Static Noncooperative Infinite Games

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Content

- 1. Review
  - 1.1 Matrix games
  - 1.2 Saddle-point equilibriums
  - 1.3. Nash equilibriums
  - 1.4. Stackelberg equilibrium
- 2. Noncooperative Infinite Games
  - 2.1 ϵ equilibrium solutions
  - 2.2 Reaction curves
  - 2.3 Robust and stable Nash solution

Review - Matrix Games

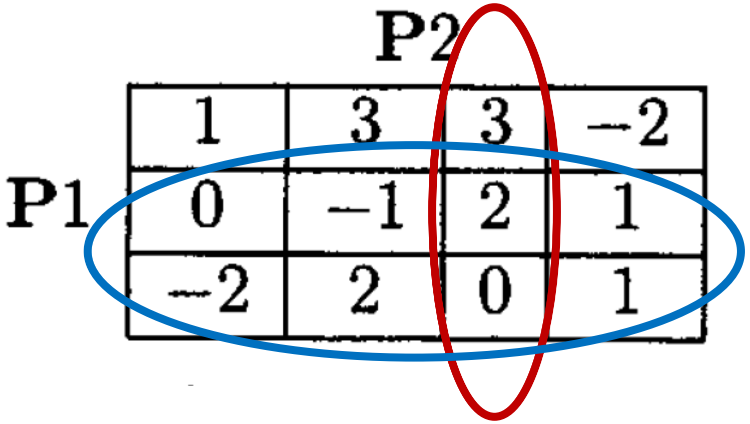
- $A = \{a_{ij}\}$, each entry is an outcome
- ith row: strategies for P1, jth column: strategies for P2
- Target of P1: find i^*th row to minimize the outcomes

$$\bar{V}(A) \triangleq \max_j a_{i^*j} \leq \max_j a_{ij}, \quad i = 1, \ldots, m,$$

  - $V(A)$ -- loss ceiling of P1 (security level for his losses)
  - $row\ i^*$ -- security strategy for P1
- Target of P2: find j^*th column to maximize the outcomes

$$\underline{V}(A) \triangleq \min_i a_{ij^*} \geq \min_i a_{ij}, \quad j = 1, \ldots, n.$$

  - $\underline{V}(A)$ -- gain-floor of P2 (security level for his gains)
  - column j^* -- security strategy for P2



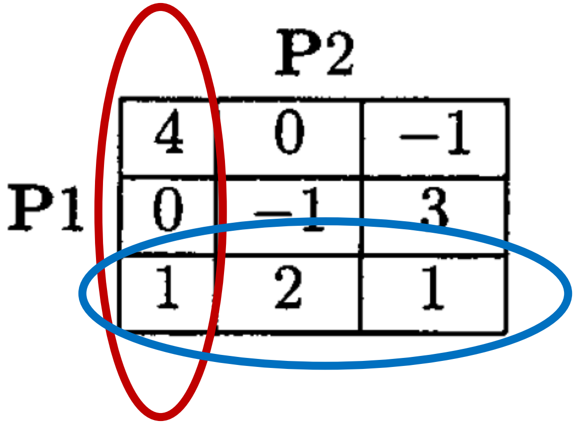
$$\begin{array}{c|cccc} & & & \mathbf{P2} & \\ \hline & 1 & 3 & 3 & -2 \\ \mathbf{P1} & 0 & -1 & 2 & 1 \\ & -2 & 2 & 0 & 1 \end{array}$$

Review - Saddle-point equilibrium

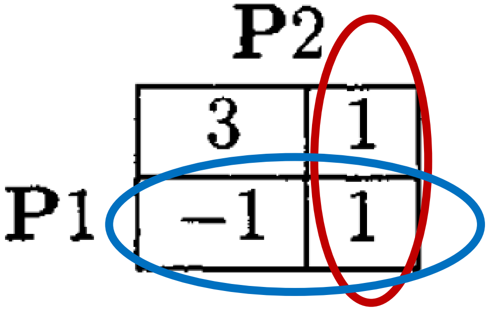
- If the pair of inequalities:

$$a_{i^*j} \leq a_{i^*j^*} \leq a_{ij^*}$$

for all i and j, then the strategies {row i^*,column j^* } are said to constitute a **saddle-point equilibrium.** And the matrix game is said to have a **saddle point** in pure strategies.



		P2	
	4	0	−1
P1	0	−1	3
	1	2	1



	P2	
	3	1
P1	−1	1

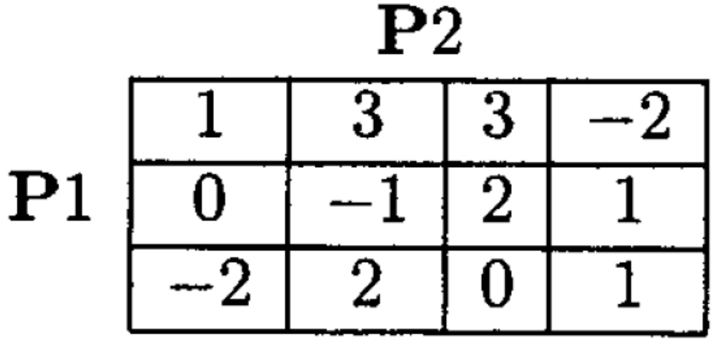
Review - Mixed Strategies

- Key idea: enlarge the strategy spaces, allow the players to base their decisions on the outcome of random events.

  E.g. $\{row1, row2, row3\}$—pure strategies space

  $\{y_1, y_2, y_3\}$—a mixed strategy, where $y_1 + y_2 + y_3 = 1$

  $Y = \{(y_1, y_2, y_3\ ), (y_1', y_2', y_3')\ \ldots\}$—the mixed strategy space of P1, comprised of all such probability distributions.



P2

	1	3	3	−2
P1	0	−1	2	1
	−2	2	0	1

Review - Mixed Strategies

- $\overline{V}_m(A) = \min_{Y} \max_{Z} y'Az$
- $\underline{V}_m(A) = \max_{Z} \min_{Y} y'Az$
- **The minimax theorem**:
  - In any matrix game A, the average security levels of the players in mixed strategies coincide, that is:

$$\overline{V}_m(A) = \underline{V}_m(A)$$

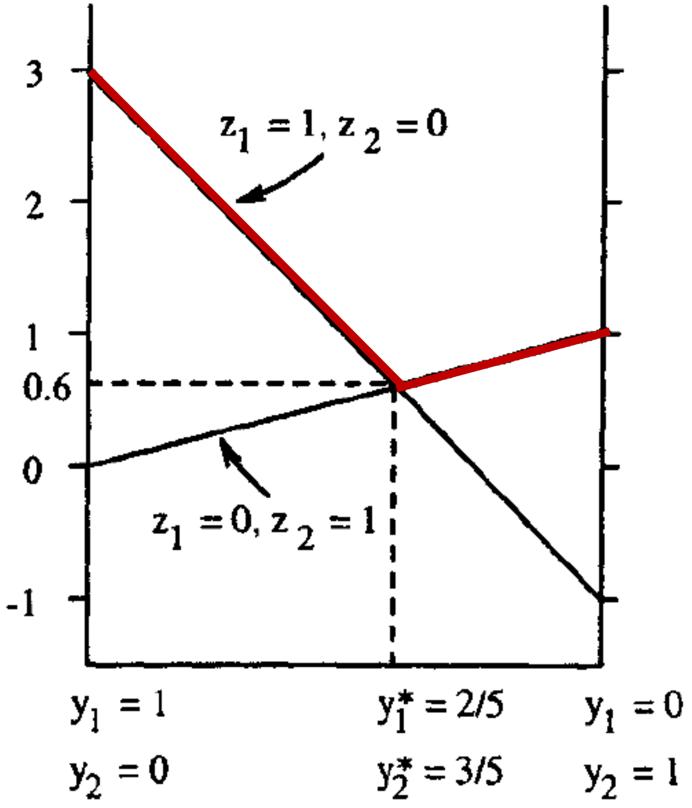
- We have thus seen that, if the players are allowed to use mixed strategies, **matrix games always admit a saddle-point solution** which, thereby, manifests itself as **the only reasonable equilibrium solution** in zero-sum two-person games of that type

Review - Mixed Equilibrium Strategies

- Graphical solution for (2 × 2) matrix games

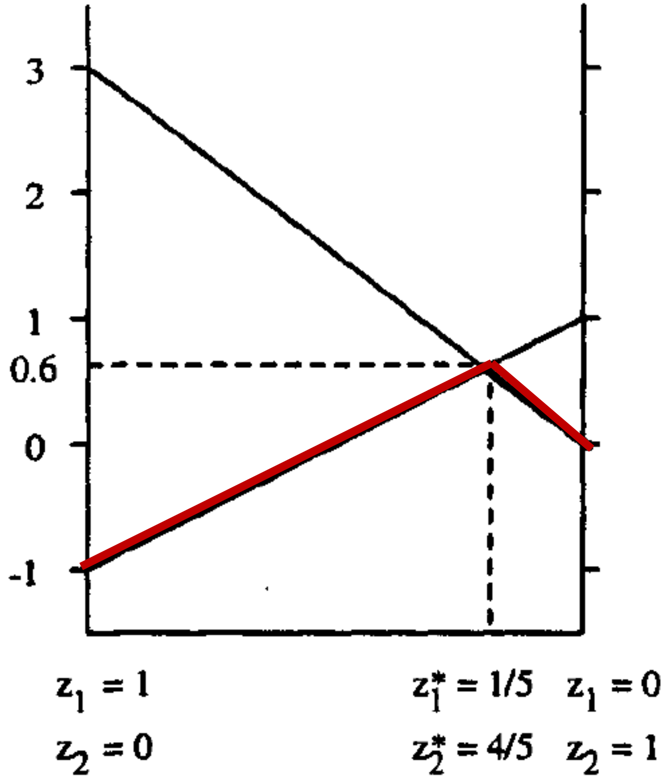
$$\begin{array}{c|cc} & \multicolumn{2}{c}{\mathbf{P2}} \\ \hline \mathbf{P1} & 3 & 0 \\ & -1 & 1 \end{array}$$

Upper envelop



Mixed security strategy of P1

Lower envelop



Mixed security strategy of P2

Review - Nash Equilibrium

- A pair of strategies {row i*, column j*} is said to constitute a noncooperative (Nash) equilibrium solution to a bimatrix game ($A = \{a_{ij}\}, B = \{b_{ij}\}$) if the following pair of inequalities is satisfied for all $i = 1, \ldots, m$ and all $j = 1, \ldots, n$:

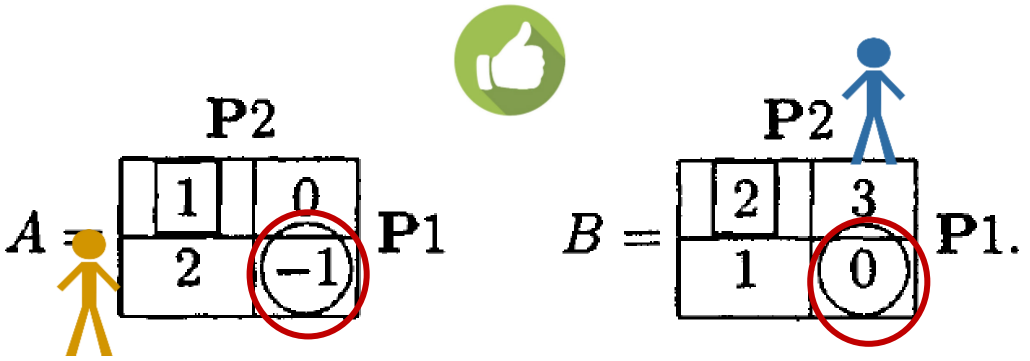
$$a_{i^*j^*} \leq a_{ij^*}$$
$$b_{i^*j^*} \leq b_{\mathrm{i}^*j}$$

- Furthermore, the pair ($a_{i^*j^*}$, $b_{i^*j^*}$) is known as a noncooperative (Nash) equilibrium outcome of the bimatrix game.
- Example: (1,2) and (-1,0) are the equilibrium outcomes.

$$A = \begin{array}{|c|c|}\multicolumn{2}{c}{\mathbf{P2}} \\ \hline \boxed{1} & 0 \\ \hline 2 & \text{\textcircled{-1}} \\ \hline \end{array}\ \mathbf{P1} \qquad B = \begin{array}{|c|c|}\multicolumn{2}{c}{\mathbf{P2}} \\ \hline \boxed{2} & 3 \\ \hline 1 & \text{\textcircled{0}} \\ \hline \end{array}\ \mathbf{P1}.$$

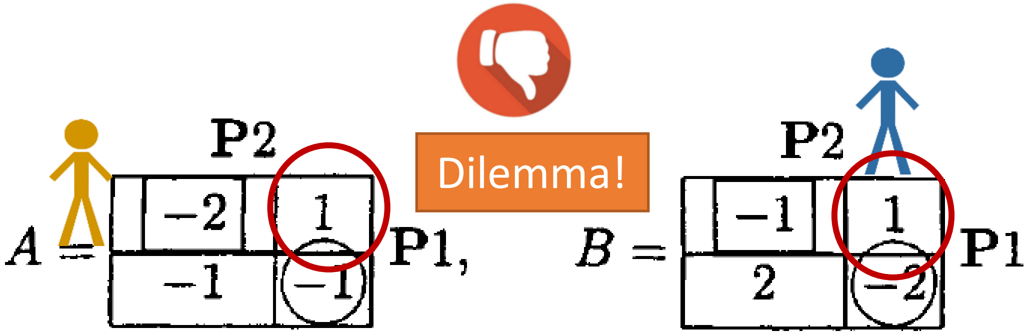
Review - Nash Equilibrium

- If a bimatrix game admits more than one admissible Nash equilibrium solution, then the equilibrium outcome of the game becomes rather ill-define.



$$A = \begin{array}{c} \textbf{P2} \\ \begin{array}{|c|c|} \hline \boxed{1} & 0 \\ \hline 2 & -1 \\ \hline \end{array} \end{array} \textbf{P1} \qquad B = \begin{array}{c} \textbf{P2} \\ \begin{array}{|c|c|} \hline \boxed{2} & 3 \\ \hline 1 & 0 \\ \hline \end{array} \end{array} \textbf{P1}.$$

(-1,0)-- most "reasonable" noncooperative equilibrium



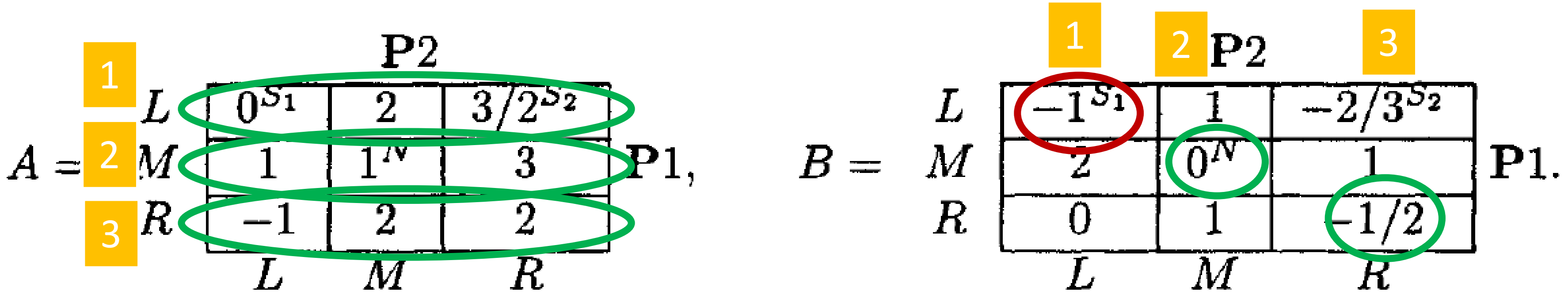
Dilemma!

$$A = \begin{array}{c} \textbf{P2} \\ \begin{array}{|c|c|} \hline \boxed{-2} & \boxed{1} \\ \hline -1 & -1 \\ \hline \end{array} \end{array} \textbf{P1}, \qquad B = \begin{array}{c} \textbf{P2} \\ \begin{array}{|c|c|} \hline \boxed{-1} & \boxed{1} \\ \hline 2 & -2 \\ \hline \end{array} \end{array} \textbf{P1}$$

Admits two admissible Nash equilibrium, no clean choice

Review - Stackelberg Equilibrium

- Hierarchical (Stackelberg) equilibrium



$$A = \begin{array}{c|c|c|c|l} & & \mathbf{P2} & & \\ L & 0^{S_1} & 2 & 3/2^{S_2} & \\ M & 1 & 1^{N} & 3 & \mathbf{P1}, \\ R & -1 & 2 & 2 & \\ & L & M & R & \end{array} \qquad B = \begin{array}{c|c|c|c|l} & & \mathbf{P2} & & \\ L & -1^{S_1} & 1 & -2/3^{S_2} & \\ M & 2 & 0^{N} & 1 & \mathbf{P1}. \\ R & 0 & 1 & -1/2 & \\ & L & M & R & \end{array}$$

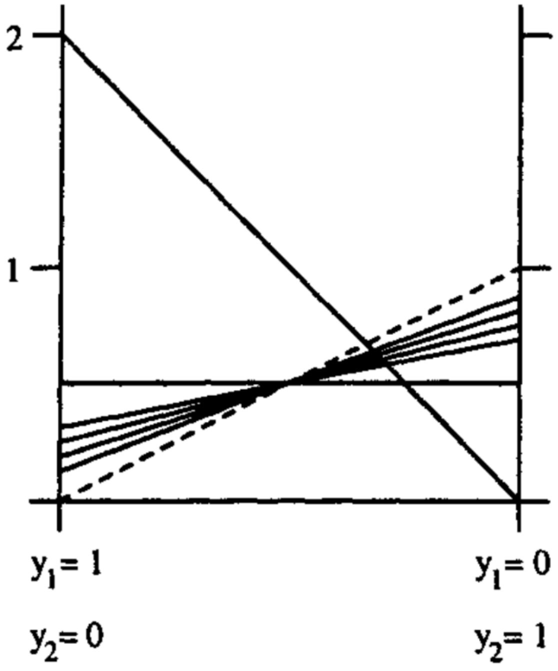
- P1 – leader, P2 – follower:
  - (L,L) is the Stackelberg solution, (0,-1) is the Stackelberg outcome with P1 as the leader.
- P2 – leader, P1 – follower:
  - (L,R) is the Stackelberg solution, (3/2,-2/3) is the Stackelberg outcome with P1 as the leader.

Noncooperative Infinite Games

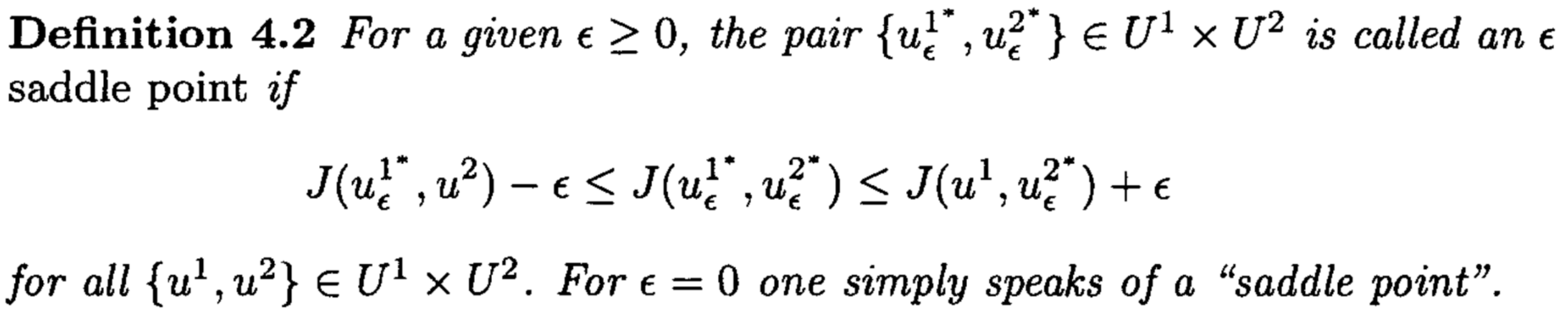
- **Infinite:** at least one of the players has at his disposal an infinite number of alternatives to choose from.

P2

P1	2	1/2	1/3	1/4	1/5	......
	0	1/2	2/3	3/4	4/5	......



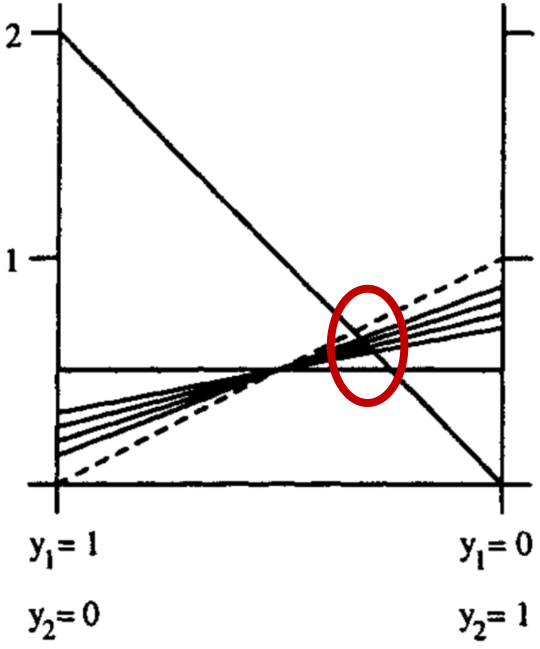
ϵ Equilibrium Solutions

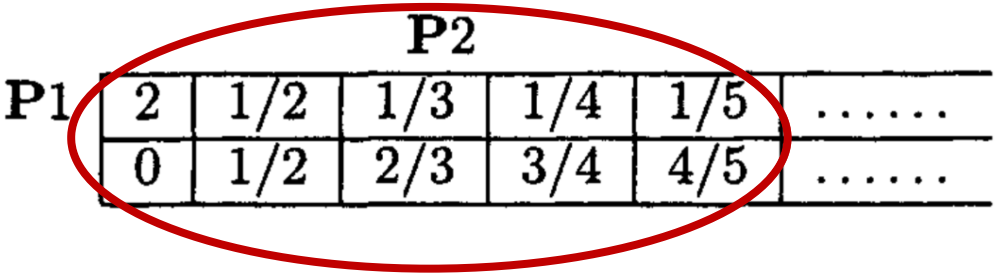


Definition 4.2 *For a given* $\epsilon \geq 0$, *the pair* $\{u_\epsilon^{1^*}, u_\epsilon^{2^*}\} \in U^1 \times U^2$ *is called an* ϵ saddle point *if*

$$J(u_\epsilon^{1^*}, u^2) - \epsilon \leq J(u_\epsilon^{1^*}, u_\epsilon^{2^*}) \leq J(u^1, u_\epsilon^{2^*}) + \epsilon$$

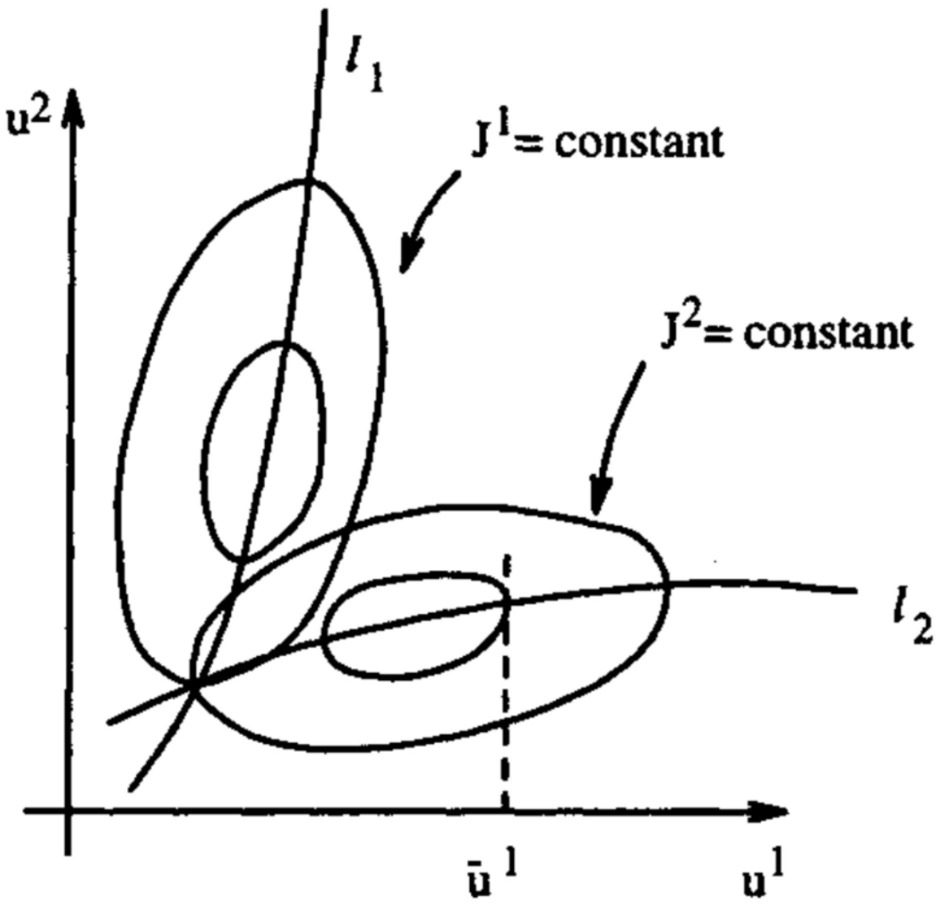
for all $\{u^1, u^2\} \in U^1 \times U^2$. *For* $\epsilon = 0$ *one simply speaks of a "saddle point".*





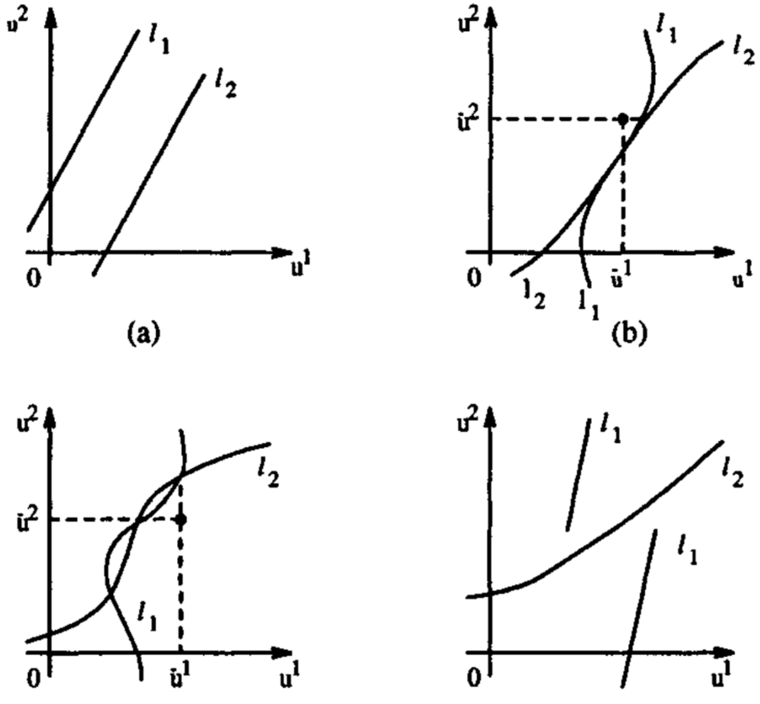
	P2				
P1	2	1/2	1/3	1/4	1/5	......
	0	1/2	2/3	3/4	4/5	......

Reaction curves



Definition 4.3 *In an N-person nonzero-sum game, let the minimum of the cost function of* **P1**, $J^1(u^1, \cdots, u^N)$, *with respect to* $u^1 \in U^1$ *be attained for each* $u_{-1} \in U_{-1}$, *where* $u_{-1} \triangleq \{u^2, \ldots, u^N\}$ *and* $U_{-1} \triangleq U^2 \times \cdots \times U^N$. *Then, the set* $R^1(u_{-1}) \subset U^1$ *defined by*

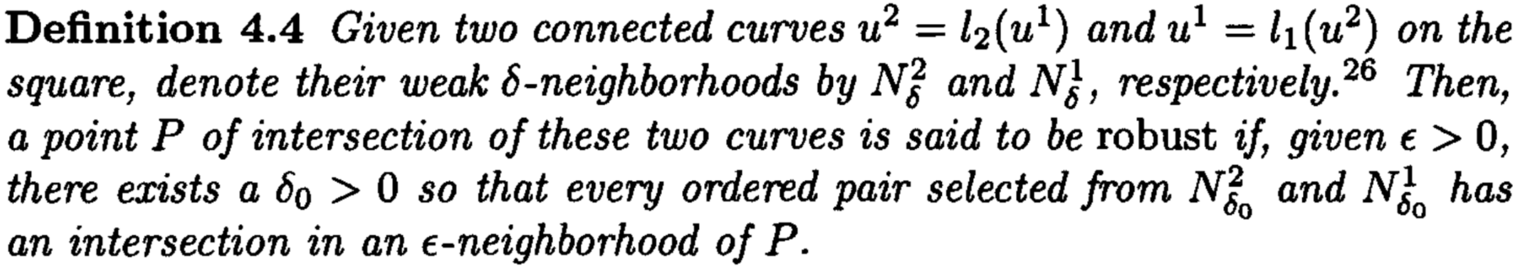
$$R^1(u_{-1}) = \{\xi \in U^1 : J^1(\xi, u_{-1}) \leq J^1(u^1, u_{-1}), \quad \forall u^1 \in U^1\}$$



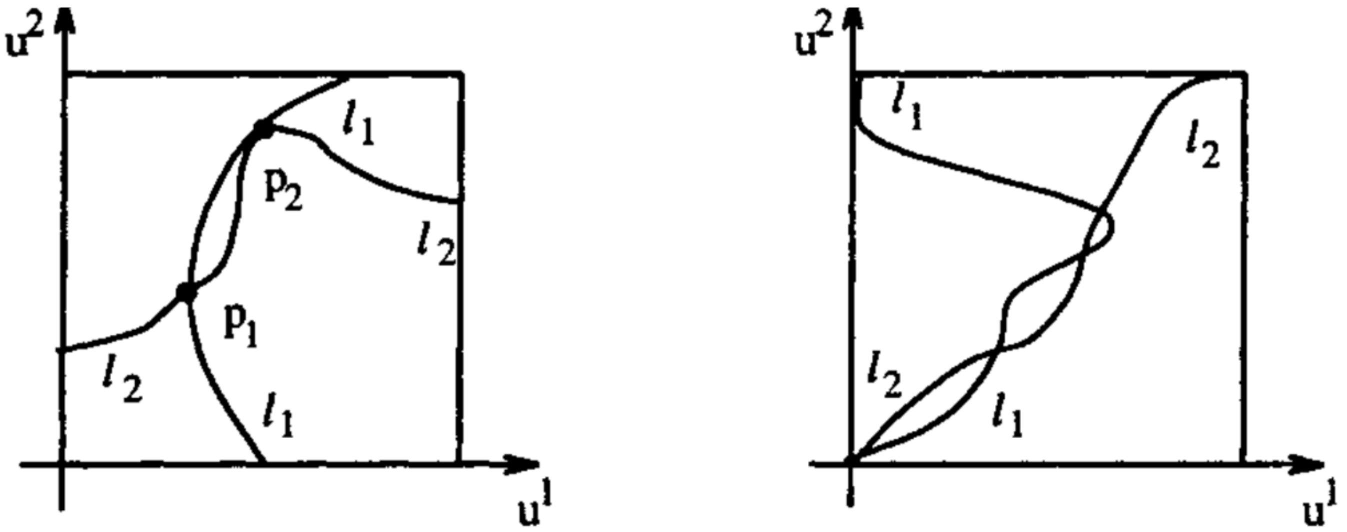
(a) (b)

(c) (d)

Robust Nash Solution



Definition 4.4 *Given two connected curves* $u^2 = l_2(u^1)$ *and* $u^1 = l_1(u^2)$ *on the square, denote their weak* δ*-neighborhoods by* N_δ^2 *and* N_δ^1*, respectively.*[26] *Then, a point* P *of intersection of these two curves is said to be* robust *if, given* $\epsilon > 0$*, there exists a* $\delta_0 > 0$ *so that every ordered pair selected from* $N_{\delta_0}^2$ *and* $N_{\delta_0}^1$ *has an intersection in an* ϵ*-neighborhood of* P*.*

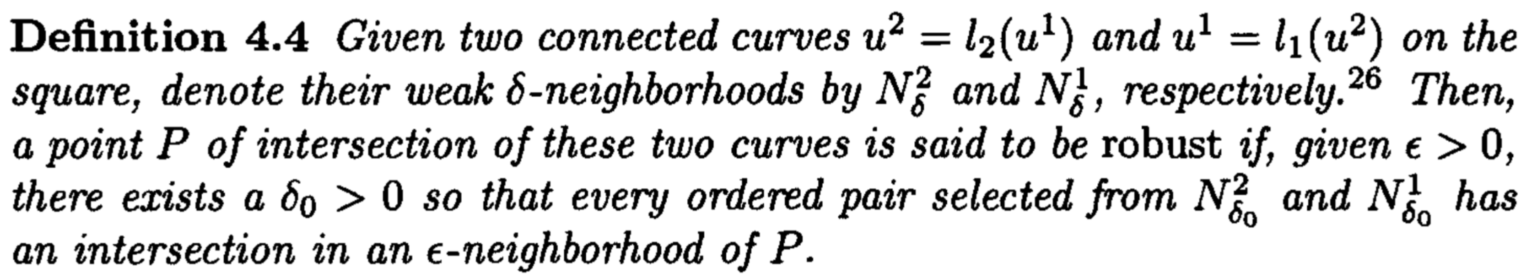


(a) (b)

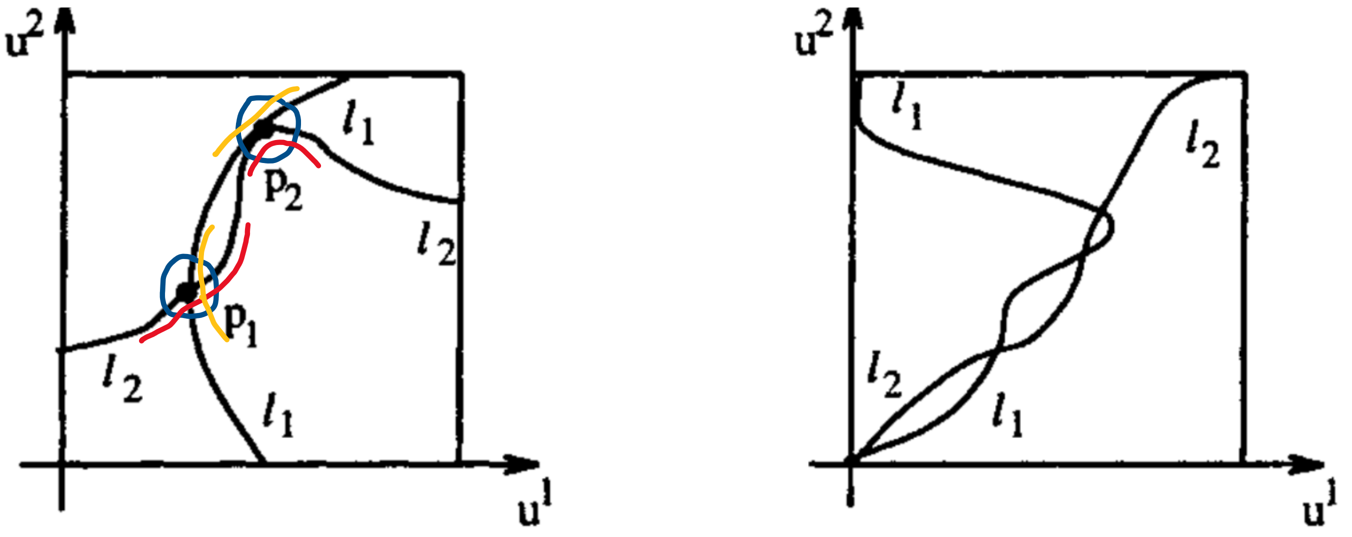
P1 is robust
P2 is not robust

All points are robust

Robust Nash Solution



Definition 4.4 *Given two connected curves* $u^2 = l_2(u^1)$ *and* $u^1 = l_1(u^2)$ *on the square, denote their weak* δ*-neighborhoods by* N^2_δ *and* N^1_δ*, respectively.*[26] *Then, a point* P *of intersection of these two curves is said to be* robust *if, given* $\epsilon > 0$*, there exists a* $\delta_0 > 0$ *so that every ordered pair selected from* $N^2_{\delta_0}$ *and* $N^1_{\delta_0}$ *has an intersection in an* ϵ*-neighborhood of* P*.*



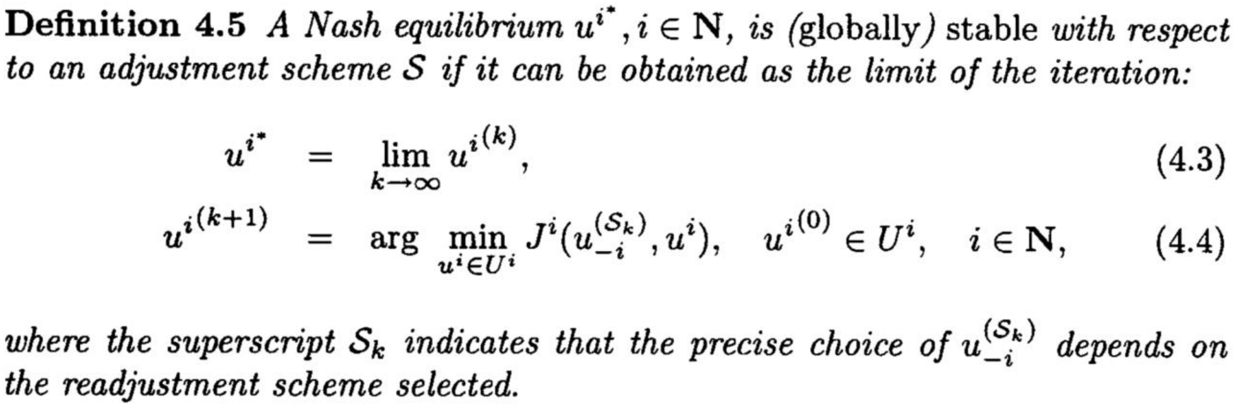
(a)

(b)

P1 is robust
P2 is not robust

All points are robust

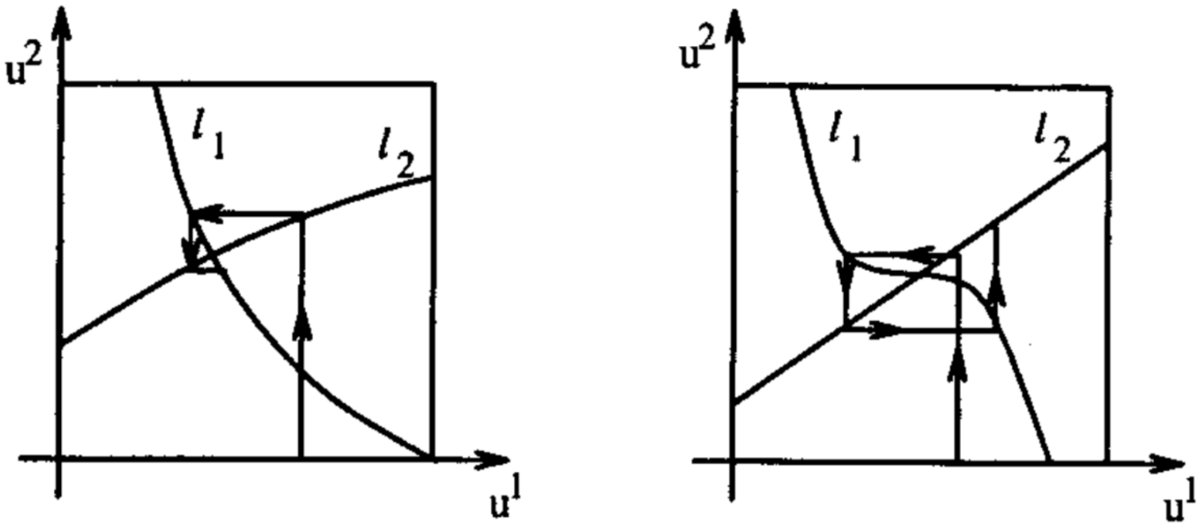
Stable Nash Solution



Definition 4.5 *A Nash equilibrium* $u^{i^*}, i \in \mathbf{N}$, *is* (globally) stable *with respect to an adjustment scheme* $\mathcal{S}$ *if it can be obtained as the limit of the iteration:*

$$
\begin{aligned}
u^{i^*} &= \lim_{k \to \infty} u^{i(k)}, & (4.3)\\
u^{i(k+1)} &= \arg \min_{u^i \in U^i} J^i(u_{-i}^{(\mathcal{S}_k)}, u^i), \quad u^{i(0)} \in U^i, \quad i \in \mathbf{N}, & (4.4)
\end{aligned}
$$

where the superscript $\mathcal{S}_k$ *indicates that the precise choice of* $u_{-i}^{(\mathcal{S}_k)}$ *depends on the readjustment scheme selected.*



(a) (b)

Thanks

Happy Halloween

Hope to meet you soon