

MAS 433: Cryptography

Lecture 7
Block Cipher (Part 3, AES)

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Lecture Outline

- Classical ciphers
- Symmetric key encryption
 - One-time pad & information theory
 - Block cipher
 - DES, Double DES, Triple DES
 - AES
 - Modes of Operation
 - Attacks
 - Stream cipher
- Hash function and Message Authentication Code
- Public key encryption
- Digital signature
- Key establishment and management
- Introduction to other cryptographic topics

Recommended Reading

- CTP Section 3.6
- FIPS 197 (complete AES specifications)
<http://csrc.nist.gov/publications/fips/fips197/fips-197.pdf>
- Wikipedia:
 - AES
http://en.wikipedia.org/wiki/Advanced_Encryption_Standard

Advanced Encryption Standard (AES)

- AES
 - Block cipher
 - 128-bit block size
 - Substitution-permutation network
 - Three different key sizes & round numbers
 - AES-128: 128-bit key + 10 rounds
 - AES-192: 192-bit key + 12 rounds
 - AES-256: 256-bit key + 14 rounds

AES: History

- 1997: NIST called for algorithm to replace DES
- 1998: 15 ciphers submitted for competition
- 2001: Rijndael was approved as AES (FIPS 197)
 - Designers: Joan Daemen, Vincent Rijmen



AES: Applications

- AES
 - Free
 - Simplicity
 - High security
 - High performance (hardware & software)
- USA government standard
 - AES-128 for SECRET information
 - AES-192 and AES-256 for TOP SECRET information
- Commercial applications
 - Too many ...

Mathematical Preliminaries

1) The finite field $\text{GF}(2^8)$

Mathematical Preliminaries

- The finite field GF(2⁸)
 - Binary notation: byte \mathbf{b} , consisting of eight bits
$$b_7 \ b_6 \ b_5 \ b_4 \ b_3 \ b_2 \ b_1 \ b_0$$
 - Polynomial notation: \mathbf{b} is considered as a polynomial with binary coefficients (either 0 or 1):
$$b_7 x^7 + b_6 x^6 + b_5 x^5 + b_4 x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0$$
 - Example: the byte with hexadecimal value {57} (denoted as 0x57, binary 01010111) corresponds with polynomial:

$$x^6 + x^4 + x^2 + x + 1$$

Bit Pattern	Character
0000	0
0001	1
0010	2
0011	3

Bit Pattern	Character
0100	4
0101	5
0110	6
0111	7

Bit Pattern	Character
1000	8
1001	9
1010	a
1011	b

Bit Pattern	Character
1100	c
1101	d
1110	e
1111	f

Mathematical Preliminaries (contd.)

- Addition in finite field GF(2⁸):

$$(x^6 + x^4 + x^2 + x + 1) + (x^7 + x + 1) = x^7 + x^6 + x^4 + x^2 \quad (\text{polynomial notation});$$

$$\{01010111\} \oplus \{10000011\} = \{11010100\} \quad (\text{binary notation});$$

$$\{57\} \oplus \{83\} = \{d4\} \quad (\text{hexadecimal notation}).$$

Bit Pattern	Character
0000	0
0001	1
0010	2
0011	3

Bit Pattern	Character
0100	4
0101	5
0110	6
0111	7

Bit Pattern	Character
1000	8
1001	9
1010	a
1011	b

Bit Pattern	Character
1100	c
1101	d
1110	e
1111	f

Mathematical Preliminaries (contd.)

- Multiplication in finite field $\text{GF}(2^8)$
 - denoted by •
 - defined as multiplication of binary polynomials modulo an irreducible binary polynomial of degree 8
 - irreducible polynomial: indivisible by any polynomial other than 1 and itself
 - In AES, the following irreducible polynomial is used:

$$m(x) = x^8 + x^4 + x^3 + x + 1$$

Mathematical Preliminaries (contd.)

- Multiplication in finite field GF(2⁸) (contd.)
 - Example:

$$\begin{aligned}
 & \frac{(x^6 + x^4 + x^2 + x + 1) (x^7 + x + 1)}{x^6 + x^4 + x^2 + x + 1} = x^{13} + x^{11} + x^9 + x^8 + x^7 + \\
 & \quad x^7 + x^5 + x^3 + x^2 + x + x^6 + x^4 + x^2 + x + 1 = x^{13} + x^{11} + x^9 + x^8 + x^6 + x^5 + x^4 + x^3 + 1 \\
 & x^{13} + x^{11} + x^9 + x^8 + x^6 + x^5 + x^4 + x^3 + 1 \pmod{(x^8 + x^4 + x^3 + x + 1)} \\
 & = x^7 + x^6 + 1. \\
 & \{57\} \bullet \{83\} = \{c1\}
 \end{aligned}$$

Mathematical Preliminaries (contd.)

- Multiplicative inverse in finite field GF(2⁸)
 - Extended Euclidean Algorithm is used to find the inverse
 - For given a and b , find x and y satisfying
$$ax + by = \gcd(a, b)$$
 - If $\gcd(a, b) = 1$, then $ax \bmod b = 1$, i.e.,
 x is the modular multiplicative inverse of a modulo b

Mathematical Preliminaries (contd.)

2) Polynomials with coefficients in GF(2^8)

Mathematical Preliminaries (contd.)

- Polynomials with coefficients in $\text{GF}(2^8)$
 - Let $[a_0, a_1, a_2, a_3]$ denote four bytes
$$a(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$
 - The above notation of polynomial is different from the polynomial notation of $\text{GF}(2^8)$
 - The same indeterminate x is used
 - But coefficients here are elements of $\text{GF}(2^8)$
 - And the multiplication of four-term polynomials uses a different reduction polynomial:
$$x^4 + 1$$
 - $x^4 + 1$ is not an irreducible polynomial over $\text{GF}(2^8)$!
 - Multiplication by a fixed polynomial is not necessarily invertible
 - In AES, a particular fixed polynomial with an inverse is used.

Mathematical Preliminaries (contd.)

- Addition of polynomials with coefficients in $\text{GF}(2^8)$

$$a(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

$$b(x) = b_3x^3 + b_2x^2 + b_1x + b_0$$

$$a(x) + b(x) = (a_3 \oplus b_3)x^3 + (a_2 \oplus b_2)x^2 + (a_1 \oplus b_1)x + (a_0 \oplus b_0)$$

Mathematical Preliminaries (contd.)

- Multiplication of polynomials with coefficients in GF(2⁸)
 - Denoted by \otimes

$$a(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

$$b(x) = b_3x^3 + b_2x^2 + b_1x + b_0$$

$$c(x) = a(x) \bullet b(x)$$

$$= c_6x^6 + c_5x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$$

$$c_0 = a_0 \bullet b_0$$

$$c_4 = a_3 \bullet b_1 \oplus a_2 \bullet b_2 \oplus a_1 \bullet b_3$$

$$c_1 = a_1 \bullet b_0 \oplus a_0 \bullet b_1$$

$$c_5 = a_3 \bullet b_2 \oplus a_2 \bullet b_3$$

$$c_2 = a_2 \bullet b_0 \oplus a_1 \bullet b_1 \oplus a_0 \bullet b_2$$

$$c_6 = a_3 \bullet b_3$$

$$c_3 = a_3 \bullet b_0 \oplus a_2 \bullet b_1 \oplus a_1 \bullet b_2 \oplus a_0 \bullet b_3$$

Mathematical Preliminaries (contd.)

- Multiplication of polynomials with coefficients in $\text{GF}(2^8)$ (contd.)

$$\begin{aligned} d(x) &= a(x) \otimes b(x) \\ &= (a(x) \bullet b(x)) \bmod (x^4 + 1) \end{aligned}$$

Since

$$x^i \bmod (x^4 + 1) = x^{i \bmod 4}$$

we have

$$d(x) = c_3 x^3 + (c_6 + c_2)x^2 + (c_5 + c_1)x + (c_4 + c_0)$$

Mathematical Preliminaries (contd.)

- Multiplication of polynomials with coefficients in $\text{GF}(2^8)$ (contd.)

Let

$$d(x) = d_3x^3 + d_2x^2 + d_1x + d_0$$

We have

$$d_0 = (a_0 \bullet b_0) \oplus (a_3 \bullet b_1) \oplus (a_2 \bullet b_2) \oplus (a_1 \bullet b_3)$$

$$d_1 = (a_1 \bullet b_0) \oplus (a_0 \bullet b_1) \oplus (a_3 \bullet b_2) \oplus (a_2 \bullet b_3)$$

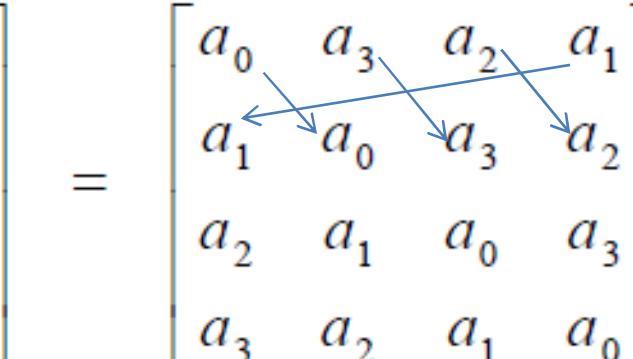
$$d_2 = (a_2 \bullet b_0) \oplus (a_1 \bullet b_1) \oplus (a_0 \bullet b_2) \oplus (a_3 \bullet b_3)$$

$$d_3 = (a_3 \bullet b_0) \oplus (a_2 \bullet b_1) \oplus (a_1 \bullet b_2) \oplus (a_0 \bullet b_3)$$

Mathematical Preliminaries (contd.)

- Multiplication of polynomials with coefficients in $\text{GF}(2^8)$ (contd.)

For a fixed polynomial $a(x)$, $d(x) = a(x) \otimes b(x)$ can be written in a matrix form:

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} a_0 & a_3 & a_2 & a_1 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_1 & a_0 & a_3 \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$


Mathematical Preliminaries (contd.)

- Multiplication of polynomials with coefficients in GF(2⁸) (contd.)

In AES, an invertible $a(x)$ is used:

$$a(x) = \{03\}x^3 + \{01\}x^2 + \{01\}x + \{02\}$$

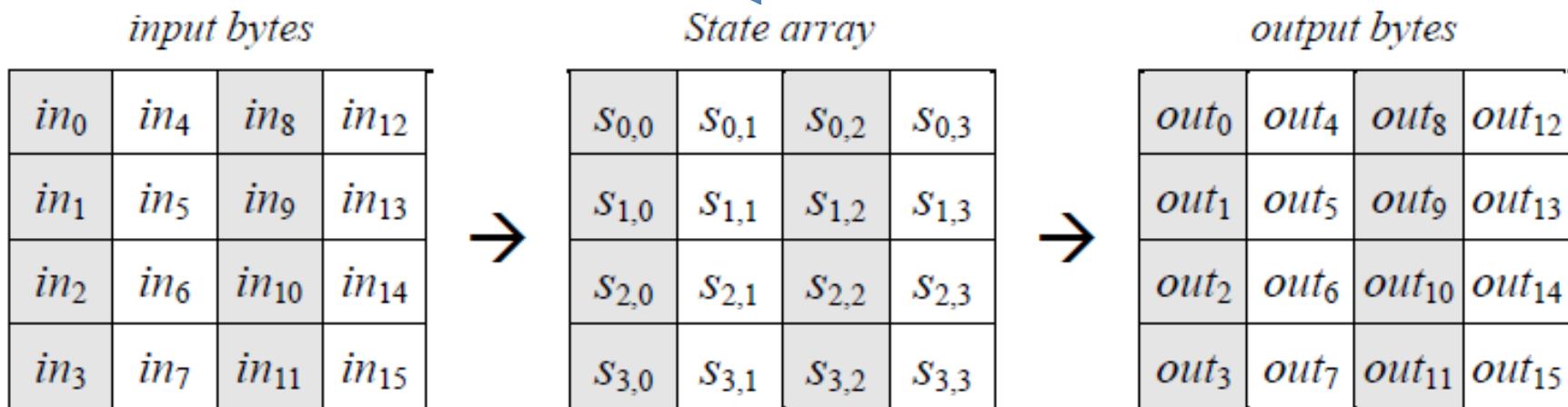
$$a^{-1}(x) = \{0b\}x^3 + \{0d\}x^2 + \{09\}x + \{0e\}$$

AES: state

16 bytes (the same as the plaintext size)

Represented as a 2D array

- Four rows, four columns



AES: overall

State = Plaintext

AddRoundKey(State, RoundKey₀)

for $i = 1$ to $r-1$,

SubBytes(State)

ShiftRows(State)

MixColumns(State)

AddRoundKey(State, RoundKey _{i})

end for;

SubBytes(State)

ShiftRows(State)

~~MixColumns(State)~~

AddRoundKey(State, RoundKey _{r})

} $r-1$ rounds

} The last round

Ciphertext = state

$r+1$ round keys are used

$S_{0,0}$	$S_{0,1}$	$S_{0,2}$	$S_{0,3}$
$S_{1,0}$	$S_{1,1}$	$S_{1,2}$	$S_{1,3}$
$S_{2,0}$	$S_{2,1}$	$S_{2,2}$	$S_{2,3}$
$S_{3,0}$	$S_{3,1}$	$S_{3,2}$	$S_{3,3}$

AES: SubByte

State = Plaintext

AddRoundKey(State, Key0)

for $i = 1$ to $r-1$,

SubBytes(State)

ShiftRows(State)

MixColumns(State)

AddRoundKey(State, RoundKey $_i$)

end for;

SubBytes(State)

ShiftRows(State)

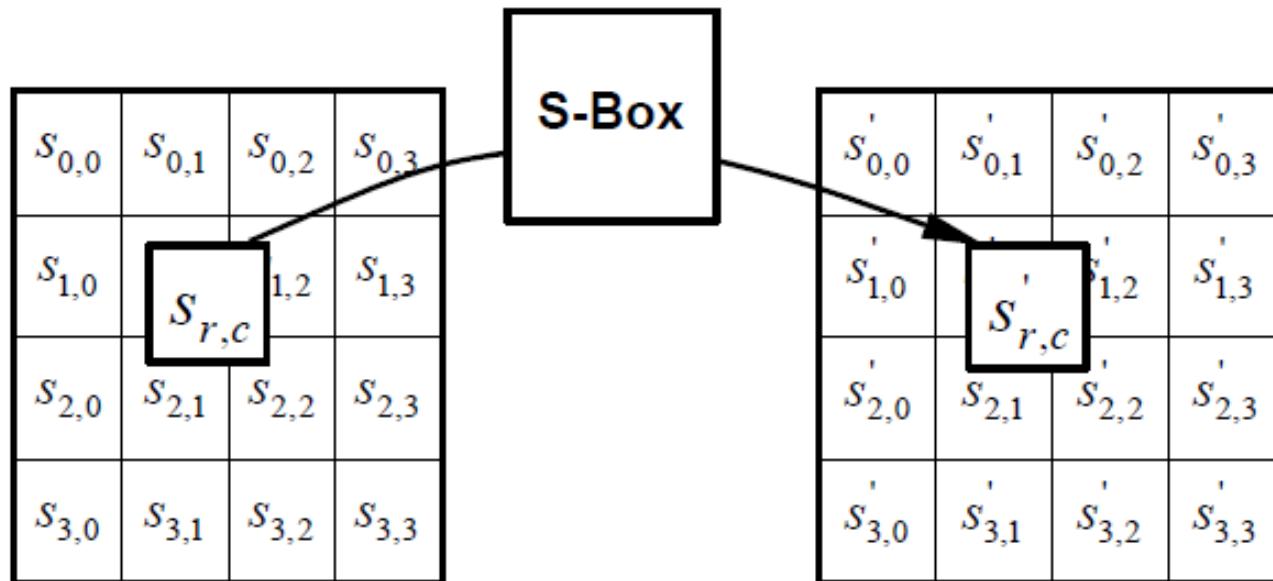
~~MixColumns(State)~~

AddRoundKey(State, RoundKey $_r$)

Ciphertext = state

AES: SubByte (contd.)

Apply Sbox to each byte in the state



AES: SubByte (contd.)

- S-Box
 - 8-bit input; 8-bit output, Invertible
 - Two steps to compute $b' = S(x)$
 - $b = x^{-1}$ in $\text{GF}(2^8)$
 - Apply the following transformation to b

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

AES: ShiftRows

State = Plaintext

AddRoundKey(State, Key0)

for $i = 1$ to $r-1$,

 SubBytes(State)

ShiftRows(State)

 MixColumns(State)

 AddRoundKey(State, RoundKey $_i$)

end for;

SubBytes(State)

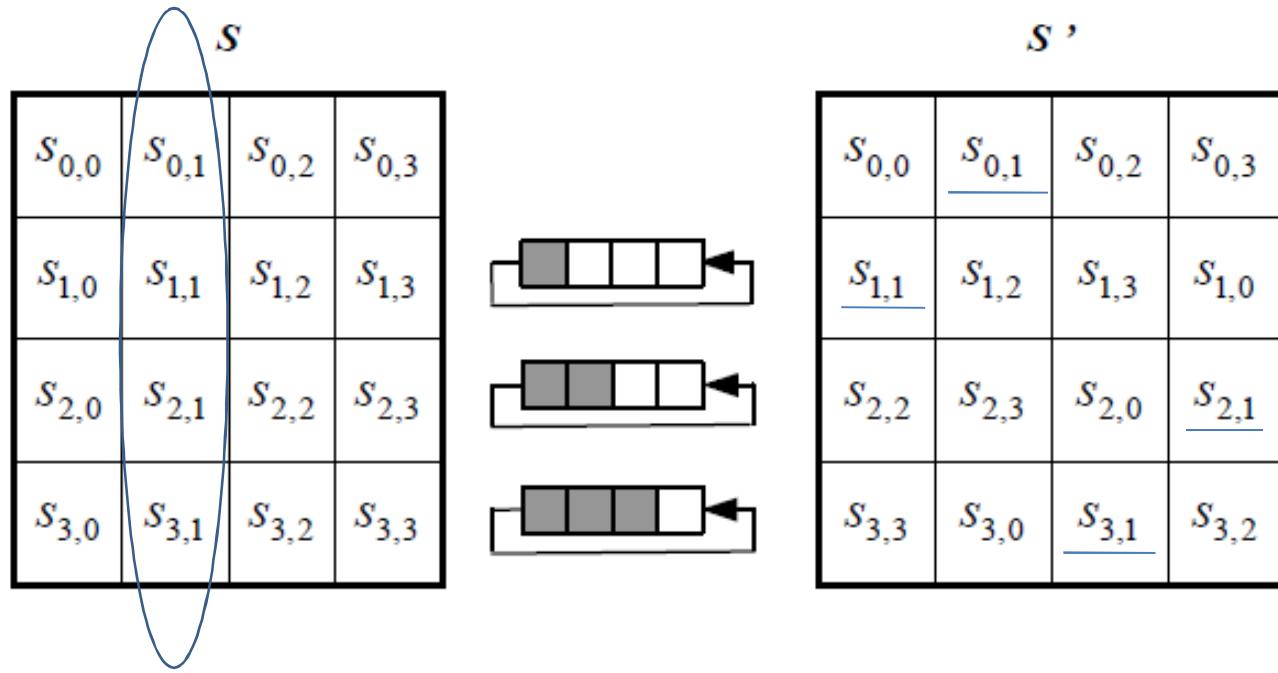
ShiftRows(State)

~~MixColumns(State)~~

AddRoundKey(State, RoundKey $_r$)

Ciphertext = state

AES: ShiftRows (contd.)



1st row: remain unchanged
 2nd row: left rotate one-byte position
 3rd row: left rotate two-byte position
 4th row: left rotate three-byte position

Reason for ShiftRows: four elements in one column relocated to 4 different columns after the ShiftRows

AES: MixColumns

State = Plaintext

AddRoundKey(State, Key0)

for $i = 1$ to $r-1$,

 SubBytes(State)

 ShiftRows(State)

MixColumns(State)

 AddRoundKey(State, RoundKey $_i$)

end for;

SubBytes(State)

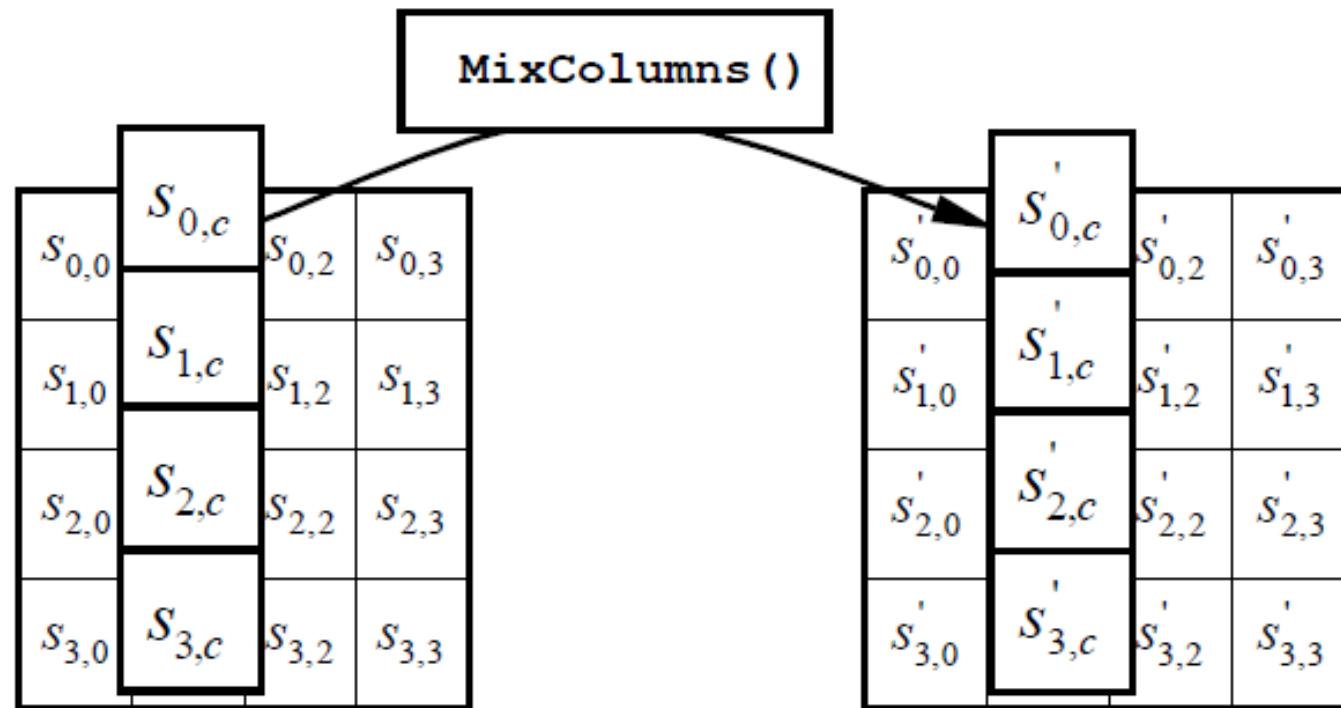
ShiftRows(State)

~~MixColumns(State)~~

AddRoundKey(State, RoundKey $_r$)

Ciphertext = state

AES: MixColumns (contd.)



$$a(x) = \{03\}x^3 + \{01\}x^2 + \{01\}x + \{02\}$$

$$s'(x) = a(x) \otimes s(x)$$

AES: MixColumns (contd.)

$$\begin{bmatrix} s_{0,c}' \\ s_{1,c}' \\ s_{2,c}' \\ s_{3,c}' \end{bmatrix} = \begin{bmatrix} 02 & 03 & 01 & 01 \\ 01 & 02 & 03 & 01 \\ 01 & 01 & 02 & 03 \\ 03 & 01 & 01 & 02 \end{bmatrix} \begin{bmatrix} s_{0,c} \\ s_{1,c} \\ s_{2,c} \\ s_{3,c} \end{bmatrix}$$

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} a_0 & a_3 & a_2 & a_1 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_1 & a_0 & a_3 \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$a(x) = \{03\}x^3 + \{01\}x^2 + \{01\}x + \{02\}$$

AES: MixColumns (contd.)

- Important properties of MixColumn
 - One input byte affects all four output bytes
 - If there are α non-zero bytes in the input;
 β non-zero bytes in the output;
then $(\alpha + \beta) \geq 5$

$$\begin{bmatrix} s_{0,c} \\ s_{1,c} \\ s_{2,c} \\ s_{3,c} \end{bmatrix} = \begin{bmatrix} 02 & 03 & 01 & 01 \\ 01 & 02 & 03 & 01 \\ 01 & 01 & 02 & 03 \\ 03 & 01 & 01 & 02 \end{bmatrix} \begin{bmatrix} s_{0,c} \\ s_{1,c} \\ s_{2,c} \\ s_{3,c} \end{bmatrix}$$

The AES MixColumn generates a maximum distance separable (MDS) code. For a 4-byte x , the distance between any $(x, \text{Mixcolumn}(x))$ is at least 5 over $\text{GF}(2^8)$.

AES: AddRoundKey

State = Plaintext

AddRoundKey(State, Key₀) $l = 0$

for $i = 1$ to $r-1$,

 SubBytes(State)

 ShiftRows(State)

 MixColumns(State)

AddRoundKey(State, RoundKey _{i}) $l = 4i$

end for;

SubBytes(State)

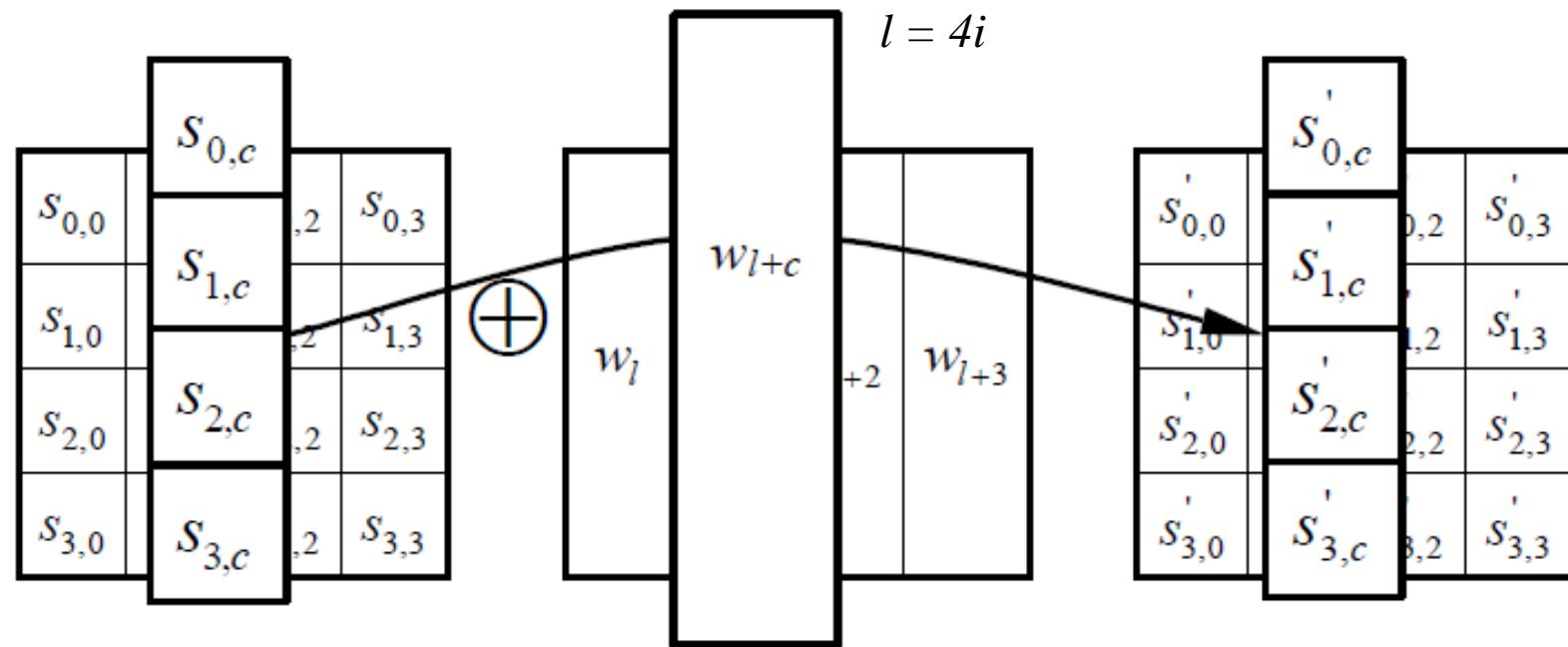
ShiftRows(State)

~~MixColumns(State)~~

AddRoundKey(State, RoundKey _{r})

Ciphertext = state

AES: AddRoundKey



AES: Key Schedule

- Each round key is 128-bit;
- Round keys are represented as an array of 32-bit words: $w[i]$
 - The first round key is $w[0], w[1], w[2], w[3]$
 - The second round key is $w[4], w[5], w[6], w[7]$
 -
- Secret key is represented as an array of bytes: $\text{key}[0], \text{key}[1], \text{key}[2], \dots$
- Two functions are used in the key schedule
 - SubWord()
 - 4-byte input
 - Apply Sbox to each input byte
 - RotWord()
 - Input: 4-byte $[a_0, a_1, a_2, a_3]$
 - Output: $[a_1, a_2, a_3, a_0]$
- A 32-bit round constant is used for generating each round key
 - $Rcon(i): [x^{i-1}, 0, 0, 0]$, where $x = 2$, x^{i-1} is the power of x in the field $GF(2^8)$
 - Different constants for different rounds to prevent slide-attack

AES: Key Schedule

- Example: AES-128

```

 $RCon[1] \leftarrow 01000000$ 
 $RCon[2] \leftarrow 02000000$ 
 $RCon[3] \leftarrow 04000000$ 
 $RCon[4] \leftarrow 08000000$ 
 $RCon[5] \leftarrow 10000000$ 
 $RCon[6] \leftarrow 20000000$ 
 $RCon[7] \leftarrow 40000000$ 
 $RCon[8] \leftarrow 80000000$ 
 $RCon[9] \leftarrow 1B000000$ 
 $RCon[10] \leftarrow 36000000$ 
for  $i \leftarrow 0$  to 3
    do  $w[i] \leftarrow (key[4i], key[4i + 1], key[4i + 2], key[4i + 3])$ 
for  $i \leftarrow 4$  to 43
    do {
         $temp \leftarrow w[i - 1]$ 
        if  $i \equiv 0 \pmod{4}$ 
            then  $temp \leftarrow \text{SUBWORD}(\text{ROTWORD}(temp)) \oplus RCon[i/4]$ 
         $w[i] \leftarrow w[i - 4] \oplus temp$ 
    }
return  $(w[0], \dots, w[43])$ 

```

Round constants

$W[0] = (K[0], K[1], K[2], K[3])$
 $W[1] = (K[4], K[5], K[6], K[7])$
 $W[2] = (K[8], K[9], K[10], K[11])$
 $W[3] = (K[12], K[13], K[14], K[15])$

Load the key into $w[]$

11 round keys

AES: key schedule

```
KeyExpansion(byte key[4*Nk], word w[Nb*(Nr+1)], Nk)
begin
    word temp

    i = 0

    while (i < Nk)
        w[i] = word(key[4*i], key[4*i+1], key[4*i+2], key[4*i+3])
        i = i+1
    end while

    i = Nk

    while (i < Nb * (Nr+1)]
        temp = w[i-1]
        if (i mod Nk = 0)
            temp = SubWord(RotWord(temp)) xor Rcon[i/Nk]
        else if (Nk > 6 and i mod Nk = 4)
            temp = SubWord(temp)
        end if
        w[i] = w[i-Nk] xor temp
        i = i + 1
    end while
end
```

	Key Length (Nk words)	Block Size (Nb words)	Number of Rounds (Nr)
AES-128	4	4	10
AES-192	6	4	12
AES-256	8	4	14

AES: Decryption

(functions are different from that in encryption)

State = ciphertext

AddRoundKey(State, RoundKey_r)

for $i = 1$ to $r-1$,

 InvShiftRows(State)

 InvSubBytes(State)

 AddRoundKey(State, RoundKey_{r-i})

 InvMixColumns(State)

end for;

 InvShiftRows(State)

 InvSubBytes(State)

 AddRoundKey(State, RoundKey₀)

~~InvMixColumns(State)~~

plaintext = state

} $r-1$ rounds

} The last round

$r+1$ round keys are used

AES: Encryption & Decryption

State = Plaintext

```
AddRoundKey(State, RoundKey0)  
  
for i = 1 to r-1,  
    SubBytes(State)  
    ShiftRows(State)  
    MixColumns(State)  
    AddRoundKey(State, RoundKeyi)  
end for;  
  
SubBytes(State)  
ShiftRows(State)  
MixColumns(State)  
AddRoundKey(State, RoundKeyr)  
  
ciphertext = state
```

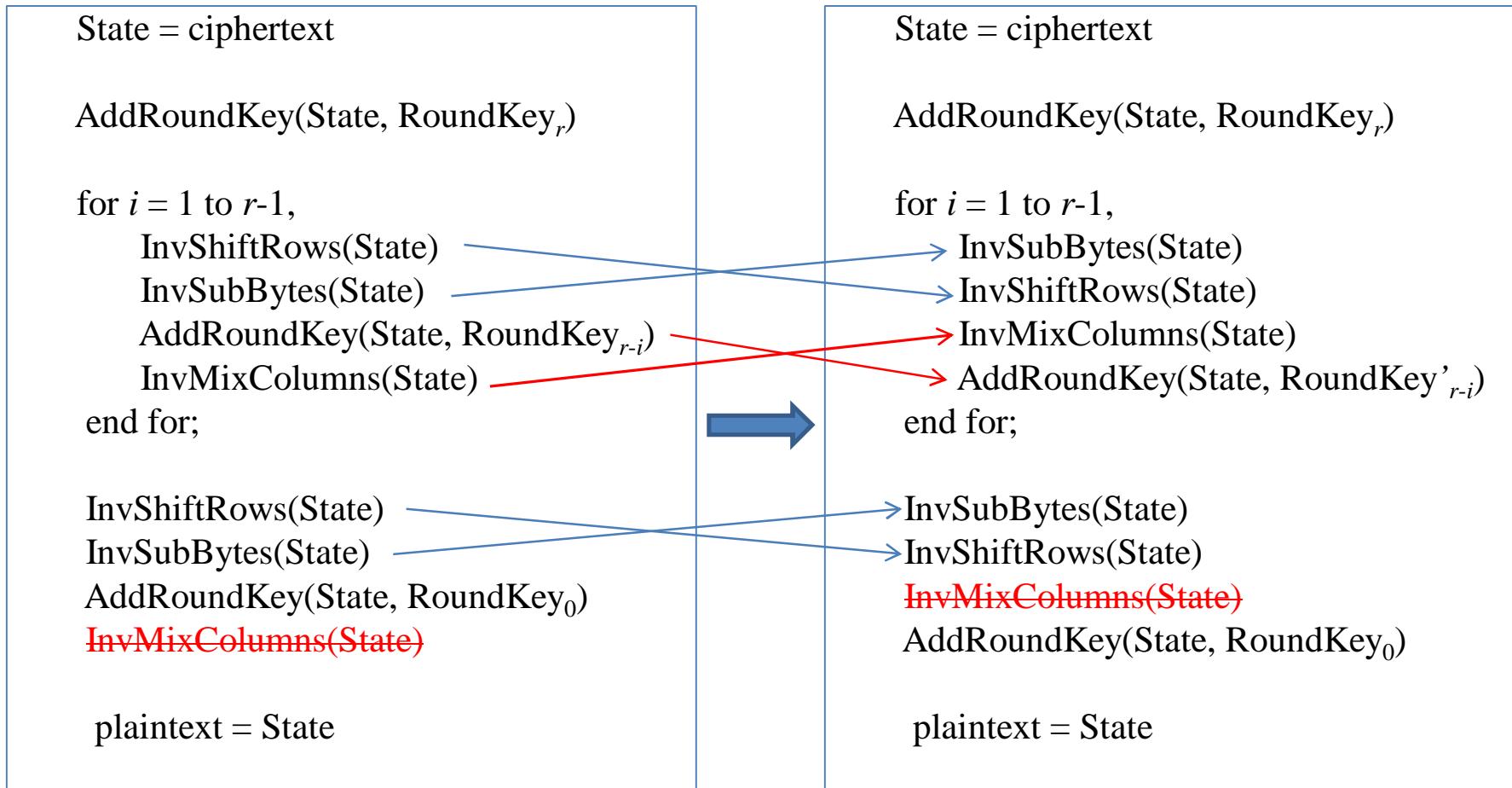
State = ciphertext

```
AddRoundKey(State, RoundKeyr)  
  
for i = 1 to r-1,  
    InvShiftRows(State)  
    InvSubBytes(State)  
    AddRoundKey(State, RoundKeyr-i)  
    InvMixColumns(State)  
end for;  
  
InvShiftRows(State)  
InvSubBytes(State)  
AddRoundKey(State, RoundKey0)  
InvMixColumns(State)  
  
plaintext = State
```

AES: Equivalent Decryption Algorithm

- The order of InvSubBytes() and InvShiftRows() can be reversed
- The order of AddRoundKey() and InvMixColumns() can also be reversed, if the columns of decryption round keys are modified using the InvMixColumns() transformation

AES: Equivalent Decryption Algorithm



$$\text{RoundKey}'_{r-i} = \text{InvMixColumns}(\text{RoundKey}_{r-i}) \text{ for } i = 1 \text{ to } r-1$$

AES Implementation

- In practice, the implementation of the cipher must be correct
 - How to check the correctness of the implementation?
 - Using the test vectors provided in the standard
<http://csrc.nist.gov/publications/fips/fips197/fips-197.pdf>

Summary

- Mathematical preliminaries
 - $\text{GF}(2^8)$
 - Polynomials with coefficients in $\text{GF}(2^8)$
- AES
 - Encryption
 - Substitution-Permutation Network
 - Round function
 - different round numbers for different key sizes
 - Key schedule
 - different for different key sizes
 - Two equivalent decryption algorithms
 - One is straight forward inverse
 - Another with modified key schedule