Shubhashree Dash

Question 1

$|\psi\rangle = 1/\sqrt{2}\ [|00\rangle - |11\rangle]$

Basis

$$|+\rangle = 1/\sqrt{2}\ [|0\rangle + |1\rangle]$$

$$|-\rangle = 1/\sqrt{2}\ [|0\rangle - |1\rangle]$$

For 2 qubits, the values of the states is

$$|++\rangle = |+\rangle \otimes |+\rangle$$
$$= 1/2\ [|00\rangle + |01\rangle + |10\rangle + |11\rangle]$$

$$|+-\rangle = |+\rangle \otimes |-\rangle$$
$$= 1/2\ [|00\rangle - |01\rangle + |10\rangle - |11\rangle]$$

$$|-+\rangle = |-\rangle \otimes |+\rangle$$
$$= 1/2\ [|00\rangle + |01\rangle - |10\rangle - |11\rangle]$$

$$|--\rangle = |-\rangle \otimes |-\rangle$$
$$= 1/2\ [|00\rangle - |01\rangle - |10\rangle + |11\rangle]$$

The state becomes

$$|\psi\rangle = 1/\sqrt{2}\ [|00\rangle - |11\rangle]$$
$$= 1/\sqrt{2}\ [|+-\rangle - |-+\rangle]\ \ldots\ \text{in new basis}$$

If the observed value of first bit is $|+\rangle$ then the second qubit is $|-\rangle$

If the observed value of first bit is $|-\rangle$ then the second qubit is $|+\rangle$

The probabilities of the states is

$|++\rangle = 1/2$

$|--\rangle = 1/2$

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Question 2

$|\psi'\rangle = 1/\sqrt{2}\ [|00\rangle - |11\rangle]$

The general basis u and v be

$$|u\rangle = a|0\rangle + b|1\rangle$$

$$|v\rangle = b*|0\rangle - a*|1\rangle$$

Where both u and v are orthonormal to each other that is $\langle u|v\rangle = 0$

$$\begin{aligned}
\langle u|v\rangle &= (b*|0\rangle - a*|1\rangle)^{\dagger} \cdot (a|0\rangle + b|1\rangle) \\
&= (b\langle 0| - a\langle 1|) \cdot (a|0\rangle + b|1\rangle) \\
&= ba\langle 0|0\rangle + b^2\langle 0|1\rangle - a^2\langle 1|0\rangle - ab\langle 1|1\rangle \\
&= ba + 0 - 0 - ab \\
&= 0
\end{aligned}$$

The 4 possible states for 2 qubits

$$\begin{aligned}
|uu\rangle &= |u\rangle \otimes |u\rangle \\
&= [a^2|00\rangle + ab|01\rangle + ab|10\rangle + b^2|11\rangle]
\end{aligned}$$

$$\begin{aligned}
|uv\rangle &= |u\rangle \otimes |v\rangle \\
&= [ab*|00\rangle - |a|^2|01\rangle + |b|^2|10\rangle - a*b|11\rangle]
\end{aligned}$$

$$\begin{aligned}
|vu\rangle &= |v\rangle \otimes |u\rangle \\
&= [ab*|00\rangle + |b|^2|01\rangle - |a|^2|10\rangle - a*b|11\rangle]
\end{aligned}$$

$$\begin{aligned}
|vv\rangle &= |v\rangle \otimes |v\rangle \\
&= [(b*)^2|00\rangle - a*b*|01\rangle - a*b*|10\rangle + (a*)^2|11\rangle]
\end{aligned}$$

Probability of measuring $|\psi\rangle$ to get $|uu\rangle$

$$\begin{aligned}
P_{\psi}(uu) &= |\langle uu|\psi\rangle|^2 \\
&= |[a^2|00\rangle + ab|01\rangle + ab|10\rangle + b^2|11\rangle][1/\sqrt{2}(|00\rangle - |11\rangle)|^2 \\
&= |1/\sqrt{2}(a^2 - b^2)|^2
\end{aligned}$$

$$P_{\psi}(uv) = |\langle uv|\psi\rangle|^2$$
$$= |[ab*|00\rangle - |a|^2|01\rangle + |b|^2|10\rangle - a*b|11\rangle][1/\sqrt{2}(|00\rangle - |11\rangle)|^2$$
$$= |1/\sqrt{2}(ab* - a*b)|^2$$

$$P_{\psi}(vu) = |\langle vu|\psi\rangle|^2$$
$$= |[ab*|00\rangle + |b|^2|01\rangle - |a|^2|10\rangle - a*b|11\rangle][1/\sqrt{2}(|00\rangle - |11\rangle)|^2$$
$$= |1/\sqrt{2}(ab* - a*b)|^2$$

$$P_{\psi}(vv) = |\langle vv|\psi\rangle|^2$$
$$= |[(b*)^2|00\rangle - a*b*|01\rangle - a*b*|10\rangle + (a*)^2|11\rangle][1/\sqrt{2}(|00\rangle - |11\rangle)|^2$$
$$= |1/\sqrt{2}((b*)^2 - (a*)^2)|^2$$

Finally,

$$|\psi'\rangle = 1/\sqrt{2}\ [|00\rangle - |11\rangle]$$
$$= 1/\sqrt{2}\ [|uv\rangle - |vu\rangle]\ \ \ldots\ \text{in new basis}$$

The representation of the state given is not affected by the magnitude of the qubits in the new basis.

Question 3

Hermitian Operation : Operator A whose adjoint is same as A.

$A^\dagger = (A*)^T = A$

$A_{i,j} = A_{j,i}*$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a* & c* \\ b* & d* \end{bmatrix}$

Unitary Operations : An operator U is said to be unitary if $U^\dagger U = I$.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a* & c* \\ b* & d* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hadamard operator :

$H = 1/\sqrt{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$H* = 1/\sqrt{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$(H*)^T = H^\dagger$

$= 1/\sqrt{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Prove $H^\dagger H = HH^\dagger = I$

$HH^\dagger = H^\dagger H = 1/2 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$= 1/2 \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= I$

As $H^\dagger = H$

It implies $HH^\dagger = H^\dagger H = HH$

Thus H is Hermitian, unitary and satisfies HH = I.