

P1. Base case: $i=2$

$$P(x_1, x_2) = P(x_1) \cdot P(x_2)$$

Induction

$$\text{Suppose } p(x_1, \dots, x_k) = p(a) = \prod_{i=1}^k P(x_i | x_1, \dots, x_{i-1})$$

$$\begin{aligned} P(x_1, \dots, x_{k+1}) &= p(a, x_{k+1}) = P(x_{k+1} | a) \cdot p(a) \\ &= p(x_{k+1} | x_1, \dots, x_k) \prod_{i=1}^k P(x_i | x_1, \dots, x_{i-1}) \\ &= \prod_{i=1}^{k+1} P(x_i | x_1, \dots, x_{i-1}) \end{aligned}$$

Conclusion:

$$\begin{aligned} p(x_1, \dots, x_k) &= \prod_{i=1}^k P(x_i | x_1, \dots, x_{i-1}) \\ \hookrightarrow p(x_1, \dots, x_{k+1}) &= \prod_{i=1}^{k+1} P(x_i | x_1, \dots, x_{i-1}). \end{aligned}$$

Problem 2

$$\begin{aligned} P(x, y | z) &= \frac{P(x, y, z)}{P(z)} = \frac{P(x|y, z) \cdot P(y|z)}{P(z)} = P(x|y, z) \times \frac{P(y|z)}{P(z)} \\ &= P(x|y, z) \times P(y|z) \end{aligned}$$

$$P(x|y, z) = P(x|y, z) \times P(y|z) = P(x|z) \times P(y|z)$$

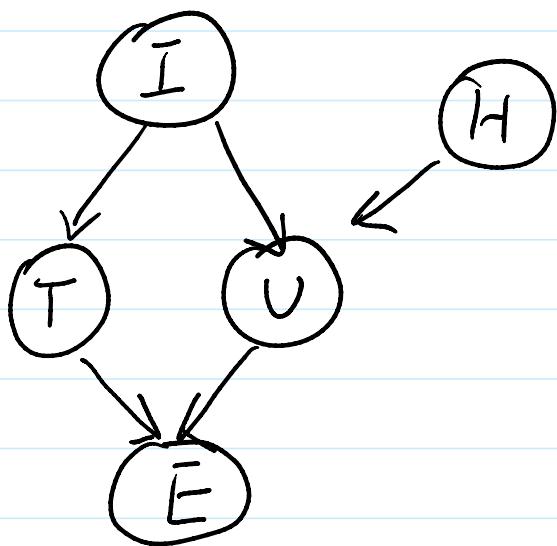
Problem 3

1): False, $x \rightarrow z \leftarrow y$, they are depended after z is given

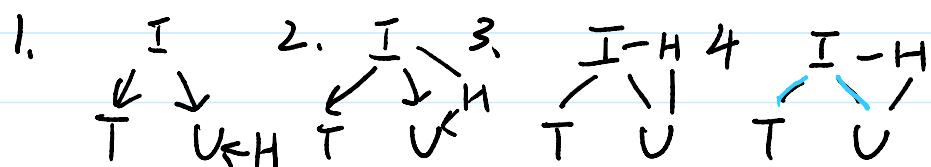
1): False , $X \rightarrow Z \leftarrow Y$, they are dependent after Z is given

2): False , $X \leftarrow Z \rightarrow Y$, they are unconditionally dependent.

Problem 4

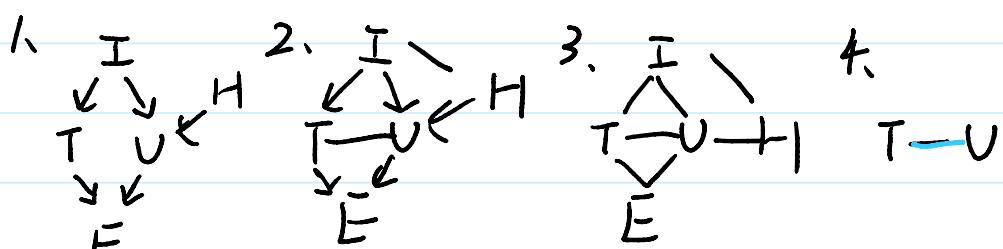


1. $T \perp U$



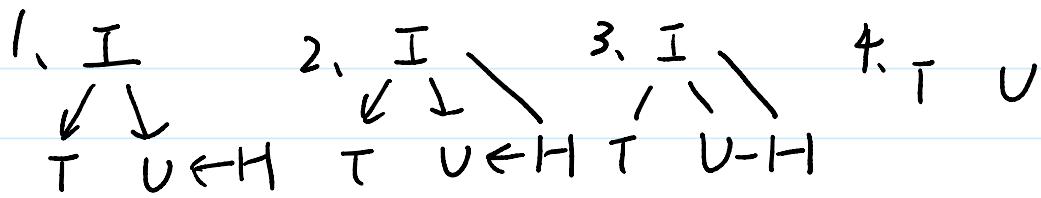
not guaranteed to be independent, $\therefore F$

2. $T \perp U | I, E, H$



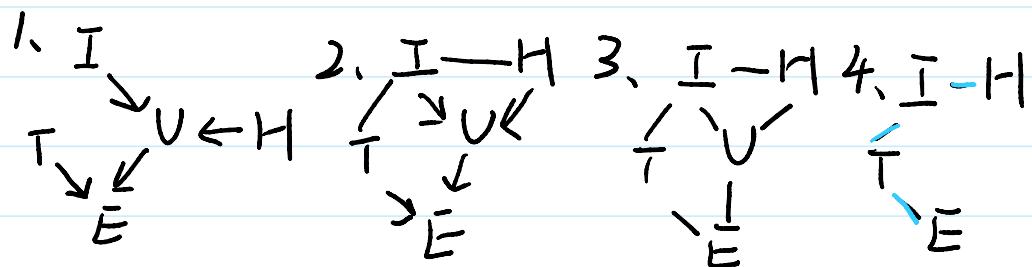
not guaranteed to be independent $\therefore F$

3. $T \perp U | I, H$



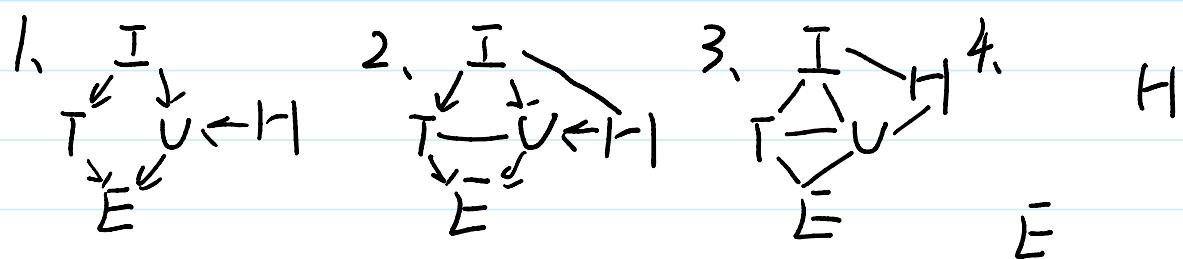
guaranteed to be independent, $\therefore T$

4. $E \perp H \mid U$



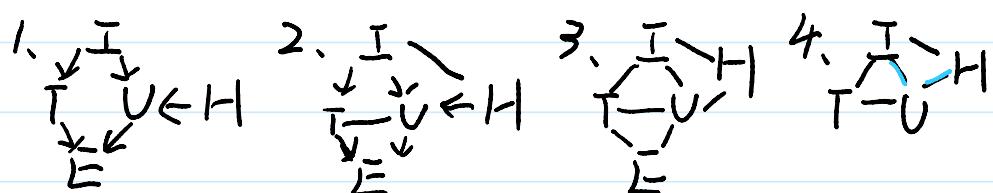
not guaranteed to be independent $\therefore F$

5. $E \perp H \mid U, I, T$



guaranteed to be independent, $\therefore T$

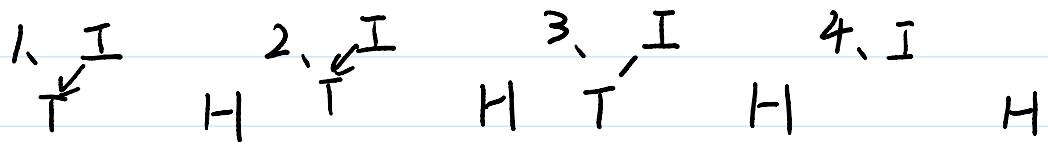
6. $I \perp H \mid E$



not guaranteed to be independent, $\therefore F$

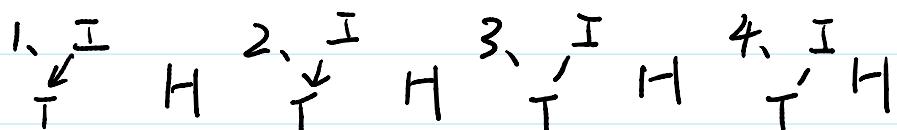
7. $I \perp H \mid T$

7. $I \perp H | T$



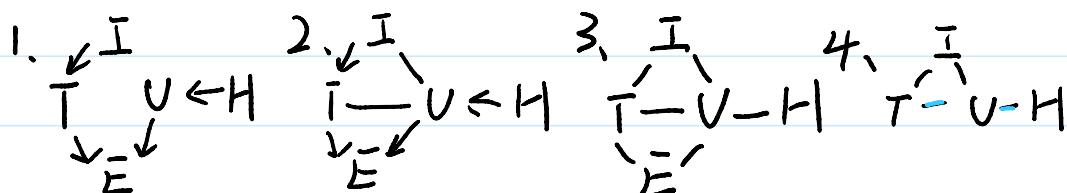
guaranteed to be independent, $\therefore T$

8. $T \perp H$



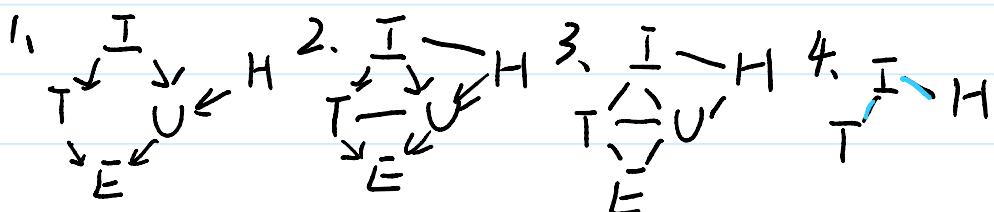
guaranteed to be independent, $\therefore T$

9. $T \perp H | E$



not guaranteed to be independent, $\therefore F$

10. $T \perp H | E, U$

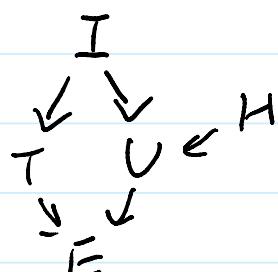


guaranteed to be independent, $\therefore F$

Problem 5.

$$P(+u | t_e) = \frac{P(+u, t_e)}{P(t_e)}$$

$P(+u, t_e) = P(t_e) P(+u | t_e)$



$$P(+u, +e) = \sum_{h, i, t} P(+u, +e, i, t, h)$$

$$= \sum_{h, i, t} P(i) \times P(t|i) \times P(+u|i, h) \times P(h) \times P(+e|t, +u)$$

$$= \sum_h P(h) \sum_i P(i) \cdot P(+u|i, h) \underbrace{\sum_t P(t|i) \times P(+e|t, +u)}_{f_2}$$

$$f_2(h, i) = \sum_t P(t|i) \times P(+e|t, +u)$$

$$\begin{aligned} f_2(h, +i) &= P(t+|+i) \times P(+e|t+, +u) + P(+e|-t, +u) \times P(-t|+i) \\ &= 0.8 \times 0.9 + 0.2 \times 0.7 = 0.86 \end{aligned}$$

$$\begin{aligned} f_2(h, -i) &= P(t+|-i) \times P(+e|t+, +u) + P(-t|-i) \times P(+e|-t, +u) \\ &= 0.5 \times 0.9 + 0.5 \times 0.7 = 0.8 \end{aligned}$$

$$f_1(h) = \sum_i P(i) \cdot P(+u|i, h) \cdot f_2(h, i)$$

$$\begin{aligned} f(+h) &= P(-i) \cdot P(+u|-i, +h) \cdot f_2(+h, i) + \\ &\quad P(+i) \cdot P(+u|+i, +h) \cdot f_2(+h, i) \\ &= 0.3 \times 0.5 \times 0.8 + 0.7 \times 0.9 \times 0.86 \\ &= 0.6618 \end{aligned}$$

$$\begin{aligned} f(-h) &= P(-i) \cdot P(+u|-i, -h) \cdot f_2(-h, i) + \\ &\quad P(+i) \cdot P(+u|+i, -h) \cdot f_2(-h, i) \\ &= 0.3 \times 0.1 \times 0.8 + 0.7 \times 0.3 \times 0.86 \\ &= 0.2046 \end{aligned}$$

$$P(+u, +e) = \sum_h P(h) \cdot f_1(h)$$

$$= P(h) \times f_1(h) + P(-h) \times f_1(-h)$$

- 0.6618 + 0.2046 = 0.4572

$$\begin{aligned}
 &= P(h)xt_i(h) + P(-h)x_{t_i(h)} \\
 &= 0.6 \times 0.6618 + 0.4 \times 0.2046 \\
 &= 0.47892
 \end{aligned}$$

$$\begin{aligned}
 P(-u, +e) &= \sum_{h,i,t} P(-u, te, i, th) \\
 &= \sum_{h,i,t} P(i) \times P(t|i) \times P(-u|i, h) \times P(h) \times P(te|t, -u) \\
 &= \sum_h P(h) \sum_i P(i) \cdot P(-u|i, h) \underbrace{\sum_t P(t|i) \times P(te|t, -u)}_{f_2}
 \end{aligned}$$

$$f_2(h, i) = \sum_t P(t|i) \times P(te|t, -u)$$

$$\begin{aligned}
 f_2(h, +i) &= P(+t|+i) \times P(te|tt, -u) + P(+te|-t, -u) \times P(-t|+i) \\
 &= 0.8 \times 0.5 + 0.2 \times 0.3 = 0.46
 \end{aligned}$$

$$\begin{aligned}
 f_2(h, -i) &= P(+t|-i) \times P(te|+t, -u) + P(-t|-i) \times P(te|-t, -u) \\
 &= 0.5 \times 0.5 + 0.5 \times 0.3
 \end{aligned}$$

$$f_1(h) = \sum_i P(i) \cdot P(-u|i, h) \cdot f_2(h, i)$$

$$\begin{aligned}
 f(th) &= P(-i) \cdot P(-u|-i, th) \cdot f_2(+h, i) + \\
 &\quad P(+i) \cdot P(-u|+i, th) \cdot f_2(+h, +i) \\
 &= 0.3 \times 0.5 \times 0.4 + 0.7 \times 0.1 \times 0.46 \\
 &= 0.0922
 \end{aligned}$$

$$\begin{aligned}
 f(-h) &= P(-i) \cdot P(-u|-i, -h) \cdot f_2(-h, i) + \\
 &\quad P(+i) \cdot P(-u|+i, -h) \cdot f_2(-h, +i) \\
 &= 0.3 \times 0.9 \times 0.4 + 0.7 \times 0.7 \times 0.46 \\
 &= 0.2334
 \end{aligned}$$

$$= 0.3 \times 0.9 \times 0.4 + 0.7 \times 0.7 \times 0.46$$
$$= 0.3334$$

$$P(-u, te) = \sum_h P(h) \cdot f_t(h)$$
$$= P(h) \times f_t(h) + P(-h) \times f_t(-h)$$
$$= 0.6 \times 0.0922 + 0.4 \times 0.3334$$
$$= 0.18868$$

$$P(te) = P(+u) \cdot P(te|+u) + P(-u) \cdot P(te| -u)$$
$$= P(+u, te) + P(-u, te)$$
$$= 0.18868 + 0.47892$$
$$= 0.6676$$

$$P(+u | te) = \frac{P(+u, te)}{P(te)} = 0.7112$$