2nd Int'l Conf. on Electrical Engineering and Information & Communication Technology (ICEEICT) 2015
Jahangirnagar University, Dhaka-1342, Bangladesh, 21-23 May 2015

# Safe Labeling of Graphs with Minimum Span

Umma Habiba[†], Md. Saidur Rahman[α], Shah Hasnat Lamia[†], Tahmima Chowdhury [†]
[†] Department of Computer Science and Engineering,
Military Institute of Science and Technology, Dhaka-1216, Bangladesh
[α] Department of Computer Science and Engineering,
Bangladesh University of Engineering and Technology (BUET)
Dhaka-1000, Bangladesh.
Email: ummahabibaema@yahoo.com, saidurrahman@cse.buet.ac.bd,
lamia.mist@ovi.com, tahmima91@gmail.com

***Abstract*—Let $G$ be a graph of $n$ vertices and $k$ be a positive integer. We wish to label the vertices of $G$ with positive integers such that each vertex receives a distinct integer and the difference of the labels of two adjacent vertices in $G$ is at least $k$. We call such a labeling of $G$ a $k$-safe labeling of $G$. We call the range from the smallest to the largest integers assigned to the vertices of $G$ in a $k$-safe labeling the span of the $k$-safe labeling. The $k$-safe labeling problem asks to find a $k$-safe labeling of a graph with the minimum span. In this paper we show that the $k$-safe labeling problem is NP-hard. We also give upper bounds on $k$ -safe labelings of trees, bipartite graphs, cycles and cactus graphs. Our proofs for upper bounds lead to linear algorithms for finding those labelings.**

**Keywords: graph labeling, safe labeling, minimum span, cactus graph.**

## I. INTRODUCTION

Graph labeling is an interesting research topic in graph theory which concerns the assignment of values to the edges and/or vertices of a graph such that the assignments meet some conditions in the graph [1]. Graph labeling has numerous practical applications in computer science and engineering [2]. In this paper we consider the following graph labeling problem. Let $G$ be a graph of $n$ vertices and $k$ be a positive integer. We wish to label the vertices of $G$ with positive integers such that each vertex receives a distinct integer and the difference of the labels of two adjacent vertices is at least $k$. We call such a labeling of $G$ a *$k$-safe labeling* of $G$. We call the range from the largest to the smallest integers assigned to the vertices of $G$ in a $k$-safe labeling the *span* of the $k$-safe labeling. Let $I_s$ and $I_l$ be the smallest and the largest integer, respectively, used for a $k$-safe labeling of a graph $G$. Then the span of the $k$-safe labeling of $G$ is $I_l - I_s + 1$.

The *$k$-safe labeling problem* asks to find a $k$-safe labeling of a graph with the minimum span. In Figure 1 we show three $k$-safe labelings of the same graph. The spans of the $k$-safe labelings in Figure 1(a), (b) and (c) are 30, 13 and 10, respectively. In fact the $k$-safe labeling in Figure 1(c) is a $k$-safe labeling with the minimum span. For a complete graph of $n$ vertices the span for a $k$-safe labeling is at least $(n-1)k+1$. (See a $k$-safe labeling of $K_5$ in Figure 2(a) with span 21 for $k$=5.). Again for a complete bipartite graph $K_{p,q}$ the span for a $k$-safe labeling is at least $p+q+k-1 = n+k-1$, since $p+q = n$. (See a $k$-safe labeling of $K_{3,3}$ in Figure 2(b) with span 10 for $k$=5.). Thus one may expect to find $k$-safe labelings of sparse graphs using narrow spans.

Reading the problem of $k$-safe labeling one may think of the anti-bandwidth problem [3], [4] and the graph coloring problem [5], [6]. The anti-bandwidth problem asks to label the vertices of a graph on $n$ vertices with the integers $0, 1, 2, \cdots, n$ in such a way that each vertex receives a distinct label and the minimum difference of the labels of adjacent vertices is maximized. The possible maximum value of the minimum difference is called the antibandwidth of the graph. Figure 3(a) illustrates an antibandwidth labeling of a binary tree. But in our safe labeling problem, we are allowed to label the vertices of a graph of $n$ vertices with labels possibly greater than $n$ such that each vertex receives a distinct label, the difference between the labels of two adjacent vertices is $k$ or more for a given $k$ and the range of the labels used is as small as possible. Note that if the minimum span for a $k$-safe labeling is $n$, the

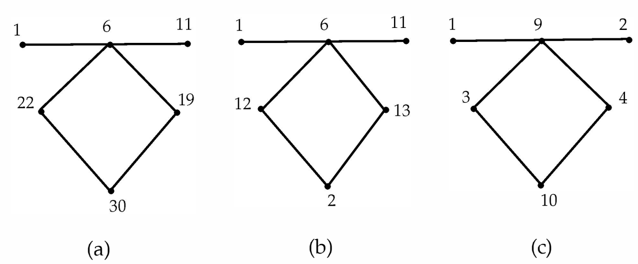



Fig. 1. For $k = 5$ (a) labeling of a graph with span 30, (b) labeling of a graph with span 13 and (c) labeling of a graph with minimum span 10.

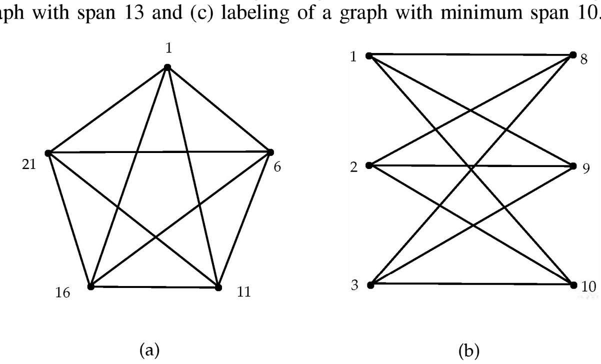



Fig. 2. For $k = 5$ (a) A $k$-safe labeling of $K_5$ with span 21, (b) a $k$-safe labeling of $K_{3,3}$ with span 10.

number of vertices, then $k$ is equal to the antibandwidth of the graph. In a vertex coloring problem, the vertices of the graph are colored in such a way that two adjacent vertices receive different colors. The objective is to minimize the number of colors used. Colors can be represented by integers and hence the objective is to minimize the span. But here same integer can be used for two or more vertices if they are not adjacent. Figure 3(b) illustrates a vertex coloring of a graph where colors are represented by integers.

The study of $k$-safe labeling is motivated by radio frequency assignment [7], [8] of transmitters in interference range. In a radio frequency allocation problem, each transmitters are assigned a frequency so that no two transmitters are assigned the same frequency for ensuring their identity. Furthermore, two transmitters within interference distance must be given two frequencies having a big difference. This problem cannot be properly addressed by antibandwidth labeling or vertex coloring.

The remaining of the paper is organized as follows. In Section 2 we give some definitions. Section 3 provides a proof for the hardness of the $k$-safe labeling problem. Section 4 deals with $k$-safe labelings of various classes of sparse graphs. Finally, Section 5 concludes the paper.

## II. Preliminaries

In this section we give some terminologies and known results that will be used throughout the paper.

Let $G = (V, E)$ be a simple graph. A $path$ is a sequence $v_1, v_2, \cdots, v_q$ of distinct vertices (except possibly $v_1$ and $v_q$) such that $v_1, \cdots v_q \in V$ and $(v_{i-1}, v_i) \in E$ for all $2 \le i \le q$. The $length$ of the path is the number of edges on the path. A path $v_1, v_2, \cdots, v_q$ is $closed$ if $v_1 = v_q$. A closed path containing at least one edge is called a *cycle*. An $odd\ cycle$ is a cycle with odd length, that is, with an odd number of edges. An $even\ cycle$ is a cycle with even length, that is, with an even number of edges.

The *contraction* of an edge $(u, v)$ of a graph $G$ is the operation of deleting the edge $(u, v)$ and identifying the two vertices $u$ and $v$. Thus to contract the edge $(u, v)$, we delete the two vertices $u$ and $v$ and add a new vertex $w$ where all edges incident to $u$ and $v$ in $G$ other than the edge $(u, v)$ are made incident to $w$. We call $w$ an *identified* vertex in the resulting graph. When we use contraction in a simple graph replace multi-edges by a single edge after contraction of each edge.

A graph $G$ is $connected$ if there is a path between every pair of vertices in $G$. A connected graph $G$ is $biconnected$ if at least two vertices of $G$ are needed to be deleted to make $G$ disconnected or a single vertex graph. A maximal biconnected subgraph of a connected graph $G$ is called a $biconnected\ component$ of $G$. A $cactus$ graph is a connected graph in which every biconnected component is a cycle. Figure 7 shows a cactus graph with 26 vertices. A $tree$ is a connected graph containing no cycle.

Let $G = (V, E)$ be a graph. A subset $V'$ of $V$ is called *independent* if the vertices in $V'$ are pair-wise non-adjacent. If the vertex set $V$ of $G$ can be partitioned into two nonempty subsets $P$ and $Q$ such that both $P$ and $Q$ are independent sets then $G$ is called a *bipartite graph*. The partition $V = P \cup Q$ is called a *bipartition* of the bipartite graph $G$. Note that $P \cup Q = V$ and $P \cap Q = \emptyset$. The following lemmas on bipartite graphs are known [9].

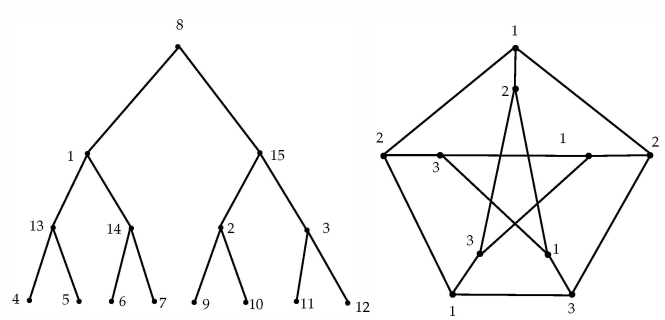
**Lemma 2.1.** *A graph $G$ is bipartite if and only if $G$ has no odd cycle.*

**Lemma 2.2.** *A graph $G$ is bipartite if and only if the vertices of $G$ can be colored by two colors such that every pair of adjacent vertices receive different colors.*

The characterization of bipartite graphs in Lemma 2.2, leads to a linear-time algorithm to determine whether a graph is bipartite or not, and also compute a bipartition of a bipartite graph, as follows. Color the vertices of a graph with two colors using DFS or BFS by giving different colors to each pair of adjacent vertices. If a conflict free coloring is obtained, then the graph is bipartite. The coloring of the vertices induces a bipartition if the graph is bipartite.

## III. Hardness of Safe Labeling Problem

In this section we prove the NP-hardness of the $k$-safe labeling problem using a trivial reduction of the antibandwidth problem [10] to the $k$-safe labeling problem. We first give the




(a) (b)

Fig. 3. (a) A antibandwidth labeling of a tree, and (b) a vetrtex coloring of a graph.

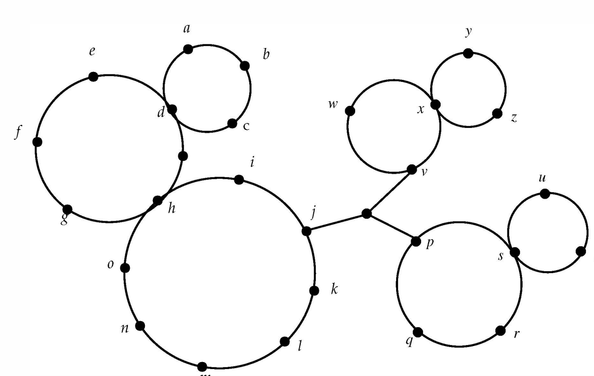



Fig. 4. A cactus graph $G$ with 26 vertices.

definition of the decision version of both the $k$-safe labeling problem and the antibandwidth problem.

**$k$-safe labeling problem:** *Let $G$ be a graph of $n$ vertices and let $k$ be a positive integer. For a given positive integer $s$, does $G$ admits a $k$-safe labeling with the span at most $s$?*

**Antibandwidth problem:** *Let $G$ be a graph of $n$ vertices. For a given integer $k$, does $G$ admit a labeling of vertices with distinct labeling ranging from 1 to $n$ such that the minimum difference of the labels of two adjacent vertices is at least $k$?*

The antibandwidth problem can be rephrased as follows as a $k$-safe labeling problem: Let $G$ be a graph of $n$ vertices and let $k$ be a positive integer. Does $G$ admit a $k$-safe labeling with the span exactly $n$?

Note that the span can not be less than $n$, since there are $n$ vertices and each vertex receives a distinct label in a $k$-safe labeling. Thus one can solve an antibandwidth problem using a solver of $k$-safe labeling problem in polynomial time. Since the antibandwidth problem is NP-hard [10], the $k$-safe labeling problem is NP-hard.

## IV. $k$-SAFE LABELINGS OF SPARSE GRAPHS

In this section we study $k$-safe labelings for some sparse graphs. We give upper bound on the span of those graphs.

We first consider $k$-safe labelings of bipartite graphs since it plays important roles in studing $k$-safe labelings of trees, cycles and cactus graphs. We have the following theorem on the $k$-safe labelings of bipartite graphs.

**Theorem 4.1.** *Let $G = (V, E)$ be a bipartite graph of $n$ vertices. Then $G$ admits a $k$-safe labeling with a span $n+k-1$.*

*Proof:* Let $V = P \cup Q$ is a bipartition of $G$. Let $P = \{u_1, u_2, \cdots, u_p\}$ and $Q = \{v_1, v_2, \cdots, v_q\}$. Then $p + q = n$. We label the vertices in $P$ from the set of consecutive integers $\{1, 2, 3, \cdots, p\}$ and the vertices in $Q$ from the set of consecutive integers $\{p+k, p+k+1, \cdots, p+k+q-1\}$. Since both $P$ and $Q$ are sets of independent vertices, the minimum difference the labels of two adjacent vertices is at least $p + k - p = k$ and the span is $p + k + q - 1 = n + k - 1$. ■

It is interesting to observe that the trivial span $n + k - 1$ is optimal for bipartite graphs, since a complete bipartite graph does not admit a $k$-safe labeling with a span smaller than $n + k - 1$.

We have the following corollary of Theorem 4.1 on $k$-safe labelings of trees.

**Corollary 4.2.** *Every tree of $n$ vertices admits a $k$-safe labeling with span $n + k - 1$. Furthermore, such a labeling can be computed in linear time.*

*Proof:* Since a tree has no cycle, by Lemma 2.1 a tree is a bipartite graph. Hence every tree of $n$ vertices has a $k$-safe labeling with span $n + k - 1$ by Theorem 4.1. We only prove our claim on time complexity. Let $T$ be a tree. Two partite sets $P$ and $Q$ can be obtained in linear time by 2-coloring of vertices in $T$ using a DFS or BFS. Then the vertices can be labeled as described in the proof of Theorem 4.1 in linear time. ■

We have the following theorem on $k$-safe labeling of cycles.

**Theorem 4.3.** *Let $G$ be a cycle of $n$ vertices. If $G$ is an even cycle then $k$-safe labeling of $G$ can be done with the span $n + k - 1$ and if $G$ is an odd cycle then $k$-safe labeling of $G$ can be done with the span $n + 2k - 2$.*

*Proof:* Since an even cycle is a bipartite graph by Lemma 2.1, the claim on every even cycle is immediate from Theorem 4.3. We only prove the claim on odd cycles. Let $G$ be an odd cycle of $n$ vertices. We can trivially label the vertices of $G$ of $n = 3$ satisfying the span. We thus assume that $n > 3$. We contract an edge $e$ of $G$ and obtain a cycle graph $G'$ of $n - 1$ vertices. (See Figure 5(a) and (b); the even cycle in Figure 5(b) is obtained from the odd cycle in Figure 5(a) by contracting the marked edge (g,a).) Note that two vertices of $G$ is unified as one vertex in $G'$ after contraction. Since $G$ is an odd cycle, $G'$ is an even cycle. By Theorem 4.3 $G'$ has a $k$-safe labeling using the lowest integer 1 and the highest integer $n-1+k-1$. (See Figure 5(b).) We now reinstate the contracted edge $e$ to get $G$. We use the label of the unified vertex in $G'$ for one of the two end vertices of $e$ and we label the other end vertices $e$ with label $n-1+k-1+k = n+2k-2$. (See Figure 5(c).) Thus the span becomes $n - 2k - 2$. ■

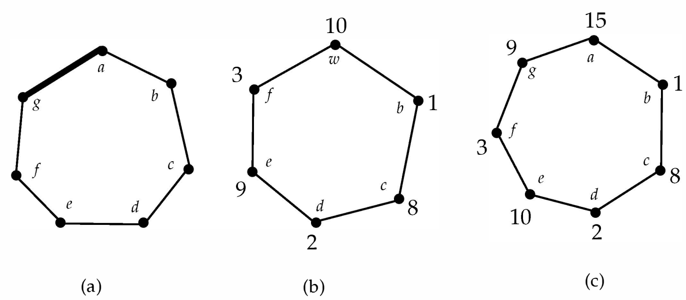



Fig. 5. (a) An odd cycle $G$ of $n = 7$ vertices, (b) an even cycle $G'$ (obtained from $G$) of 6 vertices with $k$-safe labeling with $k = 5$ and span $n - 1 + k - 1 = 10$, and (c) a $k$-safe labeling of $G$ with $k = 5$ and span $n + 2k - 2 = 15$.

We next deal with $k$-safe labelings of cactus graphs and give an upper bound on the span for $k$-safe labelings of cactus graphs. We have the following theorem on $k$-safe labeling of cactus graphs.

**Theorem 4.4.** *Let $G$ be a cactus graph of $n$ vertices. Then $G$ admits a $k$-safe labeling with span $n + 2k - 2$, and such a labeling can be computed in linear time.*

*Proof:* Let $G$ be a cactus graph. If all the cycles in $G$ are even cycles, then $G$ is a bipartite graph by Lemma 2.1, and hence $G$ has a $k$-safe labeling with the span $n + k - 1$ by Theorem 4.4. We thus assume that $G$ has an odd cycle. Let $c$ be the number of odd cycles in $G$. We obtain a graph $G'$ from $G$ by contracting an edge on each odd cycle, as illustrated in Figure 6(a) and (b). (We select exactly one edge from an odd cycle for contraction such that selected edges are vertex disjoint, as illustrated in Figure 6(a). It is not difficult to show that such a set of edges exists and can be selected in linear time.). Since two cycles in $G$ can share at most one vertex, every odd cycle in $G$ becomes an even cycle in $G'$ and $G'$ has no odd cycle. Therefore by Lemma 2.1, $G'$ is a bipartite graph of $n - c$ vertices. We obtain a $k$-safe labeling of $G'$ using the lowest integer 1 and the highest integer $n - c + k - 1$, as illustrated in Figure 6(b). We reinstate the contracted edges and obtain graph $G$ and use the label of the unified vertices of a cycle in $G'$ to label of one end vertex of the contracted vertex of the cycle. Then there $c$ unlabeled vertices in $G$. We label those vertices from the set of consecutive integers $\{n - c+k-1+k, n-c+k-1+k+1, \cdots, n-c+k-1+k+c-1\}$, as illustrated in Figure 6(c). Clearly the labeling above is a $k$-safe labeling and the span is $n-c+k-1+k+c-1 = n+2k-2$. Clearly the labeling above can be computed in linear time. ■

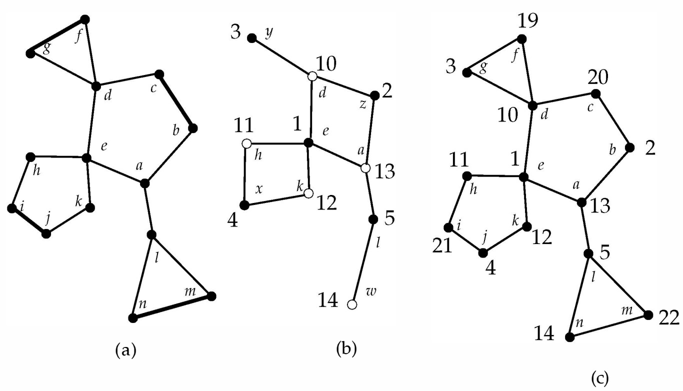



Fig. 6. (a) A cactus graph $G$ of 14 vertices where each to be contracted on an odd cycle is drawn by thick lines, (b) a $k$-safe labeling (with $k = 5$ and span $n - c + k - 1$= 14) of the bipartite graph $G'$ obtained by contracting an edge from each odd cycle of $G$, and (c) a $k$-safe labeling of graph $G$ with $k = 5$ and span $n + 2k - 2 = 22$.

## V. Conclusion

In this paper, we have introduced a new graph labeling that is a $k$-safe labeling. We have proved that the $k$-safe labeling problem is NP-hard and have also presented upper bounds on $k$-safe labelings of bipartite graphs, cycles, trees and cactus graphs. Our bound for bipartite graphs is optimal. There are several natural open problems raised by our work, as follows.

(a) We have mentioned lower bounds on $k$-safe labelings for complete graphs and complete bipartite graphs, which are trivial. It is interesting to find non-trivial lower bounds for general graphs and other subclasses of graphs.

(b) Our upper bounds for trees and cactus graphs are not tight. Finding more tight upper bounds is an interesting future work.

(c) Proving NP-hardness for subclasses of graphs can also be an interesting topic for future works.

## VI. Acknowledgement


This work was done when the second author was with the Dept. of CSE, MIST, Bangladesh in sabbatical leave. The second author thanks MIST authority for providing necessary support.


## References


[1] J. A. Gallian, "A dynamic survey of graph labeling," *The Electronic Journal of Combinatorics*, vol. 16, pp. 1–308, 2013.

[2] J. L. Gross, J. Yellen, and P. Zhang, Eds., *Handbook of graph theory*. 2nd edition, CRC Press, 2014.

[3] A. Raspaud, H. Schroder, O. Sykora, L. Török, and I. Vrťo, "Antibandwidth and cyclic antibandwidth of meshes and hypercubes," *Discrete Mathematics*, vol. 309, no. 11, pp. 3541–3552, 2009.

[4] M. S. Rahaman, T. A. Eshan, S. A. Abdullah, and M. S. Rahman, "Antibandwidth problem for itchy caterpillars," *In: Proc. of International Conference on Informatics, Electronics & Vision (ICIEV), 2014*, pp. 1–6, 2014.

[5] Y. Hu, S. G. Kobourov, and S. Veeramoni, "On maximum differential graph coloring," in *Proc. of 18th International Symposium on Graph Drawing, GD 2010, Konstanz, Germany, September 21-24, 2010. Revised Selected Papers*, 2010, pp. 274–286.

[6] D. Bozdag, Ü. V. Çatalyürek, A. H. Gebremedhin, F. Manne, E. G. Boman, and F. Özgüner, "A parallel distance-2 graph coloring algorithm for distributed memory computers," in *Proc. of First International conference on High Performance Computing and Communications, HPCC 2005, Sorrento, Italy, September 21-23, 2005, Proceedings*, 2005, pp. 796–806.

[7] M. Priseler and A. Reichman, "Time and frequency resource allocation using graph theory in ofdma wireless mesh networks," *Systemics, Cybernetics and Informatics*, vol. 12, no. 1, pp. 52–57, 2014.

[8] J. Riihijärvi, M. Petrova, and P. Mähönen, "Frequency allocation for WLANS using graph colouring techniques," in *2nd International Conference on Wireless on Demand Network Systems and Service (WONS 2005), 19-21 January 2005, St. Moritz, Switzerland*, 2005, pp. 216–222. [Online]. Available: http://doi.ieeecomputersociety.org/10.1109/WONS.2005.19

[9] D. B. West, *Introduction to Graph Theory*. 2nd Edition, Prentice Hall, (2001).

[10] J. Y. Leung, O. Vornberger, and J. D. Witthoff, "On some variants of the bandwidth minimization problem," *SIAM J. Comput.*, vol. 13, no. 3, pp. 650–667, 1984. [Online]. Available: http://dx.doi.org/10.1137/0213040