Advanced Probability Examples

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Contents

CHAPTER 1

Example Sheet 1

(i) Let Y_t be a candidate state price density. Then $Y_1(t) = [\alpha_1, \alpha_2, \alpha_3]$ must satisfy the requirement that $Y_t P_t$ is a martingale, and $\alpha_i > 0$. Thus we have the linear equations

$$4 = \frac{1}{4} \cdot \alpha_1 \cdot 3 + \frac{1}{4} \cdot \alpha_2 \cdot 6 + \frac{1}{2} \cdot \alpha_3 \cdot 6 \tag{1.1}$$
$$6 = \frac{1}{4} \cdot \alpha_1 \cdot 9 + \frac{1}{4} \cdot \alpha_2 \cdot 8 + \frac{1}{2} \cdot \alpha_3 \cdot 4 \tag{1.2}$$

Solving these set of equations, we have the augmented matrix

$$\begin{bmatrix} 3 & 6 & 12 & 16 \\ 9 & 8 & 8 & 24 \end{bmatrix} \tag{1.3}$$

which has the set of solutions

$$z = t \tag{1.4}$$
$$y = \frac{12 - 14t}{5} \tag{1.5}$$
$$x = \frac{8 + 24t}{15} \tag{1.6}$$

for $0 < t < \frac{5}{6}$.

(ii) Similar to above, we have the discount factor is unity, and so we have to solve the augmented matrix

$$\begin{bmatrix} 9 & 8 & 18 \\ 6 & 10 & 21 \\ 1 & 1 & 1 \end{bmatrix} \tag{1.7}$$

This has no solution, and thus there are no state price densities, thus there are arbitrage opportunities in this market.

(iii) Following Girsanov's theorem, consider the process

$$\tilde{W}_t = W_t - \int_0^t \alpha ds \tag{1.8}$$

with W_t a standard $\mathbb{P}$-Brownian motion. Then, under the measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^t \alpha dW_s - \int_0^t \alpha^2 ds\right) \tag{1.9}$$

$$= \exp\left(tW_s - \frac{1}{2}t\alpha^2\right) \tag{1.10}$$

, $\tilde{W}_t$ is a $\mathbb{Q}$-Brownian motion.

Thus, for $t = 1$, we have W_1 is a random vector with $N_d(0, I)$ distribution, and under the change of measure given in the question, $\tilde{W}_1 = W_1 - \alpha$ has a $N_d(0, I)$ under $\mathbb{Q}$ as required.

(iv)

(v) (i) Let a be a trading strategy, and let X represent the discounted gains from the first period, and Z be a state price density Then the theorem gives that if no arbitrage trading strategies exist, then a state price density exists.[1]

(ii) Assuming for a linearly independent set of vectors is sufficient, as each condition is invariant when a linearly independent set of random variables X is replaced by the transformation $Y = MX$ for arbitrary $M \in R^{m\times d}$.

If there exists a random variable $Z > 0$ a.s. such that $\mathbb{E}(Z|X_i|) < \infty$ and $\mathbb{E}(ZX_i) = 0$, then $Y_i = \sum_i \lambda_i X_i$ satisfies these conditions by linearity and the trivial bound.

The precondition is more involved. If $a \cdot Y = a \cdot (MX) \geq 0$ for some a, then $(M^T a) \cdot X \geq 0$, and so $M^T a = 0$ a.s, and so $(M^T a) \cdot X = a \cdot Y = 0$ a.s.

(iii)

(vi) First, some terminology. Consider a n-state, m-asset single period market model.Let $S = (S_{ij}) \in \mathbb{R}^{m\times n}$ be the discounted difference in value of the i-th asset in the j-th state between the initial and first period.

Let $P = (P_j) \in \mathbb{R}^n$ be the market probability of the j-th state.

Let $Y = (Y_j) \in \mathbb{R}^n$ be a candidate state price density for our market model.

Let $H = (H_i) \in \mathbb{R}^m$ be a candidate arbitrage for our market model.

By the first fundamental theorem of arbitrage pricing, we have exactly one of two alternatives are true.

(i) There exists a state price density - that is, there exists $Y' \in \mathbb{R}^n$ with $Y' > 0$ and $\mathbb{E}(Y'S) = 0$.

(ii) There exists an arbitrage - that is, there exists $H' \in \mathbb{R}^m$ with $(S^T H')_i \geq 0$ for all $1 \leq i \leq n$, and $(S^T H')_i > 0$ for at least one i with $P_i > 0$.

[1] What does the numeraire requirement imply?

Now, let $B = (B_{ij}) \in \mathbb{R}^{m \times n}$ be defined by

$$B_{ij} = B_{ij} \times P_j \tag{1.11}$$

The condition for existence of a state price density becomes

$$\sum_{j=1}^{n} Y_j P_j S_{ij} = \sum_{j=1}^{n} B_{ij} Y_j \tag{1.12}$$

for all $1 \leq i \leq m$.

The condition for the existence of an arbitrage becomes

$$\sum_{i=1}^{m} H_i S_{ij} P_j = \sum_{i=1}^{m} H_i B_{ij} \tag{1.13}$$

for all $1 \leq j \leq n$.

Thus, we can restate the FTAP as

(i) There exists $Y \in \mathbb{R}^n$ with $Y_i > 0$ such that $BY = 0$.

(ii) There exists $H \in \mathbb{R}^m$ with $(B^T H)_i \geq 0$ and with $B^T H \neq 0$.

which is the required result.

(vii)

(viii)

(ix)

(x)

(xi) Note that $Z_t = X_t - Y_t$ is a martingale, and thus $|Z_t|$ is a submartingale (as it is trivially bounded above by two integrable functions, and a convex function of a martingale is a submartingale by Jensen's inequality). Then for any $0 \leq t \leq T$, we have

$$\begin{aligned} 0 &= \mathbb{E}(|X_T - Y_T||\mathcal{F}_t) && (1.14) \\ &= \mathbb{E}(|Z_T||\mathcal{F}_t) && (1.15) \\ &\geq |Z_s| && (1.16) \\ &\geq 0 && (1.17) \end{aligned}$$

where the first line follows from $Z_T = 0$ almost surely and the second follows from the submartingale property. Thus the equalities are strict, and we have $Z_t = 0$ almost surely, and so

$$X_t = Y_t \tag{1.18}$$

almost surely.

(xii) Note that on the sub-σ-algebra $\mathcal{G}$, the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ is the conditional expectation $\mathbb{E}(Z|\mathcal{G})$. This is because, for arbitrary $A \in \mathcal{G}$,

$$\mathbb{E}(\mathbb{E}(Z|\mathcal{G})\,\mathbb{I}(A)) = \mathbb{E}(\mathbb{E}(Z\mathbb{I}(A)\,|\mathcal{G})) \tag{1.19}$$

$$= \mathbb{E}(Z\mathbb{I}(A)) \tag{1.20}$$

$$= \mathbb{Q}(A) \tag{1.21}$$

as required.

Now, we treat the problem in the question. Let $A \in \mathcal{G}$. Let $Y = \mathbb{E}_{\mathbb{P}}(Z|\mathcal{G})$. Then

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{I}_A \mathbb{E}_{\mathbb{P}}(ZX|\mathcal{G})) = \mathbb{E}_{\mathbb{P}}(\mathbb{I}_A Y \mathbb{E}_{\mathbb{P}}(ZX|\mathcal{G})) \tag{1.22}$$

$$= \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(Y\mathbb{I}(A)\,ZX|\mathcal{G})) \tag{1.23}$$

$$= \mathbb{E}_{\mathbb{P}}(Y\mathbb{I}(A)\,ZX) \tag{1.24}$$

$$= \mathbb{E}_{\mathbb{Q}}(Y\mathbb{I}(A)\,X) \tag{1.25}$$

$$= \mathbb{E}_{\mathbb{Q}}(\mathbb{I}(A)\,\mathbb{E}_{\mathbb{Q}}(YX|\mathcal{G})) \tag{1.26}$$

and as A was arbitrary, we have that

$$\mathbb{E}_{\mathbb{P}}(ZX|\mathcal{G}) = \mathbb{E}_{\mathbb{Q}}(YX|\mathcal{G}) \tag{1.27}$$

$$= Y\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}) \tag{1.28}$$

$$= \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G})\,\mathbb{E}_{\mathbb{P}}(Z|\mathcal{G}) \tag{1.29}$$

which is sufficient to prove our required result.

(xiii)

(xiv)

Bibliography