Solution: Problem E

Detailed solution for Problem E:

We have two squares of equal side length a.

- $\mathbf{S_1}$: A square of side length a, centered at the origin $(0,0)$ and aligned with the coordinate axes.

This means:

$$S_1 : -\frac{a}{2} \leq x \leq \frac{a}{2} \quad \text{and} \quad -\frac{a}{2} \leq y \leq \frac{a}{2}.$$

The vertices of S_1 are:

$$A = \left(\frac{a}{2}, \frac{a}{2}\right), \quad B = \left(\frac{a}{2}, -\frac{a}{2}\right), \quad C = \left(-\frac{a}{2}, -\frac{a}{2}\right), \quad D = \left(-\frac{a}{2}, \frac{a}{2}\right).$$

The area of S_1 is a^2.

- $\mathbf{S_2'}$ *before rotation*: We shall consider an axis-aligned square S_2' of side length a, placed such that one of its vertices is at the origin $(0,0)$ and the square extends into the first quadrant. Without rotation, its vertices are:

$$W' = (0,0), \quad X' = (a,0), \quad Y' = (a,a), \quad Z' = (0,a).$$

This square S_2' also has area a^2. In this configuration, S_2' is not centered at the origin. Instead, it just has a vertex at the origin.

Now, Rotate S_2' by 45° Clockwise:

We rotate the square S_2' by 45° **clockwise** about the origin as shown in the figure. A 45° clockwise rotation transforms a point (x,y) as follows:

- Rotation by 45° clockwise can be represented by the transformation:

$$(x', y') = \left(x\frac{\sqrt{2}}{2} + y\frac{\sqrt{2}}{2},\ -x\frac{\sqrt{2}}{2} + y\frac{\sqrt{2}}{2}\right).$$

We now apply this rotation to each vertex of S_2':

1. $W' = (0,0)$:

$$W = \left(0 \cdot \frac{\sqrt{2}}{2} + 0,\ -0 \cdot \frac{\sqrt{2}}{2} + 0\right) = (0,0).$$

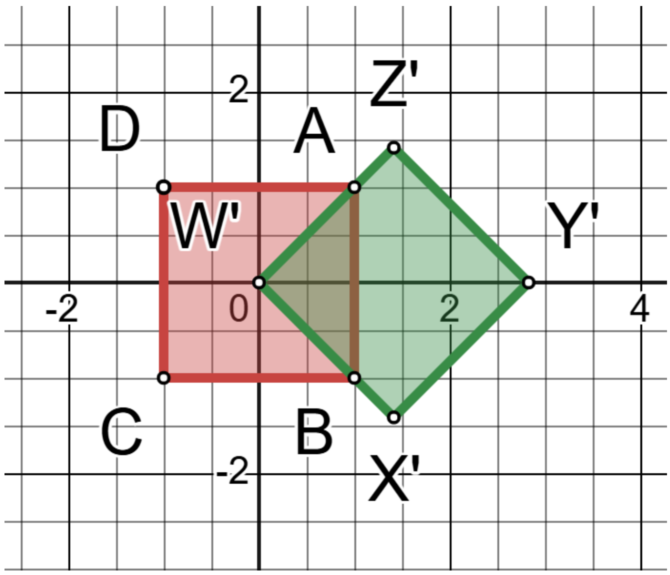

Figure 1: Desmos plot of S_1 (red) and S_2 (green) after rotating S_2' by 45° clockwise about the origin.

2. $X' = (a, 0)$:

$$X = \left(a\frac{\sqrt{2}}{2} + 0,\ -a\frac{\sqrt{2}}{2} + 0\right) = \left(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right).$$

3. $Y' = (a, a)$:

$$Y = \left(a\frac{\sqrt{2}}{2} + a\frac{\sqrt{2}}{2},\ -a\frac{\sqrt{2}}{2} + a\frac{\sqrt{2}}{2}\right).$$

We shall combine terms inside each coordinate: - For x': $a\frac{\sqrt{2}}{2} + a\frac{\sqrt{2}}{2} = a\sqrt{2}$. - For y': $-a\frac{\sqrt{2}}{2} + a\frac{\sqrt{2}}{2} = 0$.

Thus:

$$Y = (a\sqrt{2}, 0).$$

4. $Z' = (0, a)$:

$$Z = \left(0 + a\frac{\sqrt{2}}{2},\ -0 + a\frac{\sqrt{2}}{2}\right) = \left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right).$$

After rotation, the square S_2 (rotated version of S_2') has vertices:

$$W = (0, 0), \quad X = \left(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right), \quad Y = (a\sqrt{2}, 0), \quad Z = \left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right).$$

The area of S_2 remains a^2.

Note: S_2 is no longer axis-aligned nor centered at the origin. It has one vertex at the origin and is tilted 45° clockwise.

Visualizing the Overlap:

- S_1 is a square centered at the origin, with corners at $(\pm a/2, \pm a/2)$. - S_2, after the 45° clockwise rotation, has a vertex at $W = (0, 0)$, stretches to the right (positive x-direction) up to $Y = (a\sqrt{2}, 0)$, and also has vertices Z in the first quadrant and X in the fourth quadrant.

This configuration implies: - S_2 extends into both the first quadrant (Q1) and fourth quadrant (Q4), with the origin as a pivot point. - We will analyze the intersection $S_1 \cap S_2$ quadrant by quadrant.

Defining the Edges of S_2:

The rotated square S_2 has vertices in the order $W \to Z \to Y \to X \to W$.

1. Edge WZ: Connects $W = (0, 0)$ to $Z = (a/\sqrt{2}, a/\sqrt{2})$. - Slope $= \frac{a/\sqrt{2}-0}{a/\sqrt{2}-0} = 1$. Equation: $y = x$.

2. Edge ZY: Connects $Z = (a/\sqrt{2}, a/\sqrt{2})$ to $Y = (a\sqrt{2}, 0)$. Find the slope:

$$\text{slope} = \frac{0 - a/\sqrt{2}}{a\sqrt{2} - a/\sqrt{2}} = \frac{-a/\sqrt{2}}{a(\sqrt{2} - 1/\sqrt{2})}.$$

Simplify denominator: $\sqrt{2} - \frac{1}{\sqrt{2}} = \frac{2-1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$.

Thus denominator $= a/\sqrt{2}$.

Slope $= \frac{-a/\sqrt{2}}{a/\sqrt{2}} = -1$.

Line through Z:

$$y - \frac{a}{\sqrt{2}} = -1\left(x - \frac{a}{\sqrt{2}}\right) \implies y = -x + a\sqrt{2}.$$

3. Edge YX: Connects $Y = (a\sqrt{2}, 0)$ to $X = (a/\sqrt{2}, -a/\sqrt{2})$. Slope:

$$\frac{-a/\sqrt{2} - 0}{a/\sqrt{2} - a\sqrt{2}} = \frac{-a/\sqrt{2}}{a/\sqrt{2} - a\sqrt{2}}.$$

Factor out $a/\sqrt{2}$:

$$a/\sqrt{2} - a\sqrt{2} = a\left(\frac{1}{\sqrt{2}} - \sqrt{2}\right) = a\left(\frac{1-2}{\sqrt{2}}\right) = \frac{-a}{\sqrt{2}}.$$

Numerator $= -a/\sqrt{2}$, Denominator $= -a/\sqrt{2}$. Slope $= 1$.

Equation through Y:

$$y = x - a\sqrt{2}.$$

4. Edge XW: Connects $X = (a/\sqrt{2}, -a/\sqrt{2})$ to $W = (0, 0)$. Slope:

$$\frac{0 + a/\sqrt{2}}{0 - a/\sqrt{2}} = \frac{a/\sqrt{2}}{-a/\sqrt{2}} = -1.$$

Equation through W:

$$y = -x.$$

So the edges are:

$$WZ : y = x, \quad ZY : y = -x + a\sqrt{2}, \quad YX : y = x - a\sqrt{2}, \quad XW : y = -x.$$

Intersection with S_1:

Recall S_1:

$$-\frac{a}{2} \leq x \leq \frac{a}{2}, \quad -\frac{a}{2} \leq y \leq \frac{a}{2}.$$

S_2 occupies a region around the origin but rotated. Let's consider each relevant quadrant.

Intersection in the First Quadrant (Q1):

In Q1: - S_1 constraints: $0 \leq x \leq a/2,\ 0 \leq y \leq a/2$. - In Q1, S_2 near the origin is bounded below by $WZ : y = x$. To be inside S_2, we must have $y \geq x$ in this region.

The upper edges of S_2 in Q1 are very high (like $y = -x + a\sqrt{2}$), which is well above $y = a/2$ since $a\sqrt{2} > a/2$. Thus, the top boundary in Q1 intersection is actually the top of S_1, i.e. $y = a/2$.

Hence, in Q1, the intersection region is:

$$\{(x, y) : 0 \leq x \leq a/2,\ x \leq y \leq a/2\}.$$

This is a right triangle with vertices: $(0, 0), (0, a/2), (a/2, a/2)$.

Area in Q1:

$$\frac{1}{2} \times \frac{a}{2} \times \frac{a}{2} = \frac{a^2}{8}.$$

Intersection in the Second Quadrant (Q2):

We check if S_2 extends into Q2 with $y > 0$. From the vertex configuration and the shape after rotation, S_2 mainly extends into Q1 and Q4. It does not have any part that goes into Q2 above the x-axis. Thus, there is no intersection region in Q2.

Intersection in the Fourth Quadrant (Q4):

In Q4: - S_1 constraints: $0 \leq x \leq a/2,\ -a/2 \leq y \leq 0$. - In Q4, the relevant edge of S_2 near the origin is $XW : y = -x$. To be inside S_2 in Q4, we must have $y \geq -x$ because the polygon opens above the line $y = -x$ in that region.

We know $y \leq 0$ from S_1, and $-a/2 \leq y \leq 0$.

At $x = 0$, $y \geq 0$ and $y \leq 0$ implies $y = 0$.

At $x = a/2$, since $y \geq -x$ implies $y \geq -a/2$, and from S_1, $y \leq 0$. This forms another right triangle with vertices: $(0, 0), (0, -a/2), (a/2, -a/2)$.

Area in Q4:

$$\frac{1}{2} \times \frac{a}{2} \times \frac{a}{2} = \frac{a^2}{8}.$$

No Intersection in the Third Quadrant (Q3):

Since S_2 does not extend below $y = 0$ in a manner that would intersect S_1 in Q3, there is no intersection there.

Total Intersection Area:

We have: - Q1 intersection area: $a^2/8$ - Q4 intersection area: $a^2/8$ - Q2: No intersection - Q3: No intersection

Total intersection:

$$A(S_1 \cap S_2) = \frac{a^2}{8} + \frac{a^2}{8} = \frac{a^2}{4}.$$

Non-Overlapping Area of S_2:

The area of S_2 is a^2. The intersection is $a^2/4$. Thus:

$$A_{\text{non-overlap}} = A(S_2) - A(S_1 \cap S_2) = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}.$$

Final Answer:

$$\boxed{\mathbf{A(S_1 \cap S_2) = \frac{a^2}{4} \text{ and } A_{\text{non-overlap}} = \frac{3a^2}{4}}}$$

Mathematical Concepts Involved in Problem E:

- **Coordinate Plane Visualization:** Plotting squares S_1 and S_2 to see their relative positions.
- **Square Geometry:** Familiarity with side length, diagonals, and areas of squares.
- **Rotation Transformations:** A 45° clockwise rotation formula in coordinates.
- **Line Equations and Slopes:** Identifying the boundaries of S_2 after rotation by computing slopes and line equations.
- **Intersection/Overlap Computation:** Splitting the intersection region by quadrants and summing triangular areas.
- **Area Computations:** Using right triangles for partial overlaps and subtracting from the total area.
- **Final Subtraction for Non-overlap:** $a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$.

**************** END OF SOLUTION: E ****************
