数学类(13)

$f(x)\in C^2[0,1], n\in\mathbb{N}_+, \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) = -\frac{f(0)+f(1)}{2}$, 证明：

$$\left(\int_0^1 f(x)dx\right)^2 \le \frac{1}{120n^4}\int_0^1 |f''(x)|^2 dx$$

证明：

由$E-M$公式$\sum_{k=a}^{b} f(k) = \int_a^b f(x)dx + \frac{f(a)+f(b)}{2} + \int_a^b b_1(x)f'(x)dx$

就有$\sum_{k=a}^{b} f(k) = \int_a^b f(x)dx + \frac{f(a)+f(b)}{2} + b_2(b)f'(b) - b_2(a)f'(a) - \int_a^b b_2(x)f''(x)dx$

$$\sum_{k=0}^{n} f\left(\frac{k}{n}\right) = \int_0^n f\left(\frac{x}{n}\right)dx + \frac{f(0)+f(1)}{2} + b_2(0)\left[\frac{1}{n}f'(1) - \frac{1}{n}f'(0)\right] - \frac{1}{n^2}\int_0^n b_2(x)f''\left(\frac{x}{n}\right)dx$$

$x\in[0,1)$时，就有

$$b_1(x) = x - \frac{1}{2}, b_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + c, \int_0^1 b_2(x)dx = 0 \Rightarrow c = \frac{1}{12},$$

$$\sum_{k=0}^{n} f\left(\frac{k}{n}\right) = \int_0^n f\left(\frac{x}{n}\right)dx + \frac{f(0)+f(1)}{2} + \frac{f'(1)-f'(0)}{12n} - \frac{1}{n^2}\int_0^n b_2(x)f''\left(\frac{x}{n}\right)dx$$

$$\sum_{k=0}^{n} f\left(\frac{k}{n}\right) = n\int_0^1 f(x)dx + \frac{f(0)+f(1)}{2} + \frac{f'(1)-f'(0)}{12n} - \frac{1}{n}\int_0^1 b_2(nx)f''(x)dx$$

$$\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) = n\int_0^1 f(x)dx - \frac{f(0)+f(1)}{2} + \frac{f'(1)-f'(0)}{12n} - \frac{1}{n}\int_0^1 b_2(nx)f''(x)dx$$

$$\int_0^1 f(x)dx = \frac{1}{n^2}\int_0^1 b_2(nx)f''(x)dx - \frac{f'(1)-f'(0)}{12n^2} = \frac{1}{n^2}\int_0^1\left[b_2(nx) - \frac{1}{12}\right]f''(x)dx$$

$$\left(\int_0^1 f(x)dx\right)^2 = \frac{1}{n^4}\left(\int_0^1\left[b_2(nx)-\frac{1}{12}\right]f''(x)dx\right)^2 \le \frac{1}{n^4}\int_0^1\left[b_2(nx)-\frac{1}{12}\right]^2 dx\int_0^1 |f''(x)|^2 dx$$

$$\int_0^1\left[b_2(nx)-\frac{1}{12}\right]^2 dx = \frac{1}{n}\int_0^n\left[b_2(x)-\frac{1}{12}\right]^2 dx$$

$$= \frac{n}{n}\int_0^1\left[b_2(x)-\frac{1}{12}\right]^2 dx = \int_0^1\left(\frac{1}{2}x^2 - \frac{1}{2}x\right)^2 dx = \frac{1}{120}$$

$f(x)\in C[0,+\infty)$非负,$f(f(x))=x^a, a\in\mathbb{N}_+$

证明:$\int_0^1 f^2(x)dx\geq\dfrac{2a-1}{a^2+6a-3}$

证明：回忆我们提到过的结构$f(f(x))$的处理手法.

设$f(x_1)=f(x_2)\Rightarrow f(f(x_1))=x_1^a=x_2^a=f(f(x_2))\Rightarrow x_1=x_2\Rightarrow f(x)$严格递增.

$f(f(0))=0\Rightarrow f(0)=0, f(f(1))=1$, 如果$f(1)>1, 1=f(f(1))>f(1)=1$, 矛盾,

如果$f(1)<1$, 那么$1=f(f(1))<f(1)=1$, 矛盾, 因此只能有$f(1)=1$,

$f(x)=f^{-1}(x^a)$, 这些条件在疯狂的暗示$young$不等式.

$1=\int_0^1 f(x)dx+\int_0^1 f^{-1}(x)dx=\int_0^1 f(x)dx+a\int_0^1 x^{a-1}f^{-1}(x^a)dx$

$=\int_0^1(1+ax^{a-1})f(x)dx$, 因此$1\leq\int_0^1(1+ax^{a-1})^2\int_0^1 f^2(x)dx=\dfrac{a^2+6a-3}{2a-1}\int_0^1 f^2(x)dx$

因此,我们完成了证明.

$\varphi > 0$,连续严格递减,且$\lim\limits_{x\to 0^+}\varphi\left(x\right)=+\infty$，且$\int_0^{\infty}\varphi\left(x\right)dx=a$, 证明：

$$\int_0^{\infty}\varphi^2\left(x\right)dx+\int_0^{\infty}[\varphi^{-1}\left(x\right)]^2dx\geq\frac{1}{2}a^{\frac{3}{2}}$$

分析：显然$\lim\limits_{x\to+\infty}\varphi\left(x\right)=0,\int_0^{\infty}\varphi\left(x\right)dx=\int_0^{\infty}\varphi^{-1}\left(x\right)dx=a$

$$\int_0^{\infty}\varphi^2\left(x\right)dx+\int_0^{\infty}[\varphi^{-1}\left(x\right)]^2dx\geq\int_0^{p}\varphi^2\left(x\right)dx+\int_0^{q}[\varphi^{-1}\left(x\right)]^2dx$$

转化成一次积分才能回到面积角度,所以使用$cauchy$不等式：

$$\int_0^{p}\varphi^2\left(x\right)dx+\int_0^{q}[\varphi^{-1}\left(x\right)]^2dx\geq\frac{\left[\int_0^{p}\varphi\left(x\right)dx\right]^2}{p}+\frac{\left[\int_0^{q}\varphi^{-1}\left(x\right)dx\right]^2}{q}$$

$\geq\frac{2}{\sqrt{pq}}\int_0^{p}\varphi\left(x\right)dx\int_0^{q}\varphi^{-1}\left(x\right)dx\geq\frac{1}{2}a^{\frac{3}{2}}$，如果$pq=a$,只需证明：

$$\int_0^{p}\varphi\left(x\right)dx=\int_0^{q}\varphi^{-1}\left(x\right)dx\geq\frac{a}{2}$$

证明:

显然$\lim\limits_{x\to+\infty}\varphi\left(x\right)=0,\int_0^{\infty}\varphi\left(x\right)dx=\int_0^{\infty}\varphi^{-1}\left(x\right)dx=a$

构造$g\left(p\right)=\int_p^{\infty}\varphi\left(x\right)dx-\int_{\frac{a}{p}}^{\infty}\varphi^{-1}\left(x\right)dx,g\left(0\right)=a=-g\left(+\infty\right)$

存在p_0使得$g\left(p_0\right)=0$,取$q_0=\frac{a}{p_0}$,

于是$\int_0^{p_0}\varphi\left(x\right)dx=\int_0^{q_0}\varphi^{-1}\left(x\right)dx\geq\frac{a}{2}$(只有这一步涉及几何, 思考代数叙述)

$$\int_0^{\infty}\varphi^2\left(x\right)dx+\int_0^{\infty}[\varphi^{-1}\left(x\right)]^2dx\geq\int_0^{p_0}\varphi^2\left(x\right)dx+\int_0^{q_0}[\varphi^{-1}\left(x\right)]^2dx$$

$$\geq\frac{\left[\int_0^{p_0}\varphi\left(x\right)dx\right]^2}{p_0}+\frac{\left[\int_0^{q_0}\varphi^{-1}\left(x\right)dx\right]^2}{q_0}\geq\frac{2}{\sqrt{a}}\int_0^{p_0}\varphi\left(x\right)dx\int_0^{q_0}\varphi^{-1}\left(x\right)dx$$

$$=\frac{2}{\sqrt{a}}\left(\int_0^{p_0}\varphi\left(x\right)dx\right)^2\geq\frac{2}{\sqrt{a}}\frac{a^2}{4}=\frac{1}{2}a^{\frac{3}{2}}$$

$f\left(x\right)\in C^2\left[0,1\right]$,证明:$\int_0^1\left|f'(x)\right|dx\le 4\int_0^1\left|f\left(x\right)\right|dx+\int_0^1\left|f''\left(x\right)\right|dx$

特别的,若$f\left(0\right)f\left(1\right)\ge 0$,则4可以修正为2.

证明:$\left|f'(x)\right|\le 4\int_0^1\left|f\left(x\right)\right|dx+\int_0^1\left|f''\left(x\right)\right|dx$

$$\left|f'(x)\right|\le\left|f'(x)-f'(\theta)\right|+\left|f'(\theta)\right|=\left|\int_\theta^x\left|f''\left(x\right)\right|dx\right|+\left|f'\left(\theta\right)\right|\le\int_0^1\left|f''\left(x\right)\right|dx+\left|f'\left(\theta\right)\right|$$

只需证明,

存在一个$\theta\in\left[0,1\right]$,使得$\left|f'(\theta)\right|\le 4\int_0^1\left|f\left(x\right)\right|dx$,即证$\min\limits_{x\in[0,1]}\left|f'\left(x\right)\right|\le 4\int_0^1\left|f\left(x\right)\right|dx$

如果f'有零点,显然$\min\limits_{x\in[0,1]}\left|f'\left(x\right)\right|=0\le 4\int_0^1\left|f\left(x\right)\right|dx$

所以只需考虑$f'\ne 0$,

不妨设$f\left(x\right)$严格递增,若$f(x)$没零点,不妨设$f\left(x\right)>0$

$$f\left(x\right)=f\left(0\right)+xf'\left(\eta\right)\ge xf'\left(\eta\right),\int_0^1\left|f\left(x\right)\right|dx\ge\int_0^1 xf'\left(\eta\right)dx=\frac{f'(\eta)}{2}\ge\frac{\min\limits_{x\in[0,1]}\left|f'(x)\right|}{4}$$

若$f\left(t\right)=0,\left|f\left(x\right)\right|=\left|f'(\theta)\right|\left|x-t\right|$,

$$\int_0^1\left|f\left(x\right)\right|dx\ge\min_{x\in[0,1]}\left|f'(x)\right|\int_0^1\left|x-t\right|dx\ge\min_{x\in[0,1]}\left|f'\left(x\right)\right|\int_0^1\left|x-\frac{1}{2}\right|dx=\frac{\min\limits_{x\in[0,1]}\left|f'(x)\right|}{4}$$

故$\int_0^1\left|f'(x)\right|dx\le 4\int_0^1\left|f\left(x\right)\right|dx+\int_0^1\left|f''\left(x\right)\right|dx$

当$f\left(0\right)f\left(1\right)\ge 0$,不妨设$f\left(0\right)\ge 0$,请自己完成证明的书写.

$f(x)\in C[a,b], f>0, L>0, |f(x)-f(y)|\le L|x-y|, a<c<d<b$, 记

$\alpha=\int_c^d \frac{1}{f(x)}dx, \beta=\int_a^b \frac{1}{f(x)}dx$, 证明: $\int_a^b f(x)dx \le \frac{e^{2L\beta}-1}{2L\alpha}\int_c^d f(x)dx$

分析:

看到 $|f(x)-f(y)|\le L|x-y|$ 应该非常高兴, 告诉你用

$f(y)-L|x-y|\le f(x)\le f(y)+L|x-y|$ 放缩

证明:

取 $f(x_0)=m=\min_{x\in[a,b]} f(x)$

$$\int_a^b f(x)dx \le \int_a^b f(x_0)+L|x_0-x|dx=(b-a)f(x_0)+\frac{L}{2}\left[(x_0-a)^2+(b-x_0)^2\right]$$

$$\beta=\int_a^b \frac{1}{f(x)}dx \ge \int_a^b \frac{1}{f(x_0)+L|x_0-x|}dx=\frac{1}{L}\ln\left[\left(1+\frac{L(x_0-a)}{f(x_0)}\right)\left(1+\frac{L(b-x_0)}{f(x_0)}\right)\right]$$

$$e^{2L\beta}-1\ge\left(1+\frac{L(x_0-a)}{f(x_0)}\right)^2\left(1+\frac{L(b-x_0)}{f(x_0)}\right)^2-1$$

$$\frac{\int_c^d f(x)dx}{\int_c^d \frac{1}{f(x)}dx}\ge\frac{\int_c^d f(x_0)dx}{\int_c^d \frac{1}{f(x_0)}dx}=\frac{(d-c)m}{(d-c)\frac{1}{m}}=m^2$$

只需证明

$$(b-a)m+\frac{L}{2}\left[(x_0-a)^2+(b-x_0)^2\right]\le\frac{\left[\left(1+\frac{L(x_0-a)}{m}\right)^2\left(1+\frac{L(b-x_0)}{m}\right)^2-1\right]}{2L}m^2$$

$x=\frac{L(x_0-a)}{m}, y=\frac{L(b-x_0)}{m}, x+y=\frac{L(b-a)}{m}$, 只需证明:

$\left[(1+x)^2(1+y)^2-1\right]m^2\ge 2(x+y)m^2+m^2(x^2+y^2)$, 做差因式分解

$\Leftrightarrow xy(x+2)(y+2)m^2\ge 0$

因此我们完成了证明.

找积分不等式取最值时的取等条件可用$E-L$方程

但这个并不严谨,只能用于小题或者猜测取等条件以辅助思考

对于$I(f)=\int_a^b L(x,f,f')dx$,求$f$使得$I(f)$取最值的$f$满足微分方程

$$L_f(x,f,f')=\frac{d}{dx}L_{f'}$$

证明:如果$I(f)$在f_0处取最小值,则$I(f_0+t\phi)\geq I(f_0),\forall\phi\in C_c^\infty[a,b]$

即$\phi(a)=\phi(b)=0$

$$g(t)=I(f_0+t\phi)=\int_a^b L(x,f_0+t\phi,f_0'+t\phi')dx$$

$$g'(t)=\int_a^b \phi L_f(x,f_0+t\phi,f_0'+t\phi')+\phi' L_{f'}(x,f_0+t\phi,f_0'+t\phi')dx$$

$$=\int_a^b \phi L_f(x,f_0+t\phi,f_0'+t\phi')+\phi' L_{f'}(x,f_0+t\phi,f_0'+t\phi')dx$$

$$g'(0)=\int_a^b \phi L_f(x,f_0,f_0')+\phi' L_{f'}(x,f_0,f_0')dx$$

$$=\int_a^b \phi L_f(x,f_0,f_0')-\phi\frac{d}{dx}L_{f'}(x,f_0,f_0')dx=0$$

因此由ϕ的任意性,我们知道$L_f(x,f_0,f_0')=\frac{d}{dx}L_{f'}(x,f_0,f_0')$

通过变分法构造微分方程的解,是丘赛唯一考查的非线性pde部分.

$f(0)=f(1)=0,f(x)\in C[0,1]$,

$\int_0^1[f'(x)]^2dx-8\int_0^1 f(x)dx+\frac{4}{3}=0$,求$f(x)$,

$$\int_0^1[f'(x)]^2-8f(x)+\frac{4}{3}dx=0$$

$L_f=-8$,$L_{f'}=2f',\frac{d}{dx}L_{f'}=2f''$,

因此解$2f''=-8$,因此解出$f(x)=-2x^2+2x$