Question 1.

Say whether the following is true or false and support your answer by a proof:

$$(\exists m \in \mathbb{N})(\exists n \in \mathbb{N})(3m + 5n = 12)$$

Prove:

Since n and m are natural numbers and 3m + 5n =12, n can only take values of 1 and 2.

Then, if n = 2, 12 - 2n = 12 – 10 = 2 = 3m.

However, since 2 is not divisible by 3, there is no such m $\in \mathbb{N}$

If n = 1, 12 – n = 12 – 5 = 7 = 3m.

Similarly, since 7 is not divisible by 3, there is no such m $\in \mathbb{N}$

Therefore, the above proof shows the statement is false.

Question 2.

Say whether the following is true or false and support your answer by a proof:

The sum of any five consecutive integers is divisible by 5 (without remainder).

Prove:

Let n, n + 1, n + 2, n + 3, n + 4 be any five consecutive integers.

Then, the sum of these five consecutive integers is:

n + n + 1 + n + 2 + n +3 + n + 4 = 5n + 10 = 5(n + 2)

Therefore, this statement is correct.

Question 3

Say whether the following is true or false and support your answer by a proof:

For any integer n, the number $n^2 + n + 1$ is odd.

Prove:

Let's consider two cases.

If n is even, n = 2m, where m $\in \mathbb{Z}$, then:

$$(2m)^2 + 2m + 1 = 2(2m^2 + m) + 1$$

If n is odd and n = 2m + 1, where m $\in \mathbb{Z}$, then:

$$(2m + 1)^2 + 2m + 1 + 1 = 2(2m^2 + 3m + 1) + 1$$

For both cases, the number $n^2 + n + 1$ is odd.

Therefore, the above statement is true.

Question 4

Prove that every odd natural number is of one of the forms 4n + 1or 4n + 3, where n is an integer.

Prove:

Every odd natural number m can be written as 2k + 1, where k $\in \mathbb{Z}$

Then, let's consider two cases.

If k is even, then:

m = 2k +1 = 2(2z) + 1 = 4z + 1, where z $\in \mathbb{Z}$

If k is odd, then:

m = 2k + 1 = 2(2z + 1) + 1 = 4z + 3, where z $\in \mathbb{Z}$

For both cases, m is odd.

Therefore, we proved the above statement.

Question 5

Prove that for any integer *n*, at least one of the integers *n*, *n* + 2, *n* + 4 is divisible by 3.

Prove:

Let's consider two cases.

Case 1: If n is divisible by 3, then at least one of three integers is divisible by 3

Case 2: If n is not divisible by 3, then, by Euclidean algorithm, the reminder can only be 1 or 2.

This indicates that both n + 2 and n +4 are divisible by 3.

Hence, from both cases, at least one of the integers *n*, *n* + 2, *n* + 4 is divisible by 3.

Therefore, we proved the above statement.

Question 6

A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7

Prove:

Let n, n + 2, and n + 4 be primes and n >3. Then we consider two cases.

Let's consider two cases.

Case 1: If n is divisible by 3, then at least one of three integers is divisible by 3.

Case 2: If n is not divisible by 3, then, by Euclidean algorithm, the reminder can only be 1 or 2.

This indicates that both n + 2 and n +4 are divisible by 3.

Hence, from both cases, at least one of the integers *n*, *n* + 2, *n* + 4 is divisible by 3.

Therefore, n, n+2, and n+3 cannot all be primes, and the only prime triple is 3, 5, and 7.

Question 7

Prove that for any natural number *n*:

$$2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2 \qquad *$$

Prove by induction:

Step 1. For n = 1, 2 = $2^2 - 2$ = 2, which is correct.

Step 2. Assume * is correct, then let's adding 2^{n+1} to both side:

$$2 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1} = 2^{n+1} + 2^{n+1} - 2$$

Simplifying:

$$2 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1} = 2 \cdot 2^{n+1} - 2$$

Then we get:

$$2 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1} = 2^{n+2} - 2$$

This proves A (n + 1).

Question 8

Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$tends to limit *L* as *n*→∞, then for any fixed number *M*>0, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit *ML*

Prove:

Let $\varepsilon > 0$ be given, since sequence $\{a_n\}_{n=1}^{\infty}$tends to limit *L* as *n*→∞, we can find N such that for all n≥N:

$$|a_n - L| < \varepsilon$$

Then, multiply M to both side:

$$\mathrm{M} \cdot |a_n - L| < \mathrm{M} \cdot \varepsilon$$

This shows that $\{Ma_n\}_{n=1}^{\infty}$ tends to limit ML.

Question 9

Given a collection A_n, n=1, 2, ... of intervals of the real line, their intersection is defined to be

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid (\forall n)\ (x \in A_n)\}$$

Give an example of a family of intervals A_n, n=1, 2, ..., such that $A_{n+1} \subset A_n$ for all n and

$$\bigcap_{n=1}^{\infty} A_n = \phi$$

Prove that your example has the stated property.

Prove:

Suppose $A_n = (0, 1/n)$, then $A_1 = (0,1)$ and $\bigcap_{n=1}^{\infty} A_n \subseteq (0, 1)$.

However, as n → ∞, $\{1/n\}_{n=1}^{\infty}$ tends to 0, and we can always find $\frac{1}{n}$ which is smaller than x.

Therefore, x $\notin A_n$.

Therefore, x $\notin \bigcap_{n=1}^{\infty} A_n$, and $\bigcap_{n=1}^{\infty} A_n = \phi$

Question 10

Give an example of a family of intervals A_n, n =1, 2, ..., such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

Prove:

Suppose $A_n = [0, 1/n)$, then $A_1 = [0,1)$ and $\bigcap_{n=1}^{\infty} A_n \subseteq [0, 1)$.

However, as n → ∞, $\{1/n\}_{n=1}^{\infty}$ tends to 0, and we can always find $\frac{1}{n}$ which is smaller than x.

Therefore, x can only be 0.

Therefore, $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number.