$) on Earth is given by

$$\mathscr{L}_{\mathrm{E,Ref.}}(\alpha_r, \nu, \mathrm{t}) = \int\limits_{\substack{\mathrm{A\,=\,Planet\ \ area} \\ \mathrm{visible\ to\ Earth}}} \overrightarrow{\mathscr{F}_{\mathrm{E,Ref.}}(\Theta_{\mathrm{P}}, \Phi_{\mathrm{P}}, \nu, \mathrm{t})} \cdot \widehat{p_e}\ \mathrm{dA_E}$$

$$= \int\limits_{\substack{\mathrm{A\,=\,Planet\ \ area} \\ \mathrm{visible\ to\ Earth}}} \mathscr{F}_{\mathrm{E,Ref.}}(\Theta_{\mathrm{P}}, \Phi_{\mathrm{P}}, \nu, \mathrm{t})\, \hat{e} \cdot \widehat{p_e}\ \mathrm{dA_E}$$

$$= \int\limits_{\substack{\mathrm{A\,=\,Planet\ \ area} \\ \mathrm{visible\ to\ Earth}}} \mathscr{F}_{\mathrm{E,Ref.}}(\Theta_{\mathrm{P}}, \Phi_{\mathrm{P}}, \nu, \mathrm{t})\, \cos\alpha_r\, \mathrm{R_E}^2\, \sin\Theta_{\mathrm{P}}\, \mathrm{d}\Theta_{\mathrm{P}}\, \mathrm{d}\Phi_{\mathrm{P}}$$

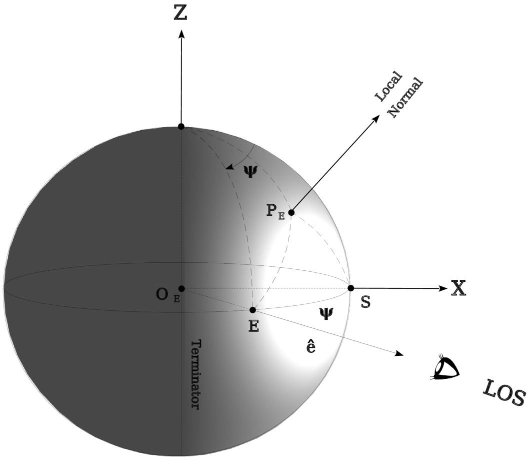
Far-Off Approximations

$$\beta \approx \beta' \approx r$$

$$Ж \approx Ж'$$

$$\zeta \approx \xi$$

$$\epsilon \to 0$$



Luminosity in the Far-Off Approximations

When the observer is along the planet's equator,

$$\begin{aligned}\cos(\alpha_{\mathrm{i}}) &= \cos(\frac{\pi}{2})\cos(\Theta_{\mathrm{P}}) + \sin(\frac{\pi}{2})\,\sin(\Theta_{\mathrm{P}})\,\cos(\frac{\pi}{2} + \psi - \Phi_{\mathrm{P}}) \\ &= -\sin(\Theta_{\mathrm{P}})\,\sin(\psi - \Phi_{\mathrm{P}})\end{aligned}$$

$$\begin{aligned}\cos(\alpha_{\mathrm{r}}) = \; &= \cos(\frac{\pi}{2})\cos(\Theta_{\mathrm{P}}) + \sin(\frac{\pi}{2})\,\sin(\Theta_{\mathrm{P}})\,\cos(\Phi_{\mathrm{P}} - \frac{\pi}{2}) \\ &= \sin(\Theta_{\mathrm{P}})\,\sin(\Phi_{\mathrm{P}})\end{aligned}$$

Then,

$$\mathscr{L}_{\mathrm{E,Ref.}}(\alpha_{\mathrm{r}}, \nu, \mathrm{t}) = \int_{\substack{\mathrm{A} = \text{Planet area} \\ \text{visible to Earth}}} \mathscr{F}_{\mathrm{E,Ref.}}(\Theta_{\mathrm{P}}, \Phi_{\mathrm{P}}, \nu, \mathrm{t})\,\cos\alpha_{\mathrm{r}}\,\mathrm{dA_E}$$

Luminosity Continued...

$$= \pi\, \mathrm{I}_0\, \mathbb{R}(\Theta_\mathrm{P}, \Phi_\mathrm{P}) \left(\frac{\mathrm{R}_\mathrm{P}\, \mathrm{R}_\mathrm{S}}{\mathrm{r}}\right)^2 \times \int_{\psi}^{\pi}\int_{0}^{\pi} \sin^3\Theta_\mathrm{P}\ \sin\Phi\ \sin\left(\Phi_\mathrm{P} - \psi\right) d\Theta_\mathrm{P}\, d\Phi_\mathrm{P}$$

$$= \pi\, \mathrm{I}_0\, \mathbb{R}(\Theta_\mathrm{P}, \Phi_\mathrm{P}) \left(\frac{\mathrm{R}_\mathrm{P}\, \mathrm{R}_\mathrm{S}}{\mathrm{r}}\right)^2 f(\psi) = \mathscr{L}_{\mathrm{E,Ref.}}(\psi, \Theta_\mathrm{P}, \Phi_\mathrm{P})$$

$$\mathscr{L}_{\mathrm{E,Ref.}}(\Theta_\mathrm{P}, \Phi_\mathrm{P}, 0, \nu, \mathrm{t}) = \pi\, \mathrm{I}_0\, \mathbb{R}(\Theta_\mathrm{P}, \Phi_\mathrm{P}) \left(\frac{\mathrm{R}_\mathrm{P}\, \mathrm{R}_\mathrm{S}}{\mathrm{r}}\right)^2 f(0, \nu, \mathrm{t})$$

Where I_0 is the intensity distribution of the stellar disk.

$$\begin{aligned}\mathscr{L}_{\mathrm{E,Ref.}}(\Theta_\mathrm{P}, \Phi_\mathrm{P}, \psi, \nu, \mathrm{t}) &= \mathscr{L}_{\mathrm{E,Ref.}}(\Theta_\mathrm{P}, \Phi_\mathrm{P}, 0, \nu, \mathrm{t}) \times \left(\frac{f(\psi, \nu, \mathrm{t})}{f(0, \nu, \mathrm{t})}\right) \\ &= \mathscr{L}_{\mathrm{E,Ref.}}(\Theta_\mathrm{P}, \Phi_\mathrm{P}, 0, \nu, \mathrm{t})\, \Psi(\psi, \nu, \mathrm{t})\end{aligned}$$

Geometric Albedo

$$\mathrm{A_G} = \frac{\mathscr{L}_{\mathrm{E,Ref.}}(\Theta_\mathrm{P}, \Phi_\mathrm{P}, 0, \nu, \mathrm{t})}{\mathscr{L}_{\mathrm{Lambert}}} = \frac{\mathscr{F}_{\mathrm{E,Ref.}}(\Theta_\mathrm{P}, \Phi_\mathrm{P}, 0, \nu, \mathrm{t})}{\mathscr{F}_{\mathrm{Lambert}}}$$

Where,

$$\mathscr{L}_{\mathrm{Lambert}} = \mathscr{F}_{\mathrm{Lambert}} \times \pi \, \mathrm{R_P}^2 = (\pi \mathrm{R_P})^2 \times \mathrm{I}_{\mathrm{Emit.}} \left(\frac{\mathrm{R_S}}{\mathrm{r}}\right)^2$$

$$\mathscr{L}_{\mathrm{E,Ref.}}(\Theta_\mathrm{P}, \Phi_\mathrm{P}, \psi, \nu, \mathrm{t}) = \mathscr{L}_{\mathrm{Lambert}} \times \mathrm{A_G} \times \Psi(\psi, \nu, \mathrm{t})$$

Disc-Integrated Flux

Reflected flux of the Planet received at Earth

$$= \frac{\mathscr{L}_{\mathsf{E,Ref.}}(\Theta_{\mathsf{P}}, \Phi_{\mathsf{P}}, \psi, \nu, \mathsf{t})}{4\pi\mathscr{R}^2}$$

Flux received from the star at Earth

$$= \frac{\mathscr{L}_{\mathsf{S}}}{4\pi\mathscr{R}^2}$$

where $\mathscr{R}$ is the distance between the planetary system and Earth.

Reflected Fractional Flux

$$\mathfrak{F}_{\mathfrak{f}} = \left(\frac{\mathrm{R_P}}{\mathrm{r}}\right)^2 \times \mathrm{A_G} \times \Psi(\psi, \nu, \mathrm{t})$$

$$f(\psi) = \frac{2}{3} \times \mathbb{R} \times \left\{\frac{(\pi - \psi)\cos\psi + \sin\psi}{\pi}\right\}$$

$$\mathfrak{F}_{\mathfrak{f}} = \left\{\frac{\mathrm{R_P}}{\mathrm{r}}\right\}^2 \times \mathrm{A_G} \times \left\{\frac{(\pi - \psi)\cos\psi + \sin\psi}{4\pi}\right\}$$

Obtaining the Phase Curve

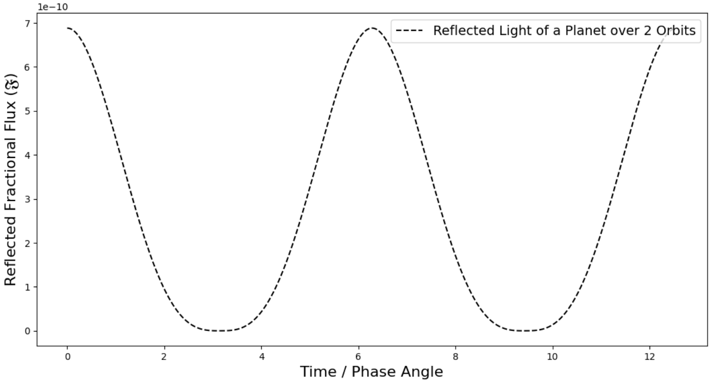


Figure: Fractional Flux

Plots Continued...

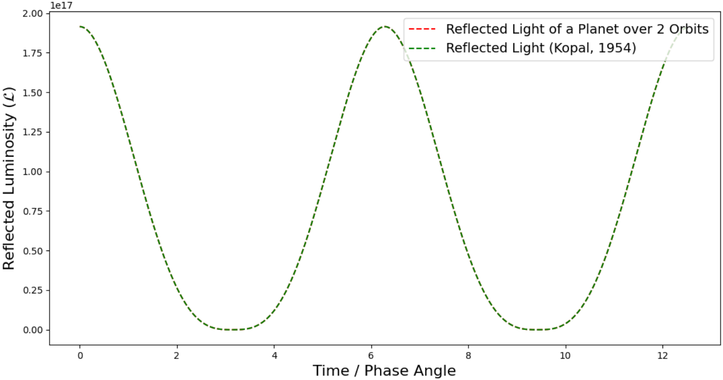


Figure: Luminosity

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