Master thesis

Singularity types of central projections and their versality

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Abstract

We discuss singularities of central projections of a regular surface in $\mathbb{R}^3$. We describe criteria of singularity types of central projections of a given surface in terms of its Monge normal form and discuss their geometric meaning, which is often not clearly understood. We consider all possible central projections of a fixed surface as a central projection unfolding and discuss their $\mathcal{A}$-versality. We obtain geometric criteria of versality for central projection unfoldings. We also observe that geometric meaning of criteria of singularity types of central projections become clear assuming the versality of central projection unfoldings.

Contents

1 Introduction

Central projections (perspective projections) have been used since the ancient Greece world. Thales of Miletus used the gnomonic projection for star charts. This projections mean the central projection of the sphere from the center onto a plane tangent to the sphere. In the Renaissance period, there was interest in central projections as the drawing in perspective. G. Desargues (see, for instance, [5, Theorem 2.32]) gave a mathematical comprehension to the central projections.

When we observe the Earth from space, we recognize the shape of the Earth as the singular set of the projection. Nowadays, computer vision (for instance, [3, Chapter 11]) motivates to study singularities of central projection. One good example is to analyze view of pinhole camera model, which is also a central projection. The theory of singularities is necessary to analyze more complex image in computer vision.

In this paper, we investigate singularities of central projections changing center as parameter. Actually, we investigate central projection unfolding of a regular surface in Euclidian 3-space. Let us prepare several notations. Let f be a parametrized regular surface

$$f = (f_1(x), f_2(x), f_3(x)) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, f(0)) : (x_1, x_2) \mapsto f(0) + x_1\boldsymbol{u} + x_2\boldsymbol{v} + Q(x)\boldsymbol{w} \quad (1.1)$$

where $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal frame of $\mathbb{R}^3$ and

$$Q(x) := \sum\nolimits_{k\geq 2} H_k(x_1, x_2) \quad \text{and} \quad H_k(x_1, x_2) := \sum\nolimits_{i+j=k} \frac{a_{ij}}{i!\,j!} {x_1}^i {x_2}^j \quad (1.2)$$

are C^∞-functions. We call the expression (1.1) Monge normal form for $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$. We denote Euclidean inner product by $\langle\ ,\ \rangle$. Then, a **central projection** of a regular surface $f(x)$ from $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3$ to a z_1z_2-plane π is defined as the following:

$$\pi : (\mathbb{R}^2, 0) \times (\mathbb{R}^3, y) \longrightarrow (\mathbb{R}^2, \pi_y(0)) : (x_1, x_2, y) \mapsto \pi_y(x_1, x_2) \quad (1.3)$$

where

$$\pi_y(x_1, x_2) := \begin{pmatrix} \langle t(x,y)f(x) + (1 - t(x,y))\, y,\ \boldsymbol{e}_1\rangle \\ \langle t(x,y)f(x) + (1 - t(x,y))\, y,\ \boldsymbol{e}_2\rangle \end{pmatrix},$$

$$t(x, y) := \frac{y_3}{y_3 - \langle f(x),\ \boldsymbol{e}_3\rangle} \quad \text{and} \quad \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\ :\ \text{the standard basis of } \mathbb{R}^3.$$

π_y is written as

$$\frac{1}{y_3 - f_3(x)} \left(y_3 \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} - f_3(x) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right). \quad (1.4)$$

The center y of projection π in often called a **viewpoint** and the line $\mathcal{L}$ through y and $f(x)$ is called a **viewline**.

We regard π as an unfolding of π_y with parameters y. We call π a **central projection unfolding**. Our main result for versality of π is the following Theorem:

Theorem 1.1. *Let f be the normal form of parametrized regular surface as in* (1.1) *at the origin and the central projection unfolding π of f be as in* (1.3)*. Then, the criteria of versality of π at each $\mathcal{A}_e$-codimension ≤ 3 singularity are given as in Table* 1*.*

We quickly review the history of mathematical research on central projection from singularity theory viewpoint (cf. [4]). C. T. C. Wall [23] started to consider central projections from a perspective of singularity theory and stated a general transversality theorem due to his student J. M. S. David. He considered "generic" projections including central projections in [6].

type	$\mathcal{A}_e$-cod.	$\mathcal{A}$-det.	criteria	geometric interpretation
fold	0	2	always	
cusp	0	3	always	
swallowtail	1	4	always	
butterfly	2	7	$2a_{31}\,k_1 - 3a_{21}^2 \neq 0$	the flecnodal curve is not singular
elder butterfly	3	7	$2a_{31}\,k_1 - 3a_{21}^2 \neq 0$ and $(a_{60}\,k_1 - 3a_{21}\,a_{50})\,p_1 - 18a_{50}\,k_1 \neq 0$	the flecnodal curve is not singular and y is not in a special position
unimodal	3	8	not versal	
lips	1	3	always	
beaks	1	3	always	
goose	2	4	$k_2 \neq 0$	f is not flat umbilic
ugly goose	3	5		
gulls	2	5	$a_{40}\,k_2 - 3a_{21}^2 \neq 0$	f is the first order blue ridge
ugly gulls	3	7		
type 12	3	6	not versal	
type 16	3	5	not versal	

Table 1: Criteria of versality of π at each singularity

J. H. Rieger and M. A. S. Ruas [19, 20] classified all corank one map germs with $\mathcal{A}_e$-codimension ≤ 3. Criteria of singularities of π_y have been given by O. A. Platnova [18] and V. I. Arnold [1, 2]. O. A. Platnova recognized that asymptotic direction lines appear as a set of viewpoints y so that π_y is not fold. She states the following paragragh ([18], p. 2798):

> The only exclusions concern some points on isolated asymptotic lines in a hyperbolic domain with fourth order contact (no more than two on a line) and on asymptotic lines passing through parabolic points of the surface (not more than one on a line).

The asymptotic line here is that we call asymptotic direction line (in §2, Definition 2.5). She called the excluded points "h-focal" ("h" for "hyperbolic") and "p-focal" ("p" for "parabolic") respectively. Since she mentioned "p'-focal" in [18, Table 1], she seemed to be aware of the following treatment : Once we fix f, π_y has the same type Σ of singularity for a viewpoint y on an asymptotic direction line except several points y. We call such a point Σ-focal point.

Y. Kabata [13] has written criteria of singularities whose $\mathcal{A}_e$-codimension is up to 4 for plane-to-plane map-germs and apply to central projections of regular surfaces in the projective space $\mathbb{P}^3$. He also gave the conditions of Σ-focal point in terms of the coefficients of the Monge normal form f. We recall these results in our terminology for criteria of singularity types of a central projection π_y in §3.

As an applications of singularities of π_y, H. Sano, Y. Kabata and J. L. Deolindo Silva and T. Ohmoto [22] classified regular surfaces on $\mathbb{P}^3$ by using the classification of singularities of central projections of them. And related to bifurcations, they have determined local topological types of binary differential equations of asymptotic curves at parabolic point in $\mathbb{P}^3$ ([7]). From these, we are motivated to investigate certain criteria of versality of central projection unfoldings.

Versality for several geometric unfoldings are already investigated in [10] and [12]. T. Fukui and M. Hasegawa show ($\mathcal{K}$)-versality of distance squared unfoldings ([10]). In [12], criteria of $\mathcal{A}$-versality of orthogonal projection unfoldings are given. Both of them are concerned with geometric interpretation of conditions of versality. In this article, we show criteria of $\mathcal{A}$-versality of the central projection in §4. The key step is to compute the $\mathcal{A}_e$-tangent space. The computation is often complicated and we completed them using the aid of computer. The source code of Maxima scripts are available at `https://github.com/Shuhei-singularity123/Versality_of_central_projection_of_regular_surface`.

In §5 we show an application of our criteria of versality of π to geometric interpretations of singularities of π_y. Versal gulls series singularity of central projections is related to contact type with a cone. J. Montaldi [16] defines the notion of contact between two submanifolds and established the relation to $\mathcal{K}$-equivalence which is introduced by J. Mather ([14, §2]). For criteria of contact types of a surface, for instance, T. Fukui, M. Hasegawa, and K. Nakagawa [11] investigated contact type of a regular surface with right circular cylinders in $\mathbb{R}^3$.

2 Preliminary

We briefly summarize the basics. In this paper, $f : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^n, f(0))$ is a smooth map germ at 0; here "smooth" means C^∞. We set $\mathcal{E}_m$ an $\mathbb{R}$-algebra of smooth map-germs $\mathbb{R}^m, 0 \to \mathbb{R}$ with a unique maximal ideal $\mathrm{m}_m := \langle x_1, \cdots, x_m \rangle_{\mathcal{E}_m}$. We define

$$\mathcal{E}_m^n := \{f : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^n, f(0)) : f \text{ is a smooth map germ at } 0\}.$$

which is an $\mathcal{E}_m$-module. In particular,

$$\mathrm{m}_m \mathcal{E}_m^n := \{f \in \mathcal{E}_m^n : f(0) = 0\}.$$

2.1 Definitions from differential geometry

We consider a regular surface f as in (1.1) We set

$$E := \langle f_{x_1}, f_{x_1} \rangle, \ F := \langle f_{x_1}, f_{x_2} \rangle, \ G := \langle f_{x_2}, f_{x_2} \rangle$$

and

$$L := \langle f_{x_1 x_1}, \mathbf{n} \rangle, \ M := \langle f_{x_1 x_2}, \mathbf{n} \rangle, \ N := \langle f_{x_2 x_2}, \mathbf{n} \rangle$$

where $\mathbf{n}$ is the unit normal vector $\frac{f_{x_1} \times f_{x_2}}{|f_{x_1} \times f_{x_2}|}$. We call E, F and G (resp. L, M and N) the first (resp. second) fundamental quantities of the regular surface f. And, we define the **Gauss curvature**

$$K := \frac{LN - M^2}{EG - F^2}.$$

Then,

- If $K > 0$ at x, we call a point $f(x)$ **elliptic point**,
- If $K = 0$ at x, we call a point $f(x)$ **parabolic point**,
- If $K < 0$ at x, we call a point $f(x)$ **hyperbolic point**.

If there is a non-zero vector $\mathbf{v}$ such that

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} \mathbf{v} = \kappa \begin{pmatrix} E & F \\ F & G \end{pmatrix} \mathbf{v} \ \text{ for some } \ \kappa,$$

we call κ a **principal curvature** and a unit eigenvector generated by $\mathbf{v}$ for κ a **principal vector** on $\mathbb{R}^3$. We set κ_1 and κ_2 the principal curvatures of f at x. If $\kappa_1 = \kappa_2$ at x, we call a point $f(x)$ **umbilic point**. We call a point $f(x)$ **flat umbilic point** if $\kappa_1 = \kappa_2 = 0$ at x.

Definition 2.1. We assume that $f(x_0)$ is not an umbilic of a regular surface f, with principal vectors $\mathbf{v_1}$ ('blue') and $\mathbf{v_2}$ ('red') corresponding to principal curvature κ_1, κ_2. We say that the point $f(x_0)$ is a $\mathbf{v_i}$**-ridge point** ('blue ridge point' for $i = 1$, 'red ridge point' for $i = 2$) if $\mathbf{v_i}\kappa_i(x_0) = 0$, where $\mathbf{v_i}\kappa_i$ is the directional derivative of κ_i in $\mathbf{v_i}$.
Moreover, $f(x_0)$ is a k**-th order ridge point** relative to $\mathbf{v_i}$ if

$$\mathbf{v_i}^{(m)}\kappa_i(x_0) = 0 \quad (1 \leq m \leq k) \qquad \text{and} \qquad \mathbf{v_i}^{(k+1)}\kappa_i(x_0) \neq 0,$$

where $\mathbf{v_i}^{(m)}\kappa_i(x_0)$ is the m-times directional derivative of κ_i in $\mathbf{v_i}$. We call the set of ridge points a **ridge line** or **ridges**.

Lemma 2.2 ([10, Lemma 2.1])**.** *Let f be a regular surface* (1.1) *with parabolic point at the origin. Then the origin is a first order blue ridge point if and only if*

$$a_{30} = 0 \quad \textit{and} \quad 3{a_{21}}^2 - a_{40}k_2 \neq 0.$$

Definition 2.3. We assume that $f(x_0)$ is not an umbilic of a regular surface f, with principal vectors $\mathbf{v_1}$ ('blue') and $\mathbf{v_2}$ ('red') corresponding to principal curvature κ_1, κ_2. We say that the point $f(x_0)$ is a $\mathbf{v_i}$**-sub-parabolic point** ('blue sub-parabolic point' for $i = 1$, 'red sub-parabolic point' for $i = 2$) if $\mathbf{v_i}\kappa_j(x_0) = 0 \quad (i \neq j)$. We call the set of sub-parabolic points a **sub-parabolic line**.

Lemma 2.4 ([10, Lemma 2.2])**.** *Let f be a regular surface* (1.1) *with parabolic point at the origin. Then the origin is not red sub-parabolic point if and only if*

$$a_{21} \neq 0.$$

Definition 2.5. We say (dx_1, dx_2) represents an **asymptotic direction** of f at x if the second fundamental form

$$II := Ldx_1^2 + 2Mdx_1dx_2 + Ndx_2^2$$

vanishes at x. The tangent space of f at $f(x)$ contains a line $\mathcal{L}$ which is generated by the corresponding direction. We call $\mathcal{L}$ **asymptotic direction line** of f at x.

Definition 2.6. Let $\alpha(t) := (x_1(t), x_2(t))$ be a regular plane curve and let β another plane curve given as the zero set of a smooth function $\Phi : \mathbb{R}^2 \to \mathbb{R}$. We say that the curve α has $(k+1)$**-point contact** (k**-th order contact**) at t_0 with the curve β if t_0 is a zero of order k of the function $g(t) = \Phi(\alpha(t)) = \Phi(x_1(t), x_2(t))$, that is,

$$g(t_0) = g'(t_0) = \cdots = g^{(k)}(t_0) = 0 \ \text{ and } \ g^{(k+1)}(t_0) \neq 0$$

where $g^{(i)}$ denotes the i^{th}-derivative of the function g.

Definition 2.7. A point p on M is a **flecnodal point** if there is an asymptotic direction line through p which has at least 4-point contact with M at p. Equivalently, p is a flecnodal point if it is in the closure of the set of points where the projection along an asymptotic direction has a swallowtail singularity. The **flecnodal curve** of M is the set of flecnodal points.

Theorem 2.8 ([12, Theorem 6.6 (ii)])**.** *We assume that a regular surface f is hyperbolic at the origin and π_y has the butterfly singularity at this point. Then, the flecnodal curve of f is not singular if and only if $2k_1a_{31} - 3a_{21}^2$ does not vanish.*

2.2 Definitions from singularity theory

In this section, let f and f_i $(i = 1, 2)$ be smooth map germs in $\mathcal{E}^n_m$. We say f_1 and f_2 are **$\mathcal{A}$-equivalent** $(f_1 \sim_{\mathcal{A}} f_2)$ if there exist diffeomorphism germs φ and ψ so that the following diagram commutes :

$$\begin{array}{ccc} \mathbb{R}^m, 0 & \xrightarrow{f_1} & \mathbb{R}^n, f_1(0) \\ \varphi\big\downarrow & & \big\downarrow\psi \\ \mathbb{R}^m, 0 & \xrightarrow{f_2} & \mathbb{R}^n, f_2(0) \end{array}.$$

Definition 2.9 ($\mathcal{A}$-stability)**.** 1. Let $F : (\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0) \longrightarrow (\mathbb{R}^n, F(0, 0))$ be a smooth map germ. If $F(x, 0) = f(x)$, F is called an unfolding of f.

2. An unfolding F is trivial if there exist germs of diffeomorphisms $h : (\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0) \to (\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0)$ and $H : (\mathbb{R}^n \times \mathbb{R}^k, 0 \times 0) \to (\mathbb{R}^n \times \mathbb{R}^k, 0 \times 0)$ such that

   (i) $h(x, 0) = (x, 0)$ and $H(X, 0) = (X, 0)$.

   (ii) The following diagram is commutative ;

$$\begin{array}{ccccc} \mathbb{R}^m \times \mathbb{R}^k, 0 \times 0 & \xrightarrow{(F,\Pi)} & \mathbb{R}^n \times \mathbb{R}^k, 0 \times 0 & \xrightarrow{\Pi'} & \mathbb{R}^k, (0) \\ h\big\downarrow & & \big\downarrow H & & \big\downarrow id \\ \mathbb{R}^m \times \mathbb{R}^k, 0 \times 0 & \xrightarrow{(f,\Pi)} & \mathbb{R}^n \times \mathbb{R}^k, 0 \times 0 & \xrightarrow{\Pi'} & \mathbb{R}^k, (0) \end{array}$$

   where $\Pi : (\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0) \to \mathbb{R}^k, 0$ is the canonical projection.

3. We call $f : \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$ is **$\mathcal{A}$-stable** if every unfolding of f is trivial.

Definition 2.10 ($\mathcal{A}_e$-versal unfolding)**.** 1. Let $F_i : (\mathbb{R}^m \times \mathbb{R}^{k_i}, 0 \times 0) \to (\mathbb{R}^n, F_i(0, 0))(i = 1, 2)$ be unfoldings of f. A triplet (s, t, φ) is an $\mathcal{A}_e$-morphism from F_1 to F_2 if $\varphi : (\mathbb{R}^{k_1}, 0) \to (\mathbb{R}^{k_2}, 0)$ is a smooth map germ, $s : (\mathbb{R}^m \times \mathbb{R}^{k_1}, 0 \times 0) \to (\mathbb{R}^m, 0)$ and $t : (\mathbb{R}^n \times \mathbb{R}^{k_1}, F_2(0, 0) \times 0) \to (\mathbb{R}^n, F_1(0, 0))$ are unfoldings of id_m and id_n respectively such that

$$F_1(x, y) = t(F_2(s(x, y), \varphi(y)), y).$$

2. Let $F : (\mathbb{R}^m \times \mathbb{R}^k, 0) \to (\mathbb{R}^n, F(0, 0))$ be unfoldings of f with parameter y in $\mathbb{R}^k$. F is called an **$\mathcal{A}_e$-versal unfolding** if for any unfolding $(\mathbb{R}^m \times \mathbb{R}^l, 0) \to (\mathbb{R}^n, G(0, 0))$ of f, there exists an $\mathcal{A}_e$-morphism from G to F.

Let $\xi : \mathbb{R}^m, 0 \to T\mathbb{R}^n$ be a smooth map germ such that $\Pi \circ \xi = f$ where Π is a projection of tangent vector bundle. We call ξ the vector field along f or infinitesimal deformation of f. We write $\theta(f)$ for the set of all the vector field along f. $\theta(f)$ is a $\mathcal{E}_m$-module. For the identity maps $id_m : \mathbb{R}^m, 0 \to \mathbb{R}^m, 0$ and $id_n : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$, we write $\theta_m = \theta(id_m)$ and $\theta_n = \theta(id_n)$ which are the module of vector field germs. We define

$$tf : \theta_m \to \theta(f) : \xi \mapsto df \circ \xi, \quad \omega f : \theta_n \to \theta(f) : \eta \mapsto \eta \circ f$$

and $\mathcal{A}_e$-tangent space of f

$$T\mathcal{A}_e(f) := tf(\theta_m) + \omega f(\theta_n) \subset \theta(f).$$

Then the $\mathcal{A}_e$-codimension of f is defined by

$$cod(\mathcal{A}_e, f) := \dim_{\mathbb{R}} \frac{\theta(f)}{T\mathcal{A}_e(f)}.$$

Definition 2.11 ($\mathcal{A}_e$-infinitesimal versal unfolding)**.** Let $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, f_i(0))$ be a smooth map germ, and $(\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0) \to (\mathbb{R}^n, F(0,0))$ be an unfolding of f with parameter y in $\mathbb{R}^k$. Then, F is called an **infinitesimal $\mathcal{A}_e$-versal unfolding** if

$$T\mathcal{A}_e(f) + \sum_{i=1}^{k} \mathbb{R}\frac{\partial F}{\partial y_i}(x, 0) = \theta(f)$$

Theorem 2.12. *Let $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, f_i(0))$ be a smooth map germ and $F : (\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0) \to (\mathbb{R}^n, F(0,0))$ be an unfolding of f. F is $\mathcal{A}_e$-versal if and only if F is infinitesimal $\mathcal{A}_e$-versal.*

Proof. See [24, Theorem 3. 3 and Theorem 3. 4 (i)]. □

Theorem 2.13. *Let $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, f_i(0))$ be a smooth map germ and $F : (\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0) \to (\mathbb{R}^n, F(0,0))$ be an unfolding of f. If f is $\mathcal{A}$-stable, any F is $\mathcal{A}_e$-versal.*

Proof. From [15, Theorem 1], we know $T\mathcal{A}_e(f) = \theta(f)$ if f is $\mathcal{A}$-stable. □

Definition 2.14 (finite $\mathcal{A}$-determinacy)**.** A germ f is said to be k-$\mathcal{A}$-determined if any g with $j^k g = j^k f$ is $\mathcal{A}$-equivalent to f. The least integer k with this property is called the degree of determinacy of f. A finitely $\mathcal{A}$-determined germ is a k-$\mathcal{A}$-determined germ for integer k.

The following Theorem for k-$\mathcal{A}$-determinacy is important to prove versality of unfoldings.

Theorem 2.15 ([17, Thorem 3.3.2 (1)])**.** *For a smooth map germ f in $\mathcal{E}_m^n$,*

$$\mathrm{m}_n^{k+1}\theta(f) \subset T\mathcal{A}_e(f)$$

if f is k-$\mathcal{A}$-determined.

Proof. From [24, Lemma 1.6] proved by C. T. C. Wall and A. A. du Plessis in [8, 24], we see that

$$\mathrm{m}_n^{k+1}\mathcal{E}_m^n/tf\,(\mathrm{m}_m\,\mathcal{E}_m^m) \subset T\mathcal{A}(f)/tf\,(\mathrm{m}_m\,\mathcal{E}_m^m).$$

□

3 Criteria of singularity types

3.1 $\mathcal{A}_e$-codimension ≤ 1 singularities

First of all, we recall several results for criteria of singularity type of smooth map germ $g : \mathbb{R}^2, 0 \longrightarrow \mathbb{R}^2, 0$ with corank one at the origin. Let (x_1, x_2) be coordinates of source. We define $\lambda(x_1, x_2) := \det\left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}\right)$ and take an arbitrary vector field η near the origin of the source such that η spans $\ker dg$ on $\lambda = 0$. We denote $\eta^k\lambda := \eta(\eta^{k-1}\lambda)$.

Theorem 3.1 (Whitney [25, §4], Saji [21, Theorem 3])**.** *For a plane-to-plane map-germ g, $\mathcal{A}$-types of fold, cusp, swallowtail, lips and beaks are characterized by the following table:*

Type	*Normal form*	*Criteria*
fold	(x_1, x_2^2)	$d\lambda(0) \neq 0$, $\eta\lambda(0) \neq 0$
cusp	$(x_1, x_1x_2 + x_2^3)$	$d\lambda(0) \neq 0$, $\eta\lambda(0) = 0$, $\eta^2\lambda(0) \neq 0$
swallowtail	$(x_1, x_1x_2 + x_2^4)$	$d\lambda(0) \neq 0$, $\eta\lambda(0) = \eta^2\lambda(0) = 0$, $\eta^3\lambda(0) \neq 0$
lips(+), beaks(-)	$(x_1, x_2^3 \pm x_1^2x_2)$	$d\lambda(0) = 0$, $\det H_\lambda(0) \neq 0$, $\eta^2\lambda(0) \neq 0$

where $\det H_\lambda(0)$ *is the Hessian of* λ *at the origin.*

We denote by $J\pi_y(x)$ the Jacobi matrix of π_y at x.

Theorem 3.2. *The projection π_y has a singular point at the origin if and only if the viewline $\mathcal{L}$ is contained in the tangent space of f at the origin.*

Proof. The Jacobi matrix of π_y is written as

$$J\pi_y(x) = \frac{y_3}{(y_3 - f_3(x))^2}\left((y_3 - f_3(x))\begin{pmatrix}(f_1)_{x_1} & (f_1)_{x_2}\\ (f_2)_{x_1} & (f_2)_{x_2}\end{pmatrix} - \begin{pmatrix}y_1 - f_1(x) & \\ & y_2 - f_2(x)\end{pmatrix}\begin{pmatrix}(f_3)_{x_1} & (f_3)_{x_2}\\ (f_3)_{x_1} & (f_3)_{x_2}\end{pmatrix}\right)$$

from (1.4). Since

$$\begin{pmatrix}J\pi_y & 0\\ * & y_3\end{pmatrix} = \frac{y_3}{y_3 - f_3(x)}\begin{pmatrix}1 & & -\frac{y_1-f_1(x)}{y_3-f_3(x)}\\ & 1 & -\frac{y_2-f_2(x)}{y_3-f_3(x)}\\ & & 1\end{pmatrix}(f_{x_1}(x), f_{x_2}(x), y - f(x)), \tag{3.1}$$

x is a singular point of π_y if and only if

$$\lambda(x) := |f_{x_1}(x)\, f_{x_2}(x)\, y - f(x)| \tag{3.2}$$

vanishes at x. This means that $y - f(x)$ expresses a tangent vector of f at x. $\square$

Now, we consider criteria of singularity types of π_y. We suppose that $\mathcal{L}$ is a tangent of f at the origin and p_1, p_2 are coefficients which satisfy

$$y - f(0) = p_1 f_{x_1}(0) + p_2 f_{x_2}(0). \tag{3.3}$$

Then, the following Theorem is easily shown.

Theorem 3.3 (cf.[1, §10.2])**.** *The projection π_y is $\mathcal{A}$-equivalent to fold singularity (x_1, x_2^2) if and only if the viewline is not an asymptotic direction line.*

Proof. The differential of λ (defined as (3.2)) at 0 is written as

$$d\lambda := \lambda_{x_1} dx_1 + \lambda_{x_2} dx_2$$

where

$$\begin{aligned}\lambda_{x_1} &= |f_{x_1x_1}\, f_{x_2}\, y - f(x)| + |f_{x_1}\, f_{x_1x_2}\, y - f(x)|\,,\\ \lambda_{x_2} &= |f_{x_1x_2}\, f_{x_2}\, y - f(x)| + |f_{x_1}\, f_{x_2x_2}\, y - f(x)|\end{aligned}$$

since $y - f(0)$ is written as (3.3) from Theorem 3.2. From $f_{x_1}(0) = \boldsymbol{u}$ and $f_{x_2}(0) = \boldsymbol{v}$, λ_{x_1} and λ_{x_2} are expressed as the following :

$$\begin{aligned}\lambda_{x_1}(0) &= -p_1 a_{20} - p_2 a_{11},\\ \lambda_{x_2}(0) &= -p_1 a_{11} - p_2 a_{02}.\end{aligned}$$

Thus,

$$d\lambda = 0 \text{ at } 0 \ \Leftrightarrow\ (p_1,\, p_2)\left(\begin{matrix}a_{20} & a_{11}\\ a_{11} & a_{02}\end{matrix}\right) = 0. \tag{3.4}$$

This condition means that f is parabolic at 0 and (p_1, p_2) is an asymptotic direction of f at 0 from the second fundamental quantities of f at 0

$$L = \langle f_{x_1x_1}, \mathbf{n}\rangle = a_{20},\ M = \langle f_{x_1x_2}, \mathbf{n}\rangle = a_{11},\ N = \langle f_{x_2x_2}, \mathbf{n}\rangle = a_{02}.$$

Furthermore, we set a kernel vector of π_y at the origin $\eta := r\,p_1\,\frac{\partial}{\partial x_1} + r\,p_2\,\frac{\partial}{\partial x_2}$ in $ker\,d\lambda$ at 0 with real number r. Then,

$$\begin{aligned} \eta\lambda(0) &= -\left|\eta f_{x_1}(0)\,p_1 f_{x_1}(0)\,f_{x_2}(0)\right| - \left|\eta f_{x_2}(0)\,f_{x_1}(0)\,p_2 f_{x_2}(0)\right| \\ &= -\left|p_1\eta f_{x_1}(0) + p_2\eta f_{x_2}(0)\,f_{x_1}(0)\,f_{x_2}(0)\right| \\ &= -r\,(p_1^2\,a_{20} + 2\,p_1\,p_2\,a_{11} + p_2^2 a_{02}) \end{aligned} \tag{3.5}$$

Therefore, we get the claim by using Theorem 3.1. □

Remark 3.4. If f is elliptic at the origin, π_y has only the fold singualrity at 0 for any y.

Next, we talk about case of hyperbolic and parabolic. We assume that the degree 2 polynomial of $Q(x)$ is written as

$$H_2(x) = k_1\,x_1\,x_2 + \frac{k_2}{2}x_2^2. \tag{3.6}$$

Then we get the following two Theorems 3.5 and 3.6 if $\mathcal{L}$ is the asymptotic direction line on $\boldsymbol{u}$-axis.

Theorem 3.5. *Let f be hyperbolic at 0, which means we may assume that $k_1 \neq 0$. Then, π_y at 0 is $\mathcal{A}$-equivalent to*

(1) cusp $(x_1, x_1x_2 + x_2^3) \Leftrightarrow f$ has the 2-nd order contact with $\mathcal{L}$ at 0.

(2) swallowtail $(x_1, x_1x_2 + x_2^4) \Leftrightarrow f$ has the 3-rd order contact with $\mathcal{L}$ at 0.

Theorem 3.6. *Let f be parabolic at 0. Then, π_y at 0 is $\mathcal{A}$-equivalent to*

lips $(x_1, x_2^3 + x_1^2x_2)$ (resp. beaks $(x_1, x_2^3 - x_1^2x_2)$) $\Leftrightarrow f$ has the 2-nd order contact with $\mathcal{L}$ at 0 and

$$\frac{k_2}{p_1} < \frac{1}{a_{30}}\begin{vmatrix} a_{30} & a_{21} \\ a_{21} & a_{12} \end{vmatrix} \quad (\textit{resp.} >).$$

Proof of Theorem 3.5 and 3.6. $\eta := (p_1, p_2)$ spans $\ker d\lambda$ at 0. We calculate $\eta^i\lambda(0)$ ($i = 2$, 3 and 4) as follows.

1. If $\eta\lambda(0) = 0$, $\eta^2\lambda(0)$ is

   $-\eta|p_1\eta f_{x_1}(0) + p_2\eta f_{x_2}(0)\,f_{x_1}(0)\,f_{x_2}(0)| = -|p_1\eta^2 f_{x_1}(0) + p_2\eta^2 f_{x_2}(0)\,f_{x_1}(0)\,f_{x_2}(0)| = -H_3(p_1, p_2).$

2. If $\eta\lambda(0) = \eta^2\lambda(0) = 0$, $\eta^3\lambda(0)$ is

$$-\left|p_1\eta^3 f_{x_1}(0) + p_2\eta^3 f_{x_2}(0)\,f_{x_1}(0)\,f_{x_2}(0)\right| = -H_4(p_1, p_2).$$

3. If $\eta\lambda(0) = \eta^2\lambda(0) = \eta^3\lambda(0) = 0$, $\eta^4\lambda(0)$ is

$$-\left|p_1\eta^4 f_{x_1}(0) + p_2\eta^4 f_{x_2}(0)\,f_{x_1}(0)\,f_{x_2}(0)\right| = -H_5(p_1, p_2).$$

We consider hyperbolic and paraboplic case. For $\lambda = |f_{x_1}\,f_{x_2}\,y - f(x)|$,

- hyperbolic case
  Since $\begin{pmatrix} a_{20} & a_{11} \\ a_{11} & a_{02} \end{pmatrix} = \begin{pmatrix} 0 & k_1 \\ k_1 & k_2 \end{pmatrix}$ with $k_1 \neq 0$, we know $d\lambda \neq 0$ at 0. Thus, asymptotic directions of a hyperbolic surface at 0 are $(1, 0)$ and $(\frac{k_2}{2}, -k_1)$ in this case.

- parabolic case
From $\begin{pmatrix} a_{20} & a_{11} \\ a_{11} & a_{02} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & k_2 \end{pmatrix}$, $d\lambda = 0$ at the origin. H_λ are given by calculating $d^2\lambda$ at the origin as follows :

$$\begin{aligned} &d^2\lambda_0(\xi, \xi') \\ &= -p_1 \left|d^2 f_{x_1}(\xi, \xi')\, f_{x_1}\, f_{x_2}\right| - p_2 \left|d^2 f_{x_2}(\xi, \xi')\, f_{x_1}\, f_{x_2}\right| + \xi_1' \left|df_{x_1}(\xi)\, f_{x_1}\, f_{x_2}\right| + \xi_2' \left|df_{x_2}(\xi)\, f_{x_1}\, f_{x_2}\right| \\ &= -\left|p_1 d^2 f_{x_1}(\xi, \xi') + p_2 d^2 f_{x_2}(\xi, \xi')\, f_{x_1}\, f_{x_2}\right| + \left|\xi_1' df_{x_1}(\xi) + \xi_2' df_{x_2}(\xi)\, f_{x_1}\, f_{x_2}\right| \\ &= -(p_1 d^2 Q_{x_1}(\xi, \xi') + p_2 d^2 Q_{x_2}(\xi, \xi') - k_2 \xi_2 \xi_2') \\ &= -(\xi_1\, \xi_2) \begin{pmatrix} p_1 a_{30} + p_2 a_{21} & p_1 a_{21} + p_2 a_{12} \\ p_1 a_{21} + p_2 a_{12} & p_1 a_{12} + p_2 a_{03} - k_2 \end{pmatrix} \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix}. \end{aligned}$$

Since $p_2 = 0$, $\det H_\lambda = 0$ if anf only if $\det \begin{pmatrix} p_1 a_{30} & p_1 a_{21} \\ p_1 a_{21} & p_1 a_{12} - k_2 \end{pmatrix} = 0$.

Therefore, using criteria of each singularity in Theorem 3.1, we get the claims. □

Using the results of Kabata [13], we obtain criteria of $\mathcal{A}$-types of $\mathcal{A}_e$-codimension 2 to 4 for corank 1 plane-to-plane map-germ. In the following subsection 3.2 and 3.3, we introduce results of criteria of $\mathcal{A}_e$-codimension 2 or 3 singularity types of π_y.

3.2 Singularities in hyperbolic case

Theorem 3.7 (Kabata [13, Proposition 3.4]). *For a corank* 1 *germ* g, $\mathcal{A}_e$-codimension 2 *or* 3 *singularities are characterized as follows:*

1. *g is $\mathcal{A}$-equivalent to $(x_1,\, x_1x_2 + x_2^k + \sum_{i+j\geq k+1} c_{ij} x_1^i x_2^j)$ if and only if*
$$d\lambda(0) \neq 0,\ \eta\lambda(0) = \eta^2\lambda(0) = \cdots = \eta^{k-2}\lambda(0) = 0 \ \ \textit{and} \ \ \eta^{k-1}\lambda(0) \neq 0.$$
2. *If g is expressed as $(x_1,\, x_1x_2 + x_2^5 + \sum_{i+j\geq 6} c_{ij} x_1^i x_2^j)$,*
$$\begin{cases} g \sim_{\mathcal{A}} \textit{butterfly } (x_1,\, x_1x_2 + x_2^5 \pm x_2^7) & \iff \quad c_{07} - \frac{5}{8}c_{06}^2 \neq 0, \\ g \sim_{\mathcal{A}} \textit{elder butterfly } (x_1,\, x_1x_2 + x_2^5) & \iff \quad c_{07} - \frac{5}{8}c_{06}^2 = 0. \end{cases}$$
3. *If g is expressed as $(x_1,\, x_1x_2 + x_2^6 + \sum_{i+j\geq 7} c_{ij} x_1^i x_2^j)$,*
$g \sim_{\mathcal{A}}$ unimodal $(x_1, x_1x_2 + x_2^6 \pm x_2^8 + \alpha x_2^9) \iff c_{08} - \frac{3}{5}c_{07}^2 \neq 0$.

Using this Theorem, we get the following result for π_y.

Theorem 3.8 (cf. [13, §4.3]). *Let $\mathcal{L}$ be the asymptotic direction line written as $f(0) + t\,\boldsymbol{u}$. This means that*
$$y - f(0) = p_1 f_{x_1}(0).$$
Then, criteria of $\mathcal{A}$-singularities of π_y at 0 *is in the following table:*

type	*normal form*	c	*position of y*
butterfly	$(x_1,\, x_1x_2 + x_2^5 \pm x_2^7)$	4	*not h-focal*
elder butterfly	$(x_1,\, x_1x_2 + x_2^5)$	4	*h-focal*
unimodal	$(x_1,\, x_1x_2 + x_2^6 \pm x_2^8 + \alpha x_2^9)$	5	*not u-focal (see remark below)*

where c is contact order of f with $\mathcal{L}$ at $x = 0$.

Remark 3.9. From criteria of butterfly and elder butterfly singualrites, it turns out that the only exclusions concern some points on isolated asymptotic direction lines in a hyperbolic domain with 4-th order contact (no more than two on a line). We call the excluded points h-**focal** ("h" for "hyperbolic"). This is introduced by Platnova [18] and is characterized by the coefficients of f as (3.7) in the following Proposition 3.10 from Kabata [13]. We often call **butterfly-focal** point which is the same as h-focal point.

In the same way, we define u-**focal** point ("u" for "unimodal") as exceptional points characterized as the formula in the following Proposition 3.10.

Proposition 3.10. *For the criteria of the table in Theorem 3.8, y is butterfly-focal point if and only if*

$$\begin{aligned}&(48a_{50}\,a_{70}\,{k_1}^2-35{a_{60}}^2\,{k_1}^2+42a_{21}\,a_{50}\,a_{60}\,k_1-1680a_{31}\,{a_{50}}^2\,k_1+2205{a_{21}}^2\,{a_{50}}^2){p_1}^2\\&+(-84a_{50}\,a_{60}\,{k_1}^2+252a_{21}\,{a_{50}}^2\,k_1)\,p_1+756{a_{50}}^2\,{k_1}^2=0\end{aligned}\tag{3.7}$$

In the same way, y is u-focal point if and only if

$$\begin{aligned}&(35a_{60}\,a_{80}\,{k_1}^2-24{a_{70}}^2\,{k_1}^2+28a_{21}\,a_{60}\,a_{70}\,k_1-1960a_{31}\,{a_{60}}^2\,k_1+2646{a_{21}}^2\,{a_{60}}^2)\,p_1^2\\&-28a_{60}\,k_1\,(2a_{70}\,k_1-7a_{21}\,a_{60})\,p_1+784{a_{60}}^2\,{k_1}^2=0.\end{aligned}$$

Proof of Theorem 3.8 and Proposition 3.10. From Theorem 3.7 which is the result of Kabata [13], we get the claim. From the assumption of f and the viewline, the 1-jet of π_y at the origin is

$$-\frac{y_3}{p_1\,u_3^2}\begin{pmatrix}0 & w_2\\ 0 & w_1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

If the tangent plane of f at the origin are not parallel to z_1z_2-plane, from the change of source $(\tilde{Y}_1,\tilde{Y}_2):=(Y_1-\frac{w_2}{w_1}Y_2,\,Y_2)$, $j^1\pi_y(0)$ is written as

$$-\frac{w_1\,y_3}{p_1\,u_3^2}\,x_2\,\boldsymbol{e}_2.$$

Thus, we suppose that

$$\boldsymbol{u}=\begin{pmatrix}\cos\theta\\ 0\\ \sin\theta\end{pmatrix},\ \boldsymbol{v}=\begin{pmatrix}0\\ 1\\ 0\end{pmatrix},\ \boldsymbol{w}=\begin{pmatrix}-\sin\theta\\ 0\\ \cos\theta\end{pmatrix}$$

with θ in $(0,\frac{\pi}{2}]$. The Taylor expansion of $\pi_y(x)$ is expressed as the folloing:

$$\pi_y(x)=\sum\nolimits_{i+j\geq 1}\begin{pmatrix}c_{ij}\\ d_{ij}\end{pmatrix}\frac{x_1^i\,x_2^j}{i!\,j!}.\tag{3.8}$$

Thus, we take the change of target

$$\psi(Y_1,Y_2):=(Y_2-\langle\pi_y(0,0),\boldsymbol{e}_2\rangle,\,Y_1-\langle\pi_y(0,0),\boldsymbol{e}_1\rangle)$$

and the change of source

$$\varphi(x_1,x_2):=\left(\frac{1}{d_{01}}\left(x_2-\sum\nolimits_{i+j\geq 2}\frac{d_{ij}}{i!\,j!}x_1^j\,x_2^i\right),\,x_1\right).$$

We remark that if the new coordinate for the change of source φ is denoted as

$$(\tilde{x}_1,\tilde{x}_2):=\varphi(x_1,x_2),$$

x_1 is regarded as a function with the new variables $\tilde{x_1}$ and $\tilde{x_2}$. And it is written as the following Taylor series at the origin :

$$x_1 = \tilde{x}_1 - \tfrac{d_{11}}{d_{01}} \tilde{x}_1 \tilde{x}_2 - \tfrac{d_{03}}{6d_{01}} \tilde{x}_1^3 - \tfrac{d_{12}}{2d_{01}} \tilde{x}_1^2 \tilde{x}_2 + \left(\tfrac{{d_{11}}^2}{{d_{01}}^2} - \tfrac{d_{21}}{2d_{01}} \right) \tilde{x}_1 \tilde{x}_2^2 - \tfrac{\tilde{x}_1^4 d_{04}}{24d_{01}} + \left(\tfrac{2d_{03}\, d_{11}}{3{d_{01}}^2} - \tfrac{d_{13}}{6d_{01}} \right) \tilde{x}_1^3 \tilde{x}_2$$
$$+ \left(\tfrac{3d_{11}\, d_{12}}{2{d_{01}}^2} - \tfrac{d_{22}}{4d_{01}} \right) \tilde{x}_1^2 \tilde{x}_2^2 - \left(\tfrac{{d_{11}}^3}{{d_{01}}^3} + \tfrac{d_{31}}{6d_{01}} \right) \tilde{x}_1 \tilde{x}_2^3 + \left(\tfrac{{d_{03}}^2}{12{d_{01}}^2} - \tfrac{d_{05}}{120d_{01}} \right) \tilde{x}_1^5 + \left(\tfrac{5d_{04}\, d_{11}}{24{d_{01}}^2} + \tfrac{5d_{03}\, d_{12}}{12{d_{01}}^2} - \tfrac{d_{14}}{24d_{01}} \right) \tilde{x}_1^4 \tilde{x}_2$$
$$+ \left(-\tfrac{5d_{03}\, {d_{11}}^2}{3{d_{01}}^3} + \tfrac{{d_{12}}^2}{2{d_{01}}^2} + \tfrac{2d_{11}\, d_{13}}{3{d_{01}}^2} + \tfrac{d_{03}\, d_{21}}{3{d_{01}}^2} - \tfrac{d_{23}}{12d_{01}} \right) \tilde{x}_1^3 \tilde{x}_2^2 + \left(-\tfrac{3{d_{11}}^2\, d_{12}}{{d_{01}}^3} + \tfrac{3d_{12}\, d_{21}}{4{d_{01}}^2} + \tfrac{3d_{11}\, d_{22}}{4{d_{01}}^2} - \tfrac{d_{32}}{12d_{01}} \right) \tilde{x}_1^2 \tilde{x}_2^3$$
$$+ \left(\tfrac{{d_{11}}^4}{{d_{01}}^4} - \tfrac{3{d_{11}}^2\, d_{21}}{2{d_{01}}^3} + \tfrac{{d_{21}}^2}{4{d_{01}}^2} + \tfrac{d_{11}\, d_{31}}{3{d_{01}}^2} - \tfrac{d_{41}}{24d_{01}} \right) \tilde{x}_1 \tilde{x}_2^4 + \cdots .$$

We define $\tilde{\pi}_y := \psi \circ \pi_y \circ \varphi = (x_1, \langle \pi_y(\tilde{x}_1, \tilde{x}_2),\, \boldsymbol{e}_1 \rangle)$ and the Taylor series of $\tilde{\pi}_y$ at the origin is written as

$$\tilde{\pi}_y(x_1, x_2) = \left(x_1, \sum\nolimits_{i+j \geq 2} \frac{\tilde{c}_{ij}}{i!\, j!} x_1^i x_2^j \right).$$

Towards $\tilde{\pi}_y$, we can set $\eta = (0,1) \in ker\, d\tilde{\lambda}$ at 0. If we set $\tilde{\lambda} := det J\tilde{\pi}_y(x_1, x_2)$, we know

$$\tilde{\lambda} = \sum\nolimits_{i+j \geq 2, j \geq 1} \frac{\tilde{c}_{ij}}{i!\, j!} x_1^i x_2^{j-1}.$$

$\boxed{(i) \text{ butterfly / elder butterfly}}$

From Theorem 3.7, $\eta\tilde{\lambda}(0) = \eta^2\tilde{\lambda}(0) = \eta^3\tilde{\lambda}(0) = 0$. This conditions mean $\tilde{c}_{03} = a_{30} = 0$ and $\tilde{c}_{04} = a_{40} = 0$. Then $\eta^4\tilde{\lambda}(0)$ is

$$\tilde{c}_{05} = -\left| p_1 \eta^4 f_{x_1}(0) + p_2 \eta^4 f_{x_2}(0)\ f_{x_1}(0)\ f_{x_2}(0) \right| = -H_5(p_1, p_2) = -a_{50}.$$

Thus, $\tilde{\pi}_y$ is written as

$$\tilde{\pi}_y(x_1, x_2)$$
$$= \left(x_1, \frac{\tilde{c}_{20}}{2!} x_1^2 + \tilde{c}_{11} x_1 x_2 + \frac{\tilde{c}_{30}}{3!} x_1^3 + \frac{\tilde{c}_{21}}{2!\, 1!} x_1^2 x_2 + \frac{\tilde{c}_{12}}{1!\, 2!} x_1 x_2^2 + \frac{\tilde{c}_{40}}{4!} x_1^4 + \cdots + \frac{\tilde{c}_{13}}{1!\, 3!} x_1 x_2^3 + \frac{\tilde{c}_{50}}{5!} x_1^5 + \cdots \right).$$

From the change of target denoted as follows:

$$\tilde{\psi}(X, Y) := \left(X, Y - \left(\frac{\tilde{c}_{20}}{2!} X^2 + \frac{\tilde{c}_{30}}{3!} X^3 + \frac{\tilde{c}_{40}}{4!} X^4 + \frac{\tilde{c}_{50}}{5!} X^5 \right) \right),$$

we get

$$\tilde{\psi} \circ \tilde{\pi}_y(x_1, x_2) = \left(x_1,\, \tilde{c}_{11} x_1 x_2 + \frac{\tilde{c}_{21}}{2!\, 1!} x_1^2 x_2 + \frac{\tilde{c}_{12}}{1!\, 2!} x_1 x_2^2 + \frac{\tilde{c}_{31}}{3!\, 1!} x_1^3 x_2 + \frac{\tilde{c}_{22}}{2!\, 2!} x_1^2 x_2^2 + \frac{\tilde{c}_{13}}{1!\, 3!} x_1 x_2^3 + \cdots \right).$$

Furthermore, we consider the following change of source

$$\tilde{\varphi}(x_1, x_2) := (\tilde{x}_1, \tilde{x}_2)$$

where

$$\tilde{x}_1 := x_1 \quad \text{and} \quad \tilde{x}_2 := x_2 \left(\tilde{c}_{11} + \frac{\tilde{c}_{21}}{2!\, 1!} x_1 + \frac{\tilde{c}_{12}}{1!\, 2!} x_2 + \frac{\tilde{c}_{31}}{3!\, 1!} x_1^2 + \frac{\tilde{c}_{22}}{2!\, 2!} x_1 x_2 + \frac{\tilde{c}_{13}}{1!\, 3!} x_2^2 + \cdots + \frac{\tilde{c}_{14}}{1!\, 4!} x_2^3 \right).$$

This means that we should substitute

$$\tilde{x}_1 \quad \text{and} \quad -\frac{p_1 \sin^2\theta}{k_1\, y_3} \tilde{x}_2 + \frac{(a_{12}\, p_1 - k_2) \sin^2\theta}{2{k_1}^2\, y_3} \tilde{x}_1 \tilde{x}_2 - \frac{a_{21}\, {p_1}^2 \sin^4\theta}{2{k_1}^3\, {y_3}^2} \tilde{x}_2^2 + O_3$$

for x_1 and x_2 respectively. Then,

$$\tilde{\psi} \circ \tilde{\pi}_y \circ \tilde{\varphi}(x_1, x_2) = \left(x_1,\ x_1 x_2 + \frac{\tilde{c}_{05}}{5!} x_2^5 + O_6 \right).$$

Using the following change of target and source, $\tilde{\pi}_y$ is written as

$$\tilde{\psi} \circ \tilde{\pi}_y \circ \tilde{\varphi}(x_1, x_2) = (x_1,\ x_1 x_2 + x_2^5 + O_6)$$

where

$$\tilde{\psi}(X, Y) := (X,\, Y) / \left(\frac{\tilde{c}_{05}}{5!} \right) \quad \text{and} \quad \tilde{\varphi}(x_1, x_2) := \left(\frac{\tilde{c}_{05}}{5!}\, x_1,\, x_2 \right).$$

From the result of [13] in Theorem 3.7, we get the claim as the following formula :

$$\tfrac{\tilde{c}_{07}}{7!} - \tfrac{5}{8} \left(\tfrac{\tilde{c}_{06}}{6!} \right)^2 = (48 a_{50}\, a_{70}\, {k_1}^2\, {p_1}^2 - 35 {a_{60}}^2\, {k_1}^2\, {p_1}^2 + 42 a_{21}\, a_{50}\, a_{60}\, k_1\, {p_1}^2 - 1680 a_{31}\, {a_{50}}^2\, k_1\, {p_1}^2$$
$$+ 2205 {a_{21}}^2\, {a_{50}}^2 {p_1}^2 - 84 a_{50}\, a_{60}\, {k_1}^2\, p_1 + 252 a_{21}\, {a_{50}}^2\, k_1\, p_1 + 756 {a_{50}}^2\, {k_1}^2) \tfrac{\sin^4\theta}{2016 {a_{50}}^2\, {k_1}^4\, {y_3}^2}.$$

$\boxed{(ii) \text{ unimodal}}$

Given $\eta\tilde{\lambda}(0) = \eta^2\tilde{\lambda}(0) = \eta^3\tilde{\lambda}(0) = \eta^4\tilde{\lambda}(0) = 0$, we know $\tilde{c}_{03} = a_{30} = 0$, $\tilde{c}_{04} = a_{40} = 0$ and $\tilde{c}_{05} = a_{50} = 0$ and $\eta^5\tilde{\lambda}(0)$ is

$$\tilde{c}_{06} = - \left| p_1 \eta^5 f_{x_1}(0) + p_2 \eta^5 f_{x_2}(0)\ f_{x_1}(0)\ f_{x_2}(0) \right| = -H_6(p_1, p_2).$$

The Taylor series of π_y at the origin is expressed as the following:

$$\left(x_1, \frac{\tilde{c}_{20}}{2!} x_1^2 + \tilde{c}_{11} x_1 x_2 + \frac{\tilde{c}_{30}}{3!} x_1^3 + \frac{\tilde{c}_{21}}{2!\,1!} x_1^2 x_2 + \frac{\tilde{c}_{12}}{1!\,2!} x_1 x_2^2 + \frac{\tilde{c}_{40}}{4!} x_1^4 + \cdots + \frac{\tilde{c}_{14}}{1!\,4!} x_1 x_2^4 + \frac{\tilde{c}_{60}}{6!} x_1^6 + \cdots \right).$$

Using the change of target

$$\tilde{\psi}(X, Y) := \left(X, Y - \left(\frac{\tilde{c}_{20}}{2!} X^2 + \frac{\tilde{c}_{30}}{3!} X^3 + \frac{\tilde{c}_{40}}{4!} X^4 + \frac{\tilde{c}_{50}}{5!} X^5 + \frac{\tilde{c}_{60}}{6!} X^6 \right) \right),$$

$\tilde{\pi}_y$ is $\mathcal{A}$-equivalent to

$$\tilde{\psi} \circ \tilde{\pi}_y(x_1, x_2) = (x_1,\, \tilde{c}_{11} x_1 x_2 + \frac{\tilde{c}_{21}}{2!\,1!} x_1^2 x_2 + \frac{\tilde{c}_{12}}{1!\,2!} x_1 x_2^2 + \frac{\tilde{c}_{31}}{3!\,1!} x_1^3 x_2 + \frac{\tilde{c}_{22}}{2!\,2!} x_1^2 x_2^2 + \frac{\tilde{c}_{13}}{1!\,3!} x_1 x_2^3 + \cdots).$$

Furthermore, we set the following change of source

$$\tilde{\varphi}(x_1, x_2) := (\tilde{x}_1, \tilde{x}_2)$$

where

$$\tilde{x}_1 := x_1 \quad \text{and} \quad \tilde{x}_2 := x_2 \left(\tilde{c}_{11} + \frac{\tilde{c}_{21}}{2!\,1!} x_1 + \frac{\tilde{c}_{12}}{1!\,2!} x_2 + \frac{\tilde{c}_{31}}{3!\,1!} x_1^2 + \frac{\tilde{c}_{22}}{2!\,2!} x_1 x_2 + \frac{\tilde{c}_{13}}{1!\,3!} x_2^2 + \cdots + \frac{\tilde{c}_{15}}{1!\,5!} x_2^4 \right).$$

This means that we should substitute

$$\tilde{x}_1 \quad \text{and} \quad - \frac{1}{\tilde{c}_{11}} \tilde{x}_2 - \frac{\tilde{c}_{21}}{2\tilde{c}_{11}^2} \tilde{x}_1\, \tilde{x}_2 - \frac{\tilde{c}_{12}}{2\tilde{c}_{11}^3} \tilde{x}_2^2 + O_3$$

for x_1 and x_2 respectively. Then,

$$\tilde{\psi}\circ\tilde{\pi}_y\circ\tilde{\varphi}(x_1,x_2)=\left(x_1,x_1x_2+\frac{\tilde{c}_{06}}{6!}x_2^6+O_7\right).$$

Using the following change of target and source, we get

$$\tilde{\psi}\circ\tilde{\pi}_y\circ\tilde{\varphi}(x_1,x_2)=(x_1,x_1x_2+x_2^6+O_7)$$

where

$$\tilde{\psi}(X,Y):=(X,\,Y)/\left(\frac{\tilde{c}_{06}}{6!}\right)\quad\text{and}\quad\tilde{\varphi}(x_1,x_2):=\left(\frac{\tilde{c}_{06}}{6!}\,x_1,\,x_2\right).$$

Therefore, we get the claim from the result of Kabata [13] in Theorem 3.7. □

3.3 Singularities in parabolic case

Theorem 3.11 (Kabata [13, Proposition 3.9]). *Let g in $\mathcal{E}_2^2$ be a corank 1 germ.*

1. $g\sim_{\mathcal{A}}(x_1,x_2^3+\sum_{i+j\geq 4}c_{ij}x_1^ix_2^j)\iff d\lambda(0)=0,\ \mathrm{rk}H_\lambda(0)=1,\ \eta^2\lambda(0)\neq 0.$

2. *If g is written as* $(x_1,x_2^3+\sum_{i+j\geq 4}c_{ij}x_1^ix_2^j)$,
$$\begin{cases} g\sim_{\mathcal{A}}\text{ goose }(x_1,x_2^3+x_1^3x_2) & \iff\quad c_{31}\neq 0,\\ g\sim_{\mathcal{A}}\text{ ugly goose }(x_1,x_2^3\pm x_1^4x_2) & \iff\quad c_{31}=0,\ c_{41}-\frac{1}{3}c_{22}^2\neq 0.\end{cases}$$

3. $g\sim_{\mathcal{A}}(x_1,x_1^2x_2+x_2^4+\sum_{i+j\geq 5}c_{ij}x_1^ix_2^j)$
$\iff d\lambda(0)=0,\ \mathrm{rk}H_\lambda(0)=1,\ \eta^2\lambda(0)=0,\ \eta^3\lambda(0)\neq 0.$

4. *If g is written as* $(x_1,x_1^2x_2+x_2^4+\sum_{i+j\geq 5}c_{ij}x_1^ix_2^j)$,
$g\sim_{\mathcal{A}}$ *type*16 $(x_1,x_2^4+x_1^2x_2)\iff c_{05}\neq 0.$

Theorem 3.12 (Kabata [13, Proposition 3.6]). *Let g in $\mathcal{E}_2^2$ be a corank 1 germ.*

1. $g\sim_{\mathcal{A}}(x_1,x_1x_2^2+x_2^4+\sum_{i+j\geq 5}c_{ij}x_1^ix_2^j)\iff d\lambda(0)=0,\ \det H_\lambda(0)<0,\ \eta\lambda(0)=\eta^2\lambda(0)=0,$
$\eta^3\lambda(0)\neq 0.$

2. *If g is written as* $(x_1,x_1x_2^2+x_2^4+\sum_{i+j\geq 5}c_{ij}x_1^ix_2^j)$,
$$\begin{cases} g\sim_{\mathcal{A}}\text{ gulls }(x_1,x_1x_2^2+x_2^4+x_2^5) & \iff\quad c_{05}\neq 0,\\ g\sim_{\mathcal{A}}\text{ ugly gulls }(x_1,x_1x_2^2+x_2^4+x_2^7) & \iff\quad c_{05}=0,\ c_{07}-2c_{15}+4a_{23}\neq 0.\end{cases}$$

3. $g\sim_{\mathcal{A}}(x_1,x_1x_2^2+x_2^5+\sum_{i+j\geq 6}c_{ij}x_1^ix_2^j)$
$\iff d\lambda(0)=0,\ \det H_\lambda(0)<0,\ \eta\lambda(0)=\eta^2\lambda(0)=\eta^3\lambda(0)=0,\ \eta^4\lambda(0)\neq 0.$

   If g is written as $(x_1,x_1x_2^2+x_2^5+\sum_{i+j\geq 6}c_{ij}x_1^ix_2^j)$,
   $g\sim_{\mathcal{A}}$ *type* 12 $(x_1,x_1x_2^2+x_2^5+x_2^6)\iff c_{06}\neq 0.$

We assume that f is parabolic at 0 and $\mathcal{L}$ is an asymptotic direction line of f at 0. This means that $k_1=0$.

Theorem 3.13 (cf. [13, §4.4, §4.5]). *Let $\mathcal{L}$ be the asymptotic direction line written as $f(0)+t\,\boldsymbol{u}$. This means that*

$$y - f(0) = p_1 f_{x_1}(0).$$

Then, criteria of $\mathcal{A}$-equivalence class of π_y at 0 is as in the following table :

type	*normal form*	c	*position of y*	*other condition*
goose	$(x_1,\, x_2^3 + x_1^3 x_2)$	2	*p-focal*	$H_3(-a_{21}, a_{30}) \neq \frac{1}{2} H_{4x_1}(-a_{21}, a_{30})\, p_1$
ugly goose	$(x_1,\, x_2^3 \pm x_1^4 x_2)$	2	*p-focal*	$H_3(-a_{21}, a_{30}) = \frac{1}{2} H_{4x_1}(-a_{21}, a_{30})\, p_1$ *and* $a_{30}(H_{5x_1}(-a_{21}, a_{30})p_1 - 3H_4(-a_{21}, a_{30}))p_1$ $\neq \frac{1}{2}(H_{4x_1x_1}(-a_{21}, a_{30})p_1 - 2H_{3x_1}(-a_{21}, a_{30}))^2$
type 16	$(x_1,\, x_1^2 x_2 + x_2^4 \pm x_2^5)$	3	*not* 16*-focal*	1*-st or higher order red subparabolic*
gulls	$(x_1,\, x_1 x_2^2 + x_2^4 + x_2^5)$	3	*not p'-focal*	*not red subparabolic*
ugly gulls	$(x_1,\, x_1 x_2^2 + x_2^4 + x_2^7)$	3	*p'-focal*	*not red subparabolic* *and* $\tilde{A}_2 p_1^2 + \tilde{A}_1 p_1 + \tilde{A}_0 \neq 0$
type 12	$(x_1,\, x_1 x_2^2 + x_2^5 + x_2^6)$	4	*not* 12*-focal*	*not red-subparabolic*

where c is contact order of f with $\mathcal{L}$ at $x = 0$.

Remark 3.14.

1. As seen in Remark 3.9, we also have an exceptional point on asymptotic direction lines passing through parabolic points of the surface (not more than one on a line). If $a_{30} \neq 0$ for parabolic surface at the origin, the lips or beaks singularities appears from viewpoints on the line except for the point. The exceptional point are called ***p*-focal** point ("p" for parabolic) by Platonova [18] and characterized by (3.10) in the following Proposition 3.15 from Kabata [13]. We often call **goose-focal** point which is the same as p-focal point.

2. As seen in the above, there is an exceptional point on asymptotic direction lines passing through parabolic points of the surface (not more than one on a line) if $a_{30} = 0$ and $a_{40} \neq 0$ for parabolic surface at the origin. It is called **p'-focal** by Platonova [18]. And Kabata has characterized by (3.11) in the following Proposition 3.15 from Kabata [13]. We often call **gulls-focal** point which is the same as p'-focal point.

3. In the same way, we define **12-focal** point ("12" for "type 12 singularity") and **16-focal** point ("16" for "type 16 singularity") as exceptional points characterized as the formula in the following Proposition 3.15.

4. the coefficients of the condition of ugly gulls

$$\tilde{A}_2 p_1^2 + \tilde{A}_1 p_1 + \tilde{A}_0 \neq 0 \tag{3.9}$$

are the following :

$$\begin{aligned}\tilde{A}_2 := {} & 225 a_{21}{}^3\, a_{40}{}^2\, a_{70} - 315 a_{21}{}^2\, a_{40}\, (3a_{21}\, a_{50} - 5a_{31}\, a_{40})\, a_{60} - 1575 a_{21}{}^2\, a_{40}{}^3\, a_{51} \\ & + (756 a_{21}{}^3\, a_{50}{}^2 - 3150 a_{21}{}^2\, a_{40}\, (a_{31}\, a_{50} - a_{40}\, a_{41}) - 1575 a_{21}\, a_{22}\, a_{40}{}^3 + 4200 a_{21}\, a_{31}{}^2\, a_{40}{}^2) a_{50} \\ & - 5250 a_{21}\, a_{31}\, a_{40}{}^3\, a_{41} - 875 a_{13}\, a_{40}{}^5 + 2625 (a_{21}\, a_{32} + a_{22}\, a_{31})\, a_{40}{}^4 - 1750 a_{31}{}^3\, a_{40}{}^3,\end{aligned}$$

$$\tilde{A}_1 := -70 a_{40} \left\{ \begin{aligned} & 5a_{40}(9a_{21}{}^2\, (a_{21}\, a_{60} - 5a_{40}\, a_{41}) - 5a_{40}{}^2\, (a_{03}\, a_{40} - 9a_{21}\, a_{22})) \\ & - 3(3a_{21}\, a_{50} - 5a_{31}\, a_{40})\, (9a_{21}{}^2\, a_{50} + 5a_{12}\, a_{40}{}^2 - 20 a_{21}\, a_{31}\, a_{40}) \end{aligned} \right\}$$

and

$$\tilde{A}_0 := 3150 a_{21}\, a_{40}{}^2\, (3a_{21}{}^2\, a_{50} + 5a_{12}\, a_{40}{}^2 - 10 a_{21}\, a_{31}\, a_{40}).$$

Proposition 3.15. *For the criteria of the table in Theorem 3.13, y is*

(1) a goose-focal point if and only if

$$\frac{k_2}{p_1} = \frac{1}{a_{30}} \begin{vmatrix} a_{30} & a_{21} \\ a_{21} & a_{12} \end{vmatrix}. \tag{3.10}$$

(2) a 16-focal point if and only if

$$(a_{12}\,a_{50} - 10a_{22}\,a_{40} + 10{a_{31}}^2)p_1^2 - (a_{50}\,k_2 - 25a_{12}\,a_{40})p_1 - 5a_{40}\,k_2 = 0.$$

(3) a gulls-focal point if and only if

$$(3{a_{21}}^2\,a_{50} + 5a_{12}\,{a_{40}}^2 - 10a_{21}\,a_{31}\,a_{40})\,p_1 - 5a_{40}\,(a_{40}\,k_2 - 3{a_{21}}^2) = 0. \tag{3.11}$$

(4) a 12-focal point if and only if

$$(a_{21}\,a_{60} - 5a_{31}\,a_{50})p_1 + 6a_{21}\,a_{50} = 0.$$

Proof. Using Theorem 3.11 and 3.12, we get the claim. Since f is parabolic at 0 and $\mathcal{L}$ is the asymptotic direction line, the differential $d\tilde{\lambda} = 0$ at the origin from the equivalence 3.4. Now, we can set $\eta = (0,1)$ in $ker\, d\tilde{\lambda}_0$ that is $\eta = \frac{\partial}{\partial x_2}$.

$\boxed{(i) \text{ gulls / ugly gulls}}$

It is satisfied that $\tilde{c}_{03} = a_{30} = 0$ since the assumption $\eta\tilde{\lambda}(0) = \eta^2\tilde{\lambda}(0) = 0$. And the condition

$$\det H_{\tilde{\lambda}}(0) = \begin{vmatrix} \tilde{c}_{21} & \tilde{c}_{12} \\ \tilde{c}_{12} & \tilde{c}_{03} \end{vmatrix} = -\tilde{c}_{12}^2 = -{a_{21}}^2 < 0$$

gives $\tilde{c}_{12} = a_{21} \neq 0$.

We know π_y is $\mathcal{A}$-equivalent to

$$\tilde{\pi}_y(x_1, x_2) = \left(x_1,\ \frac{\tilde{c}_{20}}{2!}x_1^2 + \frac{\tilde{c}_{30}}{3!}x_1^3 + \frac{\tilde{c}_{21}}{2!\,1!}x_1^2x_2 + \frac{\tilde{c}_{12}}{1!\,2!}x_1x_2^2 + \sum\nolimits_{i+j\geq 4}\frac{\tilde{c}_{ij}}{i!\,j!}x_1^ix_2^j\right).$$

Using the following coordinate change of target

$$\psi(X, Y) := (X,\, Y - \tilde{c}_{20}X^2 - \tilde{c}_{30}X^3),$$

Then,

$$\tilde{\pi}_y(x_1, x_2) = \left(x_1,\ \frac{\tilde{c}_{21}}{2!\,1!}x_1^2x_2 + \frac{\tilde{c}_{12}}{1!\,2!}x_1x_2^2 + \sum\nolimits_{i+j\geq 4}\frac{\tilde{c}_{ij}}{i!\,j!}x_1^ix_2^j\right).$$

Furthermore, we set the coordinate change of source as follows:

$$\tilde{x}_1 := x_1,\ \tilde{x}_2^2 + a\tilde{x}_1^2 := \frac{\tilde{c}_{12}}{2}x_2^2 + \frac{\tilde{c}_{21}}{2}x_1x_2$$

which means x_1 and x_2 are expressed as the Taylor series at 0 with variable $\tilde{x}_1$ and $\tilde{x}_2$ as follows:

$$x_1 = \tilde{x}_1,$$

$$x_2 = \frac{2c_{02}\,d_{11} - c_{12}\,d_{01}}{2c_{21}\,d_{01}}\tilde{x}_1 + 2\sqrt{\frac{1}{|c_{21}|}}\tilde{x}_2 = -\frac{a_{12}\,p_1 - k_2}{2a_{21}\,p_1}\tilde{x}_1 + 2\sqrt{\left|\frac{p_1}{a_{21}\,y_3}\right|}\,|\sin\theta|\,\tilde{x}_2.$$

Using this, $\tilde{\pi}_y$ is written as

$$\tilde{\pi}_y(x_1, x_2) = (x_1,\, x_1(x_2^2 + ax_1^2) + \cdots) = \left(x_1,\, ax_1^3 + x_1x_2^2 + \sum\nolimits_{i+j\geq 4}\frac{\tilde{c}_{ij}}{i!\,j!}x_1^i x_2^j\right).$$

From the coordinate change of target

$$X' := X, Y' := Y - aX^3,$$

π_y is $\mathcal{A}$-equivalent to

$$\tilde{\pi}_y(x_1, x_2) = \left(x_1,\, x_1x_2^2 + \sum\nolimits_{i+j\geq 4}\frac{\tilde{c}_{ij}}{i!\,j!}x_1^i x_2^j\right).$$

Using a coordinate change of source $(\tilde{x}_1, \tilde{x}_2)$ defined by

$$x_1 = \tilde{x}_1,\, x_2 = \tilde{x}_2 - \left(\frac{\tilde{c}_{31}}{2\tilde{c}_{12}}\tilde{x}_1^2 + \frac{\tilde{c}_{22}}{2\tilde{c}_{12}}\tilde{x}_1\tilde{x}_2 + \frac{\tilde{c}_{13}}{2\tilde{c}_{12}}\tilde{x}_2^2\right),$$

$\tilde{\pi}_y$ is written as

$$\left(x_1,\, x_1x_2^2 + \frac{\tilde{c}_{04}}{4!}x_2^4 + \sum\nolimits_{i+j\geq 5}\frac{\tilde{c}_{ij}}{i!\,j!}x_1^i x_2^j\right).$$

This is $\mathcal{A}$-equivalent to

$$\left(x_1,\, x_1x_2^2 + x_2^4 + \frac{4!}{\tilde{c}_{04}}\sum\nolimits_{i+j\geq 5}\frac{\tilde{c}_{ij}}{i!\,j!}\left(\frac{\tilde{c}_{04}}{4!}x_1\right)^i x_2^j\right)$$

by the change of target $(X, Y) \mapsto \frac{4!}{\tilde{c}_{04}}(X, Y)$ and the change of source

$$\tilde{x}_1 := \frac{4!}{\tilde{c}_{04}}\,x_1,\, \tilde{x}_2 := x_2.$$

From Theorem 3.7, we get the claim.

$\boxed{(ii)\text{ type12}}$

In the same as gulls / ugly gulls singularities, $\tilde{c}_{03} = a_{30} = 0$ and $\tilde{c}_{12} = a_{21} \neq 0$. From criteria of type12 in Theorem 3.7, $\tilde{c}_{04} = a_{40} = 0$. Thus, π_y is $\mathcal{A}$-equivalent to

$$\tilde{\pi}_y(x_1, x_2) = \left(x_1,\, x_1x_2^2 + \sum\nolimits_{i+j\geq 5}\frac{\tilde{c}_{ij}}{i!\,j!}x_1^i x_2^j\right).$$

Using the following coordinate change of source $(\tilde{x}_1, \tilde{x}_2)$:

$$x_1 = \tilde{x}_1,\, x_2 = \tilde{x}_2 - \left(\frac{\tilde{c}_{41}}{2}\tilde{x}_1^3 + \frac{\tilde{c}_{32}}{2}\tilde{x}_1^2\tilde{x}_2 + \frac{\tilde{c}_{23}}{2}\tilde{x}_1\tilde{x}_2^2 + \frac{\tilde{c}_{14}}{2}\tilde{x}_2^3\right),$$

$\tilde{\pi}_y$ is written as

$$\tilde{\pi}_y(x_1, x_2) = \left(x_1,\, x_1x_2^2 + \frac{\tilde{c}_{05}}{5!}x_2^5 + \sum\nolimits_{i+j\geq 6}\frac{\tilde{c}_{ij}}{i!\,j!}x_1^i x_2^j\right).$$

This is $\mathcal{A}$-equivalent to

$$\left(x_1,\, x_1x_2^2 + x_2^5 + \frac{5!}{\tilde{c}_{05}}\sum\nolimits_{i+j\geq 5}\frac{\tilde{c}_{ij}}{i!\, j!}\left(\frac{\tilde{c}_{05}}{5!}x_1\right)^i x_2^j\right)$$

by the change of target $(X, Y) \mapsto \frac{5!}{\tilde{c}_{05}}(X, Y)$ and the change of source

$$\tilde{x}_1 := \frac{5!}{\tilde{c}_{05}}\, x_1,\ \tilde{x}_2 := x_2.$$

From Thesis of Kabata (Theorem 3.12), we get the claim.

$\boxed{(i) \text{ goose / ugly goose}}$

From Theorem 3.11, $\eta\tilde{\lambda}(0) = 0$ and $\eta^2\tilde{\lambda}(0) \neq 0$. Thus, one of the coefficient of π_y and f are satisfied $\tilde{c}_{03} = a_{30} \neq 0$. Then, $\tilde{\pi}_y$ is written as follows:

$$\tilde{\pi}_y(x_1, x_2) = \left(x_1,\, \frac{\tilde{c}_{21}}{2!\,1!}x_1^2x_2 + \frac{\tilde{c}_{12}}{1!\,2!}x_1x_2^2 + \frac{\tilde{c}_{03}}{3!}x_2^3 + \frac{\tilde{c}_{40}}{4!}x_1^4 + \cdots\right)$$

from the above conditions. Thus, we get the p-focal condition

$$(a_{30}\, a_{12} - {a_{21}}^2)\, p_1 - a_{30}k_2 = 0$$

since $detH_{\tilde{\lambda}} = \begin{vmatrix} \tilde{c}_{21} & \tilde{c}_{12} \\ \tilde{c}_{12} & \tilde{c}_{03} \end{vmatrix} = 0$ at the origin where

$$\tilde{c}_{21} = -\frac{(a_{12}\, p_1 - k_2)\, y_3}{2{p_1}^2 \sin^2\theta},\ \tilde{c}_{12} = -\frac{a_{21}\, y_3}{2p_1 \sin^2\theta}\ \text{ and }\ \tilde{c}_{03} = -\frac{a_{30}\, y_3}{6p_1 \sin^2\theta}.$$

Writing new variables as $\tilde{x}_1$, $\tilde{x}_2$, we take the change of source as follows:

$$\tilde{x}_1 = x_1,\ \tilde{x}_2 = x_2 - \frac{a_{21}}{a_{30}}x_1,$$

and some change of target. Then, we know π_y is $\mathcal{A}$-equivalent to

$$\left(x_1,\, \frac{b}{2!\,1!}x_1^2x_2 + \frac{\tilde{c}_{03}}{3!}x_2^3 + \frac{\tilde{c}_{40}}{4!}x_1^4 + \cdots\right)$$

where

$$b = -\frac{\{(a_{12}\, a_{30} - {a_{21}}^2)\, p_1 - a_{30}k_2\}\, y_3}{a_{30}\, {p_1}^2 \sin^2\theta}.$$

If $x = 0$ is goose type, $b = 0$ from the p-focal condition. Furthermore, using the change of target

$$(\tilde{X}, \tilde{Y}) = (X, Y/\tilde{c}_{03}),$$

π_y is written by

$$\left(x_1,\, \frac{x_2^3}{3!} + \frac{\tilde{c}_{40}}{4!}x_1^4 + \cdots\right).$$

Then, We check the following expressed as coefficients:

$$\tilde{c}_{31} = \frac{H_{4x_1}(-a_{21}, a_{30})p_1 - 2H_3(-a_{21}, a_{30})}{{a_{30}}^4\, p_1}$$

and

$$\tilde{c}_{41} - \frac{\tilde{c}_{22}^2}{3}$$
$$= \{2a_{30}(H_{5x_1}(-a_{21}, a_{30})p_1 - 3H_4(-a_{21}, a_{30}))p_1 - (H_{4x_1x_1}(-a_{21}, a_{30})p_1 - 2H_{3x_1}(-a_{21}, a_{30}))^2\}\frac{3}{{a_{30}}^6 {p_1}^2}$$

Applying the result of [13] for the above two formulas, we get the criteria of goose and ugly goose as follows:

(i)-1 $\tilde{c}_{31} \neq 0$ if and only if
$$H_{4x_1}(-a_{21}, a_{30})p_1 \neq 2H_3(-a_{21}, a_{30}),$$

(i)-2 If $\tilde{c}_{31} = 0$, then, $\tilde{c}_{41} - \frac{\tilde{c}_{22}^2}{3} \neq 0$ if and only if

$$a_{30}(H_{5x_1}(-a_{21}, a_{30})p_1 - 3H_4(-a_{21}, a_{30}))p_1 \neq \frac{1}{2}(H_{4x_1x_1}(-a_{21}, a_{30})p_1 - 2H_{3x_1}(-a_{21}, a_{30}))^2.$$

$\boxed{(ii)\ \text{type16}}$

Since $\eta^2\tilde{\lambda}(0) = 0$, we know $\tilde{c}_{03} = a_{30} = 0$. And, from $rank H_{\tilde{\lambda}} = 1$, we get the conditions $a_{21} = 0$ and $a_{12}\, p_1 - k_2 \neq 0$. Now, $\tilde{\pi}_y$ is written as

$$\tilde{\pi}_y(x_1, x_2) = \left(x_1,\ x_1^2 x_2 + \frac{2!\,1!}{\tilde{c}_{21}} \sum\nolimits_{i+j \geq 4} \frac{\tilde{c}_{ij}}{i!\,j!} x_1^i x_2^j\right)$$

from the coordinate change of target $(X, Y) \mapsto (X, \frac{2!\,1!}{\tilde{c}_{21}} Y)$. To eliminate other terms whose degree is 4, we use the following coordinate change of source $(\tilde{x}_1, \tilde{x}_2)$ defined by

$$\tilde{x}_1 = x_1,\ \tilde{x}_2 = x_2 + \frac{2!\,1!}{\tilde{c}_{21}}\left(\frac{\tilde{c}_{31}}{3!\,1!} x_1 x_2 + \frac{\tilde{c}_{22}}{2!\,2!} x_2^2\right).$$

Then we have
$$\tilde{\pi}_y(x_1, x_2) = (x_1,\ x_1^2 x_2 + x_2^4 + O_5).$$

From Thesis of Kabata (3.11), we can get the claim. □

4 Versality of central projection unfolding π

We set an orthonormal frame

$$\boldsymbol{u} = \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{w} = \begin{pmatrix} -\sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$$

with θ in $(0, \frac{\pi}{2}]$. Let f be given by Monge normal form and its normal as (1.1).

Theorem 4.1. *We assume that* $(x_1, x_2) = (0, 0)$ *is a singularity of* π_y *whose* $\mathcal{A}_e$-codimension *is less than* 1 *in Theorems* 3.3, 3.5 *or* 3.6. *Then* π *is* $\mathcal{A}_e$-versal unfolding *at* $(0, y)$ *in* $(\mathbb{R}^2 \times \mathbb{R}^3, (0, y))$.

Theorem 4.2. *1. If* $(x_1, x_2) = (0, 0)$ *is the butterfly singularity of* π_y *in* Theorem 3.5, *then the following two conditions* (*i*) *and* (*ii*) *are equivalent.*

(i) π *is* $\mathcal{A}_e$-versal unfolding *at* $(0, y)$ *in* $(\mathbb{R}^2 \times \mathbb{R}^3, (0, y))$,

(ii) *the flecnodal curve is not singular at* 0. *This means that*

$$2a_{31}\, k_1 - 3{a_{21}}^2 \tag{4.1}$$

does not vanish.

2. *We assume that* π_y *has the elder butterfly singularity in* Theorem 3.5 *at the origin. Then,* π *is* $\mathcal{A}_e$-versal unfolding *at* $(0, y)$ *in* $(\mathbb{R}^2 \times \mathbb{R}^3, (0, y))$ *if and only if both of two formulas* (4.1) *and*

$$(a_{60}\, k_1 - 3a_{21}\, a_{50})\, p_1 - 18a_{50}\, k_1 \tag{4.2}$$

do not vanish. The later condition means that there is a special degenerate position of a viewpoint y *for* $\mathcal{A}_e$*-versality.*

3. *We assume that* $(x_1, x_2) = (0, 0)$ *is the unimodal singularity of* π_y *in* Theorem 3.5. *Then,* π *is not* $\mathcal{A}_e$-versal unfolding *at* $(0, y)$ *in* $(\mathbb{R}^2 \times \mathbb{R}^3, (0, y))$.

Theorem 4.3. *1. We assume that* π_y *has the gulls or ugly gulls singularity in* Theorem 3.6 *at the origin. Then the following two conditions* (i) *and* (ii) *are equivalent.*

(i) π *is* $\mathcal{A}_e$-versal unfolding *at* $(0, y)$ *in* $(\mathbb{R}^2 \times \mathbb{R}^3, (0, y))$,

(ii) f *is first order blue ridge at the origin (i.e.* $a_{40}\, k_2 - 3\, {a_{21}}^2 \neq 0$*).*

2. *If* π_y *has the type* 12 *singularity in* Theorem 3.6 *at the origin, Then* π *is not* $\mathcal{A}_e$-versal unfolding *at* $(0, y)$ *in* $(\mathbb{R}^2 \times \mathbb{R}^3, (0, y))$.

Theorem 4.4. *1. If* $(x_1, x_2) = (0, 0)$ *is the goose or ugly goose singularity in* Theorem 3.6, *Then the following two conditions* (i) *and* (ii) *are equivalent.*

(i) π *is* $\mathcal{A}_e$-versal unfolding *at* $(0, y)$ *in* $(\mathbb{R}^2 \times \mathbb{R}^3, (0, y))$,

(ii) k_2 *does not vanish. This means* $f(0)$ *is not flat umbilic point.*

2. *We assume that* $(x_1, x_2) = (0, 0)$ *is the type* 16 *singularity of* π_y *whose* $\mathcal{A}_e$-codimension *is* 3 *in* Theorem 3.6. *Then* π *is not* $\mathcal{A}_e$-versal unfolding *at* $(0, y)$ *in* $(\mathbb{R}^2 \times \mathbb{R}^3, (0, y))$.

Remark 4.5. The conditions (Theorem 4.2 (ii), Theorem 4.3 (ii) and Theorem 4.4 (ii)) above have already appeared as criteria of versality of orthogonal projection (cf.[12, Theorem 6.8]).

Each singularity type in the section 3 has finitely $\mathcal{A}$-determinacy. From Theorem 2.12 and 2.15, to prove versality of π at a singularity point with k-$\mathcal{A}$-determinacy, we should show the following equality :

$$T\mathcal{A}_e\pi_y + \left\langle \frac{\partial \pi_y}{\partial y_1}, \frac{\partial \pi_y}{\partial y_2}, \frac{\partial \pi_y}{\partial y_3} \right\rangle_{\mathbb{R}} = \theta(\pi_y). \tag{4.3}$$

over $\mathbb{R}$ modulo $\mathrm{m}^k\varepsilon_2$. The Taylor series of a central projection π_y at the origin is expressed as follows :

$$\pi_y(x) = \sum\nolimits_{i+j\geq 1} \begin{pmatrix} c_{ij} \\ d_{ij} \end{pmatrix} \frac{x_1^i\, x_2^j}{i!\, j!}. \tag{4.4}$$

In the proof of the above Theorems, we assume that $H_2(x) = k_1\, x_1 x_2 + \frac{k_2}{2} x_2^2$ for (1.2) from a lotation of f and criteria of singularities. And suppose that $\mathcal{L}$ is the asymptotic direction line written as $f(0) + t\,\boldsymbol{u}$. This assumption means that $p_1 \neq 0$ and $p_2 = 0$ for (3.3). Thus coefficients of the 3-jet of π_y are written as follows :

$c_{10} = d_{10} = c_{01} = 0,\ d_{01} = c := \frac{y_3}{p_1 \sin\theta} \neq 0,$

$c_{20} = d_{20} = 0,\ c_{11} = -\frac{k_1 c}{\sin\theta},\ d_{11} = \frac{c}{p_1} \neq 0,\ c_{02} = -\frac{k_2 c}{\sin\theta},\ d_{02} = 0,$

$c_{30} = -\frac{a_{30} c}{y_3 \sin\theta},\ c_{21} = -(a_{21} p_1 + 2k_1)\frac{c}{p_1 \sin\theta},\ c_{12} = -(a_{12} p_1 + k_2)\frac{c}{p_1 \sin\theta},\ c_{03} = -\frac{a_{03} c}{\sin\theta},$
$d_{30} = 0,\ d_{21} = \frac{2c}{p_1^2} \neq 0,\ d_{12} = \frac{2k_1 \cos\theta\, c}{p_1 \sin\theta},\ d_{03} = \frac{3k_2 \cos\theta\, c}{p_1 \sin\theta}.$

And we assume that $\pi_y(0,0) = (0,0)$ from a coordinate change of target.

4.1 Proof of Theorem 4.1 for the case of $\mathcal{A}_e$-cod.$\pi_y \leq 1$

4.1.1 Fold and cusp

These singularities are stable. It is clear that the central projection unfolding π is $\mathcal{A}_e$-versal unfolding in this case by Theorem 2.13.

4.1.2 Swallowtail

Proof of Theorem 4.1 in the hyperbolic case. The swallowtail singularity is 4-$\mathcal{A}$-deternined. Thus it is enough to show (4.3) that

$$T\mathcal{A}_e\pi_y + \left\langle \frac{\partial \pi_y}{\partial y_1}, \frac{\partial \pi_y}{\partial y_2}, \frac{\partial \pi_y}{\partial y_3} \right\rangle_{\mathbb{R}} \tag{4.5}$$

spans $\theta(\pi_y)$ over $\mathbb{R}$ modulo $\mathrm{m}_2^5\mathcal{E}_2^2$. From criteria of the swallowtail singularity, $k_1 \neq 0$, $a_{40} \neq 0$ and $a_{30} = 0$. Thus we have $c_{30} = 0$ and several coefficients of degree 4 monomials of π_y at 0 as follows :

$$c_{40} = -\frac{a_{40}\, c}{\sin\theta} \ \text{ and } \ d_{40} = 0.$$

Since $\begin{pmatrix} 0 \\ O_4 \end{pmatrix} = \frac{1}{d_{01}} O_4 \frac{\partial \pi_y}{\partial x_2}$ in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^5\mathcal{E}_2^2$, we know all degree 4 monomials of the second component $\begin{pmatrix} 0 \\ O_4 \end{pmatrix}$ are contained in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^5\mathcal{E}_2^2$. Working modulo these monomils, $\begin{pmatrix} x_2\, O_3 \\ 0 \end{pmatrix} = \frac{1}{c_{11}} O_3 \frac{\partial \pi_y}{\partial x_1}$ is contained in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} \mathrm{m}_2^5 \\ \mathrm{m}_2^4 \end{pmatrix}$. This means that all degree 4 monomials except $\begin{pmatrix} x_1^4 \\ 0 \end{pmatrix}$ is spaned by $T\mathcal{A}_e\pi_y$.

Using $\begin{pmatrix} 0 \\ x_2 O_2 \end{pmatrix} = \frac{1}{d_{01}} x_2\, O_2 \frac{\partial \pi_y}{\partial x_2}$ in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2\mathrm{m}_2^3 + \mathrm{m}_2^5 \\ \mathrm{m}_2^4 \end{pmatrix}$, degree 3 monomials of the second component except $\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix}$ is spaned by $T\mathcal{A}_e\pi_y$. From $\begin{pmatrix} x_2 O_2 \\ 0 \end{pmatrix} = \frac{1}{c_{11}} O_2 \frac{\partial \pi_y}{\partial x_1}$ in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2\mathrm{m}_2^3 + \mathrm{m}_2^5 \\ x_2\mathrm{m}_2^2 + \mathrm{m}_2^4 \end{pmatrix}$, we know that the degree 3 monomial $\begin{pmatrix} x_2 O_2 \\ 0 \end{pmatrix}$ is contained in $T\mathcal{A}_e\pi_y$.

We also know that several degree 2 monomials $\begin{pmatrix} 0 \\ x_2 O_1 \end{pmatrix}$ and $\begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}$ are in $T\mathcal{A}_e\pi_y$ from

$$\begin{pmatrix} 0 \\ x_2 O_1 \end{pmatrix} = \frac{1}{d_{01}} x_2 O_1 \frac{\partial \pi_y}{\partial x_2} \ \text{ and } \ \begin{pmatrix} x_2^2 \\ \frac{d_{11}}{c_{11}} x_2^2 \end{pmatrix} = \frac{1}{c_{11}} x_2 \frac{\partial \pi_y}{\partial x_1}$$

in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2\mathrm{m}^2+\mathrm{m}^5 \\ x_2\mathrm{m}^2+\mathrm{m}^4 \end{pmatrix}$. From this, $\begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x_1^4 \\ 0 \end{pmatrix}$ are spaned by $T\mathcal{A}_e\pi_y$ from the following two vectors

$$\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_1\rangle \boldsymbol{e}_1 \\ x_1\frac{\partial \pi_y}{\partial x_1} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{40}/24 \\ c_{11} & c_{40}/6 \end{pmatrix} \begin{pmatrix} x_1x_2\boldsymbol{e}_1 \\ x_1^4\boldsymbol{e}_1 \end{pmatrix}$$

in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2+x_2\mathrm{m}^2+\mathrm{m}^5 \\ x_2\mathrm{m}+\mathrm{m}^4 \end{pmatrix}$. The determinant of the matrix in this formula is

$$\frac{a_{40}\, k_1\, {p_1}^2\, c^4}{8{y_3}^2}.$$

This does not vanish from criteria of swallowtail singularity. Working modulo these monomials, the following elements are written as

$$\begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \frac{1}{d_{01}}\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2\rangle \\ 0 \end{pmatrix}, \ \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \frac{1}{d_{01}}\begin{pmatrix} 0 \\ \langle \pi_y, \boldsymbol{e}_2\rangle \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ x_1^3 \end{pmatrix} = \frac{1}{d_{01}}x_1^3\frac{\partial \pi_y}{\partial x_2}$$

in $T\mathcal{A}_e\pi_y$ modulo $(x_2\mathrm{m}+\mathrm{m}^4)\varepsilon_2$. So three monomials $\begin{pmatrix} x_2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix}$ are contained in $\frac{T\mathcal{A}_e\pi_y}{\mathrm{m}^5\varepsilon_2}$. From this, we know $\begin{pmatrix} x_1^3 \\ 0 \end{pmatrix} = \frac{1}{c_{40}}\frac{\partial \pi_y}{\partial x_1}$ in $T\mathcal{A}_e\pi_y$ modulo $(x_2+x_2\mathrm{m})\varepsilon_2 + \begin{pmatrix} \mathrm{m}^4 \\ \mathrm{m}^3 \end{pmatrix}$.

Finally, we consider the following four vectors

$$\begin{pmatrix} \frac{\partial \pi_y}{\partial y_2} \\ \frac{\partial \pi_y}{\partial x_2} \\ x_1\frac{\partial \pi_y}{\partial x_2} \\ x_1^2\frac{\partial \pi_y}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & -c/p_1 & 0 & -c/p_1^2 \\ c_{11} & d_{11} & c_{21}/2 & d_{21}/2 \\ 0 & d_{01} & c_{11} & d_{11} \\ 0 & 0 & 0 & d_{01} \end{pmatrix} \begin{pmatrix} x_1\boldsymbol{e}_1 \\ x_1\boldsymbol{e}_2 \\ x_1^2\boldsymbol{e}_1 \\ x_1^2\boldsymbol{e}_2 \end{pmatrix}$$

in (4.5) modulo $(x_2+x_2\mathrm{m}+\mathrm{m}^3)\varepsilon_2$. The determinant of the matrix of a linear transformation of them is

$$\frac{{k_1}^2\, p_1\, c^6}{{y_3}^2}$$

and this does not vanish. So $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_1 \end{pmatrix}$, $\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}$ are contained in (4.5).
Therefore the equality (4.3) is satisfied. □

4.1.3 Lips/Beaks

Proof of Theorem 4.1 in the parabolic case. From criteria of the lips/beaks singularity, $k_1 = 0$, $k_2 \neq 0$ and $a_{30} \neq 0$. Then we know $c_{11} = 0$ and $c_{02} \neq 0$.
The lips/beaks singularities are 3-$\mathcal{A}$-deternined. Thus we need to prove the equality (4.3) where $k = 3$.

Several degree 3 monomials $\begin{pmatrix} 0 \\ O_3 \end{pmatrix}$ and $\begin{pmatrix} x_2^3 \\ 0 \end{pmatrix}$ are spaned by $\begin{pmatrix} 0 \\ O_3 \end{pmatrix} = \frac{1}{d_{01}}O_3\frac{\partial \pi_y}{\partial x_2}$ and $\begin{pmatrix} x_2^3 \\ 0 \end{pmatrix} = \frac{1}{d_{01}^3}\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2\rangle^3 \\ 0 \end{pmatrix}$ in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^4\mathcal{E}_2^2$. In the same way, $\begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}$ is spaned by $\begin{pmatrix} 0 \\ x_2^2 \end{pmatrix} = \frac{1}{d_{01}}x_2^2\frac{\partial \pi_y}{\partial x_2}$ in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^3+\mathrm{m}_2^4 \\ \mathrm{m}_2^3 \end{pmatrix}$.

The determinant of the following 6×6 matrix D defined by the following :

$$
{}^t\!\left(\begin{pmatrix}0\\ \langle\pi_y,\boldsymbol{e}_2\rangle\end{pmatrix},\ \begin{pmatrix}\langle\pi_y,\boldsymbol{e}_2\rangle^2\\ 0\end{pmatrix},\ x_1\frac{\partial\pi_y}{\partial x_1},\ x_2\frac{\partial\pi_y}{\partial x_1},\ x_2\frac{\partial\pi_y}{\partial x_2},\ x_1x_2\frac{\partial\pi_y}{\partial x_2}\right)
$$
$$
= D\,{}^t\!\left(\begin{pmatrix}0\\ x_2\end{pmatrix},\begin{pmatrix}0\\ x_1x_2\end{pmatrix},\begin{pmatrix}x_2^2\\ 0\end{pmatrix},\begin{pmatrix}x_1^3\\ 0\end{pmatrix},\begin{pmatrix}x_1^2x_2\\ 0\end{pmatrix},\begin{pmatrix}x_1x_2^2\\ 0\end{pmatrix}\right)
$$

where

$$
D:=\begin{pmatrix}
d_{01} & d_{11} & 0 & 0 & 0 & 0\\
0 & 0 & {d_{01}}^2 & 0 & 0 & 2d_{01}\,d_{11}\\
0 & 0 & 0 & 0 & c_{30}/2 & c_{21}\\
0 & d_{11} & 0 & c_{30}/2 & c_{21} & c_{12}/2\\
d_{01} & d_{11} & c_{02} & 0 & c_{21}/2 & c_{12}\\
0 & d_{01} & 0 & 0 & 0 & c_{02}
\end{pmatrix}
$$

on $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix}x_2^3+\mathrm{m}_2^4\\ x_2^2+\mathrm{m}_2^3\end{pmatrix}$ is $a_{30}\frac{(a_{12}\,a_{30}-{a_{21}}^2)p_1-k_2\,a_{30}}{4{p_1}^8\sin^{10}\theta}\,y_3^7$. From criteria of lips and beaks, this does not vanish. Thus we get monomials $\begin{pmatrix}0\\ x_2\end{pmatrix}$, $\begin{pmatrix}0\\ x_1x_2\end{pmatrix}$, $\begin{pmatrix}x_2^2\\ 0\end{pmatrix}$, $\begin{pmatrix}x_1^3\\ 0\end{pmatrix}$, $\begin{pmatrix}x_1^2x_2\\ 0\end{pmatrix}$ and $\begin{pmatrix}x_1x_2^2\\ 0\end{pmatrix}$ in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^4\mathcal{E}_2^2$.

Next, we consider the generation of degree 1 monomials and remaining degree 2 monomials. A degree 2 monomial $\begin{pmatrix}0\\ x_1^2\end{pmatrix}=\frac{1}{d_{01}}x_1^2\frac{\partial\pi_y}{\partial x_2}$ is contained in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix}x_2^2\\ x_2+x_2\mathrm{m}_2\end{pmatrix}+\mathrm{m}_2^3\mathcal{E}_2^2$. Furthemore, we consider linear independence of the following elements :

$$
\begin{pmatrix}\langle\pi_y,\boldsymbol{e}_2\rangle\boldsymbol{e}_1\\ \frac{\partial\pi_y}{\partial x_1}\\ \frac{\partial\pi_y}{\partial x_2}\\ x_1\frac{\partial\pi_y}{\partial x_2}\end{pmatrix}
=\begin{pmatrix}
0 & d_{01} & 0 & d_{11}\\
0 & 0 & c_{30}/2 & c_{21}\\
d_{11} & c_{02} & c_{21}/2 & c_{12}\\
d_{01} & 0 & 0 & c_{02}
\end{pmatrix}
\begin{pmatrix}x_1\,\boldsymbol{e}_2\\ x_2\,\boldsymbol{e}_1\\ x_1^2\,\boldsymbol{e}_1\\ x_1x_2\,\boldsymbol{e}_1\end{pmatrix}
$$

in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix}x_2^2+\mathrm{m}_2^3\\ x_2+\mathrm{m}_2^2\end{pmatrix}$. The determinant of the matrix in this equation is

$$
-\frac{(a_{12}\,a_{30}-{a_{21}}^2)p_1-k_2\,a_{30}}{2{p_1}^5\sin^6\theta}\,y_3^4.
$$

From criteria of lips and beaks, This does not vanish. So, $\begin{pmatrix}0\\ x_1\end{pmatrix}$, $\begin{pmatrix}x_2\\ 0\end{pmatrix}$, $\begin{pmatrix}x_1^2\\ 0\end{pmatrix}$ and $\begin{pmatrix}x_1x_2\\ 0\end{pmatrix}$ are spaned by $T\mathcal{A}_e\pi_y/\mathrm{m}_2^4\mathcal{E}_2^2$.

Finally, we get the remaining monomial $\begin{pmatrix}x_1\\ 0\end{pmatrix}$ in (4.5) modulo $\begin{pmatrix}x_2+\mathrm{m}_2^2\\ \mathrm{m}_2\end{pmatrix}$ since $\begin{pmatrix}x_1\\ 0\end{pmatrix}=-\frac{p_1}{c}\frac{\partial\pi_y}{\partial y_1}$. □

4.2 Hyperbolic surfaces with $\mathcal{A}_e$-cod.$\pi_y=2,3$

Using criteria of the butterfly singularity, we know two coefficients both of two coefficients k_1 and a_{50} does not vanish and $a_{30}=a_{40}=0$. Thus coefficients of the 3-jet of π_y is the same as in the case of the swallowtail singularity. And several coefficients of the 7-jet of π_y at 0 are written as follows :

$$c_{40} = d_{40} = 0,\ c_{31} = -(a_{31}\,{p_1}^2 + 3a_{21}\,p_1 + 6k_1)\frac{c^2}{p_1\,y_3},\ d_{31} = \frac{6\,c}{p_1^3},$$

$$c_{50} = -\frac{a_{50}\,p_1\,c^2}{y_3} \neq 0,\ d_{50} = 0,\ c_{41} = -(a_{41}\,{p_1}^3 + 4a_{31}\,{p_1}^2 + 12a_{21}\,p_1 + 24k_1)\frac{c^2}{{p_1}^2\,y_3},\ d_{41} = \frac{24\,c}{p_1^4},$$

$$c_{60} = -(a_{60}\,p_1 + 6a_{50})\frac{c^2}{y_3},\ d_{60} = 0,\ c_{70} = -(a_{70}\,{p_1}^2 + 7a_{60}\,p_1 + 42a_{50})\frac{c^2}{p_1\,y_3},\ d_{70} = 0.$$

4.2.1 Butterfly

Proof of 1 in Theorem 4.2. Since the butterfly singularity is 7-$\mathcal{A}$-deternined, it is enough to show that (4.5) spans $\theta(\pi_y)$ over $\mathbb{R}$ modulo $\mathrm{m}_2^8\mathcal{E}_2^2$.

From $\begin{pmatrix} 0 \\ O_7 \end{pmatrix} = \frac{1}{d_{01}} O_7 \frac{\partial \pi_y}{\partial x_2}$ in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^8\mathcal{E}_2^2$, we get all monomials $\begin{pmatrix} 0 \\ O_7 \end{pmatrix}$ in $T\mathcal{A}_e\pi_y/\mathrm{m}^8\varepsilon_2$. $\begin{pmatrix} x_2O_6 \\ 0 \end{pmatrix}$ in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^8\mathcal{E}_2^2$ is generated by $\begin{pmatrix} x_2O_6 \\ 0 \end{pmatrix} = \frac{1}{c_{11}} O_6 \frac{\partial \pi_y}{\partial x_1}$ in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^8\mathcal{E}_2^2$ over $\mathbb{R}$.

In the same way, all monomials $\begin{pmatrix} 0 \\ x_2O_k \end{pmatrix}$ and $\begin{pmatrix} x_2O_l \\ 0 \end{pmatrix}$ for $k = 3$ to 5, $l = 4$ to 6 are spanned by

$$\frac{1}{d_{01}} x_2 O_k \frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} 0 \\ x_2O_k \end{pmatrix} \quad \text{and} \quad \frac{1}{c_{11}} O_l \frac{\partial \pi_y}{\partial x_1} = \begin{pmatrix} x_2O_l \\ \frac{d_{11}}{c_{11}} x_2O_l \end{pmatrix}$$

in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2\mathrm{m}_2^6 + \mathrm{m}_2^8 \\ \mathrm{m}_2^7 \end{pmatrix}$. From $\begin{pmatrix} x_2^2O_2 \\ 0 \end{pmatrix} = \frac{1}{c_{11}} {x_2}^2 O_2 \frac{\partial \pi_y}{\partial x_1}$ in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2\mathrm{m}_2^4 + \mathrm{m}_2^8 \\ x_2\mathrm{m}_2^3 + \mathrm{m}_2^7 \end{pmatrix}$, $\begin{pmatrix} x_2^2O_2 \\ 0 \end{pmatrix}$ is contained in $T\mathcal{A}_e\pi_y$. Thus, degree 3 monomials $\begin{pmatrix} x_2^2O_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_2^2O_1 \end{pmatrix}$ are spaned by

$$\frac{1}{c_{11}} x_2 O_1 \frac{\partial \pi_y}{\partial x_1} = \begin{pmatrix} x_2^2O_1 \\ \frac{d_{11}}{c_{11}} x_2^2O_1 \end{pmatrix} \quad \text{and} \quad \frac{1}{d_{01}} x_2^2 O_1 \frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} 0 \\ x_2^2O_1 \end{pmatrix}$$

in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2\mathrm{m}_2^2 + x_2\mathrm{m}_2^4 + \mathrm{m}_2^8 \\ x_2\mathrm{m}_2^3 + \mathrm{m}_2^7 \end{pmatrix}$. Using the following linearly independent elements

$$\frac{1}{c_{11}} x_2 \frac{\partial \pi_y}{\partial x_1} = \begin{pmatrix} x_2^2 \\ \frac{d_{11}}{c_{11}} x_2^2 \end{pmatrix} \quad \text{and} \quad \frac{1}{d_{01}} x_2^2 \frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}$$

in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2\mathrm{m}_2 + x_2\mathrm{m}_2^4 + \mathrm{m}_2^8 \\ x_2^2\mathrm{m}_2 + x_2\mathrm{m}_2^3 + \mathrm{m}_2^7 \end{pmatrix}$, we know degree 2 monomials $\begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}$ are contained in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^8\mathcal{E}_2^2$.

We consider the following fifteen elements

$$\begin{aligned} &{}^t\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_1\rangle \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_1\rangle \boldsymbol{e}_2,\ \langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_2, \\ \frac{\partial \pi_y}{\partial x_1},\ x_1\frac{\partial \pi_y}{\partial x_1},\ x_2\frac{\partial \pi_y}{\partial x_2},\ x_1^2\frac{\partial \pi_y}{\partial x_1},\ x_1x_2\frac{\partial \pi_y}{\partial x_2},\ x_1^3\frac{\partial \pi_y}{\partial x_1},\ x_1^3\frac{\partial \pi_y}{\partial x_2},\ x_1^2x_2\frac{\partial \pi_y}{\partial x_2},\ x_1^4\frac{\partial \pi_y}{\partial x_2},\ x_1^5\frac{\partial \pi_y}{\partial x_2},\ x_1^6\frac{\partial \pi_y}{\partial x_2} \end{pmatrix} \\ &= D_1\,{}^t\begin{pmatrix} x_2\boldsymbol{e}_1,\ x_2\boldsymbol{e}_2,\ x_1x_2\boldsymbol{e}_1,\ x_1x_2\boldsymbol{e}_2,\ x_1^3\boldsymbol{e}_2,\ x_1^2x_2\boldsymbol{e}_1,\ x_1^2x_2\boldsymbol{e}_2, \\ x_1^4\boldsymbol{e}_1,\ x_1^4\boldsymbol{e}_2,\ x_1^3x_2\boldsymbol{e}_1,\ x_1^5\boldsymbol{e}_1,\ x_1^5\boldsymbol{e}_2,\ x_1^6\boldsymbol{e}_1,\ x_1^6\boldsymbol{e}_2,\ x_1^7\boldsymbol{e}_1 \end{pmatrix}, \end{aligned}$$

$$D_1 := \begin{pmatrix} 0 & 0 & c_{11} & 0 & 0 & c_{21}/2 & 0 & 0 & 0 & c_{31}/6 & c_{50}/120 & 0 & c_{60}/720 & 0 & c_{70}/5040 \\ 0 & 0 & 0 & c_{11} & 0 & 0 & c_{21}/2 & 0 & 0 & 0 & 0 & c_{50}/120 & 0 & c_{60}/720 & 0 \\ d_{01} & 0 & d_{11} & 0 & 0 & d_{21}/2 & 0 & 0 & 0 & d_{31}/6 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_{01} & 0 & d_{11} & 0 & 0 & d_{21}/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{11} & d_{11} & c_{21} & d_{21} & 0 & c_{31}/2 & d_{31}/2 & c_{50}/24 & 0 & c_{41}/6 & c_{60}/120 & 0 & c_{70}/720 & 0 & * \\ 0 & 0 & c_{11} & d_{11} & 0 & c_{21} & d_{21} & 0 & 0 & c_{31}/2 & c_{50}/24 & 0 & c_{60}/120 & 0 & c_{70}/720 \\ 0 & d_{01} & c_{11} & d_{11} & 0 & c_{21}/2 & d_{21}/2 & 0 & 0 & c_{31}/6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{11} & d_{11} & 0 & 0 & c_{21} & 0 & 0 & c_{50}/24 & 0 & c_{60}/120 \\ 0 & 0 & 0 & d_{01} & 0 & c_{11} & d_{11} & 0 & 0 & c_{21}/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{11} & 0 & 0 & 0 & 0 & c_{50}/24 \\ 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & c_{11} & d_{11} & 0 & c_{21}/2 & d_{21}/2 & c_{31}/6 & d_{31}/6 & c_{41}/24 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & c_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & c_{11} & d_{11} & c_{21}/2 & d_{21} & c_{31}/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & c_{11} & d_{11} & c_{21}/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & c_{11} \end{pmatrix}.$$

in $T\mathcal{A}_e\pi_y$ modulo $x_2^2\mathcal{E}_2^2 + \begin{pmatrix} x_2\mathrm{m}_2^4 + \mathrm{m}_2^8 \\ x_2\mathrm{m}_2^3 + \mathrm{m}_2^7 \end{pmatrix}$. The determinant of D_1 is

$$\left\{ \begin{pmatrix} (48a_{50}\, a_{70} - 35{a_{60}}^2)\, {k_1}^2 \\ +42(a_{21}\, a_{60} - 40a_{31}\, a_{50})\, a_{50}\, k_1 \\ +\, 2205{a_{21}}^2\, {a_{50}}^2 \end{pmatrix} p_1^2 - 84a_{50}\, k_1\, (a_{60}\, k_1 - 3a_{21}\, a_{50})\, p_1 + 756{a_{50}}^2\, {k_1}^2 \right\} \frac{{a_{50}}^2\, {k_1}^2\, {p_1}^6\, c^{23}}{10450944000{y_3}^8}.$$

Thus, the above fifteen monomials are spaned by $T\mathcal{A}_e\pi_y$ since y is not butterfly-focal point. Finally, we consider the following five elements

$${}^t\left(\frac{\partial \pi_y}{\partial y_1}, \frac{\partial \pi_y}{\partial y_2}, \frac{\partial \pi_y}{\partial x_2}, x_1\frac{\partial \pi_y}{\partial x_2}, x_1^2\frac{\partial \pi_y}{\partial x_2}\right) = D_2\,{}^t\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} x_1^3 \\ 0 \end{pmatrix} \right)$$

where

$$D_2 := \begin{pmatrix} -c/p_1 & 0 & -c/{p_1}^2 & 0 & c/{p_1}^3 \\ 0 & -c/p_1 & 0 & -c/{p_1}^2 & 0 \\ c_{11} & d_{11} & c_{21}/2 & d_{21}/2 & c_{31}/6 \\ 0 & d_{01} & c_{11} & d_{11} & c_{21}/2 \\ 0 & 0 & 0 & d_{01} & c_{11} \end{pmatrix},$$

$$\frac{\partial^2 \pi_y}{\partial y_1 \partial x_1} = \begin{pmatrix} -\frac{c}{p_1} \\ 0 \end{pmatrix}, \quad \frac{1}{2}\frac{\partial^3 \pi_y}{\partial y_1 \partial x_1^2} = \begin{pmatrix} -\frac{c}{{p_1}^2} \\ 0 \end{pmatrix} \quad \text{and} \quad \frac{1}{6}\frac{\partial^3 \pi_y}{\partial y_1 \partial x_1^3} = \begin{pmatrix} -\frac{c}{{p_1}^3} \\ 0 \end{pmatrix}.$$

The determinant of D_2 is

$$(2a_{31}\, k_1 - 3{a_{21}}^2)\frac{c^7}{12{y_3}^2}.$$

Therefore, the above five monomials are spaned by (4.5) if and only if $2a_{31}k_1 - 3{a_{21}}^2$ does not vanish. □

Remark 4.6. Our source code for computation the determinant of D_1 and D_2 is available at `https://github.com/Shuhei-singularity123/Versality_of_central_projection_of_regular_surface/blob/master/versality_of_butterfly.mac`.

4.2.2 Elder butterfly

Proof of 2 in Theorem 4.2. The elder butterfly singularity is 7-$\mathcal{A}$-determined which is equal to the determinacy of the butterfly singularity. Thus we should prove the equality (4.3) holds for $k = 7$. We know the fifteen elements expressed by D_1 in the subsection 4.2.1 are not linearly independent since y is butterfly-focal. The other elements used in the subsection 4.2.1 of (4.5) are linearly independent if $2a_{31}k_1 - 3{a_{21}}^2$ does not vanish. Thus, we retake the following fifteen elements in (4.5) :

$$
{}^t\begin{pmatrix} \frac{\partial \pi_y}{\partial y_3} + \frac{\partial \pi_y}{\partial y_1}\tan\theta, \\ \langle \pi_y, \boldsymbol{e}_1\rangle \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_1\rangle \boldsymbol{e}_2,\ \langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_2, \\ \frac{\partial \pi_y}{\partial x_1},\ x_1\frac{\partial \pi_y}{\partial x_1},\ x_1^2\frac{\partial \pi_y}{\partial x_1},\ x_1x_2\frac{\partial \pi_y}{\partial x_2},\ x_1^3\frac{\partial \pi_y}{\partial x_1},\ x_1^3\frac{\partial \pi_y}{\partial x_2},\ x_1^2x_2\frac{\partial \pi_y}{\partial x_2},\ x_1^4\frac{\partial \pi_y}{\partial x_2},\ x_1^5\frac{\partial \pi_y}{\partial x_2},\ x_1^6\frac{\partial \pi_y}{\partial x_2} \end{pmatrix}
$$

$$
= \begin{pmatrix} \boldsymbol{d}_{11} \\ D_{12} \end{pmatrix} {}^t\begin{pmatrix} x_2\boldsymbol{e}_1,\ x_2\boldsymbol{e}_2,\ x_1x_2\boldsymbol{e}_1,\ x_1x_2\boldsymbol{e}_2,\ x_1^3\boldsymbol{e}_2,\ x_1^2x_2\boldsymbol{e}_1,\ x_1^2x_2\boldsymbol{e}_2, \\ x_1^4\boldsymbol{e}_1,\ x_1^4\boldsymbol{e}_2,\ x_1^3x_2\boldsymbol{e}_1,\ x_1^5\boldsymbol{e}_1,\ x_1^5\boldsymbol{e}_2,\ x_1^6\boldsymbol{e}_1,\ x_1^6\boldsymbol{e}_2,\ x_1^7\boldsymbol{e}_1 \end{pmatrix},
$$

in $T\mathcal{A}_e\pi_y$ modulo $x_2^2\mathcal{E}_2^2 + \begin{pmatrix} x_2\mathrm{m}_2^4 + \mathrm{m}_2^8 \\ x_2\mathrm{m}_2^3 + \mathrm{m}_2^7 \end{pmatrix}$ where the $(14, 15)$-matrix D_{12} is

$$
\begin{pmatrix}
0 & 0 & c_{11} & 0 & 0 & c_{21}/2 & 0 & 0 & 0 & c_{31}/6 & c_{50}/120 & 0 & c_{60}/720 & 0 & c_{70}/5040 \\
0 & 0 & 0 & c_{11} & 0 & 0 & c_{21}/2 & 0 & 0 & 0 & 0 & c_{50}/120 & 0 & c_{60}/720 & 0 \\
d_{01} & 0 & d_{11} & 0 & 0 & d_{21}/2 & 0 & 0 & 0 & d_{31}/6 & 0 & 0 & 0 & 0 & 0 \\
0 & d_{01} & 0 & d_{11} & 0 & 0 & d_{21}/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_{11} & d_{11} & c_{21} & d_{21} & 0 & c_{31}/2 & d_{31}/2 & c_{50}/24 & 0 & c_{41}/6 & c_{60}/120 & 0 & c_{70}/720 & 0 & * \\
0 & 0 & c_{11} & d_{11} & 0 & c_{21} & d_{21} & 0 & 0 & c_{31}/2 & c_{50}/24 & 0 & c_{60}/120 & 0 & c_{70}/720 \\
0 & 0 & 0 & 0 & 0 & c_{11} & d_{11} & 0 & 0 & c_{21} & 0 & 0 & c_{50}/24 & 0 & c_{60}/120 \\
0 & 0 & 0 & d_{01} & 0 & c_{11} & d_{11} & 0 & 0 & c_{21}/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{11} & 0 & 0 & 0 & 0 & c_{50}/24 \\
0 & 0 & 0 & 0 & d_{01} & 0 & 0 & c_{11} & d_{11} & c_{02} & c_{21}/2 & d_{21}/2 & c_{31}/6 & d_{31}/6 & c_{41}/24 \\
0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & c_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & c_{11} & d_{11} & c_{21}/2 & d_{21}/2 & c_{31}/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & c_{11} & d_{11} & c_{21}/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & c_{11}
\end{pmatrix}
$$

and

$$
\boldsymbol{d}_{11} := \frac{c^3}{{y_3}^3}\begin{pmatrix} 0,\ -\frac{f_{03}\,y_3}{c},\ f_{03}\,k_1\,p_1,\ -\frac{(y_3+f_{03})\,y_3}{p_1\,c},\ 0, \\ \frac{(2k_1\,(y_3+f_{03})+a_{21}\,f_{03}\,p_1)}{2},\ -\frac{(2y_3+f_{03})\,y_3}{{p_1}^2\,c},\ 0,\ 0,\ \frac{3a_{21}\,p_1\,(y_3+f_{03})+6k_1\,(2y_3+f_{03})+a_{31}\,f_{03}\,{p_1}^2}{6\,p_1}, \\ \frac{a_{50}\,f_{03}\,p_1}{120},\ 0,\ \frac{6a_{50}\,(y_3+f_{03})+a_{60}\,f_{03}\,p_1}{720},\ 0,\ \frac{7a_{60}\,p_1\,(y_3+f_{03})+42a_{50}\,(2a_{50}\,y_3+f_{03})+a_{70}\,f_{03}\,{p_1}^2}{5040\,p_1} \end{pmatrix}.
$$

The determinant of D_1 is

$$
{a_{50}}^3\,{k_1}^3\,((a_{60}\,k_1 - 3a_{21}\,a_{50})\,p_1 - 18a_{50}\,k_1)\frac{{p_1}^6\,c^{24}}{62208000\,{y_3}^9}.
$$

Our source code for the computation of the determinant of D_1 is available at `https://github.com/Shuhei-singularity123/Versality_of_central_projection_of_regular_surface/blob/master/versality_of_elder_butterfly.mac`. Therefore, π is versal at the origin in the case of elder butterfly if and only if

$$
2a_{31}k_1 - 3{a_{21}}^2 \ \text{ and } \ (a_{60}\,k_1 - 3a_{21}\,a_{50})\,p_1 - 18a_{50}\,k_1
$$

do not vanish. □

4.2.3 Unimodal

Proof of 3 in Theorem 4.2. From the assumption and criteria, $a_{50} = 0$ and a_{60} does not vanish. The unimodal singularity is 8-$\mathcal{A}$-determined. If we know versality of this type, we check the equality (4.3) where $k = 8$. We consider whether seven monomials $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_1 \end{pmatrix}$, $\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}$,

$\begin{pmatrix} x_1^3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix}$ and $\begin{pmatrix} x_1^4 \\ 0 \end{pmatrix}$ are generated by several elements in (4.5) modulo $\mathfrak{m}_2^9\mathcal{E}_2^2$. However, we can only choose the following elements in (4.5) modulo $\mathfrak{m}_2^9\mathcal{E}_2^2$ to generate the above monomials :

$$
{}^t\left(\frac{\partial \pi_y}{\partial y_1},\quad \frac{\partial \pi_y}{\partial y_2},\quad \frac{\partial \pi_y}{\partial y_3},\quad \frac{\partial \pi_y}{\partial x_2},\quad x_1\frac{\partial \pi_y}{\partial x_2},\quad x_1^2\frac{\partial \pi_y}{\partial x_2},\quad x_1^3\frac{\partial \pi_y}{\partial x_2}\right)
$$

$$
= \begin{pmatrix}
-\frac{y_3}{{p_1}^2\sin\theta} & 0 & -\frac{y_3}{{p_1}^3\sin\theta} & 0 & -\frac{y_3}{{p_1}^4\sin\theta} & 0 & -\frac{y_3}{{p_1}^5\sin\theta} \\
0 & -\frac{y_3}{{p_1}^2\sin\theta} & 0 & -\frac{y_3}{{p_1}^3\sin\theta} & 0 & -\frac{y_3}{{p_1}^4\sin\theta} & 0 \\
\frac{y_3\ \cos\theta}{{p_1}^2\sin^2\theta} & 0 & \frac{y_3\ \cos\theta}{{p_1}^3\sin^2\theta} & 0 & \frac{y_3\ \cos\theta}{{p_1}^4\sin^2\theta} & 0 & \frac{y_3\ \cos\theta}{{p_1}^5\sin^2\theta} \\
c_{11} & d_{11} & c_{21} & d_{21} & c_{31} & d_{31} & c_{41} \\
0 & d_{01} & c_{11} & d_{11} & c_{21} & d_{21} & c_{31} \\
0 & 0 & 0 & d_{01} & c_{11} & d_{11} & c_{21} \\
0 & 0 & 0 & 0 & 0 & d_{01} & c_{11}
\end{pmatrix}
\begin{pmatrix} x_1\boldsymbol{e}_1 \\ x_1\boldsymbol{e}_2 \\ x_1^2\boldsymbol{e}_1 \\ x_1^2\boldsymbol{e}_2 \\ x_1^3\boldsymbol{e}_1 \\ x_1^3\boldsymbol{e}_2 \\ x_1^4\boldsymbol{e}_1 \end{pmatrix} + \cdots .
$$

in (4.5) modulo $\mathfrak{m}_2^9\mathcal{E}_2^2$. From $\frac{\partial \pi_y}{\partial y_1}$ and $\frac{\partial \pi_y}{\partial y_3}$ are not linearly independent in this part, these monomials can not generate the seven elements. Therefore we know that an unfolding π is not versal at the unimodal singularity. $\square$

4.3 Parabolic surfaces so that π_y has gulls series singularities with $\mathcal{A}_e$-cod.$\pi_y \leq 3$

The Taylor series of central projection π_y is (4.4) where $c_{21} \neq 0$ from criteria of the gulls series singularities which are $a_{30} = 0$ and $a_{21} \neq 0$. Several coefficients of the 7-jet of π_y are expressed as follows :

$c_{40} = -\frac{a_{40}\,p_1\,c^2}{y_3},\ d_{40} = 0,\ c_{31} = -(a_{31}\,p_1 + 3a_{21})\frac{c^2}{2y_3},\ d_{31} = \frac{6\,c}{{p_1}^3},$
$c_{22} = -(a_{22}\,{p_1}^2 + 2a_{12}\,p_1 + 2k_2)\frac{c^2}{p_1\,y_3},\ c_{13} = -(a_{13}\,p_1 + a_{03})\frac{c^2}{y_3},$

$c_{50} = -(a_{50}\,p_1 + 5a_{40})\frac{c^2}{y_3},\ c_{41} = -(a_{41}\,{p_1}^2 + 4a_{31}\,p_1 + 12a_{21})\frac{c^2}{p_1\,y_3},$
$c_{32} = -(a_{32}\,{p_1}^3 + 3a_{22}\,{p_1}^2 + 6a_{12}\,p_1 + 6k_2)\frac{c^2}{{p_1}^2\,y_3},$

$c_{60} = -(a_{60}\,{p_1}^2 + 6a_{50}\,p_1 + 30a_{40})\frac{c^2}{p_1\,y_3},\ c_{51} = -(a_{51}\,{p_1}^3 + 5a_{41}\,{p_1}^2 + 20a_{31}\,p_1 + 60a_{21})\frac{c^2}{{p_1}^2\,y_3},$

$c_{70} = -(a_{70}\,{p_1}^3 + 7a_{60}\,{p_1}^2 + 42a_{50}\,p_1 + 210a_{40})\frac{c^2}{{p_1}^2\,y_3}.$

4.3.1 Gulls

Proof of 1 in Theorem 4.3 at gulls singularity. Since gulls type is 5-$\mathcal{A}$-determined, we should show that the equality (4.3) holds for $k = 5$. From criteria of gulls singularity, $a_{40} \neq 0$ and $c_{40} \neq 0$. The element $\frac{1}{d_{01}}O_5\frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} 0 \\ O_5 \end{pmatrix}$ in $T\mathcal{A}_e\pi_y/\mathfrak{m}_2^6\mathcal{E}_2^2$ gives all degree 5 monomial of second component expressed as $\begin{pmatrix} 0 \\ O_5 \end{pmatrix}$ in $T\mathcal{A}_e\pi_y/\mathfrak{m}_2^6\mathcal{E}_2^2$. From this, we know a monomial $\begin{pmatrix} 0 \\ x_2^4 \end{pmatrix} = \frac{1}{d_{01}^4}\begin{pmatrix} 0 \\ \langle \pi_y, \boldsymbol{e}_2\rangle^4 \end{pmatrix}$ in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} \mathfrak{m}_2^6 \\ \mathfrak{m}_2^5 \end{pmatrix}$. As same, we know a monomial $\begin{pmatrix} x_2^5 \\ 0 \end{pmatrix}$ is generated by $\begin{pmatrix} x_2^5 \\ 0 \end{pmatrix} = \frac{1}{d_{01}^5}\langle \pi_y, \boldsymbol{e}_2\rangle^5\boldsymbol{e}_1$ in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} \mathfrak{m}_2^6 \\ x_2^4 + \mathfrak{m}_2^5 \end{pmatrix}$. Using the following linearly independent elements of $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^5 + \mathrm{m}_2^6 \\ x_2^4 + \mathrm{m}_2^5 \end{pmatrix}$:

$$x_1 O_3 \frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} c_{02} x_1 x_2 O_3 \\ d_{01} x_1 O_3 \end{pmatrix},\ x_1 O_2 \frac{\partial \pi_y}{\partial x_1} = \begin{pmatrix} c_{21} x_1^2 x_2 O_2 + \frac{c_{12}}{2} x_1 x_2^2 O_2 \\ d_{11} x_1 x_2 O_2 \end{pmatrix} \text{ and } x_2^3 \frac{\partial \pi_y}{\partial x_1} = \begin{pmatrix} c_{21} x_1 x_2^4 \\ 0 \end{pmatrix},$$

we get monomials $\begin{pmatrix} 0 \\ x_1 O_3 \end{pmatrix}$ and $\begin{pmatrix} x_1 x_2 O_3 \\ 0 \end{pmatrix}$ respectively. From this, we get $\begin{pmatrix} x_2^4 \\ 0 \end{pmatrix}$ by $\begin{pmatrix} x_2^4 \\ 0 \end{pmatrix} = \frac{1}{d_{01}^4} \langle \pi_y, \boldsymbol{e}_2 \rangle^4 \boldsymbol{e}_1$ in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2 \mathrm{m}_2^4 + \mathrm{m}_2^6 \\ \mathrm{m}_2^4 \end{pmatrix}$. An degree 3 monomial $\begin{pmatrix} 0 \\ x_2^3 \end{pmatrix} = \frac{1}{d_{01}} \begin{pmatrix} 0 \\ \langle \pi_y, \boldsymbol{e}_2 \rangle^3 \end{pmatrix}$ is spaned by $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^4 + x_2 \mathrm{m}_2^4 + \mathrm{m}_2^6 \\ \mathrm{m}_2^4 \end{pmatrix}$.

We consider the following fourteen elements

$$\begin{aligned} & {}^t\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_1 \rangle \boldsymbol{e}_2,\ \langle \pi_y, \boldsymbol{e}_2 \rangle \boldsymbol{e}_2,\ \langle \pi_y, \boldsymbol{e}_2 \rangle^2 \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_2 \rangle^2 \boldsymbol{e}_2,\ \langle \pi_y, \boldsymbol{e}_2 \rangle^3 \boldsymbol{e}_1, \\ x_1 \frac{\partial \pi_y}{\partial x_1},\ x_2 \frac{\partial \pi_y}{\partial x_1},\ x_2 \frac{\partial \pi_y}{\partial x_2},\ x_1^2 \frac{\partial \pi_y}{\partial x_1},\ x_1 x_2 \frac{\partial \pi_y}{\partial x_1},\ x_2^2 \frac{\partial \pi_y}{\partial x_1},\ x_1 x_2 \frac{\partial \pi_y}{\partial x_2},\ x_2^2 \frac{\partial \pi_y}{\partial x_2} \end{pmatrix} \\ & = D\, {}^t\begin{pmatrix} x_2 \boldsymbol{e}_2,\ x_1 x_2 \boldsymbol{e}_2,\ x_2^2 \boldsymbol{e}_1,\ x_2^2 \boldsymbol{e}_2,\ x_1^2 x_2 \boldsymbol{e}_1,\ x_1^2 x_2 \boldsymbol{e}_2,\ x_1 x_2^2 \boldsymbol{e}_1,\ x_1 x_2^2 \boldsymbol{e}_2,\ x_2^3 \boldsymbol{e}_1, \\ x_1^4 \boldsymbol{e}_1,\ x_1^3 x_2 \boldsymbol{e}_1,\ x_1^2 x_2^2 \boldsymbol{e}_1,\ x_1 x_2^3 \boldsymbol{e}_1,\ x_1^5 \boldsymbol{e}_1 \end{pmatrix} \end{aligned}$$

where

$$D := \begin{pmatrix}
0 & 0 & c_{02}/2 & 0 & c_{21}/2 & 0 & c_{12}/2 & 0 & c_{03}/6 & c_{40}/24 & c_{31}/6 & c_{22}/4 & c_{13}/6 & c_{50}/120 \\
0 & 0 & 0 & c_{02}/2 & 0 & c_{21}/2 & 0 & c_{12}/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_{01} & d_{11} & 0 & 0 & 0 & d_{21}/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & {d_{01}}^2 & 0 & 0 & 0 & 2d_{01}\, d_{11} & 0 & 0 & 0 & 0 & d_{01}\, d_{21}+{d_{11}}^2 & 0 & 0 \\
0 & 0 & 0 & {d_{01}}^2 & 0 & 0 & 0 & 2d_{01}\, d_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {d_{01}}^3 & 0 & 0 & 0 & 3{d_{01}}^2\, d_{11} & 0 \\
0 & d_{11} & 0 & 0 & c_{21} & d_{21} & c_{12}/2 & 0 & 0 & c_{40}/6 & c_{31}/2 & c_{22}/2 & c_{13}/6 & c_{50}/24 \\
0 & 0 & 0 & d_{11} & 0 & 0 & c_{21} & d_{21} & c_{12}/2 & 0 & c_{40}/6 & c_{31}/2 & c_{22}/2 & 0 \\
d_{01} & d_{11} & c_{02} & 0 & c_{21}/2 & d_{21}/2 & c_{12} & 0 & c_{03}/2 & 0 & c_{31}/6 & c_{22}/2 & c_{13}/2 & 0 \\
0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & c_{21} & c_{12}/2 & 0 & c_{40}/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & c_{21} & c_{12}/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{21} & 0 \\
0 & d_{01} & 0 & 0 & 0 & d_{11} & c_{02} & 0 & 0 & 0 & c_{21}/2 & c_{12} & c_{03}/2 & 0 \\
0 & 0 & 0 & d_{01} & 0 & 0 & 0 & d_{11} & c_{02} & 0 & 0 & c_{21}/2 & c_{12} & 0
\end{pmatrix}.$$

The determinant of the matirx D is

$$-\frac{{a_{21}}^4\, a_{40} (3{a_{21}}^2\, a_{50}\, p_1 + 5a_{12}\, {a_{40}}^2\, p_1 - 10a_{21}\, a_{31}\, a_{40}\, p_1 - 5{a_{40}}^2\, k_2 + 15{a_{21}}^2\, a_{40})\, {p_1}^6\, c^{26}}{23040{y_3}^8}$$

Our source code for computation of the determinant of D is available at `https://github.com/Shuhei-singularity123/Versality_of_central_projection_of_regular_surface/blob/master/versality_of_gulls.mac`. Thus we can get the above monomials at gulls type.
$\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix}$ is spaned by $\frac{1}{d_{01}} x_1^3 \frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} 0 \\ x_1^3 \end{pmatrix}$. in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2 + x_2 \mathrm{m}_2^2 \\ x_2 \varepsilon_1 \end{pmatrix} + \mathrm{m}_2^4 \mathcal{E}_2^2$. In the same way, we know $\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix} = \frac{1}{d_{01}} x_1^2 \frac{\partial \pi_y}{\partial x_2}$ in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2 + x_2 \mathrm{m}_2^2 + \mathrm{m}_2^4 \\ x_2 \varepsilon_1 + \mathrm{m}_2^3 \end{pmatrix}$. A degree 1 monomial $\begin{pmatrix} 0 \\ x_1 \end{pmatrix} = -\frac{p_1}{c} \frac{\partial \pi_y}{\partial y_2}$ is spaned by (4.5) modulo $\begin{pmatrix} x_2^2 + x_2 \mathrm{m}_2^2 + \mathrm{m}_2^4 \\ x_2 + \mathrm{m}_2^2 \end{pmatrix}$.

We have no other way to generate two monomials $\begin{pmatrix} x_1 x_2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x_1^3 \\ 0 \end{pmatrix}$ which is to use pair of elements

$$\left(\frac{\partial \pi_y}{\partial x_1},\ x_1 \frac{\partial \pi_y}{\partial x_2} \right) = \left(\begin{pmatrix} x_1 x_2 \\ 0 \end{pmatrix},\ \begin{pmatrix} x_1^3 \\ 0 \end{pmatrix} \right) \begin{pmatrix} c_{21} & c_{02} \\ \frac{c_{40}}{6} & \frac{c_{21}}{2} \end{pmatrix}$$

in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2 + x_2\mathrm{m}_2^2 + \mathrm{m}_2^4 \\ \mathrm{m}_2 \end{pmatrix}$. This two elements are linearly independent to each other if and only if f is the 1-st order blue ridge point at the origin (this means $a_{40}\,k_2 - 3\,{a_{21}}^2 \neq 0$). Finally, we get remaining monomials $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$ by linearly independent elements

$$\frac{\partial \pi_y}{\partial y_1} = \begin{pmatrix} -\frac{c}{p_1}x_1 - \frac{c}{{p_1}^2}x_1^2 \\ 0 \end{pmatrix}, \ \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle \\ 0 \end{pmatrix} = \begin{pmatrix} d_{01}x_2 \\ 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} c_{02}x_2 + \frac{c_{21}}{2}x_1^2 \\ 0 \end{pmatrix}$$

in (4.5) modulo $\begin{pmatrix} x_2\mathrm{m}_2 + \mathrm{m}_2^3 \\ \mathrm{m}_2 \end{pmatrix}$ respectively.

□

4.3.2 Ugly gulls

Proof of 1 in Theorem 4.3 at ugly gulls singularity. The ugly gulls singularity is 7-$\mathcal{A}$-deternined. Thus, if we know versality of this type, we check the equality (4.3) in the case of $k = 7$. The 4-jet of each derivative of central projection π_y is the same as the case of gulls singularity. If $a_{40}\,k_2 - 3\,{a_{21}}^2$ vanishes, π_y is not versal at 0 from the same reason in the gulls case. We assume that f is 1-st order blue ridge at the origin.

Since d_{01} is non 0, degree 7 monomials of the second component are spaned by $T\mathcal{A}_e\pi_y/\mathrm{m}_2^8\mathcal{E}_2^2$ as $\begin{pmatrix} 0 \\ O_7 \end{pmatrix} = \frac{1}{d_{01}}O_7\frac{\partial \pi_y}{\partial x_2}$. And we get degree 7 monomials of the first component except $\begin{pmatrix} x_1^7 \\ 0 \end{pmatrix}$ and all of degree 6 monomials $\begin{pmatrix} 0 \\ O_6 \end{pmatrix}$ from the following linearly independent vectors

$$O_5\frac{\partial \pi_y}{\partial x_1} = O_5\begin{pmatrix} c_{21}x_1x_2 + c_{12}x_2^2/2 \\ d_{11}x_2 \end{pmatrix}, \ \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle^7 \\ 0 \end{pmatrix} = \begin{pmatrix} d_{01}^7x_2^7 \\ 0 \end{pmatrix} \quad \text{and} \quad O_6\frac{\partial \pi_y}{\partial x_2} = O_6\begin{pmatrix} c_{02}x_2 \\ d_{01} \end{pmatrix}$$

in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} \mathrm{m}_2^8 \\ \mathrm{m}_2^7 \end{pmatrix}$. In the same way, we get monomials $\begin{pmatrix} x_2^2O_4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_2O_4 \end{pmatrix}$ from the following linearly independent vectors

$$x_2O_3\frac{\partial \pi_y}{\partial x_1} = x_2O_3\begin{pmatrix} c_{21}x_1x_2 + c_{12}x_2^2/2 \\ d_{11}x_2 \end{pmatrix}, \ \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle^6 \\ 0 \end{pmatrix} = \begin{pmatrix} d_{01}^6x_2^6 \\ 0 \end{pmatrix} \quad \text{and} \quad x_2O_4\frac{\partial \pi_y}{\partial x_2} = x_2O_4\begin{pmatrix} c_{02}x_2 \\ d_{01} \end{pmatrix}$$

in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2\mathrm{m}_2^6 + \mathrm{m}_2^8 \\ \mathrm{m}_2^6 \end{pmatrix}$ respectively. Furthermore, we get $\begin{pmatrix} x_2^3O_2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_2^2O_2 \end{pmatrix}$ from the following linearly independent vectors

$$x_2^2O_1\frac{\partial \pi_y}{\partial x_1} = x_2^2O_1\begin{pmatrix} c_{21}x_1x_2 + c_{12}x_2^2/2 \\ d_{11}x_2 \end{pmatrix}, \ \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle^5 \\ 0 \end{pmatrix} = \begin{pmatrix} d_{01}^5x_2^5 \\ 0 \end{pmatrix} \quad \text{and} \quad x_2^2O_2\frac{\partial \pi_y}{\partial x_2} = x_2^2O_2\begin{pmatrix} c_{02}x_2 \\ d_{01} \end{pmatrix}$$

in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2\mathrm{m}_2^4 + x_2\mathrm{m}_2^6 + \mathrm{m}_2^8 \\ x_2\mathrm{m}_2^4 + \mathrm{m}_2^6 \end{pmatrix}$ respectively. We know two elements $\begin{pmatrix} x_2^4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_2^3 \end{pmatrix}$ contained in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^8\mathcal{E}_2^2$ from

$$\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle^4 \\ 0 \end{pmatrix} = \begin{pmatrix} d_{01}^4x_2^4 \\ 0 \end{pmatrix} \quad \text{and} \quad x_2^3\frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} c_{02}x_2^4 \\ d_{01}x_2^3 \end{pmatrix}$$

in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^3\mathrm{m}_2^2 + x_2^2\mathrm{m}_2^4 + x_2\mathrm{m}_2^6 + \mathrm{m}_2^8 \\ x_2^2\mathrm{m}_2^2 + x_2\mathrm{m}_2^4 + \mathrm{m}_2^6 \end{pmatrix}$.

To show that $\begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ and remaining monomials whose degree is degree 2 or more except $\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}$, $\begin{pmatrix} x_1 x_2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x_1^3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix}$ are spaned by (4.5), we consider the elements of (4.5) given by the following elemnts:

$$
{}^t\begin{pmatrix} \frac{\partial \pi_y}{\partial y_3} + \frac{1}{\tan\theta}\frac{\partial \pi_y}{\partial y_1} + \frac{f_3(0)}{y_3\, p_1\, \sin\theta}\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_1 \rangle \\ \langle \pi_y, \boldsymbol{e}_2 \rangle \end{pmatrix}, \\ \langle \pi_y, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_1 \rangle \boldsymbol{e}_2,\ \langle \pi_y, \boldsymbol{e}_2 \rangle \boldsymbol{e}_2,\ \langle \pi_y, \boldsymbol{e}_1 \rangle \langle \pi_y, \boldsymbol{e}_2 \rangle \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_2 \rangle^2 \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_2 \rangle^3 \boldsymbol{e}_1, \\ x_1 \frac{\partial \pi_y}{\partial x_1},\ x_2 \frac{\partial \pi_y}{\partial x_1},\ x_2 \frac{\partial \pi_y}{\partial x_2},\ x_1^2 \frac{\partial \pi_y}{\partial x_1},\ x_1 x_2 \frac{\partial \pi_y}{\partial x_1},\ x_2^2 \frac{\partial \pi_y}{\partial x_1},\ x_1 x_2 \frac{\partial \pi_y}{\partial x_2},\ x_2^2 \frac{\partial \pi_y}{\partial x_2}, \\ x_1^2 x_2 \frac{\partial \pi_y}{\partial x_1},\ x_1^2 x_2 \frac{\partial \pi_y}{\partial x_2},\ x_1 x_2^2 \frac{\partial \pi_y}{\partial x_2},\ x_1^4 \frac{\partial \pi_y}{\partial x_1},\ x_1^4 \frac{\partial \pi_y}{\partial x_2},\ x_1^3 x_2 \frac{\partial \pi_y}{\partial x_2},\ x_1^5 \frac{\partial \pi_y}{\partial x_2} \end{pmatrix}
$$

$$
= \begin{pmatrix} \boldsymbol{d} \\ D_1 \end{pmatrix} {}^t\begin{pmatrix} x_2\, \boldsymbol{e}_2,\ x_1 x_2\, \boldsymbol{e}_2,\ x_2^2\, \boldsymbol{e}_1,\ x_2^2\, \boldsymbol{e}_2,\ x_1^2 x_2\, \boldsymbol{e}_1, \\ x_1^2 x_2\, \boldsymbol{e}_2,\ x_1 x_2^2\, \boldsymbol{e}_1,\ x_1 x_2^2\, \boldsymbol{e}_2,\ x_2^3\, \boldsymbol{e}_1, \\ x_1^4\, \boldsymbol{e}_1,\ x_1^4\, \boldsymbol{e}_2,\ x_1^3 x_2\, \boldsymbol{e}_1,\ x_1^3 x_2\, \boldsymbol{e}_2,\ x_1^2 x_2^2\, \boldsymbol{e}_1,\ x_1 x_2^3\, \boldsymbol{e}_1, \\ x_1^5\, \boldsymbol{e}_1,\ x_1^5\, \boldsymbol{e}_2,\ x_1^4 x_2\, \boldsymbol{e}_1,\ x_1^3 x_2^2\, \boldsymbol{e}_1,\ x_1^6\, \boldsymbol{e}_1,\ x_1^5 x_2\, \boldsymbol{e}_1,\ x_1^7\, \boldsymbol{e}_1 \end{pmatrix}
$$

where D_1 is the $(21, 22)$-matrix expressed as follows :

$$
\begin{pmatrix}
0 & 0 & \frac{c_{02}}{2} & 0 & \frac{c_{21}}{2} & 0 & \frac{c_{12}}{2} & 0 & \frac{c_{03}}{6} & \frac{c_{40}}{24} & 0 & \frac{c_{31}}{6} & 0 & \frac{c_{22}}{4} & \frac{c_{13}}{6} & \frac{c_{50}}{120} & 0 & \frac{c_{41}}{24} & \frac{c_{32}}{12} & \frac{c_{60}}{720} & \frac{c_{51}}{120} & \frac{c_{70}}{5040} \\
0 & 0 & 0 & \frac{c_{02}}{2} & 0 & \frac{c_{21}}{2} & 0 & \frac{c_{12}}{2} & 0 & 0 & \frac{c_{40}}{24} & 0 & \frac{c_{31}}{6} & 0 & 0 & 0 & \frac{c_{50}}{120} & 0 & 0 & 0 & 0 & 0 \\
d_{01} & d_{11} & 0 & 0 & 0 & \frac{d_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{d_{31}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{c_{02}\, d_{01}}{2} & 0 & 0 & 0 & 0 & \frac{c_{21}\, d_{01}}{2} & \frac{\alpha_{13}}{2} & 0 & 0 & \frac{c_{40}\, d_{01}}{24} & \frac{\alpha_{32}}{6} & 0 & \frac{\alpha_{51}}{120} & 0 \\
0 & 0 & {d_{01}}^2 & 0 & 0 & 0 & 2d_{01}\, d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & \beta_{22} & 0 & 0 & 0 & 0 & \frac{\beta_{32}}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {d_{01}}^3 & 0 & 0 & 0 & 0 & 0 & 3{d_{01}}^2\, d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_{11} & 0 & 0 & c_{21} & d_{21} & \frac{c_{12}}{2} & 0 & 0 & \frac{c_{40}}{6} & 0 & \frac{c_{31}}{2} & \frac{d_{31}}{2} & \frac{c_{22}}{2} & \frac{c_{13}}{6} & \frac{c_{50}}{24} & 0 & \frac{c_{41}}{6} & \frac{c_{32}}{4} & \frac{c_{60}}{120} & \frac{c_{51}}{24} & \frac{c_{70}}{720} \\
0 & 0 & 0 & d_{11} & 0 & 0 & c_{21} & d_{21} & \frac{c_{12}}{2} & 0 & 0 & \frac{c_{40}}{6} & 0 & \frac{c_{31}}{2} & \frac{c_{22}}{2} & 0 & 0 & \frac{c_{50}}{24} & \frac{c_{41}}{6} & 0 & \frac{c_{60}}{120} & 0 \\
d_{01} & d_{11} & c_{02} & 0 & \frac{c_{21}}{2} & \frac{d_{21}}{2} & c_{12} & 0 & \frac{c_{03}}{2} & 0 & 0 & \frac{c_{31}}{6} & \frac{d_{31}}{6} & \frac{c_{22}}{2} & \frac{c_{13}}{2} & 0 & 0 & \frac{c_{41}}{24} & \frac{c_{32}}{6} & 0 & \frac{c_{51}}{120} & 0 \\
0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & 0 & c_{21} & d_{21} & \frac{c_{12}}{2} & 0 & \frac{c_{40}}{6} & 0 & \frac{c_{31}}{2} & \frac{c_{22}}{2} & \frac{c_{50}}{24} & \frac{c_{41}}{6} & \frac{c_{60}}{120} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & 0 & c_{21} & \frac{c_{12}}{2} & 0 & 0 & \frac{c_{40}}{6} & \frac{c_{31}}{2} & 0 & \frac{c_{50}}{24} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{21} & 0 & 0 & 0 & \frac{c_{40}}{6} & 0 & 0 & 0 \\
0 & d_{01} & 0 & 0 & 0 & d_{11} & c_{02} & 0 & 0 & 0 & 0 & \frac{c_{21}}{2} & \frac{d_{21}}{2} & c_{12} & \frac{c_{03}}{2} & 0 & 0 & \frac{c_{31}}{6} & \frac{c_{22}}{2} & 0 & \frac{c_{41}}{24} & 0 \\
0 & 0 & 0 & d_{01} & 0 & 0 & 0 & d_{11} & c_{02} & 0 & 0 & 0 & 0 & \frac{c_{21}}{2} & c_{12} & 0 & 0 & 0 & \frac{c_{31}}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{21} & 0 & \frac{c_{40}}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & c_{02} & 0 & 0 & 0 & \frac{c_{21}}{2} & c_{12} & 0 & \frac{c_{31}}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & 0 & 0 & 0 & 0 & c_{02} & 0 & 0 & 0 & \frac{c_{21}}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{21} & \frac{c_{40}}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & 0 & 0 & 0 & d_{11} & c_{02} & 0 & \frac{c_{21}}{2} & c_{12} & \frac{c_{31}}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & 0 & 0 & 0 & c_{02} & 0 & \frac{c_{21}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & 0 & c_{02} & \frac{c_{21}}{2}
\end{pmatrix},
$$

$\alpha_{13} := c_{02}\, d_{11} + c_{12}\, d_{01}$, $\quad \alpha_{32} := d_{01}\, c_{31} + 3c_{21}\, d_{11}$, $\quad \alpha_{51} := 5c_{40}\, d_{11} + c_{50}\, d_{01}$,
$\beta_{22} := d_{01}\, d_{21} + {d_{11}}^2$, $\quad \beta_{32} := d_{01}\, d_{31} + 3d_{11}\, d_{21}$
and

$$
\boldsymbol{d} := \frac{c^2}{p_1\, y_3}\begin{pmatrix} 0,\ -1,\ 0,\ 0,\ 0,\ -\frac{2}{p_1},\ \frac{k_2}{2\sin\theta},\ 0,\ 0,\ 0,\ 0,\ \frac{a_{21}}{2\sin\theta},\ -\frac{3}{{p_1}^2},\ \frac{a_{12}\, p_1 + 2k_2}{2p_1 \sin\theta},\ \frac{a_{03}}{6\sin\theta},\ \frac{a_{40}}{24\sin\theta},\ 0, \\ \frac{a_{31}\, p_1 + 6a_{21}}{6p_1 \sin\theta},\ \frac{a_{22}\, {p_1}^2 + 4a_{12}\, p_1 + 6k_2}{4{p_1}^2 \sin\theta},\ \frac{a_{50}\, p_1 + 10a_{40}}{120 p_1 \sin\theta},\ \frac{a_{41}\, {p_1}^2 + 8a_{31}\, p_1 + 36a_{21}}{24{p_1}^2 \sin\theta},\ \frac{a_{60}\, {p_1}^2 + 12a_{50}\, p_1 + 90a_{40}}{720{p_1}^2 \sin\theta} \end{pmatrix}.
$$

The determinant of $\begin{pmatrix} \boldsymbol{d} \\ D_1 \end{pmatrix}$ does not vanish from the non-degenerate condition of ugly gulls singularity. Our source code for Gauss elimination method of the determinant of $\begin{pmatrix} \boldsymbol{d} \\ D_1 \end{pmatrix}$ is available at `https://github.com/Shuhei-singularity123/Versality_of_central_projection_of_regular_surface/blob/master/versality_of_ugly_gulls.mac`.

Remaining degree 1 to 3 monomials can be spaned by the only eight elements :

$$
{}^t\left(\frac{\partial\pi_y}{\partial y_1},\ \frac{\partial\pi_y}{\partial y_2},\ \begin{pmatrix}\langle\pi_y,\boldsymbol{e}_2\rangle\\0\end{pmatrix},\ \frac{\partial\pi_y}{\partial x_1},\ \frac{\partial\pi_y}{\partial x_2},\ x_1\frac{\partial\pi_y}{\partial x_2},\ x_1^2\frac{\partial\pi_y}{\partial x_2},\ x_1^3\frac{\partial\pi_y}{\partial x_2}\right)
$$
$$
= D_2{}^t\left(\begin{pmatrix}x_1\\0\end{pmatrix},\ \begin{pmatrix}0\\x_1\end{pmatrix},\ \begin{pmatrix}x_2\\0\end{pmatrix},\ \begin{pmatrix}x_1^2\\0\end{pmatrix},\ \begin{pmatrix}0\\x_1^2\end{pmatrix},\ \begin{pmatrix}x_1x_2\\0\end{pmatrix},\ \begin{pmatrix}x_1^3\\0\end{pmatrix},\ \begin{pmatrix}0\\x_1^3\end{pmatrix}\right)
$$
where
$$
D_2 := \begin{pmatrix}
-\frac{c}{p_1} & 0 & 0 & -\frac{c}{p_1^2} & 0 & 0 & -\frac{c}{p_1^3} & 0\\
0 & -\frac{c}{p_1} & 0 & 0 & -\frac{c}{p_1^2} & 0 & 0 & -\frac{c}{p_1^3}\\
0 & 0 & d_{01} & 0 & 0 & d_{11} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & c_{21} & c_{40}/6 & 0\\
0 & d_{11} & c_{02} & c_{21}/2 & d_{21}/2 & c_{12} & c_{31}/6 & d_{31}/6\\
0 & d_{01} & 0 & 0 & d_{11} & c_{02} & c_{21}/2 & d_{21}/2\\
0 & 0 & 0 & 0 & d_{01} & 0 & 0 & d_{11}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01}
\end{pmatrix}.
$$
These are spaned by (4.5) if and only if $a_{40}\,k_2 - 3a_{21}{}^2 \neq 0$. □

4.3.3 Type 12

Proof of 2 in Theorem 4.3. From the assumption and criteria, $a_{40}=0$ and a_{50} does not vanish. The type 12 singularity is 6-$\mathcal{A}$-deternined. Thus we need to prove the equality (4.3) where $k=6$. We consider whether seven elements $\begin{pmatrix}x_1\\0\end{pmatrix}$, $\begin{pmatrix}x_2\\0\end{pmatrix}$, $\begin{pmatrix}x_1^2\\0\end{pmatrix}$, $\begin{pmatrix}0\\x_1^2\end{pmatrix}$, $\begin{pmatrix}x_1x_2\\0\end{pmatrix}$, $\begin{pmatrix}x_1^3\\0\end{pmatrix}$ and $\begin{pmatrix}x_1^4\\0\end{pmatrix}$ are generated by several seven elements in (4.5) modulo $\mathrm{m}_2^6\mathcal{E}_2^2$. However, we can only choose the following elements :
$$
{}^t\left(\frac{\partial\pi_y}{\partial y_1},\ \frac{\partial\pi_y}{\partial y_3},\ \begin{pmatrix}\langle\pi_y,\boldsymbol{e}_2\rangle\\0\end{pmatrix},\ \frac{\partial\pi_y}{\partial x_1},\ \frac{\partial\pi_y}{\partial x_2},\ x_1\frac{\partial\pi_y}{\partial x_2},\ x_1^2\frac{\partial\pi_y}{\partial x_2}\right)
$$
$$
= \begin{pmatrix}
-\frac{c}{p_1} & 0 & -\frac{c}{p_1^2} & 0 & 0 & -\frac{c}{p_1^3} & -\frac{c}{p_1{}^4}\\
\frac{c\ \cos\theta}{y_3} & 0 & \frac{c\ \cos\theta}{p_1\ y_3} & 0 & 0 & \frac{c\ \cos\theta}{p_1^2\ y_3} & \frac{c\ \cos\theta}{p_1^3\ y_3}\\
0 & d_{01} & 0 & 0 & d_{11} & 0 & 0\\
0 & 0 & 0 & 0 & c_{21} & 0 & c_{50}/24\\
0 & c_{02} & c_{21}/2 & d_{21}/2 & c_{12} & c_{31}/6 & c_{41}/24\\
0 & 0 & 0 & d_{11} & c_{02} & c_{21}/2 & c_{31}/6\\
0 & 0 & 0 & d_{01} & 0 & 0 & c_{21}/2
\end{pmatrix}
\begin{pmatrix}
x_1\boldsymbol{e}_1\\ x_2\boldsymbol{e}_1\\ x_1^2\boldsymbol{e}_1\\ x_1^2\boldsymbol{e}_2\\ x_1x_2\boldsymbol{e}_1\\ x_1^3\boldsymbol{e}_1\\ x_1^4\boldsymbol{e}_1
\end{pmatrix} + \cdots
$$
in (4.5) modulo $\mathrm{m}_2^6\mathcal{E}_2^2$ to generate the above monomials. Since $\frac{\partial\pi_y}{\partial y_1}$ and $\frac{\partial\pi_y}{\partial y_3}$ are not linearly independent in this part, these elements can not generates the seven elements and we know that an unfolding π is not versal at the type 12 singularity. □

4.4 Parabolic surfaces so that π_y has goose series singularities with $\mathcal{A}_e$-cod.$\pi_y \leq 3$

The Taylor series of central projection π_y is (4.4) where c_{30} is non 0. Thus the cofficients of terms whose degree is upto 3 are expressed as the same in the case of lips/beaks. Several coefficients of the 5-jet of π_y are written as follows :

$c_{40} = -(a_{40}\,p_1 + 4a_{30})\frac{c^2}{y_3},\ d_{40} = 0,$
$c_{31} = -(a_{31}\,p_1 + 3a_{21})\frac{c^2}{y_3},\ d_{31} = (6\sin\theta + a_{30}\,p_1{}^2\cos\theta)\frac{c^2}{p_1{}^2\,y_3},$

$c_{22} = -(a_{22}\,{p_1}^2 + 2a_{12}\,p_1 + 2k_2)\frac{c^2}{p_1\,y_3},\ d_{22} = (2a_{21}\cos\theta)\frac{c^2}{y_3},$
$c_{13} = -(a_{13}\,p_1 + a_{03})\frac{c^2}{y_3},\ d_{13} = (3(a_{12}\,p_1 + 2k_2)\cos\theta)\frac{c^2}{p_1\,y_3},$
$c_{04} = -p_1\,(a_{04} + 6{k_2}^2\,\frac{c}{y_3}\cos\theta)\frac{c^2}{y_3},\ d_{04} = 4a_{03}\cos\theta\frac{c^2}{y_3},$

$c_{50} = -(a_{50}\,{p_1}^2 + 5a_{40}\,p_1 + 20a_{30})\frac{c^2}{p_1\,y_3},\ d_{50} = 0,$
$c_{41} = -(a_{41}\,{p_1}^2 + 4a_{31}\,p_1 + 12a_{21})\frac{c^2}{p_1\,y_3},\ d_{41} = \{24\sin\theta + (a_{40}\,p_1 + 8a_{30})\,{p_1}^2\cos\theta\}\frac{c^2}{{p_1}^3\,y_3},$
$c_{32} = -\{(a_{32}\,{p_1}^3 + 3a_{22}\,{p_1}^2 + 6a_{12}\,p_1 + 6k_2)\sin\theta + 2a_{30}\,k_2\,{p_1}^2\cos\theta\}\frac{c^3}{p_1\,{y_3}^2},$
$d_{32} = 2(a_{31}\,p_1 + 6a_{21})\cos\theta\frac{c^2}{p_1\,y_3},$
$c_{23} = -\{(a_{23}\,{p_1}^2 + 2a_{13}\,p_1 + 2a_{03})\sin\theta + 6a_{21}\,k_2\,p_1\cos\theta\}\frac{c^3}{{y_3}^2},$
$d_{23} = 3\,(a_{22}\,{p_1}^2 + 4a_{12}\,p_1 + 6k_2)\,\cos\theta\frac{c^2}{{p_1}^2\,y_3},$
$c_{14} = -12k_2\,(a_{12}\,p_1 + k_2)\cos\theta\frac{c^3}{y_3^2} - (a_{14}\,p_1 + a_{04})\frac{c^2}{y_3},\ d_{14} = 4(a_{13}\,p_1 + 2a_{03})\,\cos\theta\frac{c^2}{p_1\,y_3},$
$c_{05} = -20a_{03}\,k_2\cos\theta\frac{p_1\,c^3}{{y_3}^2} - a_{05}\frac{p_1\,c^2}{y_3}.$

4.4.1 Goose

Proof of 1 in Theorem 4.4 at goose singularity. Since the goose singularity is 4-A-determined, we should show the equality (4.3) holds for $k = 4$. We consider whether all monomial bases of $\mathrm{m}_2\mathcal{E}_2^2/\mathrm{m}_2^5\mathcal{E}_2^2$ are contained in $T\mathcal{A}_e\pi_y$ modulo $\mathrm{m}_2^5\mathcal{E}_2^2$. First, we assume that the surface f is not flat umbilic at the origin. Since d_{01} is non 0, degree 4 monomials $\begin{pmatrix}0\\O_4\end{pmatrix}$ and $\begin{pmatrix}x_2^4\\0\end{pmatrix}$ are spaned by $\begin{pmatrix}0\\O_4\end{pmatrix} = \frac{1}{d_{01}}O_4\frac{\partial\pi_y}{\partial x_2}$ and $\begin{pmatrix}x_2^4\\0\end{pmatrix} = \frac{1}{d_{01}^4}\begin{pmatrix}\langle\pi_y,\boldsymbol{e}_2\rangle^4\\0\end{pmatrix}$ in $T\mathcal{A}_e\pi_y$ modulo $\mathrm{m}_2^5\mathcal{E}_2^2$. And $\begin{pmatrix}0\\x_2^3\end{pmatrix} = \frac{1}{d_{01}}x_2^3\frac{\partial\pi_y}{\partial x_2}$ is contained in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix}x_2^4+\mathrm{m}_2^5\\\mathrm{m}_2^4\end{pmatrix}$.

To show that other monomials except a monomial $\begin{pmatrix}x_1\\0\end{pmatrix}$ are contained in (4.5) at goose singularity, we consider the elements of (4.5) modulo $\begin{pmatrix}x_2^4+\mathrm{m}_2^5\\x_2^3+\mathrm{m}_2^4\end{pmatrix}$ expressed as the following table :

	$x_1\,\boldsymbol{e}_2$	$x_2\,\boldsymbol{e}_1$	$x_2\,\boldsymbol{e}_2$	$x_1^2\,\boldsymbol{e}_1$	$x_1^2\,\boldsymbol{e}_2$	$x_1x_2\,\boldsymbol{e}_1$	$x_1x_2\,\boldsymbol{e}_2$	$x_2^2\,\boldsymbol{e}_1$	$x_2^2\,\boldsymbol{e}_2$
$\frac{\partial \pi_y}{\partial y_2}$	$-\frac{c}{p_1}$	0	0	0	$-\frac{c}{{p_1}^2}$	0	0	0	$-\frac{k_2\,c^2\,\cos\theta}{2y_3}$
$\langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_1$	0	d_{01}	0	0	0	d_{11}	0	0	0
$\langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_2$	0	0	d_{01}	0	0	0	d_{11}	0	0
$\langle \pi_y, \boldsymbol{e}_2\rangle^2 \boldsymbol{e}_1$	0	0	0	0	0	0	0	${d_{01}}^2$	0
$\langle \pi_y, \boldsymbol{e}_2\rangle^3 \boldsymbol{e}_1$	0	0	0	0	0	0	0	0	0
$\frac{\partial \pi_y}{\partial x_1}$	0	0	d_{11}	$c_{30}/2$	0	c_{21}	d_{21}	$c_{12}/2$	0
$\frac{\partial \pi_y}{\partial x_2}$	d_{11}	c_{02}	0	$c_{21}/2$	$d_{21}/2$	c_{12}	0	$c_{03}/2$	$d_{03}/2$
$x_1\,\frac{\partial \pi_y}{\partial x_1}$	0	0	0	0	0	0	d_{11}	0	0
$x_2\,\frac{\partial \pi_y}{\partial x_1}$	0	0	0	0	0	0	0	0	d_{11}
$x_1\,\frac{\partial \pi_y}{\partial x_2}$	d_{01}	0	0	0	d_{11}	c_{02}	0	0	0
$x_2\,\frac{\partial \pi_y}{\partial x_2}$	0	0	d_{01}	0	0	0	d_{11}	c_{02}	0
$x_1^2\,\frac{\partial \pi_y}{\partial x_1}$	0	0	0	0	0	0	0	0	0
$x_1x_2\,\frac{\partial \pi_y}{\partial x_1}$	0	0	0	0	0	0	0	0	0
$x_2^2\,\frac{\partial \pi_y}{\partial x_1}$	0	0	0	0	0	0	0	0	0
$x_1^2\,\frac{\partial \pi_y}{\partial x_2}$	0	0	0	0	d_{01}	0	0	0	0
$x_1x_2\,\frac{\partial \pi_y}{\partial x_2}$	0	0	0	0	0	0	d_{01}	0	0
$x_2^2\,\frac{\partial \pi_y}{\partial x_2}$	0	0	0	0	0	0	0	0	d_{01}
$x_1^3\,\frac{\partial \pi_y}{\partial x_2}$	0	0	0	0	0	0	0	0	0
$x_1^2x_2\,\frac{\partial \pi_y}{\partial x_2}$	0	0	0	0	0	0	0	0	0
$x_1x_2^2\,\frac{\partial \pi_y}{\partial x_2}$	0	0	0	0	0	0	0	0	0

	$x_1^3\,\boldsymbol{e}_1$	$x_1^3\,\boldsymbol{e}_2$	$x_1^2x_2\,\boldsymbol{e}_1$	$x_1^2x_2\,\boldsymbol{e}_2$	$x_1x_2^2\,\boldsymbol{e}_1$	$x_1x_2^2\,\boldsymbol{e}_2$	$x_2^3\,\boldsymbol{e}_1$	$x_1^4\,\boldsymbol{e}_1$	$x_1^3x_2\,\boldsymbol{e}_1$	$x_1^2x_2^2\,\boldsymbol{e}_1$	$x_1x_2^3\,\boldsymbol{e}_1$
	0	$-d_{31}/6$	0	$-d_{22}/4$	0	$-d_{13}/6$	0	0	0	0	0
	0	0	$d_{21}/2$	0	0	0	$d_{03}/6$	0	$d_{31}/6$	$d_{22}/4$	$d_{13}/6$
	0	0	0	$d_{21}/2$	0	0	0	0	0	0	0
	0	0	0	0	$2d_{01}\,d_{11}$	0	0	0	0	$d_{01}\,d_{21} + {d_{11}}^2$	0
	0	0	0	0	0	0	${d_{01}}^3$	0	0	0	$3{d_{01}}^2\,d_{11}$
	$c_{40}/6$	0	$c_{31}/2$	$d_{31}/2$	$c_{22}/2$	$d_{22}/2$	$c_{13}/6$	$c_{50}/24$	$c_{41}/6$	$c_{32}/4$	$c_{23}/6$
	$c_{31}/6$	$d_{31}/6$	$c_{22}/2$	$d_{22}/2$	$c_{13}/2$	$d_{13}/2$	$c_{04}/6$	$c_{41}/24$	$c_{32}/6$	$c_{23}/4$	$c_{14}/6$
	$c_{30}/2$	0	c_{21}	d_{21}	$c_{12}/2$	0	0	$c_{40}/6$	$c_{31}/2$	$c_{22}/2$	$c_{13}/6$
	0	0	$c_{30}/2$	0	c_{21}	d_{21}	$c_{12}/2$	0	$c_{40}/6$	$c_{31}/2$	$c_{22}/2$
	$c_{21}/2$	$d_{21}/2$	c_{12}	0	$c_{03}/2$	$d_{03}/2$	0	$c_{31}/6$	$c_{22}/2$	$c_{13}/2$	$c_{04}/6$
	0	0	$c_{21}/2$	$d_{21}/2$	c_{12}	0	$c_{03}/2$	0	$c_{31}/6$	$c_{22}/2$	$c_{13}/2$
	0	0	0	d_{11}	0	0	0	$c_{30}/2$	c_{21}	$c_{12}/2$	0
	0	0	0	0	0	d_{11}	0	0	$c_{30}/2$	c_{21}	$c_{12}/2$
	0	0	0	0	0	0	0	0	0	$c_{30}/2$	c_{21}
	0	d_{11}	c_{02}	0	0	0	0	$c_{21}/2$	c_{12}	$c_{03}/2$	0
	0	0	0	d_{11}	c_{02}	0	0	0	$c_{21}/2$	c_{12}	$c_{03}/2$
	0	0	0	0	0	d_{11}	c_{02}	0	0	$c_{21}/2$	c_{12}
	0	d_{01}	0	0	0	0	0	0	c_{02}	0	0
	0	0	0	d_{01}	0	0	0	0	0	c_{02}	0
	0	0	0	0	0	d_{01}	0	0	0	0	c_{02}

where $\alpha_{30} := -\frac{\left(6\sin\theta + a_{30}\,{p_1}^2\cos\theta\right)c^2}{{p_1}^2\,y_3}$, $\alpha_{21} := -\frac{a_{21}\,c^2\,\cos\theta}{y_3}$ and $\alpha_{12} := -\frac{(a_{12}\,p_1 + 2k_2)\,c^2\,\cos\theta}{p_1\,y_3}$.
From Gauss elimination method of which our source code is available on Github at `https://github.com/Shuhei-singularity123/Versality_of_central_projection_of_regular_surface/blob/master/versality_of_goose.mac`, the condition of fullrank of this matrix expressed as the above table is the same as criteria of goose singularity. Thus they are spaned by elements of (4.5).

We know that the remaining monomial $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ is contained in (4.5) modulo $\mathfrak{m}_2^5\mathcal{E}_2^2$ since $\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = -\frac{p_1}{c}\frac{\partial \pi_y}{\partial y_1}$ modulo $\begin{pmatrix} x_2 + \mathfrak{m}_2^2 \\ \mathfrak{m}_2 \end{pmatrix}$. Thus π is a versal unfolding of π_y at $x = 0$ if $f(0)$ is not flat umbilic.

Next, we consider in the case of flat umbilic. This means that a coefficient k_2 vanish. In this case, we have only way to generate monomials $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}$. That is using the following five elements which form the 5×4 - matrix :

$$ {}^t\left(\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle \\ 0 \end{pmatrix}, \frac{\partial \pi_y}{\partial y_1}, \frac{\partial \pi_y}{\partial y_3}, \frac{\partial \pi_y}{\partial x_1}, \frac{\partial \pi_y}{\partial x_2}\right) = D\, {}^t\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}\right), $$

$$ D := \begin{pmatrix} 0 & d_{01} & 0 & d_{11} \\ -\frac{c}{p_1} & 0 & -\frac{c}{p_1^2} & 0 \\ \frac{c}{p_1 \tan\theta} & 0 & \frac{c}{{p_1}^2 \tan\theta} & 0 \\ 0 & 0 & c_{30}/2 & c_{21} \\ 0 & 0 & c_{21}/2 & c_{12} \end{pmatrix}. $$

From criteria of goose singularity and non linearly independency of $\frac{\partial \pi_y}{\partial y_1}$ and $\frac{\partial \pi_y}{\partial y_3}$ in this part, the rank of this matrix is less than 4.
Therefore, we get criteria of versality of π at goose singularity. □

4.4.2 Ugly goose

Proof of 1 in Theorem 4.4 at ugly goose singularity. From assumption and criteria, $a_{30} \neq 0$. The ugly goose singularity is 5-$\mathcal{A}$-deternined. We should show the equality (4.3) holds for $k = 5$. The 3-jet of each derivative of central projection π_y is the same in the case of the goose singularity. In the same way of proof at goose singularity, we know π is not $\mathcal{A}_e$-versal if $f(x)$ is flat umbilic. We enough to consider in the case of not flat umbilic.
From $d_{01}$is non 0, we get $\begin{pmatrix} 0 \\ O_5 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^4 \end{pmatrix}$ and $\begin{pmatrix} x_2^5 \\ 0 \end{pmatrix}$ in $T\mathcal{A}_e\pi_y$ modulo $\mathrm{m}_2^6\mathcal{E}_2^2$ since

$$ \begin{pmatrix} 0 \\ O_5 \end{pmatrix} = \frac{1}{d_{01}} O_5 \frac{\partial \pi_y}{\partial x_2}, \begin{pmatrix} 0 \\ x_2^4 + \frac{2d_{01}^3 d_{11}}{d_{01}^4} x_1 x_2^4 \end{pmatrix} = \frac{1}{d_{01}^4} \begin{pmatrix} 0 \\ \langle \pi_y, \boldsymbol{e}_2 \rangle^4 \end{pmatrix} \text{ and } \begin{pmatrix} x_2^5 \\ 0 \end{pmatrix} = \frac{1}{d_{01}^5} \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle^5 \\ 0 \end{pmatrix} $$

in $T\mathcal{A}_e\pi_y$ modulo $\mathrm{m}_2^6\mathcal{E}_2^2$ respectively.

To show that the remaining monomials except $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ are contained in (4.5) modulo $\mathrm{m}_2^6\mathcal{E}_2^2$, we consider the elements in (4.5) modulo $\begin{pmatrix} x_2^5 + \mathrm{m}_2^6 \\ x_2^4 + \mathrm{m}_2^5 \end{pmatrix}$ expressed as the following table :

	$x_1\,\boldsymbol{e}_2$	$x_2\,\boldsymbol{e}_1$	$x_2\,\boldsymbol{e}_2$	$x_1^2\,\boldsymbol{e}_1$	$x_1^2\,\boldsymbol{e}_2$	$x_1x_2\,\boldsymbol{e}_1$	$x_1x_2\,\boldsymbol{e}_2$	$x_2^2\,\boldsymbol{e}_1$	$x_2^2\,\boldsymbol{e}_2$
$\frac{\partial\pi_y}{\partial y_2}$	$-d_{11}$	0	0	0	$-d_{21}/2$	0	0	0	$-d_{03}/6$
$\boldsymbol{d}$	0	0	$-\frac{c^2\,f_{03}}{{y_3}^2}$	0	0	0	$-\frac{c^2\,(y_3+f_{03})}{p_1\,{y_3}^2}$	$\langle\boldsymbol{d}_{02},\boldsymbol{e}_1\rangle$	0
$\langle\pi_y,\boldsymbol{e}_2\rangle\boldsymbol{e}_1$	0	d_{01}	0	0	0	d_{11}	0	0	0
$\langle\pi_y,\boldsymbol{e}_2\rangle\boldsymbol{e}_2$	0	0	d_{01}	0	0	0	d_{11}	0	0
$\langle\pi_y,\boldsymbol{e}_2\rangle^2\boldsymbol{e}_1$	0	0	0	0	0	0	0	${d_{01}}^2$	0
$\langle\pi_y,\boldsymbol{e}_2\rangle^3\boldsymbol{e}_1$	0	0	0	0	0	0	0	0	0
$\langle\pi_y,\boldsymbol{e}_2\rangle^4\boldsymbol{e}_1$	0	0	0	0	0	0	0	0	0
$\frac{\partial\pi_y}{\partial x_1}$	0	0	d_{11}	$c_{30}/2$	0	c_{21}	d_{21}	$c_{12}/2$	0
$\frac{\partial\pi_y}{\partial x_2}$	d_{11}	c_{02}	0	$c_{21}/2$	$d_{21}/2$	c_{12}	0	$c_{03}/2$	$d_{03}/2$
$x_1\,\frac{\partial\pi_y}{\partial x_1}$	0	0	0	0	0	0	d_{11}	0	0
$x_2\,\frac{\partial\pi_y}{\partial x_1}$	0	0	0	0	0	0	0	0	d_{11}
$x_1\,\frac{\partial\pi_y}{\partial x_2}$	d_{01}	0	0	0	d_{11}	c_{02}	0	0	0
$x_2\,\frac{\partial\pi_y}{\partial x_2}$	0	0	d_{01}	0	0	0	d_{11}	c_{02}	0
$x_1^2\,\frac{\partial\pi_y}{\partial x_1}$	0	0	0	0	0	0	0	0	0
$x_1x_2\,\frac{\partial\pi_y}{\partial x_1}$	0	0	0	0	0	0	0	0	0
$x_2^2\,\frac{\partial\pi_y}{\partial x_1}$	0	0	0	0	0	0	0	0	0
$x_1^2\,\frac{\partial\pi_y}{\partial x_2}$	0	0	0	0	d_{01}	0	0	0	0
$x_1x_2\,\frac{\partial\pi_y}{\partial x_2}$	0	0	0	0	0	0	d_{01}	0	0
$x_2^2\,\frac{\partial\pi_y}{\partial x_2}$	0	0	0	0	0	0	0	0	d_{01}
$x_1^3\,\frac{\partial\pi_y}{\partial x_1}$	0	0	0	0	0	0	0	0	0
$x_1^2x_2\,\frac{\partial\pi_y}{\partial x_1}$	0	0	0	0	0	0	0	0	0
$x_1x_2^2\,\frac{\partial\pi_y}{\partial x_1}$	0	0	0	0	0	0	0	0	0
$x_2^3\,\frac{\partial\pi_y}{\partial x_1}$	0	0	0	0	0	0	0	0	0
$x_1^3\,\frac{\partial\pi_y}{\partial x_2}$	0	0	0	0	0	0	0	0	0
$x_1^2x_2\,\frac{\partial\pi_y}{\partial x_2}$	0	0	0	0	0	0	0	0	0
$x_1x_2^2\,\frac{\partial\pi_y}{\partial x_2}$	0	0	0	0	0	0	0	0	0
$x_2^3\,\frac{\partial\pi_y}{\partial x_2}$	0	0	0	0	0	0	0	0	0
$x_1^4\,\frac{\partial\pi_y}{\partial x_2}$	0	0	0	0	0	0	0	0	0
$x_1^3x_2\,\frac{\partial\pi_y}{\partial x_2}$	0	0	0	0	0	0	0	0	0
$x_1^2x_2^2\,\frac{\partial\pi_y}{\partial x_2}$	0	0	0	0	0	0	0	0	0
$x_1x_2^3\,\frac{\partial\pi_y}{\partial x_2}$	0	0	0	0	0	0	0	0	0

	$x_1^3\,\boldsymbol{e}_1$	$x_1^3\,\boldsymbol{e}_2$	$x_1^2x_2\,\boldsymbol{e}_1$	$x_1^2x_2\,\boldsymbol{e}_2$	$x_1x_2^2\,\boldsymbol{e}_1$	$x_1x_2^2\,\boldsymbol{e}_2$	$x_2^3\,\boldsymbol{e}_1$	$x_2^3\,\boldsymbol{e}_2$
	0	$-d_{31}/6$	0	$-d_{22}/4$	0	$-d_{13}/6$	0	$-d_{04}/24$
	$\langle\boldsymbol{d}_{30},\boldsymbol{e}_1\rangle$	0	$\langle\boldsymbol{d}_{21},\boldsymbol{e}_1\rangle$	$\langle\boldsymbol{d}_{21},\boldsymbol{e}_2\rangle$	$\langle\boldsymbol{d}_{12},\boldsymbol{e}_1\rangle$	0	$\langle\boldsymbol{d}_{03},\boldsymbol{e}_1\rangle$	$\langle\boldsymbol{d}_{03},\boldsymbol{e}_2\rangle$
	0	0	$d_{21}/2$	0	0	0	$d_{03}/6$	0
	0	0	0	$d_{21}/2$	0	0	0	$d_{03}/6$
	0	0	0	0	$2d_{01}\,d_{11}$	0	0	0
	0	0	0	0	0	0	${d_{01}}^3$	0
	0	0	0	0	0	0	0	0
	$c_{40}/6$	0	$c_{31}/2$	$d_{31}/2$	$c_{22}/2$	$d_{22}/2$	$c_{13}/6$	$d_{13}/6$
	$c_{31}/6$	$d_{31}/6$	$c_{22}/2$	$d_{22}/2$	$c_{13}/2$	$d_{13}/2$	$c_{04}/6$	$d_{04}/6$
	$c_{30}/2$	0	c_{21}	d_{21}	$c_{12}/2$	0	0	0
	0	0	$c_{30}/2$	0	c_{21}	d_{21}	$c_{12}/2$	0
	$c_{21}/2$	$d_{21}/2$	c_{12}	0	$c_{03}/2$	$d_{03}/2$	0	0
	0	0	$c_{21}/2$	$d_{21}/2$	c_{12}	0	$c_{03}/2$	$d_{03}/2$
	0	0	0	d_{11}	0	0	0	0
	0	0	0	0	0	d_{11}	0	0
	0	0	0	0	0	0	0	d_{11}
	0	d_{11}	c_{02}	0	0	0	0	0
	0	0	0	d_{11}	c_{02}	0	0	0
	0	0	0	0	0	d_{11}	c_{02}	0
	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0
	0	d_{01}	0	0	0	0	0	0
	0	0	0	d_{01}	0	0	0	0
	0	0	0	0	0	d_{01}	0	0
	0	0	0	0	0	0	0	d_{01}
	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0

$x_1^4\,\boldsymbol{e}_1$	$x_1^4\,\boldsymbol{e}_2$	$x_1^3x_2\,\boldsymbol{e}_1$	$x_1^3x_2\,\boldsymbol{e}_2$	$x_1^2x_2^2\,\boldsymbol{e}_1$	$x_1^2x_2^2\,\boldsymbol{e}_2$	$x_1x_2^3\,\boldsymbol{e}_1$	$x_1x_2^3\,\boldsymbol{e}_2$	$x_2^4\,\boldsymbol{e}_1$
0	$-d_{41}/24$	0	$-d_{32}/12$	0	$-d_{23}/12$	0	$-d_{14}/24$	0
$\langle \boldsymbol{d}_{40}, \boldsymbol{e}_1\rangle$	0	$\langle \boldsymbol{d}_{31}, \boldsymbol{e}_1\rangle$	$\langle \boldsymbol{d}_{31}, \boldsymbol{e}_2\rangle$	$\langle \boldsymbol{d}_{22}, \boldsymbol{e}_1\rangle$	$\langle \boldsymbol{d}_{22}, \boldsymbol{e}_2\rangle$	$\langle \boldsymbol{d}_{13}, \boldsymbol{e}_1\rangle$	$\langle \boldsymbol{d}_{13}, \boldsymbol{e}_2\rangle$	$\langle \boldsymbol{d}_{04}, \boldsymbol{e}_1\rangle$
0	0	$d_{31}/6$	0	$d_{22}/4$	0	$d_{13}/6$	0	$d_{04}/24$
0	0	0	$d_{31}/6$	0	$d_{22}/4$	0	$d_{13}/6$	0
0	0	0	0	$d_{01}\,d_{21} + {d_{11}}^2$	0	0	0	$d_{01}\,d_{03}/3$
0	0	0	0	0	0	$3{d_{01}}^2\,d_{11}$	0	0
0	0	0	0	0	0	0	0	${d_{01}}^4$
$c_{50}/24$	0	$c_{41}/6$	$d_{41}/6$	$c_{32}/4$	$d_{32}/4$	$c_{23}/6$	$d_{23}/6$	$c_{14}/24$
$c_{41}/24$	$d_{41}/24$	$c_{32}/6$	$d_{32}/6$	$c_{23}/4$	$d_{23}/4$	$c_{14}/6$	$d_{14}/6$	$c_{05}/24$
$c_{40}/6$	0	$c_{31}/2$	$d_{31}/2$	$c_{22}/2$	$d_{22}/2$	$c_{13}/6$	$d_{13}/6$	0
0	0	$c_{40}/6$	0	$c_{31}/2$	$d_{31}/2$	$c_{22}/2$	$d_{22}/2$	$c_{13}/6$
$c_{31}/6$	$d_{31}/6$	$c_{22}/2$	$d_{22}/2$	$c_{13}/2$	$d_{13}/2$	$c_{04}/6$	$d_{04}/6$	0
0	0	$c_{31}/6$	$d_{31}/6$	$c_{22}/2$	$d_{22}/2$	$c_{13}/2$	$d_{13}/2$	$c_{04}/6$
$c_{30}/2$	0	c_{21}	d_{21}	$c_{12}/2$	0	0	0	0
0	0	$c_{30}/2$	0	c_{21}	d_{21}	$c_{12}/2$	0	0
0	0	0	0	$c_{30}/2$	0	c_{21}	d_{21}	$c_{12}/2$
$c_{21}/2$	$d_{21}/2$	c_{12}	0	$c_{03}/2$	$d_{03}/2$	0	0	0
0	0	$c_{21}/2$	$d_{21}/2$	c_{12}	0	$c_{03}/2$	$d_{03}/2$	0
0	0	0	0	$c_{21}/2$	$d_{21}/2$	c_{12}	0	$c_{03}/2$
0	0	0	d_{11}	0	0	0	0	0
0	0	0	0	0	d_{11}	0	0	0
0	0	0	0	0	0	0	d_{11}	0
0	0	0	0	0	0	0	0	0
0	d_{11}	c_{02}	0	0	0	0	0	0
0	0	0	d_{11}	c_{02}	0	0	0	0
0	0	0	0	0	d_{11}	c_{02}	0	0
0	0	0	0	0	0	0	d_{11}	c_{02}
0	d_{01}	0	0	0	0	0	0	0
0	0	0	d_{01}	0	0	0	0	0
0	0	0	0	0	d_{01}	0	0	0
0	0	0	0	0	0	0	d_{01}	0

$x_1^5\,\boldsymbol{e}_1$	$x_1^4x_2\,\boldsymbol{e}_1$	$x_1^3x_2^2\,\boldsymbol{e}_1$	$x_1^2x_2^3\,\boldsymbol{e}_1$	$x_1x_2^4\,\boldsymbol{e}_1$
0	0	0	0	0
$\langle \boldsymbol{d}_{50}, \boldsymbol{e}_1\rangle$	$\langle \boldsymbol{d}_{41}, \boldsymbol{e}_1\rangle$	$\langle \boldsymbol{d}_{32}, \boldsymbol{e}_1\rangle$	$\langle \boldsymbol{d}_{23}, \boldsymbol{e}_1\rangle$	$\langle \boldsymbol{d}_{14}, \boldsymbol{e}_1\rangle$
0	$d_{41}/24$	$d_{32}/12$	$d_{23}/12$	$d_{14}/24$
0	0	0	0	0
0	0	$(d_{01}\,d_{31} + 3d_{11}\,d_{21})/3$	$d_{01}\,d_{22}/2$	$(d_{01}\,d_{13} + d_{03}\,d_{11})/3$
0	0	0	$3d_{01}\,(d_{01}\,d_{21} + 2{d_{11}}^2)/2$	0
0	0	0	0	$4{d_{01}}^3\,d_{11}$
*	*	*	*	*
*	*	*	*	*
$c_{50}/24$	$c_{41}/6$	$c_{32}/4$	$c_{23}/6$	$c_{14}/24$
0	$c_{50}/24$	$c_{41}/6$	$c_{32}/4$	$c_{23}/6$
$c_{41}/24$	$c_{32}/6$	$c_{23}/4$	$c_{14}/6$	$c_{05}/24$
0	$c_{41}/24$	$c_{32}/6$	$c_{23}/4$	$c_{14}/6$
$c_{40}/6$	$c_{31}/2$	$c_{22}/2$	$c_{13}/6$	0
0	$c_{40}/6$	$c_{31}/2$	$c_{22}/2$	$c_{13}/6$
0	0	$c_{40}/6$	$c_{31}/2$	$c_{22}/2$
$c_{31}/6$	$c_{22}/2$	$c_{13}/2$	$c_{04}/6$	0
0	$c_{31}/6$	$c_{22}/2$	$c_{13}/2$	$c_{04}/6$
0	0	$c_{31}/6$	$c_{22}/2$	$c_{13}/2$
$c_{30}/2$	c_{21}	$c_{12}/2$	0	0
0	$c_{30}/2$	c_{21}	$c_{12}/2$	0
0	0	$c_{30}/2$	c_{21}	$c_{12}/2$
0	0	0	$c_{30}/2$	c_{21}
$c_{21}/2$	c_{12}	$c_{03}/2$	0	0
0	$c_{21}/2$	c_{12}	$c_{03}/2$	0
0	0	$c_{21}/2$	c_{12}	$c_{03}/2$
0	0	0	$c_{21}/2$	c_{12}
0	c_{02}	0	0	0
0	0	c_{02}	0	0
0	0	0	c_{02}	0
0	0	0	0	c_{02}

where $\boldsymbol{d} := \frac{\partial \pi_y}{\partial y_3} + \frac{1}{\tan\theta}\frac{\partial \pi_y}{\partial y_1}$ and $\boldsymbol{d}_{ij} := \frac{1}{i!\,j!}\frac{\partial^{(1+i+j)}\boldsymbol{d}}{\partial x_1^i\,\partial x_2^j}(0)$.
From Gauss elimination method, we know that the matrix expressed as the above table is of fullrank from criteria of ugly goose singularity and the assumption $k_2 \neq 0$. Our source code is available on GitHub at `https://github.com/Shuhei-singularity123/Versality_of_central_projection_of_regular_surface/blob/master/versality_of_ugly_goose.mac`.

The degree 1 monomial $\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = -\frac{p_1}{c}\frac{\partial \pi_y}{\partial y_1}$ is contained in (4.5). Therefore, if $f(0)$ is not flat umbilic, π is versal unfolding of π_y at $x = 0$. □

4.4.3 Type 16

Proof of 2 in Theorem 4.4. We assume that $p_2 = 0$. From assumption and criteria, $a_{30} = a_{21} = 0$ and a_{40} non 0. The type 16 singularity is 5-$\mathcal{A}$-deternined. Thus we need to check the equality (4.3) holds for $k = 5$.

We consider whether two elements $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$ are generated by several elements in (4.5). However, we can only choose the following elements to generate above elements in $j^5\theta(\pi_y)$:

$$ {}^t\left(\frac{\partial \pi_y}{\partial y_1},\ \frac{\partial \pi_y}{\partial y_3}\right) = \begin{pmatrix} -\frac{c}{p_1} & -\frac{c}{{p_1}^2} \\ \frac{c}{p_1\ \tan\theta} & \frac{c}{{p_1}^2 \tan\theta} \end{pmatrix} {}^t\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix},\ \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}\right) + \cdots $$

in (4.5). From $\frac{\partial \pi_y}{\partial y_1}$ and $\frac{\partial \pi_y}{\partial y_3}$ are not linearly independent, these elements can not generates the seven elements and we know that an unfolding π is not versal at type 16. $\square$

5 Geometric conditions for versal singularities

5.1 Contact with cones

We consider contact of regular surfaces with cones. Since cones in $\mathbb{R}^3$ are determined by their vertex, direction of central axis and ungle, the moduli space of cones is of six dimensional. We show that $A_{\leq 6}$-contact with cones is related to versal gulls series singularities of central projections.

We consider a cone which has a vertex $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, a direction vector of central axis $\boldsymbol{d} = (d_1, d_2, d_3)$ in S^2 and a angle θ in $(0, \pi/2) \subset \mathbb{R}$. Then, the implicit function of that cone is defined by

$$ C_{y,\boldsymbol{d},\theta}(z_1, z_2, z_3) := \langle \boldsymbol{d}, z - y\rangle^2 - |z - y|^2 \cos^2\theta \tag{5.1} $$

where $\langle \boldsymbol{d}, y\rangle \neq 0$ and d is not parallel to the position vector of y. We measure contact between the cone (5.1) and a regular surface

$$ f(x_1, x_2) = (x_1, x_2, Q(x)) \ \text{ where } \ Q(x) \text{ is defined as in (1.2) and (3.6)} \tag{5.2} $$

as following function :

$$ \begin{aligned} C(x_1, x_2) \ &:= \ C_{y,\boldsymbol{d},\theta}(x_1, x_2, Q(x)) \\ &= \ \sum\nolimits_{k\geq 2} C_k(x_1, x_2) \ \text{ where } \ C_k(x_1, x_2) := \sum\nolimits_{i+j=k} \frac{c_{ij}}{i\,j} {x_1}^i {x_2}^j. \end{aligned} \tag{5.3} $$

We call $C(x_1, x_2)$ a **contact function with cones**.

5.2 Contact type of functions

According to [16], we define the notion of contact type.

Theorem 5.1 ([16]). *For $i = 1, 2$, let $g_i : (X_i, x_i) \to (\mathbb{R}^m, 0)$ be immersion-germs and $f_i : (\mathbb{R}^m, 0) \mapsto (\mathbb{R}^n, 0)$ be submersion-germs, with $Y_i = f_i^{-1}(0)$. Then the pairs (X_1, Y_1) and (X_2, Y_2) have the same contact type if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

Here we define that two map-germs $f, g : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ are $\mathcal{K} - equivalent$ if there are adiffeomorphism $\varphi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ and a smooth map $A : (\mathbb{R}^m, 0) \to GL(\mathbb{R}^n)$ such that $g(\varphi(x)) = A(x)\, f(x)$. In this section, we consider the A_k (or $A_k^{\pm}$)-contact types which are expressed as

$$x_1^2 \pm x_2^{k+1}$$

corresponding to $\mathcal{K}$-equivalent classes of $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$.

We set a function

$$\Phi(x_1, x_2) := \sum\nolimits_{k \geq 2} \Phi_k(x_1, x_2) \quad \text{where} \quad \Phi_k(x_1, x_2) := \sum\nolimits_{i+j=k} \frac{c_{ij}}{i!\, j!} x_1^i x_2^j. \tag{5.4}$$

in this subsection.

Theorem 5.2 (Fukui [9, § 1.1])**.** *For a function Φ defined by* (5.4)*, the following three conditions are equivalent ;*

(*i*) *rank* $\begin{pmatrix} c_{20} & c_{11} \\ c_{11} & c_{02} \end{pmatrix} = 1$.

(*ii*) *There is some directions $(dx_1, dx_2) \neq (0, 0)$ such that*

$$(dx_1, dx_2) \begin{pmatrix} c_{20} & c_{11} \\ c_{11} & c_{02} \end{pmatrix} = (0, 0).$$

(*iii*) *There is some directions $(dx_1, dx_2) \neq (0, 0)$ and $s \neq 0$ such that*

$$\begin{pmatrix} c_{20} & c_{11} \\ c_{11} & c_{02} \end{pmatrix} = s \begin{pmatrix} dx_2^2 & -dx_1 dx_2 \\ -dx_1 dx_2 & dx_1^2 \end{pmatrix}.$$

If f has none of the conditions of Theorem 5.2 at the origin, f has regular (A_1-type) at the origin. After this, we consider f such that *rank* $\begin{pmatrix} c_{20} & c_{11} \\ c_{11} & c_{02} \end{pmatrix} = 1$ and let (dx_1, dx_2) be the direction satisfied conditions of Theorem 5.2.

Theorem 5.3 (Fukui [9, Theorem 1.1 to 1.4])**.** *For a function $\Phi(x_1, x_2)$ defined by* (5.4) *and fulfilled the condition (iii) in Theorem 5.2. We define $\Phi_{kij} := \frac{\partial^{i+j}}{\partial x_1{}^i\, \partial x_2{}^j} \Phi_k$. Then, the following claims hold:*

(1) Φ has A_2-singularity type at $(0, 0)$ if and only if $\Phi_3(dx_1, dx_2)$ is non zero.

We assume that $\Phi_3(dx_1, dx_2) = 0$ and dx_1 is not zero. Then,

(2) Φ has A_3-singularity type at $(0, 0)$ if and only if

$$\Phi_4(dx_1, dx_2) - \frac{1}{2s\, dx_1^2} \Phi_{301}(dx_1, dx_2)^2 \neq 0. \tag{5.5}$$

(3) Φ has A_4-singularity type at $(0, 0)$ if and only if (5.5) *vanishes and*

$$\Phi_5(dx_1, dx_2) - \frac{1}{s\, dx_1^2} \Phi_{301}(dx_1, dx_2)\, \Phi_{401}(dx_1, dx_2) + \frac{1}{2s^2\, dx_1^4} \Phi_{302}(dx_1, dx_2)\, \Phi_{301}(dx_1, dx_2)^2 \tag{5.6}$$

does not vanish.

(4) Φ *has* A_5*-singularity type at* $(0,0)$ *if and only if both of* (5.5) *and* (5.6) *vanish and*

$$\begin{aligned}
&\Phi_6(dx_1, dx_2)\\
&-\frac{1}{2s\,dx_1^2}\{2\,\Phi_{301}(dx_1,dx_2)\,\Phi_{501}(dx_1,dx_2)+\Phi_{401}(dx_1,dx_2)^2\}\\
&+\frac{1}{2\,s^2\,dx_1^4}\Phi_{301}(dx_1,dx_2)\{\Phi_{301}(dx_1,dx_2)\Phi_{402}(dx_1,dx_2)+2\,\Phi_{302}(dx_1,dx_2)\Phi_{401}(dx_1,dx_2)\}\\
&-\frac{1}{6\,s^3\,dx_1^6}\Phi_{301}(dx_1,dx_2)^2\{\Phi_{303}(dx_1,dx_2)\Phi_{301}(dx_1,dx_2)+3\,\Phi_{302}(dx_1,dx_2)^2\}
\end{aligned} \tag{5.7}$$

does not vanish.

(5) Φ *has* A_6*-singularity type at* $(0,0)$ *if and only if* (5.5), (5.6) *and* (5.7) *vanish and*

$$\begin{aligned}
&\Phi_7(dx_1, dx_2)\\
&-\frac{1}{{dx_1}^2 s}\{\Phi_{301}(dx_1,dx_2)\Phi_{601}(dx_1,dx_2)+\Phi_{401}(dx_1,dx_2)\Phi_{501}(dx_1,dx_2)\}\\
&+\frac{1}{2{dx_1}^4\,s^2}\left\{\begin{array}{l}\Phi_{301}(dx_1,dx_2)^2\Phi_{502}(dx_1,dx_2)+2\Phi_{301}(dx_1,dx_2)\Phi_{302}(dx_1,dx_2)\Phi_{501}(dx_1,dx_2)\\ +\,2\Phi_{301}(dx_1,dx_2)\Phi_{401}(dx_1,dx_2)\Phi_{402}(dx_1,dx_2)+\Phi_{302}(dx_1,dx_2)\Phi_{401}(dx_1,dx_2)^2\end{array}\right\}\\
&-\frac{1}{6{dx_1}^6\,s^3}\,\Phi_{301}(dx_1,dx_2)\left\{\begin{array}{l}\Phi_{301}(dx_1,dx_2)^2\Phi_{403}(dx_1,dx_2)\\ +\,6\Phi_{301}(dx_1,dx_2)\Phi_{302}(dx_1,dx_2)\Phi_{402}(dx_1,dx_2)\\ +\,3\Phi_{301}(dx_1,dx_2)\Phi_{303}(dx_1,dx_2)\Phi_{401}(dx_1,dx_2)\\ +\,6\Phi_{302}(dx_1,dx_2)^2\Phi_{401}(dx_1,dx_2)\end{array}\right\}\\
&+\frac{1}{2{dx_1}^8\,s^4}\,\Phi_{301}(dx_1,dx_2)^2\Phi_{302}(dx_1,dx_2)\,\{\Phi_{301}(dx_1,dx_2)\Phi_{303}(dx_1,dx_2)+\Phi_{302}(dx_1,dx_2)^2\}
\end{aligned}$$

does not vanish.

5.3 Results

We introduce some results of $A_{\leq 6}-$contact with cones of a regular surface which is parabolic at the origin. Before stating the main results, we check the following Lemma 5.4. We need to know the condition which generatrices are included in the tangent plane at the origin of a regular surface. In this section, we assume that the vertex of cones is not origin in $\mathbb{R}^3$.

Lemma 5.4. *One of generatrix is passing through the origin in* $\mathbb{R}^3$ *if and only if the angle of cones* θ *is equal to the angle between the position vector of the vertex of cones* y *and the unit direction vector of the central axis of cones.*

Theorem 5.5. *We assume that one of generatrix is passing through the origin in* $\mathbb{R}^3$*. Then, the contact function* $C(x_1, x_2)$ *with cones has critical point at* 0 *if and only if*

$$y_3 = d_1\,y_2 - d_2\,y_1 = 0. \tag{5.8}$$

This condition means that the vertex $y = (y_1, y_2, y_3)$ *is contained in the tangent plane of regular surfaces and the orthogonal projection of the direction of central axis* $d = (d_1, d_2, d_3)$ *belong to* $v = (0, 0, 1)$ *is parallel to the position vector of* y*.*

Proof. From the assumption, the angle of cones θ is fixed by

$$\cos\theta = \frac{\langle \boldsymbol{d}, y\rangle}{|y|}.$$

We consider the 1-jet of $C(x_1, x_2)$ at 0 expressed as follows :

$$j^1 C(0,0) = -\frac{\langle \boldsymbol{d}, y\rangle}{|y|^2}\begin{pmatrix} y_3(d_1\, y_3 - d_3\, y_1) + y_2(d_1\, y_2 - d_2\, y_1) \\ y_3(d_2\, y_3 - d_3\, y_2) - y_1(d_1\, y_2 - d_2\, y_1)\end{pmatrix}(x_1,\, x_2)\,.$$

From the resultant for d_3, if one of the two equation

$$y_3 = 0 \ \text{ and } \ d_1\, y_2 - d_2\, y_1 = 0$$

is satisfied, two coefficients have a common factor. If $y_3 = 0$ is satisfied, two cofficients of $j^1 C(0,0)$ are

$$y_2\,(d_1\, y_2 - d_2\, y_1) \ \text{ and } \ -y_1\,(d_1\, y_2 - d_2\, y_1).$$

Thus, if $d_1\, y_2 - d_2\, y_1 = 0$, both of them are zero. It is the same way in the case of $d_1\, y_2 - d_2\, y_1 = 0$. □

Theorem 5.6. *We consider a cone $C_{y,\boldsymbol{d},\theta}$ whose vertex is satisfied* (5.8) *and is not origin in $\mathbb{R}^3$. We measure contact between this cone with the regular surface f defined by* (5.2) *as follows:*

1. *The cone $C_{y,\boldsymbol{d},\theta}$ has A_1-contact with the regular surface f if and only if none of the following conditions hold.*

   (A_2a) *the origin is flat umbilic.*

   (A_2b) *the origin is parabolic but not flat umbilic and the vertex y is contained in an asymptotic direction line of a regular surface at 0 i.e. $y_2 = 0$ as same $d_2 = 0$.*

2. *Suppose that f is parabolic but not flat umbilic at the origin. The cone $C_{y,\boldsymbol{d},\theta}$ has A_2-contact with f at the origin if and only if the condition (A_2b) holds and none of the following conditions hold.*

   (D_4a) *the vertex y is contained in the asymptotic direction line of f at 0 and*

   $$d_3 = -\frac{d_1}{k_2\, y_1}.$$

   This is the condition in which the rank of the Hesse matrix is 0.

   (A_3b) *the vertex y is contained in the asymptotic direction line of f at 0, $d_3 \neq -\frac{d_1}{k_2\, y_1}$ and the asymptotic direction line is 3-rd or higher order contact with f at the origin.*

Proof. 1. The Hesse matrix of $C(x_1, x_2)$ is

$$\begin{pmatrix} -\frac{2y_2\, d_2\,(d_2\, y_2 + d_1\, y_1)}{{y_1}^2 + {y_2}^2} & -2(d_3\, k_1\,(d_2\, y_2 + d_1\, y_1) - d_1\, d_2) \\ -2(d_3\, k_1\,(d_2\, y_2 + d_1\, y_1) - d_1\, d_2) & -\frac{2(d_3\, k_2\,({y_1}^2 + {y_2}^2) + d_1\, y_1)\,(d_1\, y_1 + d_2\, y_2)}{{y_1}^2 + {y_2}^2}\end{pmatrix}.$$

If the rank of this matrix is less than 2, a critical point is more degenerate. We should check degenerate condition. We can set $(y_1, y_2) = r_y\,(d_1, d_2)$ from $d_1\, y_2 - d_2\, y_1 = 0$ where d is neither $(d_1, d_2, 0)$ nor $(0, 0, d_3)$.

(A_2a). If $(0,0)$ is flat umbilic, The Hesse matrix of $C(x_1, x_2)$ is

$$-2\begin{pmatrix} {d_2}^2 & -d_1\, d_2 \\ -d_1\, d_2 & {d_1}^2 \end{pmatrix}.$$

This is a rank 1 matrix.

(A_2b). If $(0,0)$ is parabolic but not flat umbilic, The Hesse matrix of $C(x_1,x_2)$ is

$$\begin{pmatrix} -2{d_2}^2 & 2d_1\,d_2 \\ 2d_1\,d_2 & -2\left(\left({d_1}^2+{d_2}^2\right)d_3\,k_2\,r_y+{d_1}^2\right) \end{pmatrix}.$$

The determinant of this matrix is

$$4{d_2}^2\,\left({d_1}^2+{d_2}^2\right)\,d_3\,k_2\,r_y.$$

Therefore, the rank of Hesse matrix is less than 2 if and only if $d_2 = y_2 = 0$.

2. From criteria, we check the rank of the Hesse matrix and $H_3(dx_1,dx_2)$ for $C(x_1,x_2)$. In the condition of (A_2b), the Hesse matrix is expressed as follows :

$$\begin{pmatrix} 0 & 0 \\ 0 & -2d_1\,\left(d_3\,k_2\,y_1+d_1\right) \end{pmatrix}.$$

So, the rank of Hesse matrix is 0 if and only if $d_3 = -\frac{d_1}{k_2\,y_1}$. If this does not hold, we can set $s = -2d_1\,\left(d_3\,k_2\,y_1+d_1\right)$ and $(dx_1,dx_2)=(1,0)$. The 3-degree homogeneous polynomial of $C(x_1,x_2)$ is

$$-\frac{a_{30}\,{d_1}^4\,d_3\,y_1}{3}.$$

□

We consider more degenerate A_k-contact in the case of (A_2b) in Theorem 5.6. It is relevant to gulls series singularity of π_y if π is versal at gulls series singularity.

Theorem 5.7. *Assume that the origin of the regular surface f is parabolic but not flat umbilic and the vertex of the cone y is contained in the asymptotic direction line of f at 0.*

1. *The cone $C_{y,\boldsymbol{d},\theta}$ has A_3-contact with f if and only if f is the 3-rd or higher order contact with the asymptotic direction line at the origin and*

$$\left(k_2\,a_{40}-3{a_{21}}^2\right)d_3\,y_1+a_{40}\,d_1 \neq 0. \tag{5.9}$$

After this, we assume that both of a_{40} and $k_2\,a_{40}-3{a_{21}}^2$ do not vanish.

2. *The cone $C_{y,\boldsymbol{d},\theta}$ has A_4-contact with f if and only if f is the 3-rd contact with the asymptotic direction line at the origin, (5.9) vanishes and*

$$\left(3{a_{21}}^2\,a_{50}+5a_{12}\,{a_{40}}^2-10a_{21}\,a_{31}\,a_{40}\right)y_1-5k_2\,{a_{40}}^2+15{a_{21}}^2\,a_{40} \neq 0. \tag{5.10}$$

3. *The cone $C_{y,\boldsymbol{d},\theta}$ has A_5-contact with f if and only if f is the 3-rd contact with the asymptotic direction line at the origin, both of (5.9) and (5.10) vanish and*

   $AC_5 := (45{a_{21}}^3\,a_{40}\,a_{60}-54{a_{21}}^3\,{a_{50}}^2+180{a_{21}}^2\,a_{31}\,a_{40}\,a_{50}-225{a_{21}}^2\,{a_{40}}^2\,a_{41}-25a_{03}\,{a_{40}}^4+225a_{21}\,a_{22}\,{a_{40}}^3-150a_{21}\,{a_{31}}^2\,{a_{40}}^2)y_1-270{a_{21}}^3\,a_{40}\,a_{50}-450a_{12}\,a_{21}\,{a_{40}}^3+900{a_{21}}^2\,a_{31}\,{a_{40}}^2$ *does not vanish.*

4. The cone $C_{y,\boldsymbol{d},\theta}$ has A_6-contact with f if and only if f is the 3*-rd contact with the asymptotic direction line at the origin,* (5.9), (5.10) *and AC_5 vanish and*

$$AC_6 := \begin{pmatrix} 225a_{21}{}^3\,a_{40}{}^2\,a_{70} - 945a_{21}{}^3\,a_{40}\,a_{50}\,a_{60} + 1575a_{21}{}^2\,a_{31}\,a_{40}{}^2\,a_{60} - 1575a_{21}{}^2\,a_{40}{}^3\,a_{51} \\ +756a_{21}{}^3\,a_{50}{}^3 - 3150a_{21}{}^2\,a_{31}\,a_{40}\,a_{50}{}^2 + 3150a_{21}{}^2\,a_{40}{}^2\,a_{41}\,a_{50} - 1575a_{21}\,a_{22}\,a_{40}{}^3\,a_{50} \\ +4200a_{21}\,a_{31}{}^2\,a_{40}{}^2\,a_{50} - 5250a_{21}\,a_{31}\,a_{40}{}^3\,a_{41} - 875a_{13}\,a_{40}{}^5 + 2625a_{21}\,a_{32}\,a_{40}{}^4 \\ +2625a_{22}\,a_{31}\,a_{40}{}^4 - 1750a_{31}{}^3\,a_{40}{}^3 \end{pmatrix} y_1{}^2$$
$$+ 210a_{40}\,(3a_{21}\,a_{50} - 5a_{31}\,a_{40})\,(3a_{21}{}^2\,a_{50} + 5a_{12}\,a_{40}{}^2 - 10a_{21}\,a_{31}\,a_{40})y_1$$
$$- 3150a_{21}\,a_{40}{}^2\,(3a_{21}{}^2\,a_{50} + 5a_{12}\,a_{40}{}^2 - 10a_{21}\,a_{31}\,a_{40})$$

does not vanish.

Remark 5.8. Suppose that f is not red subparabolic at the origin, namely $a_{21} \neq 0$.

1. The non degenerate condition of A_4-contact in Theorem 5.7 means that criteria of gulls singularity type of the central projection π_y.
2. It follows from *3 and 4 in Theorem 5.7 that the sum of the non degenerate conditions of A_5-contact and A_6-contact*

$$AC_6 - 70\,a_{40}\,AC_5 \tag{5.11}$$

is equal to the non degenerate condition (3.9) *of ugly gulls singularity of π_y. We call* (5.11) *the ug-***focal** *condition.*

Proof. We assume conditions (A_3b) on Theorem 5.6.

1. The Hesse matrix of $C(x_1, x_2)$ is

$$\begin{pmatrix} 0 & 0 \\ 0 & -2d_1\,(k_2\,d_3\,y_1 + d_1) \end{pmatrix}.$$

Thus, we set $s := -2d_1\,(k_2\,d_3\,y_1 + d_1)$. From criteria of A_3-singularity, we check

$$C_4(1,0) - \partial x_2 C_3(1,0)^2/2s = -\frac{d_1\,d_3\,y_1\,(k_2\,a_{40}\,d_3\,y_1 - 3a_{21}{}^2\,d_3\,y_1 + a_{40}\,d_1)}{12(k_2\,d_3\,y_1 + d_1)}.$$

If the formula $(k_2\,a_{40} - 3a_{21}{}^2)\,d_3\,y_1 + a_{40}\,d_1$ does not vanish, the above condition is not equal 0.

2. We assume that (5.9) vanishes. From this assumption, d_1 is fixed as the following formula:

$$-\frac{(k_2\,a_{40} - 3a_{21}{}^2)\,d_3\,y_1}{a_{40}}.$$

From criteria of A_4-singularity, if the next formula does not vanish, $C(x_1, x_2)$ has A_4-singularity at 0 ;

$$\frac{(k_2\,a_{40} - 3a_{21}{}^2)\,d_3{}^2\,y_1\,\{(3a_{21}{}^2\,a_{50} + 5a_{12}\,a_{40}{}^2 - 10a_{21}\,a_{31}\,a_{40})\,y_1 - 5a_{40}\,(k_2\,a_{40} - 3a_{21}{}^2)\}}{180a_{21}{}^2\,a_{40}}.$$

3. We assume that both of (5.9) and (5.10) vanish. From this, k_2 is substituted for

$$k_2 = \frac{(3a_{21}{}^2\,a_{50} + 5a_{12}\,a_{40}{}^2 - 10a_{21}\,a_{31}\,a_{40})\,y_1 + 15a_{21}{}^2\,a_{40}}{5a_{40}{}^2}.$$

We remark that

$$k_2\, a_{40} - 3{a_{21}}^2 = \frac{(3{a_{21}}^2\, a_{50} + 5a_{12}\, {a_{40}}^2 - 10a_{21}\, a_{31}\, a_{40})\, y_1}{5a_{40}} \neq 0.$$

This means

$$3{a_{21}}^2\, a_{50} + 5a_{12}\, {a_{40}}^2 - 10a_{21}\, a_{31}\, a_{40}$$

does not vanish. From the criteria of A_5-singularity, if the following formula does not vanish, $C(x_1, x_2)$ has A_5-singularity at 0 ;

$(45{a_{21}}^3\, a_{40}\, a_{60} - 54{a_{21}}^3\, {a_{50}}^2 + 180{a_{21}}^2\, a_{31}\, a_{40}\, a_{50} - 225{a_{21}}^2\, {a_{40}}^2\, a_{41} - 25a_{03}\, {a_{40}}^4 + 225a_{21}\, a_{22}\, {a_{40}}^3 - 150a_{21}\, {a_{31}}^2\, {a_{40}}^2)y_1 - 90a_{21}\, a_{40}\, (3{a_{21}}^2\, a_{50} + 5a_{12}\, {a_{40}}^2 - 10a_{21}\, a_{31}\, a_{40})$.

Therefore, we get the condition which is that $C(x_1, x_2)$ has A_5-singularity at 0.

4. In the same way, suppose that (5.9), (5.10) and AC_5 vanish. From criteria of A_6-singularity, if the following does not vanish, $C(x_1, x_2)$ has A_6-singularity at 0 ;

$$\left\{\begin{array}{l} 225{a_{21}}^6\, {a_{40}}^2\, a_{70} - 1575{a_{21}}^5\, {a_{40}}^3\, a_{51} - 378{a_{21}}^6\, {a_{50}}^3 + 2520{a_{21}}^5\, a_{31}\, a_{40}\, {a_{50}}^2 \\ +(-1575{a_{21}}^5\, {a_{40}}^2\, a_{41} - 525a_{03}\, {a_{21}}^3\, {a_{40}}^4 + 3150{a_{21}}^4\, a_{22}\, {a_{40}}^3 - 5250{a_{21}}^4\, {a_{31}}^2\, {a_{40}}^2)a_{50} \\ + 2625{a_{21}}^4\, a_{31}\, {a_{40}}^3\, a_{41} + 875{a_{21}}^2\, (a_{03}\, a_{31} - a_{13}\, a_{21}){a_{40}}^5 + 3500{a_{21}}^3\, {a_{31}}^3\, {a_{40}}^3 \\ + 2625{a_{21}}^3\, (a_{21}\, a_{32} - 2a_{22}\, a_{31}){a_{40}}^4 \end{array}\right\} y_3^3/25{a_{40}}^2$$
$$- \{315{a_{21}}^6\, a_{40}\, a_{60} + 378{a_{21}}^6\, {a_{50}}^2 + (1260a_{12}\, {a_{21}}^4\, {a_{40}}^2 - 2520{a_{21}}^5\, a_{31}\, a_{40})a_{50} - 1575{a_{21}}^5\, {a_{40}}^2\, a_{41}$$
$$- 175a_{03}\, {a_{21}}^3\, {a_{40}}^4 + (1575{a_{21}}^4\, a_{22} - 2100a_{12}\, {a_{21}}^3\, a_{31}){a_{40}}^3 + 3150{a_{21}}^4\, {a_{31}}^2\, {a_{40}}^2 y_3^2\}/5a_{40}.$$

We substitute the non-degenerate formula of A_5-singularity AC_5 for the above condition to eliminate a coefficient a_{03}, and we get AC_6 as in the claim.

□

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