Solution: Problem B

Given Problem: Find all $x \in \mathbb{R}$ that solve the equation:

$$x^4 + x^2 - x - 1 = 1 - x - x^2 - x^4.$$

Deatiled solution for Problem B...

1. **We start with the given equation:**

$$x^4 + x^2 - x - 1 = 1 - x - x^2 - x^4.$$

2. **Bringing all the terms to one side:** Subtract $(1 - x - x^2 - x^4)$ from both sides:

$$x^4 + x^2 - x - 1 - (1 - x - x^2 - x^4) = 0.$$

Distributing the minus sign:

$$x^4 + x^2 - x - 1 - 1 + x + x^2 + x^4 = 0.$$

3. **We then combine like terms:** - x^4 terms: $x^4 + x^4 = 2x^4$. - x^2 terms: $x^2 + x^2 = 2x^2$. - x terms: $-x + x = 0$. - Constants: $-1 - 1 = -2$.
So we have:

$$2x^4 + 2x^2 - 2 = 0.$$

4. **Factoring out the common factor:** Divide through by 2:

$$x^4 + x^2 - 1 = 0.$$

5. **We then use a substitution to solve the quartic:** Let $y = x^2$. Then $y \geq 0$ for real x. Substitute:

$$y^2 + y - 1 = 0.$$

We have reduced to a quadratic equation in y.

6. **Now, we solve the quadratic equation in y:**

Using the quadratic formula for $ay^2 + by + c = 0$: $y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Here, $a = 1$, $b = 1$, $c = -1$. Thus:

$$y = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

So:

$$y_1 = \frac{-1+\sqrt{5}}{2}, \quad y_2 = \frac{-1-\sqrt{5}}{2}.$$

7. **We now need to check which y is valid:** Since $y = x^2 \geq 0$, discard negative solutions.

Evaluate sign: - $\sqrt{5} \approx 2.236$, $-1+\sqrt{5} > 0$, so $y_1 = \frac{-1+\sqrt{5}}{2} > 0$. - $-1-\sqrt{5} < 0$, so y_2 is negative.

Discard y_2. Thus:

$$y = \frac{-1+\sqrt{5}}{2}.$$

8. **Find x:** Since $y = x^2$, we have:

$$x^2 = \frac{-1+\sqrt{5}}{2}.$$

Take the square root (considering both positive and negative roots):

$$x = \pm\sqrt{\frac{-1+\sqrt{5}}{2}}.$$

9. **Final Solution:**

$$\boxed{x = \pm\sqrt{\frac{-1+\sqrt{5}}{2}}}.$$

These are all the real solutions to the given equation.

Mathematical concepts involved in Problem B

- **Combining and rearranging polynomial equations:** Moving all terms to one side to combine like terms and reduce the equation to a simpler form.
- **Factorization and simplification:** Factoring out common factors to simplify the polynomial.
- **Substitution method for quartic equations:** Using $y = x^2$ to convert a quartic equation in x into a quadratic equation in y.
- **Quadratic formula:** Applying the formula $y = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ to solve for y.
- **Checking validity of roots:** Ensuring that the solutions for $y = x^2$ are non-negative before taking the square root, which is crucial for identifying valid real solutions.
- **Recognizing extraneous or invalid solutions:** Discarding negative results for y since $y = x^2 \geq 0$.

************* **END OF SOLUTION: B** *************
