COMPUTING VEECH GROUPS

BRANDON EDWARDS, SLADE SANDERSON, AND THOMAS A. SCHMIDT

ABSTRACT. We present an algorithm, based on work of the first named author, for computing the Veech group, $\mathrm{SL}(X, \omega)$, of a translation surface (X, ω). This presentation is informed by an implementation of the algorithm by the second named author.

For each stratum of the space of translation surfaces, we introduce an infinite translation surface containing in an appropriate manner a copy of every translation surface of the stratum. We show that a matrix is in $\mathrm{SL}(X, \omega)$ if and only if an associated affine automorphism of the infinite surface sends each of a finite set, the "marked" *Voronoi staples*, arising from orientation-paired segments appropriately perpendicular to Voronoi 1-cells, to another pair of orientation-paired "marked" segments. ("Segments" are also known as "saddle connections".)

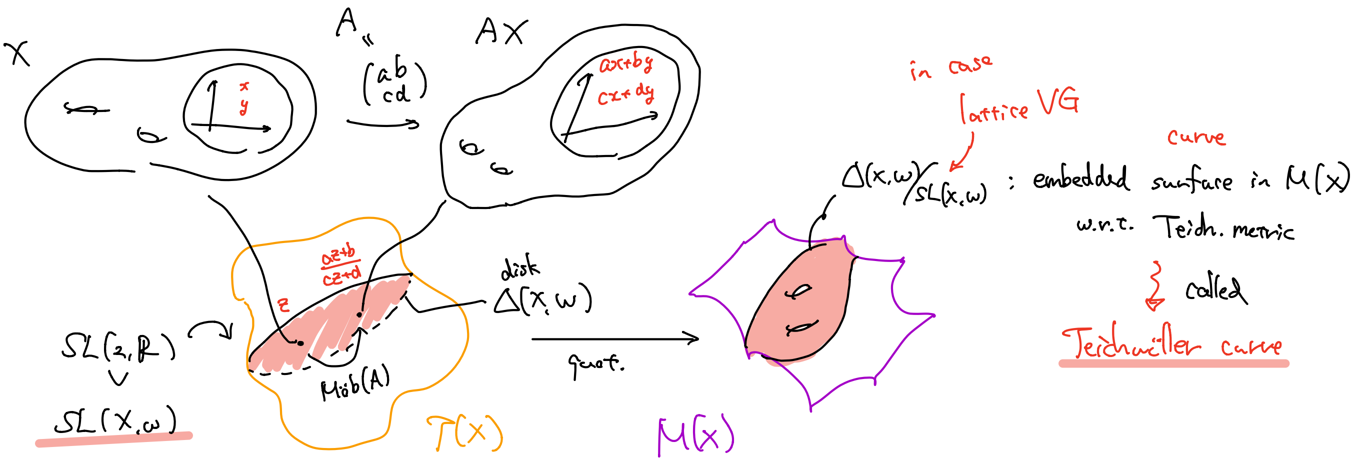
We prove a result that is of independent interest. For each real $a \geq \sqrt{2}$ there is an explicit hyperbolic ball such that for any Fuchsian group trivially stabilizing i, the Dirichlet domain centered at i of the group already agrees within the ball with the intersection of the hyperbolic half-planes determined by the group elements whose Frobenius norm is at most a. When $\mathrm{SL}(X, \omega)$ is a lattice we use this to give a condition guaranteeing that the full group $\mathrm{SL}(X, \omega)$ has been computed.

arXiv. 2012.12444 (2020, Dec)

1. Intro

- Veech group $SL(X, \omega)$ of a translation surface (X, ω)

  … the stabilizer of isom. class under the affine-deform. action of $SL(2, \mathbb{R})$.



in case lattice VG

$\Delta(X,\omega)/SL(X,\omega)$: embedded ~~surface~~ curve in $M(X)$ w.r.t. Teich. metric

called Teichmüller curve

(Veech 1989) $^\forall$Veech group is a non-cocpt Fuchsian group.

Certain triangle grps arises as VG.

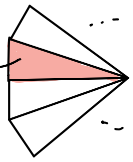
(Bouw-Möller 2010) up to finite index, every non-cocpt Fuchsian triangle grp arises as VG.

On the other hand, (Möller 2009) for a.e. trans. surf. $VG = \{I\}$.

(Kenyon, Smillie 2000 / Puchta 2001)

the only 3 non-isosceles acute triangles unfold to induce Teich. curves.

w/ rational angle



unfolding ⇝ trans. surf.

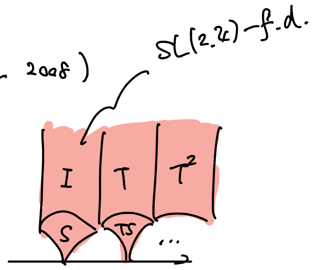
Open Prob. which non-cocpt Fuchsian grps can be VG?

(Kenyon, Smillie 2000) certain equality of trace field for VG containing a hyp. element.

In this paper, we give a new algorithm for computing VGs.

- for lattice VG, our algorithm completely calculates VG.
- approach different from
  - (Bowm. 2010)
  - special case algorithms (Schmithüsen 2004 / Freidinger 2008)

As w/ most of others, our algorithm
- determines VG elements
- builds a fundamental domain in $\mathbb{H}$. like



To determine elements, we view a trans. surf.

as being $\bigsqcup$(Voronoi 2-cells) w/ shared edges identified.

We introduce model surface for each stratum

into which the Voronoi 2-cells inject.

§2. background

translation surface
- i) Surface w/ atlas outside finite set Σ whose transitions are $z \mapsto z + c$
- ii) Surface w/ an abelian differential ω.
- iii) collection of Euclidian polygons whose edges are glued by translations.

$$\boxed{\begin{array}{l} z_\omega(p) = \int^p \omega \\ \Sigma = \mathrm{Crit}(\omega) \end{array}}$$

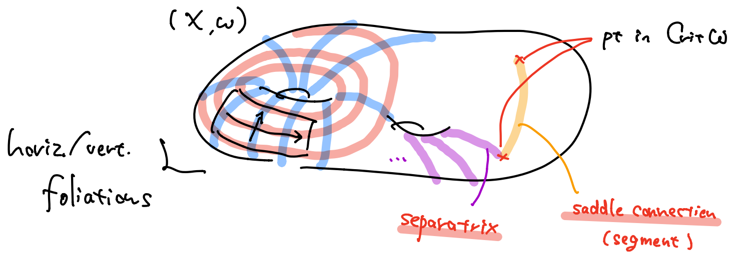
affine diffeomorphism : $f : (X, \omega) \xrightarrow{\text{homeo}} (X, \omega)$: loc. affine w.r.t. z_ω

Jacobian derivative of —— : constant matrix in $SL(2, \mathbb{Z})$.

$\rightsquigarrow$ $SL(X, \omega) = \{ \mathrm{der}(f) \mid f : \text{affine diffeo on } (X, \omega) \}$

… **Veech group**

- The horizontal foliation on $\mathbb{R}^2$ induces a foliation on $X \setminus \Sigma$.



For a saddle conn. γ, $\mathrm{hol}_\gamma = \int_\gamma \omega$: **holonomy vector**

- **Stratum** $\mathcal{H}(\overset{d}{d_1, d_2, \dots, d_s}) := \{ (R, \omega) : \text{trans. surf.} / \{ \mathrm{ord}_p \omega \mid p \in \mathrm{Crit}\, \omega \} = \{ d_1, \dots, d_s \} \}$

$\rightarrow$ by Riemann-Roch's or Gauss Bonnet's,

$\forall (X, \omega) \in \mathcal{H}(d)$ share the same genus.

sometimes denoted by $\Omega M_g(d)$.

- **Voronoi decomposition**

pull back of Euclidian metric on $\mathbb{R}^2$ via z_ω.

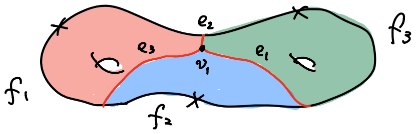
Let $(X, \omega) \in \mathcal{H}(d)$. We take the natural flat metric on X.

Voronoi 2-cell f_p corresponding to $p \in \mathrm{Crit}\, \omega$ is defined by

$$f_p = \{ q \in X \mid d(p, q) \leq d(p', q) \ \forall p' \in \mathrm{Crit}\, \omega \}^{\mathrm{int}}.$$

Bdries of Voronoi 2-cells are unions of
- **Voronoi 1-cells** : open geodesics
- **Voronoi 0-cells** : bdry pts of 1-cells

exm



§3. A model surface, Voronoi staples, and a VG membership criterion.

given $\mathcal{H}(d_1, d_2, \cdots, d_s)$, for each $i = 1, 2, \cdots, s$. (let $d_1 \le d_2 \le \cdots \le d$)

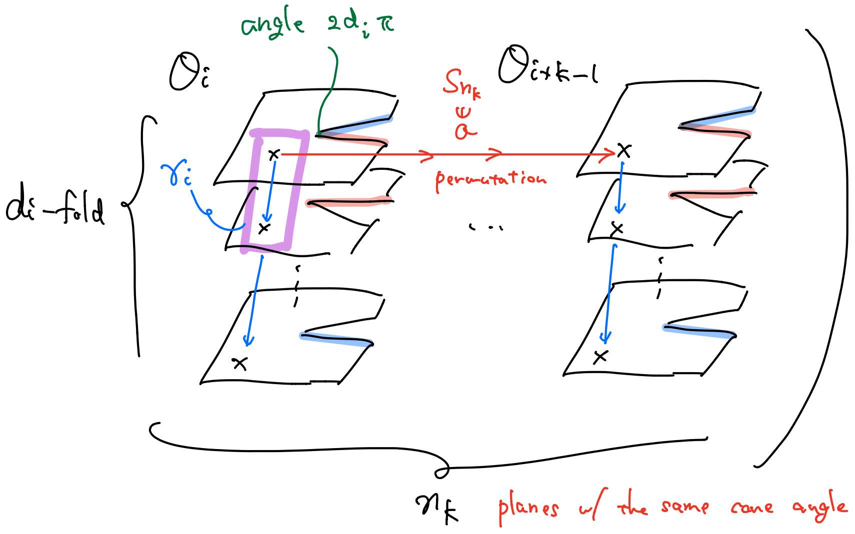
Let $\mathcal{O}_i := (\mathbb{R}^2, z^{d_i}\, dz)$ & $\mathcal{O} = \coprod_{i=1}^{s} \mathcal{O}_i$.

We partition $\{1, \cdots, s\}$ w/ $n_1, n_2, \cdots, n_t$ ($\Sigma n_k = s$.)

where $d_1 = d_2 = \cdots = d_{n_1}$, $d_{n_1+1} = \cdots = d_{n_1+n_2}$, $\cdots$, and so on. (corr. to # repeated values.)

Lemma 1 $\underset{\mathrm{Ker}(\mathrm{der}|_{\mathcal{O}_i})}{\underset{\shortparallel}{\mathrm{Trans}(\mathcal{O}_i)}} = \langle \gamma_i \rangle$ where $\gamma_i^{d_i+1} = 1$.

$$\mathrm{Trans}(\mathcal{O}) \cong \Big\langle \overbrace{S_{n_1}, S_{n_2}, \cdots, S_{n_t}}^{\text{permutation}}, \overbrace{\gamma_1, \gamma_2, \cdots, \gamma_s}^{\text{rotation}} \Big\rangle$$



Trans($\mathcal{O}$) is generated by those elements.

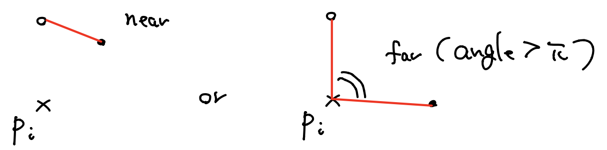
Lem singularities $0_{\mathcal{O}_i}$ should be fixed.

Lemma 2 $\forall A \in GL_2(\mathbb{R})$. $\exists f_A \in \mathrm{Aff}^+(\mathcal{O})$ s.t. $\mathrm{der} f_A = A$.

pf) $\zeta \mapsto A\zeta$ ($\zeta_{(p)} = \int_0^p z^{d_i}\, dz$: natur. coord.) belongs to $\mathrm{Aff}^+(\mathcal{O}_i)$ for $\forall i$. ⊗

Lemma 3 $\forall$ two distinct pts on $\mathcal{O}_i$ $\exists !$ geod. connecting them

which is :



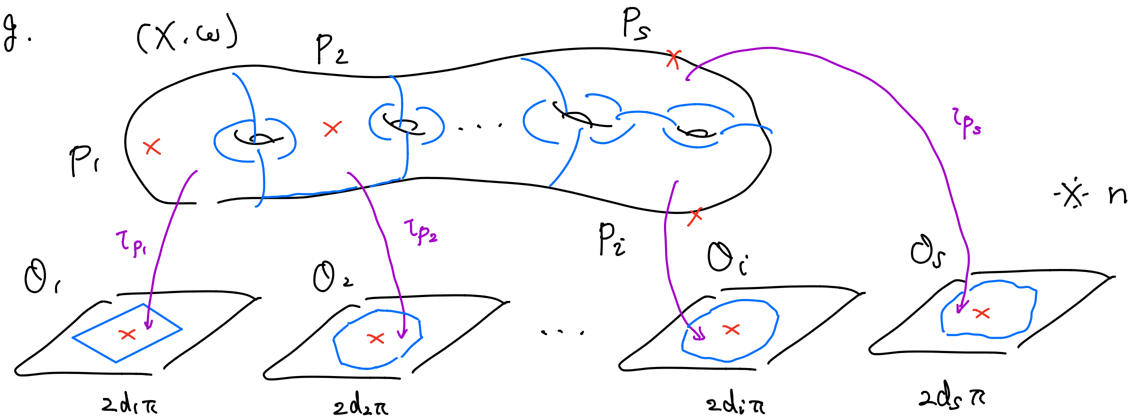
cf) Strebel, 1984

Lemma 4 Let $(X, \omega) \in \mathcal{H}(d_1, d_2, \cdots, d_s)$

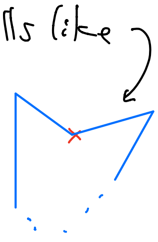
Then there is an isometric injection

$\coprod_i$ (Voronoi 2-cells $\sim d_i$) $\hookrightarrow \mathcal{O} = \coprod_i \mathcal{O}_i$ where $f_i \hookrightarrow \mathcal{O}_i$.

e.g.



※ no cells like



Def 5 We will assume throughout that an isometric embedding ι as above,

For each $p \in \mathrm{Crit}\,\omega$ ~ order d_i,

We denote by $\left\{ \begin{array}{l} \mathbb{O}_p : \mathbb{O}_i, \\ C_p : \text{the Voronoi 2-cell centred at } p\text{, and} \\ \iota_p : C_p \hookrightarrow \mathbb{O}_p \;:\; \text{the restriction of } \iota. \end{array} \right.$

Note that

$$(X.\omega) \underset{\text{isom}}{\cong} \bigsqcup_{p\in\mathrm{Crit}\,\omega} \overline{C_p} \Big/ \sim \quad \text{for some gluing rule} \sim \text{of Voronoi 1-cells.}$$

From the data of $\overline{C_{p_1}}, \cdots, \overline{C_{p_s}}$ for X,

$\iota(C_{p_1}), \cdots, \iota(C_{p_s})$ have a collection of segments which are glued to recover $(X.\omega)$,

as in the following:

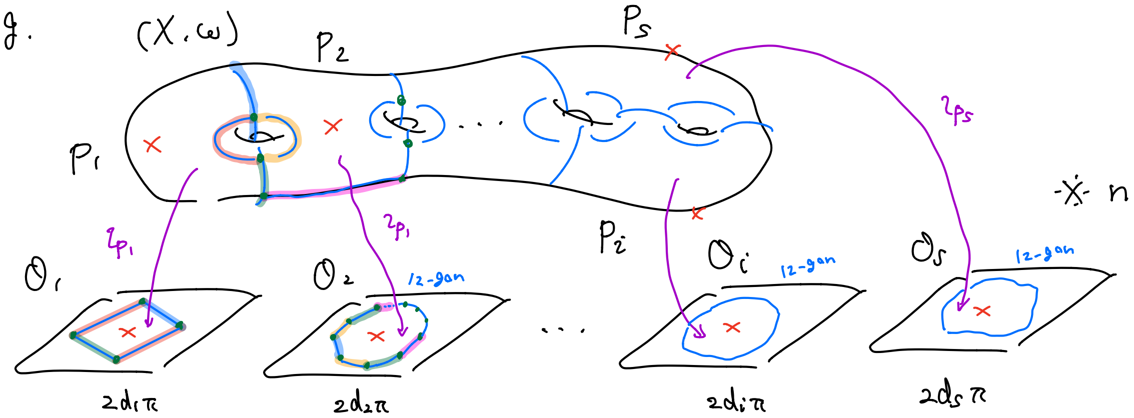
Prop 6 Let $\Upsilon := \bigsqcup_{p\in\mathrm{Crit}\,\omega} \overline{\iota(C_p)}$.

${}^{\forall}p \in \mathrm{Crit}\,\omega$, ${}^{\forall}\gamma \subset \partial\iota(C_p)$: bdry segment,

${}^{\exists !}p' \in \mathrm{Crit}\,\omega$ ${}^{\exists !}\gamma' \subset \partial\iota(C_{p'})$: ≈

s.t. the relation ($\gamma \sim \gamma'$ for ${}^{\forall}\gamma$ as above) makes a trans. surf. $\Upsilon/\sim$ isom. to $(X.\omega)$.

e.g.



Def 7. Let $j : Y = \sqcup \mathcal{O}_p/\sim \to X$ be the isometry constructed from $(\tau_p^{-1})_{p \in \mathrm{Crit}\,\omega}$.

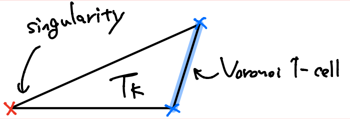
$\hat{z} := j^{-1}(z)$, $\hat{e} := j^{-1}(e)$ for $z \in R$: point & $e \subset R$: edge (segment)

For $z \in \mathcal{O}_p \setminus \{0\}$, let $D(z) := B(z : d(0,z)) \subset \mathcal{O}_p$: open disk.

Prop 8 Let $e \subset \overline{C_p}$: edge & $z \in e$: point.

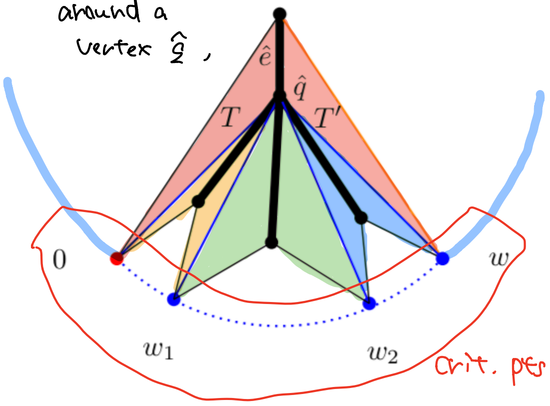
Then $D(\hat{z})$ is contained in $\bigcup_{k=1}^{\exists m} T_k$: union of triangles s.t.

say: "Voronoi triangle"



In particular, (as a bdry behavior) j maps $D(\hat{z})$ to an open set of $X \setminus \Sigma$.

pf) around a vertex $\hat{q}$,



by Prop 6 & dfn of Voronoi decomposition,

the reflection T' of a Voronoi triangle T

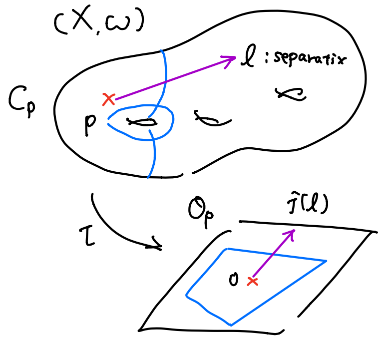
is again ______ .

Thus $D(\hat{q}) \subset \underset{\substack{\| \\ P(\hat{q})}}{\bigcup} \{ T \cup T' \mid T : \text{Voronoi triangle neighboring } \hat{q}. \}$

& for $\forall \hat{z} \in \hat{q}_1 \hat{q}_2$: edge $D(\hat{z}) \subset P(\hat{q}_1) \cup P(\hat{q}_2)$ holds. ⊠

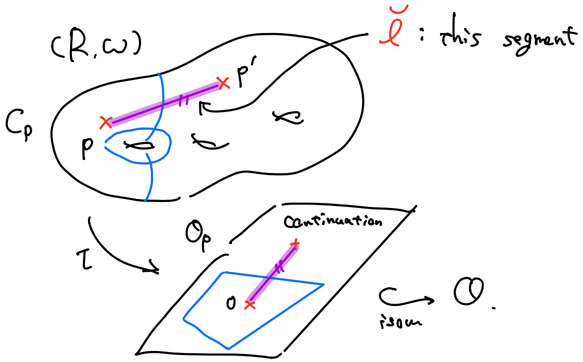
3.2. Closed Valonoi 2-cells as convex bodies on $\mathbb{C}$.

Def 9 (X, ω)



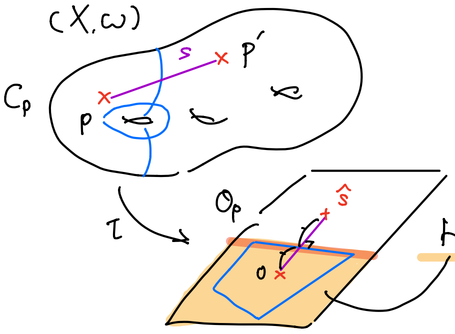
$\rightsquigarrow$

if ℓ meets w/ some singularity,



We denote the embedded image of $\breve{\ell}$ by $\hat{\imath}(\ell)$.

Prop 11 $\forall s$: saddle conn. of (X, ω).



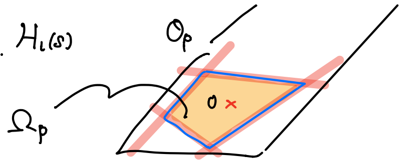
$H_0(s) := \{ z \in \mathcal{O}_p \mid d(0,z) \leq d(\hat{s}, z) \}$ ⋯ the half space of s

Then, $\partial H_0(s)$ is the prependicular bisector of s.

Prop 13 For $p \in \mathrm{Crit}\,\omega$, we define the convex body

$$\Omega_p := \bigcap_{s \ni p : s.c.} H_\iota(s)$$

Then, $\Omega_p = \overline{\iota(C_p)}$.



3.3. Voronoi staples determine (X,ω)

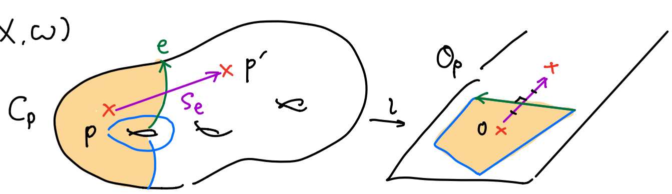
Def 14 For each saddle-conn. s, we fix an orientation

Let :

$$M(X,\omega) := \{(s,s^-) \mid s : \text{saddle-conn. on } (X,\omega)\}$$

$$\hat{M}(X,\omega) := \{(\hat\iota(s), \hat\iota(s^-)) \mid \quad 〃 \quad \}$$

Consider : (X,ω)



Every edge e of C_p is associated to a saddle-conn. s_e which is a prependicular bisector in O_p.

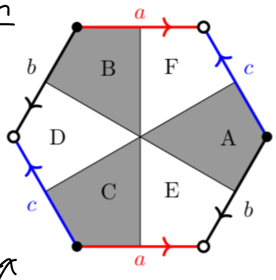
Let $\mathcal{E}_p^\perp := \{ s_e : \text{saddle conn.} \mid e : \text{edge of } C_p \} \subset SC(X,\omega)$

$$S(X,\omega) := \{(s,s^-) \in M(X,\omega) \mid s \in \mathcal{E}_p^\perp \ \ p \in \mathrm{Crit}\,\omega\}$$

$$\hat{S}(X,\omega) := \{(\hat\iota(s), \hat\iota(s^-)) \in \hat{M}(X,\omega) \mid \quad 〃 \quad \}$$

set of marked Voronoi staples

exm



regular hexagon torus.

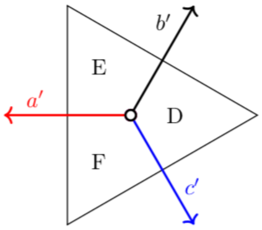


image in $\mathcal{O}_1$

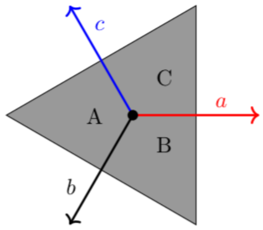
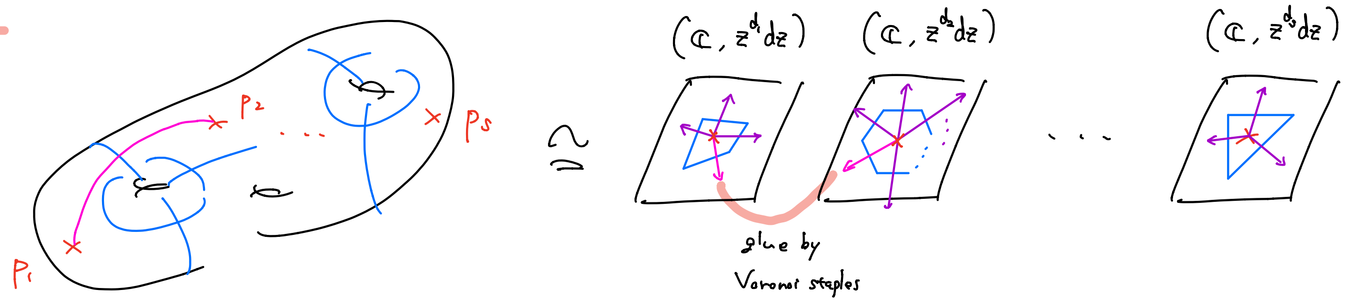


image in $\mathcal{O}_2$

there are two Voronoi 2-cells and three Voronoi staples.

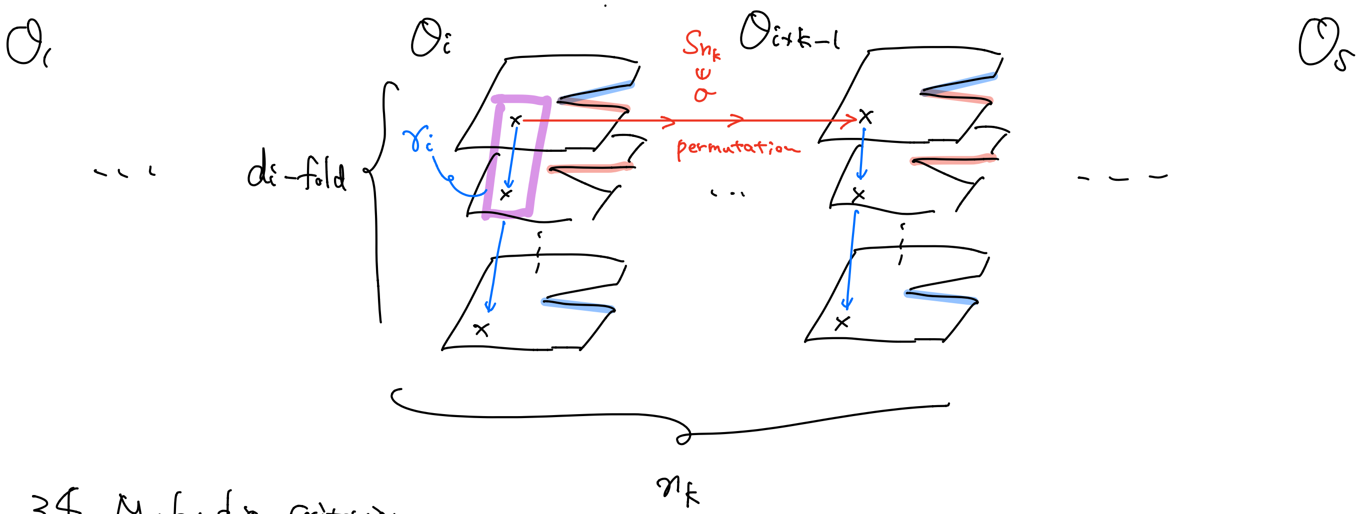
We obtain (X,ω) by gluing Voronoi 2 cells by 'staples.'

Summary



Theorem 15. *Suppose that both $(X_1, \omega_1), (X_2, \omega_2)$ belong to $\mathcal{H}(d_1, \ldots, d_s)$ and choose maps ι_1, ι_2 to $\mathcal{O}$. Then $(X_1, \omega_1), (X_2, \omega_2)$ are equivalent translation surfaces if and only if $\widehat{\mathcal{S}}(X_1, \omega_1)$ and $\widehat{\mathcal{S}}(X_2, \omega_2)$ are in the same* Trans($\mathcal{O}$)*-orbit.*

recall $\quad \mathrm{Trans}(\mathcal{O}) \simeq \langle \overbrace{S_{n_1}, S_{n_2}, \cdots, S_{n_t}}^{\text{permutation}}, \overbrace{\gamma_1, \gamma_2, \cdots, \gamma_s}^{\text{rotation}} \rangle$



3.4 Membership Criterion

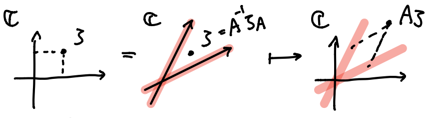
"$SL(2,\mathbb{R}) \ni A$ is membership in $SL(X,\omega)$"

recall $\quad f_A : \mathbb{C} \to \mathbb{C}$: the natural affine deformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \quad \text{where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

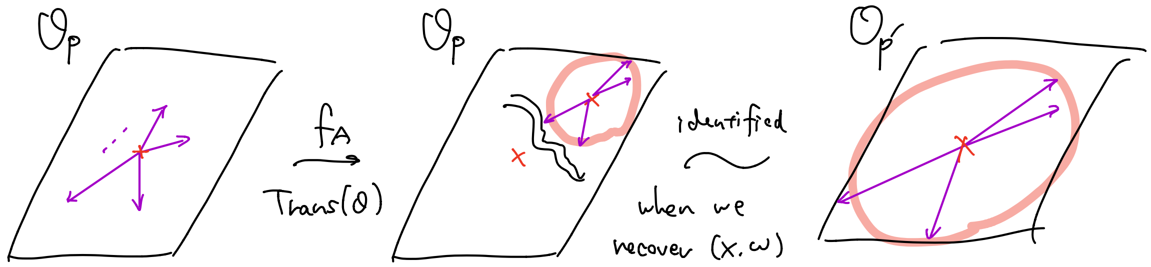
Proposition 16 (Membership Criterion, initial form)**.** *Fix (X, ω) and suppose $A \in \mathrm{SL}_2\mathbb{R}$. Then* $A \in \mathrm{SL}(X, \omega)$ *if and only if f_A sends $\widehat{\mathcal{S}}(X, \omega)$ to a subset of $\widehat{\mathcal{M}}(X, \omega)$, up to some element of* Trans($\mathcal{O}$).

$$\begin{tikzcd}
\mathcal{M}(X,\omega) \arrow[d, "\iota"] \arrow[dr, hook] & & & & \mathcal{M}(A\cdot(X,\omega)) \arrow[llll, "\mathrm{Id}_X"'] \arrow[d, "\iota_A"] \arrow[dl] \\
& (X,\omega) \arrow[rr, bend left] \arrow[d, "\hat{\iota}"] & \circlearrowleft & A\cdot(X,\omega) \arrow[d, "\hat{\iota}_A"] & \\
\widehat{\mathcal{M}}(X,\omega) \arrow[r, dashed, leftrightarrow] & \mathcal{O} \arrow[r, "\mathrm{Id}_{\mathcal{O}}"] \arrow[rr, bend left, "f_A"] & A\cdot\mathcal{O} \arrow[r, "\widehat{F}_A"] & \mathcal{O} & \widehat{\mathcal{M}}(A\cdot(X,\omega)) \arrow[l, dashed, leftrightarrow]
\end{tikzcd}$$



$\zeta_A = A\cdot\zeta$

($\Leftarrow$)



For each $p \in \mathrm{Crit}\,\omega$, f_A sends each origin O_p & staples emanating from O_p

to a point & segments emanating from it

which will be identified w/ $O_{p'}$ & staples emanating from $O_{p'}$

for some $p' \in \mathrm{Crit}\,\omega$.

So, if $\sigma = (f_A(a), f_A(a')) \in \hat{M}(X,\omega)$ for $(a,a') \in \hat{S}(X,\omega)$

it implies that $\sigma \in \hat{S}(X,\omega)$.

Thus "$f_A(\hat{S}(X,\omega)) \subset \hat{M}(X,\omega)$ up to Trans O"

implies that f_A keeps the same Voronoi staples picture up to permutation.

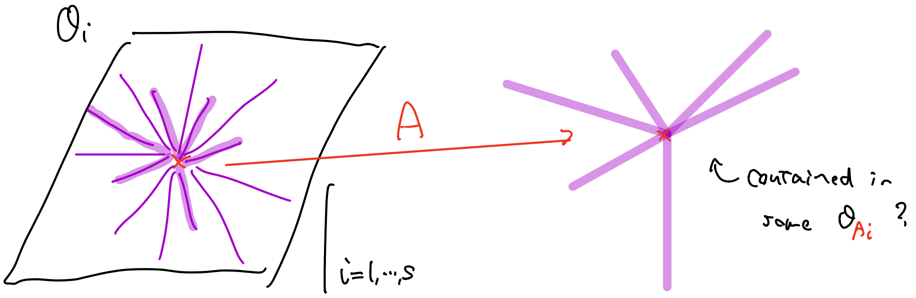
Since Voronoi staple set $\hat{S}(X,\omega)$ has enough data to recover (X,ω),

we have the conclusion $\otimes$

We may apply this criterion

by observing the planar geometry on $O = \bigsqcup O_i$

concerning $\hat{S}(X,\omega)$ & $\hat{M}(X,\omega)$



$$\boxed{A : \begin{array}{ccc}\{1,2,\dots,s\} & \to & \{1,2,\dots,s\} \\ \cup & & \cup \\ i & \longmapsto & Ai\end{array}}$$

rotation & permutation ?

(i) build $\hat{M}(X,\omega)$ w/ $\hat{S}(X,\omega)$

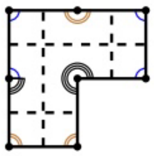
(ii) check whether $A \cdot \hat{S}(X,\omega)\big|_{O_i} \subset \hat{M}(X,\omega)$

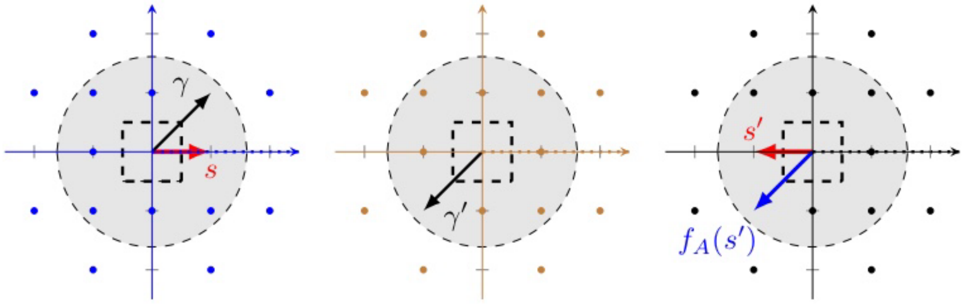
(iii) check whether A acts on $\hat{S}(X,\omega)$ as Trans(O).

We may use some geometric data

to improve the membership criterion.

- bounds of lengths of $\hat{S}(X,\omega)$





<u>Corollary 19</u> Let $\ell = \max\{\mathrm{len}(s) \mid (s,s') \in S(X,\omega)\}$

Suppose $A \in SL(2,\mathbb{R})$ has maximum singular value $\leq \nu \in [1,\infty)$
(eigenvalue ??)

$\Rightarrow A \in SL(X,\omega)$ iff f_A maps $\hat{S}(X,\omega)$ into $\hat{\mu}(X,\omega)$ w/ max length $\leq \nu\ell$

upto $\mathrm{Trans}(O)$.

- Frobenius norm of $A \in SL(2,\mathbb{R})$

… $\|A\| = \sqrt{\mathrm{tr}(A^t A)}$

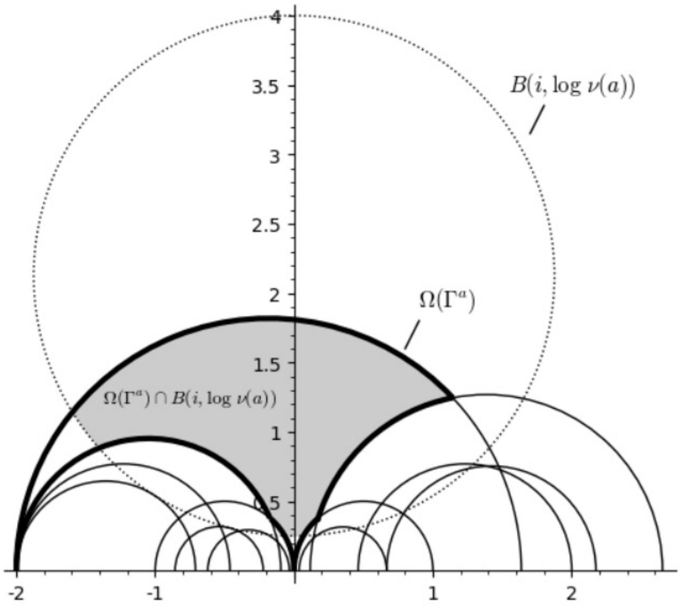
For $\Gamma < SL(2,\mathbb{Z})$, $\Gamma^a := \{A \in \Gamma \mid \|A\| \leq a\}$

hyp. ball

Theorem 25. *Suppose that $a \geq \sqrt{2}$ is such that*

$$\mu_{\mathbb{H}}(\Omega(\Gamma^a)) < 2\,\mu_{\mathbb{H}}(\Omega(\Gamma^a) \cap B(i, \log\nu(a))). \qquad \nu(a) = \sqrt{\frac{a^2 + \sqrt{a^2-4}}{2}}$$

Then the subset Γ^a generates Γ. In particular, Γ is a lattice.



shaded region shows where the computed intersection of half-
ments of small Frobenius norm must agree with the Dirichlet
ll group, as guaranteed by Proposition 24. Here, $\Gamma = \mathrm{SL}(M \cdot$
$X,\omega)$ is given in Example 20, see also Figure 6, and $M = \left(\begin{smallmatrix} 1 & 0 \\ 1/2 & 1 \end{smallmatrix}\right)$
n a surface whose Veech group has trivial stabilizer of $z = i$.
'4.

· Basic steps of algorithm

(I) Given a polygonal presented (X, ω),

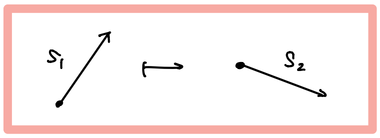
one can calculate the Voronoi decomposition.

And also $\hat{S}(X,\omega)$ & ℓ : maximal length are knowable.

We test membership using Cor 19.

(II) We may take candidate matrices

by choosing two distinct saddle-conns.



(III) Beginning w/ $a = \sqrt{2}$.

$P := \{ s :$ saddle-conn w/ length $\leq \nu(a)\,\ell \}$: finite

We then check whether

f_A maps $\hat{S}(X,\omega)$ into P modulo $\mathrm{Trans}\,\mathcal{O}$.

This test is sufficient to obtain generating set of $SL(X,\omega)$ $\boxtimes$.