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# ADVANCED PROBABILITY EXAMPLES

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# 1

## Example Sheet 1

- (i) Let  $Y_t$  be a candidate state price density. Then  $Y_1(t) = [\alpha_1, \alpha_2, \alpha_3]$  must satisfy the requirement that  $Y_t P_t$  is a martingale, and  $\alpha_i > 0$ . Thus we have the linear equations

$$4 = \frac{1}{4} \cdot \alpha_1 \cdot 3 + \frac{1}{4} \cdot \alpha_2 \cdot 6 + \frac{1}{2} \cdot \alpha_3 \cdot 6 \quad (1.1)$$

$$6 = \frac{1}{4} \cdot \alpha_1 \cdot 9 + \frac{1}{4} \cdot \alpha_2 \cdot 8 + \frac{1}{2} \cdot \alpha_3 \cdot 4 \quad (1.2)$$

Solving these set of equations, we have the augmented matrix

$$\begin{bmatrix} 3 & 6 & 12 & 16 \\ 9 & 8 & 8 & 24 \end{bmatrix} \quad (1.3)$$

which has the set of solutions

$$z = t \quad (1.4)$$

$$y = \frac{12 - 14t}{5} \quad (1.5)$$

$$x = \frac{8 + 24t}{15} \quad (1.6)$$

for  $0 < t < \frac{5}{6}$ .

- (ii) Similar to above, we have the discount factor is unity, and so we have to solve the augmented matrix

$$\begin{bmatrix} 9 & 8 & 18 \\ 6 & 10 & 21 \\ 1 & 1 & 1 \end{bmatrix} \quad (1.7)$$

This has no solution, and thus there are no state price densities, thus there are arbitrage opportunities in this market.

(iii) Following Girsanov's theorem, consider the process

$$\tilde{W}_t = W_t - \int_0^t \alpha ds \quad (1.8)$$

with  $W_t$  a standard  $\mathbb{P}$ -Brownian motion. Then, under the measure  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left( \int_0^t \alpha dW_s - \int_0^t \alpha^2 ds \right) \quad (1.9)$$

$$= \exp \left( tW_s - \frac{1}{2}t\alpha^2 \right) \quad (1.10)$$

,  $\tilde{W}_t$  is a  $\mathbb{Q}$ -Brownian motion.

Thus, for  $t = 1$ , we have  $W_1$  is a random vector with  $N_d(0, I)$  distribution, and under the change of measure given in the question,  $\tilde{W}_1 = W_1 - \alpha$  has a  $N_d(0, I)$  under  $\mathbb{Q}$  as required.

(iv)

(v) (i) Let  $a$  be a trading strategy, and let  $X$  represent the discounted gains from the first period, and  $Z$  be a state price density. Then the theorem gives that if no arbitrage trading strategies exist, then a state price density exists.<sup>1</sup>

<sup>1</sup> What does the numeraire requirement imply?

(ii) Assuming for a linearly independent set of vectors is sufficient, as each condition is invariant when a linearly independent set of random variables  $X$  is replaced by the transformation  $Y = MX$  for arbitrary  $M \in \mathbb{R}^{m \times d}$ .

If there exists a random variable  $Z > 0$  a.s. such that  $\mathbb{E}(Z|X_i|) < \infty$  and  $\mathbb{E}(ZX_i) = 0$ , then  $Y_i = \sum_i \lambda_i X_i$  satisfies these conditions by linearity and the trivial bound.

The precondition is more involved. If  $a \cdot Y = a \cdot (MX) \geq 0$  for some  $a$ , then  $(M^T a) \cdot X \geq 0$ , and so  $M^T a = 0$  a.s., and so  $(M^T a) \cdot X = a \cdot Y = 0$  a.s.

(iii)

(vi) First, some terminology. Consider a  $n$ -state,  $m$ -asset single period market model. Let  $S = (S_{ij}) \in \mathbb{R}^{m \times n}$  be the discounted difference

in value of the  $i$ -th asset in the  $j$ -th state between the initial and first period.

Let  $P = (P_j) \in \mathbb{R}^n$  be the market probability of the  $j$ -th state.

Let  $Y = (Y_j) \in \mathbb{R}^n$  be a candidate state price density for our market model.

Let  $H = (H_i) \in \mathbb{R}^m$  be a candidate arbitrage for our market model.

By the first fundamental theorem of arbitrage pricing, we have exactly one of two alternatives are true.

- (i) There exists a state price density - that is, there exists  $Y' \in \mathbb{R}^n$  with  $Y' > 0$  and  $\mathbb{E}(Y'S) = 0$ .
- (ii) There exists an arbitrage - that is, there exists  $H' \in \mathbb{R}^m$  with  $(S^T H')_i \geq 0$  for all  $1 \leq i \leq n$ , and  $(S^T H')_i > 0$  for at least one  $i$  with  $P_i > 0$ .

Now, let  $B = (B_{ij}) \in \mathbb{R}^{m \times n}$  be defined by

$$B_{ij} = B_{ij} \times P_j \quad (1.11)$$

The condition for existence of a state price density becomes

$$\sum_{j=1}^n Y_j P_j S_{ij} = \sum_{j=1}^n B_{ij} Y_j \quad (1.12)$$

for all  $1 \leq i \leq m$ .

The condition for the existence of an arbitrage becomes

$$\sum_{i=1}^m H_i S_{ij} P_j = \sum_{i=1}^m H_i B_{ij} \quad (1.13)$$

for all  $1 \leq j \leq n$ .

Thus, we can restate the FTAP as

- (i) There exists  $Y \in \mathbb{R}^n$  with  $Y_i > 0$  such that  $BY = 0$ .
- (ii) There exists  $H \in \mathbb{R}^m$  with  $(B^T H)_i \geq 0$  and with  $B^T H \neq 0$ .

which is the required result.

(vii)

(viii)

(ix)

(x)

(xi) Note that  $Z_t = X_t - Y_t$  is a martingale, and thus  $|Z_t|$  is a submartingale (as it is trivially bounded above by two integrable functions, and a convex function of a martingale is a submartingale by Jensen's inequality). Then for any  $0 \leq t \leq T$ , we have

$$0 = \mathbb{E}(|X_T - Y_T| | \mathcal{F}_t) \quad (1.14)$$

$$= \mathbb{E}(|Z_T| | \mathcal{F}_t) \quad (1.15)$$

$$\geq |Z_t| \quad (1.16)$$

$$\geq 0 \quad (1.17)$$

where the first line follows from  $Z_T = 0$  almost surely and the second follows from the submartingale property. Thus the equalities are strict, and we have  $Z_t = 0$  almost surely, and so

$$X_t = Y_t \quad (1.18)$$

almost surely.

(xii) Note that on the sub- $\sigma$ -algebra  $\mathcal{G}$ , the Radon-Nikodym derivative of  $\mathbb{Q}$  with respect to  $\mathbb{P}$  is the conditional expectation  $\mathbb{E}(Z | \mathcal{G})$ . This is because, for arbitrary  $A \in \mathcal{G}$ ,

$$\mathbb{E}(\mathbb{E}(Z | \mathcal{G}) \mathbb{I}(A)) = \mathbb{E}(\mathbb{E}(Z \mathbb{I}(A) | \mathcal{G})) \quad (1.19)$$

$$= \mathbb{E}(Z \mathbb{I}(A)) \quad (1.20)$$

$$= \mathbb{Q}(A) \quad (1.21)$$

as required.

Now, we treat the problem in the question. Let  $A \in \mathcal{G}$ . Let  $Y =$



$\mathbb{E}_{\mathbb{P}}(Z|\mathcal{G})$ . Then

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{I}_A \mathbb{E}_{\mathbb{P}}(ZX|\mathcal{G})) = \mathbb{E}_{\mathbb{P}}(\mathbb{I}_A Y \mathbb{E}_{\mathbb{P}}(ZX|\mathcal{G})) \quad (1.22)$$

$$= \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(Y \mathbb{I}(A) ZX|\mathcal{G})) \quad (1.23)$$

$$= \mathbb{E}_{\mathbb{P}}(Y \mathbb{I}(A) ZX) \quad (1.24)$$

$$= \mathbb{E}_{\mathbb{Q}}(Y \mathbb{I}(A) X) \quad (1.25)$$

$$= \mathbb{E}_{\mathbb{Q}}(\mathbb{I}(A) \mathbb{E}_{\mathbb{Q}}(YX|\mathcal{G})) \quad (1.26)$$

and as  $A$  was arbitrary, we have that

$$\mathbb{E}_{\mathbb{P}}(ZX|\mathcal{G}) = \mathbb{E}_{\mathbb{Q}}(YX|\mathcal{G}) \quad (1.27)$$

$$= Y \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}) \quad (1.28)$$

$$= \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}) \mathbb{E}_{\mathbb{P}}(Z|\mathcal{G}) \quad (1.29)$$

which is sufficient to prove our required result.

(xiii)

(xiv)



## *Bibliography*