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ADVANCED PROBABILITY EXAMPLES

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Contents

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Example Sheet 1

1.1 Conditional Expectation

(i) For $c \in \mathbb{R}$, let

$$\begin{aligned} A_c &= \{Y \leq c < X\} \\ B_c &= \{c < Y, c < X\} \\ C_c &= \{Y \leq c, X \leq c\} \end{aligned}$$

Then

$$\begin{aligned} A_c \cup B_c &= \{Y \leq c, X > C\} \cup \{c < X, c < Y\} = \{X > c\} \\ A_c \cup C_c &= \{Y \leq c, c < X\} \cup \{Y \leq c, c \geq X\} = \{Y \leq c\} \end{aligned}$$

and these unions are clearly disjoint.

Note that $\mathbb{E}((X - Y)\mathbb{I}(A_c)) \geq 0$ as $(X > Y)$ on A_c. Then, we have

$$\begin{aligned} 0 &= \mathbb{E}((X - Y)\mathbb{I}(A_c)) + \mathbb{E}((X - Y)\mathbb{I}(B_c)) && (1.1) \\ &= \mathbb{E}((X - Y)\mathbb{I}(A_c)) + \mathbb{E}((X - Y)\mathbb{I}(C_c)) && (1.2) \end{aligned}$$

Summing these equalities, we obtain

$$\mathbb{E}((X - Y)\mathbb{I}(B_c \cup C_c)) \leq 0 \qquad (1.3)$$

and by symmetry (as $B_c \cup C_c$ is symmetric with respect to X, Y), we have

$$\mathbb{E}((X - Y)\mathbb{I}(B_c \cup C_c)) = 0 \qquad (1.4)$$

and by (1.1), we have

$$\mathbb{E}((X - Y)\mathbb{I}(A_c)) = 0 \tag{1.5}$$

However, $X > Y$ on A_c, and thus by the pigeonhole principle, we must have A_c has measure zero. Taking the the countable union over all rational c, we obtain that $\mathbb{P}(X > Y) = 0$. By symmetry, we have $\mathbb{P}(X < Y) = 0$. This completes the proof.

(ii) Note that $Z \in \{-2, 0, 2\}$. We have

$$\mathbb{E}(X|Z = z) = \begin{cases} 1 & z = 2 \\ B(0, p) & z = 0 \\ -1 & z = -1 \end{cases} \tag{1.6}$$

(iii) As all distribution functions are continuous on $[0, \infty)$, by direct calculation, we have for $z \geq 0$,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(y) f_Y(z-y) dy && (1.7) \\ &= \int_0^z \theta e^{-\theta y} \theta e^{-\theta(z-y)} dy && (1.8) \\ &= \theta^2 \int_0^z e^{-\theta z} dy && (1.9) \\ &= \theta^2 z e^{-\theta z} && (1.10) \end{aligned}$$

and 0 if $z < 0$ as required.

For the second half, we have for $c \in \mathbb{R}$

$$\begin{aligned}
\mathbb{P}(X \leq c) &= \mathbb{E}(\mathbb{I}(X \leq c)) && (1.11)\\
&= \mathbb{E}(\mathbb{E}(\mathbb{I}(X \leq c)\,|Z)) && (1.12)\\
&= \mathbb{E}\left(\frac{1}{Z}\int_0^Z \mathbb{I}(u \leq c)\,du\right) && (1.13)\\
&= \int_0^\infty \theta^2 z e^{-\theta z}\int_0^z \mathbb{I}(u \leq c)\,du dz && (1.14)\\
&= \int_0^\infty \int_u^\infty \theta^2 e^{-\theta z} dz \mathbb{I}(u \leq c)\,du && (1.15)\\
&= \int_0^c [-\theta e^{-\theta z}]_u^\infty du && (1.16)\\
&= \int_0^c \theta e^{-\theta u} du && (1.17)\\
&= 1 - e^{-\theta c} && (1.18)
\end{aligned}$$

for $c \geq 0$, and 0 for $c < 0$. By uniqueness, we have the X is exponentially distributed with parameter θ.

Similarly, we have that $\mathbb{E}(Z - X|Z)$ is uniformly distributed on $[0, Z]$. An identical calculation gives that $Z - X$ is exponentially distributed with parameter θ.

We have

$$f_{X|Z}(x,z) = \frac{1}{z}\mathbb{I}(0 \leq x \leq z) = \frac{f_{X,Z}(x,z)}{f_Z(z)} = \theta^2 e^{-\theta z}$$

for $z, x > 0$ and zero elsewhere.

We then have[1]

[1] This is kind of dubious?

$$\begin{aligned}
f_X(x) f_{Z-X}(z-x) &= \left(\theta e^{-\theta x}\right)\left(\theta e^{-\theta(z-x)}\right) && (1.19)\\
&= \theta^2 e^{-\theta z} && (1.20)\\
&= f_{X,(Z-X)} f_{X,(Z-X)}(x, z-x) && (1.21)
\end{aligned}$$

and thus the random variables X and $Z - X$ are independent.

(iv) (i) Assume the proposition is false. Let $Y = \mathbb{E}(X|\mathcal{G})$. Then there exists $A \in \mathcal{G}$ with positive measure such that $\mathbb{E}(X\mathbb{I}(A)) \leq 0$. Then

$$0 \geq \mathbb{E}(X\mathbb{I}(A)) = \mathbb{E}(Y\mathbb{I}(A)) \quad (1.22)$$

as $X > 0$ on $\mathbb{I}(A)$.[2]

[2] Is this some pigeonhole argument needed?

(ii) Complete

(v) We have

$$f_{X,Y}(n,y) = b\frac{(ay)^n}{n!}e^{-(a+b)y} \tag{1.23}$$

by differentiating with respect to t.

$$\begin{aligned} f_X(x) &= b\int_0^\infty \frac{(ay)^n}{n!}e^{-(a+b)y}dy && (1.24) \\ &= \frac{ba^n}{n!}\int_0^\infty y^{\alpha-1}e^{-\beta y}dy && (1.25) \\ &= \frac{ba^n\Gamma(n+1)}{n!(a+b)^{n+1}} && (1.26) \\ &= \frac{ba^n}{(a+b)^{n+1}} && (1.27) \end{aligned}$$

Then we can compute

$$\begin{aligned} \mathbb{E}(h(Y)|X=n) &= \int_0^\infty f_{Y|X}(y,n)h(y)dy && (1.28) \\ &= \int_0^\infty \frac{f_{X,Y}(n,y)}{f_X(n)}h(y)dy && (1.29) \\ &= \int_0^\infty \frac{b\frac{(ay)^n}{n!}e^{-(a+b)y}}{\frac{ba^n}{(a+b)^{n+1}}}h(y)dy && (1.30) \\ &= \int_0^\infty \frac{h(y)y^ne^{-(a+b)y}(a+b)^{n+1}}{n!}dy && (1.31) \end{aligned}$$

as required.

We can now compute $\mathbb{E}\left(\frac{Y}{X+1}\right)$. We have (by Fubini's theorem)

$$\begin{aligned} \mathbb{E}\left(\frac{Y}{X+1}\right) &= \int_0^\infty \sum_{n=0}^\infty f_{X,Y}(n,y)dy && (1.32) \\ &= \int_0^\infty \sum_{n=0}^\infty \frac{y}{n+1}\frac{(ay)^n}{n!}e^{-(a+b)y}dy && (1.33) \\ &= \int_0^\infty \sum_{n=0}^\infty \left[\frac{(ay)^{n+1}}{(n+1)!}\right]\frac{1}{a}e^{-(a+b)y}dy && (1.34) \\ &= \int_0^\infty (e^{ay}-ay-1)\frac{1}{a}e^{-(a+b)y}dy && (1.35) \\ &= \int_0^\infty \frac{e^{-by}}{a}dy - \int_0^\infty ye^{-(a+b)y}dy - \int_0^\infty \frac{1}{a}e^{-(a+b)y}dy && (1.36) \\ &= \frac{1}{ab} - \frac{1}{(a+b)^2} - \frac{1}{a(a+b)} && (1.37) \end{aligned}$$

We now compute $f_Y(y)$. We have

$$f_Y(y) = \sum_{n=0}^{\infty} f_{X,Y}(n,y) \tag{1.38}$$
$$= \sum_{n=0}^{\infty} \frac{b(ay)^n}{n!} e^{-(a+b)y} \tag{1.39}$$
$$= be^{-(a+b)y} \sum_{n=0}^{\infty} \frac{(ay)^n}{n!} \tag{1.40}$$
$$= be^{-(a+b)y} e^{ay} \tag{1.41}$$
$$= be^{-by} \tag{1.42}$$

and thus the unconditional distribution of Y is exponential with parameter b.

We now compute $\mathbb{E}(\mathbb{I}(X = n|Y)) = \mathbb{P}(X = n|Y) = f_{X|Y}(n,y)$. We have

$$f_{X|Y}(n,y) = \frac{f_{X,Y}(n,y)}{f_Y(y)} \tag{1.43}$$
$$= \frac{\frac{b(ay)^n}{n!} e^{-(a+b)y}}{be^{-by}} \tag{1.44}$$
$$= \frac{(ay)^n e^{-ay}}{n!} \tag{1.45}$$

We now compute $\mathbb{E}(X|Y)$. We have

$$\mathbb{E}(X|Y = y) = \sum_{n=0}^{\infty} n f_{X|Y}(n,y) \tag{1.46}$$
$$= \sum_{n=0}^{\infty} n \frac{(ay)^n e^{-ay}}{n!} \tag{1.47}$$
$$= \sum_{n=0}^{\infty} \frac{(ay)^n e^{-ay}}{(n-1)!} \tag{1.48}$$
$$= \sum_{n=0}^{\infty} e^{-ay} \sum_{n=0}^{\infty} \frac{(ay)^n}{(n-1)!} \tag{1.49}$$
$$= e^{-ay}(aye^{ay}) \tag{1.50}$$
$$= ay \tag{1.51}$$

(vi) For example, constant functions are trivially independent on all σ-algebras $\mathcal{G}$.

We need to show the following are equivalent. Let $f, g \geq 0$ and measurable. Let $Z \geq 0$ and $\mathcal{G}$-measurable.

(i)

$$\mathbb{E}(f(X)g(Y)|\mathcal{G}) = \mathbb{E}(f(X)|\mathcal{G})\,\mathbb{E}(g(Y)|\mathcal{G}) \tag{1.52}$$

(ii)

$$\mathbb{E}(f(X)g(Y)Z) = \mathbb{E}(f(X)Z\mathbb{E}(g(Y)|\mathcal{G})) \tag{1.53}$$

(iii)

$$\mathbb{E}(g(Y)|\mathcal{G} \vee \sigma(X)) = \mathbb{E}(g(Y)|\mathcal{G}) \tag{1.54}$$

Complete proof

(vii)

1.2 *Discrete-time Martingales*

(i) Recall the definition of the natural filtration F_t^X as $\sigma(X_s, s \leq t)$. We need to show that

$$\mathbb{E}\left(X_t|\mathcal{F}_s^T\right) = X_s \iff \mathbb{E}(X_t|\sigma(X_s, s \leq t)) = X_s \tag{1.55}$$

(ii) Note that if C_n is bounded then we can trivially bound $|Y_n|$, and so Y_n is integrable. We need to show that for $n > 0$, $\mathbb{E}(Y_n|\mathcal{F}_{n-1}) = Y_{n-1}$.

$$\mathbb{E}(Y_n|\mathcal{F}_{n-1}) = \mathbb{E}\left(\sum_{k\leq n} C_k(X_k - X_{k-1})|\mathcal{F}_{n-1}\right) \tag{1.56}$$

$$= \mathbb{E}(c_n(X_n - X_{n-1})|\mathcal{F}_{n-1}) + \mathbb{E}\left(\sum_{k\leq n-1} c_k(X_k - X_{k-1})|\mathcal{F}_{n-1}\right) \tag{1.57}$$

$$= c_n(\mathbb{E}(X_n|\mathcal{F}_{n-1}) - X_{n-1}) + Y_{n-1} \tag{1.58}$$

$$= Y_{n-1} \tag{1.59}$$

If X is a supermartingale, then we have $\mathbb{E}(X_n|\mathcal{F}_{n-1}) \leq X_{n-1}$, and if c_n is non-negative, we can replace the equality in (1.59) with

$$c_n(\underbrace{\mathbb{E}(X_n|\mathcal{F}_{n-1}) - X_{n-1}}_{\leq 0}) + Y_{n-1} \leq Y_{n-1} \tag{1.60}$$

(iii) Note first that $\mathbb{E}(X_n) = 0, \mathbb{E}(|X_n) = 2$. Let $Y_n = \frac{S_n}{n}$ and that

$$\mathbb{E}(Y_n|\mathcal{F}_{n-1}) = \mathbb{E}\left(\frac{X_n + S_{n-1}}{n}|\mathcal{F}_{n-1}\right) \tag{1.61}$$

$$= \frac{1}{n}\mathbb{E}(X_n) + \frac{S_{n-1}}{n} \tag{1.62}$$

$$= \frac{n-1}{n}Y_{n-1} \tag{1.63}$$

$$\leq Y_{n-1} \tag{1.64}$$

and

$$\mathbb{E}(|Y_n|) = \frac{1}{n}\mathbb{E}\left(|\sum_{i=1}^{n} X_n\right) \tag{1.65}$$

$$\leq \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(|X_i|) \tag{1.66}$$

$$= \frac{1}{n}2n \qquad = 2 \tag{1.67}$$

so Y_n is a supermartingale bounded in L^1.

By the martingale convergence theorem, Y_n converges almost surely to some limit Y_∞ for some $Y \in L^1(\mathcal{F}_\infty)$. By Kolmogrov's 0-1 law, we can infer that $\frac{S_n}{n}$ converges to some constant limit $c \in \mathbb{R}$, and so $Y_n = \frac{S_n}{n} \to c$.

Show $c = 1$.

(iv) Note that $T = m\mathbb{I}(A) + m' A^c$ is simply the stopping time

$$T(\omega) = \begin{cases} m & \omega \in A \\ m' & \omega \notin A \end{cases} \tag{1.68}$$

We must show that for all $k \in N$, the event $\{T = k\}$ is $\mathcal{F}_k$ measurable. As $\{\omega \in A\}$ is $\mathcal{F}_n$ measurable, and $\mathcal{F}_m, \mathcal{F}_{m'} \supseteq \mathcal{F}_n$, and $\emptyset \in \mathcal{F}_0$ we have our required result.

Let X be a martingale and let T be a bounded stopping time. Recall that $X_{T\wedge n}$ is a martingale. Then by the dominated convergence theorem, we have

$$\mathbb{E}(X_0) = (\leq) \lim_{n\to\infty} \mathbb{E}(X_{T\wedge n}) \tag{1.69}$$
$$= \mathbb{E}\Big(\lim_{n\to\infty} X_{T\wedge n}\Big) \tag{1.70}$$
$$= \mathbb{E}(X_T) \tag{1.71}$$

where the application of dominated convergence theorem is justified as there exists $K \in \mathbb{N}$ such that $T \leq K$, and so $|X_{T\wedge n}|$ is dominated by $\sup_{k\leq K} |X_k| < \infty$.

Now, let X be an integrable adapted process and with the property that for every bounded stopping time T, $\mathbb{E}(X_T) = \mathbb{E}(X_0)$.

We must show that for $m \geq n$, $\mathbb{E}(X_m|\mathcal{F}_n) = X_n$. Let $m \geq n$ be given, and $A \in \mathcal{F}_n$. Let $T = mIA + nIA^c$, and $T' = n$. Then we have

$$\mathbb{E}(X_T) = \mathbb{E}(X_m\mathbb{I}(A)) + \mathbb{E}(X_n\mathbb{I}(A^c)) = \mathbb{E}(X_0) \tag{1.72}$$
$$\mathbb{E}(X_{T'}) = \mathbb{E}(X_n) = \mathbb{E}(X_n\mathbb{I}(A)) + \mathbb{E}(X_n\mathbb{I}(A^c)) = \mathbb{E}(X_0) \tag{1.73}$$

Thus, $\mathbb{E}(X_m\mathbb{I}(A)) = \mathbb{E}(X_n\mathbb{I}(A))$. By properties of conditional expectation, we then have $\mathbb{E}(X_m|\mathcal{F}_n) = X_n$, and thus X is a martingale.

(v) Let X be bounded. Then

$$\mathbb{E}(X_0) = \lim_{n\to\infty} \mathbb{E}(X_{T\wedge n}) \tag{1.74}$$
$$= \mathbb{E}\Big(\lim_{n\to\infty} X_{T\wedge n}\Big) \tag{1.75}$$
$$= \mathbb{E}(X_T) \tag{1.76}$$

by the dominated convergence theorem, as (1.74) is dominated by M.

Let X have bounded increments. Then we can write $X_{T\wedge n}(\omega) - X_0(\omega) = \sum_{k=1}^{T(\omega)\wedge n} X_k(\omega) - W_{k-1}(\omega)$. Note that we can bound the right hand side by M, and thus $|X_{T\wedge n} - X_0| \leq MT$, and as X_0 is integrable (X is a martingale), we can conclude that $X_{T\wedge n}$ is dominated as $n \to \infty$.

Note also that the right hand side is integrable, as T is integrable, and so the whole side is bounded by $M\mathbb{E}(T) < \infty$ Thus, we can apply the dominated convergence theorem to $X_{T\wedge n}$ and conclude that

$$\mathbb{E}(X_0) = \lim_{n\to\infty} \mathbb{E}(X_{T\wedge n}) = \mathbb{E}(X_T) \tag{1.77}$$

as required.

(vi) Note that for a discrete random variable X, we have $\mathbb{E}(X) = \sum_{i=1}^{\infty} \mathbb{P}(X \geq i)$. From the given equation, we have

$$\mathbb{P}(T \leq N + n | \mathcal{F}_n) \geq \epsilon, \tag{1.78}$$

and so by taking $A \in \mathcal{F}_n = \{T > n\}$, we have

$$\mathbb{E}(\mathbb{I}(T \leq n + N)\,\mathbb{I}(T > n)) \geq \mathbb{E}(\epsilon\mathbb{I}(T > n)) \tag{1.79}$$

$$\Rightarrow \mathbb{P}(n < T \leq n + N) \geq \epsilon\mathbb{P}(T > n) \tag{1.80}$$

and in particular,

$$\mathbb{P}(kN < T \leq (k+1)N) \geq \epsilon\mathbb{P}(T \geq kN) \tag{1.81}$$

Then, we have

$$1 \geq \mathbb{P}(T \leq mN) \tag{1.82}$$

$$\geq \sum_{k=0}^{m-1} \epsilon\mathbb{P}(T \geq kN) \tag{1.83}$$

, and so we have the bound

$$\sum_{k=0}^{m-1} \mathbb{P}(T \geq kN) \leq \frac{1}{\epsilon} \tag{1.84}$$

and so we have the bound

$$\mathbb{E}(X) \leq N \sum_{k=0}^{m-1} \mathbb{P}(T \geq kN) \leq \frac{N}{\epsilon} < \infty \tag{1.85}$$

as required.

(vii)

Finish this.

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Bibliography