数学类(27)

判断$\sum_{n=1}^{\infty}\frac{(-1)^{\left[n^{\alpha}\right]}}{n^{\beta}}$的收敛性

证明：

$\beta\le 0$, 发散, $\beta>1$绝对收敛

故下面在$\beta\in\left(0,1\right]$考虑

当$\alpha\in\mathbb{N}_{+},(-1)^{\left[n^{\alpha}\right]}=(-1)^{n}$,此时条件收敛

$\alpha\le 0,(-1)^{\left[n^{\alpha}\right]}$是常数,因此发散

$\alpha>1,\alpha\notin\mathbb{Z},\lim\limits_{k\to\infty}\left[\left(k+1\right)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}\right]=0.$

即当n充分大,对于$k\le n^{\alpha}<\left(k+1\right)$的$n$至多只有一项

即需要考虑$\left[n^{\alpha}\right]$的分布, 不要求在竞赛范围内掌握.

核心只需要掌握$\beta\in\left(0,1\right],\alpha\in\left(0,1\right)$

对$m\in\mathbb{N}$,设$k\le m^{\alpha}<\left(k+1\right)$, $\sum\limits_{1\le n<k^{\frac{1}{\alpha}}}\frac{(-1)^{\left[n^{\alpha}\right]}}{n^{\beta}}+\sum\limits_{k^{\frac{1}{\alpha}}\le n<m}\frac{(-1)^{\left[n^{\alpha}\right]}}{n^{\beta}}$

为了$\sum\limits_{n=1}^{\infty}\frac{(-1)^{\left[n^{\alpha}\right]}}{n^{\beta}}=\sum\limits_{k=1}^{\infty}\sum\limits_{k^{\frac{1}{\alpha}}\le n<(k+1)^{\frac{1}{\alpha}}}\frac{(-1)^{\left[n^{\alpha}\right]}}{n^{\beta}}=\sum\limits_{k=1}^{\infty}\sum\limits_{k^{\frac{1}{\alpha}}\le n<(k+1)^{\frac{1}{\alpha}}}\frac{(-1)^{k}}{n^{\beta}}$合理

主要到$\left|\sum\limits_{k^{\frac{1}{\alpha}}\le n<m}\frac{(-1)^{\left[n^{\alpha}\right]}}{n^{\beta}}\right|\le\sum\limits_{k^{\frac{1}{\alpha}}\le n<(k+1)^{\frac{1}{\alpha}}}\frac{1}{n^{\beta}}$

$$\lim_{k\to\infty}\sum_{k^{\frac{1}{\alpha}}\le n<(k+1)^{\frac{1}{\alpha}}}\frac{1}{n^{\beta}}\le\lim_{k\to\infty}\sum_{k^{\frac{1}{\alpha}}\le n<(k+1)^{\frac{1}{\alpha}}}\frac{1}{k^{\frac{\beta}{\alpha}}}=\lim_{k\to\infty}\frac{\left(k+1\right)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}+1}{k^{\frac{\beta}{\alpha}}}$$

同阶于$\frac{k^{\frac{1}{\alpha}-1}}{k^{\frac{\beta}{\alpha}}}=\frac{1}{k^{\frac{\beta-1+\alpha}{\alpha}}}$

$$\lim_{k\to\infty}\sum_{k^{\frac{1}{\alpha}}\le n<(k+1)^{\frac{1}{\alpha}}}\frac{1}{n^{\beta}}\ge\lim_{k\to\infty}\sum_{k^{\frac{1}{\alpha}}\le n<(k+1)^{\frac{1}{\alpha}}}\frac{1}{(k+1)^{\frac{\beta}{\alpha}}}=\lim_{k\to\infty}\frac{(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}+1}{(k+1)^{\frac{\beta}{\alpha}}}$$

同阶于$\dfrac{k^{\frac{1}{\alpha}-1}}{k^{\frac{\beta}{\alpha}}}=\dfrac{1}{k^{\frac{\beta-1+\alpha}{\alpha}}}$

因此当$\alpha+\beta>1$,上述变形合理,若$\alpha+\beta\le 1$,注意末项都不趋于0，所以发散.

因此,当$\alpha+\beta>1$,我们再断言(当k充分大)$\displaystyle\sum_{k^{\frac{1}{\alpha}}\le n<(k+1)^{\frac{1}{\alpha}}}\frac{1}{n^{\beta}}$递减,即得到我们要的结果.

从阶的角度

$$\sum_{k^{\frac{1}{\alpha}}\le n<(k+1)^{\frac{1}{\alpha}}}\frac{1}{n^{\beta}}-\sum_{(k+1)^{\frac{1}{\alpha}}\le n<(k+2)^{\frac{1}{\alpha}}}\frac{1}{n^{\beta}}\sim\sum_{0\le n<(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}}\frac{1}{\left(n+k^{\frac{1}{\alpha}}\right)^{\beta}}-\sum_{0\le n<(k+2)^{\frac{1}{\alpha}}-(k+1)^{\frac{1}{\alpha}}}\frac{1}{\left(n+(k+1)^{\frac{1}{\alpha}}\right)^{\beta}}\sim$$

$$\sum_{0\le n<(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}}\frac{1}{\left(n+k^{\frac{1}{\alpha}}\right)^{\beta}}-\sum_{0\le n<(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}}\frac{1}{\left(n+(k+1)^{\frac{1}{\alpha}}\right)^{\beta}}-\sum_{(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}\le n<(k+2)^{\frac{1}{\alpha}}-(k+1)^{\frac{1}{\alpha}}}\frac{1}{\left(n+(k+1)^{\frac{1}{\alpha}}\right)^{\beta}}$$

$$\sim\sum_{0\le n<(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}}\left[\frac{1}{\left(n+k^{\frac{1}{\alpha}}\right)^{\beta}}-\frac{1}{\left(n+(k+1)^{\frac{1}{\alpha}}\right)^{\beta}}\right]-\sum_{(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}\le n<(k+2)^{\frac{1}{\alpha}}-(k+1)^{\frac{1}{\alpha}}}\frac{1}{\left(n+(k+1)^{\frac{1}{\alpha}}\right)^{\beta}}$$

$$\sum_{0\le n<(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}}\left[\frac{1}{\left(n+k^{\frac{1}{\alpha}}\right)^{\beta}}-\frac{1}{\left(n+(k+1)^{\frac{1}{\alpha}}\right)^{\beta}}\right]\ge\sum_{0\le n<(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}}\left[\frac{1}{(k+1)^{\frac{\beta}{\alpha}}}-\frac{1}{\left(2(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}\right)^{\beta}}\right]$$

$$\text{同阶}\left[(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}\right]\left[\frac{1}{(k+1)^{\frac{\beta}{\alpha}}}-\frac{1}{\left(2(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}\right)^{\beta}}\right]$$

$$\text{同阶于}\frac{k^{\frac{1}{\alpha}-1}}{k^{\frac{\beta}{\alpha}+1}}$$

$$\sum_{(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}\le n<(k+2)^{\frac{1}{\alpha}}-(k+1)^{\frac{1}{\alpha}}}\frac{1}{\left(n+(k+1)^{\frac{1}{\alpha}}\right)^{\beta}}\le\sum_{(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}\le n<(k+2)^{\frac{1}{\alpha}}-(k+1)^{\frac{1}{\alpha}}}\frac{1}{\left(2(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}\right)^{\beta}}$$

$$\text{同阶于}\frac{(k+2)^{\frac{1}{\alpha}}-2(k+1)^{\frac{1}{\alpha}}+k^{\frac{1}{\alpha}}}{\left(2(k+1)^{\frac{1}{\alpha}}-k^{\frac{1}{\alpha}}\right)^{\beta}}\text{同阶于}\frac{k^{\frac{1}{\alpha}-2}}{k^{\frac{\beta}{\alpha}}}$$

上课时，本题得到的两边的阶是相同的，所以上面还得明确的放缩,
采用相同的放缩方法, 和复杂的计算, 可以解得到递减性.

计算过于复杂，为了计算简便可以只考虑$\alpha=\frac{1}{2},\beta=1$的情形（考场要求）

$\sum_{n=1}^{\infty} a_n$=s条件收敛,$\sum_{n=1}^{\infty} a_{f(n)} = t$是重排,$t \neq s$, 证明

$\forall N \geq 1$,存在n,使得$\left|n - f\left(n\right)\right| > N$

证明：

首先$f\left(n\right)$是一一映射,其次若对某个N，使得对任意N，

$\left|n - f\left(n\right)\right| \leq N$，那么考虑

$\sum_{n=1}^{m+N} a_{f(n)} - \sum_{n=1}^{m} a_n$,显然其中不含有项$a_j, 1 \leq j \leq m$

同时也不会含有$a_{f(j)}, j = 1, 2, \cdots m - N$这种项

让m充分大,使得$\left|a_n\right| \leq \varepsilon, \forall n \geq m$,那么此时

$\left|\sum_{n=1}^{m+N} a_{f(n)} - \sum_{n=1}^{m} a_n\right| \leq 2N\varepsilon$,从而令$M \to \infty$，就有$s = t$,矛盾！

设$F(x)$是$(0,+\infty)$正函数,$\dfrac{F(x)}{x}$递增,对某个$d>0$,$\dfrac{F(x)}{x^{1+d}}$递减

若存在数列$\lambda_n>0, a_n>0$,满足

$(1)\displaystyle\sum_{n=1}^{\infty}\lambda_n F\left(a_n\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_n}\right)<\infty$或者

$(2)\displaystyle\sum_{n=1}^{\infty}\lambda_n F\left(\sum_{k=1}^{n}\frac{a_k\lambda_k}{\lambda_n}\right)<\infty$

证明$\displaystyle\sum_{n=1}^{\infty}a_n$收敛

证明:

由单调性,不妨设$F(1)=1$,

首先$F(x)\ge x, x\ge 1, F(x)\ge x^{1+d}, x\le 1$

$$\sum_{a_n\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_n}>1}\lambda_n F\left(a_n\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_n}\right)\ge\sum_{a_n\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_n}>1}\lambda_n a_n\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_n}=\sum_{a_n\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_n}>1}a_n\sum_{k=1}^{n}\lambda_k$$

$$\ge\lambda_1\sum_{a_n\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_n}>1}a_n$$

$$\sum_{a_n\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_n}\le 1}\lambda_n F\left(a_n\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_n}\right)\ge\sum_{a_n\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_n}\le 1}a_n\lambda_n^{-d}a_n^{d}\left(\sum_{k=1}^{n}\lambda_k\right)^{d+1}$$

对$a_n>0$,那么$\displaystyle\sum_{n=1}^{\infty}\frac{a_n}{S_n^d}$收敛充分条件$d>1$(见之前上课视频)

$$\sum_{a_n\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_n}\le 1, a_n\ge\lambda_n\left(\sum_{k=1}^{n}\lambda_k\right)^{-\frac{1+d}{d}}}a_n\lambda_n^{-d}a_n^{d}\left(\sum_{k=1}^{n}\lambda_k\right)^{d+1}\ge\sum_{a_n\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_n}\le 1, a_n\ge\lambda_n\left(\sum_{k=1}^{n}\lambda_k\right)^{-\frac{1+d}{d}}}a_n$$

接下来只需考虑$S=\left\{n:\ a_n<\lambda_n\left(\sum_{k=1}^{n}\lambda_k\right)^{-\frac{1+d}{d}}\right\}$,$\displaystyle\sum_{n\in S}a_n$的收敛性

$$\sum_{n\in S}a_n\le\sum_{n\in S}\frac{\lambda_n}{\left(\sum_{k=1}^{n}\lambda_k\right)^{1+\frac{1}{d}}}<\infty$$

$(2):\sum_{n=1}^{\infty}\lambda_n F\left(\sum_{k=1}^{n}\frac{a_k\lambda_k}{\lambda_n}\right)<\infty$

证明：

首先说明,a_n有上界, 若不然, 存在充分大的$M>1$, 考虑最小的n_0, 使得$a_{n_0}>M>1$,

此时$\sum_{n=1}^{\infty}\lambda_n F\left(\sum_{k=1}^{n}\frac{a_k\lambda_k}{\lambda_n}\right)\geq\sum_{a_n>M}\lambda_n\sum_{k=1}^{n}\frac{a_k\lambda_k}{\lambda_n}=\sum_{a_n>M}\sum_{k=1}^{n}a_k\lambda_k\geq\sum_{a_n>M}a_{n_0}\lambda_{n_0}$

因此$\{n:a_n>M\}$是有限集

即a_n有上界,

$$\sum_{n=1}^{\infty}\lambda_n F\left(\sum_{k=1}^{n}\frac{a_k\lambda_k}{\lambda_n}\right)=\frac{1}{a_n}\sum_{n=1}^{\infty}a_n\lambda_n F\left(a_n\sum_{k=1}^{n}\frac{a_k\lambda_k}{a_n\lambda_n}\right)\geq\frac{1}{M}\sum_{n=1}^{\infty}a_n\lambda_n F\left(a_n\sum_{k=1}^{n}\frac{a_k\lambda_k}{a_n\lambda_n}\right)$$

把$a_n\lambda_n$看成整体,即运用(1)知,$\sum a_n$收敛.