Time Series and Monte Carlo Inference

Andrew Tulloch

Contents

CHAPTER 1

Time Series Analysis

1. Introduction

References:

(i) Brockwell and Davis [2009]
(ii) Brockwell and Davis [2002]

Definition 1.1 (Time Series). A set of observations (X_t), each being recorded at a predictable time $t \in T_0$.

In a continuous time series, T_0 is continuous. In a discrete time series, T_0 is discrete.

Definition 1.2 (Time Series Model). Specification of joint distribution (or only means and covariances) of a sequence of random variables of which X_t is a realization.

Remark 1.3. *A complete probability model specifies the joint distribution of all the random variables X_t, $t \in T$.*

This often requires too many estimators, so we only specify the first and second order moments.

Example 1.4. *When X_t is multivariate IID -*

$$\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=1}^{n} F(x_i) \tag{1.1}$$

Example 1.5. *First order moving average model*

Example 1.6. *Trend and seasonal component.*

2. Stationary Processes

Intuitively, a stationary time series is one where the joint distribution is invariant to time shifts.

Definition 1.7 (Mean, Covariance function). Define the mean function $\mu_X(t) = \mathbb{E}(X_t)$.
Define the covariance function $\gamma_X(t,s) = \mathrm{Cov}(X_t, X_s) = \mathbb{E}((X_t - \mu_X(t))(X_s - \mu_X(s)))$.

Definition 1.8 (Weak Stationarity). A time series X_t is stationary if

(i) $\mathbb{E}\big(|X_t|^2\big) < \infty$ for all $t \in \mathbb{Z}$
(ii) $\mathbb{E}(X_t) = c$ for all $t \in \mathbb{Z}$

(iii) $\gamma_X(t,s) = \gamma_X(t+h, s+h))$ for all $t, s, h \in \mathbb{Z}$

Definition 1.9 (Strict Stationarity). A time series X_t is said to be strict stationary if the joint distributions of $X_{t_1,\ldots,X_{t_k}}$ and $X_{t_1+h}, \ldots, X_{t_k+h}$ are identical for all k and for all $t_1, \ldots, t_k, h \in Z$.

Definition 1.10 (Autocovariance function). For a stationary time series X_t, define the autocovariance function

$$\gamma_X(t) = \mathrm{Cov}(X_{t+h}, X_t)\,. \tag{1.2}$$

and the autocorrelation function

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)}. \tag{1.3}$$

Lemma 1.11 (Properties of the autocovariance function).

$$\gamma(0) \geq 0 \tag{1.4}$$

$$|\gamma(h)| \leq \gamma(0) \tag{1.5}$$

$$\gamma(h) = \gamma(-h) \tag{1.6}$$

for all h.

Note that these all hold for the autocorrelation function ρ, with the additional condition that $\rho(0) = 1$.

Theorem 1.12. *A real-valued function defined on the integers is the autocovariance function of a stationary time series if and only if it is even and nonnegative definite.*

Example 1.13. *Consider a white noise, with X_t a time series with X_t uncorrelated with mean zero and variance σ^2.*

Then

$$\gamma_X(h) = \sigma^2 \mathbb{I}(h = 0) \tag{1.7}$$

$$\rho_X(h) = \mathbb{I}(h = 0) \tag{1.8}$$

Example 1.14 (First order moving average MA(1)).

$$X_t = Z_t + \theta Z_{t-1} \tag{1.9}$$

with $Z_t \sim WN(0, \sigma^2)$*. Then*

$$\gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2) & h = 0 \\ \sigma^2\theta & |h| = 1 \\ 0 & otherwise \end{cases} \tag{1.10}$$

$$\rho_X(h) = \begin{cases} 1 & h = 0 \\ \frac{\theta}{1+\theta} & |h| = 1 \\ 0 & otherwise \end{cases} \tag{1.11}$$

Definition 1.15 (Sample Autocovariance). The sample autocovariance function of $\{x_1, \ldots, x_n\}$ is defined by

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{j=1}^{n-h}(x_{j+h} - \bar{x})(x_j - \bar{x}), 0 \leq h < n \tag{1.12}$$

and $\hat{\gamma}(h) = \hat{\gamma}(-h)$, $-n < h \leq 0$.

Note that the divisor is n rather than $n - h$ since this ensures that the sample autocovariance matrix

$$\hat{\Gamma}_n = (\hat{\gamma}(i-j))_{i,j} \tag{1.13}$$

is positive semidefinite.

3. State Space Modesl

Definition 1.16. The observation equation is

$$Y_t = G_t X_t + W_t. \tag{1.14}$$

The state equation is

$$X_{t+1} = F_t X_t + V_t \tag{1.15}$$

$\{Y_t\}$ has a state-space representation if there exists a state-space model for $\{Y_t\}$ as specified by the previous equations.

Theorem 1.17 (De Finitte). *If* $\{X_1, V_1, V_2, \ldots\}$ *are independent, then* $\{X_t\}$ *has the Markov property - that is,* $X_{t+1}|X_t, X_{t-1}, \cdots = X_{t+1}|X_t$.

[1]

In the stable case, there is a unique stationary solution, given by

$$X_t = \sum_{j=0}^{\infty} F^j V_{t-j-1} \tag{1.16}$$

[1]All of Section 8.1 in Introduction to Time Series and Forecasting

DEFINITION 1.18. The state equation is said to be "stable" if the matrix F has all it's eigenvalues in the interior of the unit circle .

4. Stationary Processes

4.1. Linear Processes.

DEFINITION 1.19 (Wold Decomposition). If X_t is a nondeterministic stationary time series, then

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} + V_t \tag{1.17}$$

where

(i) $\psi_0 = 1$ and $\sum_{j=0}^{\infty} \psi_j^2 < \infty$,
(ii) $Z_t \sim WN(0, \sigma^2)$,
(iii) $\mathrm{Cov}(Z_s, V_t) = 0$ for all s, t,
(iv) $Z_t = \tilde{P}_t Z_t$ for all t,
(v) $V_t = \tilde{P}_s V_t$ for all s, t,
(vi) V_t is deterministic.

The sequences Z_t, ψ_j, V_t are unique and can be written explicitly as

$$Z_t = X_t - \tilde{P}_{t-1} X_t \tag{1.18}$$

$$\psi_j = \frac{\mathbb{E}(X_t Z_{t-j})}{\mathbb{E}(Z_t)^2} \tag{1.19}$$

$$V_t = X_t - \sum_{j=0}^{\infty} \psi_j Z_{t-j}. \tag{1.20}$$

DEFINITION 1.20. A times series $\{X_t\}$ is a **linear process** if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \tag{1.21}$$

where $Z_t \sim WN(0, \sigma^2)$ and $\{\psi_j\}$ is a sequence of constants with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$.

A linear process is called a **moving average** or MA(∞) if $\psi_j = 0$ for all $j < 0$, so

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}. \tag{1.22}$$

PROPOSITION 1.21. *Let Y_t be a stationary time series with mean zero and coavariance function γ_Y. If $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, then the time series*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j} = \psi(B) Y_t \tag{1.23}$$

is stationary with mean zero and autocovariance function

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h+k-j). \tag{1.24}$$

In the special case where X_t is a linear process,

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2. \tag{1.25}$$

4.2. Forecasting Stationary Time Series. Our goal is to find the linear combination of $1, X_n, X_{n-1}, \ldots, X_1$ that forecasts X_{n+h} with minimum mean squared error. The best linear predictor in terms of $1, X_n, \ldots, X_1$ will be deonted by $P_n X_{n+h}$ and clearly has the form

$$P_n X_{n+h} = a_0 + a_1 X_n + \cdots + a_n X_1. \tag{1.26}$$

To find these equations, we solve the convex problem by setting derivatives to zero, and obtain the result given below.

Theorem 1.22 (Properties of h-step best linear predictor $P_n X_{n+h}$). *(i)*

$$P_n X_{n+h} = \mu + \sum_{i=1}^{n} a_i (X_{n+1-i} - \mu) \tag{1.27}$$

where $\mathbf{a}_n = (a_1, \ldots, a_n)$ satisfies

$$\Gamma_n \mathbf{a}_n = \gamma_n(h) \tag{1.28}$$

$$\Gamma_n = [\gamma(i-j)]_{i,j=1}^{n} \tag{1.29}$$

$$\gamma_n(h) = (\gamma(h), \gamma(h+1), \ldots, \gamma(h+n-1)) \tag{1.30}$$

(ii)

$$\mathbb{E}\big((X_{n+h} - P_n X_{n+h})^2\big) = \gamma(0) - \langle \mathbf{a}_n, \gamma_n(h) \rangle \tag{1.31}$$

(iii)

$$\mathbb{E}(X_{n+h} - P_n X_{n+h}) = 0 \tag{1.32}$$

(iv)

$$\mathbb{E}((X_{n+h} - P_n X_{n+h}) X_j) = 0 \tag{1.33}$$

for $j = 1, \ldots, n$.

Definition 1.23 (Prediction Operator $P(\cdot|\mathbf{W})$). Suppose that $\mathbb{E}(U^2) < \infty$, $\mathbb{E}(V^2) < \infty$, $\Gamma = \mathrm{Cov}(\mathbf{W}, \mathbf{W})$, and $\beta, \alpha_1, \ldots, \alpha_n$ are constants.

(i)

$$P(U|\mathbf{W}) = \mathbb{E}(U) = \mathbf{a}'(\mathbf{W} - \mathbb{E}(\mathbf{W})) \tag{1.34}$$

where $\Gamma \mathbf{a} = \mathrm{Cov}(U, \mathbf{W})$.

(ii)

$$\mathbb{E}((U - P(U|\mathbf{W}))\mathbf{W}) = 0 \tag{1.35}$$

and

$$\mathbb{E}(U - P(U|\mathbf{W})) = 0 \tag{1.36}$$

(iii)

$$\mathbb{E}\big((U - P(U|\mathbf{W}))^2\big) = \mathbb{V}(U) - \mathbf{a}'\mathrm{Cov}(U, \mathbf{W}) \tag{1.37}$$

(iv)

$$P\alpha_1 + \alpha_2 V + \beta|\mathbf{W} = \alpha_1 P(U|\mathbf{W}) + \alpha_2 P(V|\mathbf{W}) + \beta \tag{1.38}$$

(v)

$$P(\sum_{i=1}^{n} \alpha_i W_i + \beta|\mathbf{W}) = \sum_{i=1}^{n} \alpha_i W_i + \beta \tag{1.39}$$

(vi)

$$P(U|\mathbf{W}) = EU \tag{1.40}$$

if $\mathrm{Cov}(U, \mathbf{W}) = 0$.

4.3. Innovation Algorithm.

THEOREM 1.24. *Suppose X_t is a zero-mean series with $\mathbb{E}\big(|X_t|^2\big) < \infty$ for each t and $\mathbb{E}(X_i X_j) = \kappa(i, j)$. Let $\hat{X}_n = 0$ if $n = 1$, and $P_{n-1}X_n$ if $n = 2, 3, \dots,$ and let $v_n = \mathbb{E}\big((X_{n+1} - P_n X_{n+1})^2\big)$. Define the innovations, or one-step prediction errors, as $U_n = X_n - \hat{X}_n$. Then we can write*

$$\hat{X}_{n+1} = \begin{cases} 0 & n = 0 \\ \sum_{j=1}^{n} \theta_{nj}(X_{n+1-j} - \hat{X}_{n+1-j}) & \end{cases} \tag{1.41}$$

where the coefficients $\theta_{n1}, \dots, \theta nn$ can be computed recursively from the equations

$$v_0 = \kappa(1, 1) \tag{1.42}$$

$$\theta_{n,n-k} = \frac{1}{v_k}(\kappa(n+1, k+1) - \sum_{j=0}^{k-1} \theta_{k,k-j}\theta_{n,n-j}v_j) \tag{1.43}$$

for $0 \leq k < n$, and

$$v_n = \kappa(n+1, n+1) - \sum_{j=0}^{n-1} \theta_{n,n-j}^2 v_j. \tag{1.44}$$

5. ARMA Processes

DEFINITION 1.25. X_t is an ARMA(p, q) process if X_t is stationary and if for every t,

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} \tag{1.45}$$

where $Z_t \sim WN(0, \sigma^2)$ and the polynomials $(1 - \phi_1 z - \cdots - \phi_p z^p)$ and $(1 + \theta_1 z + \cdots + \theta_q z^q)$ have no common factors.

It can be more convenient to write this in the form

$$\phi(B)X_t = \theta(B)Z_t \tag{1.46}$$

with B the back-shift operator.

ARMA$(0, q)$ is a moving average process of order q (MA(q)). ARMA$(p, 0)$ is an autoregressive process of order p (AR(p)).

THEOREM 1.26. *A stationary solution of* (1.45) *exists (and is the unique stationary solution) if and only if*

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0 \tag{1.47}$$

for all $|z| = 1$

DEFINITION 1.27. An ARMA(p, q) process X_t is causal (or a causal function of Z_t) if there exists constants ψ_j such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \tag{1.48}$$

for all t.

THEOREM 1.28. *An ARMA(p, q) process is causal if and only if*

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0 \tag{1.49}$$

for all $|z| \leq 1$.

Note that the coefficients ψ_j *are determined by*

$$\psi_j - \sum_{k=1}^{p} \theta_k \psi_{j-k} = \theta_j \tag{1.50}$$

for $j = 0, 1, \ldots.$ *and* $\theta_0 = 1$, $\theta_j = 0$ *for* $j > q$, *and* $\psi_j = 0$ *for* $j < 0$.

DEFINITION 1.29. An ARMA(p, q) is invertible if there exist constants π_j such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} \tag{1.51}$$

for all t.

The coefficients π_j are determined by the equations

$$\pi_j + \sum_{k=1}^{q} \theta_k \pi_{j-k} = -\phi_j \tag{1.52}$$

where $\phi_0 = -1$, $\theta_j = 0$ for $j > p$, and $\pi_j = 0$ for $j < 0$.

THEOREM 1.30. *Invertibility is equivalent to the condition*

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0 \tag{1.53}$$

for all $|z| \leq 1$.

5.1. ACF and PACF of an ARMA(p, q) Process.

THEOREM 1.31. *For a causal ARMA(p,q) process defined by*

$$\phi(B)X_t = \theta(B)Z_t \tag{1.54}$$

we know we can write

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \tag{1.55}$$

where $\sum_{j=0}^{\infty} \psi_j z^j = \theta(z)/\phi(z)$ *for* $|z| \leq 1$.

Thus, the ACVF γ *is given as*

$$\gamma(h) = \mathbb{E}(X_{t+h}X_t) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|} \tag{1.56}$$

A second approach is to multiple each side by X_{t_k} *and take expectations, and obtain a sequence of* m *homogenous linear difference equations with constant coefficients. These can be solved to obtain the* $\gamma(h)$ *values.*

DEFINITION 1.32 (PACF). The partial autocorrelation function (PACF) of an AMRA process X is the function $\alpha(\cdot)$ defined by

$$\alpha(0) = 1 \tag{1.57}$$

$$\alpha(h) = \phi_{hh}, h \geq 1 \tag{1.58}$$

where ϕ_{hh} is the last component of $\phi_h = \Gamma_h^{-1}\gamma_h$, where $\Gamma_h = [\gamma(i-j)]_{i,j=1}^{h}$, and $\gamma_h = [\gamma(1), \gamma(2), \ldots, \gamma(h)]$.

THEOREM 1.33. *For an AR(p) process, the sample PACF values at lags greater than p are approximately independent* $N(0, \frac{1}{n})$ *random variables. Thus, if we have a sample PACF satisfying*

$$|\hat{\alpha}(h)| > \frac{1.96}{\sqrt{n}} \tag{1.59}$$

for $0 \leq h \leq p$ *and*

$$|\hat{\alpha}(h)| < \frac{1.96}{\sqrt{n}} \tag{1.60}$$

for $h > p$, *this suggests an AR(p) model for the data.*

THEOREM 1.34 (PACF summary). *For an AR(p) process* X_t, *the PACF* $\alpha(\cdot)$ *has the properties that* $\alpha(p) = \phi_p$, *and* $\alpha(h) = 0$ *for* $h > p$. *For* $h < p$ *we can compute numerically from the expression that* $\phi_h = \Gamma_h^{-1}\gamma_h$.

5.2. Forecasting ARMA Processes. For the causal ARMA(p, q) process

$$\phi(B)X_t = \theta(B)Z_t, Z_t \sim WN(0, \sigma^2) \tag{1.61}$$

we can avoid using the full innovations algorithm.

If we apply the algorithm to the transformed process W_t given by

$$W_t = \begin{cases} \frac{1}{\sigma}X_t & t = 1, \dots, m \\ \frac{1}{\sigma}\phi(B)X_t & t > m \end{cases} \tag{1.62}$$

where $m = \max(p, q)$.

For notational convenience, take $\theta_0 = 1$, $\theta_j = 0$ for $j > q$.

LEMMA 1.35. *The autocovariances $\kappa(i, j) = \mathbb{E}(W_i W_j)$ are found from*

$$\kappa(i,j) = \begin{cases} \sigma^2\gamma_X(i-j) & 1 \le i, j \le m \\ \sigma^2(\gamma_X(i-j) - \sum_{r=1}^{p}\phi_r\gamma_X(r-|i-j|)) & \min(i,j) \le m < \max(i,j) \le 2m \\ \sum_{r=0}^{q}\theta_r\theta_{r+|i-j|} & \min(i,j) > m \\ 0 & \textit{otherwise} \end{cases} \tag{1.63}$$

Applying the innovations algorithm to the process W_t, we obtain

$$\hat{W}_{n+1} = \begin{cases} \sum_{j=1}^{n}\theta_{nj}(W_{n+1-j} - \hat{W}_{n+1-j}) & 1 \le n < m \\ \sum_{j=1}^{q}\theta_{nj}(W_{n+1-j} - \hat{W}_{n+1-j}) & n \ge m \end{cases} \tag{1.64}$$

where the coefficients θ_{nj} and MSE $r_n = \mathbb{E}\Big((W_{n+1} - \hat{W}_{n+1})^2\Big)$ are found recursively using the innovations algorithm.

Since the equations (1.62) allow us to write X_n as a linear combination of $W_j, 1 \le j \le n$, and conversely, each $W_n, n \ge 1$ to be written as a linear combination of $X_j, 1 \le j \le n$. Thus the best linear predictor of the random variable Y in terms of $\{1, X_1, \dots, X_n\}$ is the same as the best linear predictor of Y in terms of $\{1, W_1, \dots, W_n$. Thus, by linearity of $\hat{P}_n$, we have

$$\hat{W}_t = \begin{cases} \frac{1}{\sigma}\hat{X}_t & t = 1, \dots, m \\ \frac{1}{\sigma}(\hat{X}_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p}) & t > m \end{cases} \tag{1.65}$$

which shows that

$$X_t - \hat{X}_t = \sigma(W_t - \hat{W}_t) \tag{1.66}$$

Substituting into (1.63) and (1.64), we obtain

$$\hat{X}_{n+1} = \begin{cases} \sum_{j=1}^{n}\theta_{nj}(X_{n+1-j} - \hat{X}_{n+1-j}) & 1 \le n < m \\ \phi_1 X_n + \cdots + \phi_p X_{n+1-p} + \sum_{j=1}^{q}\theta_{nj}(X_{n+1-j} - \hat{X}_{n+1-j}) & n \ge m \end{cases} \tag{1.67}$$

and

$$\mathbb{E}\Big((X_{n+1}-\hat{X}_{n+1})^2\Big)=\sigma^2\mathbb{E}\Big((W_{n+1}-\hat{W}_{n+1})^2\Big)=\sigma^2 r_n \tag{1.68}$$

where θ_{nj} and r_n are found using the innovation algorithm.

6. Estimation of ARMA Processes

6.1. Yule-Walker Equations. Consider estimating a causal AR(p) process. We can write

$$X_t=\sum_{j=0}^{\infty}\psi_j Z_{t-j} \tag{1.69}$$

where $\sum_{j=0}^{\infty}\psi_j z^j=\frac{1}{\phi(z)}$ for $z\leq 1$.

Multiplying each side by Z_{t-j}, and taking expectations, we obtain the Yule-Walker equations

$$\Gamma_p\phi=\gamma_p \tag{1.70}$$

and $\sigma^2=\gamma(0)-\langle\phi,\gamma_p\rangle$ where $\Gamma_p=[\gamma(i-j)]_{i,j=1}^{p}$ and $\gamma_p=(\gamma(1),\gamma(2),\ldots,\gamma(p))$.

If we replace the covariances by the sample covariances $\hat{\gamma}(j)$, we obtain a set of equations for the so-called Yule-Walker estimators $\hat{\phi}$ and $\hat{\sigma}^2$, given by

$$\hat{\Gamma}_p\hat{\phi}=\hat{\gamma}_p \tag{1.71}$$

and $\hat{\sigma}^2=\hat{\gamma}(0)-\left\langle\hat{\phi},\hat{\gamma}_p\right\rangle$

Theorem 1.36. *If X_t is the causal AR(p) process and $\hat{\phi}$ is the Yule-Walker estimator of ϕ, then*

$$n^{\frac{1}{2}}(\hat{\phi}-\phi)\xrightarrow{d}N(0,\sigma^2\Gamma_p^{-1}) \tag{1.72}$$

Moreover, $\hat{\sigma}^2\xrightarrow{p}\sigma^2$.

Theorem 1.37. *If X_t is a causal AR(p) process and $\hat{\phi}_m$ is the Yule-Walker estimate of order $m>p$, then*

$$n^{\frac{1}{2}}(\hat{\phi}_m-\phi_m)\xrightarrow{d}N(0,\sigma^2\Gamma_m^{-1}) \tag{1.73}$$

where $\hat{\phi}_m$ is the coefficient vector of the best linear predictor $\langle\phi_m,\mathbf{X}_m\rangle$ of X_{m+1} based on $X_m,\ldots,X_1$. So $\phi_m=R_m^{-1}\rho_m$. In particular, for $m>p$,

$$n^{\frac{1}{2}}\hat{\phi}_{mm}\xrightarrow{d}N(0,1) \tag{1.74}$$

Theorem 1.38 (Durbin-Levinson Algorithm for AR models). *Consider fitting an AR(m) process*

$$X_t-\hat{\theta}_{m1}X_{t-1}-\cdots-\hat{\theta}_{mm}X_{t-m}=Z_t \tag{1.75}$$

with $Z_t\sim WN(0,\hat{v}_m)$.

If $\hat{\gamma}(0) > 0$, then the fitted autoregressive models for $m = 1, 2, \ldots, n-1$ can be determined recursively from the relations

$$\hat{\phi}_{11} = \hat{\rho}(1) \tag{1.76}$$

$$\hat{v}_1 = \hat{\gamma}(0)(1 - \hat{\rho}^2)(1) \tag{1.77}$$

$$\hat{\phi}_{mm} = \frac{\hat{\gamma}(m) - \sum_{j=1}^{m-1} \hat{\phi}_{m-1,j}\hat{\gamma}(m-j)}{\hat{v}_{m-1}} \tag{1.78}$$

$$\begin{Bmatrix} \hat{\phi}_{m1} \\ \vdots \\ \hat{\phi}_{m,m-1} \end{Bmatrix} = \hat{\phi}_{m-1} - \hat{\phi}_{mm} \begin{Bmatrix} \hat{\phi}_{m-1,m-1} \\ \vdots \\ \hat{\phi}_{m-1,1} \end{Bmatrix} \tag{1.79}$$

$$\hat{v}_m = \hat{v}_{m-1}(1 - \hat{\phi}_{mm}^2) \tag{1.80}$$

Theorem 1.39 (Confidence intervals for AR(p) estimation). *Under the assumption that the order p of the fitted model is the correct value, for large sample-size n, the region*

$$\{\phi \in \mathbb{R}^p | (\phi - \hat{\phi}_p)' \hat{\Gamma}_p (\phi - \hat{\phi}_p) \leq \frac{1}{n} \hat{v}_p \chi^2_{1-\alpha}(p)\} \tag{1.81}$$

contains ϕ_p with probability close to $1-\alpha$ where $\chi^2_{1-\alpha}(p)$ is the $(1-\alpha)$ quantile of the chi-squared distribution with p degrees of freedom.

Similarly, if $\Phi_{1-\alpha}$ is the $(1-\alpha)$ quantile of the standard normal distribution and $\hat{v}_{jj}$ is the j-th diagonal element of $\hat{v}_p \hat{\Gamma}_p^{-1}$, then for large n

$$\{\hat{\phi}_{pj} \pm \Phi_{1-\frac{\alpha}{2}} \frac{1}{n^{\frac{1}{2}}} \hat{v}_{jj}^{\frac{1}{2}}\} \tag{1.82}$$

contains ϕ_{pj} with probability close to $(1-\alpha)$.

6.2. Estimation for Moving Average Processes Using the Innovations Algorithm. Consider estimating

$$X_t = Z_t + \hat{\theta}_{m1} Z_{t-1} + \cdots + \hat{\theta}_{mm} Z_{t-m} \tag{1.83}$$

with $Z_t \sim WN(0, \hat{v}_m)$.

Theorem 1.40. *We can apply the innovation estimates by applying the recursive relations*

$$\hat{v}_0 = \hat{\gamma}(0) \tag{1.84}$$

$$\hat{\theta}_{m,m-k} = \frac{1}{\hat{v}_k}(\hat{\gamma}(m-k) - \sum_{j=0}^{k-1} \hat{\theta}_{m,m-j}\hat{\theta}_{k,k-j}\hat{v}_j) \tag{1.85}$$

for $k = 0, \ldots, m-1$, and

$$\hat{v}_m = \hat{\gamma}(0) - \sum_{j=0}^{m-1} \hat{\theta}^2_{m,m-j}\hat{v}_j. \tag{1.86}$$

THEOREM 1.41. *Let X_t be the causal invertible ARMA process $\phi(B)X_t = \theta(B)Z_t$ with $Z_t \sim WN(0,\sigma^2)$, $\mathbb{E}(Z_t^4) < \infty$, and let $\psi(z) = \sum_{j=0}^{\infty}\psi_j z^j = \frac{\theta(z)}{\phi(z)}$ for $|z| \leq 1$, and $\psi_0 = 1$ and $\psi_j = 0$ for $j < 0$.*

Then for any sequence of positive integers m_n, such that $m < n$, $m \to \infty$, and $m = o(n^{\frac{1}{3}})$ as $n \to \infty$, we have for each k,

$$\frac{n^{\frac{1}{2}}}{(}\hat{\theta}_{m1} - \psi_1, \ldots, \hat{\theta}_{mk} - \psi_k) \xrightarrow{d} N(0, A) \tag{1.87}$$

where $A = [a_{ij}]_{i,j=1}^{k}$ and

$$a_{ij} = \sum_{r=1}^{\min(i,j)} \psi_{i-r}\psi_{j-r} \tag{1.88}$$

and

$$\hat{v}_m \xrightarrow{p} \sigma^2. \tag{1.89}$$

REMARK 1.42. *Note that for the AR(p) process, the Yule-Walker estimator is a consistent estimator of ϕ_p. However, for an MA(q) process, the estimator $\hat{\theta}_q$ is not consistent for the true parameter vector as $n \to \infty$. For consistency, it is necessary to use the estimators with m satisfying the conditions given in Theorem 1.41.*

THEOREM 1.43 (Asymptotic confidence regions for the θ_q).

$$\{\theta \in R | |\theta - \hat{\theta}_{mj}| \leq \Phi_{1-\frac{\alpha}{2}} \frac{1}{n^{\frac{1}{2}}} (\sum_{k=0}^{j-1} \hat{\theta}_{mk}^2)^{\frac{1}{2}}\} \tag{1.90}$$

is an $(1-\alpha)$ confidence interval for θ_{mj}.

6.3. Maximum Likelihood Estimation. Consider X_t a gaussian time series with zero mean and autocovariance function $\kappa(i,j) = \mathbb{E}(X_i X_j)$. Let $\hat{X}_j = P_{j-1}X_j$. Let Γ_n be the covariance matrix and assume Γ_n is nonsingular. The likelihood of X_n is

$$L(\Gamma_n) = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{(\det \Gamma_n)^{\frac{1}{2}}} \exp(-\frac{1}{2}\mathbf{X}_n' \Gamma_n^{-1} \mathbf{X}_n) \tag{1.91}$$

THEOREM 1.44. *The likelihood of the vector $\mathbf{X}_n$ reduces to*

$$L(\Gamma_n) = \frac{1}{\sqrt{(2\pi)^n \prod_{i=0}^{n-1} r_i}} \exp(-\frac{1}{2}\sum_{j=1}^{n} \frac{(X_j - \hat{X}_j)^2}{r_{j-1}}) \tag{1.92}$$

REMARK 1.45. *Even if X_t is not Gaussian, the large sample estimates are the same for $Z_t \sim IID(0,\sigma^2)$, regardless of whether or not Z_t is Gaussian.*

THEOREM 1.46 (Maximum Likelihood Estimators for ARMA processes).

$$\hat{\sigma}^2 = \frac{1}{n} S(\hat{\phi}, \hat{\theta}) \tag{1.93}$$

where $\hat{\phi}, \hat{\theta}$ are the values of ϕ, θ that minimize

$$\ell(\phi, \theta) = \ln(\frac{1}{n} S(\theta, \theta)) + \frac{1}{n} \sum_{j=0}^{n-1} \ln r_j \tag{1.94}$$

and

$$S(\hat{\phi}, \hat{\theta}) = \sum_{j=1}^{n} \frac{(X_j - \hat{X}_j)^2}{r_{j-1}} \tag{1.95}$$

THEOREM 1.47 (Asyptotic Distribution of Maximum Likelihood Estimators). *For a large sample from an ARMA(p, q) process,*

$$\hat{\beta} = N(\beta, \frac{1}{n} V\beta) \tag{1.96}$$

where

$$V(\beta) = \sigma^2 \begin{bmatrix} \mathbb{E}(U_t U_t') & \mathbb{E}(U_t V_t') \\ \mathbb{E}(V_t U_t') & \mathbb{E}(V_t V_t') \end{bmatrix}^{-1} \tag{1.97}$$

and U_t are the autoregressive process $\phi(B)U_t = Z_t$ and $\theta(B)V_t = Z_t$.

Note that for $p = 0$, $V(\beta) = \sigma^2[\mathbb{E}(V_t V_t')]^{-1}$, and for $q = 0$, $V(\beta) = \sigma^2[\mathbb{E}(U_t U_t')]^{-1}$.

6.4. Order Selection.

DEFINITION 1.48 (Kullback-Leibler divergence). The Kullback-Leibler (KL) divergence between $f(\cdot; \psi)$ and $f(\cdot; \theta)$ is defined as

$$d(\psi|\theta) = \Delta(\psi|\theta) - \Delta(\theta|\theta) \tag{1.98}$$

where

$$\Delta(\psi|\theta) = \mathbb{E}_\theta(-2 \ln f(X; \psi)) \tag{1.99}$$

is the Kullback-Leibler index of $f(\cdot; \psi)$ relative to $f(\cdot; \theta)$.

THEOREM 1.49 (AICC of ARMA(p, q) process).

$$AICC(\beta) = -2 \ln L_X(\beta, \frac{S_X(\beta)}{n}) + \frac{2(p+q+1)n}{n-p-q-2} \tag{1.100}$$

THEOREM 1.50 (AIC of ARMA(p, q) process).

$$AIC(\beta) = -2 \ln L_X(\beta, \frac{S_X(\beta)}{n}) + 2(p+q+1) \tag{1.101}$$

THEOREM 1.51 (BIC of ARMA(p, q) process).

$$BIC(\beta) = (n-p-q) \ln \frac{n\hat{\sigma}^2}{n-p-q} + n(1 + \ln \sqrt{2\pi}) + (p+q) \ln \frac{\sum_{t=1}^{n} X_t^2 - n\hat{\sigma}^2}{p+q} \tag{1.102}$$

where $\hat{\sigma}^2$ is the MLE estimate of the white noise variance.

7. Spectral Analysis

Let X_t be a zero-mean stationary time series with autocovariance function $\gamma(\cdot)$ satisfying $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$.

Definition 1.52. The spectral density of X_t is the function $f(\cdot)$ defined by

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} y(h) \tag{1.103}$$

The summability implies that the series converges absolutely.

Theorem 1.53. *(i) f is even*
(ii) $f(\lambda) \geq 0$ for all $\lambda \in (-\pi, \pi]$.
(iii) $\gamma(k) = \int_{-\pi}^{\pi} e^{-k\lambda} f(\lambda) d\lambda = \int_{-\pi}^{\pi} \cos(k\lambda) f(\lambda) d\lambda$.

Definition 1.54. A function f is the **spectral density** of a stationary time series X_t with autocovariance function $\gamma(\cdot)$ if

(i) $f(\lambda) \geq 0$ for all $\lambda \in (0, \pi]$,
(ii) $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda$ for all integers h.

Lemma 1.55. *If f and g are two spectral density corresponding to the autocovariance function γ, then f and g have the same Fourier coefficients and hence are equal.*

Theorem 1.56. *A real-valued function f defined on $(-\pi, \pi]$ is the spectral density of a stationary process if and only if*

(i) $f(\lambda) = f(-\lambda)$,
(ii) $f(\lambda) \geq 0$
(iii) $\int_{-\pi}^{\pi} f(\lambda) d\lambda < \infty$.

Theorem 1.57. *An absolutely summable function $\gamma(\cdot)$ is the autocovariance function of a stationary time series if and only if it is even and*

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) \geq 0 \tag{1.104}$$

for all $\lambda \in (-\pi, \pi]$, in which case $f(\cdot)$ is the spectral density of $\gamma(\cdot)$.

Theorem 1.58 (Spectral Representation of the ACVF). *A function $\gamma(\cdot)$ defined on the integers is the ACVF of a stationary time series if and only if there exists a right-continuous, nondecreasing, bounded function F on $[-\pi, \pi]$ with $F(-\pi) = 0$ such that*

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} dF(\lambda) \tag{1.105}$$

for all integers h.

REMARK 1.59. *The function F is a* ***generalized distribution function*** *on $[-\pi,\pi]$ in the sense that $G(\lambda)=\frac{F(\lambda)}{F(\pi)}$ is a probability distribution function on $[-\pi,\pi]$. Note that since $F(\pi)=\gamma(0)=\mathbb{V}(X_1)$, the ACF of X_t has the spectral representation function*

$$\rho(h)=\int_{-\pi}^{\pi} e^{ih\lambda}dG(\lambda) \tag{1.106}$$

The function F is called the spectral distribution function of $\gamma(\cdot)$. If $F(\lambda)$ can be expressed as $F(\lambda)=\int_{-\pi}^{\lambda} f(y)dy$ for all $\lambda\in[-\pi,\pi]$, then f is the spectral density function and the time series is said to have a continuous spectrum. If F is a discrete function, then the time series is said to have a discrete spectrum.

THEOREM 1.60. *A complex valued function $\gamma(\cdot)$ is the autocovariance function of a stationary process X_t if and only if either*

- *(i)* $\gamma(h)=\int_{-\pi}^{\pi} e^{-ihv}dF(v)$ *for all $h=0,\pm1,\dots$ where F is a right-continuous, non-decreasing, bounded function on $[-\pi,\pi]$ with $F(-\pi)=0$, or*
- *(ii)* $\sum_{i,j=1}^{n} a_i\gamma(i-j)\overline{a}_j\geq 0$ *for all positive integers n and all $a=(a_1,\dots,a_n\in\mathbb{C}^n)$.*

7.1. The Spectral Density of an ARMA Process.

THEOREM 1.61. *If Y_t is any zero-mean, possibly complex-valued stationary process with spectral distribution function $F_Y(\cdot)$ and X_t is the process $X_t=\sum_{j=-\infty}^{\infty}\psi_jY_{t-j}$ where $\sum_{j=-\infty}^{\infty}|\psi_j|<\infty$, then X_t is stationary with spectral distribution function $F_X(\lambda)=\int_{-\pi,\lambda}|\sum_{j=-\infty}^{\infty}\psi_je^{-ijv}|^2dF_Y(v)$ for $-\pi\leq\lambda\leq\pi$.*

If Y_t has a spectral density $f_Y(\cdot)$, then X_t has a spectral density $f_X(\cdot)$ given by $f_X(\lambda)=|\Psi(e^{-i\lambda})|^2f_Y(\lambda)$ where $\Psi(e^{-i\lambda})=\sum_{j=-\infty}^{\infty}\psi_je^{-ij\lambda}$.

THEOREM 1.62. *Let X_t be an ARMA(p,q) process, not necessarily causal or invertible satisfying $\phi(B)X_t=\theta(B)Z_t$, $Z_t\sim WN(0,\sigma^2)$ where $\phi(z)=1-\phi_1z-\cdots-\phi_pz^p$ and $\theta(z)=1+\theta_1z+\cdots+\theta_qz^q$ have no common zeroes and $\phi(z)$ has no zeroes on the unit circle. Then X_t has spectral density*

$$f_X(\lambda)=\frac{\sigma^2}{2\pi}\frac{|\theta(e^{-i\lambda})|^2}{||\phi(e^{-i\lambda})|^2} \tag{1.107}$$

for $-\pi\leq\lambda\leq\pi$.

THEOREM 1.63. *The spectral density of the white noise process is constant, $f(\lambda)=\frac{\sigma^2}{2\pi}$.*

7.2. The Periodogram.

DEFINITION 1.64. The periodogram of $(x_1,\dots,x_n)$ is the function

$$I_n(\lambda)=\frac{1}{n}|\sum_{t=1}^{n}x_te^{-it\lambda}|^2 \tag{1.108}$$

THEOREM 1.65. *If $x_1, \dots, x_n$ are any real numbers and ω_k is any of the nonzero Fourier Frequencies $\frac{2\pi k}{n}$ in $(-\pi, \pi]$, then $I_n(\omega_k) = \sum_{|h|<n} \hat{\gamma}(h) e^{-ih\omega_k}$ where $\hat{\gamma}(h)$ is the sample ACVF of $x_1, \dots, x_n$.*

THEOREM 1.66. *Let X_t be the linear process $X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$, $Z_t \sim IID(0, \sigma^2)$, with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$. Let $I_n(\lambda)$ be the periodogram of $X_1, \dots, X_n$, and let $f(\lambda)$ be the spectral density of X_t.*

(i) If $f(\lambda) > 0$for all $\lambda \in [-\pi, \pi]$ and if $0 < \lambda_1 < \dots < \lambda_m < \pi$, then the random vector $(I_n(\lambda_1), \dots, I_n(\lambda_m))$ converges in distribution to a vector of independent and exponentially distributed random variables, the i-th component which has mean $2\pi f(\lambda_i)$, $i = 1 \dots, m$.

(ii) If $\sum_{j=-\infty}^{\infty} |\psi_j||j|^{\frac{1}{2}} < \infty$, $\mathbb{E}(Z_1^4) = \nu\sigma^4 < \infty$, $\omega_j = \frac{2\pi j}{n} \geq 0$, and $\omega_k = \frac{2\pi k}{n} \geq 0$, then

$$Cov(I_n(\omega_j), I_n(\omega_k), -NoValue-) = \begin{cases} 2(2\pi)^2 f^2(\omega_j) + O(n^{-\frac{1}{2}}) & \omega_j = \omega_k = \{0, \pi\} \\ (2\pi)^2 f^2(\omega_j) + O(n^{-\frac{1}{2}}) & 0 < \omega_j = \omega_k < \pi \\ O(n^{-1}) & \omega_j \neq \omega_k \end{cases} \tag{1.109}$$

DEFINITION 1.67. The estimator $\hat{f}(\omega) = \hat{f}(g(n, \omega))$ with $\hat{f}(\omega_j)$ defined by $\frac{1}{2\pi} \sum_{|k| \leq m_n} W_n(k) I_n(w_{j+k})$ with $m \to \infty$, $\frac{m}{n} \to 0$, $W_n(k) = W_n(-k)$, $W_n(k) \geq 0$ for all k, and $\sum_{|k| \leq m} W_n(k) = 1$, and $\sum_{|k|} W_n^2(k) \to 0$ as $n \to \infty$ is called a **discrete spectral average estimator** of $f(w)$.

THEOREM 1.68. *Let X_t be the linear process $X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$, $Z_t \sim IID(0, \sigma^2)$, with $\sum_{j=-\infty}^{\infty} |\psi_j||j|^{\frac{1}{2}} < \infty$ and $\mathbb{E}(Z_1^4) < \infty$. If $\hat{f}$is a discrete spectral average estimator of the spectral density f, then for $\lambda, \omega \in [0, \pi]$,*

(i) $\lim_{n\to\infty} \mathbb{E}\left(\hat{f}(\omega)\right) = f(\omega)$

(ii)

$$\lim_{n\to\infty} \frac{1}{\sum_{|j| \leq m} W_n^2(j)} Cov\left(\hat{f}(\omega), \hat{f}(\lambda), -NoValue-\right) = \begin{cases} 2f^2(\omega) & w = \lambda = \{0, \pi\} \\ f^2(\omega) & 0 < \omega = \lambda < \pi \\ \\ 0 & \omega \neq \lambda. \end{cases} \tag{1.110}$$

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