

Penn State CSE
Spring 2023
Theory Qualifying Exam
02/24/2023
Time Limit: 170 Minutes

Name: _____

This exam contains 14 pages (including this cover page, double-sided) and 6 questions.
Total of points is 60.
This will contribute to 40 % of your total grade

Grade Table (for grader use only)

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. (10 points) (**Induction**) Prove that $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9 for all positive integers n .

2. (10 points) (**Logic**) Formalize the following sentences in propositional logic using the provided phrase associated with each prime proposition.

(a) (5 points) If it is sunny tomorrow, then I will play golf, provided that I am relaxed.

- s It is sunny tomorrow.
- g I will play golf.
- r I am relaxed.

- (b) (5 points) If I pass the qualifying exam, I will focus on my research; otherwise, I will spend more time studying theory.
- q I pass the qualifying exam.
 - r I will focus on my research.
 - t I will spend more time studying theory.

3. (10 points) (**Modular arithmetic**) Let $m > 2$ be a positive integer. Suppose there exist *prime* numbers p and q such that $m = pq$. For an $a \in \{1, 2, \dots, m-1\}$ such that $\gcd(a, m) = 1$, let r be the *minimum* integer satisfying $a^r = 1 \pmod{m}$, and assume r is even. Answer the following questions.

- (a) (3 points) Prove that it is impossible to have m divides $a^{r/2} - 1$.

(b) (7 points) Prove that as long as m does not divide $a^{r/2} - 1$, it holds that

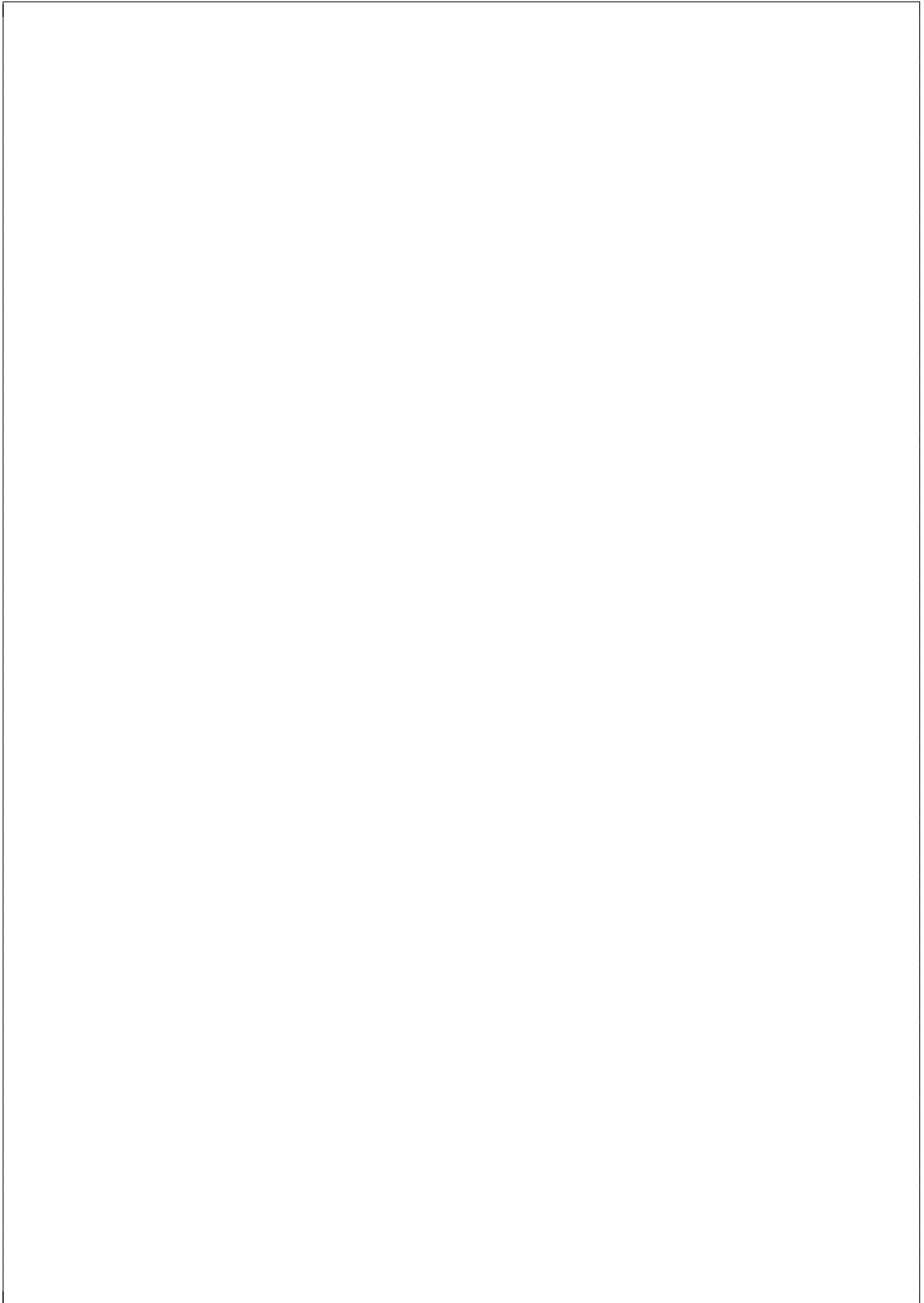
$$\gcd(a^{r/2} + 1, m) = p \text{ or } q.$$

(Hint: consider the fact that $a^r - 1 = (a^{r/2} + 1)(a^{r/2} - 1)$)

4. (10 points) (**Graph Algorithm**)

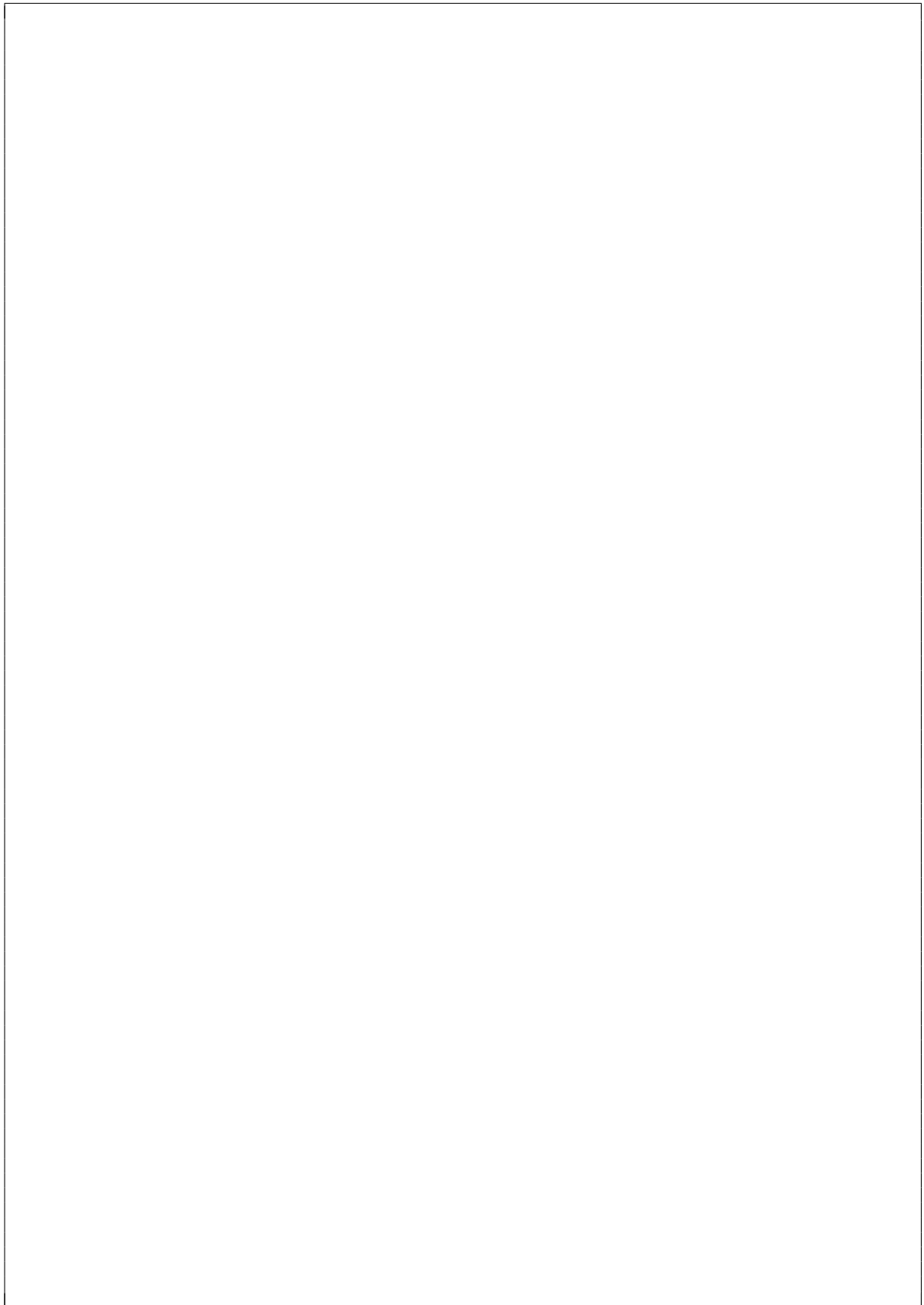
Given an undirected graph G , determine whether it is bipartite using DFS.

A bipartite graph G is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V .



5. (10 points) (**Divide and Conquer**)

You have 9 balls, equally big, equally heavy - except for one, which is a little heavier. You are also given a pan balance with no weights. How would you identify the heavier ball if you could use the pan balance scale only twice?



6. (10 points) (**Dynamic Program**)

The shortest common supersequence (SCS) of two sequences x and y is the shortest sequence which has both x and y as subsequences. Given x of length m and y of length n , write $O(mn)$ time algorithm to find the SCS of x, y .

(a) (2 points) Define the subproblems for the DP algorithm.

(b) (2 points) Write the base case.

(c) (4 points) Define the recurrence relation.

(d) (2 points) Analyze the running time.