Noncooperative Finite Games: Nonzero-Sum

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Review: Matrix Games

- $A = \{a_{ij}\}$, each entry is an outcome
- ith row: strategies for P1, jth column: strategies for P2
- Target of P1: find i^*th row to minimize the outcomes

$$\bar{V}(A) \triangleq \max_j a_{i^*j} \leq \max_j a_{ij}, \quad i = 1, \ldots, m,$$

  - $V(A)$ -- loss ceiling of P1 (security level for his losses)
  - $row\ i^*$ -- security strategy for P1

- Target of P2: find j^*th column to maximize the outcomes

$$\underline{V}(A) \triangleq \min_i a_{ij^*} \geq \min_i a_{ij}, \quad j = 1, \ldots, n.$$

  - $\underline{V}(A)$ -- gain-floor of P2 (security level for his gains)
  - column j^* -- security strategy for P2

$$\begin{array}{cc} & \mathbf{P2} \\ \mathbf{P1} & \begin{array}{|c|c|c|c|} \hline 1 & 3 & 3 & -2 \\ \hline 0 & -1 & 2 & 1 \\ \hline -2 & 2 & 0 & 1 \\ \hline \end{array} \end{array}$$

Review: Saddle-point Equilibrium

- If the pair of inequalities:

$$a_{i^*j} \leq a_{i^*j^*} \leq a_{ij^*}$$

for all i and j, then the strategies {row i^*,column j^* } are said to constitute a **saddle-point equilibrium.** And the matrix game is said to have a **saddle point** in pure strategies.

$$\begin{array}{cc} & \textbf{P2} \\ \textbf{P1} & \begin{array}{|c|c|c|} \hline 4 & 0 & -1 \\ \hline 0 & -1 & 3 \\ \hline 1 & 2 & 1 \\ \hline \end{array} \end{array} \qquad \begin{array}{cc} & \textbf{P2} \\ \textbf{P1} & \begin{array}{|c|c|} \hline 3 & 1 \\ \hline -1 & 1 \\ \hline \end{array} \end{array}$$

Review: Mixed Strategies

- Key idea: enlarge the strategy spaces, allow the players to base their decisions on the outcome of random events.

  E.g. $\{row1, row2, row3\}$—pure strategies space

  $\{y_1, y_2, y_3\}$—a mixed strategy, where $y_1 + y_2 + y_3 = 1$

  $Y = \{(y_1, y_2, y_3\ ), (y_1', y_2', y_3')\ \dots\}$—the mixed strategy space of P1, comprised of all such probability distributions.

P2

	1	3	3	−2
P1	0	−1	2	1
	−2	2	0	1

Nash Equilibrium - Definition

- A pair of strategies {row i*, column j*} is said to constitute a noncooperative (Nash) equilibrium solution to a bimatrix game ($A = \{a_{ij}\}, B = \{b_{ij}\}$) if the following pair of inequalities is satisfied for all $i = 1, \dots, m$ and all $j = 1, \dots, n$:

$$a_{i^*j^*} \leq a_{ij^*}$$
$$b_{i^*j^*} \leq b_{i^*j}$$

- Furthermore, the pair ($a_{i^*j^*}$, $b_{i^*j^*}$) is known as a noncooperative (Nash) equilibrium outcome of the bimatrix game.
- Example: (1,2) and (-1,0) are the equilibrium outcomes.

$$A = \begin{array}{c} \mathbf{P2} \\ \begin{array}{|c|c|} \hline \boxed{1} & 0 \\ \hline 2 & -1 \\ \hline \end{array} \end{array} \ \mathbf{P1} \qquad B = \begin{array}{c} \mathbf{P2} \\ \begin{array}{|c|c|} \hline \boxed{2} & 3 \\ \hline 1 & 0 \\ \hline \end{array} \end{array} \ \mathbf{P1}.$$

Nash Equilibrium - Definition

- A pair of strategies $\{row\ i_1, column\ j\}$ is said to be better than another pair of strategies $\{row\ i_2$ $columnj_2\}$ if $a_{i_1j_1} \leq a_{i_2j_2}, b_{i_1j_1} \leq b_{i_2j_2}$, and if at least one of these inequalities is strict.

- Nash equilibrium strategy pair is said to be admissible if there exists no better Nash equilibrium strategy pair.

$$A = \begin{array}{c} \mathbf{P2} \\ \begin{bmatrix} \boxed{1} & 0 \\ 2 & -1 \end{bmatrix} \end{array} \mathbf{P1} \qquad B = \begin{array}{c} \mathbf{P2} \\ \begin{bmatrix} \boxed{2} & 3 \\ 1 & 0 \end{bmatrix} \end{array} \mathbf{P1}.$$

(-1,0)-- most “reasonable” noncooperative equilibrium

$$A = \begin{array}{c} \mathbf{P2} \\ \begin{bmatrix} \boxed{-2} & 1 \\ -1 & -1 \end{bmatrix} \end{array} \mathbf{P1}, \qquad B = \begin{array}{c} \mathbf{P2} \\ \begin{bmatrix} \boxed{-1} & 1 \\ 2 & -2 \end{bmatrix} \end{array} \mathbf{P1}$$

Admits two admissible Nash equilibrium, no clean choice

Nash Equilibrium - Definition

- If a bimatrix game admits more than one admissible Nash equilibrium solution, then the equilibrium outcome of the game becomes rather ill-define.

$$
\begin{array}{cc} & \text{P2} \\ A = & \begin{bmatrix} \boxed{1} & 0 \\ 2 & -1 \end{bmatrix} \end{array} \text{P1} \qquad \begin{array}{cc} & \text{P2} \\ B = & \begin{bmatrix} \boxed{2} & 3 \\ 1 & 0 \end{bmatrix} \end{array} \text{P1}.
$$

(-1,0)-- most “reasonable” noncooperative equilibrium

Dilemma!

$$
\begin{array}{cc} & \text{P2} \\ A = & \begin{bmatrix} \boxed{-2} & \boxed{1} \\ -1 & -1 \end{bmatrix} \end{array} \text{P1}, \qquad \begin{array}{cc} & \text{P2} \\ B = & \begin{bmatrix} \boxed{-1} & \boxed{1} \\ 2 & -2 \end{bmatrix} \end{array} \text{P1}
$$

Admits two admissible Nash equilibrium, no clean choice

Minimax Solution

- A pair of strategies {row i, column j} is known as a pair of minimax strategies for the players in a bimatrix game (A, B) if the former is a security strategy for P1 in the matrix game A, and the latter is a security strategy for P2 in the matrix game B. The corresponding security levels of the players are known as the minimax values of the bimatrix game.

- Minimax values of a bimatrix game are definitely not lower (in an ordered way) than the pair of values of any Nash equilibrium outcome.

Nash $\leq$ Minimax

$$A = \begin{array}{c} \mathbf{P2} \\ \begin{array}{|c|c|} \hline \boxed{1} & 0 \\ \hline 2 & -1 \\ \hline \end{array} \end{array} \ \mathbf{P1} \qquad B = \begin{array}{c} \mathbf{P2} \\ \begin{array}{|c|c|} \hline \boxed{2} & 3 \\ \hline 1 & 0 \\ \hline \end{array} \end{array} \ \mathbf{P1}.$$

Minimax Solution

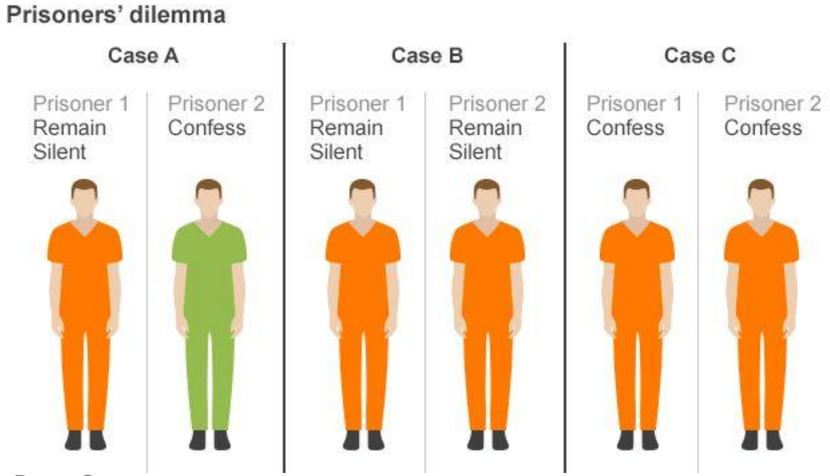
Prisoners' dilemma!

$$A = \begin{array}{c} \text{P2} \\ \begin{bmatrix} \boxed{1} & 0 \\ 2 & -1 \end{bmatrix} \end{array} \text{P1} \qquad B = \begin{array}{c} \text{P2} \\ \begin{bmatrix} \boxed{2} & 3 \\ 1 & 0 \end{bmatrix} \end{array} \text{P1}.$$

The security strategies of the players correspond to one of the Nash equilibrium solutions, but not to the admissible one.

$$A = \begin{array}{c} \text{P2} \\ \begin{bmatrix} 8 & 0 \\ 30 & 2 \end{bmatrix} \end{array} \text{P1}, \quad \begin{array}{l} \text{Confess} \\ \text{Remain silent} \end{array} \quad B = \begin{array}{c} \text{P2} \\ \begin{bmatrix} 8 & 30 \\ 0 & 2 \end{bmatrix} \end{array} \text{P1}.$$

Unique pair of Nash equilibrium strategies also the security of the players.



Minimax Solution – Mixed Strategies

- Nash equilibrium might not exist in pure strategies:

$$A = \begin{array}{c} \textbf{P2} \\ \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 2 & -1 \\ \hline \end{array} \end{array} \textbf{P1}, \qquad B = \begin{array}{c} \textbf{P2} \\ \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 0 & 1 \\ \hline \end{array} \end{array} \textbf{P1}.$$

No Nash equilibrium solution, but the minimax strategies exist

- For mixed Nash equilibrium strategy:

$$\begin{aligned} -2y_1^* z_1^* + y_1^* &\leq -2y_1 z_1^* + y_1, \quad 0 \leq y_1 \leq 1, \\ 2y_1^* z_1^* - z_1^* &\leq 2y_1^* z_1 - z_1, \quad 0 \leq z_1 \leq 1. \end{aligned}$$

$$\begin{aligned} y &= (y_1, (1 - y_1))', \\ z &= (z_1, (1 - z_1))', \end{aligned}$$

$$\{y^* = (\tfrac{1}{2}, \tfrac{1}{2})', z^* = (\tfrac{1}{2}, \tfrac{1}{2})'\}.$$

Minimax Solution – Mixed Strategies

- A pair $\{y^* \in Y, z^* \in Z\}$ is said to constitute a noncooperative (Nash) equilibrium solution to a bimatrix game (A, B) in mixed strategies, if the following inequalities are satisfied for all $y \in Y$ and $z \in Z$:

$$
\begin{aligned}
y^{*\prime} A z^* &\leq y' A z^*, \quad y \in Y, \\
y^{*\prime} B z^* &\leq y^{*\prime} B z, \quad z \in Z.
\end{aligned}
$$

- Here, the pair $(y^{*\prime} A z^*, y^{*\prime} B z^*)$ is known as a noncooperative (Nash) equilibrium outcome of the bimatrix game in mixed strategies.

- Every bimatrix game has at least one Nash equilibrium solution in mixed strategies.

Minimax Solution – Mixed Strategies

- A pair $\{y^*, z^*\}$constitutes a mixed-strategy Nash equilibrium solution to a bimatrix game (A, B) if, and only if, there exists a pair (p^*, q^*)such that $\{y^*, z^*,\ p^*, q^*\}$is a solution of the following bilinear programming problem:

$$\min_{y,z,p,q} \; [y'Az + y'Bz + p + q]$$

subject to

$$\left.\begin{array}{ll} Az \geq -pl_m, & B'y \geq -ql_n, \\ y \geq 0, z \geq 0, & y'l_m = 1, z'l_n = 1. \end{array}\right\}$$

The Stackelberg Equilibrium Solution

- Hierarchical (Stackelberg) equilibrium

$$A = \begin{array}{c|c|c|c|l} & & \mathbf{P2} & & \\ L & 0^{S_1} & 2 & 3/2^{S_2} & \\ M & 1 & 1^{N} & 3 & \mathbf{P1}, \\ R & -1 & 2 & 2 & \\ & L & M & R & \end{array} \qquad B = \begin{array}{c|c|c|c|l} & & \mathbf{P2} & & \\ L & -1^{S_1} & 1 & -2/3^{S_2} & \\ M & 2 & 0^{N} & 1 & \mathbf{P1}. \\ R & 0 & 1 & -1/2 & \\ & L & M & R & \end{array}$$

- P1 – leader, P2 – follower:
  - (L,L) is the Stackelberg solution, (0,1) is the Stackelberg outcome with P1 as the leader.
- P2 – leader, P1 – follower:
  - (L,R) is the Stackelberg solution, (3/2,-2/3) is the Stackelberg outcome with P1 as the leader.

The Stackelberg Equilibrium Solution

- Hierarchical (Stackelberg) equilibrium

$$
A = \begin{array}{c} \\ L \\ M \\ R \\ \\ \end{array}
\begin{array}{c} \mathbf{P2} \\ \left[\begin{array}{c|c|c} 0^{S_1} & 2 & 3/2^{S_2} \\ \hline 1 & 1^{N} & 3 \\ \hline -1 & 2 & 2 \end{array}\right] \\ \begin{array}{ccc} L & M & R \end{array} \end{array} \mathbf{P1},
\qquad
B = \begin{array}{c} \\ L \\ M \\ R \\ \\ \end{array}
\begin{array}{c} \mathbf{P2} \\ \left[\begin{array}{c|c|c} -1^{S_1} & 1 & -2/3^{S_2} \\ \hline 2 & 0^{N} & 1 \\ \hline 0 & 1 & -1/2 \end{array}\right] \\ \begin{array}{ccc} L & M & R \end{array} \end{array} \mathbf{P1}.
$$

- P1 – leader, P2 – follower:
  - (L,L) is the Stackelberg solution, (0,1) is the Stackelberg outcome with P1 as the leader.
- P2 – leader, P1 – follower:
  - (L,R) is the Stackelberg solution, (3/2,-2/3) is the Stackelberg outcome with P1 as the leader.

The Stackelberg Equilibrium Solution

- Hierarchical (Stackelberg) equilibrium

$$A = \begin{array}{c|c|c|c|c} & \multicolumn{3}{c}{\mathbf{P2}} & \\ L & 0^{S_1} & 2 & 3/2^{S_2} & \\ M & 1 & 1^{N} & 3 & \mathbf{P1}, \\ R & -1 & 2 & 2 & \\ & L & M & R & \end{array} \qquad B = \begin{array}{c|c|c|c|c} & \multicolumn{3}{c}{\mathbf{P2}} & \\ L & -1^{S_1} & 1 & -2/3^{S_2} & \\ M & 2 & 0^{N} & 1 & \mathbf{P1}. \\ R & 0 & 1 & -1/2 & \\ & L & M & R & \end{array}$$

- P1 – leader, P2 – follower:
  - (L,L) is the Stackelberg solution, (0,1) is the Stackelberg outcome with P1 as the leader.
- P2 – leader, P1 – follower:
  - (L,R) is the Stackelberg solution, (3/2,-2/3) is the Stackelberg outcome with P1 as the leader.

The Stackelberg Equilibrium Solution

- The follower's response to every strategy of the leader should be unique. If this requirement is not satisfied, then there is ambiguity in the possible responses of the follower and thereby in the possible attainable cost levels of the leader

$$A = \begin{array}{c|ccc|} & & \mathbf{P2} & \\ L & 0 & 1 & 3 \\ R & 2 & 2 & -1 \\ & L & M & R \end{array} \ \mathbf{P1}, \qquad B = \begin{array}{c|ccc|} & & \mathbf{P2} & \\ L & 0 & 0 & 1 \\ R & -1 & 0 & -1 \\ & L & M & R \end{array} \ \mathbf{P1}$$

How to figure out the Stackelberg equilibrium for P1 (leader)?

The Stackelberg Equilibrium Solution

Definition 3.26 *In a two-person finite game, the set $R^2(\gamma^1) \subset \Gamma^2$ defined for each $\gamma^1 \in \Gamma^1$ by*

$$R^2(\gamma^1) = \{\xi \in \Gamma^2 : J^2(\gamma^1, \xi) \leq J^2(\gamma^1, \gamma^2), \quad \forall \gamma^2 \in \Gamma^2\} \tag{3.36}$$

is the optimal response (rational reaction) set *of* **P2** *to the strategy $\gamma^1 \in \Gamma^1$ of* **P1**.

Definition 3.27 *In a two-person finite game with* **P1** *as the leader, a strategy $\gamma^{1*} \in \Gamma^1$ is called a* Stackelberg equilibrium strategy *for the leader, if*

$$\max_{\gamma^2 \in R^2(\gamma^{1*})} J^1(\gamma^{1*}, \gamma^2) = \min_{\gamma^1 \in \Gamma^1} \max_{\gamma^2 \in R^2(\gamma^1)} J^1(\gamma^1, \gamma^2) \triangleq J^{1*}. \tag{3.37}$$

The quantity J^{1*} is the Stackelberg cost *of the leader. If, instead,* **P2** *is the leader, the same definition applies with only the superscripts 1 and 2 interchanged.*

The Stackelberg Equilibrium Solution

- The follower's response to every strategy of the leader should be unique. If this requirement is not satisfied, then there is ambiguity in the possible responses of the follower and thereby in the possible attainable cost levels of the leader

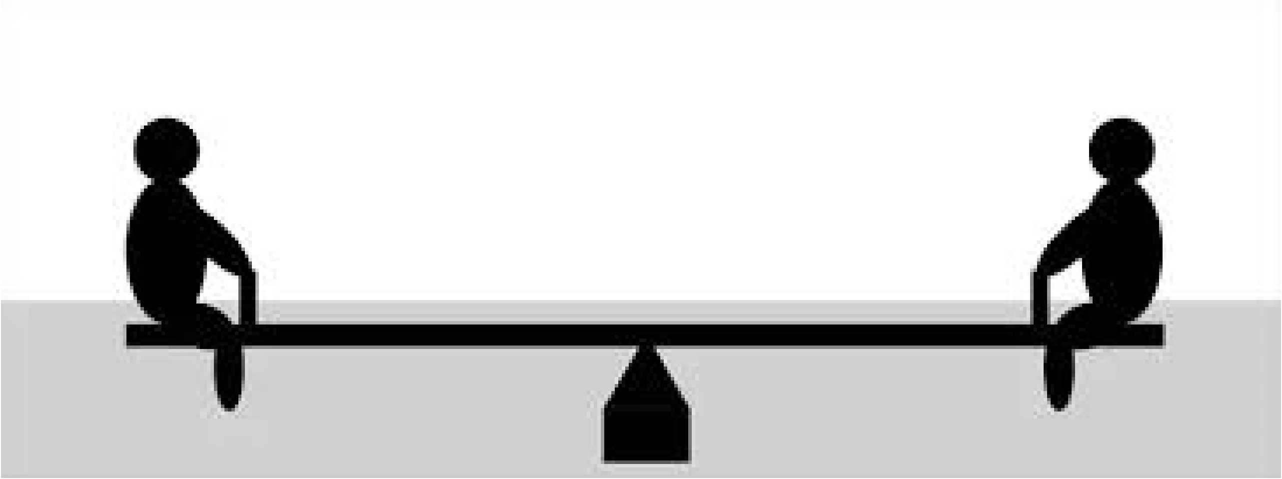
$$A = \begin{array}{c|c|c|c|l} & & \mathbf{P2} & & \\ L & 0 & 1 & 3 & \\ R & 2 & 2 & -1 & \mathbf{P1}, \\ & L & M & R & \end{array} \qquad B = \begin{array}{c|c|c|c|l} & & \mathbf{P2} & & \\ L & 0 & 0 & 1 & \\ R & -1 & 0 & -1 & \mathbf{P1} \\ & L & M & R & \end{array}$$

$\gamma^{1*} = L$- unique Stackelberg strategy for P1, $J^{1*} = 1$

$\gamma^{2} = L/M$

Noncooperative games

Saddle-Point Equilibrium	Safety strategy, concern self-situation, without decision order
Nash Equilibrium	A “stable strategy”, concern both sides, without decision order
Stackelberg Equilibrium	Concern both sides, with decision order



Thanks!