THE DEFICIENCY OF BEING A CONGRUENCE GROUP FOR VEECH GROUPS OF ORIGAMIS

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Abstract. We study "how far away" a finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is from being a congruence group. For this we define its *deficiency of being a congruence group*. We show that the index of the image of Γ in $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is the biggest, if n is the general Wohlfahrt level. We furthermore show that the Veech groups of origamis (or square-tiled surfaces) in $\mathcal{H}_2(2)$ are far away from being congruence groups and that in each genus one finds an infinite family of origamis such that they are "as far as possible" from being a congruence group.

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§1. Intro (§2-3 included)

(half) translation structure : G-manifold structure w/ $G = \text{trans}\ \mathbb{C}$ (resp. $\langle z \mapsto -z, \text{trans}\ \mathbb{C} \rangle$)

Teich curve induced from a (half) translation surface (R, μ)

: image of $\mathbb{D} \hookrightarrow T_{g,n} \longrightarrow M_{g,n}$

$\tau \in \mathbb{D} \mapsto [R_\tau, f_\tau]$

where $R_\tau = (I_{g,n}, \mu_\tau = \{(U, z + \tau\bar{z}) \mid (U, z) \in \mu\})$

$f_\tau = id_{I_{g,n}} : R \to R_\tau$

affine map : G-morphism on (half) translation surface

Veech grp : the grp of Jacobian derivatives of all affine maps on (half) translation surface

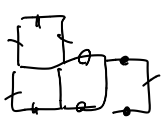
Teich curves & Veech groups : 1989 (Veech) ~.

THM (Veech) Veech grp is a discrete subgrp of $SL(2,\mathbb{R})$ ($PSL(2,\mathbb{R})$), and never cocpt.

Teichmüller curve $\cong \mathbb{D}/$Veech group

Origami (sq-tiled surfs.) : Special class of translation surfaces, consisting of just finitely many unit squares.

e.g.



… been studied 2004 ~

Prop Veech grp of an origami is a finite index subgrp of $SL(2,\mathbb{Z})$.

→ Teich curve is a cov. of $\mathbb{H}/PSL(2,\mathbb{Z})$

THM $\displaystyle\bigcup_{O:\text{origami of type}(g,n)} C(O) \subset M_{g,n}$ is dense.

$C(O)$: Teich. curve

↑ information found in [P.177: I. Coskun, T. Farkas "Surveys on Recent Development in Algebraic Geometry"]

∘ $\Gamma < SL(2,\mathbb{Z})$ is a congruence group $\iff$ Γ contains $\Gamma(n) = \{SL(2,\mathbb{Z}) \ni A \equiv \begin{pmatrix}1&0\\0&1\end{pmatrix} \bmod n\}$ (elementwise)

: the principal congruence group of level n.

Author (2005) : many | congruence grp occurs as VG of origami.
Ellenberg & McReynolds (2010) : every | congruence grp occurs as VG of origami.

However, when we focus on a stratum $\mathcal{H}_g$ of holo, unit area differentials

VGs of origamis are more likely to non-congruence.

e.g. Hubert & Lelièvre (2005) : all but 1 origami in $\mathcal{H}_2(2)$ has noncongruence VG.

genus 2 ; with single zero

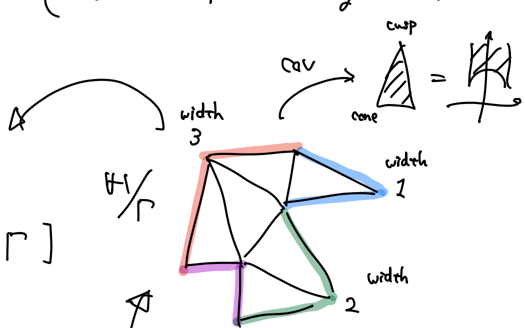
Theme in this article : "how far from a congruence grp" VG of origamis are.

ingredients $\forall \Gamma < SL(2,\mathbb{Z})$,

- minimal congruence level : $cl_\Gamma = \min\{\ell \mid \Gamma \supset \Gamma(\ell)\}$ $(= \infty \overset{\text{def}}{\iff} \Gamma : \text{non congruence})$
- index : $d_\Gamma = [SL(2,\mathbb{Z}) : \Gamma]$

key → · Wohlfahrt level : $\ell_\Gamma = \mathrm{lcm}(\{\mathrm{width}(v) \mid v : \text{cusp of } \Gamma\})$

- deficiency w.r.t. m : $f_m = f_{m,\Gamma} = [\Gamma(m) : \Gamma(m) \cap \Gamma]$
  of Γ : $f_\Gamma = f_{\ell_\Gamma,\Gamma}$
- level index w.r.t. m : $e_m = [SL_2(\mathbb{Z}/m\mathbb{Z}) : p_m(\Gamma)]$
  of Γ : $e_\Gamma = e_{\ell_\Gamma}$



$\rightsquigarrow \ell_\Gamma = 6$

※ for a cusp x on F.D of Γ, we also say $x \in \Gamma x$ a cusp.

$$\begin{tikzcd}
1 \arrow[r, "f_m"] & \Gamma(m) \arrow[r, hook, "\boxed{d_\Gamma}"] & SL_2(\mathbb{Z}) \arrow[r, "p_m"] & SL_2(\mathbb{Z}/m\mathbb{Z}) \arrow[r, "\triangle e_m"] & 1, \\
1 \arrow[r, "1"] & \Gamma(m)\cap\Gamma \arrow[r, hook, "\boxed{1}"] \arrow[u, hook] & \Gamma \arrow[r, "p_m"] \arrow[u, hook] & \overline{\Gamma} \arrow[r, "\triangle 1"] \arrow[u, hook] & 1
\end{tikzcd}$$

THM A (Wohlfahrt, '64)

If Γ is a congruence grp, $cl_\Gamma = \ell_\Gamma$ holds.

→ in particular,

Γ : congr. $\Rightarrow$ $f_\Gamma = 1$, $d_\Gamma = e_\Gamma$

$$\left(\begin{array}{c}
SL_2(\mathbb{Z}/m\mathbb{Z}) \cong SL_2(\mathbb{Z})/\Gamma(m) \\
p_m(\Gamma) = \overline{\Gamma} \cong \Gamma/\Gamma(m)\cap\Gamma \\
SL_2(\mathbb{Z}/m\mathbb{Z})/\overline{\Gamma} \cong SL_2(\mathbb{Z})/\Gamma \Big/ \Gamma(m)/\Gamma(m)\cap\Gamma \\
e_m = d_\Gamma / f_m \\
e_m \,\&\, f_m \text{ are divisor of } d_\Gamma.
\end{array}\right)$$

Def $\Gamma < SL_2(\mathbb{Z})$ is called totally non-congruence

if $f_\Gamma = d_\Gamma$, $e_\Gamma = 1$.

Theorem 1 (Proof in Section 3). *Let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$ and $l \in \mathbb{N}$. The deficiency $f_{\Gamma,l}$ becomes minimal if l is the Wohlfahrt level of Γ.*

In particular one has the following conclusion.

Corollary 1.1. *For a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with Wohlfahrt level l we have: If Γ is a totally non-congruence group, i.e. $e_{\Gamma,l} = 1$, then Γ surjects to $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ for each natural number n.*

i.e. $\forall A \in SL_2(\mathbb{Z})$, $\exists B \in \Gamma$ s.t. $A \equiv B \bmod n$ for $\forall n \in \mathbb{N}$

◦ nice criterion for being a totally non-congruence grp

Theorem 2 (Proof in Section 3). *Let Γ_1 and Γ_2 be conjugated finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Suppose that Γ_1 has width a_1 at the cusp 0 and width b_1 at the cusp ∞ and Γ_2 has width a_2 at the cusp 0 and width b_2 at the cusp ∞. If*

(1) $$n_1 = \mathrm{lcm}(a_1, b_1) \text{ and } n_2 = \mathrm{lcm}(a_2, b_2) \text{ are relatively prime,}$$

then Γ_1 and Γ_2 are totally non-congruence groups, i.e. they surject onto $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ for each natural number n.

(Seems to not an iff-condition...)

◦ the deficiency of the VG of an origami in $\mathcal{H}_2(2)$

Theorem 3. *Let O be an origami in $\mathcal{H}_2(2)$ with j squares and let $\Gamma(O)$ be its Veech group. We distinguish the two different cases that O is in the orbit A_j or B_j in the classification of orbits in $\mathcal{H}_2(2)$ by McMullen and Hubert/Lelièvre (see Section 4).*

i) *If j is odd, $j \geq 5$ and O is in B_j, then $\Gamma(O)$ is a totally non-congruence group, i.e. it surjects onto $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ for each $n \in \mathbb{N}$.*

ii) *If j is even, or j is odd and O is in A_j, or $j = 3$, then the deficiency $f_{\Gamma,l}$ with respect to the Wohlfahrt level l of $\Gamma(O)$ is equal to $\frac{d}{3}$. I.e. the index of its image in $\mathrm{SL}_2(\mathbb{Z}/l\mathbb{Z})$ is 3.*

Corollary 1.2. *The Veech groups of all origamis in $\mathcal{H}_2(2)$ form an* honest family of non-congruence groups, *i.e. an infinite family such that the level index e_l is bounded by a constant. More precisely their level index is bounded by 3.* ※

※ a family $\{\Gamma_n\}$ w/ level indices $\{e_{\Gamma_n}\}$ uniformly bounded (⇔ NOT accumulating to congruence grp)

Rem. Similar approach is used for the arguments on twisted Teich curves arrised from L-shaped surf. L_D by C. Weiß (2012).

$\rightsquigarrow$ VG: subgrp of $SL_2(\mathcal{O}_D)$

○ An application of thm 2: infinite families of origamis w/ totally non-congr. VG

$$\boxed{\begin{array}{l}\underline{\text{Theorem 4}}\ \text{For each } g\geq 3,\ \text{the stratum } \mathcal{H}(2g-2) \text{ contains an infinite family of origamis}\\ \qquad\qquad\qquad\qquad \text{whose VGs are totally non-congr. grps.}\end{array}}$$

§3. The deficiency of being a non-congr. grp.

Let Γ be a finite index subgrp of $SL(2,\mathbb{Z})$.

Lem 3.5 for $k,\ a \in \mathbb{N}$, $\underbrace{f_{ka,\Gamma}}_{[\Gamma(ka):\Gamma(ka)\cap\Gamma]}$ devides $\underbrace{f_{a,\Gamma}}_{[\Gamma(a):\Gamma(a)\cap\Gamma]}$

pf) it follows from $q(p_{ka}(\Gamma)) = p_a(\Gamma)$ where $q: SL_2(\mathbb{Z}/ka\mathbb{Z}) \to SL_2(\mathbb{Z}/a\mathbb{Z})$: natur. proj.

(q project to $\bar{q}$: $SL_2(\mathbb{Z}/ka\mathbb{Z})/p_{ka}(\Gamma) \to SL_2(\mathbb{Z}/a\mathbb{Z})/p_a(\Gamma)$ & $e_{ka} = e_a \cdot \deg \bar{q}$
$\Rightarrow f_a = f_{ka} \cdot \deg \bar{q}$)

$$\begin{tikzcd}
\Gamma(ka)\cap\Gamma \arrow[r, hook] \arrow[dddd, bend right, "f_{ka}"'] & \Gamma \arrow[r, "p_{ka}"] \arrow[dr, "p_a"] \arrow[ddd, "d"'] & p_{ka}(\Gamma) \arrow[r, hook, "e_{ka}"] \arrow[d, "q"] & SL(\mathbb{Z}/ka\mathbb{Z}) \arrow[d, "q"] \\
\Gamma(a)\cap\Gamma \arrow[ur, hook] \arrow[d, "f_a"] & & p_a(\Gamma) \arrow[r, hook, "e_a"] & SL(\mathbb{Z}/a\mathbb{Z}) \\
\Gamma(a) \arrow[drr] & & & \\
& & SL_2(\mathbb{Z}) \arrow[uuur] \arrow[uur] & \\
\Gamma(ka) \arrow[urr] & & &
\end{tikzcd}$$

Theorem 1 (Proof in Section 3). *Let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$ and $l \in \mathbb{N}$. The deficiency $f_{\Gamma,l}$ becomes minimal if l is the Wohlfahrt level of Γ.*

pf of Thm 1) from Lemma 3 we have $f_m \geq f_{ml}$ for $\forall m \in \mathbb{N}$ & $l = l_\Gamma$ (Wohlfahrt level Γ)

we show: $f_{ml} = f_l$

Consider $\Gamma' := p_{ml}^{-1}(p_{ml}(\Gamma)) < SL(2,\mathbb{Z})$.

$\longrightarrow$ (i) $p_{ml}(\Gamma) = p_{ml}(\Gamma')$

(ii) $\Gamma' \supset \Gamma$.

(iii) Γ' is a congruence grp. ($\ker p_{ml} = \Gamma(ml)$)

Let $\ell' := \ell_{\Gamma'}$: min. cong. level $\overset{\text{Thm A}}{=}$ Wohlfahrt level

$$\begin{array}{ccc} \text{min. cong. level} & & \text{Wohlfahrt level} \\ \downarrow & & \downarrow \ (\text{def : lcm}) \\ \Gamma(\ell')\subset\Gamma' & & \ell' \text{ divides } \ell\ ,\ \Gamma(\ell')\supset\Gamma(\ell) \end{array}$$

$$\Rightarrow \quad \Gamma'\supset\Gamma(\ell')\supset\Gamma(\ell) \overset{p_{m\ell}}{\rightsquigarrow} \boxed{\begin{array}{l} p_{m\ell}(\Gamma')\supset p_{m\ell}(\Gamma(\ell)) \\ \quad \shortparallel \\ p_{m\ell}(\Gamma) \end{array}}$$

Consider now the following diagram of exact seqs :

```tikzcd
1 \arrow[r] & p_{lm}(\Gamma(l)) \arrow[r] & \mathrm{SL}_2(\mathbb{Z}/lm\mathbb{Z}) \arrow[r] & \mathrm{SL}_2(\mathbb{Z}/l\mathbb{Z}) \arrow[r] & 1 \\
1 \arrow[r] & \boxed{p_{lm}(\Gamma(l))\cap p_{lm}(\Gamma)} \arrow[u, hook] \arrow[r] & p_{lm}(\Gamma) \arrow[u, hook] \arrow[r] & p_l(\Gamma) \arrow[u, hook] \arrow[r] & 1
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$$\shortparallel$$
$$p_{m\ell}(\Gamma(\ell))$$

i.e. $e_{m\ell} = [SL_2(\mathbb{Z}/m\ell\mathbb{Z}) : p_{m\ell}(\Gamma)]$

$\quad = [SL_2(\mathbb{Z}/\ell\mathbb{Z}) : p_\ell(\Gamma)] = e_\ell$, thus $f_{m\ell} = f_\ell$ holds. ⊠

Def 3.7 $\Gamma < SL_2(\mathbb{Z})$: finite index w/ cusps $\begin{matrix}\infty\\0\end{matrix}$ of width $\begin{matrix}a\\b\end{matrix}$

$\rightsquigarrow$ we call (a,b) a <u>normalized cusp-width pair</u> of Γ.

Prop 3.8 (a,b) : normalized cusp-width pair of $\Gamma < SL_2(\mathbb{Z})$: fin. ind.

Suppose the Wohlfahrt level ℓ_Γ can be decomposed

as $\ell_\Gamma = M\cdot N$

rel. prime   devided by $\mathrm{lcm}(a,b)$

$$\Longrightarrow \quad \begin{array}{ccc} p_\ell & : SL_2(\mathbb{Z}) \longrightarrow & SL_2(\mathbb{Z}/\ell\mathbb{Z}) \\ & & \wr\shortparallel \\ & \cup & SL_2(\mathbb{Z}/N\mathbb{Z})\times SL_2(\mathbb{Z}/M\mathbb{Z}) \\ & & \cup \\ & \Gamma \longrightarrow & \boxed{\begin{array}{c} \{I\}\times SL_2(\mathbb{Z}/M\mathbb{Z}) \\ \cup \\ p_\ell(\Gamma) \end{array}} \end{array}$$

(outline)

Let $N = nn'$, $n = a\cdot b' = a'b$, $kN + k'M = 1$
$(kN \equiv 1 \mod M)$

∞ : cusp w/ width $a \Rightarrow \Gamma \ni T^a$, in particular $(T^a)^{b'n'k} = T^{Nk}$ ※ $T = \begin{pmatrix}1&1\\0&1\end{pmatrix}$, $T' = \begin{pmatrix}1&0\\1&1\end{pmatrix}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\downarrow$ cusp ∞ $\qquad$ $\downarrow$ cusp 0

$\therefore\ p_\ell(\Gamma) \ni p_\ell(T^{Nk}) = \begin{pmatrix}1&0\\0&1\end{pmatrix}_{\mathrm{mod}\,N}\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix}_{\mathrm{mod}\,M} = (I,T) \in SL_2(\mathbb{Z}/N\mathbb{Z})\times SL_2(\mathbb{Z}/M\mathbb{Z})$

in the same way for T'^{b}, we have $p_\ell(\Gamma) \ni (I,T') \in SL_2(\mathbb{Z}/N\mathbb{Z})\times SL_2(\mathbb{Z}/M\mathbb{Z})$

$\therefore\ p_\ell(\Gamma) \supset \langle (I,T), (I,T')\rangle = SL_2(\mathbb{Z}/N\mathbb{Z})\times SL_2(\mathbb{Z}/M\mathbb{Z})$, qed ⊠

Theorem 2 (Proof in Section 3). *Let Γ_1 and Γ_2 be conjugated finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Suppose that Γ_1 has width a_1 at the cusp 0 and width b_1 at the cusp ∞ and Γ_2 has width a_2 at the cusp 0 and width b_2 at the cusp ∞. If*

$$(1) \qquad n_1 = \mathrm{lcm}(a_1, b_1) \text{ and } n_2 = \mathrm{lcm}(a_2, b_2) \text{ are relatively prime,}$$

then Γ_1 and Γ_2 are totally non-congruence groups, i.e. they surject onto $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ for each natural number n.

pf of Thm 2) Let $l = l_{\Gamma_1} = l_{\Gamma_2}$: common Wohlfahrt level.

$\rightarrow$ by def, n_1 & n_2 divides l.

$\rightarrow$ we can take prime factorisations $\begin{cases} n_1 = p_1^{k_1} \cdots p_r^{k_r} \\ l = \underbrace{p_1^{m_1} \cdots p_r^{m_r}}_{N} \cdot \underbrace{p_{r+1}^{m_{r+1}} \cdots p_{r+s}^{m_{r+s}}}_{M} \end{cases}$ $(m_1 \geq k_1 \cdots m_r \geq k_r)$

$\xrightarrow{\text{Prop 3.8}} p_l(\Gamma_1) \supset I \times SL_2(\mathbb{Z}/M\mathbb{Z})$

$\xrightarrow{\text{conj.}} p_l(\Gamma_2) \supset I \times SL_2(\mathbb{Z}/M\mathbb{Z})$

n_1 & n_2 : rel. prime $\Rightarrow \gcd(n_2, N) = 1$

by def n_2 devides $l = l_{\Gamma_2}$

$\Big\}$ n_2 devides M !

$\xrightarrow{\text{Prop 3.8}} p_l(\Gamma_2) \supset SL_2(\mathbb{Z}/N\mathbb{Z}) \times I$

$\xrightarrow{\text{conj}} p_l(\Gamma_1) \supset SL_2(\mathbb{Z}/N\mathbb{Z}) \times I$

$\rightarrow$ We finally obtain that $p_l(\Gamma_1)$ & $p_l(\Gamma_2)$ contain the full grp $SL_2(\mathbb{Z}/N\mathbb{Z}) \times SL_2(\mathbb{Z}/M\mathbb{Z}) = SL_2(\mathbb{Z}/l\mathbb{Z})$ $\otimes$

~~Def~~ 3.9 let $n | N$.

$t = \max\{t' \mid t' | N,\ t' \& n \text{ share the same prime divisors}\} =: \mathrm{mpde}_N(n)$

e.g. $N = p_1^{r_1} \cdots p_k^{r_k}$ $n = p_1^{s_1} \cdots p_k^{s_k}$ $(r_1 \geq s_1 \cdots r_k \geq s_k)$

$\Rightarrow t = p_1^{u_1} \cdots p_k^{u_k}$ where $u_i = \begin{cases} r_i & \text{if } s_i \neq 0 \\ 0 & \text{if } s_i = 0 \end{cases}$ $\longrightarrow$ $\underline{\text{rem}}$ $n | t$, $\gcd(n, t) = n$, $\gcd(t, N/t) = 1$

Lemma 3.10. *Suppose that we are in the situation of Theorem 2 except that in (1) it is not given that n_1 and n_2 are relatively prime. I.e. we have two conjugated groups Γ_1 and Γ_2 with normalised cusp-width pairs (a_1, b_1) and (a_2, b_2), respectively, and $n_1 = \mathrm{lcm}(a_1, b_1)$. Define $N = \mathrm{mpde}_l(n_1)$ (see Definition 3.9) and $M = l/N$ as in the proof of the theorem. Let furthermore $g_1 = \gcd(a_2, N)$ and $g_2 = \gcd(b_2, N)$. Then we have:*

$$\begin{pmatrix} 1 & g_1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ g_2 & 1 \end{pmatrix} \text{ are in } p_l(\Gamma_2).$$

§4. the L-origamis

goal: We focus on origamis in $\mathcal{H}_2(2)$ and...

Theorem 3. *Let O be an origami in $\mathcal{H}_2(2)$ with j squares and let $\Gamma(O)$ be its Veech group. We distinguish the two different cases that O is in the orbit A_j or B_j in the classification of orbits in $\mathcal{H}_2(2)$ by McMullen and Hubert/Lelièvre (see Section 4).*

i) *If j is odd, $j \geq 5$ and O is in B_j, then $\Gamma(O)$ is a totally non-congruence group, i.e. it surjects onto $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ for each $n \in \mathbb{N}$.*

ii) *If j is even, or j is odd and O is in A_j, or $j = 3$, then the deficiency $f_{\Gamma,l}$ with respect to the Wohlfahrt level l of $\Gamma(O)$ is equal to $\frac{d}{3}$. I.e. the index of its image in $\mathrm{SL}_2(\mathbb{Z}/l\mathbb{Z})$ is 3.*

Corollary 1.2. *The Veech groups of all origamis in $\mathcal{H}_2(2)$ form an* honest family of non-congruence groups, *i.e. an infinite family such that the level index e_l is bounded by a constant. More precisely their level index is bounded by 3.*

- classification the of $SL_2(\mathbb{R})$-orbits in $\mathcal{H}_2(2)$

Hubert/Lelièvre (2006, prime degree) & McMullen (2005, general)

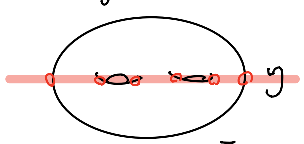
└ Theorem B The set of primitive origamis (cells are unit squares) of degree n forms the following $SL_2(\mathbb{R})$-orbits:

(i) if $n=3$ or even: one single orbit.

(ii) if $n \neq 3$ and odd: Two orbits A_n w/ one integer Weierstraß point
B_n w/ three integer Weierstraß point

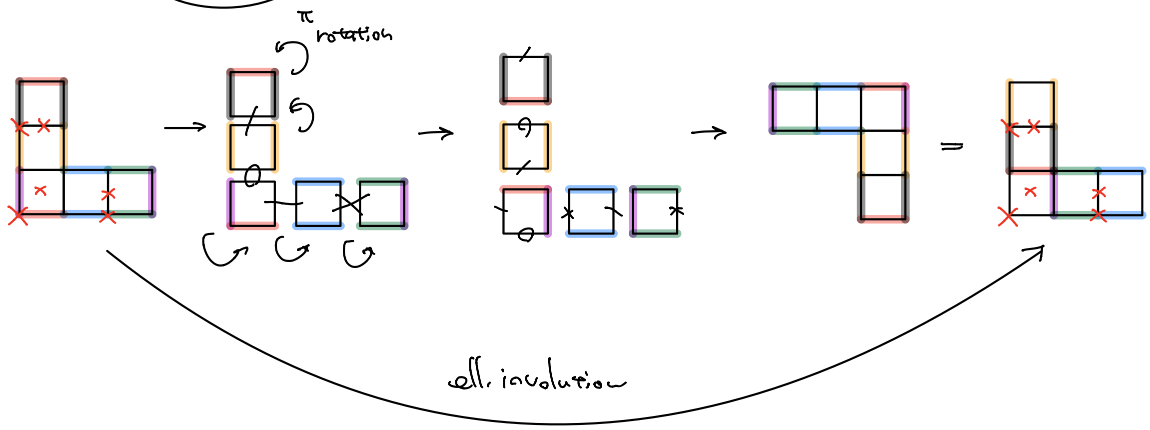
Def in genus two the Weierstraß pts are six fixed pts of the hyper elliptic involution

e.g.



$L_{3,3}$

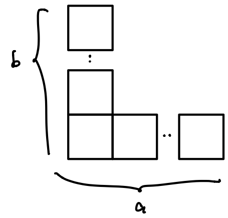
in B_5



$L_{2,4}$ is in A_5.

Weierstraß pt at a vertice of a cell is called an integer Weierstraß pt.

⊖ L-shaped origami $L_{a,b} = L(a,b)$



$\leftarrow$ of degree $n = a+b-1$

· One can directly observe that $\Gamma(L_{a,b})$ contains T^a & T'^b.

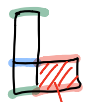
Furthermore, $\langle T^a, T'^b \rangle < \Gamma(L_{a,b})$ is the maximal parab. grp in $\Gamma(L_{a,b})$.

☹ $\forall A \in \Gamma(L_{a,b})$: parab. w/ eigenvect. $\binom{1}{0}$

$D \uparrow$

f : affine map on $L_{a,b}$.

$\Rightarrow$ f has to permute the three horiz. saddle conns.



a unique one lies on the bdry of only one cylinder.

$\uparrow$ this property is preserved by affine maps.

in particular, the saddle conn. should be preserved pointwise.

$\rightarrow$ A is a power of T^a

the same argument is valid for T'^b

⊗ ?

Theorem 3. *Let O be an origami in $\mathcal{H}_2(2)$ with j squares and let $\Gamma(O)$ be its Veech group. We distinguish the two different cases that O is in the orbit A_j or B_j in the classification of orbits in $\mathcal{H}_2(2)$ by McMullen and Hubert/Lelièvre* (*see Section* 4).

i) *If j is odd, $j \geq 5$ and O is in B_j, then $\Gamma(O)$ is a totally non-congruence group, i.e. it surjects onto $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ for each $n \in \mathbb{N}$.*

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Corollary 1.2. *The Veech groups of all origamis in $\mathcal{H}_2(2)$ form an* honest family of non-congruence groups, *i.e. an infinite family such that the level index e_l is bounded by a constant. More precisely their level index is bounded by 3.*

pf : (i) We distinguish cases : (A) $j \equiv 1 \mod 4$ & $j \neq 5$

(B) $j \equiv 3 \mod 4$ & $j \neq 7$

(C) $j = 5$ or 7

<u>1st step</u> (A) $j \equiv 1 \mod 4$ & $j \geq 9$.

$\Rightarrow j = 2a-1$ w/ $a \geq 5$; odd

by <u>THM B</u> (H.-L. & McMullen's thm), we know that

$L_{a,a}$ & $L_{a+2,a-2}$ lies in the orbit B_j.

we need $a-2 \geq 2$ (satisfied)

$\rightsquigarrow$ $\Gamma_1 = \Gamma(L_{a,a})$ & $\Gamma_2 = \Gamma(L_{a+2,a-2})$ are conjugated.

($\because$ they belongs to one $SL_2(\mathbb{R})$-orbit B_j.)

The normalized cusp-width $= \begin{cases} \Gamma_1 : (a,a) \to n_1 = \mathrm{lcm}(a,a) = a \\ \Gamma_2 : (a+2, a-2) \to n_2 = \mathrm{lcm}(a-2, a+2) \end{cases}$

since $a \geq 5$: odd, n_1 & n_2 are rel. prime !

By thm 2, we have $p_\ell(\Gamma_1) = p_\ell(\Gamma_2) = SL_2(\mathbb{Z}/\ell\mathbb{Z})$ where $\ell = \ell_{\Gamma_1} = \ell_{\Gamma_2}$: Wohlfahrt level

$\boxtimes$

<u>2nd step</u> $j \equiv 3 \mod 4$ & $j \geq 11$

$j = 2a-1$ w/ $a \geq 6$: even

again by <u>THM B</u>, $L_{a+1,a-1}$ & $L_{a+3,a-3} \in B_j$

$\Gamma_1 = \Gamma(L_{a+1,a-1})$ & $\Gamma_2 = \Gamma(L_{a+3,a-3})$: conjugated.

The normalized cusp-width $= \begin{cases} \Gamma_1 : (a+1, a-1) \to n_1 = \mathrm{lcm}(a+1, a-1) \\ \Gamma_2 : (a+3, a-3) \to n_2 = \mathrm{lcm}(a+3, a-3) \end{cases}$

since $a \geq 6$: even, n_1 & n_2 are rel. prime !

By thm 2, we have $p_\ell(\Gamma_1) = p_\ell(\Gamma_2) = SL_2(\mathbb{Z}/\ell\mathbb{Z})$ where $\ell = \ell_{\Gamma_1} = \ell_{\Gamma_2}$: Wohlfahrt level

$\boxtimes$

<u>3rd step</u> $j = 5$ or 7 : it can be concretely calculated !

$\boxtimes$. q.e.d.

(ii) is proved in similar way.

§5. One zero strata

Thm 2 & methods used in §4 are quite general.

We exemplary use them to construct an infinite family of of origamis of genus $^\forall g \geq 3$

w/ totally non-congruence Veech gps.

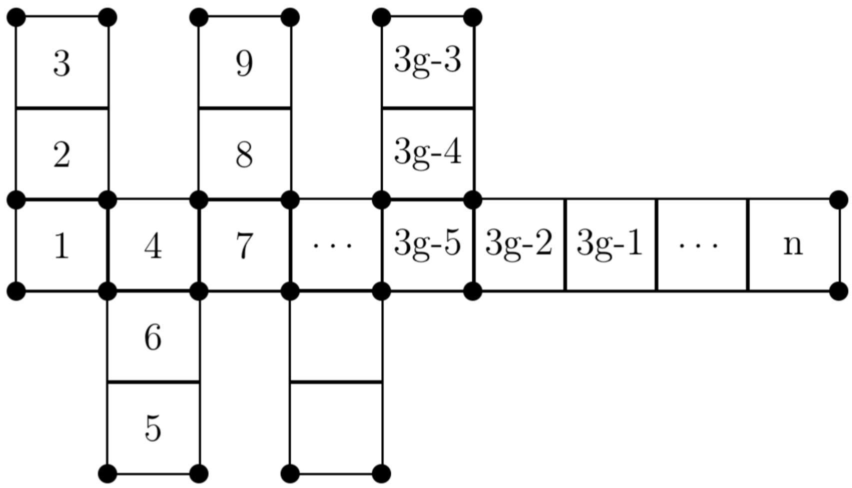

Figure 5: The origami $O_{g,n}$. Opposite edges are glued

We observe that there is an origami $A \cdot O_{g,n}$ $(A \in SL_2(\mathbb{Z}))$ in the same orbit

which has only one cylinder w.r.t. horizontal & vertical.

↓ then we may see ...

Corollary 5.4. *The Veech group $\Gamma(A \cdot O_{g,n})$ contains T^n and T'^n. Thus for its normalised cusp-width pair (a', b') we have that a' and b' divide n.*

We now can conclude Theorem 4 from the following proposition.

Proposition 5.5. *If n is coprime to 3 and coprime to $2g-2$, then the Veech group $\Gamma(O_{g,n})$ is a totally non congruence group.*

<u>Theorem 4</u> For each $g \geq 3$, the stratum $\mathcal{H}(2g-2)$ contains an infinite family of origamis whose VGs are totally non-congr. gps.