Noncooperative Finite Games: Two-Person Zero-Sum

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Little history

- Game theory did not exist as a unique field until John von Neumann published the paper *On the Theory of Games of Strategy* in 1928.
- In 1950, the first mathematical discussion of the prisoner's dilemma appeared by RAND because of its possible applications to global nuclear strategy.
- In 1950, John Nash developed Nash equilibrium, applicable to a wider variety of games.

John von Neumann

RAND CORPORATION

John Nash

Introduction

- Game theory: the study of mathematical models of strategic interactions among rational agents -- Roger Myerson.
- Dynamic noncooperative finite game:
    - Dynamic: the order in which the decisions are made is important
    - Noncooperative: each person involved pursues his own interests which are partly conflicting with others'.
    - Finite: action space is a finite number.

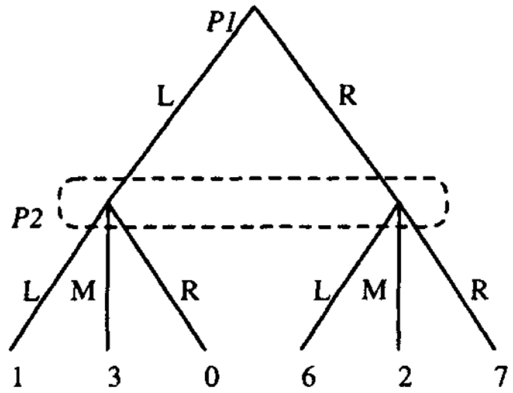
Table 1.1: The place of dynamic game theory.

	One player	Many players
Static	Mathematical programming	(Static) game theory
Dynamic	Optimal control theory	Dynamic (and/or differential) game theory

Framework

- Two-person Zero-sum
  - **Normal form (matrix form)**
    - Pure strategies
      - Saddle-point equilibrium
    - Mixed strategies
      - Saddle-point solution
  - Extensive form (tree form)
    - Behavior strategies

	P2		
	4	0	−1
P1	0	−1	3
	1	2	1



Matrix Games

- $A = \{a_{ij}\}$, each entry is an outcome
- ith row: strategies for P1, jth column: strategies for P2
- Target of P1: find i^*th row to minimize the outcomes

$$\bar{V}(A) \triangleq \max_j a_{i^*j} \le \max_j a_{ij}, \quad i = 1, \ldots, m,$$

  - $V(A)$ -- loss ceiling of P1 (security level for his losses)
  - $row\ i^*$ -- security strategy for P1

- Target of P2: find j^*th column to maximize the outcomes

$$\underline{V}(A) \triangleq \min_i a_{ij^*} \ge \min_i a_{ij}, \quad j = 1, \ldots, n.$$

  - $\underline{V}(A)$ -- gain-floor of P2 (security level for his gains)
  - column j^* -- security strategy for P2

	P2			
P1	1	3	3	−2
	0	−1	2	1
	−2	2	0	1

Saddle-point equilibrium

- What's the difference between the following two cases:

$$\begin{array}{cc} & \textbf{P2} \\ \textbf{P1} & \begin{array}{|c|c|c|} \hline 4 & 0 & -1 \\ \hline 0 & -1 & 3 \\ \hline 1 & 2 & 1 \\ \hline \end{array} \end{array}
\qquad\qquad
\begin{array}{cc} & \textbf{P2} \\ \textbf{P1} & \begin{array}{|c|c|} \hline 3 & 1 \\ \hline -1 & 1 \\ \hline \end{array} \end{array}$$

Saddle-point equilibrium

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P1:row 3 -> P2: column 1
P1 regrets– "I know P2 will choose column 1, why not choose row 2 to enjoy the 0 loss?"
P2 regrets– "P1 choose row 3, why not choose column 2 instead?"
-> no equilibrium!

$$\begin{array}{cc} & \mathbf{P2} \\ \mathbf{P1} & \begin{array}{|c|c|}\hline 3 & 1 \\ \hline -1 & 1 \\ \hline \end{array} \end{array}$$

P1:row 2 -> P2: column 2
$\overline{V} = \underline{V} = 1$ in equilibrium!

Saddle-point equilibrium

- If the pair of inequalities:

$$a_{i^* j} \leq a_{i^* j^*} \leq a_{i j^*}$$

for all i and j, then the strategies {row i^*,column j^* } are said to constitute a **saddle-point equilibrium.** And the matrix game is said to have a **saddle point** in pure strategies.

$$\begin{array}{cc} & \textbf{P2} \\ \textbf{P1} & \begin{array}{|c|c|c|} \hline 4 & 0 & -1 \\ \hline 0 & -1 & 3 \\ \hline 1 & 2 & 1 \\ \hline \end{array} \end{array}$$

$$\begin{array}{cc} & \textbf{P2} \\ \textbf{P1} & \begin{array}{|c|c|} \hline 3 & 1 \\ \hline -1 & 1 \\ \hline \end{array} \end{array}$$

Mixed Strategies

- Key idea: enlarge the strategy spaces, allow the players to base their decisions on the outcome of random events.

  E.g. $\{row1, row2, row3\}$—pure strategies space

  $\{y_1, y_2, y_3\}$—a mixed strategy, where $y_1 + y_2 + y_3 = 1$

  $Y = \{(y_1, y_2, y_3\ ), (y_1', y_2', y_3')\ \ldots\}$—the mixed strategy space of P1, comprised of all such probability distributions.

P2

	1	3	3	−2
P1	0	−1	2	1
	−2	2	0	1

Mixed Strategies

- Two player's mixed strategies space:
  - $Y = \{y \in R^m : y \geq 0, \sum_{i=1}^{m} y_i = 1\}$
  - $Z = \{z \in R^n : z \geq 0, \sum_{j=1}^{n} z_j = 1\}$
- Average value of the outcome of the game:
  - $J(y, z) = \sum_{i=1}^{m} \sum_{j=1}^{n} y_i a_{ij} z_j = y' A z$
- Mixed security strategy for P1 $y^* \in Y$ and P2 $z^* \in Z$
  - $\overline{V}_m(A) \triangleq \max_{z \in Z} y^{*\prime} A z \leq \max_{z \in Z} y' A z, \quad y \in Y$
  - $\underline{V}_m(A) \triangleq \min_{y \in Y} y' A z^* \geq \min_{y \in Y} y' A z, \quad z \in Z$
  - Here $\overline{V}_m$ called average security level for P1 (average upper value of the game)
  - $\underline{V}_m$ called average security level for P2 (average lower value of the game)
- Saddle point for a matrix game A (y^*, z^*)
  - $y^{*\prime} A z \leq y^{*\prime} A z^* \leq y' A z^*$
  - $V_m(A) = y^{*\prime} A z^*$ – Saddle-point value

Mixed Strategies

- $\overline{V}_m(A) = \min_{Y} \max_{Z} y'Az$
- $\underline{V}_m(A) = \max_{Z} \min_{Y} y'Az$
- **The minimax theorem**:
  - In any matrix game A, the average security levels of the players in mixed strategies coincide, that is:

$$\overline{V}_m(A) = \underline{V}_m(A)$$

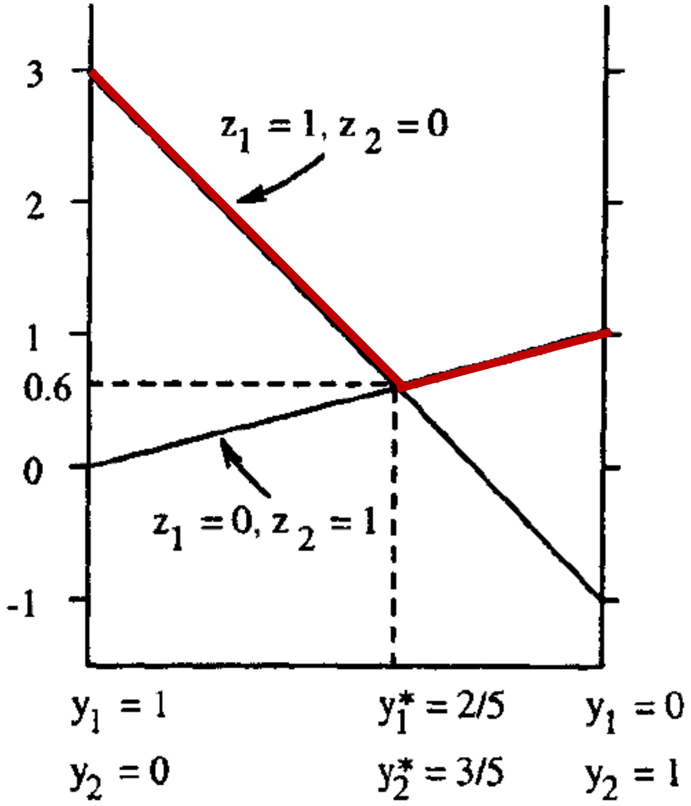
- We have thus seen that, if the players are allowed to use mixed strategies, **matrix games always admit a saddle-point solution** which, thereby, manifests itself as **the only reasonable equilibrium solution** in zero-sum two-person games of that type

Computation of mixed equilibrium strategies

- Graphical solution for (2×2) matrix games

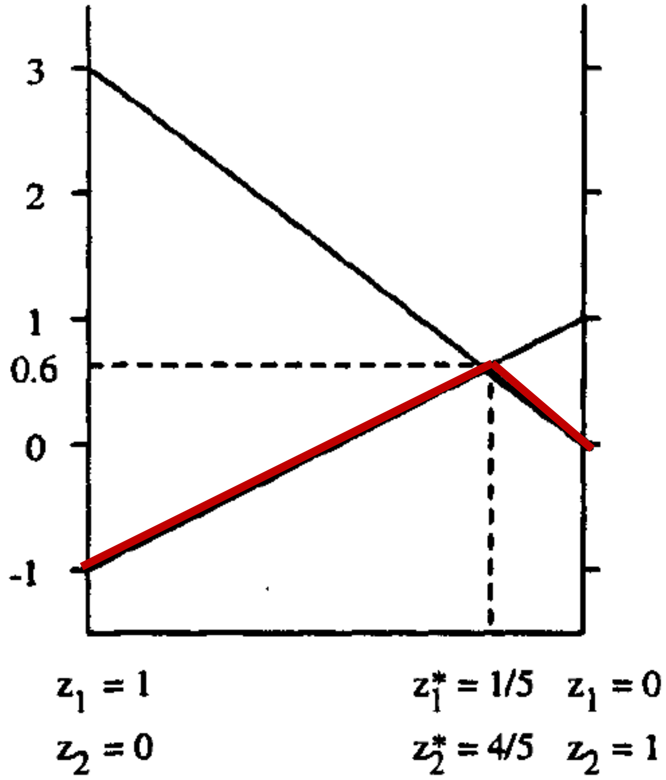
	P2	
P1	3	0
	−1	1

Upper envelop



Mixed security strategy of P1

Lower envelop

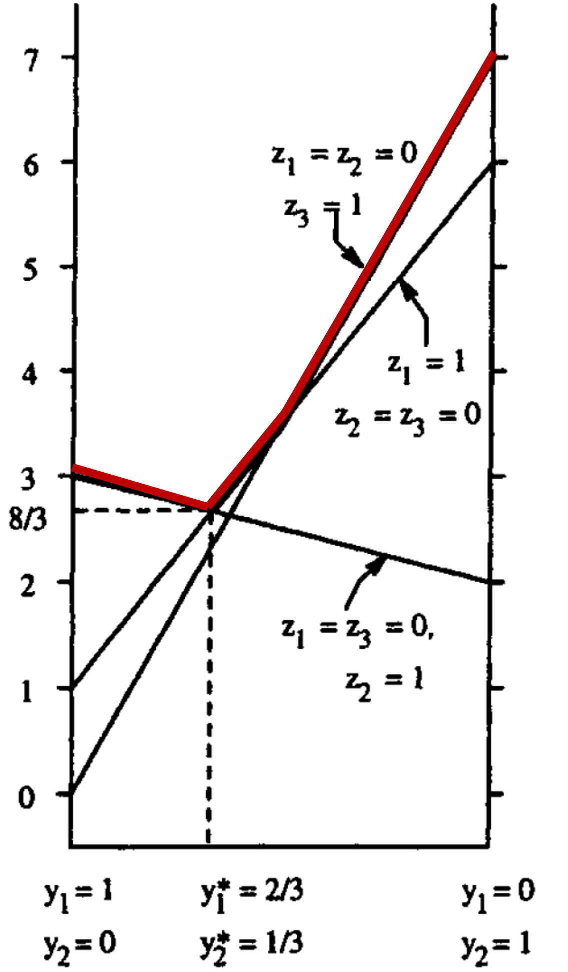


Mixed security strategy of P2

Computation of mixed equilibrium strategies

- Graphical solution for $(2 \times n)$ matrix games

	P2		
P1	1	3	0
	6	2	7



Mixed security strategy of P1

	P2	
P1	1	3
	6	2

Reduced strategy space of P2

Computation of mixed equilibrium strategies

- Linear programming (LP), to solve $(m \times n)$ matrix game.

$$\left.\begin{array}{l} \max y' 1_m \\ A' y \leq 1_n \\ y \geq 0 \end{array}\right\} \text{ "primal problem"},$$

$$\left.\begin{array}{l} \min z' 1_n \\ A z \geq 1_m \\ z \geq 0 \end{array}\right\} \text{ "dual problem"},$$

*(i) Both **LP** problems admit a solution, and* $V_\mathrm{p} = V_\mathrm{d} = 1/V_\mathrm{m}(A)$.

(ii) If (y^*, z^*) *is a mixed saddle-point solution of the matrix game* B*, then* $y^*/V_\mathrm{m}(A)$ *solves (2.28a), and* $z^*/V_\mathrm{m}(A)$ *solves (2.28b).*

(iii) If $\tilde{y}^*$ *is a solution of (2.28a), and* $\tilde{z}^*$ *is a solution of (2.28b), the pair* $(\tilde{y}^*/V_\mathrm{p}, \tilde{z}^*/V_\mathrm{d})$ *constitutes a mixed saddle-point solution for matrix game* B*. Furthermore,* $V_\mathrm{m}(B) = (1/V_\mathrm{p}) - c$.

Derivation for the LP form

$$V_m(A) = \min_Y \max_Z y'Az = \max_Z \min_Y y'Az, \tag{2.23}$$

which is necessarily a positive quantity by our positivity assumption on A. Let us now consider the min-max operation used in (2.23). Here, first a $y \in Y$ is given, and then the resulting expression is maximized over Z; that is, the choice of $z \in Z$ can depend on y. Hence, the middle expression of (2.23) can also be written as

$$\min_{y \in Y} v_1(y),$$

where $v_1(y)$ is a positive function of y, defined by

$$v_1(y) = \max_Z y'Az \geq y'Az \quad \forall z \in Z. \tag{2.24}$$

Since Z is the n-dimensional simplex as defined by (2.13b), the inequality in (2.24) becomes equivalent to the vector inequality

$$A'y \leq 1_n v_1(y),$$

where

$$1_n \triangleq (1, \ldots, 1)' \in \mathbf{R}^n.$$

$$1_n' z = 1$$
$$(y'Az)' \leq [(1_n' z) v_1(y)]'$$
$$z'Ay \leq z' 1_n v_1(y)$$

Further introducing the notation $\tilde{y} = y/v_1(y)$ and recalling the definition of Y from (2.13a), we observe that the optimization problem faced by **P1** in determining his mixed security strategy is

$$\text{minimize } v_1(y) \text{ over } \mathbf{R}^m$$

subject to

$$\begin{aligned} A'\tilde{y} &\leq 1_n, \\ \tilde{y}' 1_m &= [v_1(y)]^{-1}, \\ \tilde{y} \geq 0, &\quad y = \tilde{y} v_1(y). \end{aligned}$$

This is further equivalent to the maximization problem

$$\max \quad \tilde{y}' 1_m \tag{2.25a}$$

subject to

$$A'\tilde{y} \leq 1_n, \tag{2.25b}$$

$$\tilde{y} \geq 0, \tag{2.25c}$$

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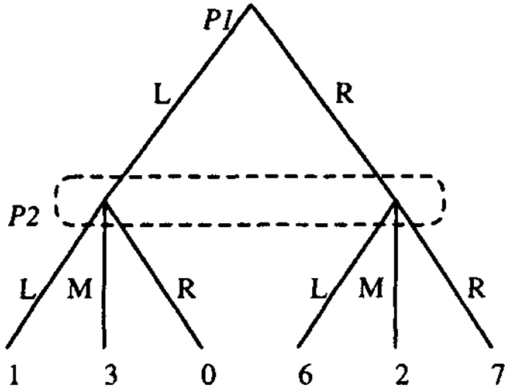
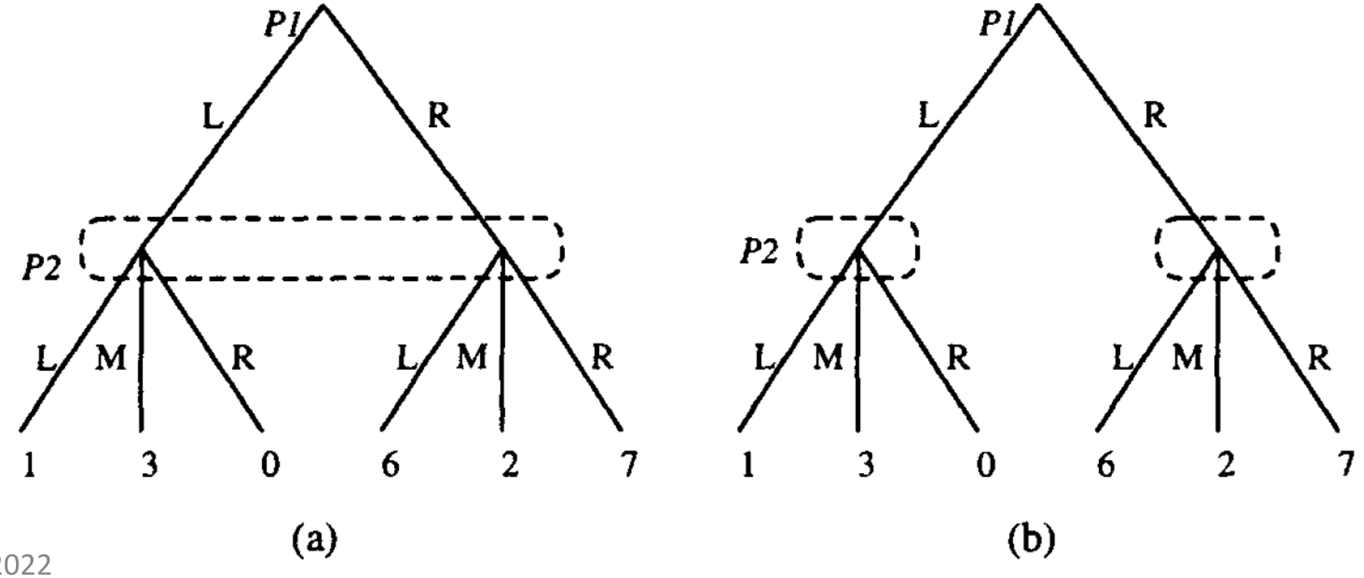


Extensive forms

- (a): P2 doesn't know where it is, or P1 and P2 play simultaneously.
- (b): P2 play after P1, know where it is.

$$\gamma^{2*}(u^1) = \begin{cases} M & \text{if} \quad u^1 = L, \\ R & \text{if} \quad u^1 = R. \end{cases} \qquad \gamma^{1*} \equiv u^{1*} = L$$

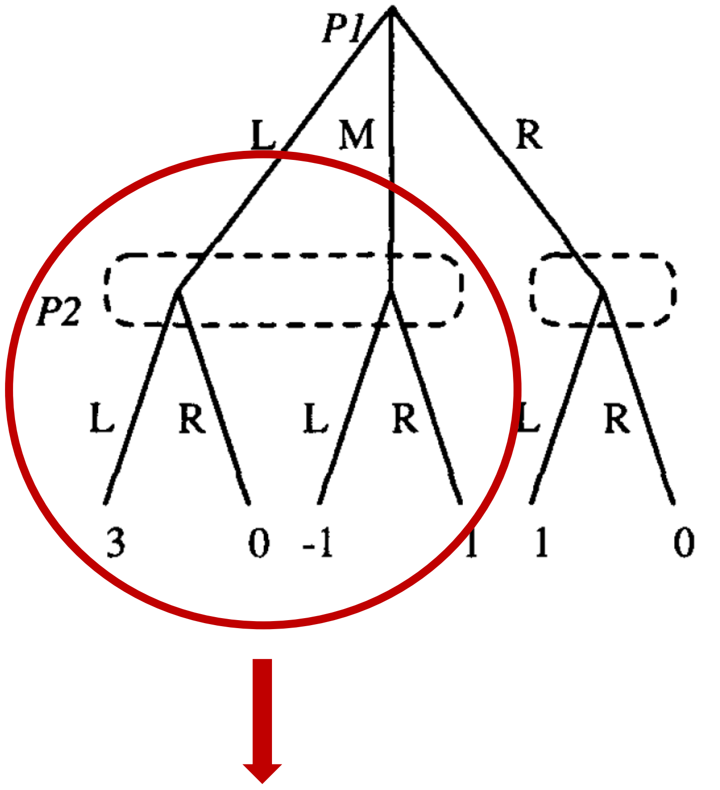


Extensive forms

- Behavior strategies

$$\hat{\gamma}^{2*} = \begin{cases} L & \text{w.p.} \quad 1 & \text{if } u^1 = R, \\ L & \text{w.p.} \quad 1/5, & \\ R & \text{w.p.} \quad 4/5 & \text{otherwise} , \end{cases}$$

$$\hat{\gamma}^{1*} = \begin{cases} L & \text{w.p.} \quad 2/5, \\ M & \text{w.p.} \quad 3/5, \\ R & \text{w.p.} \quad 0, \end{cases}$$



$$\begin{array}{cc} & \mathbf{P2} \\ \mathbf{P1} & \begin{array}{|c|c|} \hline 3 & 0 \\ \hline -1 & 1 \\ \hline \end{array} \end{array}$$

Thanks!