Advanced Probability

Andrew Tulloch

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CHAPTER 1

Conditional Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Ω is a set, \mathcal{F} is a σ -algebra on Ω , and \mathbb{P} is a probability measure on (Ω, \mathcal{F}) .

DEFINITION 1.1. \mathcal{F} is a σ -algebra on Ω if it satisfies

- $\emptyset, \Omega \in \mathcal{F}$
- $A \in \mathcal{F} ==> A^c \in \mathcal{F}$
- $(A_n)_{n\geq 0}$ is a collection of sets in \mathcal{F} then $\cup_n A_n \in \mathcal{F}$.

Definition 1.2. \mathbb{P} is a probability measure on (Ω, \mathcal{F}) if

- $\mathbb{P}: \mathcal{F} \to [0,1]$ is a set function.
- $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$,
- $(A_n)_{n\geq 0}$ is a collection of pairwise disjoint sets in \mathcal{F} , then $\mathbb{P}(\cup_n A_n) = \sum_n \mathbb{P}(A_n)$.

DEFINITION 1.3. The Borel σ -algebra $\mathbb{B}(\mathbb{R})$ is the σ -algebra generated by the open sets of \mathbb{R} . Call \mathcal{O} the collection of open subsets of \mathbb{R} , then

$$\mathbb{B}(\mathbb{R}) = \bigcap \{ \xi : \xi \text{ is a sigma algebra containing } \mathcal{O} \}$$
 (1.1)

DEFINITION 1.4. \mathcal{A} a collection of subsets of Ω , then we write $\sigma(\mathcal{A}) = \bigcap \{ \xi : \xi \text{ a sigma algebra containing } \mathcal{A} \}$

DEFINITION 1.5. X is a random variable on (Ω, \mathcal{F}) if $X : \Omega - > \mathbb{R}$ is a function with the property that $X^{-1}(V) \in \mathcal{F}$ for all V open sets in \mathbb{R} .

EXERCISE 1.6. If X is a random variable then $\{B \subseteq \mathbb{R}, X^{-1}(B) \in \mathcal{F}\}$ is a σ -algebra and contains $\mathbb{B}(\mathbb{R})$.

If $(X_i, i \in I)$ is a collection of random variables, then we write $\sigma(X_i, i \in I) = \sigma(\{\omega \in \Omega : X_{i(\omega) \in \mathcal{B}\}, i \in I, B \in \mathbb{B}(\mathbb{R})\})}$ and it is the smallest σ -algebra that makes all the X_i 's measurable.

DEFINITION 1.7. First we define it for the positive simple random variables.

$$\mathbb{E}\left(\sum_{i=1}^{n} c_i \mathbf{1}(A_i)\right) = \sum_{i=1}^{n} \mathbb{P}(A_i). \tag{1.2}$$

with c_i positive constants, $(A_i) \in \mathcal{F}$.

We can extend this to any positive random variable $X \ge 0$ by approximation X as the limit of piecewise constant functions.

For a general X, we write $X = X^+ - X^-$ with $X^+ = \max(X, 0), X^- = \max(-X, 0)$.

If at least one of $\mathbb{E}(X^+)$ or $\mathbb{E}(X^-)$ is finite, then we define $\mathbb{E}(X) = \mathbb{E}(X^+) + \mathbb{E}(X^-)$. We call X integrable if $\mathbb{E}(|X|) < \infty$.

DEFINITION 1.8. Let $A, B \in \mathcal{F}, \mathbb{P}(B) > 0$. Then

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$
$$\mathbb{E}[X|B] = \frac{\mathbb{E}(X \mathbb{1}(B))}{\mathbb{P}(B)}$$

Goal - we want to define $\mathbb{E}(X|\mathcal{G})$ that is a random variable measurable with respect to the σ -algebra \mathcal{G} .

1. Discrete Case

Suppose \mathcal{G} is a σ -algebra countably generated $(B_i)_{i\in\mathbb{N}}$ is a collection of pairwise disjoint sets in \mathcal{F} with $\cup B_i = \Omega$. Let $\mathcal{G} = \sigma(B_i, i \in \mathbb{N})$.

It is easy to check that $\mathcal{G} = \{ \bigcup_{j \in J} B_j, J \subseteq N \}.$

Let X be an integrable random variable. Then

$$X' = \mathbb{E}(X|\mathcal{G}) = \sum_{i \in \mathbb{N}} \mathbb{E}(X|\mathcal{B}_i) \, \mathbb{I}(B_i)$$

- (i) X' is \mathcal{G} -measurable (check).
- (ii)

$$\mathbb{E}(|X'|) \le \mathbb{E}(|X|) \tag{1.3}$$

and so X' is integrable.

(iii) $\forall G \in \mathcal{G}$, then

$$\mathbb{E}(X\mathbb{I}(G) = \mathbb{E}(X'\mathbb{I}(G))) \tag{1.4}$$

(check).

2. Existence and Uniqueness

DEFINITION 1.9. $A \in \mathcal{F}$, A happens almost surely (a.s.) if $\mathbb{P}(A) = 1$.

THEOREM 1.10 (Monotone Convergence Theorem). If $X_n \geq 0$ is a sequence of random variables and $X_n \uparrow X$ as $n \to \infty$ a.s, then

$$\mathbb{E}(X_n) \uparrow \mathbb{E}(X) \tag{1.5}$$

almost surely as $n \to \infty$.

THEOREM 1.11 (Dominated Convergence Theorem). If (X_n) is a sequence of random variables such that $|X_n| \leq Y$ for Y an integrable random variable, then if $X_n \stackrel{as}{\to} X$ then $\mathbb{E}(X_n) \stackrel{as}{\to} \mathbb{E}(X)$.

DEFINITION 1.12. For $p \in [1, \infty)$, f measurable functions, then

$$||f||_p = E[|f|^p]^{\frac{1}{p}} \tag{1.6}$$

$$||f||_{\infty} = \inf\{\lambda : |f| \le \lambda a.e.\}$$
(1.7)

Definition 1.13.

$$L^p = L^p(\Omega, \mathcal{F}, \mathbb{P}) = \{ f : ||f||_p < \infty \}$$

Formally, L^p is the collection of equivalence classes where two functions are equivalent if they are equal a.e. We will just represent an element of L^p by a function, but remember that equality is a.e.

THEOREM 1.14. The space $(L^2, \|\cdot\|_2)$ is a Hilbert space with $\langle U, V \rangle >= \mathbb{E}(UV)$.

Suppose \mathcal{H} is a closed subspace, then $\forall f \in L^2$ there exists a unique $g \in \mathcal{H}$ such that $||f - g||_2 = \inf\{||f - h||_2, h \in \mathcal{H} \text{ and } \langle f - g, h \rangle = 0 \text{ for all } h \in \mathcal{H}.$

We call g the orthogonal projection of f onto \mathcal{H} .

THEOREM 1.15. Let $(\Omega, \mathcal{F}, \P)$ be an underlying probability space, and let X be an integrable random variable, and let $\mathcal{G} \subset \mathcal{F}$ sub σ -algebra. Then there exists a random variable Y such that

- (i) Y is \mathcal{G} -measurable
- (ii) If $A \in \mathcal{G}$,

$$\mathbb{E}(X\mathbb{I}(A) = \mathbb{E}(Y\mathbb{I}(A))) \tag{1.8}$$

and Y is integrable.

Moreover, if Y' also satisfies the above properties, then Y = Y' a.s.

Remark 1.16. Y is called a version of the conditional expectation of X given \mathcal{G} and we write $\mathcal{G} = \sigma(Z)$ as $Y = \mathbb{E}(X|\mathcal{G})$.

Remark 1.17. (b) could be replaced by the following condition: for all Z \mathcal{G} -measurable, bounded random variables,

$$\mathbb{E}(XZ) = \mathbb{E}(YZ) \tag{1.9}$$

PROOF. Uniqueness - let Y' satisfy (a) and (b). If we consider $\{Y' - Y > 0\} = A$, A is \mathcal{G} measurable. From (b),

$$\mathbb{E}((Y'-Y)\mathbb{I}(A)) = \mathbb{E}(X\mathbb{I}(A)) - \mathbb{E}(X\mathbb{I}(A)) = 0$$

and hence $\mathbb{P}(Y'-Y>0)=0$ which implies that $Y'\leq Y$ a.s. Similarly, $Y'\geq Y$ a.s.

Existence - Complete the following three steps:

(i) $X \in L^2(\Omega, \mathcal{F}, \P)$ is a Hilbert space with $\langle U, V \rangle = \mathbb{E}(UV)$. The space $L^2(\Omega, \mathcal{G}, \P)$ is a closed subspace.

$$X_n \to X(L^2) => X_n \xrightarrow{p} X => \exists subseq X_{n_k} \xrightarrow{as} X => X' = \limsup X_{n_k}$$
 (1.10)

We can write

$$L^{2}(\Omega, \mathcal{F}, \P) = L^{2}(\Omega, \mathcal{G}, \P) + L^{2}(\Omega, \mathcal{G}, \P)^{\perp}$$
$$X = Y + Z$$

Set $Y = \mathbb{E}(X|\mathcal{G})$, Y is \mathcal{G} -measurable, $A \in \mathcal{G}$.

$$\mathbb{E}(X\mathbb{I}(A)) = EY\mathbb{I}(A) + \underbrace{EZ\mathbb{I}(A)}_{=0}$$

(ii) If $X \ge 0$ then $Y \ge 0$ a.s. Consider $A = \{Y < 0\}$, then

$$0 \le \mathbb{E}(X\mathbb{I}(A)) = \mathbb{E}(Y\mathbb{I}(A)) \le 0 \tag{1.11}$$

Thus $\mathbb{P}(A) = 0 \Rightarrow Y \geq 0$ a.s.

Let $X \geq 0$, Set $0 \leq X_n = max(X, n) \leq n$, so $X_n \in L^2$ for all n. Write $Y_n = \mathbb{E}(X_n | \mathcal{G})$, then $Y_n \geq 0$ a.s., Y_n is increasing a.s.. Set $Y = \limsup Y_n$. So Y is \mathcal{G} -measurable. We will show $Y = \mathbb{E}(X|\mathcal{G})$ a.s. For all $A \in \mathcal{G}$, we need to check $\mathbb{E}(X\mathbb{I}(A)) = \mathbb{E}(Y\mathbb{I}(A))$. We know that $\mathbb{E}(X_n\mathbb{I}(A)) = \mathbb{E}(Y_n\mathbb{I}(A))$, and $Y_n \uparrow Y$ a.s. Thus, by monotone convergence theorem, $\mathbb{E}(X\mathbb{I}(A)) = \mathbb{E}(Y\mathbb{I}(A)).$

If X is integrable, setting $A = \Omega$, we have Y is integrable.

(iii) X is a general random variable, not necessarily in L^2 or ≥ 0 . Then we have that X = $X^+ + X^-$. We define $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X^+|\mathcal{G}) - \mathbb{E}(X^-|\mathcal{G})$. This satisfies (a), (b).

REMARK 1.18. If $X \geq 0$, we can always define $Y = \mathbb{E}(X|\mathcal{G})$ a.s. The integrability condition of Y may not be satisfied.

DEFINITION 1.19. Let $\mathcal{G}_0, \mathcal{G}_1, \ldots$ be sub σ -algebras of \mathcal{F} . Then they are called independent if for all $i, j \in \mathbb{N}$,

$$\mathbb{P}(G_i \cap \dots \cap G_j) = \prod_{i=1}^n \mathbb{P}(G_i)$$
(1.12)

(i) If $X \geq 0$ then $\mathbb{E}(X|\mathcal{G}) \geq 0$ THEOREM 1.20.

- (ii) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$ $(A = \Omega)$
- (iii) X is \mathcal{G} -measurable implies $\mathbb{E}(X|\mathcal{G}) = X$ a.s.
- (iv) X is independent of \mathcal{G} , then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$.

Theorem 1.21 (Fatau's lemma). $X_n \ge 0$, then for all n,

$$\mathbb{E}(\liminf X_n) \le \liminf \mathbb{E}(X_n) \tag{1.13}$$

Theorem 1.22 (Conditional Monotone Convergence). Let $X_n \geq 0$, $X_n \uparrow X$ a.s. Then

$$\mathbb{E}(X_n|\mathcal{G}) \uparrow \mathbb{E}(X|\mathcal{G}) \ a.s. \tag{1.14}$$

PROOF. Set $Y_n = \mathbb{E}(X_n | \mathcal{G})$. Then $Y_n \geq 0$ and Y_n is increasing. Set $Y = \limsup Y_n$. Then Y is \mathcal{G} -measurable.

Theorem 1.23 (Conditional Fatau's Lemma). $X_n \ge 0$, then

$$\mathbb{E}(\liminf X_n | \mathcal{G}) \le \liminf \mathbb{E}(X_n | \mathcal{G}) \ a.s. \tag{1.15}$$

PROOF. Let X denote the limit inferior of the X_n . For every natural number k define pointwise the random variable $Y_k = \inf_{n \geq k} X_n$. Then the sequence Y_1, Y_2, \ldots is increasing and converges pointwise to X. For $k \leq n$, we have $Y_k \leq X_n$, so that

$$\mathbb{E}(Y_k|\mathcal{G}) \le \mathbb{E}(X_n|\mathcal{G}) \ a.s \tag{1.16}$$

by the monotonicity of conditional expectation, hence

$$\mathbb{E}(Y_k|\mathcal{G}) \le \inf_{n \ge k} \mathbb{E}(X_n|\mathcal{G}) \ a.s. \tag{1.17}$$

because the countable union of the exceptional sets of probability zero is again a null set. Using the definition of X, its representation as pointwise limit of the Y_k , the monotone convergence theorem for conditional expectations, the last inequality, and the definition of the limit inferior, it follows that almost surely

$$\mathbb{E}\left(\liminf_{n\to\infty} X_n | \mathcal{G}\right) = \mathbb{E}(X|\mathcal{G})$$
(1.18)

$$= \mathbb{E}\left(\lim_{k \to \infty} Y_k | \mathcal{G}\right) \tag{1.19}$$

$$= \lim_{k \to \infty} \mathbb{E}(Y_k | \mathcal{G}) \tag{1.20}$$

$$\leq \lim_{k \to \infty} \inf_{n \geq k} \mathbb{E}(X_n | \mathcal{G}) \tag{1.21}$$

$$= \liminf_{n \to \infty} \mathbb{E}(X_n | \mathcal{G}) \tag{1.22}$$

Theorem 1.24 (Conditional dominated convergence). TODO

3. Conditional Jensen's Inequalities

Let X be an integrable random variable such that $\phi(x)$ is integrable of ϕ is non-negative. Suppose $\mathcal{G} \subset \mathcal{F}$ is a σ -algebra. Then

$$\mathbb{E}(\phi(X)|\mathcal{G}) \ge \phi(\mathbb{E}(X|\mathcal{G})) \tag{1.23}$$

almost surely. In particular, if $1 \le p < \infty$, then

$$\|\mathbb{E}(X|\mathcal{G})\|_p \le \|X\|_p \tag{1.24}$$

PROOF. Every convex function can be written as $\phi(x) = \sup_{i \in \mathbb{N}} (a_i x + b_i), a_i, b_i \in \mathbb{R}$. Then

$$\mathbb{E}(\phi(X)|\mathcal{G}) \ge a\mathbb{E}(X|\mathcal{G}) + b_i$$

$$\mathbb{E}(\phi(X)|\mathcal{G}) \ge \sup_{i \in \mathbb{N}} (a_i\mathbb{E}(X|\mathcal{G}) + b_i)$$

$$= \phi(\mathbb{E}(X|\mathcal{G})$$

The second part follows from

$$\|\mathbb{E}(X|\mathcal{G})\|_p^p = \mathbb{E}(|\mathbb{E}(X|\mathcal{G})|^p) \le \mathbb{E}(\mathbb{E}(|X|^p|\mathcal{G})) = \mathbb{E}(|X|^p) = \|X\|_p^p$$
(1.25)

PROPOSITION 1.25 (Tower Property). Let $X \in L^1$, $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ be sub- σ -algebras. Then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H}) \tag{1.26}$$

almost surely.

PROOF. Clearly $\mathbb{E}(X|\mathcal{H})$ is \mathcal{H} -measurable. Let $A \in \mathcal{H}$. Then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{H})\,\mathbb{I}(A)) = \mathbb{E}(X\mathbb{I}(A)) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})\,\mathbb{I}(A)) \tag{1.27}$$

PROPOSITION 1.26. Let $X \in L^1$, $\mathcal{G} \subset \mathcal{F}$ be sub- σ -algebras. Suppose that Y is bounded, \mathcal{G} -measurable. Then

$$\mathbb{E}(XY|\mathcal{G}) = Y\mathbb{E}(X|\mathcal{G}) \tag{1.28}$$

almost surely.

PROOF. Clearly $Y\mathbb{E}(X|\mathcal{G})$ is \mathcal{G} -measurable. Let $A \in \mathcal{G}$. Then

$$\mathbb{E}(Y\mathbb{E}(X|\mathcal{G})\,\mathbb{I}(A)) = \mathbb{E}\left(\mathbb{E}(X|\mathcal{G})\,\underbrace{(Y\mathbb{I}(A))}_{\mathcal{G}\text{-measurable, bounded}}\right) = \mathbb{E}(XY\mathbb{I}(A)) \tag{1.29}$$

DEFINITION 1.27. A collection \mathcal{A} of subsets of Ω is called a π -system if for all $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

PROPOSITION 1.28 (Uniqueness of extension). Suppose that ξ is a σ -algebra on E. Let μ_1, μ_2 be two measures on (E, ξ) that agree on a π -system generating ξ and $\mu_1(E) = \mu_2(E) < \infty$. Then $\mu_1 = \mu_2$ everywhere on ξ .

THEOREM 1.29. Let $X \in L^1$, $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ two sub- σ -algebras. If $\sigma(X, \mathcal{G})$ is independent of \mathcal{H} , then

$$\mathbb{E}(X|\sigma(\mathcal{G},\mathcal{H})) = \mathbb{E}(X|\mathcal{G}) \tag{1.30}$$

almost surely.

PROOF. Take $A \in \mathcal{G}, B \in \mathcal{H}$.

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})\mathbb{I}(A)\mathbb{I}(B)) = \mathbb{P}(B)\mathbb{E}(\mathbb{E}(X|\mathcal{G})\mathbb{I}(A))$$

$$= \mathbb{P}(B)\mathbb{E}(X\mathbb{I}(A))$$

$$= \mathbb{E}(X\mathbb{I}(A)\mathbb{I}(B))$$

$$= \mathbb{E}(\mathbb{E}(X|\sigma(\mathcal{G},\mathcal{H}))\mathbb{I}(A\cap B))$$

Assume $X \geq 0$, the general case follows by writing $X = X^+ - X^-$.

Now, letting $F \in \mathcal{F}$, we have that $\mu(F) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})\mathbb{I}(F))$, and if μ, ν are two measures on $(\Omega, p\mathcal{F})$, setting $\mathcal{A} = \{A \cap B, A \in \mathcal{G}, B \in \mathcal{H}\}$. Then \mathcal{A} is a π -system.

 μ, ν are two measurables that agree on the π -system \mathcal{A} and $\mu(\Omega) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X) = \nu\Omega < \infty$, since X is integrable. Note that \mathcal{A} generates $\sigma(\mathcal{G}, \mathcal{H})$.

So, by the uniqueness of extension theorem, μ, ν agree everywhere on $\sigma(\mathcal{G}, \mathcal{H})$.

Remark 1.30. If we only had X independent of \mathcal{H} and \mathcal{G} independent of \mathcal{H} , the conclusion can fail. For example, consider coin tosses X, Y independent 0, 1 with probability $\frac{1}{2}$, and $Z = \mathbb{I}(X = Y)$.

4. Product Measures and Fubini's Theorem

DEFINITION 1.31. A measure space (E, ξ, μ) is called σ -finite if there exists sets $(S_n)_n$ with $\cup S_n = E$ and $\mu(S_n) < \infty$ for all n.

Let (E_1, ξ_1, μ_1) and (E_2, ξ_2, μ_2) be two σ -finite measure spaces, with $\mathcal{A} = \{A_1 \times A_2 : A_1 \in \xi_1, A_2 \in \xi_2\}$ a π -system of subsets of $E = E_1 \times E_2$. Define $\xi = \xi_1 \otimes \xi_2 = \sigma(A)$.

DEFINITION 1.32 (Product measure). Let (E_1, ξ_1, μ_1) and (E_2, ξ_2, μ_2) be two σ -finite measure spaces. Then there exists a unique measure μ on (E, ξ) ($\mu = \mu_1 \otimes \mu_2$) satisfying

$$\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2) \tag{1.31}$$

for all $A_1 \in \xi_1, A_2 \in \xi_2$.

THEOREM 1.33 (Fubini's Theorem). Let (E_1, ξ_1, μ_1) and (E_2, ξ_2, μ_2) be σ -finite measure spaces. Let $f \geq 0$, f is ξ -measurable. Then

$$\mu(f) = \int_{E_1} \left(\int_{E_2} f(x_1, x_2) \mu_2(dx_2) \right) \mu_1(dx_1)$$
(1.32)

If f is integrable, then $x_2 \mapsto f(x_1, x_2)$ is u_2 -integrable for u_1 -almost all x.

Moreover, $x_1 \mapsto \int_{E_2} f(x_1, x_2 \mu_2(dx_2))$ is μ_1 -integrable and $\mu(f)$ is given by (1.32).

5. Examples of Conditional Expectation

DEFINITION 1.34. A random vector $(X_1, X_2, \dots, X_n) \in \mathbb{R}^n$ is called a Gaussian random vector if and only if for all $a_1, \dots, a_n \in \mathbb{R}$,

$$a_1 X_1 + \dots + a_n X_n \tag{1.33}$$

is a Gaussian random variable.

 $(X_t)_{t\geq 0}$ is called a Gaussian process if for all $0\leq t_1\leq t_2\leq \cdots \leq t_n$, the vector X_{t_1},\ldots,X_{t_n} is a Gaussian random vector.

Example 1.35 (Gaussian case). Let (X,Y) e a Gaussian vector in \mathbb{R}^2 . We want to calculate

$$\mathbb{E}(X|Y) = \mathbb{E}(X|\sigma(Y)) = X' \tag{1.34}$$

where X' = f(Y) with f a Borel function. Let's try f of a linear function $X' = aY = b, a, b \in \mathbb{R}$ to be determined.

Note that $\mathbb{E}(X) = \mathbb{E}(X')$ and $E(X' - X)Y = 0 \Rightarrow Cov(X - X', Y) = 0$ by laws of conditional expectation. Then we have that

$$a\mathbb{E}(Y) + b = \mathbb{E}(X) \ Cov(X, Y) = a\mathbb{V}(X) \tag{1.35}$$

TODO - continue inference

6. Notation for Example Sheet 1

- (i) $G \vee H = \sigma(G, H)$.
- (ii) Let X, Y be two random variables taking values in R with joint density $f_{X,Y}(x,y)$ and $h: \mathbb{R} \to \mathbb{R}$ be a Borel function such that h(X) is integrable. We want to calculate

$$\mathbb{E}(h(X)|Y) = \mathbb{E}(h(X)|\sigma(Y)) \tag{1.36}$$

Let g be bounded and measurable. Then

$$\mathbb{E}(h(X)g(Y)) = \int \int h(x)g(y)f_{X,Y}(x,y)dxdy$$
 (1.37)

$$= \int \int h(x)g(y)\frac{f_{X,Y}(x,y)}{f_{Y}(y)}f_{Y}(y)dxdy$$
 (1.38)

$$= \int \left(\int h(x) \frac{f_{X,Y}(x,y)}{f_Y(y)} dx \right) g(y) f_Y(y) dy \tag{1.39}$$

with 0/0 = 0

Set $\phi(y) = \int h(x) \frac{f_{X,Y}(x,y)}{f_Y(y)} dx$ if $f_Y(y) > 0$, and 0 otherwise. Then we have

$$\mathbb{E}(h(X)|Y) = \phi(Y) \tag{1.40}$$

almost surely, and

$$\mathbb{E}(h(X)|Y) = \int h(x)\nu(Y, dx)$$
 (1.41)

with $\nu(y,dx) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \mathbb{I}(f_Y(y) > 0) dx = f_{X|Y}(x|y) dx$. $\nu(y,dx)$ is called the conditional distribution of X given Y = y and $f_{X|Y}(x|y)$ is the conditional density of X given Y = y.

CHAPTER 2

Discrete Time Martingales

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (E, ξ) a measurable space. Usually $E = \mathbb{R}, \mathbb{R}^d, \mathbb{C}$. For us, $E = \mathbb{R}$. A sequence $X = (X_n)_{n \geq 0}$ of random variables taking values in E is called a **stochastic process**.

A filtration is an increasing family $(\mathcal{F}_n)_{n\geq 0}$ of sub- σ -algebras of \mathcal{F}_n , so $\mathcal{F}_n\subseteq \mathcal{F}_{n+1}$.

Intuitively, \mathcal{F}_n is the information available to us at time n. To every stochastic process X we associate a filtration called the natural filtration

$$(\mathcal{F}_n^X)_{n\geq 0}, \mathcal{F}_n^X = \sigma(X_k, k \leq n)$$
(2.1)

A stochastic process X is called adapted to $(\mathcal{F}_n)_{n\geq 0}$ if X_n is \mathcal{F}_n -measurable for all n.

A stochastic process X is called integrable if X_n is integrable for all n.

DEFINITION 2.1. An adapted integrable process $(X_n)_{n>0}$ taking values in \mathbb{R} is called a

- (i) martingale if $\mathbb{E}(X_n|\mathcal{F}_m) = X_m$ for all $n \geq m$.
- (ii) super-martingale if $\mathbb{E}(X_n|\mathcal{F}_m) \leq X_m$ for all $n \geq m$.
- (iii) sub-martingale if $\mathbb{E}(X_n|\mathcal{F}_m) \geq X_m$ for all $n \geq m$.

REMARK 2.2. A (sub, super)-martingale with respect to a filtration \mathcal{F}_n is also a (sub, super)-martingale with respect to the natural filtration of X_n (by the tower property)

EXAMPLE 2.3. Suppose (ξ_i) are IID random variables with $\mathbb{E}(\xi_i) = 0$. Set $X_n = \sum_{i=1}^n \xi_i$. Then (X_n) is a martingale.

Example 2.4. As above, but let (ξ_i) be IID with $\mathbb{E}(\xi_i) = 1$. Then $X_n = \prod_{i=1}^n \xi_i$ is a martingale.

DEFINITION 2.5. A random variables $T: \Omega \to \mathbb{Z}_+ \cup \{\infty\}$ is called a stopping time if $\{T \leq n\} \in \mathcal{F}_n$ for all n. Equivalently, $\{T = n\} \in \mathcal{F}_n$ for all n.

Example 2.6. (i) Constant times are trivial stopping times.

(ii) $A \in \mathcal{B}(\mathbb{R})$. Define $T_A = \inf\{n \geq 0 | X_n \in A\}$, with $\inf \emptyset = \infty$. Then T_A is a stopping time.

PROPOSITION 2.7. Let $S, T, (T_n)$ be stopping times on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$. Then $S \wedge T, S \vee T, \inf_n T_n$, $\liminf_n T_n$, $\limsup_n T_n$ are stopping times. NOTATION. T stopping time, then $X_T(\omega) = X_{T(\omega)}(\omega)$. The stopped process X^T is defined by $X_t^T = X_{T \wedge t}$.

$$\mathcal{F}_T = \{ A \in \mathcal{F} | A \cap T \le T \in \mathcal{F}_t, \forall t \}.$$

PROPOSITION 2.8. $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P}), X = (X_n)_{n \geq 0}$ is adapted.

- (i) $S \leq T$, stopping times, then $\mathcal{F}_S \subseteq \mathcal{F}_T$
- (ii) $X_T \mathbb{I}(T < \infty)$ is \mathcal{F}_T -measurable.
- (iii) T a stopping time, then X^T is adapted
- (iv) If X is integrable, then X^T is integrable.

PROOF. Let $A \in \xi$. Need to show that $\{X_T \mathbb{I}(T < \infty) \in A\} \in \mathcal{F}_T$.

$$\{X_T \mathbb{I}(T < \infty)\} \cap \{T \le t\} = \bigcup_{s \le t} \left(\underbrace{\{T = s\}}_{\mathcal{F}_s \subseteq \mathcal{F}_t} \cap \underbrace{\{X_s \in A\}}_{\in \mathcal{F}_s \subseteq \mathcal{F}_t}\right) \in \mathcal{F}_t$$
 (2.2)

1. Optional Stopping

Theorem 2.9. Let X be a martingale.

- (i) If T is a stopping time, then X^T is also a martingale. In particular, $\mathbb{E}(X_{T \wedge t}) = \mathbb{E}(X_0)$ for all t.
- (ii)
- (iii)
- (iv)

PROOF. By the tower property, it is sufficient to check

$$\mathbb{E}(X_{T \wedge t} | \mathcal{F}_{t-1}) = \mathbb{E}\left(\sum_{i=1}^{t-1} X_s \underbrace{\mathbb{I}(T=s)}_{\in \mathcal{F}_s \subseteq \mathcal{F}_{t-1}} | \mathcal{F}_{t-1}\right) + \mathbb{E}(X_t \mathbb{I}(T>t-1) | \mathcal{F}_{t-1})$$

$$= \sum_{s=0}^{t-1} \mathbb{I}(T=s) X_s + \mathbb{I}(t>t-1) X_{t-1} = X_{T \wedge (t-1)}$$

Since it is a martingale, $\mathbb{E}(X_{T \wedge t}) = \mathbb{E}(X_0)$.

Theorem 2.10. Let X be a martingale.

(i) If T is a stopping time, then X^T is also a martingale, so in particular

$$\mathbb{E}(X_{T \wedge t}) = \mathbb{E}(X_0) \tag{2.3}$$

(ii) If $X \leq T$ are bounded stopping times, then $\mathbb{E}(X_T | \mathcal{F}_S) = X_S$ almost surely.

PROOF. Let $S \leq T \leq n$. Then $X_T = (X_T - X_{T-1}) + (X_{T-1} - X_{T-2}) + \dots + (X_{S+1} - X_S) + X_S = X_s + \sum_{k=0}^n (X_{k+1} - X_k) \mathbb{I}(S \leq k < T)$.

Let $A \in \mathcal{F}_s$. Then

$$\mathbb{E}(X_T \mathbb{I}(A)) = \mathbb{E}(X_s \mathbb{I}(A)) + \sum_{k=0}^n \mathbb{E}((X_{k+1} - X_k) \mathbb{I}(S \le k < T) \mathbb{I}(A))$$
(2.4)

$$= \mathbb{E}(X_s \mathbb{I}(A)) \tag{2.5}$$

Remark 2.11. The optimal stopping theorem also holds for super/sub-martingales with the respective martingale inequalities in the statement.

Example 2.12. Suppose that $(\xi_i)_i$ are random variables with

$$\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = \frac{1}{2} \tag{2.6}$$

Set $X_0 = 0, X_n = \sum_{i=1}^n \xi_i$. This is a simply symmetric random walk on X_n . Let $T = \inf\{n \ge 0 : X_n = 1\}$. Then $\P T < \infty = 1$, but T is not bounded.

PROPOSITION 2.13. If X is a positive supermartingale and T is a stopping time which is finite almost surely ($\mathbb{P}(T < \infty) = 1$), then

$$\mathbb{E}(X_T) \le \mathbb{E}(X_0) \tag{2.7}$$

Proof.

$$\mathbb{E}(X_T) = \mathbb{E}\left(\liminf_{t \to \infty} X_{t \wedge T}\right) \le \liminf_{t \to \infty} \mathbb{E}(X_{t \wedge T}) \le \mathbb{E}(X_0)$$
(2.8)

2. Hitting Probabilities for a Simple Symmetric Random Walk

Let (ξ_i) be IID ± 1 equally likely. Let $X_0 = 0, X_n = \sum_{i=1}^n \xi_i$. For all $x \in Z$ let

$$T_x = \inf\{n \ge 0 : X_n = x\}$$
 (2.9)

which is a stopping time. We want to explore hitting probabilities $(\mathbb{P}(T_{-a} < T_b))$ for a, b > 0. If $\mathbb{E}(T) < \infty$, then by (iv) in Theorem 2.10, $\mathbb{E}(X_T) = \mathbb{E}(X_0) = 0$.

$$\mathbb{E}(X_T) = -a\mathbb{P}(T_{-a} < T_b) + b\mathbb{P}(T_b < T_{-a}) = 0$$
(2.10)

and thus obtain that

$$\mathbb{P}(T_{-a} < T_b) = \frac{b}{a+b}.\tag{2.11}$$

Remains to check $\mathbb{E}(T) < \infty$. We have $\mathbb{P}(\xi_1 = 1, \xi_{a+b} = 1) = \frac{1}{2^{a+b}}$.

3. Martingale Convergence Theorem

THEOREM 2.14. Let $X = (X_n)_{n\geq 0}$ be a (super-)-martingale bounded in L^1 , that is, $\sup_{n\geq 0} \mathbb{E}(|X_n|) < \infty$. Then X_n converges as $n \to \infty$ almost surely towards an a.s. finite limit $X \in L^1(\mathcal{F}_{\infty})$ with $\mathcal{F}_{\infty} = \sigma(\mathcal{F}_n, n \geq 0)$. To prove it we will use Doob's trick which counts up-crossings of intervals with rational endpoints.

COROLLARY 2.15. Let X be a positive supermartingale. Then it converges to an almost surely finite limit as $n \to \infty$.

Proof.

$$\mathbb{E}(|X_n|) = \mathbb{E}(X_n) \le \mathbb{E}(X_0) < \infty \tag{2.12}$$

PROOF. Let $x = (x_n)_n$ be a sequence of real numbers, and let a < b be two real numbers. Let $T_0(x) = 0$ and inductively for $k \ge 0$,

$$S_{k+1}(x) = \inf\{n \ge T_k(x) : x_n \le a\} T_{k+1}(x) = \inf\{n \ge S_{k+1}(x) : x_n \ge b\}$$
(2.13)

with the usual convention that $\inf \emptyset = \infty$.

Define $N_n([a,b],x) = \sup\{k \geq 0 : T_k(x) \leq n\}$ - the number of up-crossings of the interval [a,b] by the sequence x by the time n. As $n \to \infty$, we have

$$N_n([a, b], x) \uparrow N([a, b], x) = \sup\{k \ge 0 : T_k(x) < \infty\},$$
 (2.14)

the total number of up-crossings of the interval [a, b].

LEMMA 2.16. A sequence of rationals $x = (x_n)_n$ converges in $\mathbb{R} = \mathbb{R} \cup \{\pm \infty\}$ if and only if $N([a,b],x) < \infty$ for all rationals a,b.

PROOF. Assume x converges. Then if for some a < b we had that $N([a, b], x) = \infty$, then $\lim \inf_n x_n \le a < b \le \limsup_n x_n$, which is a contradiction.

Then, suppose that x does converge. Then $\lim \inf_n x_n > \lim \sup_n x_n$, and so taking a, b rationals between these two numbers gives that $N([a, b], x) = \infty$ as required.

Theorem 2.17 (Doob's up-crossing inequality). Let X be a supermartingale and a < b be two real numbers. Then for all $n \ge 0$,

$$(b-a)\mathbb{E}(N_n([a,b],X)) \le \mathbb{E}((X_n-a)^-)$$
(2.15)

PROOF. For all k,

$$X_{T_b} - X_{S_b} \ge b - a$$
 (2.16)

4. Uniform Integrability

Theorem 2.18. Suppose $X \in L^1$. Then the collection of random variables

$$\{\mathbb{E}(X|\mathcal{G})\}\tag{2.17}$$

for $\mathcal{G} \subseteq \mathcal{F}$ a sub- σ -algebra is uniformly integrable.

PROOF. Since $X \in L^1$, for all $\epsilon > 0$ there exists S > 0 such that if $A \in \mathcal{F}$ and $\mathbb{P}(A) < \delta$, then $\mathbb{E}(|X|\mathbb{I}(A)) \leq \epsilon$.

Set $Y = \mathbb{E}(X|\mathcal{G})$. Then $\mathbb{E}(|Y|) \leq \mathbb{E}(|X|)$. Choose $\lambda < \infty$ such that $\mathbb{E}(|X|) \leq \lambda \delta$. Then

$$\mathbb{P}(|Y| \ge \lambda) \le \frac{\mathbb{E}(|Y|)}{\lambda} \le \delta \tag{2.18}$$

by Markov's inequality.

Then

$$\mathbb{E}(|Y|\mathbb{I}(|Y| \ge \lambda)) \le \mathbb{E}(\mathbb{E}(|X||\mathcal{G})\,\mathbb{I}(|Y| \ge \lambda)) \tag{2.19}$$

$$= \mathbb{E}(|X|\mathbb{I}(|Y| \ge \lambda)) \tag{2.20}$$

$$\leq \epsilon$$
 (2.21)

DEFINITION 2.19. A process $X = (X_n)_{n \ge 0}$ is called a uniformly integrable martingale if it is a martingale and the collection (X_n) is uniformly integrable.

THEOREM 2.20. Let X be a martingale. Then the following are equivalent.

- (i) X is a uniformly integrable martingale.
- (ii) X converges almost surely and in L^1 to a limit X_{∞} as $n \to \infty$.
- (iii) There exists a random variable $Z \in L^1$ such that $X_n = \mathbb{E}(Z|\mathcal{F}_n)$ almost surely for all $n \geq 0$.

THEOREM 2.21 (Chapter 13 of Williams). Let $X_n, X \in L^1$ for all $n \geq 0$ and suppose that $X_n \stackrel{as}{\to} X$ as $n \to \infty$. Then X_n converges to X in L^1 if and only if (X_n) is uniformly integrable.

PROOF. We proceed as follows.

- $(i) \Rightarrow (ii)$ Since X is uniformly integrable, it is bounded in L^1 and by the martingale convergence theorem, we get that X_n converges almost surely to a finite limit X_{∞} . By the previous theorem, Theorem 2.21 gives L^1 convergence.
- (ii) \Rightarrow (iii) Set $Z = X_{\infty}$. We need to show that $X_n = \mathbb{E}(Z|\mathcal{F}_n)$ almost surely for all $n \geq 0$. For all $m \geq n$ by the martingale property we have

$$||X_n - \mathbb{E}(X_\infty | \mathcal{F}_n)||_1 = ||\mathbb{E}(X_m - X_\infty | \mathcal{F}_n)||_1 \le ||X_m - X_\infty||_1 \to 0$$
 (2.22)

as $m \to \infty$.

 $(iii) \Rightarrow (i) \mathbb{E}(Z|\mathcal{F}_n)$ is a martingale by the tower property of conditional expectation. Uniform integrability follows from Theorem 2.18.

Remark 2.22. If X is UI then $X_{\infty} = \mathbb{E}(Z|\mathcal{F}_{\infty})$ a.s where $F_{\infty} = \sigma(\mathcal{F}_n, n \geq 0)$.

REMARK 2.23. If X is a super/sub-martingale UI, then it converges almost surely and in L¹ to a finite limit X_{∞} with $\mathbb{E}(X_{\infty}|\mathcal{F}_n)$ (\geq)(\leq) X_n almost surely.

EXAMPLE 2.24. Let $X_1, X_2, ...$ be IID random variables with $\mathbb{P}(X = 0) = \mathbb{P}(X = 2) = \frac{1}{2}$. Set $Y_n = X_1 \cdot ... \cdot X_n$. Then Y_n is a martingale.

As $\mathbb{E}(Y_n) = 1$ for all n, we have (Y_n) is bounded in L^1 , and it converges almost surely to 0. But $\mathbb{E}(Y_n) = 1$ for all n, and hence it does not converge in L^1 .

If X is a UI martingale and T is a stopping time, then we can unambiguously define

$$X_T = \sum_{n=0}^{\infty} X_n \mathbb{I}(T=n) + X_{\infty} \mathbb{I}(T=\infty)$$
 (2.23)

Theorem 2.25 (Optional stopping for UI martingales). Let X be a UI martingale and let S,T be stopping times with $S \leq T$. Then

$$\mathbb{E}(X_T|\mathcal{F}_S) = X_S \tag{2.24}$$

almost surely.

PROOF. We first show that $\mathbb{E}(X_{\infty}|\mathcal{F}_T) = X_T$ almost surely for any stopping time T. First, check that $X_T \in L^1$. Since $|X_n| \leq \mathbb{E}(|X_{\infty}||\mathcal{F}_n)$, we have

$$\mathbb{E}(|X_T|) = \sum_{n=0}^{\infty} \mathbb{E}(|X_n|\mathbb{I}(T=n) + \mathbb{E}(|X_\infty|\mathbb{I}(T=\infty)))$$
(2.25)

$$\leq \sum_{n \in \mathbb{Z}^+ \cup \{\infty\}} \mathbb{E}(|X_{\infty}|\mathbb{I}(T=n)) \qquad (2.26)$$

$$= \mathbb{E}(|X_{\infty}|) \tag{2.27}$$

Let $B \in \mathcal{F}_T$. Then

$$\mathbb{E}(\mathbb{I}(B) X_T) = \sum_{n \in \mathbb{Z}^+ \cup \{\infty\}} \mathbb{E}(\mathbb{I}(B) \mathbb{I}(T=n) X_n)$$
(2.28)

$$= \sum_{n \in Z^+ \cup \{\infty\}} \mathbb{E}(\mathbb{I}(B) \,\mathbb{I}(T=n) \,X_{\infty}) \tag{2.29}$$

$$= \mathbb{E}(\mathbb{I}(B) X_{\infty}) \tag{2.30}$$

where for the second equality we used that $\mathbb{E}(X_{\infty}|\mathcal{F}_n) = X_n$ almost surely.

Clearly $X_T is \mathcal{F}_T$ -measurable, and hence $\mathbb{E}(X_\infty | \mathcal{F}_T) = X_T$ almost surely. Using the tower property of conditional expectation, we have for stopping times $S \leq T$ (as $\mathcal{F}_S \subseteq \mathcal{F}_T$),

$$\mathbb{E}(X_T | \mathcal{F}_S) = \mathbb{E}(\mathbb{E}(X_\infty | \mathcal{F}_T) | \mathcal{F}_S) \tag{2.31}$$

$$= \mathbb{E}(X_{\infty}|\mathcal{F}_S) \tag{2.32}$$

$$=X_S \tag{2.33}$$

almost surely.

5. Backwards Martingales

Let $... \subseteq \mathcal{G}_{-2} \subseteq G_{-1} \subseteq G_0$ be a sequence of

Fill in proof from lecture notes

6. Applications of Martingales

THEOREM 2.26 (Kolmogrov's 0-1 law). Let $(X_i)_{i\geq 1}$ be a sequence of IID random variables. Let $\mathcal{F}_n = \sigma(X_k, k \geq n)$ and $\mathcal{F}_{\infty} = \cap_{n\geq 0} \mathcal{F}_n$. Then \mathcal{F}_{∞} is trivial - that is, every $A \in \mathcal{F}_{\infty}$ has probability $\mathbb{P}(A) \in \{0, 1\}$.

PROOF. Let $G_n = \sigma(X_k, k \leq n)$ and $A \in \mathcal{F}_{\infty}$. Since \mathcal{G}_n is independent of \mathcal{F}_{n+1} , we have that

$$\mathbb{E}(\mathbb{I}(A) | \mathcal{G}_n) = \mathbb{P}(A) \tag{2.34}$$

Theorem 2.26 (LN) gives that $\mathbb{P}(A) = \mathbb{E}(\mathbb{I}(A) | \mathcal{G}_n)$ converges to $\mathbb{E}(\mathbb{I}(A) | \mathcal{G}_\infty)$ as $n \to \infty$, where $\mathcal{G}_\infty = \sigma(\mathcal{G}_n, n \ge 0)$. Then we deduce that $\mathbb{E}(\mathbb{I}(A) | \mathcal{G}_n) = \mathbb{I}(A) = \mathbb{P}(A)$ as $\mathcal{F}_\infty \subseteq \mathcal{G}_\infty$. Therefore, $\mathbb{P}(A) = \mathbb{P}(A) = \mathbb{P}(A)$

link to correct theorem

THEOREM 2.27 (Strong law of large numbers). Let $(X_i)_{i\geq 1}$ be a sequence of IID random variables in L^1 with $\mu = \mathbb{E}(X_i)$. Let $S_n = \sum_{i=1}^n X_i$ and $S_0 = 0$. Then $\frac{S_n}{n} \to \mu$ as $n \to \infty$ almost surely and in L^1 .

Proof.

fill in, this is somewhat in-

Theorem 2.28 (Kakutani's product martingale theorem). Let $(X_n)_{n\geq 0}$ be a sequence of independent non-negative random variables of mean 1. Let $M_0=1$, $M_n=\prod_{i=1}^n X_i$ for $n\in\mathbb{N}$. Then $(M_n)_{n\geq 0}$ is a non-negative martingale and $M_n\to M_\infty$ a.s. as $n\to\infty$ for some random variable M_∞ . We set $a_{n=\mathbb{E}\left(\sqrt{X_n}\right)}$, then $a_n\in(0,1]$. Moreover,

- (i) If $\prod_n a_n > 0$, then $M_n \to M_\infty$ in L^1 and $\mathbb{E}(M_\infty) = 1$,
- (ii) If $\prod_n a_n = 0$, then $M_{\infty} = 0$ almost surely.

Proof. □ rdfill in

- **6.1.** Martingale proof of the Radon-Nikodym theorem. Let \mathbb{P}, \mathbb{Q} be two probability measures on the measurable space Ω, \mathcal{F} . Assume that \mathcal{F} is countably generated, that is, there exists a collection of sets $(F_n)_{n\in\mathbb{N}}$ such that $\mathcal{F} = \sigma(F_N, n \in \mathbb{N})$. Then the following are equivalent.
 - (i) $\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A)$ for all $A \in \mathcal{F}$. That is, \mathbb{Q} is absolutely continuous with respect to \mathbb{P} and write $\mathbb{Q} << \mathbb{P}$
 - (ii) For all $\epsilon > 0$, there exists $\delta > 0$ such that $\mathbb{P}(A) \leq \delta \Rightarrow \mathbb{Q}(A) \leq \epsilon$.
 - (iii) There exists a non-negative random variable X such that

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(X\mathbb{I}(A)) \tag{2.35}$$

PROOF. (i) \to (ii). If (ii) does not hold, then there exists $\epsilon > 0$ such that for all $n \ge 1$ there exists a set A_n with $\mathbb{P}(A_n) \le \frac{1}{n^2}$ and $\mathbb{Q}(A_n) \ge \epsilon$. By Borel-Cantelli, we get that $\mathbb{P}(A_n i.o) = 0$. Therefore from (i) we get that $\mathbb{Q}(A_n i.o) = 0$. But

$$\mathbb{Q}(A_n i.o) = \mathbb{Q}(\cap_n \cup_{k \ge n} A_k) = \lim_{n \to \infty} Q(\cup_{k \ge n} A_k) \ge \epsilon$$
 (2.36)

which is a contradiction.

 $(ii) \rightarrow (iii)$. Consider the filtration $\mathcal{F}_n = \sigma(F_k, k \leq n)$. Let

$$\mathcal{A}_n = \{ H_1 \cap \dots \cap H_n | H_i = F_i \text{ or } F_i^c \}$$
(2.37)

then it is easy to see that $\mathcal{F}_n = \sigma(A_n)$. Note also that sets in A_n are disjoint.

□ continue proof

CHAPTER 3

Stochastic Processes in Continuous Time

Our setting is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space with $t \in J \subseteq \mathbb{R}_+ = [0, \infty)$

DEFINITION 3.1. A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing collection of σ -algebras $(\mathcal{F}_t)_{t \in J}$, satisfying $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $t \geq s$. A stochastic process in continuous time is an ordered collection of random variables on Ω .

Bibliography