MAS 433: Cryptography

Lecture 10

Birthday Attack

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Lecture Outline

- Classical ciphers
- Symmetric key encryption
- Hash function and Message Authentication Code
  - **Birthday attack**
  - Hash function
  - Message Authentication Code
- Public key encryption
- Digital signature
- Key establishment and management
- Introduction to other cryptographic topics

Recommended Reading

- CTP Section 4.2.2
- HAC Section 9.7.1, 3.2.2
- Wikipedia
  - Birthday problem

    http://en.wikipedia.org/wiki/Birthday_problem
  - Floyd’s cycle finding algorithm

    (tortoise and hare algorithm)
    http://en.wikipedia.org/wiki/Floyd%27s_cycle-finding_algorithm#Tortoise_and_hare

Birthday Problem (Birthday Paradox)

- In a set of randomly chosen people, what is the probability that at least two of them have the same birthday? (assume 365 days/year)
  - 100% if there are 366 people
    - Pigeonhole principle
  - 99% if there are 57 people
  - 50% if there are 23 people

$$\left.\begin{array}{l}\text{99\% if there are 57 people}\\ \text{50\% if there are 23 people}\end{array}\right\}\ \text{Why?}$$

Birthday Problem (Birthday Paradox)

- Example: 23 randomly chosen people
  - There are $\binom{23}{2} = \frac{23 \times 22}{2} = 253$ possible pairs
  - For each pair, the probability that their birthday are the same is about 1/365   (assume 356 days)
    - But the probability is not exactly 1/365 since those 253 pairs are not independent
  - With so many pairs from just 23 people, the chance for the birthday "collision" is higher than that expected based on intuition

Birthday Problem

- How to compute the probability *p* that two people (among *n* randomly chosen person) have the same birthday ?
  - We first compute the probability that any two people do not have the same birthday

    (denoted as *p'*)
  - Then $p = 1 - p'$

Birthday Problem

- How to compute the probability p' that any two people do not have the same birthday
  - Randomly choose two people,
    - the birthday of the second people should be different from the first one:

$$p' = (365-1)/365$$

  - Randomly choose three people,
    - the birthday of the second should be different from the first
    - the birthday of the third should be different from the previous two:

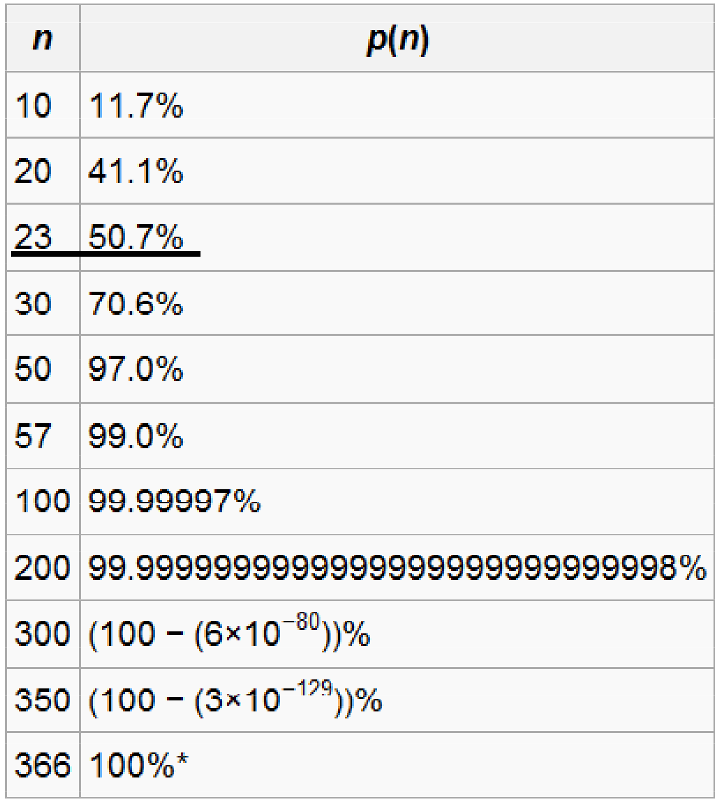
$$p' = (365-1)/365 \times (365-2)/365$$

  - Randomly choose n people,

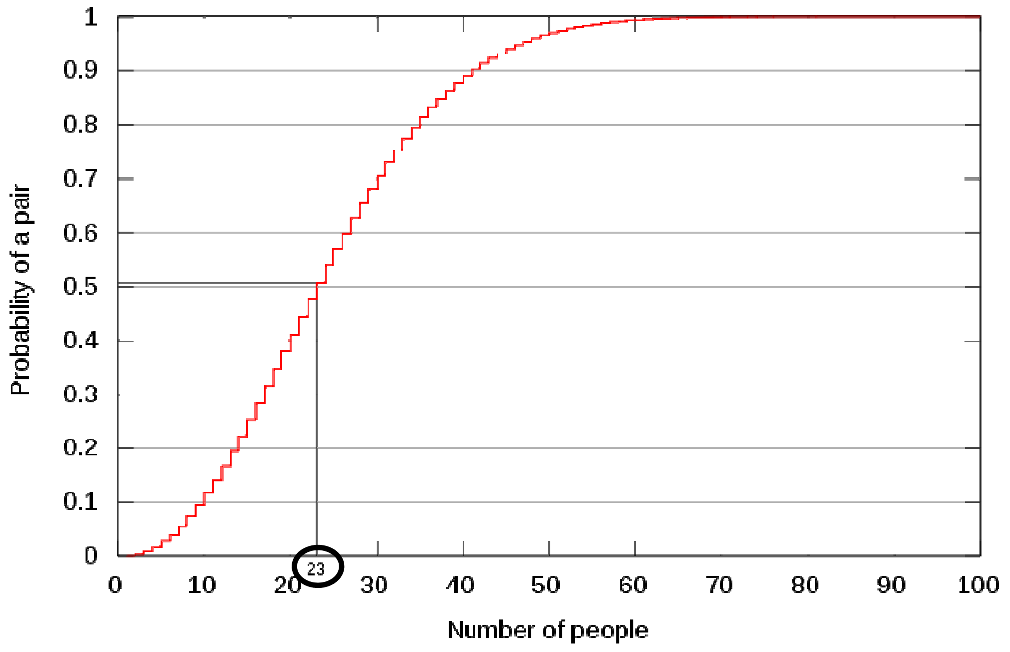
$$p' = (365-1)/365 \times (365-2)/365 \times \ldots\ldots \times (365-n+1)/365$$

Birthday Problem

- The values of $p = 1 - p'$



n	*p*(*n*)
10	11.7%
20	41.1%
23	50.7%
30	70.6%
50	97.0%
57	99.0%
100	99.99997%
200	99.999999999999999999999999998%
300	$(100 - (6\times10^{-80}))\%$
350	$(100 - (3\times10^{-129}))\%$
366	100%*



Birthday Problem

- General form for calculating *p* = 1- *p’*
  - Suppose that there are *M* different values, randomly generate *Q* of those *M* values, compute the probability that at least two values are identical?

$$p' = \left(1 - \frac{1}{M}\right)\left(1 - \frac{2}{M}\right)\cdots\left(1 - \frac{Q-1}{M}\right) = \prod_{i=1}^{Q-1}\left(1 - \frac{i}{M}\right)$$

  - But it is difficult to compute *p* for large *Q* and *M*
    - In cryptography, we will encounter $M = 2^{128}$ or 2^{256} or 2^{512}
    - Approximation is needed ….

Birthday Problem

- Approximation
  - For small *x*, $1 - x \approx e^{-x}$
    - Reason: $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots$
  - Thus

$$\prod_{i=1}^{Q-1}\left(1 - \frac{i}{M}\right) \approx \prod_{i=1}^{Q-1} e^{\frac{-i}{M}}$$

$$= e^{-\sum_{i=1}^{Q-1} \frac{i}{M}}$$

$$= e^{\frac{-Q(Q-1)}{2M}}.$$

$$p = 1 - e^{\frac{-Q(Q-1)}{2M}}$$

Birthday Problem

- Approximation (contd.)
  - For a given value of p, Q can be estimated as

$$Q \approx \sqrt{2M \ln \frac{1}{1-p}}$$

  - For $p = 0.5,\ Q \approx 1.17\sqrt{M}$

Birthday Attack

- Based on birthday problem
- For a given random function f
  - function f is easy to compute,
  - but function f is difficult to invert
  - function f is non-injective,
  - try to find two inputs s_1 and s_2 so that $f(s_1) = f(s_2)$, i.e., to find a collision of function f

Birthday Attack

- Direct method
  - Suppose that the output of function f is uniformly distributed, and the size of its output space is M
  - After generating $Q \approx 1.17\sqrt{M}$ outputs, two outputs would be equal with probability 0.5

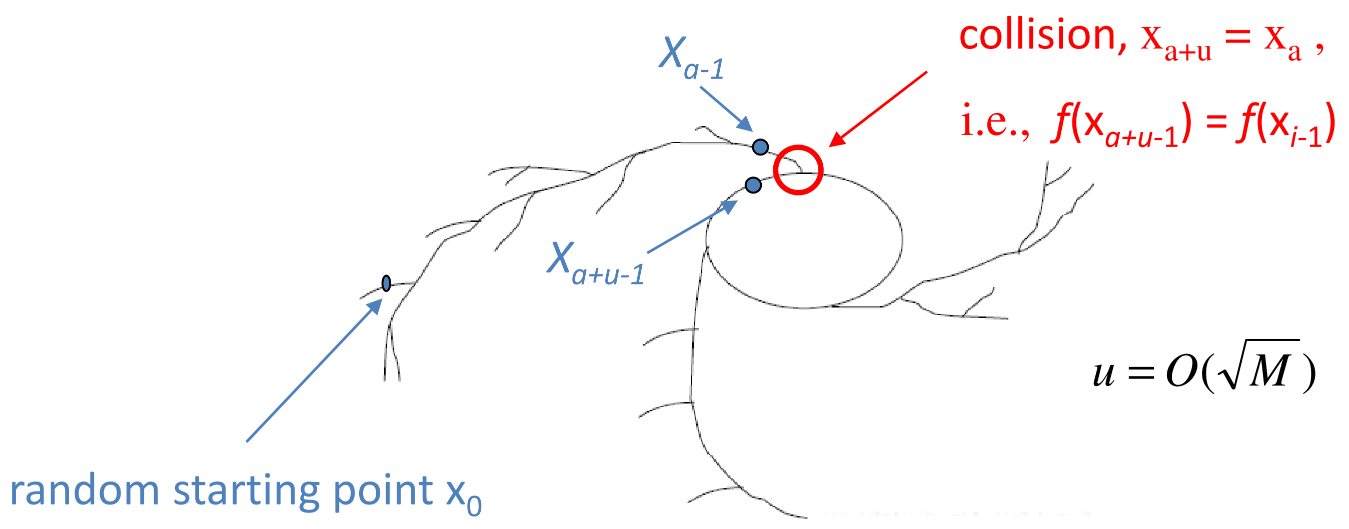
- Complexity of the above attack
  - Computational complexity: $1.17\sqrt{M}$ computations of f
  - Memory complexity: $1.17\sqrt{M}$ input-output pairs of f

Birthday Attack

- How to reduce the memory in the attack?
  - Observe that for a function f, if we calculate

    $$x_0,\ x_1 = f(x_0),\ x_2 = f(x_1),\ x_3 = f(x_2),\ x_4 = f(x_3)\ \ldots$$

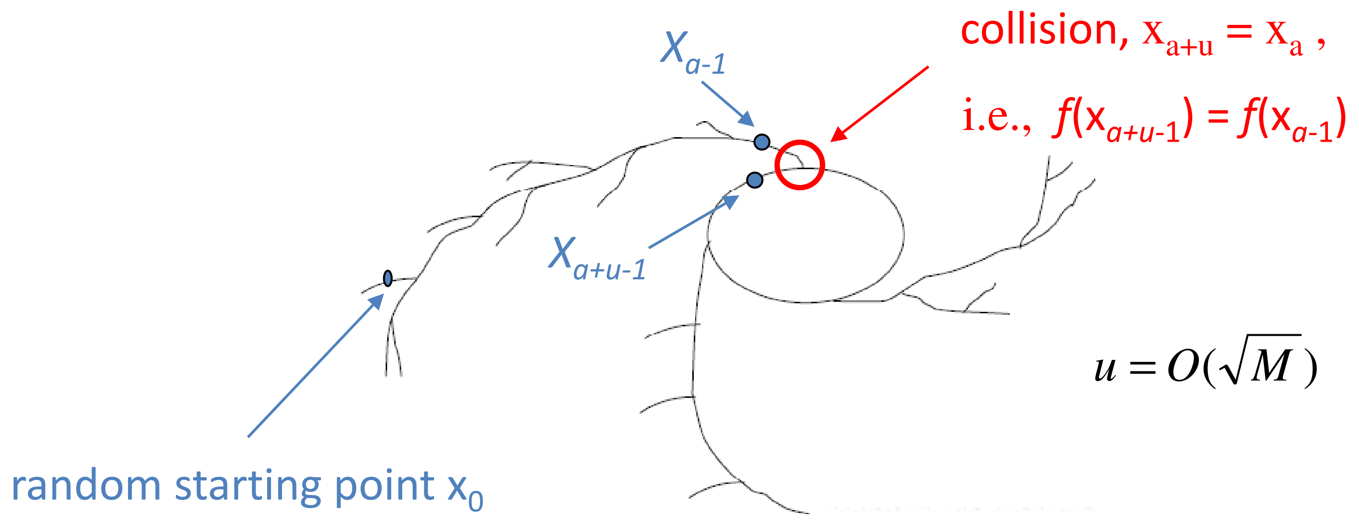
    eventually we will get $x_{a+u} = x_a$
  - if f is non-injective, then:



X_{a-1}
collision, $x_{a+u} = x_a$,
i.e., $f(x_{a+u-1}) = f(x_{i-1})$
X_{a+u-1}
$u = O(\sqrt{M})$
random starting point x_0

Birthday Attack

- How to reduce the memory in the attack? (contd.)
  - How to find out *a* and *u*?
    - Pollard's Rho method
      - Based on the "tortoise and hare" algorithm
        - » Used to detect cycle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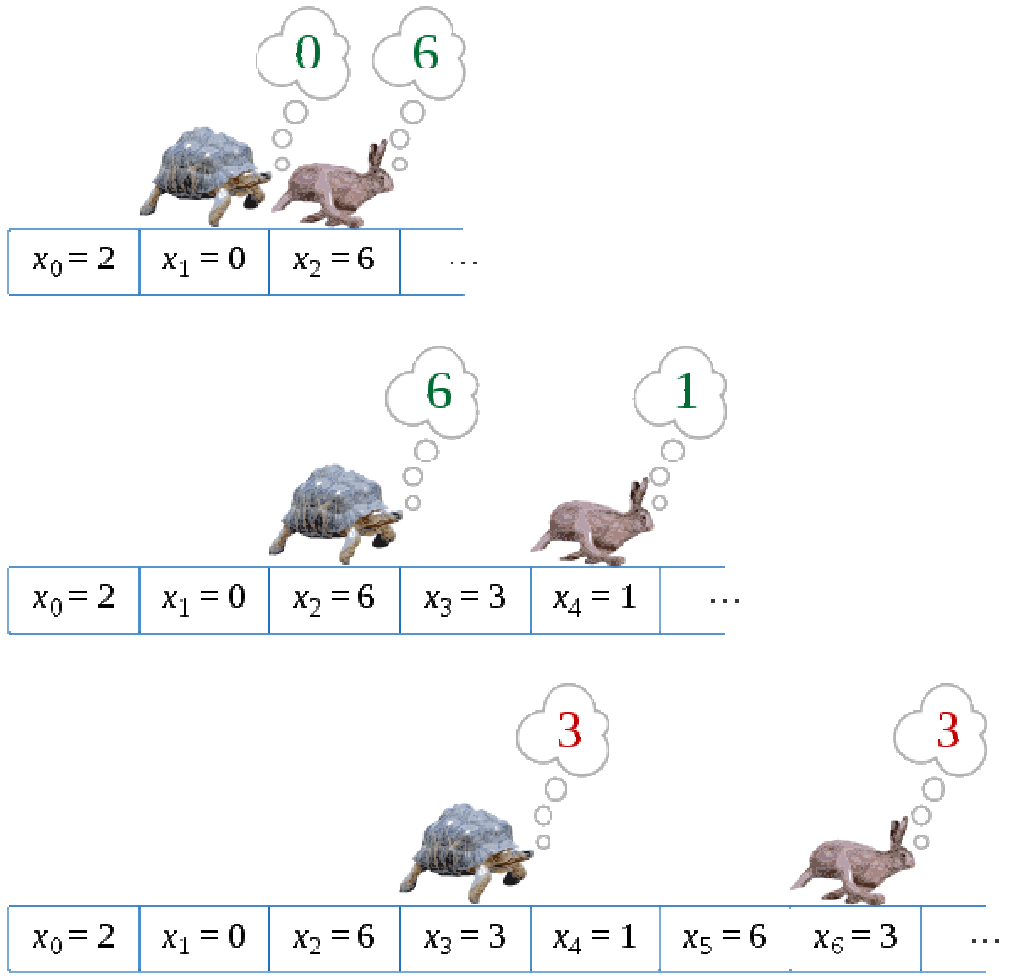
$u = O(\sqrt{M})$
random starting point x_0

Tortoise & Hare

Sequence:
2, 0, 6, 3, 1, 6, 3, 1, 6, 3, 1 ...



http://en.wikipedia.org/wiki/Floyd%27s_cycle-finding_algorithm#Tortoise_and_hare

Birthday Attack

- "tortoise and hare" algorithm
  - Invented by Floyd in the 1960s
  - For a sequence of numbers x_0, x_1, x_2, x_3 …, this algorithm is to find the cycle in the sequence (its starting point and period)
    - One pointer x_i (tortoise), each step index increases by 1
    - Another pointer x_{2i} (hare), each step index increases by 2
    - The distance between x_i and x_{2i} is i
      - the distance i keeps increasing
    - Observation:
      - If x_i enters the cycle, and i is the multiple of the period of the cycle, we will get $x_{2i} = x_i$

Birthday Attack

- "tortoise and hare" algorithm (contd.)
  - In the algorithm, we detect whether x_{2i} is the same as x_i, once we obtained $x_{2i}=x_i$, we know that $2i - i = i$ is the multiple of the period of the cycle
  - To find out the starting point of the cycle:
    - Observation: We know that *i* is the multiple of period, so $x_j = x_{j+i}$ as long as x_j is on the cycle.
    - Now we start with two pointers
      - one from x_0, another from x_i
      - Each pointer increased by 1 at each step
      - Now for the first $x_j = x_{j+i}$, we know that x_j just enters the cycle, i.e., $a = j$
  - To find out the period of the cycle
    - Starting from x_j, we increase a pointer to find the first $x_{j+k} = x_j$, then the cycle period $u = k$

Birthday Attack

- Pollard's Rho method
  - reduces the memory complexity of the birthday attack
  - but increases the computational complexity for a few times
  - There were further improvements on Rho method to reduce computational complexity
- There was further improvement on birthday attack so that it
  - Requires little memory
  - Achieves parallel processing

Summary

- Birthday problem
  - The probability that at least two elements of n random elements are the same
- Birthday attack
  - Find a collision of a function f
    - Function f is non-injective
  - Methods:
    - Direct birthday attack
      - computational & memory complexity $1.17\sqrt{M}$
    - *Rho method
      - Reduce the memory complexity