Assignment 2

To go through the following interest rate models.

Vasicek Model

Cox-Ingersoll-Ross (CIR) Model

Heath-Jarrow-Morton (HJM) Model

Hull-White Model

Solution-

For designing forward curve using the above interest rates model, we have the dataset of Fed rate from 1994-2023 and ED dataset whch contains the maturity of ED bonds of each working day from Jan2000. These data contain 40 maturity containing data for bonds price for next 10 years with 3 month in forward move. It is to be notes that bond price only till Dec 2022 is contained so expiry of bond after 2012 is lesser than 10 years, meaning less than 40 maturities.

Assumption-

- Forward curve at particular date is made for twelve maturities i.e for next 3 years.
- Initial interest taken is of Fed lower bound for month of which we are drawing forward curve.
- The parameters for model are derived from bond prices of days before the given days (it is optional to choose any number of days to train)
- Parameters for each maturity would be different as ED prices for diferent maturity are different.
- For each formula, discrete form of formula are used.
- It is assumed that forward curve is drawn only at working date. If non-working date is entered as values of vriable- 'end_date' code will not run.
- Date before December 2019 should be entered as yield curve for 3 years can be obtained only for days befre this due to data constraint.

About fixed Parameters of each Model

1. **Mean-Reverting Speed(a):** The models assumes that the short-term interest rate follows a mean-reverting process, aiming to capture the tendency of interest rates to revert to a long-term mean. The mean-reversion feature is controlled by a parameter called the speed of mean reversion.
2. **Stochastic Volatility(σ):** Volatility for the interest rate process at particular maturity
3. **Random Shocks (dz)** -: The model incorporates random shocks or disturbances using a Wiener process or Brownian motion. These random shocks represent the unpredictable and instantaneous changes in interest rates.

4. **Equilibrium Interest Rate (b):** Models includes an equilibrium or long-term mean interest rate level. The model assumes that interest rates will tend to revert to this equilibrium rate over time.

How calculated these parameters

As ED data is arranged according to maturity at each working date one after other. So it is acceptable to save the position of every occurrence of a particular date in the ED data Excel sheet. Every occurrence of date corresponds to the ED price for a specific term maturity bond in increasing order of occurrence. Training the model so each occurrence with dates above that particular date of whose forward curve we are drawing gives us the expected values of parameters.

I Converted the ED price to the labor rate by subtracting it by 100. The price of each trained data is stored and fitted linearly with the change in interest rate corresponding to the interest rate at each value(substracted previous date rate with current data to give change in an interest rate at a particular date.). The slope of the linear fit line shows the mean reversion speed, intercept/slope provides the reversion with rate, and standard deviation gives volatility.

It is worth noting that we trained the model only for 12 maturities, and there would be 12 times the training of the data set to obtain 12 different value parameters, each of corresponding maturity.

Note- It is important to note that these parameters used by me are different for different maturity because we are making forward curve by predicting the rates at interval of three month maturity. For every maturity, we train model for dates before the particular date where we are building forward curve with prices of that particular maturity of trained dates

Vasicek Model

It is a one-factor stochastic model commonly used for pricing fixed income securities and analyzing interest rate derivatives. The model assumes that the short-term interest rate follows a mean-reverting stochastic process.

$$dr = a(b - r)dt + \sigma dz$$

As stated above, discrete from is used.

So $R_T=R_{T-1}+A_T*(B_T-R_{T-1})dT$

Where dT implies changes in time.

For T=0, we take Fed rate of that month as R_0

Cox-Ingersoll Ross Model

The Cox-Ingersoll-Ross model is a popular interest rate model that was introduced by John C. Cox, Jonathan E. Ingersoll, and Stephen A. Ross in 1985. The CIR

model is an extension of the Vasicek model that addresses one of its limitations - the potential for negative interest rates. The model assumes that the short-term interest rate follows a mean-reverting stochastic process while ensuring that the interest rate stays non-negative.

$$dr = a(b-r)\,dt + \sigma\sqrt{r}\,dz$$

So $R_T=R_{T-1}+A_T*(B_T-R_{T-1})dT$

Where dT implies changes in time.

For T=0, we take Fed rate of that month as R_0

As R term is there inside square root, so negative interest rate, which is a very rar e phenomenon is avoided here.

Hull-White Model

It extends the Vasicek model by allowing the volatility of interest rates to be time-dependent, capturing the term structure of interest rates more accurately**.** The Hull-White model is designed within a no-arbitrage framework, ensuring that it produces interest rate processes consistent with the absence of arbitrage opportunities in the market**.** The mathematical formulation of the Hull-White model involves a stochastic differential equation (SDE) that describes the dynamics of interest rates over time. The precise form of the SDE depends on the specific implementation and assumptions made in the model.

Formula used- $dr=a_t*(b_t-r(t))*dt+\sigma(t)dz - a*r(t)$

$R_t=R_{t-1}+dr$

HJM Model

Unlike the Vasicek, Cox-Ingersoll-Ross (CIR), and Hull-White models, the HJM model is a multi-factor model that captures the entire term structure of interest rates. The HJM model is based on the assumption that the dynamics of the entire yield curve are determined by a set of underlying stochastic factors. These factors can represent various sources of risk, such as changes in short-term rates, changes in the slope of the yield curve, and changes in the curvature of the yield curve. The mathematical formulation of the HJM model involves a system of partial differential equations (PDEs) or stochastic differential equations (SDEs) to describe the dynamics of the yield curve.

Volatility is the key in this model. Model make assumption that the current volatility of particular maturity is the only fixed factor which determines the future interest rate.

I used discrete formula for building this model only at terms of each maturity

Formula used-

$R=R_0+ \sigma(t)*M(t)*M(t)/2*dT+\sigma(t)dW$

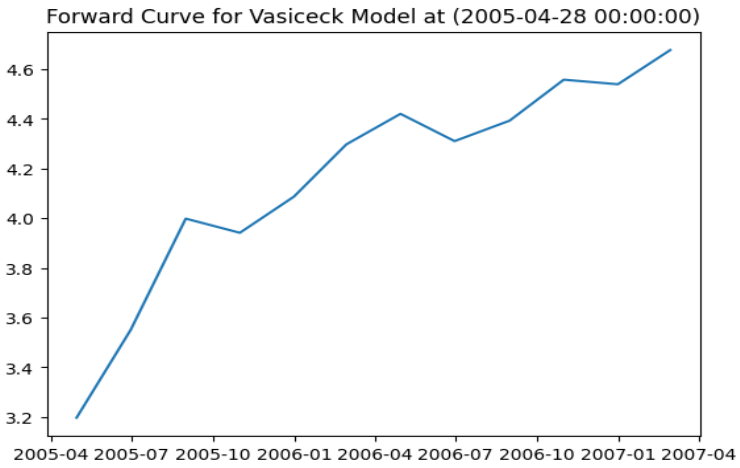
Where M(t) is the maturity number , ie 3month bond has maturity number equall to 1, 6 month has maturity number equal to 2 and so on.

Forward Curve-

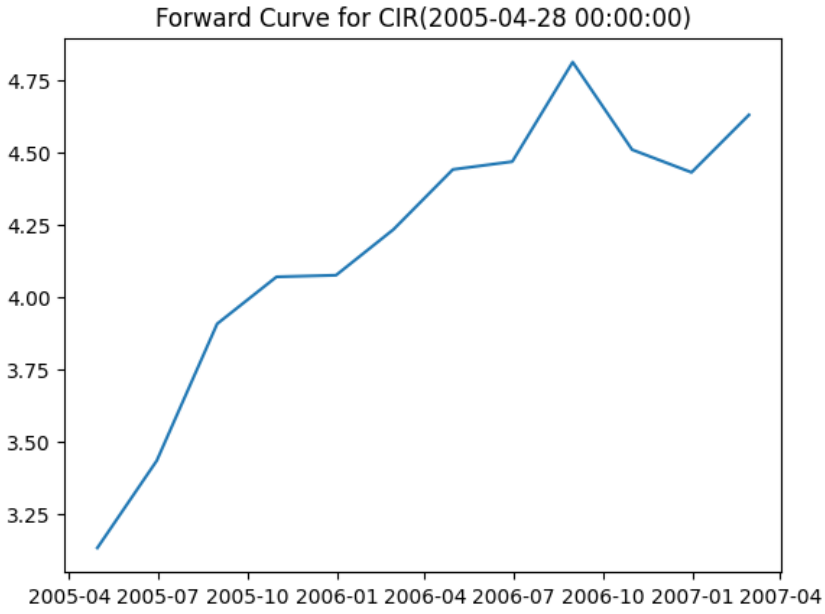
Forward curve for each model at date = 28/4/2005 is drawn.

Date is randomly chosen as by analyzing FED data, this date signify when economy so in recovery state and condition of economy was neither too good nor bad at this time

Vasicek Model

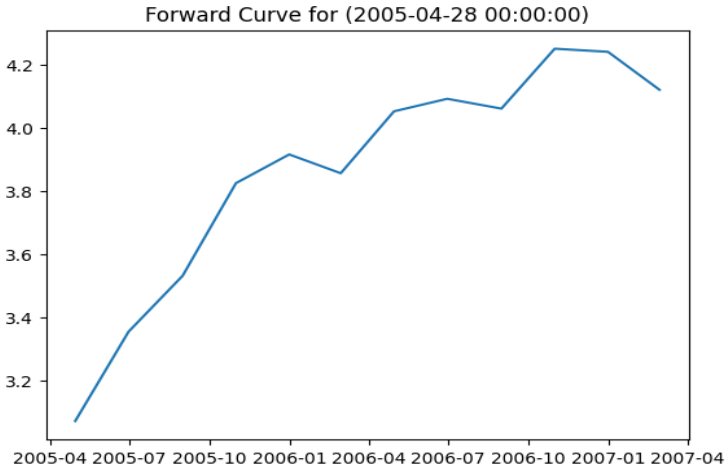


Cox- Ingroll Model

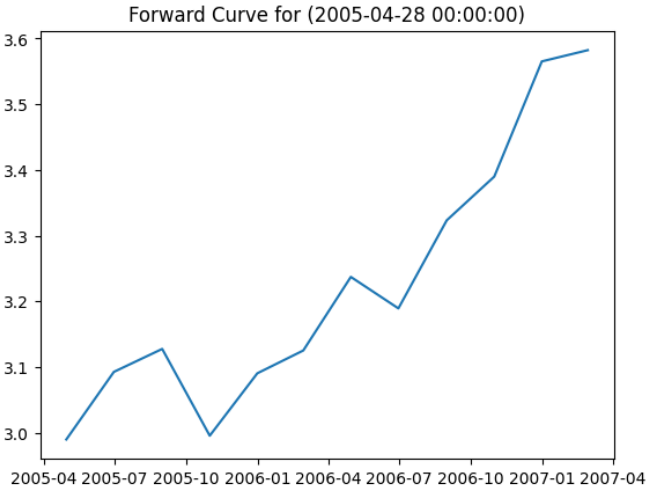


This is the forward curve for 28-04-2005

Hull-White Model



HJM Model



Backtesting the Models

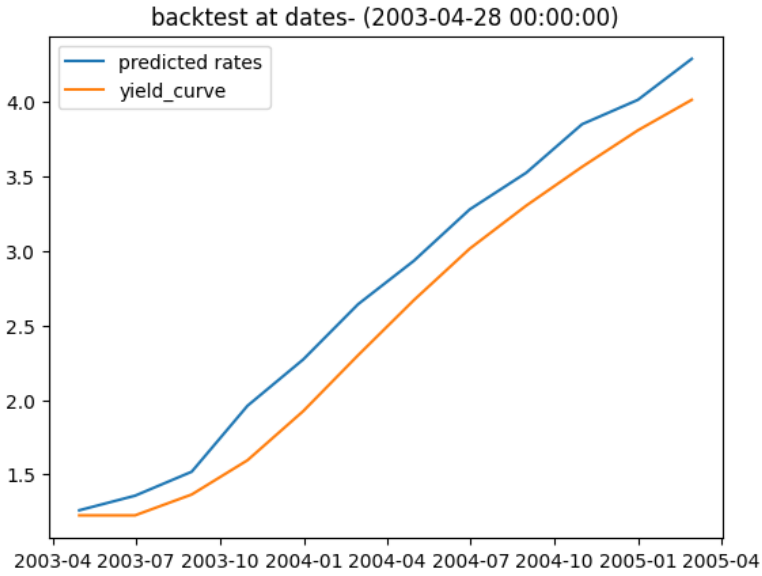
Backtesting is a crucial process in evaluating the performance and effectiveness of a model or trading strategy using historical data. It involves applying the model or strategy to past data to assess its ability to generate accurate predictions or trading signals. The goal of backtesting is to gain insights into the model's performance, validate its assumptions, and identify any areas for improvement.

Backtesting provides a valuable tool for assessing the viability and effectiveness of a model or trading strategy

We can randomly choose dates to see where our model behaves accordingly and check how can we set trade signals

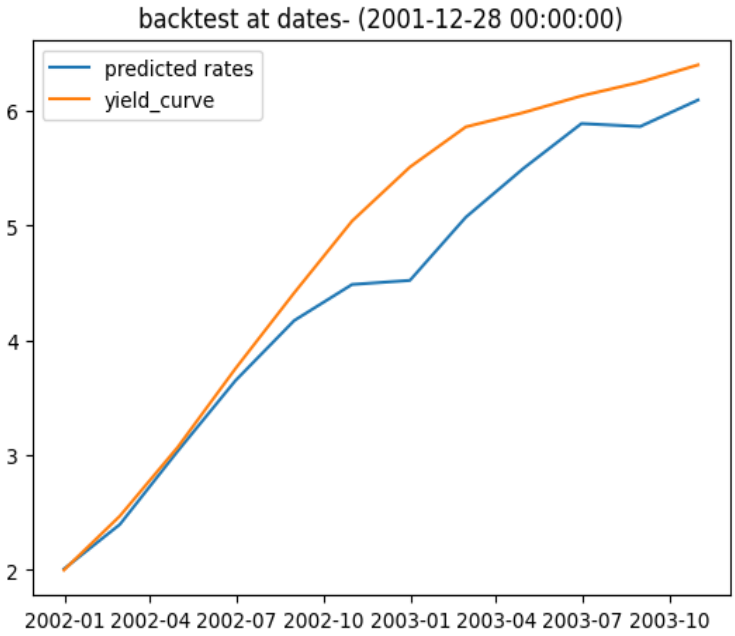
Vasicek Model

Date1- 28/4/2003



We see that at this date, forward curve is above yield curve here

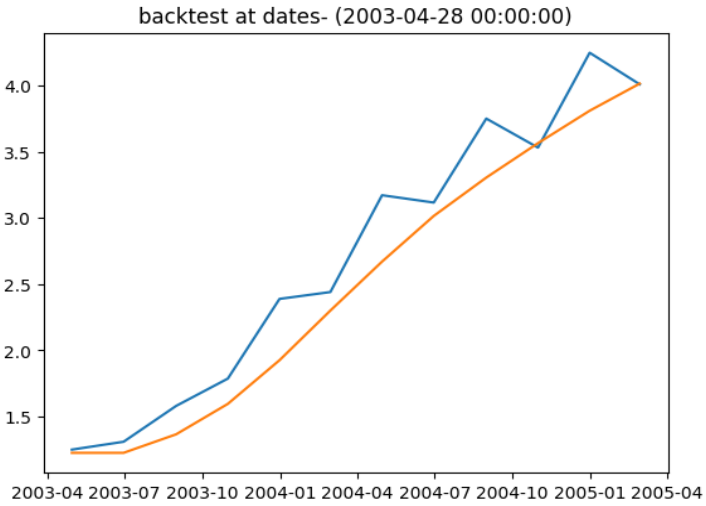
Date 2- 28/12/2001



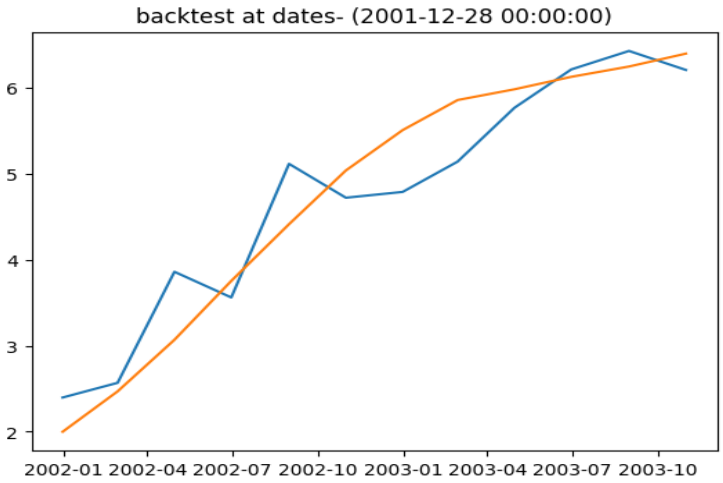
It is observed that forward curve remain on either side of yield curve

CIR Model

Date1- 28/4/2003

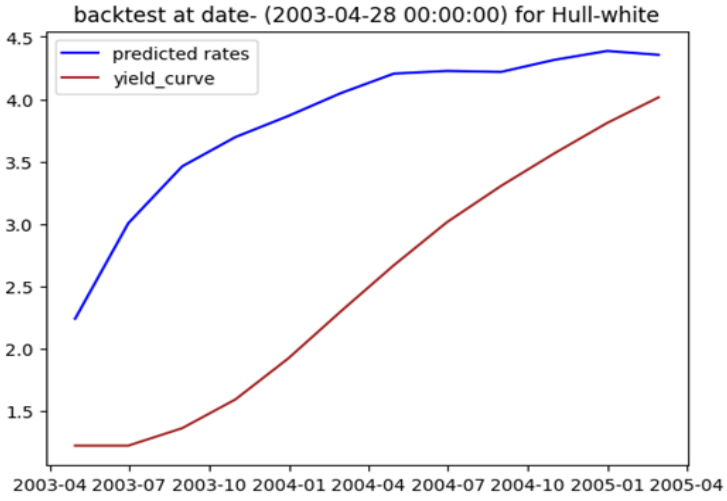


Date 2- 28/12/2001

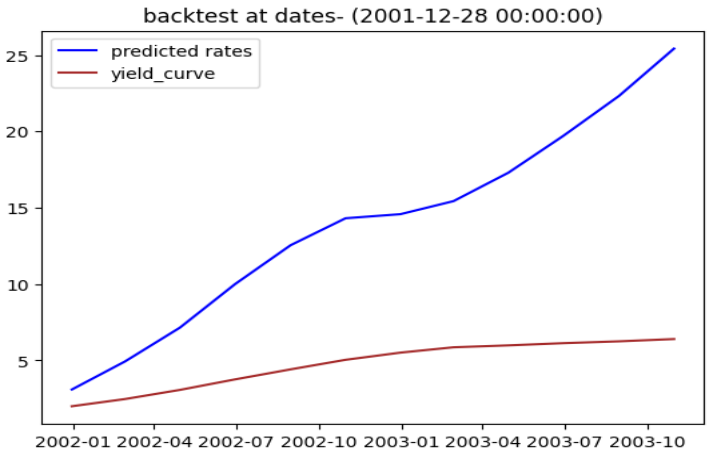


We observe that yield curve is around forward curve and this model is giving better result to build up signals

Hull – White model

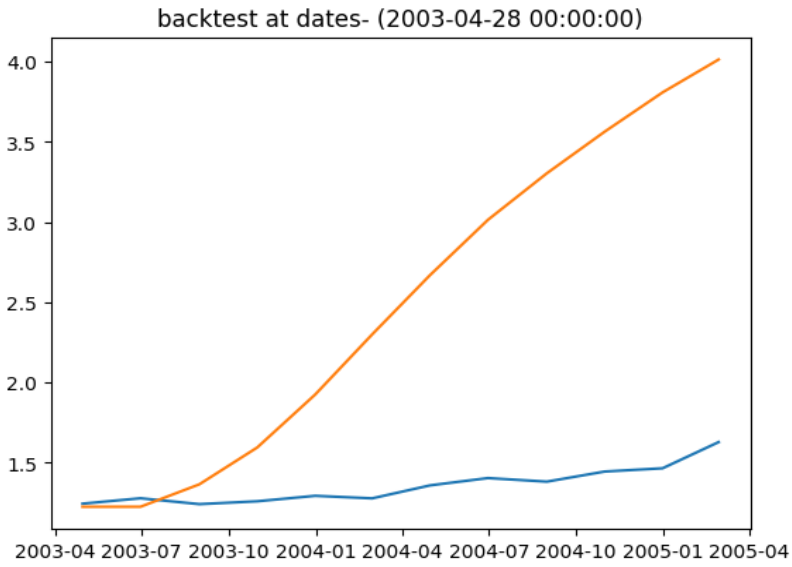


Date2 – - 28/12/2001

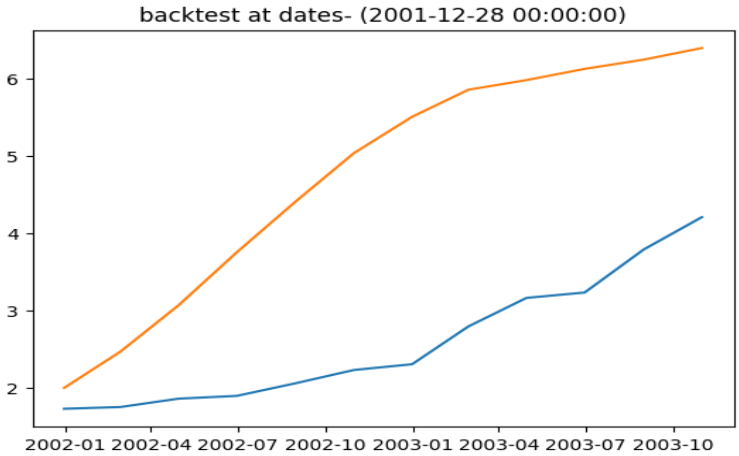


HJM model

Date1- 28/4/2003



Date2- 28/12/2001



Forward curve is bearish compared to yield curve

So looking at Backtest of all models, forward curve of which we are ploting for Date- 28/4/2005

We can make out that CIR model revolve around yield curve, vasiceck model also which less frequency. While HullWhite model appears to be bullish and HJM model is bearish, ie it expects interest rates to be lower than expected.

Trade signals

For Trade signals,we set up the model to vary within **+2σ and -2σ** of yield curve for that date. This rule is said to be followed when model forward curve predicted is similar to yield curve on backtest dates

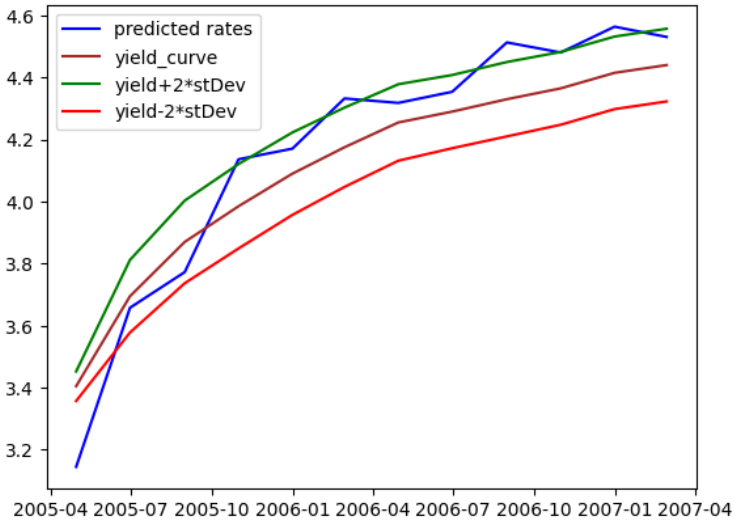
But as we see the regular pattern in our model that on backtest dates, we find the relations between forward curve and yield curve and to give improved trade signals, I decided to set my own parameter.

- For Vasicek and CIR model, Generate buy signals when forward curve above yield+2σ[i] curve, and sell when forward curve below yield curve -2σ[i]
- For HJM model, I took t**hreshold** value as average of difference between yield and forward curve and signal for buy/sell is =threshold+/- 5*σ[i]
- For Hull-White model, as on backtesting we observe that forward curve is not very much below yield curve, so in generating signal we can take only 1 multiple of standard deviation.- signal for buy/sell generated at value= threshold+/- σ[i]

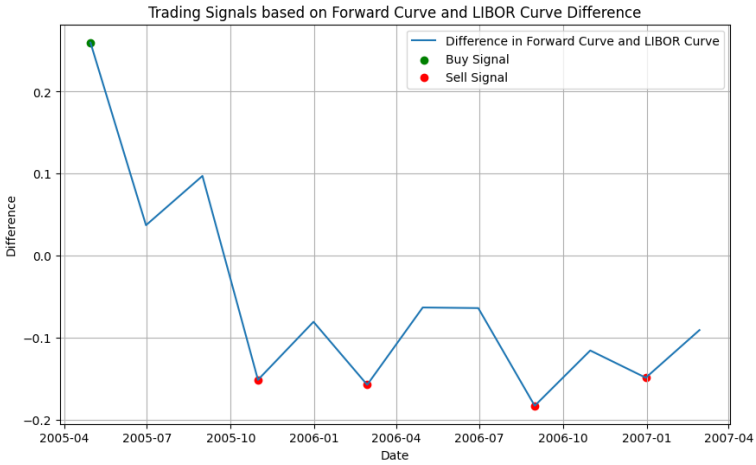
Note- σ[i] denotes the volatillty of i^{th} maturity.

Signal Curves for Date- 28/4/2005

Vasicek Model

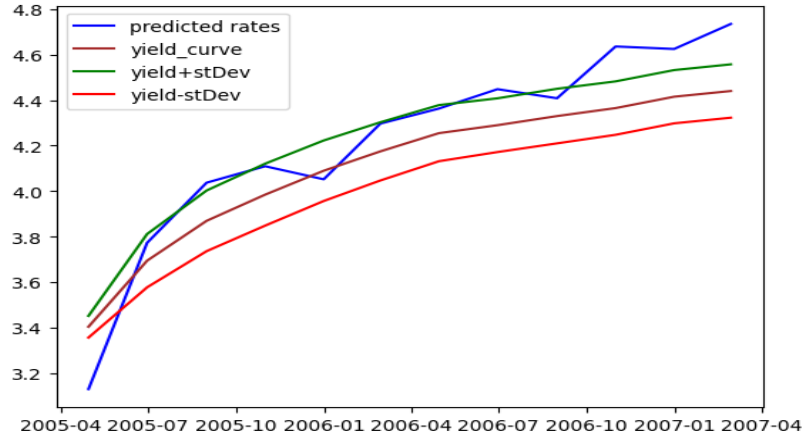


Forward curve around yield curve with std Deviation

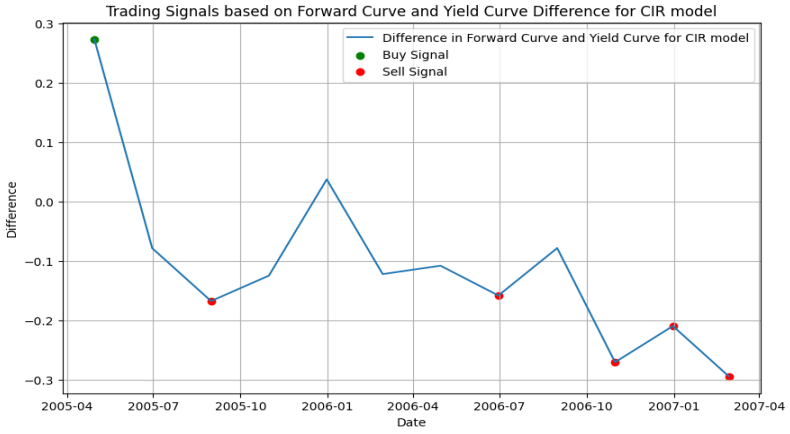


Buy and Sell signals Generated at particular maturity, (Hold where there is no signal)

CIR Model

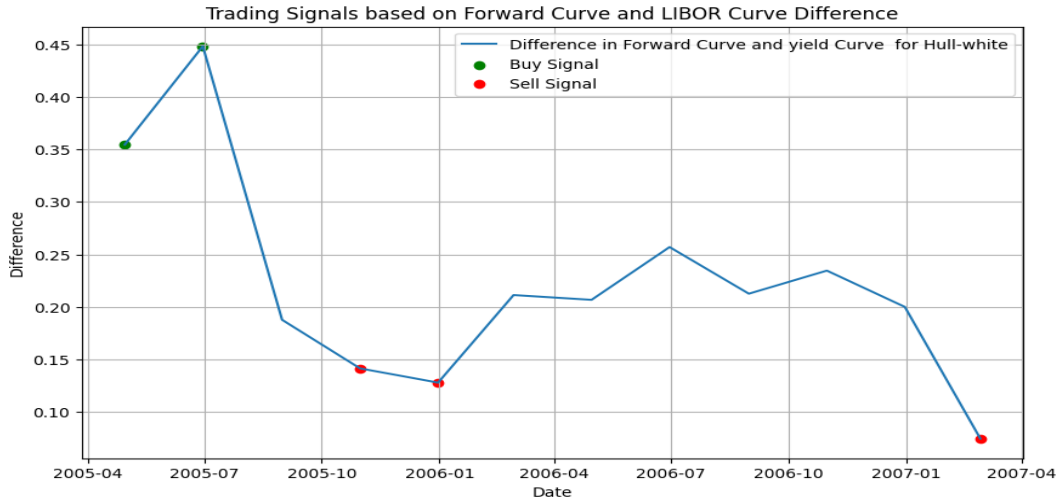


*CIR model difference around yield curve with 2*std difference*



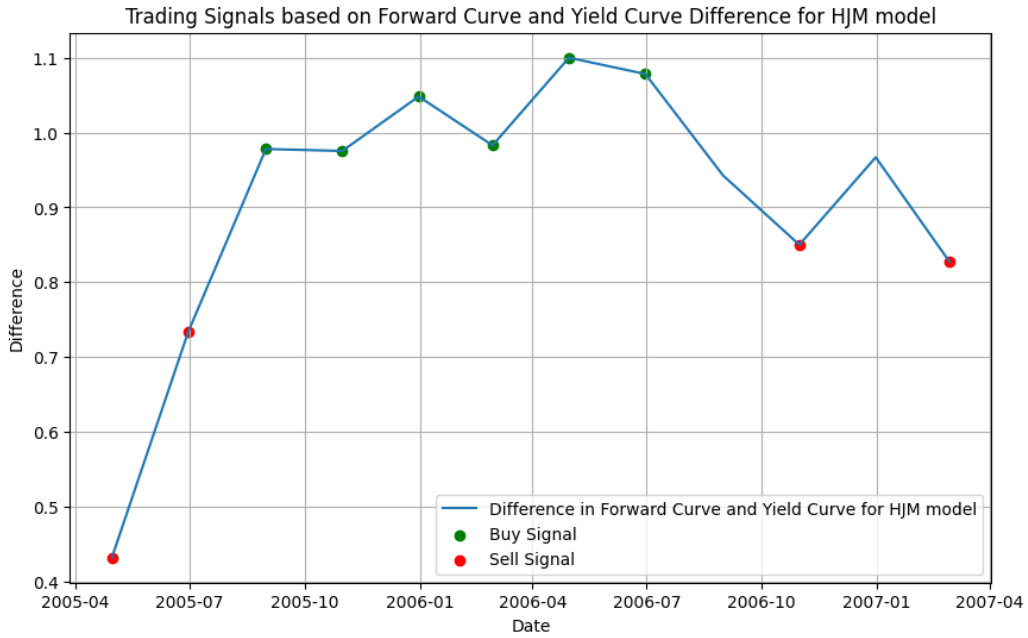
Buy and Sell signals Generated at particular maturity, (Hold where there is no signal)

Hull-White Model



Buy and Sell signals Generated at particular maturity, (Hold where there is no signal)

HJM Model



Buy and Sell signals Generated at particular maturity, (Hold where there is no signal)

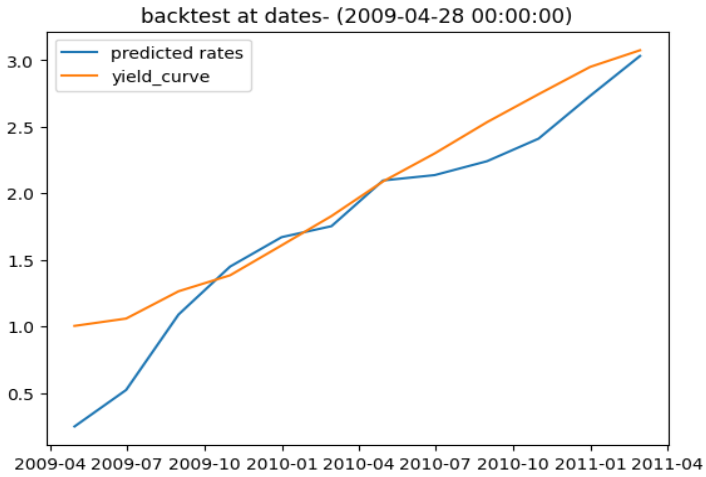
As observed with this date, the trade signal is similar for first three model, while differ in 4rd model. This may be due to short term rate preference by HJM model

Comparison between Yield and Forward curve for models during recession

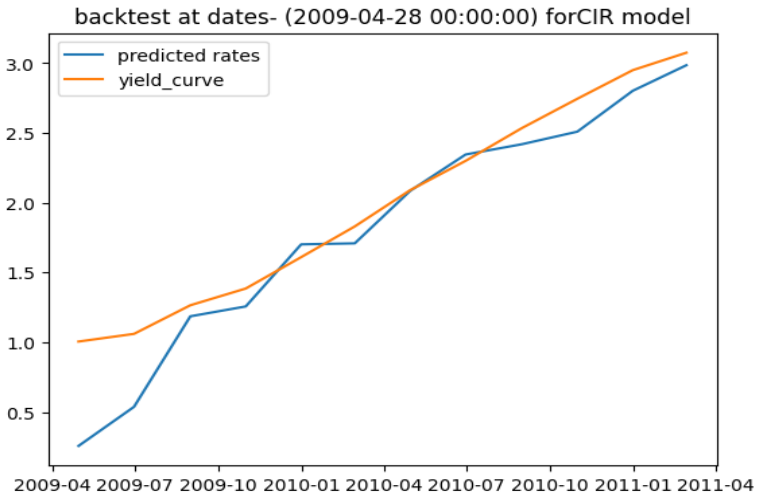
Test of model is best during the extreme economy period. So in order to compare libor yield curve and my forward curve, it took date of April 2009 , when US was in the mid of recession . As there were a lot of events around that period so it is acknowledged to train the dataset for longer time. I trained dataset for 60 days.

Curve comparison for all models are as follow

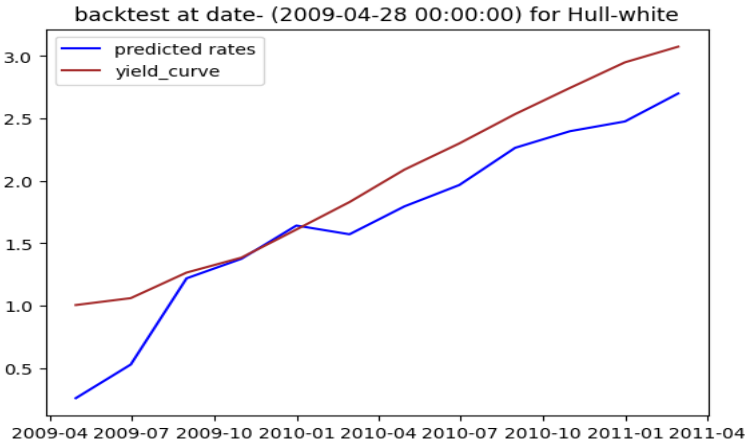
Vasicek Model



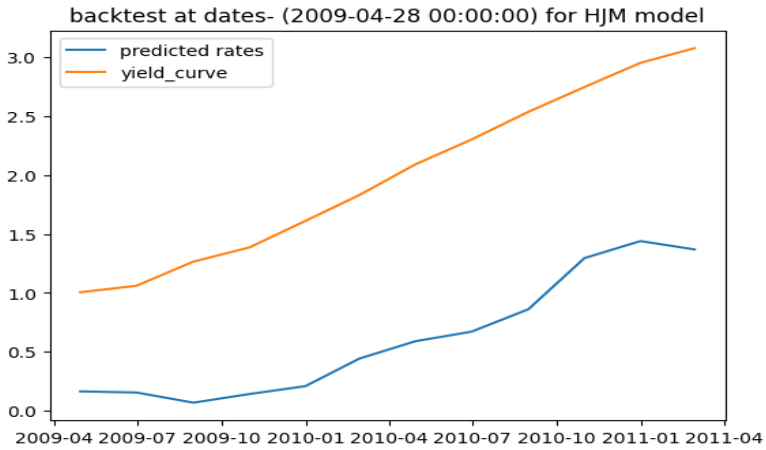
CIR Model



Hull-White Model



HJM Model

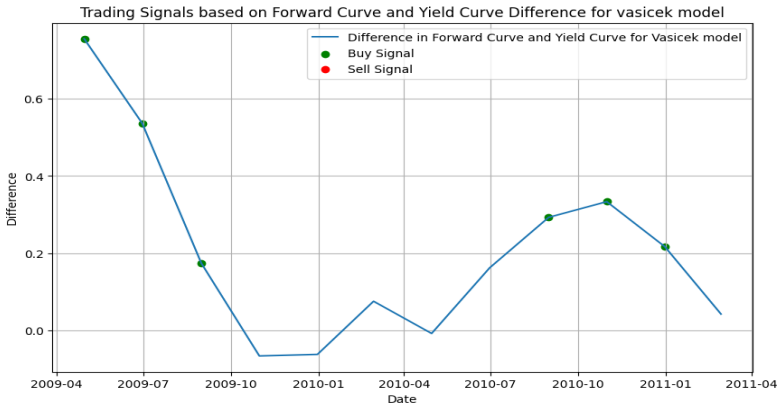


Trade signals during Recession

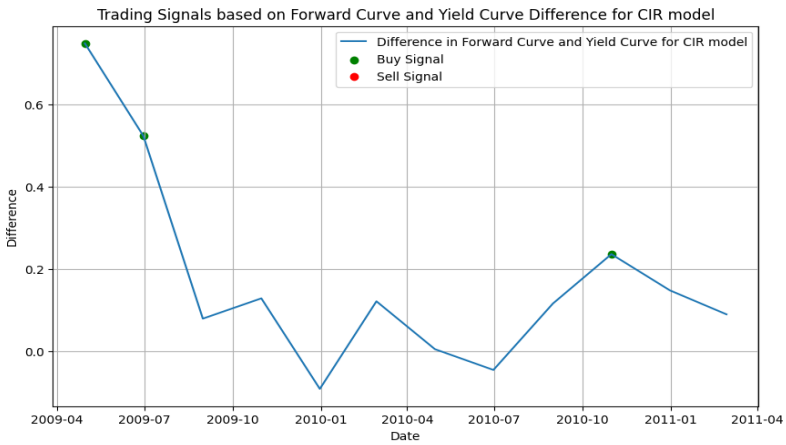
For Trade signal during recession I took date of April 2009. As there were a lot of events around that period so it is acknowledged to train the dataset for longer time. I trained dataset for 60 days.

Trade signal for models are as follow

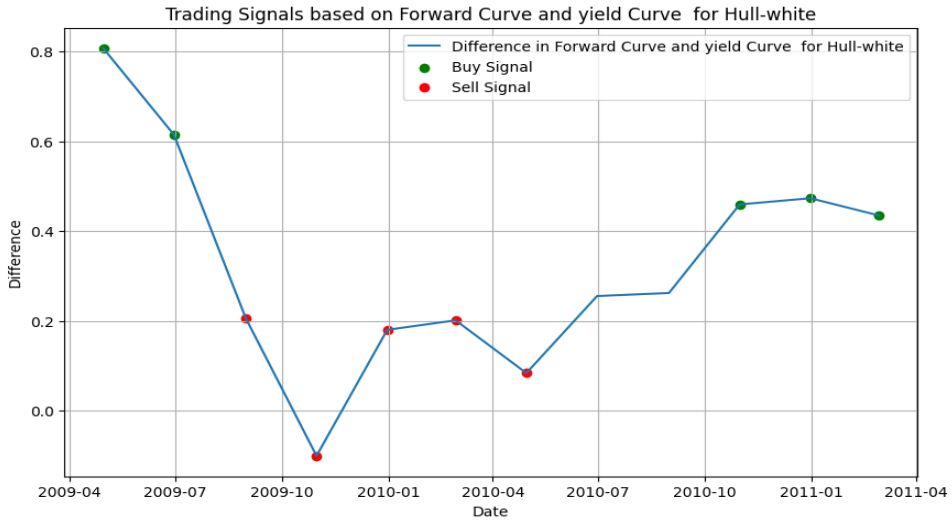
Vasicek Model



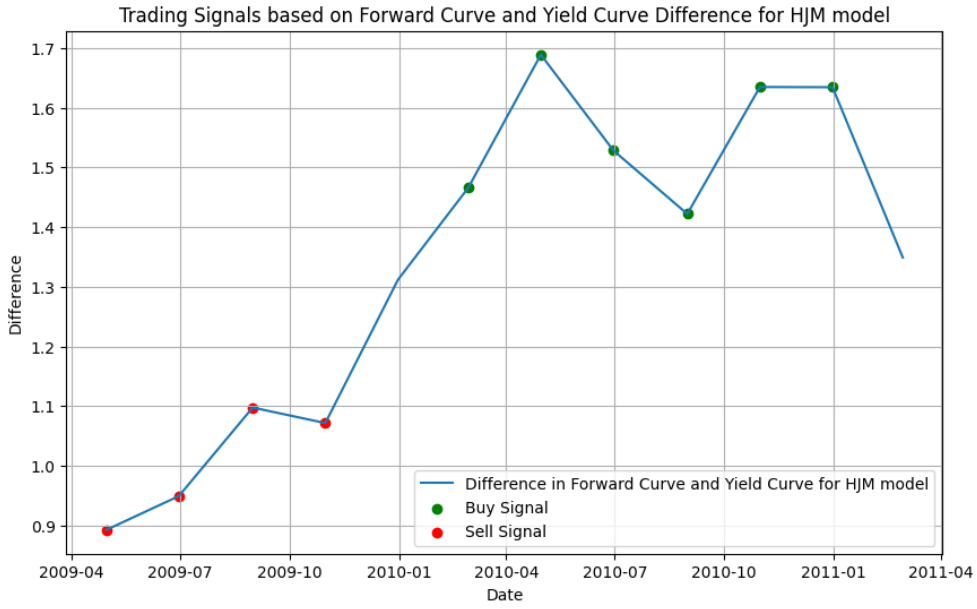
CIR Model



HULL-White Model



HJM model

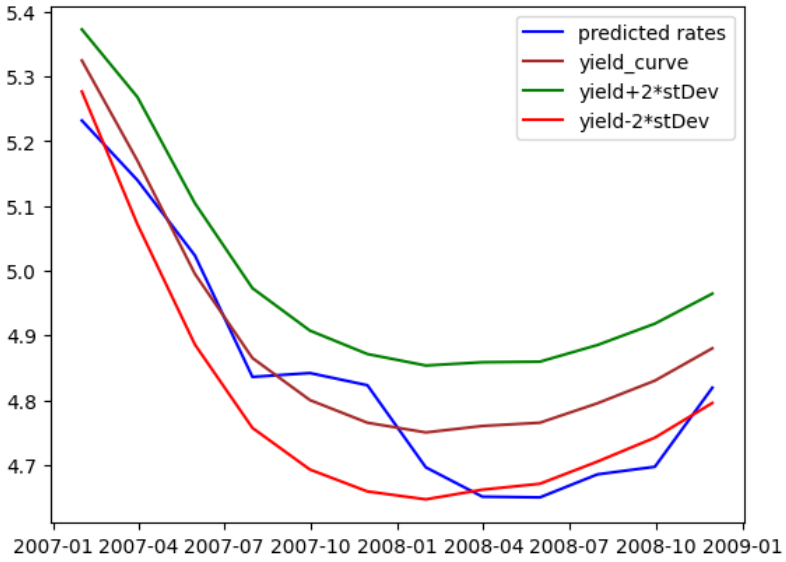


Rate of return is expected to be decreased initially by initial 3 models so the signal to buy is given by these models. Vasicek model and CIR models were conservative for some maturity due to volatility of market at that time. HJM model looked to short term event. There was QE applied that time to it predicted initial increase in rate and then decrease, thus generating mixed signals.

Date when 3-month libor rate similar to FED rate-

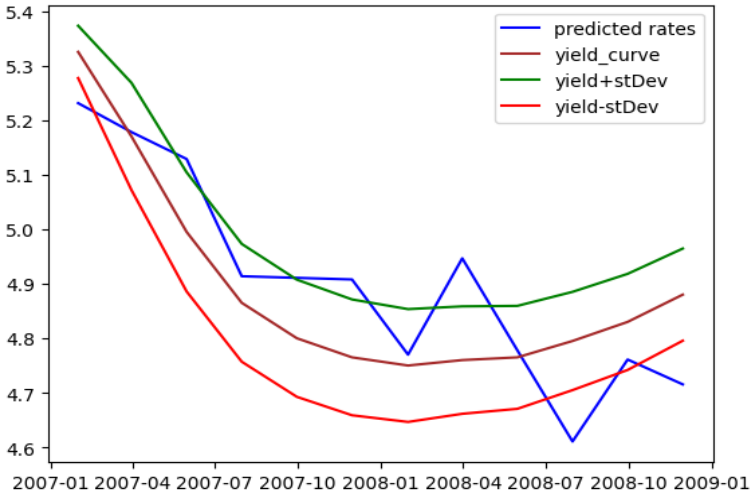
This comparison is used to see our model behavior at condition when ED market is synchronous to fed in terms of its short term prediction

Vasicek Model



*See the plot of predicted forward rate is around the 2*stdev factor of yield curve*

CIR Model



CIR model have spikes outside expected range, thereby generating trade signals

Comparision of all models

Vasicek	CIR	Hull- White	HJM
Can produce negative interest rates.	Extends the Vasicek model to prevent negative interest rates.	Extends the Vasicek model with time-varying volatility.	Multi-factor model that captures the entire yield curve dynamics.

Single-factor model with constant volatility.	Still a single-factor model with time-varying volatility.	Incorporates a mean-reverting process and stochastic volatility.	Incorporates multiple stochastic factors.
Relatively simple to implement and calibrate.	Suitable for short- to medium-term interest rate modeling.	Commonly used for pricing and risk management of interest rate derivatives.	More complex and computationally demanding compared to single-factor models.
Does not capture term structure dynamics.	Does not capture term structure dynamics.	Provides more accurate results than single-factor models.	Provides a more realistic representation of interest rate dynamics.

In our implementation, we used discrete formula rather than doing stochastic differential for each process. This is the reason that as we increase our term maturity, Hull-White and HJM model were not performing the perfect results.

Moreover, as our model is for medium term interest rates(3 years) so CIR model seem to produce the best matched signals. Also incorporating the root term which nullify the possibility of negative interest rate further suites the model.

Vasicek model is also a simpler model which gave correct forward curves for most of the duration of economic cycle. This model can be used to generate a safer trade plans as a many dates, forward curve of this model signify to ol the stocks and refrain from active market participation**.**

All models have been iterated at many dates, a few is only added in the report. One can perform iterations for any date before 2019 in the code attached