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Dear IYMC Organizers,

I am Shubham Nahata, currently a pre-final year undergraduate student pursuing M.Sc.(Hons.) Mathematics and B.E.(Hons.) Electronics & Communication Engineering at Birla Institute of Technology and Science Pilani, Pilani. It is my pleasure to submit my solutions to the International Youth Math Challenge in the Senior Category. I greatly appreciate the opportunity to participate in such a challenging and rewarding experience. It was really fun attempting the problems provided.

Honestly, it was a trip down memory lane to my high school days. My favorite problems among all the five problems were Problem A and Problem E. I have always loved Geometry and Sequences & Series, and I could solve both problems in very little time. I also chose the route of typing the solutions digitally since I wanted to learn LaTeX, and now I can appreciate how intricate and involving digitally typing mathematical equations can be. I am grateful to the organizers for providing this facility, as now I can proudly say I have learnt a new and crucial skill.

I would also like to share the approach I followed while solving the problems. I adopted the mindset of teacher-learner pedagogy while solving and writing my solution. I assumed that the reader of my solution needs to learn from scratch about that particular problem and should feel confident about the understanding of the problem. I wrote down all the solutions by hand first and then converted the same to LaTeX when I had verified all my solutions and the steps involved. I have also highlighted the breakdown of skillsets and mathematical concepts involved for every problem. I believe that every problem can be broken down into a collection of interconnected concepts and ideas, and the beauty behind all this is that even the most difficult problems are made up of elementary and fundamental concepts.

I would also like to highlight that a few problems could be solved by multiple solutions, but in the interest of simplicity and utilizing fewer pages, I have presented my best and most favorite solution — the one that I could think about first.

In conclusion, I would like to reiterate my gratefulness to the organizers for this incredible opportunity. I hope and pray that I qualify for the next round.

Thanking You,

Yours sincerely,
Shubham Nahata,
Math Lover

Solution: Problem A

The Problem Statement:
We are given two sequences of numbers:

n	1	2	3	4	5	6
a_n	2	5	10	17	26	37
b_n	1	2	8	48	384	3840

The problem states to analyze each sequence, continue the pattern, and find a closed-form (an explicit formula) for the n-th term of each sequence, a_n and b_n.

Detailed solution for Problem A...

Part 1: Deriving the Formula for a_n

1. **We inspect the given terms for a_n:**
The sequence is:

$$a_1 = 2, \quad a_2 = 5, \quad a_3 = 10, \quad a_4 = 17, \quad a_5 = 26, \quad a_6 = 37.$$

We have six terms. To find a pattern, a common method and my first instinct was to look at the differences between consecutive terms.

2. **First Differences Δa_n:**
We compute $\Delta a_n = a_{n+1} - a_n$ for each pair of consecutive terms:

$$a_2 - a_1 = 5 - 2 = 3, \quad a_3 - a_2 = 10 - 5 = 5, \quad a_4 - a_3 = 17 - 10 = 7, \quad a_5 - a_4 = 26 - 17 = 9, \quad a_6 - a_5 = 37 -$$

So the first differences are:

$$\Delta a_n : 3, 5, 7, 9, 11, \ldots$$

We observe that these differences form an arithmetic progression with a common difference of 2.

3. **Second Differences $\Delta^2 a_n$:**
Now, the second differences are checked which are the differences of the first differences:

$$5 - 3 = 2, \quad 7 - 5 = 2, \quad 9 - 7 = 2, \quad 11 - 9 = 2.$$

All second differences are constant and equal to 2. A sequence with a constant second difference indicates that a_n can be expressed as a quadratic polynomial in n.

4. **We will assume a Quadratic Form:**
Since we have constant second differences, we assume:

$$a_n = An^2 + Bn + C,$$

where A, B, C are constants we need to determine.

5. **Plugging in the known terms:**
Using the first three terms of the sequence (a_1, a_2, a_3) to form three equations in terms of A, B, C.

- For $n = 1$, $a_1 = 2$:

$$A(1)^2 + B(1) + C = A + B + C = 2.$$

So:

$$A + B + C = 2. \quad (1)$$

- For $n = 2$, $a_2 = 5$:

$$A(2^2) + B(2) + C = 4A + 2B + C = 5.$$

So:

$$4A + 2B + C = 5. \quad (2)$$

- For $n = 3$, $a_3 = 10$:

$$A(3^2) + B(3) + C = 9A + 3B + C = 10.$$

So:

$$9A + 3B + C = 10. \quad (3)$$

Now we have three linear equations:

$$(1)\ A + B + C = 2$$

$$(2)\ 4A + 2B + C = 5$$

$$(3)\ 9A + 3B + C = 10$$

6. **We now solve the system for A, B, C:**
To solve, we eliminate variables step-by-step:

Step A: Subtract Equation (1) from Equation (2):
$(2) - (1)$:

$$(4A + 2B + C) - (A + B + C) = 5 - 2.$$

On the left: - The C cancels. - $4A - A = 3A$. - $2B - B = B$. On the right: $5 - 2 = 3$.
Thus:

$$3A + B = 3. \quad (4)$$

Step B: Subtract Equation (1) from Equation (3):
$(3) - (1)$:

$$(9A + 3B + C) - (A + B + C) = 10 - 2.$$

On the left: - C cancels. - $9A - A = 8A$. - $3B - B = 2B$. On the right: $10 - 2 = 8$.
We have:

$$8A + 2B = 8. \quad (5)$$

Step C: Simplify Equation (5):
Divide (5) by 2:

$$4A + B = 4. \quad (6)$$

Now compare (4) and (6): From (4): $3A + B = 3$. From (6): $4A + B = 4$.

Step D: Subtract (4) from (6):

$$(4A + B) - (3A + B) = 4 - 3.$$

Left side: - B cancels. - $4A - 3A = A$. Right side:

$$4 - 3 = 1.$$

So:

$$A = 1.$$

With $A = 1$, substitute into (4) $3A + B = 3$:

$$3(1) + B = 3 \implies 3 + B = 3 \implies B = 0.$$

Now $A = 1, B = 0$. From (1): $A + B + C = 2$:

$$1 + 0 + C = 2 \implies C = 1.$$

Thus:

$$A = 1,\ B = 0,\ C = 1.$$

7. **Final Quadratic Formula for a_n:**

$$a_n = n^2 + 1.$$

$$\boxed{a_n = n^2 + 1.}$$

Part 2: Deriving the Formula for b_n

1. **We first inspecting the given terms for b_n:**
The sequence is:

$$b_1 = 1, \quad b_2 = 2, \quad b_3 = 8, \quad b_4 = 48, \quad b_5 = 384, \quad b_6 = 3840.$$

The terms grow very fast which is key hint. We will hendforth look at the ratio of consecutive terms.

2. **Check Ratios $\frac{b_{n+1}}{b_n}$:**
Compute:

$$\frac{b_2}{b_1} = \frac{2}{1} = 2, \quad \frac{b_3}{b_2} = \frac{8}{2} = 4, \quad \frac{b_4}{b_3} = \frac{48}{8} = 6, \quad \frac{b_5}{b_4} = \frac{384}{48} = 8, \quad \frac{b_6}{b_5} = \frac{3840}{384} = 10.$$

The ratios are:

$$2, 4, 6, 8, 10, \ldots$$

These ratios form an arithmetic sequence starting at 2 and increasing by 2 each time. The k-th ratio is $2k$.

3. **Constructing b_n using ratios:**
Start with $b_1 = 1$:
- To get b_2: multiply b_1 by 2:

$$b_2 = 1 \times 2.$$

- To get b_3: multiply b_2 by 4:

$$b_3 = (1 \times 2) \times 4 = 1 \times 2 \times 4.$$

- To get b_4: multiply b_3 by 6:

$$b_4 = (1 \times 2 \times 4) \times 6 = 1 \times 2 \times 4 \times 6.$$

- To get b_5: multiply b_4 by 8:

$$b_5 = (1 \times 2 \times 4 \times 6) \times 8 = 1 \times 2 \times 4 \times 6 \times 8.$$

4. **Visualizing the Pattern ("b Triangle"):**
We arrange factors in a triangular form:

$$\begin{array}{c} b_1 : \ 1 \\ \\ b_2 : \ 1 \ \times 2 \\ \\ b_3 : \ 1 \ \times 2 \ \times 4 \\ \\ b_4 : \ 1 \ \times 2 \ \times 4 \ \times 6 \\ \\ b_5 : \ 1 \ \times 2 \ \times 4 \ \times 6 \ \times 8 \end{array}$$

Each b_n (for $n > 1$) is formed by multiplying together even numbers starting from 2.

5. **Expressing b_n in General Form:**
For $n > 1$:

$$b_n = 1 \times 2 \times 4 \times 6 \times \cdots \times [2(n-1)].$$

There are $(n-1)$ even factors. Write them as $2k$:

$$b_n = \prod_{k=1}^{n-1} (2k).$$

Factor out 2 from each term:

$$b_n = 2^{n-1}(1 \cdot 2 \cdot 3 \cdots (n-1)) = 2^{n-1}(n-1)!.$$

Check for a few terms matches perfectly.

$$\boxed{b_n = 2^{n-1}(n-1)!}.$$

Mathematical concepts involved in Problem A

- **Arithmetic Progressions (AP):** Used to identify patterns in the first differences of a_n and the sequence of ratios for b_n.
- **Constant Second Differences & Quadratic Sequences:** For a_n, the constant second difference confirmed it can be represented by a quadratic polynomial.
- **Solving Linear Equations:** Found A, B, C for a_n by forming and solving a system of linear equations.
- **Factorials:** To recognize the product pattern for b_n led to a factorial expression $(n-1)!$ and a power of 2.
- **Decomposition into Basic Factors:** For b_n, factoring out 2 from every even factor gave a neat closed form.

Final Answers

$$\boxed{a_n = n^2 + 1, \quad b_n = 2^{n-1}(n-1)!.}$$

**************** END OF SOLUTION: A ****************

Solution: Problem B

Given Problem: Find all $x \in \mathbb{R}$ that solve the equation:

$$x^4 + x^2 - x - 1 = 1 - x - x^2 - x^4.$$

Deatiled solution for Problem B...

1. **We start with the given equation:**

$$x^4 + x^2 - x - 1 = 1 - x - x^2 - x^4.$$

2. **Bringing all the terms to one side:** Subtract $(1 - x - x^2 - x^4)$ from both sides:

$$x^4 + x^2 - x - 1 - (1 - x - x^2 - x^4) = 0.$$

Distributing the minus sign:

$$x^4 + x^2 - x - 1 - 1 + x + x^2 + x^4 = 0.$$

3. **We then combine like terms:** - x^4 terms: $x^4 + x^4 = 2x^4$. - x^2 terms: $x^2 + x^2 = 2x^2$. - x terms: $-x + x = 0$. - Constants: $-1 - 1 = -2$.
So we have:

$$2x^4 + 2x^2 - 2 = 0.$$

4. **Factoring out the common factor:** Divide through by 2:

$$x^4 + x^2 - 1 = 0.$$

5. **We then use a substitution to solve the quartic:** Let $y = x^2$. Then $y \geq 0$ for real x. Substitute:

$$y^2 + y - 1 = 0.$$

We have reduced to a quadratic equation in y.

6. **Now, we solve the quadratic equation in y:**

Using the quadratic formula for $ay^2 + by + c = 0$: $y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Here, $a = 1$, $b = 1$, $c = -1$. Thus:

$$y = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

So:

$$y_1 = \frac{-1+\sqrt{5}}{2}, \quad y_2 = \frac{-1-\sqrt{5}}{2}.$$

7. **We now need to check which y is valid:** Since $y = x^2 \geq 0$, discard negative solutions.

Evaluate sign: - $\sqrt{5} \approx 2.236$, $-1+\sqrt{5} > 0$, so $y_1 = \frac{-1+\sqrt{5}}{2} > 0$. - $-1-\sqrt{5} < 0$, so y_2 is negative.

Discard y_2. Thus:

$$y = \frac{-1+\sqrt{5}}{2}.$$

8. **Find x:** Since $y = x^2$, we have:

$$x^2 = \frac{-1+\sqrt{5}}{2}.$$

Take the square root (considering both positive and negative roots):

$$x = \pm\sqrt{\frac{-1+\sqrt{5}}{2}}.$$

9. **Final Solution:**

$$\boxed{x = \pm\sqrt{\frac{-1+\sqrt{5}}{2}}}.$$

These are all the real solutions to the given equation.

Mathematical concepts involved in Problem B

- **Combining and rearranging polynomial equations:** Moving all terms to one side to combine like terms and reduce the equation to a simpler form.
- **Factorization and simplification:** Factoring out common factors to simplify the polynomial.
- **Substitution method for quartic equations:** Using $y = x^2$ to convert a quartic equation in x into a quadratic equation in y.
- **Quadratic formula:** Applying the formula $y = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ to solve for y.
- **Checking validity of roots:** Ensuring that the solutions for $y = x^2$ are non-negative before taking the square root, which is crucial for identifying valid real solutions.
- **Recognizing extraneous or invalid solutions:** Discarding negative results for y since $y = x^2 \geq 0$.

************* **END OF SOLUTION: B** *************

Solution: Problem C

Given Problem: Determine the numerical value of the following expression without the use of a calculator:

$$\left(\sqrt{2}+(3^2)^{\frac{1}{4}}+\sum_{m=1}^{3}\left(\frac{1}{m!}-\sqrt{m}\right)\right)\cdot\left(2^{\log_2(8)}+\frac{1}{2^3}-\prod_{k=1}^{8}\left(1+\frac{1}{k}\right)\right).$$

We are informed the final answer is $-\frac{7}{12}$. We will verify this step-by-step, justifying each simplification thoroughly.

Detailed solution for Problem C...

1. **We first rewrite the main expression:**

Let

$$A=\left(\sqrt{2}+(3^2)^{1/4}+\sum_{m=1}^{3}\left(\frac{1}{m!}-\sqrt{m}\right)\right)$$

and

$$B=\left(2^{\log_2(8)}+\frac{1}{2^3}-\prod_{k=1}^{8}\left(1+\frac{1}{k}\right)\right).$$

Our target is to compute:

$$A\times B.$$

2. **We then simplify $(3^2)^{1/4}$:**
We have:

$$(3^2)^{1/4}=9^{1/4}.$$

Since $9=3^2$,

$$9^{1/4}=(3^2)^{1/4}=3^{2/4}=3^{1/2}=\sqrt{3}.$$

Therefore:

$$(3^2)^{1/4}=\sqrt{3}.$$

3. **Now, we substitute this into A:**
Now:

$$A=\left(\sqrt{2}+\sqrt{3}\right)+\sum_{m=1}^{3}\left(\frac{1}{m!}-\sqrt{m}\right).$$

4. **We evaluate the summation $\sum_{m=1}^{3}\frac{1}{m!}$:**

Computing each factorial:

$$1! = 1, \quad 2! = 2, \quad 3! = 6.$$

Thus:

$$\frac{1}{1!} = 1, \quad \frac{1}{2!} = \frac{1}{2}, \quad \frac{1}{3!} = \frac{1}{6}.$$

Summing these:

$$1 + \frac{1}{2} + \frac{1}{6}.$$

Finding a common denominator (6):

$$1 = \frac{6}{6}, \quad \frac{1}{2} = \frac{3}{6}, \quad \frac{1}{6} = \frac{1}{6}.$$

Adding them:

$$\frac{6}{6} + \frac{3}{6} + \frac{1}{6} = \frac{6+3+1}{6} = \frac{10}{6} = \frac{5}{3}.$$

Therefore:

$$\sum_{m=1}^{3} \frac{1}{m!} = \frac{5}{3}.$$

5. **We now evaluate $\sum_{m=1}^{3} \sqrt{m}$:**

$$\sqrt{1} = 1, \quad \sqrt{2} = \sqrt{2}, \quad \sqrt{3} = \sqrt{3}.$$

Thus:

$$\sum_{m=1}^{3} \sqrt{m} = 1 + \sqrt{2} + \sqrt{3}.$$

6. **We combine the terms in the summation:**
We have:

$$\sum_{m=1}^{3} \left(\frac{1}{m!} - \sqrt{m} \right) = \sum_{m=1}^{3} \frac{1}{m!} - \sum_{m=1}^{3} \sqrt{m}.$$

Substitute the values:

$$= \frac{5}{3} - (1 + \sqrt{2} + \sqrt{3}).$$

7. **The substitute back into A:**
Recall:

$$A = (\sqrt{2} + \sqrt{3}) + \left(\frac{5}{3} - (1 + \sqrt{2} + \sqrt{3}) \right).$$

Distribute the subtraction:

$$A = (\sqrt{2} + \sqrt{3}) + \frac{5}{3} - 1 - \sqrt{2} - \sqrt{3}.$$

Combine like terms:

$$\sqrt{2} - \sqrt{2} = 0, \quad \sqrt{3} - \sqrt{3} = 0.$$

All the radicals cancel out. Hence, we are left with:

$$A = \frac{5}{3} - 1.$$

Convert 1 to $\frac{3}{3}$:

$$A = \frac{5}{3} - \frac{3}{3} = \frac{2}{3}.$$

Hence:

$$A = \frac{2}{3}.$$

8. **We now evaluate the second bracket B:**
Recall:

$$B = 2^{\log_2(8)} + \frac{1}{2^3} - \prod_{k=1}^{8}\left(1 + \frac{1}{k}\right).$$

First, simplify $2^{\log_2(8)}$: Since $8 = 2^3$, $\log_2(8) = 3$. Thus:

$$2^{\log_2(8)} = 2^3 = 8.$$

Next, $\frac{1}{2^3} = \frac{1}{8}$.

Now, we consider the product:

$$\prod_{k=1}^{8}\left(1 + \frac{1}{k}\right) = \prod_{k=1}^{8}\frac{k+1}{k}.$$

Write out a few terms:

$$= \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} \times \frac{6}{5} \times \frac{7}{6} \times \frac{8}{7} \times \frac{9}{8}.$$

Most terms cancel out in a telescoping manner:

$$= \frac{9}{1} = 9.$$

Therefore:

$$B = 8 + \frac{1}{8} - 9.$$

Combine $8 - 9 = -1$:

$$B = -1 + \frac{1}{8}.$$

Convert $-1 = -\frac{8}{8}$:

$$B = -\frac{8}{8} + \frac{1}{8} = -\frac{7}{8}.$$

Thus:

$$B = -\frac{7}{8}.$$

9. **We now multiply A and B:**

We had found:

$$A = \frac{2}{3}, \quad B = -\frac{7}{8}.$$

Multiplying:

$$A \times B = \frac{2}{3} \times \left(-\frac{7}{8}\right) = \frac{2 \times (-7)}{3 \times 8} = \frac{-14}{24}.$$

We then simplify by dividing numerator and denominator by 2:

$$= \frac{-7}{12}.$$

Therefore:

$$\boxed{-\frac{7}{12}}.$$

Mathematical concepts involved in Problem C

- **Manipulation of fractional expressions and simplification of factorial-based fractions.**
- **Use of logarithm properties, such as $2^{\log_2(8)} = 8$.**
- **Telescoping products to simplify complex products into simpler fractions.**
- **Careful handling of sums and products involving factorials and radicals.**

*************** END OF SOLUTION: C ***************

Solution: Problem D

Given Problem: Prove that $n^5 - n$ is divisible by 5 for all positive integers n.

Detailed solution for Problem D...

Interpretation: To say an integer A is divisible by 5 means there exists an integer k such that $A = 5k$. In other words, if we perform division of A by 5, the remainder is 0. Our task is to show that for every positive integer n, the number $n^5 - n$ leaves no remainder upon division by 5.

I will present three distinct methods to prove this fact, ensuring that we understand it from multiple number-theoretic perspectives. Out of all these methods, Solution 2 is my favourite. It is also the shortest:

1. **Solution 1: Direct Verification Using Modular Arithmetic (Residue Classes)**.
2. **Solution 2: Using Fermat's Little Theorem**.
3. **Solution 3: Using Mathematical Induction**.

Each solution is correct and self-contained, but they illustrate different fundamental techniques.

Solution 1: Direct Verification Using Modular Arithmetic

Concepts Used in this approach: Modular arithmetic deals with remainders upon division by a given integer. To say $n^5 - n$ is divisible by 5 is to say:

$$n^5 - n \equiv 0 \pmod 5.$$

This notation means that when we divide $n^5 - n$ by 5, the remainder is 0. Equivalently:

$$n^5 \equiv n \pmod 5.$$

Key Idea: Any integer n can be expressed in the form $n = 5q + r$ where r is the remainder when n is divided by 5. The possible values of r are $0, 1, 2, 3$, or 4 because these are the only possible remainders when dividing by 5. By checking the congruence $n^5 \equiv n \pmod 5$ for each of these five possible remainders, we can confirm the statement for all integers n.

Detailed Steps:

1. **We shall consider the residue classes modulo 5:**

Let n be any positive integer. When we divide n by 5, there exist integers q and r such that:

$$n = 5q + r,$$

where $r \in \{0, 1, 2, 3, 4\}$. In modular arithmetic notation, $n \equiv r \pmod 5$.

2. **Case $r = 0$:**

If $n \equiv 0 \pmod 5$, this means n is actually a multiple of 5. Specifically, $n = 5q$ for some integer q. - Compute n^5: Since $n = 5q$, $n^5 = (5q)^5 = 5^5q^5$. This is clearly a multiple of 5 because 5^5q^5 has a factor of 5. - We also have $n = 5q$, which is obviously a multiple of 5.

Therefore:

$$n^5 - n = (5^5q^5) - (5q) = 5(5^4q^5 - q).$$

Since we have factored out a 5, $n^5 - n$ is divisible by 5. In modulo notation:

$$n^5 \equiv 0, \quad n \equiv 0 \pmod 5 \implies n^5 - n \equiv 0 - 0 = 0 \pmod 5.$$

3. **Case $r = 1$:**

If $n \equiv 1 \pmod 5$, then $n = 5q + 1$ for some integer q. - Consider n^5: Since we are only interested in the remainder modulo 5, we can use the fact that $1^5 = 1$. Thus:

$$n^5 \equiv 1^5 = 1 \pmod 5.$$

- We also have $n \equiv 1 \pmod 5$.

Combining these:

$$n^5 - n \equiv 1 - 1 = 0 \pmod 5.$$

Hence, in this case, $n^5 - n$ is also divisible by 5.

4. **Case $r = 2$:**

If $n \equiv 2 \pmod 5$, then:

$$n^5 \equiv 2^5 \pmod 5.$$

Compute $2^5 = 32$. Now, divide 32 by 5: $32 = 5 \cdot 6 + 2$, so $32 \equiv 2 \pmod 5$.

Hence:

$$n^5 \equiv 2 \pmod 5 \text{ and } n \equiv 2 \pmod 5.$$

Subtracting n:

$$n^5 - n \equiv 2 - 2 = 0 \pmod 5.$$

Again, divisibility by 5 is confirmed.

5. **Case $r = 3$:**

If $n \equiv 3 \pmod 5$:

$$n^5 \equiv 3^5 \pmod 5.$$

Compute $3^5 = 243$. Divide 243 by 5: $243 = 5 \cdot 48 + 3$. Thus, $243 \equiv 3 \pmod 5$.

Therefore:

$$n^5 \equiv 3 \pmod 5 \text{ and } n \equiv 3 \pmod 5.$$

Hence:

$$n^5 - n \equiv 3 - 3 = 0 \pmod 5.$$

6. **Case $r = 4$:**

If $n \equiv 4 \pmod 5$:

$$n^5 \equiv 4^5 \pmod 5.$$

Compute $4^5 = 1024$. Divide 1024 by 5: $1024 = 5 \cdot 204 + 4$, so $1024 \equiv 4 \pmod 5$.

Thus:

$$n^5 \equiv 4 \pmod 5 \text{ and } n \equiv 4 \pmod 5.$$

Therefore:

$$n^5 - n \equiv 4 - 4 = 0 \pmod 5.$$

7. **Conclusion for Solution 1:**

We have exhaustively checked all possible remainders $r = 0, 1, 2, 3, 4$ and found that in every case, $n^5 - n \equiv 0 \pmod 5$. This directly shows $n^5 - n$ is divisible by 5 for any integer n. Since we are particularly interested in positive integers, this result holds true for all $n > 0$ as well.

Solution 2: Using Fermat's Little Theorem

Concepts Used: Fermat's Little Theorem is a powerful result in number theory which states: If p is a prime number and a is an integer not divisible by p, then:

$$a^{p-1} \equiv 1 \pmod p.$$

For $p = 5$, this reads:

$$a^4 \equiv 1 \pmod 5 \quad \text{if } 5 \nmid a.$$

Idea of the Proof: If $5 \nmid n$, we apply Fermat's Little Theorem to get $n^4 \equiv 1 \pmod 5$, then multiply both sides by n to obtain $n^5 \equiv n \pmod 5$. If $5 \mid n$, the proof is even simpler because $n^5 - n$ has an obvious factor of n, which has a factor of 5.

Detailed Steps:

1. **Case 1: $5 \mid n$**

If $5 \mid n$, it means $n = 5q$ for some integer q. Then:

$$n^5 - n = (5q)^5 - (5q) = 5((5q)^4 q - q) = 5(\text{some integer}).$$

Thus, if n is a multiple of 5, $n^5 - n$ is trivially divisible by 5.

2. **Case 2: $5 \nmid n$**

If $5 \nmid n$, then $\gcd(n, 5) = 1$. Apply Fermat's Little Theorem with $p = 5$:

$$n^4 \equiv 1 \pmod 5.$$

Now multiply both sides of this congruence by n:

$$n^5 \equiv n \pmod 5.$$

From $n^5 \equiv n \pmod 5$, we immediately get:

$$n^5 - n \equiv 0 \pmod 5.$$

This shows that when $5 \nmid n$, $n^5 - n$ is divisible by 5.

3. **Combine both cases:**

We have shown: - If $5 \mid n$, then $5 \mid (n^5 - n)$. - If $5 \nmid n$, Fermat's Little Theorem ensures $5 \mid (n^5 - n)$.

In all scenarios, $n^5 - n$ is divisible by 5.

Solution 3: Using Mathematical Induction

Concepts Used: Mathematical induction is a technique to prove that a statement holds for all positive integers by following two steps: 1. **Base Case:** Show the statement is true for $n = 1$. 2. **Inductive Step:** Assume the statement is true for $n = k$ and then prove it is true for $n = k + 1$.

If both steps succeed, the statement is true for all positive integers.

We define the Statement: Let us define the property $P(n)$ as:

$$P(n) : n^5 - n \text{ is divisible by 5.}$$

Our goal is to prove $P(n)$ holds for all $n \in \mathbb{Z}^+$.

Detailed Steps:

1. **Base Case $n = 1$:**

Evaluate $P(1)$:

$$1^5 - 1 = 1 - 1 = 0.$$

The number 0 is divisible by every nonzero integer, including 5. Thus $P(1)$ is true. We have established the property holds for the first positive integer.

2. **Inductive Hypothesis:**

Assume that for some positive integer $k \geq 1$, $P(k)$ is true. This means we assume:

$$k^5 - k \text{ is divisible by 5.}$$

Concretely, there exists an integer m such that:

$$k^5 - k = 5m.$$

This assumption is our "inductive hypothesis." We want to use it to prove $P(k + 1)$.

3. **Inductive Step: To prove $P(k + 1)$:**

We must show $P(k+1)$ is true, i.e., we must show $(k+1)^5 - (k+1)$ is divisible by 5. Start with $(k+1)^5$. We expand this using the **Binomial Theorem**:

$$(k+1)^5 = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1.$$

The Binomial Theorem states:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i,$$

and for $n = 5$, $x = k$, $y = 1$, we have the binomial coefficients $\binom{5}{0} = 1$, $\binom{5}{1} = 5$, $\binom{5}{2} = 10$, $\binom{5}{3} = 10$, $\binom{5}{4} = 5$, $\binom{5}{5} = 1$. Thus:

$$(k+1)^5 - (k+1) = (k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) - (k+1).$$

We then distribute the subtraction of $(k+1)$:

$$= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1.$$

Combining the like terms: - The $+1$ and -1 cancel out. - The terms involving k are $5k$ and $-k$, which combine to $4k$.

After simplification, we get:

$$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 4k.$$

Now, we notice we can rewrite this expression to separate $k^5 - k$ (which we know is divisible by 5 from our inductive hypothesis):

$$k^5 - k \text{ appears naturally if we rewrite } 4k \text{ as } 5k - k.$$

Let's add and subtract k carefully:

$$(k+1)^5 - (k+1) = k^5 - k + 5k^4 + 10k^3 + 10k^2 + (5k).$$

We started with $k^5 + 5k^4 + 10k^3 + 10k^2 + 4k$. To introduce $k^5 - k$, observe:

$$k^5 + 5k^4 + 10k^3 + 10k^2 + 4k = (k^5 - k) + (5k^4 + 10k^3 + 10k^2 + 5k) - k.$$

Actually, an even clearer way: Since we know $k^5 - k$ is divisible by 5, let's isolate it directly:

$$(k+1)^5 - (k+1) = (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k.$$

How did we do this? Notice that $k^5 - k$ is already part of $k^5 + 4k$. If we rewrite $4k$ as $5k - k$, then:

$$k^5 + 4k = (k^5 - k) + 5k.$$

Substitute this back in:

$$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 4k$$

$$= (k^5 - k) + (5k^4 + 10k^3 + 10k^2 + 5k)$$

since we replaced $4k$ by $(5k - k)$ and added the $-k$ to k^5 giving $k^5 - k$, and the extra $5k$ got absorbed in the bracket with other multiples of 5.

Now we have:

$$(k + 1)^5 - (k + 1) = (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k.$$

We know from our inductive hypothesis that $k^5 - k = 5m$ for some integer m. Also, note that $5k^4$, $10k^3$, $10k^2$, and $5k$ are all obviously multiples of 5 because each term has a factor of 5: - $5k^4 = 5 \cdot k^4$ - $10k^3 = 5 \cdot (2k^3)$ - $10k^2 = 5 \cdot (2k^2)$ - $5k = 5 \cdot k$

Combining them all:

$$(k + 1)^5 - (k + 1) = 5m + 5k^4 + 10k^3 + 10k^2 + 5k.$$

We factor out the 5:

$$= 5(m + k^4 + 2k^3 + 2k^2 + k).$$

This is clearly a multiple of 5. Hence, $P(k + 1)$ is true.

4. **Conclusion of Induction:**

We showed: - Base case $P(1)$ is true. - If $P(k)$ is true, then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive integers n. Thus, for all positive integers n, $n^5 - n$ is divisible by 5.

Final Conclusion

All three approaches (direct verification, Fermat's Little Theorem, and mathematical induction) show that:

$$\boxed{5 \mid (n^5 - n) \text{ for all positive integers } n.}$$

This completes the proof.

Intricate Mathematical Concepts Involved in Problem D

- **Modular Arithmetic and Congruences:** In the first solution, we extensively used the concept of residue classes modulo 5 to directly verify the statement by checking all possible remainders.

- **Case Analysis on Residue Classes:** By enumerating the cases $r = 0, 1, 2, 3, 4$, we covered all integers since every integer is congruent to one of these residues modulo 5.

- **Fermat's Little Theorem:** In the second solution, we used a deep number-theoretic result that $n^4 \equiv 1 \pmod 5$ if 5 does not divide n. Multiplying by n then gives $n^5 \equiv n \pmod 5$, neatly proving the divisibility.

- **Mathematical Induction:** In the third solution, we used the principle of induction to prove the statement holds for all positive integers. This required us to handle the base case and then show that if it holds for $n = k$, it must hold for $n = k + 1$. The inductive step leveraged the binomial theorem to express $(k + 1)^5$ and identify a factor of 5.

- **Binomial Theorem:** The binomial theorem was crucial in the induction solution to expand $(k + 1)^5$ and identify the structure that allows factoring out 5.

************* **END OF SOLUTION: D** *************

Solution: Problem E

Detailed solution for Problem E:

We have two squares of equal side length a.

- $\mathbf{S_1}$: A square of side length a, centered at the origin $(0,0)$ and aligned with the coordinate axes.

This means:

$$S_1 : -\frac{a}{2} \leq x \leq \frac{a}{2} \quad \text{and} \quad -\frac{a}{2} \leq y \leq \frac{a}{2}.$$

The vertices of S_1 are:

$$A = \left(\frac{a}{2}, \frac{a}{2}\right), \quad B = \left(\frac{a}{2}, -\frac{a}{2}\right), \quad C = \left(-\frac{a}{2}, -\frac{a}{2}\right), \quad D = \left(-\frac{a}{2}, \frac{a}{2}\right).$$

The area of S_1 is a^2.

- $\mathbf{S_2'}$ *before rotation*: We shall consider an axis-aligned square S_2' of side length a, placed such that one of its vertices is at the origin $(0,0)$ and the square extends into the first quadrant. Without rotation, its vertices are:

$$W' = (0,0), \quad X' = (a,0), \quad Y' = (a,a), \quad Z' = (0,a).$$

This square S_2' also has area a^2. In this configuration, S_2' is not centered at the origin. Instead, it just has a vertex at the origin.

Now, Rotate S_2' by 45° Clockwise:

We rotate the square S_2' by 45° **clockwise** about the origin as shown in the figure. A 45° clockwise rotation transforms a point (x, y) as follows:

- Rotation by 45° clockwise can be represented by the transformation:

$$(x', y') = \left(x\frac{\sqrt{2}}{2} + y\frac{\sqrt{2}}{2},\ -x\frac{\sqrt{2}}{2} + y\frac{\sqrt{2}}{2}\right).$$

We now apply this rotation to each vertex of S_2':

1. $W' = (0,0)$:

$$W = \left(0 \cdot \frac{\sqrt{2}}{2} + 0,\ -0 \cdot \frac{\sqrt{2}}{2} + 0\right) = (0,0).$$

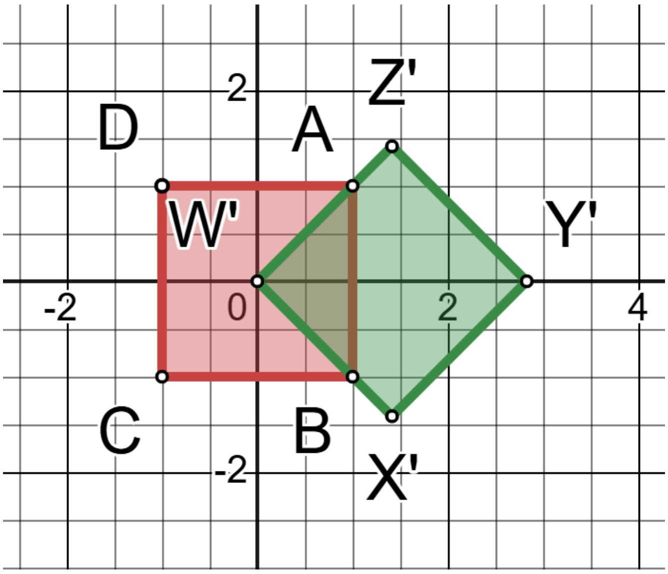

Figure 1: Desmos plot of S_1 (red) and S_2 (green) after rotating S_2' by 45° clockwise about the origin.

2. $X' = (a, 0)$:

$$X = \left(a\frac{\sqrt{2}}{2} + 0,\ -a\frac{\sqrt{2}}{2} + 0\right) = \left(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right).$$

3. $Y' = (a, a)$:

$$Y = \left(a\frac{\sqrt{2}}{2} + a\frac{\sqrt{2}}{2},\ -a\frac{\sqrt{2}}{2} + a\frac{\sqrt{2}}{2}\right).$$

We shall combine terms inside each coordinate: - For x': $a\frac{\sqrt{2}}{2} + a\frac{\sqrt{2}}{2} = a\sqrt{2}$. - For y': $-a\frac{\sqrt{2}}{2} + a\frac{\sqrt{2}}{2} = 0$.

Thus:

$$Y = (a\sqrt{2}, 0).$$

4. $Z' = (0, a)$:

$$Z = \left(0 + a\frac{\sqrt{2}}{2},\ -0 + a\frac{\sqrt{2}}{2}\right) = \left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right).$$

After rotation, the square S_2 (rotated version of S_2') has vertices:

$$W = (0,0), \quad X = \left(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right), \quad Y = (a\sqrt{2}, 0), \quad Z = \left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right).$$

The area of S_2 remains a^2.

Note: S_2 is no longer axis-aligned nor centered at the origin. It has one vertex at the origin and is tilted 45° clockwise.

Visualizing the Overlap:

- S_1 is a square centered at the origin, with corners at $(\pm a/2, \pm a/2)$. - S_2, after the $45°$ clockwise rotation, has a vertex at $W = (0, 0)$, stretches to the right (positive x-direction) up to $Y = (a\sqrt{2}, 0)$, and also has vertices Z in the first quadrant and X in the fourth quadrant.

This configuration implies: - S_2 extends into both the first quadrant (Q1) and fourth quadrant (Q4), with the origin as a pivot point. - We will analyze the intersection $S_1 \cap S_2$ quadrant by quadrant.

Defining the Edges of S_2:

The rotated square S_2 has vertices in the order $W \to Z \to Y \to X \to W$.

1. Edge WZ: Connects $W = (0, 0)$ to $Z = (a/\sqrt{2}, a/\sqrt{2})$. - Slope $= \frac{a/\sqrt{2}-0}{a/\sqrt{2}-0} = 1$. Equation: $y = x$.

2. Edge ZY: Connects $Z = (a/\sqrt{2}, a/\sqrt{2})$ to $Y = (a\sqrt{2}, 0)$. Find the slope:

$$\text{slope} = \frac{0 - a/\sqrt{2}}{a\sqrt{2} - a/\sqrt{2}} = \frac{-a/\sqrt{2}}{a(\sqrt{2} - 1/\sqrt{2})}.$$

Simplify denominator: $\sqrt{2} - \frac{1}{\sqrt{2}} = \frac{2-1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$.

Thus denominator $= a/\sqrt{2}$.

Slope $= \frac{-a/\sqrt{2}}{a/\sqrt{2}} = -1$.

Line through Z:

$$y - \frac{a}{\sqrt{2}} = -1\left(x - \frac{a}{\sqrt{2}}\right) \implies y = -x + a\sqrt{2}.$$

3. Edge YX: Connects $Y = (a\sqrt{2}, 0)$ to $X = (a/\sqrt{2}, -a/\sqrt{2})$. Slope:

$$\frac{-a/\sqrt{2} - 0}{a/\sqrt{2} - a\sqrt{2}} = \frac{-a/\sqrt{2}}{a/\sqrt{2} - a\sqrt{2}}.$$

Factor out $a/\sqrt{2}$:

$$a/\sqrt{2} - a\sqrt{2} = a\left(\frac{1}{\sqrt{2}} - \sqrt{2}\right) = a\left(\frac{1-2}{\sqrt{2}}\right) = \frac{-a}{\sqrt{2}}.$$

Numerator $= -a/\sqrt{2}$, Denominator $= -a/\sqrt{2}$. Slope $= 1$.

Equation through Y:

$$y = x - a\sqrt{2}.$$

4. Edge XW: Connects $X = (a/\sqrt{2}, -a/\sqrt{2})$ to $W = (0, 0)$. Slope:

$$\frac{0 + a/\sqrt{2}}{0 - a/\sqrt{2}} = \frac{a/\sqrt{2}}{-a/\sqrt{2}} = -1.$$

Equation through W:

$$y = -x.$$

So the edges are:

$$WZ : y = x, \quad ZY : y = -x + a\sqrt{2}, \quad YX : y = x - a\sqrt{2}, \quad XW : y = -x.$$

Intersection with S_1:

Recall S_1:

$$-\frac{a}{2} \leq x \leq \frac{a}{2}, \quad -\frac{a}{2} \leq y \leq \frac{a}{2}.$$

S_2 occupies a region around the origin but rotated. Let's consider each relevant quadrant.

Intersection in the First Quadrant (Q1):

In Q1: - S_1 constraints: $0 \leq x \leq a/2,\ 0 \leq y \leq a/2$. - In Q1, S_2 near the origin is bounded below by $WZ : y = x$. To be inside S_2, we must have $y \geq x$ in this region.

The upper edges of S_2 in Q1 are very high (like $y = -x + a\sqrt{2}$), which is well above $y = a/2$ since $a\sqrt{2} > a/2$. Thus, the top boundary in Q1 intersection is actually the top of S_1, i.e. $y = a/2$.

Hence, in Q1, the intersection region is:

$$\{(x, y) : 0 \leq x \leq a/2,\ x \leq y \leq a/2\}.$$

This is a right triangle with vertices: $(0, 0), (0, a/2), (a/2, a/2)$.

Area in Q1:

$$\frac{1}{2} \times \frac{a}{2} \times \frac{a}{2} = \frac{a^2}{8}.$$

Intersection in the Second Quadrant (Q2):

We check if S_2 extends into Q2 with $y > 0$. From the vertex configuration and the shape after rotation, S_2 mainly extends into Q1 and Q4. It does not have any part that goes into Q2 above the x-axis. Thus, there is no intersection region in Q2.

Intersection in the Fourth Quadrant (Q4):

In Q4: - S_1 constraints: $0 \leq x \leq a/2,\ -a/2 \leq y \leq 0$. - In Q4, the relevant edge of S_2 near the origin is $XW : y = -x$. To be inside S_2 in Q4, we must have $y \geq -x$ because the polygon opens above the line $y = -x$ in that region.

We know $y \leq 0$ from S_1, and $-a/2 \leq y \leq 0$.

At $x = 0$, $y \geq 0$ and $y \leq 0$ implies $y = 0$.

At $x = a/2$, since $y \geq -x$ implies $y \geq -a/2$, and from S_1, $y \leq 0$. This forms another right triangle with vertices: $(0, 0), (0, -a/2), (a/2, -a/2)$.

Area in Q4:

$$\frac{1}{2} \times \frac{a}{2} \times \frac{a}{2} = \frac{a^2}{8}.$$

No Intersection in the Third Quadrant (Q3):

Since S_2 does not extend below $y = 0$ in a manner that would intersect S_1 in Q3, there is no intersection there.

Total Intersection Area:

We have: - Q1 intersection area: $a^2/8$ - Q4 intersection area: $a^2/8$ - Q2: No intersection - Q3: No intersection

Total intersection:

$$A(S_1 \cap S_2) = \frac{a^2}{8} + \frac{a^2}{8} = \frac{a^2}{4}.$$

Non-Overlapping Area of S_2:

The area of S_2 is a^2. The intersection is $a^2/4$. Thus:

$$A_{\text{non-overlap}} = A(S_2) - A(S_1 \cap S_2) = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}.$$

Final Answer:

$$\boxed{\mathbf{A(S_1 \cap S_2) = \frac{a^2}{4} \text{ and } A_{\text{non-overlap}} = \frac{3a^2}{4}}}$$

Mathematical Concepts Involved in Problem E:

- **Coordinate Plane Visualization:** Plotting squares S_1 and S_2 to see their relative positions.
- **Square Geometry:** Familiarity with side length, diagonals, and areas of squares.
- **Rotation Transformations:** A 45° clockwise rotation formula in coordinates.
- **Line Equations and Slopes:** Identifying the boundaries of S_2 after rotation by computing slopes and line equations.
- **Intersection/Overlap Computation:** Splitting the intersection region by quadrants and summing triangular areas.
- **Area Computations:** Using right triangles for partial overlaps and subtracting from the total area.
- **Final Subtraction for Non-overlap:** $a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$.

************* **END OF SOLUTION: E** *************
