

CS5234: Combinatorial and Graph Algorithms

Problem Set 1

Due: August 23th, 6:30pm

Instructions. The *exercises* at the beginning of the problem set do not have to be submitted—they are for your review. This problem set contains one main problem, which involves coming up with a fast algorithm to decide whether a 2-dimensional data set can be classified into two sets. The goal is to develop an efficient algorithm, and the solution relies on the sampling techniques we covered in class in Week 1. (I think it is really neat how much we can determine about a data set, even when we are only willing to look at a small part of it!)

- Please submit the problem set on LumiNUS in the appropriate folder. (Typing the solution using latex is recommended.) If you want to do the problem set by hand, please submit it at the beginning of class.
- Start each problem on a separate page.
- If you submit the problem set on paper, make sure your name is on each sheet of paper (and legible).
- If you submit the problem set on paper, staple the pages together.

Remember, that when a question asks for an algorithm, you should:

- First, give an overview of your answer. Think of this as the executive summary.
- Second, describe your algorithm in English, giving pseudocode if helpful.
- Third, give an example showing how your algorithm works. Draw a picture.

You may then give a proof of correctness, or explanation, of why your algorithm is correct, an analysis of the running time, and/or an analysis of the approximation ratio, depending on what the question is asking for.

Advice. Start the problem set early—questions may take time to think about. Come talk to me about the questions. Talk to other students about the problems.

Collaboration Policy. The submitted solution must be your own unique work. You may discuss your high-level approach and strategy with others, but you must then: (i) destroy any notes; (ii) spend 30 minutes on facebook or some other non-technical activity; (iii) write up the solution on your own; (iv) list all your collaborators. Similarly, you may use the internet to learn basic material, but do not search for answers to the problem set questions. You may not use any solutions that you find elsewhere, e.g. on the internet. Any similarity to other students' submissions will be treated as cheating.

Exercises and Review (*Do not submit.*)

Exercise 1. Imagine you are taking a poll of students at NUS. (Alas, you do not know exactly how many students there are at NUS.) Your goal is to determine what fraction of students like Starbucks. The polling error is allowed to be 2%, i.e., your answer should be within 2% of the correct answer. Your survey should be correct (i.e., within the desired polling error) with probability 90%. The university will give you a list of randomly students to call or e-mail. How big a list of students do you need to get a sufficiently precise answer? Give an exact (integer) numeric answer. (Notice that the total number of students at NUS does not matter!)

Solution: Let f be the actual fraction of students who like Starbucks. Let X_i be the samples, and return as your answer $\sum(X_i)/s$. Notice that $E[\sum X_i] = f$, so we have the correct expectation. We can solve for the size of s using a Hoeffding Bound. For sample size s , where X_i are the samples and $X = \sum(X_i)$, we want $|(\sum X_i/s - f)| \leq 0.02$, i.e., $|\sum(X_i) - fs| \leq 0.02s$. Applying a Hoeffding Bound:

$$\Pr [|X - E[x]| > s \cdot 0.02] \leq e^{-2(0.0004)s} \leq e^{-0.0008 \cdot s} \leq 0.1$$

Solving for s , we conclude that $s \geq 2,878.2$, i.e., choose $s = 2,879$.

Exercise 2. Imagine you have an array $A[1..n]$. Each value in the array is an integer between 1 and M . Consider the following algorithm for finding the approximate sum of the values in the array: Fix $s = 1/\epsilon^2$. Your boss claims that this is a good algorithm, and gives the following proof:

Algorithm 1: Sum(A, n, s)

```

1 sum = 0
2 repeat  $s$  times
3   Choose a random  $i \in [1, n]$ .
4   sum = sum +  $A[i]$ .
5 return  $n(\text{sum}/s)$ 
```

Let x_i be the value of the i th random sample, and $X = \sum(x_i)$. Let $A = \sum_i A[i]/n$, i.e., A is the average value and nA is the sum of the array (i.e., what we want to find). We know that $E[X] = sA$. Then, using a Hoeffding Bound, we know that:

$$\Pr [|X - E[X]| \geq \epsilon E[X]] = \Pr [|X - sA| \geq \epsilon sA] \leq 2e^{-2(sA\epsilon)^2/s} .$$

We know that $A \geq 1$, so by choosing $s = 1/\epsilon^2$, we conclude that:

$$\begin{aligned} 2e^{-2(sA\epsilon)^2/s} &\leq 2e^{-2s\epsilon^2} \\ &\leq 2e^{-2} \end{aligned}$$

Thus, the probability of error is $\leq 1/3$. Thus, with probability at least 2/3, $|X - sA| < \epsilon sA$, which implies that: $|nX/s - nA| < \epsilon nA$, i.e.,

$$nA(1 - \epsilon) \leq \text{sum} \leq nA(1 + \epsilon) .$$

We thus conclude that the algorithm returns a $(1 \pm \epsilon)$ estimate of the sum of the values in the array.

What is wrong with this algorithm and proof? Can you give an example of an array A where this algorithm will almost certainly give the wrong answer?

Solution: The values in the array are between 1 and M , not between 0 and 1. In this case, you cannot use this Hoeffding Bound. (You can divide each value in the array by M , and then proceed. However, in that case your error margin will increase by a factor of M also!)

Imagine an array where 1 value is M and the remainder are 1. The probability of finding one of the M -values in the array is $1/n$ for each sample, and hence the probability that you find none of the M -values is at least $(1 - 1/n)^{1/\epsilon^2} \geq e^{-2/(n\epsilon^2)} > 1/2$ for $\epsilon > 2/\sqrt{n}$. Thus, the answer returned by the algorithm will be n , while the correct answer will be $(n-1) + M$. If $M > n$, then the resulting algorithm is not even a 2-approximation of the correct answer.

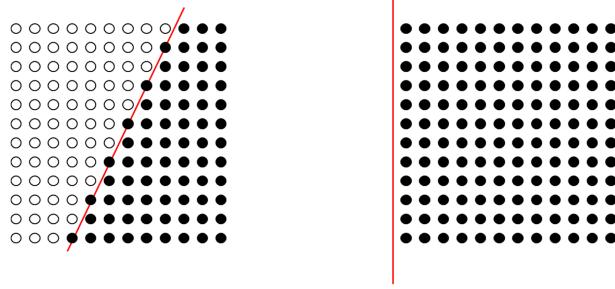
Standard Problems (to be submitted)

Problem 1. Classifying Pixels

A classic problem in data analysis is classification: given a set of points where each point is either a 0 or a 1, can we draw a line (or a hyperplane) that separates the points into two sets containing only (or mostly) 0's and only (or mostly) 1's.

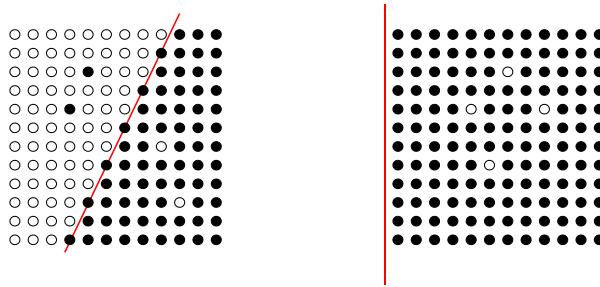
Here we look at a simplified version of the problem. Imagine you are given a two-dimensional n by n array A where $A[i, j]$ is either 0 or 1. If we think of the array A as a 2-dimensional display of pixels, we want to decide whether it is possible to draw a line separating all the 0's from all the 1's.

We say that array A is divisible if it is possible to draw such a line, where one side of the line is all 0's and the other side of the line is all 1's. Here are two examples of divisible arrays, where a white circle represents zero and a black circle represents one. The red line represents the separator.



Unfortunately, this might be very slow: we really have to look at all $\Theta(n^2)$ array locations to decide whether it is possible. Instead we ask whether an array A is either divisible or far from divisible. We say that array A is ϵ -close to being divisible if we only need to change $\epsilon \cdot n^2$ entries to make it divisible. (Equivalently, there are only $\epsilon \cdot n^2$ points on the wrong side of the separating line.)

In the following example, both arrays are 0.03-close to divisible: the array is 12 by 12 (i.e., 144 pixels), and each contains 4 mislabeled pixels. Since $4/144 < 0.03$, this is 0.03-close to divisible.



If array A is not ϵ -close to divisible, then it is ϵ -far from divisible. (Recall that it is only ϵ -far from divisible if it is not ϵ -close for any possible partition.)

Our goal in this problem is to develop an algorithm that has the following properties:

- If array A is divisible, then it returns TRUE.
- If array A is ϵ -far from divisible, then it returns FALSE.
- Otherwise, it can return either true or false.

The algorithm should return the correct answer with probability at least $2/3$. In the following parts, we will develop such an algorithm and prove it correct. We will measure the running time by counting how many times the array A is queried. All other computational costs will be ignored.

Problem 1.a. First, look at the four sides of the square: top, bottom, left, right. We say that a side of the square is *mismatched* if the endpoints are a different color. For example, in the first array given above (on the left), the top and bottom edges are mismatched while the left and right edges are not mismatched (i.e., they are matched).

Prove that for every array A , one of the following three cases holds: (i) 0 edges are mismatched, (ii) 2 edges are mismatched, or (iii) 4 edges are mismatched. (That is, you cannot have only one edge mismatched or three edges mismatched.)

Solution: Imagine starting at one corner and walking around the square, remembering track of the color of the most recent corner. Initially, the color is equal to the starting corner. When you cross a mismatched edge, you flip the color. When your return back to the first corner (completing your tour), the color is again equal to the starting color. Therefore, you must have changed the color an even number of times, i.e., 0 or 2 or 4.

Problem 1.b. Prove that if there are four edges mismatched, then A is not divisible.

Solution: Assume that it is divisible. The separator must cross the top edge, since the edge is mismatched and hence the top endpoints must be on opposite sides of the separator. Similarly, the separator must cross the bottom edge, since it is also mismatched. That implies that both ends of the left edge are on the same side of the separator, which violates the divisibility requirement as the left edge is also mismatched. Therefore, the square is not divisible.

Problem 1.c. Assume A has zero edges mismatched and all the corners are zero. There is only ONE possible array A that is divisible and satisfies this condition! (Hint: think about where you would have to draw a separator if A is divisible.)

Give an algorithm such that if A has zero edges mismatched, then it returns TRUE if A is divisible and FALSE if A is ϵ -far from divisible. (Hint: you can use an algorithm we discussed in class.)

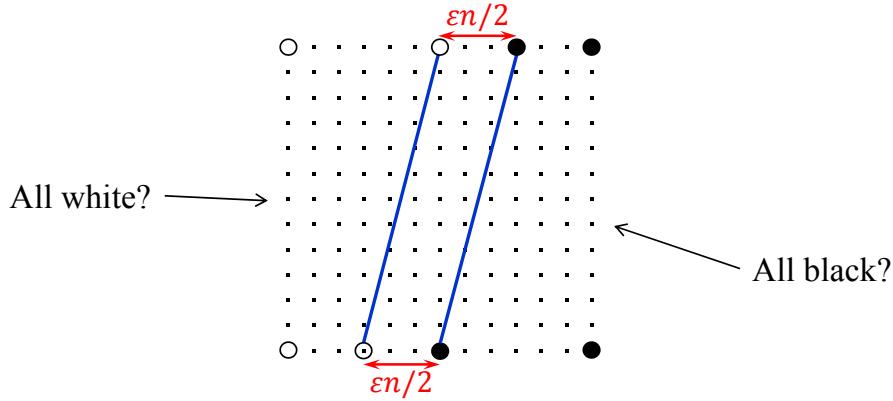
Prove (briefly) that your algorithm is correct. Your algorithm should run in sublinear time that does not depend on n (only on ϵ).

Solution: Assume that A is divisible. In this case, the separator cannot cross any of the edges (since they are not mismatched). Thus the only possible separator lies completely outside the square, i.e., all the points in the square are on the same side of the square. That is, if A is divisible, it must be the case that all the points in A are the same color.

Assume (without loss of generality) that all the corners are white. Then our goal is to find an algorithm that behaves as follows: (i) if all the points in the square are white, then return TRUE; (ii) if at least ϵn^2 points in the square are black, then return FALSE; (iii) otherwise, return either TRUE or FALSE. Notice that if A is divisible, this means that it returns true; and if the algorithm returns TRUE, then it is ϵ -close to divisible. (Hence if it is not ϵ -close to divisible, it must return FALSE.)

To solve this problem, then, we use the algorithm from class wherein we determine whether an array is all-zero or has at least ϵn ones. That is, we sample $\Theta(1/\epsilon)$ points in the square; if they are all zero, we return TRUE and if any of them are one, then we return FALSE. By the analysis from class, this is correct with probability at least $2/3$.

Problem 1.d. Assume A has two edges that are mismatched. Give an algorithm to find two points on each mismatched side that are of opposite colors and are within distance $\epsilon n/2$ of each other, as in the picture below.



On a mismatched side with corners x_1 and x_2 , your algorithm will find points y_1 and y_2 . If the points appear in order x_1, y_1, y_2, x_2 , then x_1 and y_1 should be the same color, and x_2 and y_2 should be the same color. See the diagram for an example.

(Hint: binary search.) What is the running time of your algorithm? (The running time of your algorithm should not depend on n .)

Solution: For each mismatched edge, do a binary search as follows: Begin with x and y as the two (mismatched) endpoints. Assume, without loss of generality, that x is white and y is black. Check the point p at location $\lfloor(x+y)/2\rfloor$. If the point is white, then set $x = p$; if the point is black, then set $y = p$. If $|x - y| \leq \epsilon n/2$, then stop.

Notice that for each mismatched edge, this takes at most $\log(n/(\epsilon n/2)) = \log(2/\epsilon)$ time.

Problem 1.e. Assume A is divisible and has two mismatched edges. In the picture above, we have identified two points on each mismatched side that are of opposite colors and are within distance $\epsilon n/2$ of each other. Imagine we connect these lines as shown above. If A is divisible, then it must be the case that all the points to the left of the line connecting the white points are white, and all the points to the right of the line connecting the black points are black.

Give an algorithm for the situation where A has two mismatched edges, and we have already identified two nearby points for each mismatched edge as described above. The algorithm should guarantee that:

- If A is divisible, then it returns TRUE.

- If A is ϵ -far from divisible, then it returns FALSE.

Prove that your algorithm is correct. What is the running time of your algorithm?

Solution: Assume we have two mismatched edges, and we have identified the two lines as described above, which we will refer to as line A and line B . Without loss of generality, assume that line A is to the left of line B , all the corners to the left of line A are white, and all the corners to the right of line B are black.

If A is divisible, this means that all the points to the left of A are white and all the points to the right of B are black. Hence we will check these two conditions. Use the algorithm from class, sampling $\Theta(1/\epsilon)$ points to the left of A and $\Theta(1/\epsilon)$ points to the right of B . If all the samples points to the left of A are white and all the sampled points to the right of B are black, then return TRUE. Otherwise, return FALSE. Choose the sample size so that if $> \epsilon n^2/4$ nodes are miscolored, the sampling algorithm returns FALSE.

Notice that if A is divisible, then the algorithm returns true. The lines A and B will correctly divide the square A , and so the sampling procedure will always find correctly colored points.

Assume that the algorithm does return TRUE. In that case, it means that there are at most $\epsilon n^2/4$ miscolored nodes on the left of A and at most $\epsilon n^2/4$ miscolored nodes on the right of B . There are at most $\epsilon n^2/2$ miscolored nodes between A and B . Thus, in total, there are at most ϵn^2 miscolored nodes, i.e., array A is ϵ -close to divisible.

Thus we conclude that if array A is not ϵ -close to divisible, then the algorithm returns FALSE. The total running time of the algorithm is $O(1/\epsilon)$.

Problem 1.f. Given all of the above, briefly sketch out the entire algorithm. What is the overall running time of the algorithm? (Remember, we only care about how many times array A is accessed; other computational costs can be ignored.)

Solution: The overall algorithm is as follows. First, examine the four corners and decide how many mismatched edges there are. Then:

- If there are four mismatched edges, return FALSE.
- If there are zero mismatched edges, run the sampling algorithm for time $O(1/\epsilon)$ and return TRUE if all samples are the same color as the corners and FALSE otherwise.
- If there are two mismatched edges, then use binary search (running in time $O(\log(1/\epsilon))$) to identify lines A and B , and then take $O(1/\epsilon)$ samples in the regions outside A and B . If the samples all match the corners in their region, then return TRUE. Otherwise, return FALSE.

We have proved above that in each case, if the array A is divisible, then it returns TRUE and if the array A is ϵ -far from divisible, then it returns FALSE. The total running time is $O(1/\epsilon)$.