Mathematics of Operations Research Examples

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Contents

CHAPTER 1

Example Sheet 1

(i) The Lagrangian of the problem is

$$L(x, \lambda) = -2x_1^2 - x_2^2 + x_1x_2 - \lambda(3x_1 + x_2 - 10) \tag{1.1}$$

with partial derivatives

$$\frac{\partial L}{\partial x_1} = -4x_1 + x_2 + 8 - 3\lambda \tag{1.2}$$

$$\frac{\partial L}{\partial x_2} = -2x_2 + x_1 + 3 - \lambda \tag{1.3}$$

$$\tag{1.4}$$

$$\frac{\partial L}{\partial \lambda} = 3x_1 + x_2 - 10 \tag{1.5}$$

and second partial derivatives

$$\frac{\partial^2 L}{\partial x_1^2} = -4 \tag{1.6}$$

$$\frac{\partial^2 L}{\partial x_2^2} = -2 \tag{1.7}$$

$$\frac{\partial^2 L}{\partial x_1 \partial x_2} = 1 \tag{1.8}$$

Setting (1.2), (1.3), (1.5) to zero and solving the set of equations gives

$$x_1 = \frac{69}{28} \tag{1.9}$$

$$x_2 = \frac{73}{28} \tag{1.10}$$

The Hessian matrix is

$$\begin{vmatrix} -4 & 1 \\ 1 & -2 \end{vmatrix} = 7 > 0 \tag{1.11}$$

and $\frac{\partial^2 L}{\partial x_1^2} < 0$, so the solution is the maximizer of L. Thus, by the Lagrangian Sufficiency theorem, the solution is a maximizer of original problem.

(ii)

(iii)

(iv) Let $A \overset{\star}{\equiv} B$ indicate that A is the dual of B. Then we have

$$\min\{c^T x | Ax \geq b, x \geq 0\} \overset{\star}{\equiv} \max\{b^T x | A^T x \leq c, x \geq 0\} \tag{1.12}$$

Let a linear program be given in the general form. Then

$$\begin{aligned} \min\{c^T x | Ax \geq b, x \geq 0\} &\overset{\star}{\equiv} \max\{b^T x | A^T x \leq c, x \geq 0\} && (1.13)\\ &= \min\{(-b)^T x | (-A)^T x \geq (-c), x \geq 0\} && (1.14)\\ &\overset{\star}{\equiv} \{(-c)^T x | ((-A)^T)^T x \leq -b, x \geq 0\} && (1.15)\\ &= \min\{c^T x | Ax \geq b, x \geq 0\} && (1.16) \end{aligned}$$

and thus the dual of the dual of a linear program is identical to the linear program.

(v) (i) The general linear program

$$\min\{c^T x | Ax \geq b, x \geq 0\} \overset{\star}{\equiv} \max\{b^T x | A^T x \leq c, x \geq 0\} \tag{1.17}$$

We then have

$$\begin{aligned} \max\{0^T x | Ax = b, x \geq 0\} &\equiv \min\{(-0)^T x | Ax = b, x \geq 0\} && (1.18)\\ &\overset{\star}{\equiv} \max\{z^T b | z^T A \leq 0^T\} && (1.19)\\ &\equiv \min\{y^T b | y^T A \geq 0^T\} && (1.20) \end{aligned}$$

Thus, the dual of (1.18) is (1.20) as required.

(ii) By weak duality, we have for any feasible solutions y of (1.18) and x of (1.20), that

$$0^T x = 0 \leq y^T b \tag{1.21}$$

If (1.18) is feasible, then by weak duality, (1.20) is bounded below by zero (by (1.21)). (1.20) is obviously feasible, since $y = 0$ is a feasible solution. If (1.20) is bounded, we can solve it using the Simplex method, which also gives us an optimal and therefore feasible solution for (1.18).

(iii) The two cases correspond to the two cases of the above result. (1.18) is feasible if and only if there exists $x \in \mathbb{R}^n$ with $Ax = b$ and $x \geq 0$. (1.18) is infeasible if and only if (1.20) is unbounded if and only if there exists $y \in \mathbb{R}^m$ such that $y^T A \geq 0$ and $y^T b < 0$.

(vi)

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(viii)

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(x)

(xi) (i) Trivial to verify this problem is in NP, as given a certificate, simply follow the path through the nodes, which is linear in the number of nodes.

(ii) Pretty tricky, just following construction in HTTP://AMA.epfl.ch/~moustafa/Other/Complexityslides/lec7.pdf.

(iii) Traveling salesman problem - given an instance of undirected Hamiltonian cycle $G(V,E)$, construct an instance of TSP $T(V',d)$ by placing a city at each $v \in V$ (so $V' = V$), and let

$$d(v_1, v_2) = \begin{cases} 1 & (v_1, v_2) \in E \\ 2 & \text{otherwise} \end{cases} \tag{1.22}$$

There exists a TSP path of length $|V|$ if and only if there exists an undirected Hamiltonian cycle in the original graph.

We now reduce directed Hamiltonian cycle to undirected Hamiltonian cycle. For an instance $G(V,E)$ of the directed problem, construct the undirected graph $G'(V',E')$ where each vertex v_i is replaced by three vertices $v'_{i1}, v'_{i2}, v'_{i3}$, with edges $v'_{i1} \leftrightarrow v'_{i2}, v'_{i2} \leftrightarrow v'_{i3}$, and $v'_{i3} \leftrightarrow v'_{j1}$ if $v'_i \to v'_j$.

We then trivially have that any undirected Hamiltonian cycle in G' can be mapped to a directed Hamiltonian cycle in G, as required.