数学类(14)

$f(x)\in C[0,1],\int_0^1 x^k f(x)dx=1,k=0,1,2,\ldots,n-1$, 证明: $\int_0^1 f^2(x)dx\ge n^2$

证明：

$$\int_0^1 f^2(x)dx\int_0^1\left(\sum_{k=0}^{n-1}a_kx^k\right)^2dx\ge\left[\int_0^1 f(x)\sum_{k=0}^{n-1}a_kx^kdx\right]^2$$

$$=\left[\sum_{k=0}^{n-1}a_k\int_0^1 f(x)x^kdx\right]^2=\left[\sum_{k=0}^{n-1}a_k\right]^2,\text{故有}\int_0^1 f^2(x)dx\ge\frac{\left[\sum_{k=0}^{n-1}a_k\right]^2}{\int_0^1\left(\sum_{k=0}^{n-1}a_kx^k\right)^2dx}$$

$$\frac{\left[\sum_{k=0}^{n-1}a_k\right]^2}{\int_0^1\left(\sum_{k=0}^{n-1}a_kx^k\right)^2dx}=\frac{\left[\sum_{k=0}^{n-1}a_k\right]^2}{\int_0^1\left(\sum_{k=0}^{n-1}a_kx^k\right)\left(\sum_{j=0}^{n-1}a_ky^j\right)dx}=\frac{\left[\sum_{k=0}^{n-1}a_k\right]^2}{\int_0^1\left(\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}a_ka_jx^{k+j}\right)dx}$$

$$=\frac{\left[\sum_{k=0}^{n-1}a_k\right]^2}{\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}a_ka_j\int_0^1 x^{k+j}dx}=\frac{\left[\sum_{k=0}^{n-1}a_k\right]^2}{\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\frac{a_ka_j}{k+j+1}},$$

于是问题变成了计算$\frac{\left[\sum_{k=0}^{n-1}a_k\right]^2}{\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\frac{a_ka_j}{k+j+1}}=\frac{\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}a_ka_j}{\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\frac{a_ka_j}{k+j+1}}=\frac{a^TJa}{a^THa}$可能的最大值.

$$J=\begin{pmatrix}1&1&\cdots&1\\1&1&\cdots&1\\\vdots&\vdots&\ddots&\vdots\\1&1&\cdots&1\end{pmatrix},H=\begin{pmatrix}1&\frac{1}{2}&\cdots&\frac{1}{2n-1}\\\frac{1}{2}&\frac{1}{3}&\cdots&\frac{1}{2n}\\\vdots&\vdots&\ddots&\vdots\\\frac{1}{n}&\frac{1}{n+1}&\cdots&\frac{1}{2n-1}\end{pmatrix},H\text{叫做}hilbert\text{矩阵}$$

上面已经蕴含了H正定的经典证明，这里顺便给出行列式的方法.

使用$cauchy$行列式(百度百科搜索,记忆,读证明,直接使用)

$$\left|\begin{pmatrix}\frac{1}{a_1+b_1}&\frac{1}{a_1+b_2}&\cdots&\frac{1}{a_1+b_n}\\\frac{1}{a_2+b_1}&\frac{1}{a_2+b_2}&\cdots&\frac{1}{a_2+b_n}\\\vdots&\vdots&\ddots&\vdots\\\frac{1}{a_n+b_1}&\frac{1}{a_n+b_2}&\cdots&\frac{1}{a_n+b_n}\end{pmatrix}\right|=\frac{\prod\limits_{1\le i<j\le n}\left(a_i-a_j\right)\prod\limits_{1\le i<j\le n}\left(b_i-b_j\right)}{\prod\limits_{1\le i,j\le n}\left(a_i+b_j\right)}$$

对H来说,$a_i=i+1,b_j=j$,因此$|H|=\dfrac{\prod\limits_{1\le i<j\le n}(j-i)\prod\limits_{1\le i<j\le n}(j-i)}{\prod\limits_{1\le i,j\le n}(i+1+j)}>0,$

所有顺序主子式同理，所以H是正定的.

回到原题,$\dfrac{a^TJa}{a^THa}\le\lambda$恒成立,$\lambda$的最小值是?

$a^TJa\le\lambda a^THa\Leftrightarrow a^T[\lambda H-J]a\ge 0,\forall a,$求$\lambda$最小值.

$\Leftrightarrow\lambda H-J$半正定,求$\lambda$最小值.

当$\lambda H-J$顺序主子式非负. 用1_n表示元素全为1的n维列向量

$|\lambda H-J|=|H||\lambda E-H^{-1}J|=|H||\lambda E-H^{-1}1_n1_n^T|=|H||\lambda E-1_n^TH^{-1}1_n|$

这里用到了经典结论$|E-AB|=|E-BA|$,这里A,B可以不是方阵

$|\lambda H-J|=|H|\left(\lambda-1_n^TH^{-1}1_n\right)$,$1_n^TH^{-1}1_n$是$H^{-1}$所有元素之和, 因此要保证

$\lambda H-J$顺序主子式非负, 必须要有$\lambda\geq 1_n^TH^{-1}1_n$,

当$\lambda>1_n^TH^{-1}1_n$,此时$\lambda H-J$所有顺序主子式>0, 因此$\lambda H-J$正定.

$a^T\left(\lambda H-J\right)a>0,\forall a\neq 0$,再令$\lambda\to 1_n^TH^{-1}1_n$,由连续性,$a^T\left(\left(1_n^TH^{-1}1_n\right)H-J\right)a\geq 0$

故最好的(最小的)$\lambda=1_n^TH^{-1}1_n$

结论：$hilbert$矩阵逆矩阵元素之和是n^2(经典处理手法)

$$J=\begin{pmatrix}1&1&\cdots&1\\1&1&\cdots&1\\\vdots&\vdots&\ddots&\vdots\\1&1&\cdots&1\end{pmatrix},H=\begin{pmatrix}1&\frac{1}{2}&\cdots&\frac{1}{2n-1}\\\frac{1}{2}&\frac{1}{3}&\cdots&\frac{1}{2n}\\\vdots&\vdots&\ddots&\vdots\\\frac{1}{n}&\frac{1}{n+1}&\cdots&\frac{1}{2n-1}\end{pmatrix}$$

$$\frac{i}{i+j-1}+\frac{j-1}{i+j-1}=1,\text{取}A=\begin{pmatrix}1&0&\cdots&0\\0&2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&n\end{pmatrix},B=\begin{pmatrix}0&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&n-1\end{pmatrix}$$

$$AH+HB=J,1_n^TH^{-1}1_n=tr\left(1_n^TH^{-1}1_n\right)=tr\left(H^{-1}1_n1_n^T\right)=tr\left(H^{-1}J\right)$$

$$=tr\left(H^{-1}AH+B\right)=tr\left(A\right)+tr\left(B\right)=\frac{n(n+1)}{2}+\frac{n(n-1)}{2}=n^2.$$

因此我们完成了证明.

第八届数学类预赛压轴，本题颇具技巧. 不容易想到

f,g是$[0,1]$递增函数,$0 \le f,g \le 1$, 且$\int_0^1 f-gdx=0$, 证明:$\int_0^1 |f-g|dx \le \frac{1}{2}$

本题是标准的实分析思想.

证明：

注意：答案的阶梯函数逼近只是想强行数分化, 变成黎曼积分,
但是阶梯函数逼近本身也属于实分析, 所以我们索性就在勒贝格积分框架下解决.

记$h(x)=f(x)-g(x)$

考虑$I_1=\{x\in[0,1]: f(x)>g(x)\}, I_2=\{x\in[0,1]: f(x)\le g(x)\}$

故有$\int_{I_1} hdx=\int_{I_2} -hdx, \int_0^1 |f-g|dx = \int_{I_1} hdx+\int_{I_2} -hdx=2\int_{I_1} hdx$

因此需要证明$4\int_{I_1} hdx \le 1$.

(核心步骤, 想到就解决, 想不到就做不出),$(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)\ge 4, x,y>0$

$$4\int_{I_1} hdx \le \int_{I_1} hdx\left[\frac{1}{|I_1|}+\frac{1}{|I_2|}\right]=\frac{\int_{I_1} hdx}{|I_1|}+\frac{\int_{I_1} hdx}{|I_2|}=\frac{\int_{I_1} hdx}{|I_1|}+\frac{\int_{I_2} -hdx}{|I_2|}\le \sup h-\inf h\le 1$$

证毕!

$putnam$,阿里巴巴

$m>1, a\in\mathbb{R}, n\in\mathbb{N}$,如果$\int_{-\infty}^{+\infty}\left(\sum_{j=1}^{n}\frac{1}{1+\left|x-a_j\right|^m}\right)^2 dx\le n^{a}$, 证明:

存在常数$C(m)>0$,使得$\sum_{i,j=1}^{n}\left[1+\left|a_i-a_j\right|^m\right]\ge C(m)n^{(2-a)(m+2)}$

证明:

$$\int_{-\infty}^{+\infty}\left(\sum_{j=1}^{n}\frac{1}{1+\left|x-a_j\right|^m}\right)^2 dx=\int_{-\infty}^{+\infty}\left(\sum_{j=1}^{n}\frac{1}{1+\left|x-a_j\right|^m}\right)\left(\sum_{i=1}^{n}\frac{1}{1+\left|x-a_i\right|^m}\right)dx$$

$$=\int_{-\infty}^{+\infty}\left(\sum_{i,j=1}^{n}\frac{1}{1+\left|x-a_j\right|^m}\frac{1}{1+\left|x-a_i\right|^m}\right)dx=\sum_{i,j=1}^{n}\int_{-\infty}^{+\infty}\frac{1}{1+\left|x+a_i-a_j\right|^m}\frac{1}{1+\left|x\right|^m}dx$$

考虑$f(y)=\int_{-\infty}^{+\infty}\frac{1}{1+\left|x+y\right|^m}\frac{1}{1+\left|x\right|^m}dx$是关于$y$的偶函数, 因此

$$f(y)\ge\int_0^1\frac{1}{1+\left|y+1\right|^m}\frac{1}{1+\left|x\right|^m}dx\ge\frac{1}{2}\int_0^1\frac{1}{1+\left|y+1\right|^m}dx=\frac{1}{2}\frac{1}{1+\left|y+1\right|^m}\ge\frac{C_1(m)}{1+\left|y\right|^m}$$

因此$n^a\ge\sum_{i,j=1}^{n}\frac{C_1(m)}{1+\left|a_i-a_j\right|^m}\Rightarrow\sum_{i,j=1}^{n}\frac{1}{1+\left|a_i-a_j\right|^m}\le C_2(m)n^a$.

非数学专业同学到这一步就够了,

$$\sum_{i,j=1}^{n}\frac{1}{1+\left|a_i-a_j\right|^m}\le C_2(m)n^a$$, 找常数 $C(m)>0$, 使得 $\sum_{i,j=1}^{n}\left[1+\left|a_i-a_j\right|^m\right]\ge C(m)n^{(2-a)(m+2)}$

(不妨设 $0=a_1\le a_2\le a_3\ldots\le a_n$, 记 b_q 为落入 $[q,q+1)$ 里面 $\{a_i\}_{i=1}^n$ 的元素个数, $q\ge 0, q\in\mathbb{N}$,

$\sum_{q\ge 0}b_q=n$, 设 b_q 有 k 个非 0, 估计 k 有多少.

$$C_2(m)n^a\ge\sum_{i,j=1}^{n}\frac{1}{1+\left|a_i-a_j\right|^m}\ge\sum_{q,q'}\frac{b_q b_{q'}}{1+\max\{q+1-q',q'+1-q\}^m}$$

$$\ge\sum_{q=q'}\frac{b_q^2}{2}=\sum_q\frac{b_q^2}{2}=\frac{\sum_{b_q\ne 0}1\sum_q b_q^2}{2k}\ge\frac{\left(\sum_{b_q\ne 0}b_q\right)^2}{2k}=\frac{n^2}{2k}\Rightarrow k\ge C_3(m)n^{2-a}$$

接下来估计我们需要的结果.

$$\sum_{i,j=1}^{n}\left|a_i-a_j\right|^m\ge C_4(m)\sum_{q,q'}b_q b_{q'}\left|q-q'\right|^m\ge C_4(m)\sum_{q,q',b_q\ne 0,b_{q'}\ne 0}\left|q-q'\right|^m$$

设 $b_{q_i}\ne 0$, 不妨设 q_i 严格递增, 所以 $\left|q_i-q_j\right|\ge\left|i-j\right|$

$$C_4(m)\sum_{q,q',b_q\ne 0,b_{q'}\ne 0}\left|q-q'\right|^m=C_4(m)\sum_{j=1}^{k}\sum_{i=1}^{k}\left|q_i-q_j\right|^m\ge C_4(m)\sum_{j=1}^{k}\sum_{i=1}^{k}\left|i-j\right|^m$$)

括号内的步骤没兴趣可以省略.接下来的这一步, 所有人都需要掌握

只需证明 $\sum_{j=1}^{k}\sum_{i=1}^{k}\left|i-j\right|^m$ 是 k^{m+2} 的量, $k\to+\infty$

计算 $\lim_{k\to\infty}\frac{\sum_{j=1}^{k}\sum_{i=1}^{k}\left|i-j\right|^m}{k^{m+2}}=\frac{2}{(m+2)(m+1)}$ (留作习题), 于是我们完成了证明.