数学类(3)

再次强调:禁止记忆具体各种方法使用条件和结果,而是理解其方法.

$laplace$方法严格处理为例$\left(\text{十分广泛并蕴含拟合法}\right)$:

设$m\geq 1$，$f\left(x\right)\in R\left[a,b\right],g\left(x\right)\in C^{m}\left[a,b\right]$，且有$g\left(x\right)\geq 0,g\left(x\right)$不恒为0,

设$g\left(x\right)$所有最大值点为$x_1,x_2,...,x_n,\forall i=1,2,...,n,\exists n_i,$

$1\leq n_i\leq m,g^{(n_i)}\left(x_i\right)\neq 0,$对任意$i,$当$n_i\geq 2,$还有$,g^{(k)}\left(x_i\right)=0,k=1,2...,n_i-1$

$f\left(x\right)$在$x_1,x_2,...,x_n$连续

则极限$\lim\limits_{N\to\infty}\dfrac{\int_a^b f\left(x\right)g^N\left(x\right)dx}{\int_a^b g^N\left(x\right)dx}=\dfrac{\sum\limits_{i=1}^{n}\dfrac{c_i f\left(x_i\right)}{\left|g^{(n_i)}\left(x_i\right)\right|^{\frac{1}{n_i}}}}{\sum\limits_{i=1}^{n}\dfrac{c_i}{\left|g^{(n_i)}\left(x_i\right)\right|^{\frac{1}{n_i}}}}$,这里

$$c_i=\begin{cases}2,n_i=\max\left\{n_1,n_2,...,n_n\right\},x_i\in\left\{a,b\right\}\\1,n_i=\max\left\{n_1,n_2,...,n_n\right\},x_i\in\left(a,b\right)\\0,n_i<\max\left\{n_1,n_2,...,n_n\right\}\end{cases}$$

例:

$$\int_0^1\left(1-x^2+x^3\right)^n f\left(x\right)dx$$

$$\ln\left(1-x^2+x^3\right)=-x^2+o\left(x^2\right)$$

$$\ln\left(1-x^2+x^3\right)=x-1+o\left(x-1\right)$$

最大值点0,1,即证

$$\lim_{n\to\infty}\sqrt{n}\int_0^1\left(1-x^2+x^3\right)^n f\left(x\right)dx=\frac{\sqrt{\pi}}{2}f\left(0\right)$$

事实上:

$\forall\delta>0,\left|\int_{\delta}^{1-\delta}\left(1-x^2+x^3\right)^n f\left(x\right)dx\right|\le\sup\limits_{x\in[0,1]}\left|f\left(x\right)\right|C^n\left(\delta\right)$

这里$C\left(\delta\right)=\sup\limits_{x\in[\delta,1-\delta]}\left|1-x^2+x^3\right|<1,$

故$\lim\limits_{n\to\infty}\sqrt{n}\int_{\delta}^{1-\delta}\left(1-x^2+x^3\right)^n f\left(x\right)dx=0$

再计算$\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}n\int_{1-\delta}^{1}\left(1-x^2+x^3\right)^n f\left(x\right)dx$

故

$\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}n\left|\int_{1-\delta}^{1}e^{n\ln\left(1-x^2+x^3\right)}f\left(x\right)dx\right|\le\sup\limits_{x\in[0,1]}\left|f\left(x\right)\right|\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}n\int_{1-\delta}^{1}e^{n\ln\left(1-x^2+x^3\right)}dx$

$\forall\varepsilon>0$,取δ充分小,有$\ln\left(1-x^2+x^3\right)\le x-1+\varepsilon\left(x-1\right),\forall x\in\left[1-\delta,1\right]$成立

且$\left|f\left(x\right)-f\left(0\right)\right|<\varepsilon,\forall x\in\left[0,\delta\right]$

$-x^2-\varepsilon x^2\le\ln\left(1-x^2+x^3\right)\le-x^2+\varepsilon x^2,\forall x\in\left[0,\delta\right]$

于是有

$\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}n\int_{1-\delta}^{1}e^{n\ln\left(1-x^2+x^3\right)}dx\le\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}n\int_{1-\delta}^{1}e^{-(1+\varepsilon)n(1-x)}dx=\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}n\int_{1-\delta}^{1}e^{-(1+\varepsilon)n(1-x)}dx$

$=\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}n\frac{1-e^{-\delta(1+\varepsilon)n}}{\left(1+\varepsilon\right)n}=0$

$\lim\limits_{n\to\infty}\sqrt{n}\left|\int_{0}^{\delta}e^{n\ln\left(1-x^2+x^3\right)}\left[f\left(x\right)-f\left(0\right)\right]dx\right|$

$\le\lim\limits_{n\to\infty}\varepsilon\sqrt{n}\int_{0}^{\delta}e^{n\ln\left(1-x^2+x^3\right)}dx\le\lim\limits_{n\to\infty}\varepsilon\sqrt{n}\int_{0}^{\delta}e^{-n(1+\varepsilon)x^2}dx$

$=\lim\limits_{n\to\infty}\varepsilon\sqrt{n}\frac{\sqrt{\pi}}{2\sqrt{n\left(1+\varepsilon\right)}}\le C\varepsilon$

$\lim\limits_{n\to\infty}\sqrt{n}\int_{0}^{\delta}e^{-n(1-\varepsilon)x^2}dx\le\lim\limits_{n\to\infty}\sqrt{n}\int_{0}^{\delta}e^{n\ln\left(1-x^2+x^3\right)}dx\le\lim\limits_{n\to\infty}\sqrt{n}\int_{0}^{\delta}e^{-n(1+\varepsilon)x^2}dx$

故$\frac{\sqrt{\pi}}{2\sqrt{\left(1-\varepsilon\right)}}\le\lim\limits_{n\to\infty}\sqrt{n}\int_{0}^{\delta}e^{n\ln\left(1-x^2+x^3\right)}dx\le\frac{\sqrt{\pi}}{2\sqrt{\left(1+\varepsilon\right)}}$, 故

$\lim\limits_{n\to\infty}\sqrt{n}\int_{0}^{\delta}e^{n\ln\left(1-x^2+x^3\right)}dx=\frac{\sqrt{\pi}}{2}$,故$\lim\limits_{n\to\infty}\sqrt{n}\int_{0}^{1}\left(1-x^2+x^3\right)^n f\left(x\right)dx=\frac{\sqrt{\pi}}{2}f\left(0\right)$

特别的

$$\lim_{N\to\infty}\frac{\int_0^1 \ln\left(x+2\right)\left(1-x^2+x^3\right)^N dx}{\int_0^1\left(1-x^2+x^3\right)^N dx}=\ln 2$$

特别的

$f\left(x\right)\in R\left[a,b\right]$且$f\left(x\right)$在$x=1$连续，我们有

$$\lim_{n\to\infty} n\int_0^1 x^n f\left(x\right)dx=f\left(1\right)\left(\text{拟和法基本模型}\right)$$

拉普拉斯方法的无穷阶渐进$\left(waston\text{公式}\right)$

再次强调,禁止记忆公式,甚至根本别去看这些公式.

我们以两个例子为例:

$$\ln\left(1+x^2\right)=y\Rightarrow x=\sqrt{e^y-1}$$

$$\int_0^{\infty}\frac{\cos x}{\left(1+x^2\right)^n}dx=\int_0^{\infty}e^{-n\ln\left(1+x^2\right)}\cos xdx$$

$$=\int_0^{\infty}e^{-ny}\cos\left(\sqrt{e^y-1}\right)d\sqrt{e^y-1}$$

$$=\int_0^{\infty}e^{-ny}\frac{e^y\cos\left(\sqrt{e^y-1}\right)}{2\sqrt{e^y-1}}dy$$

$$taylor\text{展开}\frac{e^y\cos\left(\sqrt{e^y-1}\right)}{2\sqrt{e^y-1}}=\frac{1}{2\sqrt{y}}+\frac{\sqrt{y}}{8}-\frac{31}{192}y^{\frac{3}{2}}+o\left(y^{\frac{3}{2}}\right)$$

计算方法:每一项$taylor$或者软件只要形式计算出即可.

然后求极限严格证明你得到的余项估计,即:

$$\lim_{y\to0^+}\frac{\frac{e^y\cos\left(\sqrt{e^y-1}\right)}{2\sqrt{e^y-1}}-\left(\frac{1}{2\sqrt{y}}+\frac{\sqrt{y}}{8}\right)}{y^{\frac{3}{2}}}=-\frac{31}{192}$$

我们可以证明如上积分的渐进的确是局部的.

$\forall\delta>0,$

$$\left|\int_{\delta}^{\infty}\frac{1}{\left(1+x^2\right)^n}\cos xdx\right|\le\int_{\delta}^{\infty}\frac{1}{\left(1+\delta^2\right)^{n-1}\left(1+x^2\right)}dx\le\frac{\pi}{2\left(1+\delta^2\right)^{n-1}}$$

故这个问题的确是局部的

$$\int_0^{\delta} e^{-ny} y^s dy = \frac{1}{n^{s+1}} \int_0^{n\delta} e^{-y} y^s dy$$

$$\int_{n\delta}^{\infty} e^{-y} y^s dy \le \int_{n\delta}^{\infty} e^{-\frac{n\delta}{2}} e^{-\frac{y}{2}} y^s dy \le \left(e^{-\frac{\delta}{2}} \right)^n \int_0^{\infty} e^{-\frac{y}{2}} y^s dy$$

所以$\forall \delta > 0$,有

$$\int_0^{\infty} \frac{\cos x}{\left(1+x^2\right)^n} dx = \int_0^{\delta} \frac{\cos x}{\left(1+x^2\right)^n} dx + O\left(\frac{1}{\left(1+\delta^2\right)^n} \right)$$

$$= \int_0^{\delta} e^{-ny} \frac{e^y \cos\left(\sqrt{e^y - 1}\right)}{2\sqrt{e^y - 1}} dy + O\left(\frac{1}{\left(1+\delta^2\right)^n} \right)$$

$$= \int_0^{\delta} e^{-ny} \left[\frac{1}{2\sqrt{y}} + \frac{\sqrt{y}}{8} + o\left(y^{\frac{1}{2}} \right) \right] dy + O\left(\frac{1}{\left(1+\delta^2\right)^n} \right)$$

$$= \int_0^{\delta} e^{-ny} \frac{1}{2\sqrt{y}} dy + \int_0^{\delta} e^{-ny} \frac{\sqrt{y}}{8} dy + \int_0^{\delta} e^{-ny} o\left(y^{\frac{1}{2}} \right) dy + O\left(\frac{1}{\left(1+\delta^2\right)^n} \right)$$

$$= \int_0^{\infty} e^{-ny} \frac{1}{2\sqrt{y}} dy + \int_0^{\infty} e^{-ny} \frac{\sqrt{y}}{8} dy + \int_0^{\infty} e^{-ny} o\left(y^{\frac{1}{2}} \right) dy + O\left(\text{公比小于1指数级别}\right)$$

$$= \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{n}} + \frac{\sqrt{\pi}}{16} \frac{1}{n^{\frac{3}{2}}} + o\left(\frac{1}{n^{\frac{3}{2}}} \right)$$

留个习题:$\int_0^{\infty} e^{-ny} o\left(y^{\frac{1}{2}} \right) dy = o\left(\frac{1}{n^{\frac{3}{2}}} \right)$是不严谨的,严格按照定义计算一下.

拉格朗日反演:

设$w=f\left(z\right)=w_0+a_k\left(z-z_0\right)^k+a_{k+1}\left(z-z_0\right)^{k+1}...,a_k\neq 0$

$$则z=z_0+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^{n-1}}{d\zeta^{n-1}}\left\{\left(\frac{\zeta-z_0}{\left(f\left(\zeta\right)-w_0\right)^{\frac{1}{k}}}\right)^n\right\}_{\zeta=z_0}\left(w-w_0\right)^{\frac{n}{k}}$$

(1):本质上我们是反函数的级数展开,并且公式是局部成立的.

(2):本结果严格掌握证明需要复分析知识,但在数学分析框架下可以如此使用:
无脑套用公式猜出结果,然后按求极限验证结果,然后完成证明.

对刚才$f\left(y\right)=-y-\ln\left(1-y\right)=-y^2+o\left(y^2\right)$

$$即k=2,g\left(x\right)=\lim_{\zeta\to0^+}\frac{\zeta}{\sqrt{-\zeta-\ln\left(1-\zeta\right)}}\sqrt{x}+\frac{1}{2}\lim_{\zeta\to0^+}\left(\frac{\zeta^2}{-\zeta-\ln\left(1-\zeta\right)}\right)'x$$

$$=\sqrt{2x}-\frac{2}{3}x+o\left(x\right)$$

$g'\left(x\right)=\frac{1}{\sqrt{2x}}-\frac{2}{3}+o\left(1\right)$,于是我们如下严格证明这个等式:

$$\lim_{x\to0^+}g'\left(x\right)-\frac{1}{\sqrt{2x}}=\lim_{y\to0^+}g'\left(f\left(y\right)\right)-\frac{1}{\sqrt{2f\left(y\right)}}=\lim_{y\to0^+}\frac{1}{f'\left(y\right)}-\frac{1}{\sqrt{2f\left(y\right)}}$$

$$g\left(f\left(x\right)\right)=x,g'\left(f\left(x\right)\right)f'\left(x\right)=1$$

因此对于显式的$\lim_{y\to0^+}\frac{1}{f'\left(y\right)}-\frac{1}{\sqrt{2f\left(y\right)}}=-\frac{2}{3}$,是课内基础知识,自行完成.

估计$\int_0^1 e^{-n\left(-y-\ln(1-y)\right)}dy$无穷阶渐进,

记$f\left(y\right)=-y-\ln\left(1-y\right)$,反函数$g\left(x\right)$

$$\int_0^1 e^{-n\left(-y-\ln(1-y)\right)}dy=\int_0^1 e^{-nx}g'\left(x\right)dx$$

本质上需要计算反函数的泰勒公式,

事实上我们有拉格朗日反演:

$$g'\left(x\right)=\frac{1}{\sqrt{2x}}-\frac{2}{3}+o\left(1\right)$$

问题是局部的,显然这个是更容易地说明

$$\int_0^1 e^{-n\left(-y-\ln(1-y)\right)}dy=\int_0^1 e^{-nx}g'\left(x\right)dx=\int_0^1 e^{-nx}\left[\frac{1}{\sqrt{2x}}-\frac{2}{3}+o\left(1\right)\right]dx$$

$$=\int_0^1 e^{-nx}\left[\frac{1}{\sqrt{2x}}-\frac{2}{3}+o\left(1\right)\right]dx$$

自己完成计算我们不再计算.

$f(x)\in C[0,1], f'(0)$存在，则有

$$\int_0^1 f(x^n)dx = f(0)+\sum_{k=0}^{m-1}\frac{1}{n^{k+1}}\int_0^1\frac{f(x)-f(0)}{x}\frac{\ln^k x}{k!}dx + O\left(\frac{1}{n^{m+1}}\right)$$

证明:

$$\int_0^1 f(x^n)-f(0)dx = \frac{1}{n}\int_0^1[f(x)-f(0)]x^{\frac{1}{n}-1}dx$$

$$=\frac{1}{n}\int_0^1\frac{f(x)-f(0)}{x}x^{\frac{1}{n}}dx = \frac{1}{n}\int_0^1\frac{f(x)-f(0)}{x}e^{\frac{\ln x}{n}}dx$$

$$=\frac{1}{n}\int_0^1\frac{f(x)-f(0)}{x}\sum_{k=0}^{\infty}\frac{\ln^k x}{k!n^k}dx$$

$$=\frac{1}{n}\int_0^1\frac{f(x)-f(0)}{x}\sum_{k=0}^{m-1}\frac{\ln^k x}{k!n^k}dx+\frac{1}{n}\int_0^1\frac{f(x)-f(0)}{x}\sum_{k=m}^{\infty}\frac{\ln^k x}{k!n^k}dx$$

$$=\sum_{k=0}^{m-1}\frac{1}{n^{k+1}}\int_0^1\frac{f(x)-f(0)}{x}\frac{\ln^k x}{k!}dx+\frac{1}{n}\int_0^1\frac{f(x)-f(0)}{x}\sum_{k=m}^{\infty}\frac{\ln^k x}{k!n^k}dx$$

$$\left|\int_0^1\frac{f(x)-f(0)}{x}\sum_{k=m}^{\infty}\frac{\ln^k x}{k!n^k}dx\right|\le M\int_0^1\sum_{k=m}^{\infty}\frac{(-\ln x)^k}{k!n^k}dx$$

由$levi$定理, $M\int_0^1\sum_{k=m}^{\infty}\frac{(-\ln x)^k}{k!n^k}dx = M\sum_{k=m}^{\infty}\int_0^1\frac{(-\ln x)^k}{k!n^k}dx$

$$M\sum_{k=m}^{\infty}\frac{1}{k!n^k}\int_0^{\infty}y^k e^{-y}dy = M\sum_{k=m}^{\infty}\frac{1}{n^k}=M\frac{\frac{1}{n^m}}{1-\frac{1}{n}}=O\left(\frac{1}{n^m}\right)$$

故我们完成了证明.

$$\int_0^1\sum_{k=m}^{\infty}\frac{(-\ln x)^k}{k!n^k}dx=\sum_{k=m}^{\infty}\int_0^1\frac{(-\ln x)^k}{k!n^k}dx$$

这一步是广义积分和无穷级数换序，由于收敛速度相当快所以可以验证广义积分一致收敛性和无穷级数一致收敛性来换序而不必使用$levi$定理,同时$levi$定理的优势也在此体现！

特别的:经典习题:

$$\lim_{n\to\infty} n\left(1-\int_0^1 \frac{1}{1+x^n}dx\right)=\ln 2$$

我们还有加强:

$$\int_0^1 \frac{x^n}{1+x^n}dx=\frac{\ln 2}{n}+\sum_{k=1}^{m-1}\frac{(-1)^k\left(1-\frac{1}{2^k}\right)\zeta(k+1)}{n^{k+1}}+o\left(\frac{1}{n^m}\right)$$

黎曼$zete$函数$\zeta(k)=\sum_{n=1}^{\infty}\frac{1}{n^k}, k=2,3,\ldots$