Advanced Financial Models

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Contents

Chapter 1. Discrete Time Models 4
1. Standing Assumptions 4
2. Setup 4
3. A Detour into Martingales 5
4. Contingent Claims 11
5. American Claims 14

Chapter 2. Continuous Time Models 17
1. Diversion into Stochastic Calculus 17
2. Itô's Formula 19
3. Arbitrage Theory in Continuous Time 22

Chapter 3. Black-Scholes 28
1. Black-Scholes Volatility 29
2. Calibration 29
3. Robustness 29

Chapter 4. Local Volatility Models 31
1. Computing Moment Generating Functions 32
2. The Heston Model 34
3. American Options (Guest Lecture) 35

Chapter 5. Bond Markets and Interest Rates 38
1. The Heath et al. [1992] Model 39

Bibliography 42

CHAPTER 1

Discrete Time Models

1. Standing Assumptions

(i) Zero dividends
(ii) Zero tick size
(iii) Zero transaction costs
(iv) Infinitely divisible transactions
(v) No short-selling constraints
(vi) No bid-ask spread
(vii) No market impact (infinitely deep market)

2. Setup

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.1. A random variable is a measurable map $X : \Omega \to \mathbb{R}$

Definition 1.2. A stochastic process $Y = (Y_t)_{t \in I}$ is a collection of random variables. For us, $I = \{0, 1, \dots\}$ or $[0, \infty)$.

Definition 1.3. A filtration $\mathbb{F} = (\mathcal{F})_{t \geq 0}$ is a collection of sub-σ-algebras on $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $0 \leq s \leq t$ (discrete and continuous time).

Example 1.4. *Tossing coins.*

(i) $\Omega = \{HH, HT, TH, TT\}$
(ii) $\mathcal{F}$ *is all 16 subsets of* Ω
(iii) $\mathbb{P}(A) = \frac{|A|}{4}$

Possible filtration

(i) $\mathcal{F}_0 = \{\emptyset, \Omega\}$
(ii) $\mathcal{F}_1 = \{\emptyset, \Omega, \{HH, HT\}, \{TH, TT\}\}$
(iii) $\mathcal{F}_2 = \mathcal{F}$

Definition 1.5. A process Y is adapted if and only if Y_t is $\mathcal{F}_t$-measurable.

Throughout the course, $\mathcal{F}_0$ is assumed trivial.

DEFINITION 1.6. Given a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ in discrete time, a process $X = (X_t)_{t\geq 1}$ is predictable if and only if X_t is $\mathcal{F}_{t-1}$-measurable.

Sometimes we need X_0 to be defined, so we just ask that X_0 is $\mathcal{F}_0$-measurable.

DEFINITION 1.7. Given $P = (P_t)_{t\geq 0}$ prices process in discrete time. An investment/consumption strategy is a predictable process (H, c) where H_t takes values in R^n and $c_t \geq 0$ and satisfies the **self-financing condition**

$$H_{t-1} - P_{t-1} = H_t \cdot P_t + c_t \tag{1.1}$$

for all $t \geq 1$.

H_t models the portfolio during $(t-1, t]$, and c_t models the consumption during $(t-1, t]$.

NOTATION. *$X_t(H) = H_t \cdot P_t$ is the wealth at time t. Note that given H, we can find C by solving the self-financing condition.*

If $c_t = 0$ a.s. for all t then H is a pure investment strategy.

EXAMPLE 1.8. *Given an initial wealth $x > 0$, find (H, c) to maximize*

$$\sum_{i=1}^{T} \mathbb{E}(U(c_t))) \tag{1.2}$$

subject to $X_T(H) = 0$ where $T > 0$ is not random.

Assume that U is strictly increasing, strongly concave, and bounded from above.

3. A Detour into Martingales

PROPOSITION 1.9. *Let X be integrable and $\mathcal{G} \subseteq \mathcal{F}$. Then there exists an integrable, $\mathcal{G}$-measurable random variable $\bar{X}$ such that*

$$\mathbb{E}(X\mathbb{I}(G)) = \mathbb{E}\big(\bar{X}\mathbb{I}(G))\big) \tag{1.3}$$

for all $G \in \mathcal{G}$. Moreover, it is unique in the sense that if $\bar{\bar{X}}$ has the same property, then $\bar{X} = \bar{\bar{X}}$.

DEFINITION 1.10. Such $\bar{X}$ is written $\mathbb{E}(X|\mathcal{G})$, the conditional expectation of X given $\mathcal{G}$.

Useful properties of conditional expectation:

(i) If X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$.

(ii) If X is independent of $\mathcal{G}$ (that is, X and $\mathbb{I}(G)$ are independent for all $G \in \mathcal{G}$), then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$.

(iii) (Tower property) If $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$, then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})\,|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})\,|\mathcal{G}) = \mathbb{E}(X|\mathcal{H}) \tag{1.4}$$

(iv) (Slot property) If Y is $\mathcal{G}$-measurable and XY is integrable, then

$$\mathbb{E}(XY|\mathcal{G}) = Y\mathbb{E}(X|\mathcal{G}) \tag{1.5}$$

Definition 1.11. A martingale $(X_t)_{t\geq 0}$ with respect to a filtration $\mathbb{F}$ has the properties

- $\mathbb{E}(|X_t|) < \infty$ for all t,
- $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ for all $0 \leq s \leq t$.

Note that X is automatically adapted.

Exercise 1.12. *Suppose X is an integrable discrete-time process such that $\mathbb{E}(X_t|\mathcal{F}_{t-1}) = X_{t-1}$ for all $t \geq 1$. Show that X is a martingale.*

Example 1.13. *Let ξ_i, $i = 1, 2, ...$ be independent, integrable random variables with $\mathbb{E}(\xi_i) = 0$. Let $\mathcal{F}_t = \sigma(\xi_1, \dots, \xi_t), X_t = \xi_1 + \xi_2 + \cdots + \xi_t$.*

Then X is a martingale.

Example 1.14. *Let ξ be integrable and let $\mathbb{F}$ be a filtration, and $X_t = \mathbb{E}(\xi|\mathcal{F}_t)$*

Proof. Integrability comes from integrability of conditional expectations.

$$\begin{aligned}\mathbb{E}(X_t|\mathcal{F}_s) &= \mathbb{E}(\mathbb{E}(\xi|\mathcal{F}_t)\,|\mathcal{F}_s)\\ &= \mathbb{E}(\xi|\mathcal{F}_s)\\ &= X_s\end{aligned}$$

□

Example 1.15. *Suppose X is a discrete-time martingale and Y is predictable and bounded. Let $Z_t = \sum_{s=1}^{t} Y_s(X_s - X_{s-1})$. Then Z is a martingale.*

Proof. Integrability checked by integrability of X and boundedness of Y.

Z_{t-1} is $\mathcal{F}_{t-1}$ measurable since measurability respects algebraic operations.

$$\begin{aligned}\mathbb{E}(Z_t|\mathcal{F}_{t-1}) &= \mathbb{E}(Z_{t-1} + Y_t(X_t - X_{t-1})|\mathcal{F}_{t-1})\\ &= Z_{t-1} + \underbrace{Y_t}_{\text{slot property}}\ \mathbb{E}\left(\underbrace{X_t - X_{t-1}}_{=0}|\mathcal{F}_{t-1}\right)\end{aligned}$$

□

Theorem 1.16. *Suppose $u : [0, \infty) \to \mathbb{R}$ is strictly increasing, strictly concave, differentiable, bounded from above. Suppose there exists investment strategy $H^\star$ and consumption $c_t^\star = (H_{t-1}^\star - H_t^\star) \cdot P_{t-1}$, and a state price density $Y^\star$ such that $u'(c_t^\star) = Y_{t-1}^\star$. Then $(H^\star, c^\star)$ is optimal for the problem $\max \sum_{t=1}^{T} \mathbb{E}(u(c_t))$, subject to $X_0(H) = x, X_T(H) = 0$.*

PROOF. We consider the case where Ω is finite.

Let $L(H,c,Y) = \mathbb{E}\Big(\sum_{t=1}^{T}\left(u(c_t) + Y_{t+1}(H_{t+1}P(t+1) - c_t - H_t \cdot P_{t-1})\right)\Big)$ Note that $L(H,c,Y)$ is the objective when (H,c) is feasible. Then

$$L(H,c,Y) = \mathbb{E}\left(\sum_{t=1}^{T}(u(c_t) - c_t Y_{t-1})\right) + Y_0 X - Y_{t-1}H_t P_{t-1} + \sum_{t=1}^{T-1} H_t(Y_t P_t - Y_{t-1}P_{t-1}) \quad (1.6)$$

First note that $u(c_t^\star) - Y_{t-1}^\star c_t^\star \geq u(c_t) - Y_{t-1}^\star c_t$ since $u'(c_t^\star) = Y_{t-1}^\star$ (first order condition for the maximum of the concave function $c \mapsto u(c) - yc$).

Second, by definition, YP is a martingale, and by finiteness of Ω, the predictable process H is bounded. Therefore, $M_t = \sum_{s=1}^{t} H_s(Y_s P_s - Y_{s-1}P_{s-1})$ is a martingale and $E(M_t) = M_s = 0$.

Putting this together, $L(H,c,Y^\star) \leq L(H^\star, c^\star, Y^\star)$. □

THEOREM 1.17. *An absolute arbitrage is an investment/consumption strategy* (H,c) *such that* $X_0(H) = 0, X_T(H) = 0$, *at some non-random time horizon* $T > 0$, *and* $\mathbb{P}\Big(\sum_{t=1}^{T} c_t > 0\Big) > 0$.

DEFINITION 1.18. A numeraire asset is one whose price is strictly positive almost surely.

EXAMPLE 1.19. *Here is a market without a numeraire.* $P_0 = 1, P_0 = -1, P_2 = 1$.
Arbitrage:

$$H_1 = -1, c_1 = 1 X_1 = 1, c_2 = 1, H_2 = 0 X_2 = 0$$

EXERCISE 1.20. *Suppose* H_1 *is an arbitrage and the market has a numeraire. Then there exists a pure investment strategy* H' *and a time horizon* T' *such that* $X_0(H') = 0, X_{T'}(H') \geq 0$ *a.s., and* $\mathbb{P}(X_{T'}(H') > 0) > 0$.

THEOREM 1.21. *A market model has no arbitrage if and only if there exists a state price density.*

PROOF. $T = 1$ case. Suppose there exists a state price density $(Y_t)_{t=0,1}$ without loss $Y_0 = 1$. Let $Y = Y_1$ for clarity, $Y > 0$ a.s.

Suppose $(H_t)_{t=1} = H_1 = H$ (non-random vector) is a candidate arbitrage, so $H \cdot P_0 \leq 0$ and $H \cdot P_1 \geq 0$ a.s. We must show $H \cdot P_0 = 0 = H \cdot P_1$ a.s.

Since $Y > 0$, $H \cdot P_1 \geq 0 \Rightarrow \mathbb{E}(YHP_1) \geq 0$, but $H \quad \underbrace{\mathbb{E}(YP_1)}_{\text{state price density}} \quad = HP_0 \leq 0$.

By the pigeonhole principle, if $Z \geq 0$ a.s and $E(Z) = 0$, then $Z = 0$ a.s.

Thus, $YH \cdot P_1 = 0$ a.s., and since $Y > 0$ a.s., $H_0 P_1 = 0 = HP_0 = 0$ a.s.

Now consider the other direction. Let $\mathcal{Y} = \{Y > 0 a.s, \mathbb{E}(Y\|P_1\|) < a\}$. $\mathcal{Y}$ is non-empty since $Y_0 = e^{-\|P_1\|} \in \mathcal{Y}$ and $\mathcal{Y}$ is convex. Let $\mathcal{C} = \{\mathbb{E}(YP_1), y \in \mathcal{Y}\}$. Suppose $P_0 \notin \mathcal{C}$.

By the separating hyperplane theorem, there exists $H \in \mathbb{R}^n$ such that

(i) For all $c \in \mathcal{C}$, $H(c - P_0) \geq 0$.
(ii) There exists $c^\star \in \mathcal{C}$, $H(c^\star - P_0) > 0$.

This implies

(i) For all $Y \in \mathcal{Y}$, $\mathbb{E}(YH \cdot P_1) \geq H \cdot P_0$
(ii) There exists $Y^\star \in \mathcal{Y}$, $\mathbb{E}(YH \cdot P_1) > H \cdot P_0$.

Let $y = \{Y > 0 : \mathbb{E}(Y\|P_1\|)\infty\}$. Let $\mathcal{P} = \{\mathbb{E}(YP_1) : Y \in \mathcal{Y}\} \subseteq \mathbb{R}^n$. Suppose $P_0 \notin \mathcal{P}$.

By the **separating/supporting hyperplane theorem** there exists a vector $H \in \mathbb{R}^n$ such that

(i) For all $p \in \mathcal{P}$, $H \cdot (p - P_0) \geq 0$,
(ii) There exists $p^\star \in \mathcal{P}$ such that $H \cdot (p^\star - P_0) > 0$.

If $p \in \mathcal{P}$ then $p = \mathbb{E}(YP_1)$ for some Y. Then

$$H \cdot p = \mathbb{E}\left(Y \underbrace{H \cdot P_1}_{X,\ \text{time 1 wealth}}\right), H \cdot P_0 = \underbrace{-c}_{\text{consumption in } (0,1]} \tag{1.7}$$

Restating, we then have:

(i) For all $Y \in \mathcal{Y}$, $\mathbb{E}(YH \cdot P_1) \geq H \cdot P_0$
(ii) There exists $Y^\star \in \mathcal{Y}$, $\mathbb{E}(YH \cdot P_1) > H \cdot P_0$.

We need to show that $X \geq 0$ a.s., $c \geq 0$, $\mathbb{P}(X + c > 0) > 0$. Let $Y_0 = e^{-\|P_0\|} \in \mathcal{Y}$. For $\epsilon > 0$, let $Y = \epsilon Y_0$ in (i), then $\epsilon\mathbb{E}(Y_0 X) \geq c \Rightarrow c \geq 0$ by taking $\epsilon \to 0$.

Let $Y = (\frac{1}{\epsilon}\mathbb{I}(X < 0) + 1)Y_0$ in (i), which implies

$$\mathbb{E}(Y_0 X \mathbb{I}(X < 0)) \geq -\epsilon(\mathbb{E}(X_0 Y) + c) \to 0 \tag{1.8}$$

as $\epsilon \to 0$.

Then $Y_0 > 0$, $X\mathbb{I}(X < 0) \leq 0$ by pigeonhole principle,

$$\mathbb{P}(X < 0) = 0 \Rightarrow X \geq 0 \tag{1.9}$$

a.s.

By (ii), $\mathbb{P}(X = 0, c = 0) < 1$. $\square$

DEFINITION 1.22. An integrable adapted process X is a supermartingale is a supermartingale if

$$\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s \tag{1.10}$$

for all $0 \leq s \leq t$.

PROPOSITION 1.23. *If X is a supermartingale and $\mathbb{E}(X_T) = X_0$ for some non-random $T > 0$, then $(X_t)_{0 \leq t \leq T}$ is a martingale.*

PROOF. Let $Y_{s,t} = X_s - \mathbb{E}(X_t|\mathcal{F}_s) \geq 0$ by assumption. Then

$$\begin{aligned}\mathbb{E}(Y_{s,t}) &= \mathbb{E}(X_s - \mathbb{E}(\mathbb{E}(X_T|\mathcal{F}_s))) \\ &= \mathbb{E}(X_s) - \mathbb{E}(X_T) \\ &\leq \underbrace{X_0}_{\text{supermartingale}} - \underbrace{X_0}_{\text{by assumption}}\end{aligned}$$

By pigeonhole, $Y_{s,T} = 0$ a.s. Then $X_s = \mathbb{E}(X_T|\mathcal{F}_s)$ for all $0 \leq s \leq T$. So by the tower property, $(X_s)_{0\leq s\leq T}$ is a martingale. □

PROOF (Easy direction of 1FTAP). Let $T > 1$, and finite sample space. Let H be a strategy, and $X = X(H)$ be a corresponding wealth process. Let Y be a state price density. Then XY is a supermartingale, as[1]

$$\begin{aligned}\mathbb{E}(X_tY_t|\mathcal{F}_{t-1}) &= \mathbb{E}(H_t \cdot P_tY_t|\mathcal{F}_{t-1}) \\ &= \underbrace{H_t}_{\text{slot property}} \cdot\mathbb{E}(P_tY_t|\mathcal{F}_{t-1}) \\ &= H_t \cdot P_{t-1}Y_{t-1} \\ &= (H_{t-1}P_{t-1} - c_t)Y_{t-1} \\ &\leq X_{t-1}Y_{t-1}.\end{aligned}$$

Suppose H is such that $X_0 = 0$ and $X_T = 0$ a.s. for some non-random $T > 0$. Then

$$\mathbb{E}(Y_TX_T) = 0 = Y_0X_0 \tag{1.11}$$

and so XY is a martingale by the previous proposition. This implies $Y_tX_t = \mathbb{E}(Y_tX_t|\mathcal{F}_t) = 0$, which implies $X_t = 0$ for all t.

By the calculation,

$$\begin{aligned}\mathbb{E}(X_tY_t|\mathcal{F}_{t-1}) &= (X_{t-1} + c_t)Y_{t-1} \\ &\Rightarrow c_t = 0\end{aligned}$$

for all t. □

DEFINITION 1.24. A stopping time for a filtration $(F_t)_{t\in\mathbb{T}}$ is a random variable $\tau : \Omega \to \mathbb{T}\cup\{\infty\}$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{T}$ (discrete or continuous time).

NOTATION. *$M_{t\wedge\tau} = M_t^\tau$ is the martingale M stopped at τ.*

PROPOSITION 1.25. *Let M be a martingale and τ a stopping time, and let $N_t = M_{t\wedge\tau}$. Then N is also a martingale.*

[1]This relies on the finiteness of Ω since this guarantees that H is bounded, and so we call use the slot property

PROOF.

$$N_t = M_0 + \sum_{s=1}^{t} \mathbb{I}(s \leq \tau)\,(M_s - M_{s-1}) \tag{1.12}$$

and $\mathbb{I}(\tau \leq s-1)$ is $\mathcal{F}_{s-1}$-measurable and bounded. □

DEFINITION 1.26. A local martingale is an adapted process X such that there exists an increasing sequence of stopping times $\tau_n \uparrow \infty$ such that X^{τ_n} is a martingale for all n.

REMARK 1.27. *Martingales are local martingales.*

PROPOSITION 1.28. *Let X be a local martingale (discrete time). Let K be predictable and let $Y_t = \sum_{s=1}^{t} K_s(X_s - X_{s-1})$. Then Y is a local martingale.*

PROOF. Since X is a local martingale, there exists a sequence $\sigma_n \to \infty$ stopping times such that X^{σ_n} is a martingale. Let

$$\tau_n = \inf\{t \geq 0 : |K_{t+1}| > N\} \tag{1.13}$$

Then we have

$$X_{t\wedge(\underbrace{\sigma_n \wedge \tau_n}_{\text{stopping time}})} = \sum_{s=1}^{t} \underbrace{K_s \mathbb{I}(s \leq \tau_n)}_{\text{bounded and predictable}} (\underbrace{X_s^{\tau_n} - X_{s-1}^{\tau_n}}_{\text{martingale difference}}) \tag{1.14}$$

□

EXAMPLE 1.29. *Let ν, ξ be random variables with ξ integrable and $\mathbb{E}(\xi) = 0$. Let $\mathcal{F}_1 = \sigma(\nu), \mathcal{F}_2 = \sigma(\nu, \xi)$. Let $X_1 = 0$, $X_2 = \nu\xi$. Then X is a local martingale.*

If the product $\nu\xi$ is also integrable, then X is a true martingale, otherwise $\mathbb{E}(X_2|\mathcal{F}_1)$ is not defined.

PROPOSITION 1.30. *Let X be a local martingale such that there exists an integrable process Y such that $Y_t \geq |X_s|$ for all $0 \leq s \leq t$. Then X is a true martingale.*

PROOF. By assumptions there exists a sequence $\tau_N \to \infty$ such that X^{τ_N} is a martingale. Also, $|X_{t\wedge\tau_N} \leq Y_t$ which is integrable. Then

$$\begin{aligned}
\mathbb{E}(X_t|\mathcal{F}_s) &= \mathbb{E}\Big(\lim_{N\to\infty} X_{t\wedge\tau_N}|\mathcal{F}_s\Big) && (1.15)\\
&= \lim_{N\to\infty} \mathbb{E}(X_{t\wedge\tau_N}|\mathcal{F}_s) && (1.16)\\
&= \lim_{N\to\infty} X_{s\wedge\tau_N} && (1.17)\\
&= X_s && (1.18)
\end{aligned}$$

□

COROLLARY 1.31. ***In discrete time***, *if X is a local martingale and $\mathbb{E}(|X_t|) < \infty$ for all $t \geq 0$ then X is a martingale.*

PROOF. Let $Y_t = \sum_{s=0}^{t} |X_s|$, and Y is integrable by assumption. □

PROPOSITION 1.32. *If X is a local martingale (in discrete or continuous time) and $X_t \geq 0$ almost surely for all t, then X is a supermartingale.*

PROOF. First, X_t is integrable, since

$$\begin{aligned}
\mathbb{E}(|X_t|) &= \mathbb{E}(X_t) && (1.19)\\
&= \mathbb{E}\Big(\lim_{N\to\infty} X_{t\wedge\tau_N}\Big) && (1.20)\\
&\leq \liminf_{N\to\infty} \mathbb{E}(X_{t\wedge\tau_N}) && (1.21)\\
&= \liminf_{N\to\infty} X_{0\wedge\tau_n} && (1.22)\\
&= X_0 < \infty. && (1.23)
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbb{E}(X_t|\mathcal{F}_s) &= \mathbb{E}(\lim X_{t\wedge\tau_N}|\mathcal{F}_s) && (1.24)\\
&\leq \liminf \mathbb{E}(X_{t\wedge\tau_N}|\mathcal{F}_s) && (1.25)\\
&= \liminf X_{s\wedge\tau_N} && (1.26)\\
&= X_s && (1.27)
\end{aligned}$$

□

COROLLARY 1.33. ***In discrete time***, *non-negative local martingales in discrete time are martingales.*

PROOF. Let X be the local martingale. Then $\mathbb{E}(|X_t|) < \infty$ for all $t \geq 0$ by Fatau. The result follows from the last corollary. □

THEOREM 1.34. *Let X be a **discrete** time local martingale. Fix $T > 0$ non-random. Then $(X_t)_{0\leq t\leq T}$ is a true martingale if either*

(i) $\mathbb{E}(|X_T|) < \infty$, *or*
(ii) $X_T \geq 0$

Lecture on Wednesday 23 October

4. Contingent Claims

Setup - P is a price process (n-dimensional space, adapted).

Two types of claims

(i) European - specified by a time horizon T (maturity date or expiry) and a $\mathcal{F}_T$-measurable random variable ξ_T (the payout of the claim).
(ii) American - specified maturity date T and an adapted process $(\xi_t)_{0\le t\le T}$ where ξ_t is the payout if owner of claim chooses to exercise at time $t \le T$.

Example 1.35. *A call option is the right, but not the obligation, to buy a certain stock at a fixed price sometime in the future.*

$$\xi_T = (S_T - k)^+ \tag{1.28}$$
$$\xi_t = (S_t - k)^+ \tag{1.29}$$

for all $0 \le t \le T$.

Definition 1.36. A European contingent claim is **attainable** or **replicable** if there exists a pure investment strategy H such that $X_T(H) = \xi_T$ almost surely.

Theorem 1.37. *Suppose ξ_t is the price of attainable claim for $0 \le t \le T$. If the augmented market (P, ξ) has no arbitrage then $\xi_t = X_t(H)$ a.s.*

Proof. Let $\tau = \inf\{t \ge 0 : X_t \ne \xi_t\}$. Let $\bar{H}_t = \mathrm{sign}(\xi_t, X_t)\mathbb{I}(t \ge \tau)\,(H_t, -1)$.
Then $c_{\tau+1} = |\xi_\tau - X_\tau|$, $\bar{X}_t(\bar{H}) = \bar{H}_t \cdot (P_t, \xi_t)$, $\bar{X}_0(\bar{H}) = 0, \bar{X}_T(\bar{H}) = 0$, and $c_t = 0$ for all t if and only if there is no arbitrage. □

Theorem 1.38. *Suppose Y is a state price density of the original market with prices P. Suppose ξ_T is the payout of an attainable claim, suppose either*

(i) $\mathbb{E}(|\xi_T|Y_T) < \infty$, *or*
(ii) $\xi_T \ge 0$ *a.s.*

If the augmented market (P, ξ) has no arbitrage, then

$$\xi_t = \frac{1}{Y_t}\mathbb{E}(Y_T \xi_T | \mathcal{F}_t) \tag{1.30}$$

for all $0 \le t \le T$.

Proof. By the previous result, there exists H (pure investment strategy) such that $X_t(H) = \xi_t$ for all t. But XY is a local martingale. From before, if either $X_T Y_T$ is integrable or non-negative, the process XY is a true martingale.

$$\xi_t Y_t = X_t Y_t = \mathbb{E}(X_T Y_T | \mathcal{F}_t) = \mathbb{E}(\xi_T Y_T | \mathcal{F}_t) \tag{1.31}$$

as required. □

REMARK 1.39. *When our price process can be decomposed into a numeraire, so $P = (N, S)$, we can let $\mathbb{Q}$ be an equivalent martingale measure. If either $\mathbb{E}_{\mathbb{Q}}\left(\frac{\xi_T}{N_T}\right) < \infty$, or $\xi_T \geq 0$, then*

$$\xi_t = N_t \mathbb{E}_{\mathbb{Q}}\left(\frac{\xi_T}{N_T}|\mathcal{F}_t\right) \tag{1.32}$$

THEOREM 1.40. *Suppose ξ_t is the price of a contingent claim at time t (not necessarily attainable). Suppose that the augmented market (P, ξ) has no arbitrage. Then there exists a positive process Y such that*

$$P_t = \frac{1}{Y_t}\mathbb{E}(Y_T P_T|\mathcal{F}_t) \tag{1.33}$$

$$\xi_t = \frac{1}{Y_t}\mathbb{E}(Y_T \xi_T|\mathcal{F}_t) \tag{1.34}$$

Here, (1.33) shows Y is a state price density for the original market, and (1.34) shows Y is a state price density for the augmented market.

PROOF. The proof is just 1FTAP applied to the augmented market. □

EXAMPLE 1.41. *Let $P_t = (B_{t,T}, S_t)$. $B_{t,T}$ is price of bond maturing at T, with $B_{T,T} = 1$ almost surely. S_t is a stock with $S_t \geq 0$ for all t. Let c_t be the price of a call with payout $(S_T - K)^+$. Suppose $(B_{t,T}, S_t, C_t)_{t\in[0,T]}$ has no arbitrage.*

In general, since the payout of the call is non-negative then $c_t \geq 0$. Also, $(S_T - K)^+ \geq S_T - K = S_T - KB_{T,T} = (-K, 1)\cdot(B_{t,T}, S_t)$.

This implies

$$c_t \geq S_t - KB_{t,T} \tag{1.35}$$

Then $c_t \geq (S_t - KB_{t,T})^+$, and $(S_T - K)^+ < S_T$, thus $c_t \leq S_t$.

If there exists a state price density Y for (B, S) such that

$$c_t = \frac{1}{Y_t}\mathbb{E}\big(Y_T(S_T - K)^+|\mathcal{F}_t\big)\,. \tag{1.36}$$

EXAMPLE 1.42. *A put option is equivalent to $(K - S_T)^+ = K - S_T + (S_T - K)^+ = (K, -1, 1)\cdot(B_{T,T}, S_T, C_T)$. If p_t is a no-arbitrage price of the put, then*

$$p_t = KB_{t,T} - S_t + c_t. \tag{1.37}$$

DEFINITION 1.43. A market is **complete** if and only if every European contingent claim is attainable. A market that is not complete is **incomplete**.

THEOREM 1.44 (Second fundamental theorem of asset pricing). *A market with no arbitrage is complete if and only if there exists a unique (up to scaling) state price density.*

PROOF. Suppose the market is complete. Let Y, Y' be state price densities with $Y_0 = Y_0' = 1$. Fix $T > 0$ and let $\xi_T \geq 0$ be $\mathcal{F}_T$-measurable. By completeness, there exists a pure investment strategy H such that $X_T(H) = \xi_T$.

From before,

$$\mathbb{E}(Y_T\xi_T) = X_0(H) = \mathbb{E}(Y'_T\xi_T) \tag{1.38}$$

and thus $\mathbb{E}(\xi_T(Y_T - Y'_T)) = 0$. Let $\xi_T = \mathbb{I}(Y_T > Y'_T)$. Then $Y_T \leq Y'_T$ almost surely, and so by symmetry, $Y_T = Y'_T$.

A claim with payout $\xi_T \geq 0$ is attainable if there exists $x \geq 0$ such that $\mathbb{E}\left(\frac{Y_T\xi_T}{Y_0}\right) = x = X_0(H)$ for all state price densities.[2]

Given there exists a unique state price density, every non-negative claim is attainable. The conclusion follows by observing $\xi_T = \xi_T^+ - \xi_T^-$. $\square$

THEOREM 1.45. *Suppose that the price process P is n-dimensional and the market is complete. The for each $t \geq 0$, there are no more than n^t disjoint sets of positive probability $\mathcal{F}_t$-measurable sets of positive probability. In particular, the random vector P_t takes on at most n^tvalues.*

PROOF. Consider the $t = 1$ case. Let $A_1, \ldots, A_k$ be disjoint $\mathcal{F}_1$-measurable sets with $\mathbb{P}(A_i) > 0$. We claim the set $\{\mathbb{I}(A_i)\}$ is linearly independent.

Suppose $\sum_i a_i\mathbb{I}(A_i) = 0$. Multiplying by $\mathbb{I}(A_j)$ implies $a_j\mathbb{I}(A_j) = 0$ almost surely by disjointness. Since $\mathbb{P}(A_j) > 0$ by assumption we have $a_j = 0$.

By completeness, each $\mathbb{I}(A_i)$ is attainable, so

$$\text{span}\{\mathbb{I}(A_i)\} \subseteq \{H \cdot P_1, H \in \mathbb{R}^n\} = \text{span}\{P_1^1, \ldots, P_1^n\} \tag{1.39}$$

$\square$

5. American Claims

Recall that the payoff of an American claim is specified by an adapted process $(\xi_t)_{0 \leq t \leq T}$ where ξ_t is the payout if the claim is executed at time t.

THEOREM 1.46. *Suppose the market is complete. Then there exists a (pure investment) strategy such that $X_t(H) \geq \xi_t$ for all $0 \leq t \leq T$, and there exists a stopping time $\tau^\star$ such that $X_{\tau^\star}(H) = \xi_{\tau^\star}$.*

Furthermore, $X_0(H) = \sup_{\text{stopping time } \tau \leq T} \mathbb{E}(Y_\tau\xi_\tau)$ where Y is the unique state price density such that $Y_0 = 1$.

DEFINITION 1.47. Let Z be an adapted integrable process $(Z_t)_{0 \leq t \leq T}$. The Snell envelope of $Z$is the process U defined by $U_T = Z_T$, $U_t = \max\{Z_t, \mathbb{E}(U_{t+1}|\mathcal{F}_t)\}$ for $0 \leq t \leq T-1$.

REMARK 1.48. *Note that $U_t \geq Z_t$for all t, and U is a supermartingale since $U_t \geq \mathbb{E}(U_{t+1}|\mathcal{F}_t)$.*

THEOREM 1.49 (Doob decomposition). *Let U be a discrete-time supermartingale. Then there exists a martingale M with $M_0 = 0$, and a non-decreasing process A with $A_0 = 0$ such that $U_t = U_0 + M_t - A_t$.*

[2]Proof in example sheet

PROOF. Let $M_0 = A_0 = 0$, $M_{t+1} = M_t + U_{t+1} - \mathbb{E}(U_{t+1}|\mathcal{F}_t)$, and $A_{t+1} = A_t + U_t - \mathbb{E}(U_{t+1}|\mathcal{F}_t)$. By induction, A_t is predictable. A is non-decreasing as U is a supermartingale.

Now, we show uniqueness. Suppose $U = U_0 + M - A = U_0 + M' - A'$. Then $M - M' = A - A'$, and as $A - A'$ is predictable, we have $M - M'$ is a predictable martingale. In discrete time, predictable martingales are almost surely constant. Thus, $M_t - M_t' = M_0 - M_0' = 0$, and thus we have demonstrated uniqueness. □

THEOREM 1.50. *Let Z be integrable and adapted, U is a Snell envelope, with Doob decomposition $U = U_0 + M - A$. Let $\tau^\star = \inf\{t \geq 0 | A_{t+1} > 0\}$ with the convention $\tau^\star = T$ on $\{A_t = 0 \forall t\}$.*

Then $U_{\tau^\star} = U_0 + M_{\tau^\star} = Z_{\tau^\star}$.

REMARK 1.51. *$\tau^\star$ is a stopping time since A is predictable.*

PROOF. Note that $A_{\tau^\star} = 0$ but $A_{\tau^\star+1} > 0$. We have

$$U_t = U_0 + M_t - A_t \tag{1.40}$$

$$= \max\{Z_t, \mathbb{E}(U_{t+1}|\mathcal{F}_t)\} \tag{1.41}$$

$$= \max\{Z_t, U_0 + M_t - A_{t+1}\}. \tag{1.42}$$

So $U_0 + M_{\tau^\star} = \max\{Z_{\tau^\star}, U_0 + M_{\tau^\star} - A_{\tau^\star-1}\}$, which implies $U_0 + M_{\tau^\star} = Z_{\tau^\star} = U_{\tau^\star}$ as required. □

THEOREM 1.52. *Under the same hypothesis as before,*

$$U_0 = \sup_{\textit{stopping times } \tau \leq T} \mathbb{E}(Z_\tau)\,. \tag{1.43}$$

PROOF. By the optional stopping theorem, $U_0 \geq \mathbb{E}(U_\tau) \leq \mathbb{E}(Z_t)$ for any stopping time $\tau \leq T$, and since $U_t \geq Z_t \forall t$.

But $U_0 = \mathbb{E}(U_0 + M_{\tau^\star}) = \mathbb{E}(Z_{\tau^\star})$. □

We now give a proof of the existence of the minimal super-replicating strategy. Let U be the Snell envelope of $(Y_t\xi_t)_{0\leq t\leq T}$. Let $U = U_0 + M - A$ be its Doob decomposition.

By completeness, there exists a strategy H such that

$$X_T(H) = \frac{U_0 + M_T}{Y_T}. \tag{1.44}$$

Since XY is a martingale (XY is a local martingale in general but by completeness all processes are bounded). So

$$\begin{aligned} X_t Y_T &= U_0 + M_t && (1.45)\\ &\geq U_0 + M_t - A_t && (1.46)\\ &= U_t && (1.47)\\ &\geq Y_t \xi_t. && (1.48) \end{aligned}$$

Thus $X_t \geq \xi_t$ for all $0 \leq t \leq T$.

Also, at $\tau^\star = \inf\{t \geq 0 | A_{t+1} > 0\}$, we have

$$X_{\tau^\star} Y_{\tau^\star} = U_0 + M_{\tau^\star} = U_{\tau^\star} = Y_{\tau^\star} \xi_{\tau^\star}, \tag{1.49}$$

and so $X_{\tau^\star} = \xi_{\tau^\star}$.

Note also that $X_0 = \mathbb{E}(U_0 + M_T) = U_0 = \sup_{\tau \leq T} \mathbb{E}(\xi_\tau Y_\tau)$.

CHAPTER 2

Continuous Time Models

In discrete time, we had $X_t - X_{t-1} = H_t \cdot (P_t - P_{t-1}) - c_t$. For continuous time, we replace this with $dX_t = H_t dP_t - c_t dt$

A state price density is some stochastic process Y with $Y_t > 0$ and YP is a martingale

Lemma 2.1. *If $t \mapsto X_t(\omega)$ is differentiable and X is a martingale then X is constant.*

This can make a pricing theory quite boring!

1. Diversion into Stochastic Calculus

Definition 2.2. A (standard scalar) Brownian motion is a process $W = (W_t)_{t\geq 0}$ such that

(i) $W_0(\omega) = 0$ for all ω.
(ii) $t \mapsto W_t(w)$ is continuous for all ω
(iii) For any $0 \leq t_0 < t_1 < \cdots < t_n$, the increments $W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent, with $W_t - W_s \sim N(0, |t-s|)$.

Theorem 2.3. *The Brownian motion exists (Weiner, 1923).*

Consider a filtration $(\mathcal{F}_t)$ with the property that $W_t - W_s$ is independent of $\mathcal{F}_s$, $0 \leq s \leq t$. Our technical assumptions are usual conditions - $\mathcal{F}_t = \cap_{\epsilon>0} \mathcal{F}_{t+\epsilon}$ (right-continuity), $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

Definition 2.4. A **simple predictable process** is of the form

$$\alpha_t(\omega) = \sum_{i=1}^{n} \mathbb{I}((t_{i-1}, t_i))\, a_i(\omega), \tag{2.1}$$

where $0 \leq t_0 < \cdots < t_n$, each a_i is a bounded $\mathcal{F}_{t_{i-1}}$-measurable random variable.

Remark 2.5. *α is left-continuous, piecewise-constant, and adapted.*

Definition 2.6.

$$\int_0^\infty \alpha_s dW_s = \sum_{i=1}^{n} a_i (W_{t_i} - W_{t_{i-1}}) \tag{2.2}$$

where α is a simple predictable process.

Definition 2.7. The predictable σ-algebra on $[0, \infty) \times \Omega$ is generated by $(s, t] \times A$ where $A \in \mathcal{F}_s$.

This is the smallest σ-algebra for which simple predictable processes are measurable.

A process measurable with respect to the predictable σ-algebra is called **predictable**.

Remark 2.8. *If α is left-continuous and adapted, it is predictable.*

Proposition 2.9 (Itô's isometry). *If α is simple and predictable, then*

$$\mathbb{E}\left(\left(\int_0^\infty \alpha_s dW_s\right)^2\right) = \mathbb{E}\left(\int_0^\infty \alpha_s^2 ds\right) \tag{2.3}$$

Thus, the isometry I from simple predictable process to square integrable random variables on $L^2(\Omega, \mathcal{F}, \mathbb{P})$ (which is complete) defined by

$$I(\alpha) = \int_0^\infty \alpha_s dW_s \tag{2.4}$$

Proof.

$$\left(\int \alpha dW\right)^2 = \left(\sum a_i \Delta W_i\right)^2 \tag{2.5}$$

$$= 2\sum_{j<i} a_j a_i \Delta W_j \Delta W_i + \sum a_i^2 (\Delta W_i)^2 \tag{2.6}$$

Note that $\mathbb{E}\left(\sum a_i^{2(\Delta W_i)^2}\right) = \ldots$

□ Finish this proof

Definition 2.10. Suppose $\mathbb{E}\left(\int_0^\infty (\alpha_s^k - \alpha_s)^2 ds\right) \to 0$, where each α^k is simple and predictable. Then

$$\int_0^\infty \alpha_s dW_s = \lim_{L^2} \int_0^\infty a_s^k dW_s \tag{2.7}$$

Theorem 2.11. *If α is predictable and $\mathbb{E}\left(\int_0^t \alpha_s^2 ds\right) < \infty$ for all t, there exists a continuous martingale X such that $X_t = \int_0^\infty \alpha_s \mathbb{I}(s \leq t)\, dW_s$.*

For notation, we represent X_t as $\int_0^t \alpha_s dW_s$. Note that $\mathbb{E}(X_t) = 0$ and $\mathbb{E}(X_t^2) = \int_0^t \alpha_s^2 ds$.

Definition 2.12 (Localization). Suppose α is predictable and $\int_0^t \alpha_s^2 ds < \infty$ almost surely for all t. Let $\tau_n = \inf\{t \geq 0 | \int_0^t \alpha_s ds > n\}$.

Let $\alpha_t^{(n)} = \alpha_t \mathbb{I}(t \leq \tau_n)$, so $\int_0^t \alpha_s^{(n)} dW_s$ is well-defined by the L^2 theory, since $\mathbb{E}\left(\int_0^t (\alpha_s^{(n)})^2 ds\right) \leq N \leq \infty$ as $\int_0^t \alpha_s^2 ds < \infty$ almost surely as $\tau_n \uparrow \infty$.

Notation. *$\int_0^t \alpha_s dW_s$ as $\int_0^t \alpha_s^{(N)} dW_s$ on $\{t \leq \tau_n\}$.*

Theorem 2.13. *If α is adapted and continuous, then $\int_0^t \alpha_s dW_s$ is defined for all $t \geq 0$ - since $t \mapsto \alpha_t(\omega)$ is continuous, α is bounded on $[0, t]$ for each ω, and so $\int_0^t \alpha_s ds < \infty$ almost surely.*

If $X_t = \int_0^t \alpha_s dW_s$, *then* X *is a continuous local martingale, since* $X^{(n)} = (X_{t\wedge\tau_n})_t \geq 0$ *is a true martingale, where* $\tau_n = \inf\{\tau \geq 0, \int_0^t \alpha_s ds \geq N\}$.

2. Itô's Formula

Definition 2.14. An Itô process X is of the form

$$X_t = X_0 + \int_0^t \alpha_s dW_s + \int_0^t \beta_s ds \tag{2.8}$$

such that α, β are predictable and $\int_0^t \alpha_s ds < \infty$ and $\int_0^t |\beta_s| ds < \infty$ for all t.

Theorem 2.15. *If* X *is an Itô process and* $f \in C^2$, *then* $f(X)$ *is an Itô process. In fact,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\alpha_s dW_s + \int_0^t \left(f'(X_s)\beta_s + \underbrace{\frac{1}{2} f''(X_s)\alpha_s^2}_{\textit{Itô's correction}} \right) ds \tag{2.9}$$

Example 2.16. $f(x) = x^2$. *Then*

$$W_t^2 = \int_0^t 2W_s dW_s + t \tag{2.10}$$

$$\mathbb{E}(W_t^2) = \mathbb{E}\left(\int_0^t 2W_s dW_s\right) + t \tag{2.11}$$

and the first term is zero as it is a martingale.

This follows from

$$\mathbb{E}\left(\int_0^t W_s^2 ds\right) = \int_0^t s ds = \frac{t^2}{2} < \infty \tag{2.12}$$

so $\int_0^t W_s dW_s$ *is a martingale.*

Theorem 2.17. *Let* X *be an Itô process. Fix* $t > 0$. *Then*

$$\lim_{n\to\infty} \sum_{k=1}^n \left(X_{\frac{tk}{n}} - X_{\frac{t(k-1)}{n}} \right)^2 = \int_0^t \alpha_s^2 ds \tag{2.13}$$

Notation.

$$\langle X \rangle_t = \int_0^t \alpha_s ds \tag{2.14}$$

is called the quadratic variation of X.

Theorem 2.18 (Itô's formula). *In integral form,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2}\int_0^t f''(X_s) d\langle X \rangle_s \tag{2.15}$$

In differential form,

$$df(X_t) = f'(X_t) dX_t + \frac{1}{2} f''(X_t) d\langle X \rangle_t \tag{2.16}$$

Morally, the idea is to take Taylor expansion around $f(X_t)$.

THEOREM 2.19 (Itô's formula, multidimensional version). *let X, Y be Itô processes. Then the quadratic covariation*

$$\langle X, Y\rangle_t = \lim_{n\to\infty} \sum_{k=1}^{n} (X_{\frac{tk}{n}} - X_{\frac{t(k-1)}{n}})(Y_{\frac{tk}{n}} - Y_{\frac{t(k-1)}{n}}) \tag{2.17}$$

$$= \frac{1}{2}\langle X+Y\rangle_t - \langle X\rangle_t - \langle Y\rangle_t \tag{2.18}$$

PROPOSITION 2.20. *The quadratic covariance satisfies the following properties:*

(i) (Bilinear, symmetric)

$$\langle aX + bY, Z\rangle = a\langle X, Z\rangle + b\langle Y, Z\rangle = \langle Z, aX + bY\rangle \tag{2.19}$$

(ii) If $X_t = X_0 + \int_0^t \beta_s ds$ then $\langle X, Y\rangle_t = 0$ for any Itô process Y.
(iii) Let W^1, W^2 be two independent Brownian motions. Then $\left\langle W^1, W^2\right\rangle_t = 0$.
(iv)

$$\left\langle \int_0^t \alpha_s dW_s, \int_0^t \beta_s dW_s \right\rangle = \int_0^t \alpha_s \beta_s ds \tag{2.20}$$

Let X be an n-dimensional Itô process, and $f \in C^2(\mathbb{R}^n \to \mathbb{R})$. Then

$$\tag{2.21}$$

Fill in this multivariate Itô's result

In finance there are state price densities $\Rightarrow$ equivalent martingale measures. How to do computations under equivalent changes of measure?

Let W be an n-dimensional BM with $W = (W^1, \ldots, W^m)$ where W^i are independent standard Brownian motions. Let α be an n-dimensional predictable process and $\int_0^t \|a_s\|^2 ds < \infty$, and let

$$Z_t = e^{\int_0^t \alpha_s dW_s - \frac{1}{2}\int_0^t \|\alpha_s\|^2 ds}. \tag{2.22}$$

PROPOSITION 2.21. *Z satisfies the following properties:*

(i) Z is a local martingale.
(ii) Z is a supermartingale.
(iii) If $\mathbb{E}(Z_T) = 1$ for some $T > 0$ (non-random), then $(Z_t)_{0\le t\le T}$ is a true martingale.

PROOF. Let $dX_t = \alpha_t \cdot dW_t - \frac{1}{2}\|\alpha_t\|^2 dt$, $X_0 = 0$. Let $f(x) = e^x$. Then

$$dZ_t = df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t \tag{2.23}$$

Note that

$$d\langle X\rangle_t = d\left\langle \sum_{i=1}^{m} \int_0^t \alpha_s^2 dW_s^2 \right\rangle_t \tag{2.24}$$

$$= d\sum_{i,j} \left\langle \int \alpha_s^i dW_s^i, \int \alpha_s^j dW^j \right\rangle_t \tag{2.25}$$

$$= \sum (\alpha_t^i)^2 dt \tag{2.26}$$

$$= \|\alpha_t\|^2 dt \tag{2.27}$$

Then

$$dZ_t = Z_t\left(\alpha_t \cdot dW_t - \frac{1}{2}\|\alpha_t\|^2 dt\right) + \frac{1}{2}Z_t\|\alpha_t\|^2 dt = Z_t\alpha_t dW_t. \tag{2.28}$$

Thus

$$Z_t = 1 + \int_0^t Z_s\alpha_s \cdot dW_s \tag{2.29}$$

and so Z is a stochastic integral, and hence a local martingale.

$Z_t > 0$ almost surely, so non-negative local martingales are supermartingales by Fatou's lemma.

Z is a supermartingale and $\mathbb{E}(Z_T) = Z_0$, and so $(Z_t)_{0\leq t\leq T}$ is a martingale (pigeonhole principle).

□

Theorem 2.22 (Cameron-Martin-Girsanov theorem). *Let Zbe as before and assume $\mathbb{E}(Z_T) = 1$for some $T > 0$. Define an equivalent martingale measure $\mathbb{Q}$ by Radon-Nikodym density*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_t \tag{2.30}$$

Let $\hat{W}_t = W_t - \int_0^t \alpha_s ds$. Then $\hat{W}$ is a $\mathbb{Q}$-Brownian motion.

Theorem 2.23 (Martingale representation theorem). *Let W be an m-dimensional Brownian motion generating the filtration $(\mathcal{F}_t)_{t\geq 0}$. Let X be a continuous local martingale. Then there exists a predictable α with $\int_0^t \|\alpha_s\|^2 ds < \infty$ almost surely for all t such that $X_t = X_0 + \int_0^t \alpha_s dW_s$.*

If $X_t > 0$ a.s. for all t, then there exists a predictable process β with $\int_0^t \|\beta_s\|^2 ds < \infty$ for all t such that

$$X_t = X_0 e^{\int_0^t \beta_s dW_s - \frac{1}{2}\int_0^t \|\beta_s\|^2 ds} \tag{2.31}$$

Theorem 2.24 (Levy's characterization theorem). *Let X be a continuous local martingale (in any filtration satisfying the usual conditions) such that its quadratic variation $\langle X\rangle_t = t$. Then X is a Brownian motion.*

3. Arbitrage Theory in Continuous Time

Recall that in discrete time,

$$X_t = H_t \cdot P_t = H_{t+1} \cdot P_t - c_{t+1} \tag{2.32}$$

$$X_{t+1} = H_{t+1} \cdot P_{t+1} \Rightarrow X_{t+1} - X_t = H_{t+1} \cdot (P_{t+1} - P_t) - c_{t+1} \tag{2.33}$$

The setup is as follows:

(i) P is an m-dimensional Itô process.

Definition 2.25. A self-financing investment/consumption strategy (H, c) is a pair of predictable processes such that $c_t \geq 0$ for all t , $\int_0^t \sum (H_s^i)^2 d\langle P^i \rangle >_s < \infty$ for all t, and

$$H_t \cdot P_t = H_0 \cdot P_0 + \int_0^t H_s \cdot dP_s - \int_0^t c_s ds \tag{2.34}$$

Definition 2.26 (Incomplete). An arbitrage is an investment/consumption strategy (H, c) such that $X_0 = X_T = 0$ and $\mathbb{P}\Big(\int_0^T c_s ds > 0\Big) > 0$ for some non-random $T > 0$

This definition is flawed.

Example 2.27 (Doubling strategies). *Consider the discrete-time model* $P = (1, S_t)$ *where* $S_t = \xi_1 + \cdots + \xi_t$ *where* ξ_i *are IID with* $\mathbb{P}(\xi_i = \pm 1) = \frac{1}{2}$.

Consider a price vector $P = (1, W)$ with W a Brownian motion . Let $X_t = \int_0^t \pi_s dW_s$, and let $f : [0, 1] \to [0, \infty]$ an increasing bijection with inverse f^{-1}. For example, $f(t) = \frac{t}{1-t}$ with $f^{-1}(u) = \frac{u}{1+u}$. Consider

$$Z_u = \int_0^{f^{-1}(u)} \sqrt{f'(s)} dW_s \tag{2.35}$$

Then

$$\langle Z \rangle_u = \int_0^{f^{-1}(u)} f'(s) ds = u \tag{2.36}$$

which implies Z is a Brownian motion by Levy's characterization. Let $\tau =\in \{u \geq 0 : Z_u > K\}$ where $K > 0$ is a constant. Let $\pi_t = \sqrt{f'(t)} \mathbb{I}\big(t \leq f^{-1}(\tau)\big)$. Note that $\int_0^1 \pi_s^2 ds = \int_0^{f^{-1}(\tau)} f'(s) ds = \tau < \infty$. So $\int_0^t \pi_s dW_s$ makes sense for all $t \leq 1$. Let $X_t = \pi_s dW_s$, with $X_1 = \int_0^{f^{-1}(\tau)} \sqrt{f'(s)} dW_s = Z_\tau = K > 0$. X is a local martingale since it is a stochastic integral, but $\mathbb{E}(X_1) - K \neq X_0 = 0$.

Definition 2.28. An investment/consumption strategy (H, c) is L-admissible if $X_t(H, c) \geq -L_t$ for all t a.s. where L is given non-negative adapted process.

For most cases, $L = 0$.

Definition 2.29. A state price density is a positive Itô process such that $(Y_t P_t)_{t \geq 0}$ is a local martingale.

THEOREM 2.30. *If there exists a state price density such that YL is uniformly integrable, then there is no arbitrage among L-admissible self-financing investment/consumption strategies.*

REMARK 2.31. *Recall that $(Z_t)_{t\geq 0}$ is uniformly integrable if and only if*

$$\lim_{k\to\infty}\sup_{t\geq 0}\mathbb{E}(|Z_t|\mathbb{I}(Z_{t\geq k}))=0 \tag{2.37}$$

REMARK 2.32. *If $(Z_t)_{0\leq t\leq T}$ is a martingale then $(Z_t)_{0\leq t\leq T}$ is uniformly integrable ($T<\infty$ not random.)*

REMARK 2.33. *If $\sup_{t\geq 0}\mathbb{E}(|Z_t|^p)<\infty$ for some $p>1$ then $(Z_t)_{t\geq 0}$ is uniformly integrable.*

REMARK 2.34. *If $Z_n\to Z_\infty$ a.s. and $(Z_n)_{n\geq 1}$ is UI then $\mathbb{E}(|Z_n-Z_\infty|)\to 0$.*

PROPOSITION 2.35. *Let (H,c) be a self financing stragey and $X_t=H_t\cdot P_t$ so that $dX_t=H_t\cdot dP_t-c_t dt$. Let Y be an Itô process. Let Y be an Itô process. Then*

$$d(X_tY_t)=H_t\cdot(dY_tP_t)-Y_tc_tdt. \tag{2.38}$$

PROOF. Since $dX=H\cdot dP-cdt$, then

$$d\langle X,Y\rangle=\sum_{i=1}^n h^i d\big\langle P^i,Y^i\big\rangle \tag{2.39}$$

By Itô's formula,

$$\begin{aligned}
d(XY)&=XdY+YdX+d<X,Y> && (2.40)\\
&=H\cdot PdY+Y(H\cdot dP-cdt)+\sum H^i d\big\langle P^i,Y^i\big\rangle && (2.41)\\
&=\sum H^i(P^idY+YdP^i+d\big\langle P^i,Y\big\rangle)-Ycdt && (2.42)\\
&=\sum H^i d(P^iY)-Ycdt && (2.43)
\end{aligned}$$

□

DEFINITION 2.36. A continuous, adapted process $(Z_t)_{t\geq 0}$ is of class $\mathcal{D}$ (Doob) if $\{Z_\tau|\tau \text{ stopping times}\}$ is uniformly integrable.

REMARK 2.37. *If $\mathbb{E}\big(\sup_{t\geq 0}|Z_t|\big)<\infty$, then $(Z_t)_{t\geq 0}$ is of class $\mathcal{D}$.*

THEOREM 2.38. *If YL is of class D (at least locally), then there is no arbitrage.*

THEOREM 2.39. *If there exists a state price density Y such that YL is of class D locally, then there are no L-admissible .*

Class D locally means $\{Z_{\tau\wedge t}$ - τ a stopping time is $UI\forall t\geq 0\}$.

PROOF.

$$\int_0^t H_s \cdot d(X_s P_s) = Y_t X_t - Y_0 X_0 + \int_0^t Y_s c_s ds \tag{2.44}$$

$$\geq -Y_t L_t - Y_0 X_0 \tag{2.45}$$

if (H, c) is L-admissible. and from the lemma.

Also, since YP is a local martingale then $\int H \cdot d(YP)$ is a local martingale (by construction of the Itô integral), so there exists a sequence of stopping times $\tau_n \uparrow \infty$ such that $(\int H \cdot d(YP))^{\tau_n}$ is a true martingale.

Then

$$\mathbb{E}\left(\int_0^T H_s \cdot d(Y_s P_s) + Y_T L_T\right) = \mathbb{E}\left(\lim_{n\to\infty} \int_0^{T\wedge\tau_n} H_s \cdot d(Y_s P_s) + L_{T\wedge\tau_n} Y_{T\wedge\tau_n}\right) \tag{2.46}$$

$$\leq \liminf_{n\to\infty} \mathbb{E}\left(\int_0^{T\wedge\tau_n} H d(YP) + L_{T\wedge\tau_n} Y_{T\wedge\tau_n}\right) \tag{2.47}$$

$$= \liminf_{n\to\infty} \mathbb{E}(Y_{T\wedge\tau_n} L_{T\wedge\tau_n}) \tag{2.48}$$

$$= \mathbb{E}(Y_T L_T) \tag{2.49}$$

by Fatau's lemma (2.47), using that $(\int_0^t H \cdot d(YP))^{\tau_n}$ is a martingale starting at zero (2.48) and the assumption of uniform integrability (2.49).

So suppose $X_0 = 0 = X_T$ almost surely. Then

$$\mathbb{E}\left(\int_0^T Y_s c_s ds\right) = \mathbb{E}\left(\int_0^T H_s \cdot d(Y_s P_s)\right) \leq 0 \Rightarrow c_t(\omega) = 0 a.e. \tag{2.50}$$

which implies no arbitrage. □

Suppose $P = (N, S)$ where $N_t > 0$ for all $t \geq 0$ almost surely - e.g. the price of a numeraire.

DEFINITION 2.40. A pure investment strategy H is an arbitrage relative to the numeraire if and only if

(i) There exists a non-random $T > 0$ such that

$$\frac{X_T}{N_0} \geq \frac{N_T}{N_0} a.s. \tag{2.51}$$

and

$$\mathbb{P}\left(\frac{X_T}{N_0} > \frac{N_T}{N_0}\right) > 0 \tag{2.52}$$

REMARK 2.41. *There exists a model P, credit limit L such that there is no absolute arbitrage but there is a relative arbitrage.* To show

DEFINITION 2.42. An equivalent (local) martingale measure is a measure $\mathbb{Q} \sim \mathbb{P}$ such that $\frac{S}{N}$ is a $\mathbb{Q}$-local martingale.

THEOREM 2.43 (FTAP1 for market with a numeraire). *Suppose $\mathbb{Q}$ is an EMM and $\frac{L}{N}$ is locally class D (with respect to $\mathbb{Q}$), then there are no L-admissible relative arbitrages.*

LEMMA 2.44. *If $X_t = \phi_t N_t + \pi_t \cdot S_t$ (i.e (ψ, π) is a self-financing pure investment strategy), then*

$$d\frac{X_t}{N_t} = \pi_t d\frac{S_t}{N_t}. \tag{2.53}$$

PROOF. Ito's lemma □

PROOF (Proof of theorem). If $\mathbb{Q}$ is an EMM, X is a $\mathbb{Q}$-local martingale, since it is the stochastic integral with respect to the Q-local martingale $\frac{S}{N}$. As $\frac{X_t+L_t}{N_t} \geq 0$, we can apply Fatau's lemma as before, obtaining

$$\mathbb{E}_Q\left(\frac{X_T}{N_T}\right) \leq \frac{X_0}{N_0}. \tag{2.54}$$

Thus, if

$$\frac{X_T}{N_T} \geq \frac{X_0}{N_0} \tag{2.55}$$

$\mathbb{P}$ a.s. then

$$\frac{X_T}{N_T} \geq \frac{X_0}{N_0} \tag{2.56}$$

$\mathbb{Q}$ a.s by equivalence of $\mathbb{P}$ and $\mathbb{Q}$.

Then $\frac{X_T}{N_T} = \frac{X_0}{N_0}$ $\mathbb{Q}$ a.s. by the pigeon hole, then $\frac{X_T}{N_T} = \frac{X_0}{N_0}$ $\mathbb{P}$ a.s, since $\mathbb{P} \sim \mathbb{Q}$. □

Fill in rest of lecture content

In the framework $P = (B, S)$, $dB_t = B_t r_t dt$, $dS_t^i = S_i(\mu_t^i dt + \sum_{j=1}^m \sigma_t^{ij} dW_t^j)$.

THEOREM 2.45. *Let λ_t be predictable and $\int_0^t \|\lambda_s\|^2 ds < 0$ a.s. $\forall t \geq 0$ and satisfying $\sigma_t \lambda_t = \mu_t = r_t$. Then $dY_t = -Y_t(r_t dt + \lambda_t dW_t)$ is a state price density and if W generates the filtration then all state price densities are of this form. λ is called a market price of risk.*

PROOF. From Itô's formula,

$$d(Y_t B_t) = -Y_t B_t \lambda_t \cdot dW_t \tag{2.57}$$

is a local martingale,

$$d(Y_t S_t^i) = Y_t S_t^i(\mu_t^i + \sum \sigma^{ij} dW^j) + Y S^i(-r dt - \sum \lambda^j dW^j) - Y S^i \sum \sigma^{ij} \lambda^j dt \tag{2.58}$$

$$d(Y S^i) = Y S^i((\sigma^{ij} - \lambda) dW + (\mu^i - r - (\sigma\lambda)^i dt)) \tag{2.59}$$

Now, if the filtration is generated by W, then all positive local martingales M are of the form (by the martingale representation theorem) $dM = -M\lambda \cdot dW$ for some predictable process λ. So if

Y is a state price density then Y is of the form $Y = \frac{M}{S}$ so $dY = -Y(rdt = \lambda dW)$. If YS^i is a local martingale for all i then $\sigma\lambda = u - r1$ in order for the dt to cancel in Itô's formula. □

If Y is a state price density such that YB is a true martingale, we can define an equivalent measure $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{Y_T B_T}{Y_0 B_0}$ for some fixed $T > 0$. This $\mathbb{Q}$ is an equivalent martingale measure.

Theorem 2.46. *Suppose $dM_t = -M_t\lambda_t \cdot dW_t$ is a true martingale where λ solves $\sigma\lambda = \mu - r1$. Fix $T > 0$ and let $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{M_T}{M_0}$. Then $\mathbb{Q}$ is an EMM and $dS_t^i = S_t^i(r_t dt + \sigma^{ij} d\hat{W}_t)$ for a $\mathbb{Q}$-Brownian motion $\hat{W}$.*

Proof. By Girsanov's theorem, $\hat{W}_t = W_t + \int_0^t \lambda_s ds$ is a $\mathbb{Q}$-Brownian motion. Now, by Itô,

$$d(\frac{S_i}{B}) = \frac{S_i}{B}((\mu^i - r)dt + \sigma^{ij} dW) \tag{2.60}$$

$$= \frac{S^i}{B}\sigma^{ij}(\lambda_t dt + dW_t) \tag{2.61}$$

$$= \frac{S^i}{B}\sigma^{ij} d\hat{W}. \tag{2.62}$$

□

Theorem 2.47. *Suppose that the filtration is generated by W. Suppose $n = d$ and that the $d \times d$ matrix $\sigma^{ij}(\omega)$ is invertible for all t, ω. Let $\lambda_t = \sigma_t^{ij}(\mu_t - r_t 1)$ and $dY_t = -Y_t(r_t dt + \lambda_t dW_t)$ is the unique state price density. Let ξ_T be a $\mathcal{F}_t$-measurable non-negative random variable such that $\xi_T Y_T$ is integrable. Then there exists a 0-admissible trading strategy H such that $X_T^H = \xi_T$ and $X_0^H = \frac{\mathbb{E}(Y_T\xi_T)}{Y_0}$.*

Furthermore, if LY is locally of class D and $\tilde{H}$ is an L-admissible strategy such that $X_T(\tilde{H}) = \xi_T$, then $X_0(\tilde{H}) \geq X_0(H)$. That is, $\frac{\mathbb{E}(Y_T\xi_T)}{\xi_0}$ is the minimal replication cost of the European claim with payout ξ_T.

Proof. Let $M_t = \mathbb{E}(Y_T\xi_T|\mathcal{F}_t)$. This is a martingale. We show that there exists H such that $X_t^H = \frac{M_t}{Y_t}$for all $0 \leq t \leq T$. By the martingale representation theorem, there exists a d-dimensional predictable process α such that

$$dM_t = \alpha_t dW_t \tag{2.63}$$

By Itô's formula,

$$d\frac{M_t}{Y_t} = \frac{M_t}{Y_t} r_t dt + (\frac{M_t\lambda_t + \sigma_t}{Y_t})(dW_t + \lambda_t dt). \tag{2.64}$$

Let $\pi_t = \mathrm{diag}(S_t)^{-1}(\sigma_t^T)^{-1}(\frac{M_t\lambda_t+\sigma_t}{Y_t})$ and

$$\phi_t = \frac{\frac{M_t}{Y_t} - \pi_t S_t}{B_t}. \tag{2.65}$$

Note that $\phi_t B_t + \pi_t S_t = \frac{M_t}{Y_t}$, and

$$\pi_t dB_t + \pi_t dS_t = \frac{M_t}{Y_t} r dt + \frac{M_t \lambda_t + \alpha}{Y_t}(dW + \lambda dt) = d(\frac{M}{Y}) \tag{2.66}$$

and so $H = (\phi, \pi)$ is a self-financing strategy. It is 0-admissible since $\frac{M_t}{Y_t} > 0$. □

THEOREM 2.48. *If $\tilde{H}$ is L-admissible and LY is in class D and $X_T(\tilde{H}) = \xi_T$ then*

$$X_0(\tilde{H}) \geq \frac{\mathbb{E}(Y_T \xi_T)}{Y_0} = X_0(H) \tag{2.67}$$

PROOF. Consider

$$- Y_t(\tilde{X}_t + L_t) \geq 0 \tag{2.68}$$

and $Y_t \tilde{X}_t$ is a local martingale.

$$\mathbb{E}(Y_{T \wedge \tau_n} L_{T \wedge L_n}) \to \mathbb{E}(Y_T L_T) \tag{2.69}$$

by uniform integrability assumption. Therefore $Y\tilde{X}$ is a supermartingale by Fatau's lemma, and thus

$$\mathbb{E}(Y_T \xi_T) = \mathbb{E}\Big(Y_T \tilde{X}_T\Big) \leq Y_0 \tilde{X}_0 \tag{2.70}$$

□

EXAMPLE 2.49. *A market model with no absolute arbitrage but with a relative arbitrage.*

Consider $P = (1, S)$, where $dS_t = S_t \sigma_t dW_t$, $n = d = 1$, $\sigma_t > 0$ for all t. On the filtration generated by W and S is a strictly local martingale, $\mathbb{E}(S_T) < S_0$ (recall that all positive local martingales are supermartingales) which implies $\mathbb{E}(\max_{0 \leq t \leq T} S_t) = \infty$.

DEFINITION 2.50. Let $Y_t = 1$ for all t be a state price density. If L is of class D locally, there exist L-admissible absolute arbitrages.

DEFINITION 2.51. Let $\mathbb{Q} = \mathbb{P}$. This is an EMM for the cash numeraire. If L is of class D locally, there are no relative arbitrages.

DEFINITION 2.52. By existential replication theorem, there exists H such that $X_T(H) = S_T$. Notice that $X_0(H) = \mathbb{E}(X_T) < S_0$ (!)

Note that $\frac{X_T}{S_T} = 1$ a.s. but $\frac{X_0}{S_0} = p < 1$ (so we have a relative arbitrage). Let $\tilde{H} = H - p \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

$$X_0(\tilde{H}) = \mathbb{E}(S_T) - pS_0 = 0 \tag{2.71}$$

$$X_T(\tilde{H}) = S_T - pS_T > 0 \tag{2.72}$$

$X_t(\tilde{H})$ is **not** of class D. So only admissible if L is wild.

CHAPTER 3

Black-Scholes

Consider the market model

$$dB_t = B_t r dt \tag{3.1}$$

$$dS_t = S_t(\mu dt + \sigma dW_t) \tag{3.2}$$

Then $B_t = B_0 e^{rt}$, $S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$, and $Y_t = e^{-(r-\lambda^2 2)t - \lambda W_t}$ is the unique state price density with $Y_0 = 1$, where $\lambda = \frac{\mu - r}{\sigma}$.

Our goal is to replicate a European claim with payout $\xi_T = g(S_T)$ where $g \geq 0$ and suitably integrable. By our replication theorem, there exists a 0-admissible strategy H such that $X_t(H) = \frac{1}{Y_t}\mathbb{E}(Y_T g(S_T)|\mathcal{F}_t)$.

Let $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\frac{\lambda^2 T}{2} - \lambda W_T}$ be the unique EMM. By the Cameron-Martin-Girsanov theorem, $\hat{W}_t = W_t + \lambda t$ is a $\mathbb{Q}$-Brownian motion. Then

$$S_T = S_t e^{(\mu - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)} \tag{3.3}$$

$$= S_t e^{(-r-\sigma^2 2)(T-t) + \sigma(\hat{W}_T - \hat{W}_t)} \tag{3.4}$$

and we have

$$X_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(g(S_T)|\mathcal{F}_t) \tag{3.5}$$

$$= \int g(S_t e^{(r-\frac{\sigma^2}{2})(T-t) + \sigma\sqrt{T-t}Z}) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \tag{3.6}$$

Substituting in $g(x) = (x-K)^+$ corresponding to a call option, we obtain the price

$$C_t(T,K) = S_t \Phi(\frac{-\log\frac{K}{S_t}}{\sigma\sqrt{T-t}} + (\frac{r}{\sigma} + \frac{\sigma}{2})\sqrt{T-t})$$
$$- Ke^{-r(T-t)}\Phi(\frac{-\log\frac{K}{S_t}}{\sigma\sqrt{T-t}} + (\frac{r}{\sigma} - \frac{\sigma}{2})\sqrt{T-t}) \tag{3.7}$$

Fill in missing lecture — Black-Scholes price as a solution to BS PDE

1. Black-Scholes Volatility

Assume we observe $(S_t)_{-T\le t\le 0}$ at some discrete intervals $(\frac{t}{n}-1)T$ for $i=0,\dots,n$, with

$$Y_i = \log\frac{S_{t_i}}{S_{t_{i-1}}} \tag{3.8}$$

$$= (\mu - \frac{\sigma^2}{2})(t_i - t_{i-1}) + \sigma(W_{t_i} - W_{t_{i-1}}) \tag{3.9}$$

$$\sim N(a\frac{T}{n}, \frac{\sigma^2 T}{n}). \tag{3.10}$$

The MLE is then

$$\hat{a} = \frac{1}{T}\sum_{i=1}^{n} Y_i = \frac{1}{T}\log\frac{S_0}{S_{-T}} \tag{3.11}$$

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{i=1}^{n}(Y_i - \frac{\hat{a}T}{n}) \tag{3.12}$$

and $\mathbb{V}(\hat{\sigma}^2) = \frac{2\sigma^4}{n} \to 0$ as $n\to\infty$.

2. Calibration

Black-Scholes model prediction, a call price

$$C_t(T,K) = C^{BS}(t,T,K,S_t,r,\sigma). \tag{3.13}$$

The Black-Scholes implied volatility for strike K, maturity T at time t is the unique σ which solves (3.13), denoted $\sum_t(T,K)$.

Black-Scholes predicts there is a unique number σ such that $\sum_t(T,K)=\sigma$ for all t,T,K. This fails in most markets.

3. Robustness

Consider a payout of claim $g(S_T)$. Assume we believe in Black-Scholes, and so we believe the price

$$V(0,S,\sigma) \tag{3.14}$$

where

$$V(t,S,\sigma) = e^{-r(T-t)}\int g(Se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}z})\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz \tag{3.15}$$

for some σ. Pick $\hat{\sigma}$ to solve $V(0,S_0,\hat{\sigma})=\xi_0$, the initial price of the claim.

Now, try to replicate the claim with portfolio (ϕ, π) with

$$\pi_t = \frac{\partial V}{\partial S}(t, S, \hat{\sigma}) \tag{3.16}$$

$$\phi_t = \frac{X_t - \pi_t S_t}{B_t} \tag{3.17}$$

Notice the equation

$$X_0 = V(0, S_0, \hat{\sigma}) \tag{3.18}$$

$$dX_t = r(X_t - \pi_t S_t)dt + \pi_t ds \tag{3.19}$$

has a unique solution given by

$$X_t = X_0 e^{rt} + e^{rt} \int_0^t \pi_s d(e^{-rs} S_s) \tag{3.20}$$

so given π, we can solve for X.

In the real model,

$$dB_t = rB_t dt \tag{3.21}$$

$$dS_t = S_t(\mu dt + \sigma_t dW_t) \tag{3.22}$$

for r, μ constant but σ_t a stochastic process.

Then

$$dV(t, S_t, \hat{\sigma}) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}d\langle S\rangle \tag{3.23}$$

$$= (\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial^2 S}\sigma_t^2 S_t^2)dt + \pi_t dS_t \tag{3.24}$$

$$= (rV - rS\frac{\partial V}{\partial S} - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2\hat{\sigma}^2 + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_t^2 S_t^2)dt + \pi_t dS_t \tag{3.25}$$

and so

$$d(X_t - V(t, S_t, \hat{\sigma})) = r(X - V)dt + \frac{1}{2}S^2(\hat{\sigma}^2 - \sigma_t^2)\frac{\partial^2 V}{\partial S^2}dt \tag{3.26}$$

and so

$$X_T - V(T, S_T, \hat{\sigma}) - X_0 + V(0, S_0, \hat{\sigma}) = X_T - g(S_T) \tag{3.27}$$

$$= \frac{1}{2}\int_0^T e^{-r(T-s)} S_s^2(\hat{\sigma}^2 - \sigma_s^2)\frac{\partial^2 V}{\partial S^2}ds \tag{3.28}$$

and so we can estimate the difference between the option and the replicating portfolio by a weighted average of the gamma multiplied by the difference in implied and realized volatility over the time period.

CHAPTER 4

Local Volatility Models

Consider

$$dB_t = rB_t dt \tag{4.1}$$

$$\begin{aligned} dS_t &= S_t(\mu(t, S_t)dt + \sigma(t, S_t)dW_t) && (4.2)\\ &= S_t(rdt + \sigma(t, S_t)d\hat{W}_t) && (4.3)\end{aligned}$$

with $d\hat{W}_t = dW_t + \frac{\mu(t,S_t)-r}{\sigma(t,S_t)}dt$ is a Brownian motion under the equivalent martingale measure $\mathbb{Q}$.

THEOREM 4.1 (Dupire). *Suppose* $C_0(T,K) = \mathbb{E}_{\mathbb{Q}}\big(e^{-rT}(S_T - K)^+\big)$. *Then*

$$\frac{\partial C_0}{\partial T} + rK\frac{\partial C_0}{\partial K} = \frac{\sigma(T,K)^2}{2}K^2\frac{\partial^2 C_0}{\partial K^2} \tag{4.4}$$

with $C_0(0,K) = (S_0 - K)^+$ *with*

$$\sigma(T,K) = \sqrt{\frac{2(\frac{\partial C_0}{\partial T} + rK\frac{\partial C_0}{\partial K})}{K^2\frac{\partial^2 C}{\partial K^2}}} \tag{4.5}$$

EXERCISE 4.2. *If*

$$C_0(T,K) = C^{BS}(t = 0, \sigma, T, S_0, K, r, \sigma_0) \tag{4.6}$$

show that

$$\sigma(T,K) = \sigma_0 \tag{4.7}$$

for all T, K.

LEMMA 4.3 (Breden-Litzenberger, 1978). *Suppose* S_T *has density* f *(under* $\mathbb{Q}$*). Then*

$$\begin{aligned} C_0(T,K) &= e^{-rT}\int_K^\infty f_{S_T}(y)(y-K)dy && (4.8)\\ \frac{\partial C_0}{\partial K} &= -e^{-rT}\int_K^\infty f_{S_T}(y)dy && (4.9)\\ \frac{\partial^2 C_0}{\partial K^2} &= e^{-rT}f_{S_T}(K) && (4.10)\end{aligned}$$

PROOF (Proof of Theorem 4.1). By Itô's formula,

$$
\begin{aligned}
(S_T - K^+) &= (S_0 - K)^+ + \int_0^T \mathbb{I}(S_t \geq K)\, dS_t + \frac{1}{2}\int_0^T \delta_K d\langle S\rangle && (4.11)\\
&= (S_0 - K)^+ + \int_0^T S_t r \mathbb{I}(S_t \geq K) + \frac{1}{2} S_t^2 \sigma(t, S_t)^2 \delta_K(S_t) dt + \int_0^T S_t \sigma(t, S_t)\mathbb{I}(S_t \geq K)\, d\hat{W}_t. && (4.12)
\end{aligned}
$$

Taking $\mathbb{E}^{\mathbb{Q}}$ on both sides, we obtain

$$
e^{rT} C_0(T, K) = (S_0 - K)^+ + \int_0^T \left(\int_K^\infty f_{S_t}(y) yr dy \right) dt + \frac{1}{2}\int_0^T f_{S_t}(K) K^2 \sigma(t, K)^2 dt \quad (4.13)
$$

which gives

$$
e^{rT} \frac{\partial C_0}{\partial T} + r e^{rT} C_0 = \int_K^\infty f_{S_T}(y) yr dy + \frac{1}{2} f_{S_T}(K) K^2 \sigma(T, K)^2 \quad (4.14)
$$

Writing $y = (y - K) + K$ and applying the previous lemma, we obtain the required result. □

REMARK 4.4. *Given a call surface $\{C_0(T, K), T, K > 0\}$ where $C_0(T, \cdot)$ is smooth, we find the density of S_T by*

$$
\frac{\partial^2 C_0}{\partial K^2} = e^{-rT} f_{S_T}(K) \quad (4.15)
$$

and hence

$$
\mathbb{E}^{\mathbb{Q}}(e^{-rT} g(S_T)) = \int_0^\infty g(y) \frac{\partial^2 C_0}{\partial K^2}(T, y) dy \quad (4.16)
$$

If g is convex and smooth, then

$$
\begin{aligned}
g(S_T) &= g(a) + g'(a)(S - a) + \int_0^a g''(K)(K)(K - S_T)^+ dK + \int_a^\infty g''(K)(S_T - K)^+ dK && (4.17)\\
&= \sum_{K_i \leq a} g''(K_i)(K_i - S_T)^+ \Delta K_i + \sum_{K_i \geq a} g''(K_i)(S_T - K_i)\Delta K_i && (4.18)
\end{aligned}
$$

1. Computing Moment Generating Functions

Consider a model with $B_t = B_0 e^{rT}$, S positive such that $(e^{-rT} S_t)_{t \geq 0}$ is a $\mathbb{Q}$-martingale. Consider

$$
\Theta = \{p + qi | 0 \leq p \leq i, q \in \mathbb{R}\} \subseteq \mathbb{C} \quad (4.19)
$$

with $i = \sqrt{-1}$.

Let $M_t(\theta) = \mathbb{E}^{\mathbb{Q}} e^{\theta \log S_t}$ be the moment generating function of $\log S_t$, with $\theta = p + iq$, $0 \le p \le 1$, and so

$$\mathbb{E}^{\mathbb{Q}} |e^{\theta \log S_t}| = \mathbb{E}^{\mathbb{Q}}(S_t^p) \le (\mathbb{E}^{\mathbb{Q}} S_t)^p = (e^{rt} S_0)^p < \infty \tag{4.20}$$

and so $M_t(\theta)$ is well defined for $\theta \in \Theta$.

Theorem 4.5.

$$\mathbb{E}^{\mathbb{Q}}(e^{-rT}(S_T - K)^+) = S_0 - \frac{e^{-rT} K^{1-p}}{2\pi} \int_{-\infty}^{\infty} \frac{M_T(p+ix) e^{-ix \log K}}{(x - ip)(x + i(1-p))} dx \tag{4.21}$$

for all $0 < p < 1$.

Theorem 4.6.

$$C_0(T,K) = S_0 \frac{e^{-rT} K^{1-p}}{2} \pi \int_{-\infty}^{\infty} \frac{M_T(p+ix) e^{-ix \log K}}{(x - ip)(x + i(1-p))} dx \tag{4.22}$$

Lemma 4.7.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iax}}{x - ip} x + i(1-p) = \begin{cases} e^{-ap} & a \ge 0 \\ a^{a(1-p)} & a < 0 \end{cases} \tag{4.23}$$

which can be shown via contour integration.

Let γ_R *be the semi-circle of radius* R *above the* x*-axis in the complex plane. Then*

$$\int_{\gamma_R} \frac{e^{iax}}{(x - ip)(x + i(1-p))} dx = 2\pi \operatorname*{Res}_{x = ip} = 2\pi e^{-ap}. \tag{4.24}$$

and we have

$$\int_{-R}^{R} + \int_{\phi=0}^{\pi} \frac{e^{ia(R\cos\phi + i\sin\phi)}}{(Re^{i\phi} - ip)(Re^{i\phi} + i(1-p))} d\phi \le \frac{e^{-aR\sin\phi}}{\frac{1}{2}R} \to 0 \tag{4.25}$$

and so we obtain our required result.

Proof (Proof of 4.6). We have

$$e^{-rT}(S_T - K)^+ = e^{-rT} S_T - \frac{K^{1-p} e^{-rT}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{p \log S_T + ix \log S_T - ix \log K}}{(x - ip)(x + i(1-p))} dx \tag{4.26}$$

Now computing $\mathbb{E}^{\mathbb{Q}}$, using Fubini's theorem to justify the interchange as

$$\mathbb{E}\left(\int \left| \frac{e^{(p+ix)\log S_T - ix \log K}}{(x - ip)(x + i(1-p))} \right| dx \right) = M_T(p) \int \frac{1}{\sqrt{(x^2 + p^2)(x^2 + (1-p)^2)}} < \infty \tag{4.27}$$

□

REMARK 4.8. *By Holder's inequality, $p \mapsto \log M_T(p) = \Lambda_T(p)$ is convex. $\Lambda_T(0) = 0, \Lambda_T(1) = \log S_0 + rT$, and $p \mapsto \Lambda_T(p)$ is smooth. It has a minimal point $p = p^\star \in (0,1)$ at*

$$\Lambda_T(p^\star + ix) \approx \Lambda_T(p^\star) + \Lambda_T^{'}(p^\star)(ix) + \frac{1}{2}\underbrace{\Lambda^{''}}_{\geq 0 \text{ by convexity}}(p^\star)(ix)^2 \tag{4.28}$$

$$= ... \tag{4.29}$$

by Taylor's theorem.

Then

$$\int \frac{M_T(p^\star + ix)e^{-ix\log K}}{(x - ip)(x + i(1-p))} \approx M_T(p^\star)\int \frac{e^{-\Lambda_T^{''}(p^\star)x^2}}{p(1-p)}dx \tag{4.30}$$

$$= \frac{M_T(p^\star)}{p(1-p)}\sqrt{\frac{2\pi}{\Lambda_T^{''}(p^\star)}} \tag{4.31}$$

2. The Heston Model

$$dB_t = B_t r dt \tag{4.32}$$

$$dS_t = S_t(rdt + \sqrt{v_t dW_t^S}) \tag{4.33}$$

$$dv_t = \lambda(\overline{v} - v_t)dt + c\sqrt{v_t}dW_t^V \tag{4.34}$$

W^S, W^v are Brownian motions under some EMM $\mathbb{Q}$, with correlation ρ. For instance, $W_t^v = \rho W_t^s + \sqrt{1-\rho^2}d_t^\perp$ with $W^s, W^\perp$ independent.

$\overline{v} > 0$ is the mean-reversion level. $\lambda > 0$ is the mean reversion rate. We have $v_t \geq 0$ almost surely [Cox et al., 1985].

Our goal is fix $T > 0, \theta \in \Theta$, want to compute $\mathbb{E}\big(e^{\theta \log S_T}\big)$.

Idea: Let $(V(t, S_t, v_t))_{0 \leq t \leq T}$ be chosen so that it is a martingale with $V(T, S_T, V_T) = e^{\theta \log S_T}$. The moment generating function is then $V(t = 0, S_0, v_0)$.

By Itô,

$$dV(t, S_{t,v_t}) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}d\langle S\rangle + \frac{\partial V}{\partial v}dv + \frac{1}{2}\frac{\partial^2}{\partial v^2}d\langle v\rangle + \frac{\partial^2 V}{\partial v \partial s}d\langle S, v\rangle. \tag{4.35}$$

We seek to make the dt terms vanish. Thus,

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}rS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2 v + \frac{\partial V}{\partial v}\lambda(\overline{v} - v) + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}c^2 v + \frac{\partial^2 V}{\partial S \partial v}\rho S v c = 0. \tag{4.36}$$

The inspired idea is to look for solutions of the form

$$V(t, S, v) = e^{\theta \log S + R(T-t)v + Q(T-t)} \tag{4.37}$$

with $R(0) = Q(0) = 0$.

Substituting this functional form in, we obtain

$$R'v - Q' + r\theta + \frac{1}{2}\theta(\theta-1)v + R\lambda(\overline{v} - v) + \frac{1}{2}R^2c^2v + \theta R\rho vc = 0 \tag{4.38}$$

Collecting terms, we have

$$\begin{cases} R' = \frac{1}{2}\theta(\theta-1) + \frac{1}{2}R^2c^2 + (\theta pc - \lambda)R \\ Q' = r\theta = R\lambda\overline{v} \end{cases} \tag{4.39}$$

which are Riccati equations, which have an explicit solution.

3. American Options (Guest Lecture)

Suppose we have some assets d and our bank account B_t. The random assets evolve as

$$dS_t^i S_t^i(\mu_t^i dt + \sum_{j=1}^{d} \sigma_{ij}(t, S_t)dW_t^j) \tag{4.40}$$

The option we want to price pays $g(S_\tau)$ if exercised at time τ. The exercise time τ must be a stopping time, with $\tau \le T$, the expiration time.

For technical reasons, suppose g is bounded. For examples sake, we assume we have one sock, and consider an American put $g(S) = (K - S)^+$.

If there are d assets, we might have a min-put, we have

$$g(S) = (K - \min_{1\le i\le d} S^i)^+ = \max_{1\le i\le d}(K - S^i)^+ \tag{4.41}$$

To solve this pricing problem, write

$$\mathcal{L}f = \frac{1}{2}\sum_{i,j} S_iS_ja_{ij}(t,S)\frac{\partial^2 f}{\partial S_i \partial S_j} + \sum_i rS_i\frac{\partial f}{\partial S_i} - rf + \frac{\partial f}{\partial t} \tag{4.42}$$

where $a = \sigma\sigma^T$, and suppose we can find some $V(t,S) \in C^{1,2}$ such that

$$\max\{\mathcal{L}V, g - V\} = 0, V(T,\cdot) = g(\cdot). \tag{4.43}$$

Then

$$V(0, S_0) = \sup_{\tau \le T} \mathbb{E}\big(e^{-r\tau}g(S_\tau)|S_0\big) \tag{4.44}$$

Why is this true? Consider

$$d(V(t,S_t)e^{-rt}) = V_s(t,S_t)S_t\sigma_t dW_t + \mathcal{L}V(t,S_t)dt \tag{4.45}$$

If we let τ be any stopping time $\leq T$, and we let $T \uparrow \infty$ be a sequence of stopping times "rediscovering" the local martingale $V_S(t,S)S\sigma dW$, and we shall then have

$$V(0,S_0) = \mathbb{E}\bigg(e^{-r\tau_n}V(\tau_n, S_{\tau_n}) - \int_0^{\tau_n} \mathcal{L}V(u,S_u)du\bigg) \tag{4.46}$$

$$\geq \mathbb{E}\big(e^{-r\tau_n}V(\tau_n, S_{\tau_n})\big) \tag{4.47}$$

$$\geq \mathbb{E}\big(e^{-r\tau_n}g(S_{\tau_n})\big)\,. \tag{4.48}$$

since $\mathcal{L}V \leq 0$.

If we let $n \to \infty, \tau_n \uparrow \tau$, we must have that

$$V(0,S_0) \geq \sup_{0\leq\tau\leq T} \mathbb{E}\big(e^{-r\tau}g(S_\tau)\big)\,. \tag{4.49}$$

To show that there is equality, consider

$$\tau^\star = \inf\{t|V(t,S_t) = g(S_t)\} \tag{4.50}$$

We know that $V(T,\cdot) = g(\cdot)$, and so $\tau^\star \leq T$. We also notice that in $[0,\tau)$, $\mathcal{L}V = 0$ because in $[0,\tau)$, $g - V < 0$, and $\max\{\mathcal{L}V, g-V\} = 0$. Now going back to the first calculation, if we write $\tau_n^\star = \tau^\star \wedge T_n$.

$$V(0,S_0) = \mathbb{E}\bigg(e^{-r\tau_n^\star}V(\tau_n^\star, S_{\tau_n^\star}) - \int_0^{\tau_n^\star} \mathcal{L}V(u,S_u)du\bigg) \tag{4.51}$$

$$= \mathbb{E}\big(e^{-r\tau_n}V(\tau_n, S_{\tau_n})\big) \tag{4.52}$$

$$= \mathbb{E}\Big(e^{-r\tau^\star}V(\tau^\star, S_{\tau^\star}) : \tau^\star \leq T_n\Big) + \mathbb{E}\big(e^{-rT_n}V(T_n, S_{T_n}) : \tau^\star > T_n\big) \tag{4.53}$$

$$= \mathbb{E}\Big(e^{-r\tau^\star}g(S_{\tau^\star})|\tau^\star \leq T_n\Big) + \mathbb{E}\big(e^{-rT_n}V(T_n, S_{T_n}) : \tau^\star > T_n\big) \tag{4.54}$$

$$\to \mathbb{E}\Big(e^{-r\tau^\star}g(S_{\tau^\star})\Big)\,. \tag{4.55}$$

n We need to show that the V we found is bounded.

Example 4.9. *American puts in one dimension.*
We have an envelope V.
We find V by solving

$$0 = -rV = \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_s \tag{4.56}$$

for $S = q$ with boundary condition

$$V(q) = (K-q)^+ \tag{4.57}$$

This we can write as

$$V(S) = AS + BS^{-2r/\sigma^2} \tag{4.58}$$

with the boundary condition $V(q) = (K-q)^+$.

Suppose we let q *be a parameter of the stopping rule, work out the value and optimize over* q. *The value is*

$$V(S) = (K-q)(\frac{S}{q})^{-\frac{2r}{\sigma^2}} = S^{-\frac{2r}{\sigma^2}} q^{\frac{2r}{\sigma^2}} (K-q) \tag{4.59}$$

Optimizing over q, *we have*

$$\frac{2r}{\sigma^2 q} = \frac{1}{K-q} \Rightarrow q = \frac{2rk}{\sigma^2 + 2r}. \tag{4.60}$$

We can check, if we use this value of q, *then* $V'(q) = -1 = \frac{\partial}{\partial S}(K-S)|_{s=q}$.

It can be shown that $\sup_{0\le\tau\le T} \mathbb{E}(e^{-r\tau} g(S_\tau)) \le \min_{M\in\mathcal{M}_0} \mathbb{E}(\sup_{\dots})$

Fill in from lecture notes.?

CHAPTER 5

Bond Markets and Interest Rates

DEFINITION 5.1. A zero coupon bond is a contingent claim that pays exactly one unit of money at maturity.

We assume that ξ_T, the payment of the bond, is 1 a.s. - that is, there is no credit risk.

DEFINITION 5.2. $P(t,T)$is the price at time t for a bond maturing at time T.

DEFINITION 5.3. The yield $y(t,T)$ is defined as

$$y(t,T) = -\frac{1}{T-t}\log P(t,T) \tag{5.1}$$

or equivalently

$$P(t,T) = e^{-(T-t)y(t,T)} \tag{5.2}$$

DEFINITION 5.4. We call $\lim_{T\downarrow t} y(t,T) = r_t$ the "spot" or "short" rate.

We call $\lim_{T\uparrow\infty} y(t,T)$ if it exists.

DEFINITION 5.5. The forward rate $f(t,T)$ is defined

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T) \tag{5.3}$$

or equivalently

$$P(t,T) = -\int_t^T f(t,u)du \tag{5.4}$$

THEOREM 5.6. *There is no arbitrage in the market prices* $(P(t,T_1), P(t,T_2), \dots, P(t,T_n))$ *if* $Y_t P(t,T)_{t\in[0,T]}$ *is a local martingale for all* T*, where* Y *is a state price density.*[1]

In particular, there is no arbitrage if $P(t,T) = \frac{1}{Y_t}\mathbb{E}(Y_T|\mathcal{F}_t)$

Introduce the bank account $dB_t = B_t r_t dt \iff B_t = B_0 e^{\int_0^t r_s ds}$ where r is the short rate. Define an equivalent martingale measure with density $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{B_T Y_T}{B_0 Y_0}$. Rewrite

$$P(t,T) = B_t \mathbb{E}_{\mathbb{Q}}\left(\frac{1}{B_T}|\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}|\mathcal{F}_t\right) \tag{5.5}$$

[1]Recall relative arbitrage, admissible class D, etc.

By the law of one price,

$$f(t,T) = -\frac{\partial}{\partial T}\log \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}|\mathcal{F}_t\right) \tag{5.6}$$

$$= \frac{\mathbb{E}_{\mathbb{Q}}\left(r_T e^{-\int_t^T r_s ds}|\mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}|\Phi_t\right)}, \tag{5.7}$$

and so $f(t,T)$ can be seen as the "market weighted conditional expectation of r_T given at $\mathcal{F}_t$."

Alternatively, we have

$$\mathbb{E}_{\mathbb{Q}}\Big((f(t,T) - r_T)e^{-\int_t^T r_s ds}|\mathcal{F}_t\Big) = 0 \tag{5.8}$$

and so the forward rate is such that the claim with payout $f(t,T) - r_T$ has price 0 at time T.

Fill in missing lec-
ture from Monday
2 December

There are two approaches to bond market pricing:

(i) Let $(r_t)_{t\geq 0}$ be fundamental, derive everything else: $f(t,T)$, etc.
(ii) Model $(f(t,T))_{0\leq t\leq T}$ directly - the Heath et al. [1992] approach.

1. The Heath et al. [1992] Model

THEOREM 5.7. *Suppose* $df(t,T) = a(t,T)dt + \sigma(t,T)\cdot d\hat{W}_t$ *for a d-dimensional Brownian motion* $\hat{W}$ *where* $\sigma(t,T)$ *is suitably measurable and integrable, and*

$$a(t,T) = \sigma(t,T)\cdot\int_t^T \sigma(t,u)du \tag{5.9}$$

Define $r_t = f(t,t)$ *and* $P(t,T) = e^{-\int_t^T f(t,u)du}$. *Then*

$$\left(e^{-\int_0^t r_s ds}P(t,T)\right)_{0\leq t\leq T} \tag{5.10}$$

is a local martingale.

REMARK 5.8.

$$f(t,T) = f(0,T) + \int_0^t a(s,T)ds + \int_0^t \sigma(s,T)\cdot d\hat{W}_s. \tag{5.11}$$

PROOF. Recall that if $d\log M_t = -\frac{|b_t|^2}{2}dt + b_t\cdot d\hat{W}_t$, then M is a local martingale if and only if $M_t = M_0 e^{-\frac{1}{[}2]\int_0^t |b_s|^2 ds + \int_0^t b_s\cdot d\hat{W}_s}$.

By differentiation, we have

$$d\left(-\int_0^t r_s ds - \int_t^T f(t,u)du\right) = -r_t dt + f(t,t)dt - \int_t^T df(t,u)du \tag{5.12}$$

$$= -(\int_t^T a(t,u)du)dt - (\int_t^T \sigma(t,u)du)\cdot d\hat{W}_t. \tag{5.13}$$

noting that

$$\int_t^T a(t,u)du = \frac{1}{2}\| \int_t^T \sigma(t,u)du\|^2 \tag{5.14}$$

gives the required result. □

EXAMPLE 5.9 (Ho and Lee [1986]). *Assume $d = 1$, $\sigma(t,T) = \sigma_0$ constant. Then*

$$df(t,T) = ((T-t)\sigma_0^2)dt + \sigma_0 d\hat{W}_t \tag{5.15}$$

$$f(t,T) = f(0,T) + \int_0^t (T-s)\sigma_0^2 ds + \sigma_0 d\hat{W}_t \tag{5.16}$$

$$r_t = f(0,t) + \frac{1}{2}\sigma_0^2 t^2 + \sigma_0 \hat{W}_t \tag{5.17}$$

EXAMPLE 5.10 (Hull and White [1990]). *Again, assume $d = 1$, $\sigma(t,T) = \sigma_0 e^{-\lambda(T-t)}$.*

$$df(t,T) = \sigma_0^2 e^{-\lambda(T-t)}(1 - e^{-\lambda(T-t)})dt + \sigma_0 e^{-\lambda(T-t)} d\hat{W}_t \tag{5.18}$$

$$dr_t = \lambda\left(\frac{f_0'(t)}{\lambda} + f_0(t) + \frac{\sigma_0^2}{2\lambda^2}(1 - e^{-\lambda t}) - r_t\right) + \sigma_0 d\hat{W}_t. \tag{5.19}$$

EXAMPLE 5.11 (Kennedy [1997]). *This is a Gaussian random field model. Suppose $\sigma(t,T)$ is not random, so*

$$f(t,T) = f(0,T) + \int_0^T a(s,T)ds + \int_0^t \sigma(s,T)d\hat{W}_s \tag{5.20}$$

is Gaussian. Then

$$\mathbb{E}_{\mathbb{Q}}(f(t,T)) = f(0,T) + \int_0^t a(s,T)ds \tag{5.21}$$

$$Cov(f(s,S), f(t,T)) = \int_0^{s\wedge t} \sigma(u,S)\cdot\sigma(u,T)du \tag{5.22}$$

Turning this around, we can model

$$(f(t,T))_{0\le t\le T} \tag{5.23}$$

as a Gaussian random field with

$$Cov(f(s,S), f(t,T)) = c_{s\wedge t}(S,T) \tag{5.24}$$

$$\mathbb{E}(f(t,T)) = f(0,T) + \int_0^T c_{s\wedge t}(s,T)ds, \tag{5.25}$$

and thus there is no need to introduce a Brownian motion. For instance,

$$d\langle f(t,S), f(t,T)\rangle = \sigma(t,S)\cdot\sigma(t,T)dt \tag{5.26}$$

$$= \sigma_0 e^{-\beta|T-S|} \tag{5.27}$$

and so we have an exponentially decaying correlation between forward rates of different maturities.

EXAMPLE 5.12. *The HJM equation*

$$df(t,T) = a(t,T)dt + \sigma(t,T)dW_t \tag{5.28}$$

$$T = t + x, f_t(x) = f(t, t+x) \tag{5.29}$$

$$df_t(x) = (\frac{\partial f}{\partial x} + a_t(x))dt + \sigma_t(x)dW_t \tag{5.30}$$

Fix a separable Hilbert space $F = \{f : \mathbb{R}_+ \to \mathbb{R}\}$. Then (dropping the x),

$$df_t = (Af_t + \alpha_t)\, dt + \sigma_t dW_t \tag{5.31}$$

can be interpreted as an evolution equation in this function space. In the simplest case, σ_t is a constant vector $F \otimes \mathbb{R}^d$, α_t is a constant vector in F, then $(f_t)_{t\geq 0}$ is an F-valued Ornstein-Uhlenbeck process.

We can apply techniques from statistics (e.g. PCA) if this model has an invariant measure — shown in early 2000's.

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