Question 1

$\overrightarrow{v} = (cos\theta, sin\theta, 0)$
$A = \overrightarrow{v} \cdot \overrightarrow{\sigma}$

$$= \begin{bmatrix} 0 & cos\theta - i\ sin\theta \\ cos\theta - i\ sin\theta & 0 \end{bmatrix}$$

The eigenvalues for A are +1 and -1.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & cos\theta - i\ sin\theta \\ cos\theta - i\ sin\theta & -\lambda \end{vmatrix} = \lambda^2 - cos^2\theta + sin^2\theta$$

$$= \lambda^2 + I = 0$$

The eigenvectors can be calculated from the equation $A|a\rangle = \lambda_a|a\rangle$
$A|u\rangle = (+1)|u\rangle$

$$|u\rangle = \begin{bmatrix} cos\theta - i\ sin\theta \\ 1 \end{bmatrix}$$

Similarly,

$A|v\rangle = (+1)|v\rangle$

$$|v\rangle = \begin{bmatrix} cos\theta - i\ sin\theta \\ -1 \end{bmatrix}$$

The normalised eigenbasis of the observable A are

$$|u\rangle = 1/\sqrt{2} \begin{bmatrix} cos\theta - i\ sin\theta \\ 1 \end{bmatrix}$$

$$|v\rangle = 1/\sqrt{2} \begin{bmatrix} cos\theta - i\ sin\theta \\ -1 \end{bmatrix}$$

Question 2

Probability of measuring λ, measuring with observable A on the state $|u\rangle$.
Probability of measuring μ, measuring with observable B on the state $|v\rangle$.

$$p(A, \lambda) = \langle u|A^\dagger A|u\rangle$$
$$p(B, \mu) = \langle v|B^\dagger B|v\rangle$$

Probability of measuring (λ, μ), measuring with observable $M_A \otimes M_B$ on the state $|u\rangle \otimes |v\rangle$.

$$
\begin{aligned}
p(A, B, \lambda, \mu) &= \langle v | \langle u | M_A \otimes M_B | u \rangle | v \rangle \\
&= \langle v | \langle u | B^\dagger A^\dagger A B | u \rangle | v \rangle \\
&= \langle u | A^\dagger A | u \rangle \langle v | B^\dagger B | v \rangle \\
&= p(A, \lambda) p(B, \mu)
\end{aligned}
$$

Question 3

A)

Let the projectors for the observable A be $|\lambda\rangle\langle\lambda|$, and the projectors for the observable B be $|\mu\rangle\langle\mu|$

Apply the measurement operator $M_A \otimes M_B$ to $|\psi\rangle$, where $|\psi\rangle = 1/\sqrt{2}\ [|01\rangle - |10\rangle]$

$$
\begin{aligned}
M_A \otimes M_B |\psi\rangle &= M_A \otimes M_B\ (1/\sqrt{2}\ (|01\rangle - |10\rangle)) \\
&= |\lambda\mu\rangle\langle\lambda\mu|\ (1/\sqrt{2}\ (|01\rangle - |10\rangle)) \\
&= 1/\sqrt{2}\, |\lambda\mu\rangle\ (\langle\lambda\mu|01\rangle - \langle\lambda\mu|10\rangle)
\end{aligned}
$$

Which is an unentangled state of form $|\phi_1\rangle|\phi_2\rangle$

B)

From the result of question 2,

$E(i, j) = \langle A_i \rangle \langle B_j \rangle$

$$
\begin{aligned}
E(1,1) + E(2,1) + E(2,2) - E(1,2) &= \langle A_1 \rangle \langle B_1 \rangle + \langle A_2 \rangle \langle B_1 \rangle + \langle A_2 \rangle \langle B_2 \rangle - \langle A_1 \rangle \langle B_2 \rangle \\
&= (\langle A_1 \rangle + \langle A_2 \rangle) \langle B_1 \rangle + (\langle A_2 \rangle - \langle A_1 \rangle) \langle B_2 \rangle
\end{aligned}
$$

The observables has eigenvalues +1 and -1 for both A and B.
Either

$(\langle A_1 \rangle + \langle A_2 \rangle) \langle B_1 \rangle = 0$

Or,

$(\langle A_2 \rangle - \langle A_1 \rangle) \langle B_2 \rangle = 0$

And in either case

$$
\begin{aligned}
E(1,1) + E(2,1) + E(2,2) - E(1,2) &= (\langle A_1 \rangle + \langle A_2 \rangle) \langle B_1 \rangle + (\langle A_2 \rangle - \langle A_1 \rangle) \langle B_2 \rangle \\
&\leq 2
\end{aligned}
$$