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RIMS Kôkyûroku (2012)
(Lect. Note)

Calculation of Grothendieck dessin of genus 1.

種数 1 の Grothendieck dessin の計算[†]

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0. Intro

Grothendieck's dessin d'enfants describe coverings of $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}$ = Belyi curves = arithmetic alg. curves $\curvearrowleft G_{\mathbb{Q}} = \mathrm{Gal}\,\bar{\mathbb{Q}}/\mathbb{Q}$

Calculation to obtain $\begin{pmatrix} G_{\mathbb{Q}}\text{-orbit} \\ \text{alg. equation} \end{pmatrix}$ from given dessin is systematically studied in the case $g=0$ by Malle, Schneps, et. al.

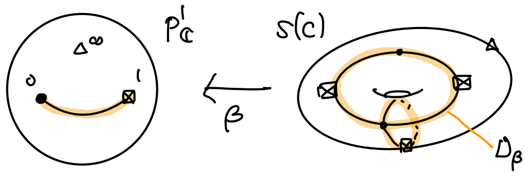
In this paper we deal w/ exms of such a calculation in the case $g=1$.

1. Belyi morphism & Grothendieck dessin

Def 1.1 Let C : complete, non singular (via homogenization) algebraic curve $/\mathbb{C}$

A (covering) morphism $C \to \mathbb{P}^1_{\mathbb{C}}$ ramified over $0, 1, \infty \in \mathbb{P}^1_{\mathbb{C}}$ only is called a Belyi morphism. (A pair (C, β) : Belyi pair)

Then, $D_\beta := \beta^{-1}([0,1]) \subset S(C)$: bipartite graph embedded in $S(C)$



- $\# \mathcal{E}(D_\beta) = \deg \beta$.
- $\mathcal{V}(D_\beta) = \beta^{-1}(0) \sqcup \beta^{-1}(1)$: bicoloured $<$ • : pt in $\beta^{-1}(0)$, ⊠ : $\beta^{-1}(1)$ resp.
- $\mathcal{F}(D_\beta) = \beta^{-1}(\infty)$.
- $^\forall$face : $^\forall$conn. component of $S(C) \setminus D_\beta$ is homeo. to a disk. —(☆)
- (valency at a vertex : # edges emanating from it) = ord. β
  (" face : # edges around it $/2$) = $-$ord. β

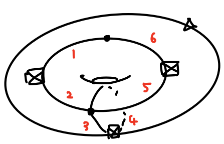
Conversely,

Def 1.2 For a bipartite graph D embedded in a surface S satisfying (☆), there exists a Belyi pair (C, β) s.t. $(S(C), D_\beta) \overset{\text{homeo.}}{\cong} (S, D)$

Such a pair (S, D) is called a (Grothendieck) dessin d'enfants.

We denote the valency list of dessin by the form $[\underbrace{m_1^\infty, m_2^\infty, \cdots}_{\text{valencies}/\infty} ; \underbrace{m_1^0, m_2^0, \cdots}_{\text{val.}/0} ; \underbrace{m_1^1, m_2^1, \cdots}_{\text{val.}/1}]$ $\left(\sum_j m_j^i = d \text{ for } ^\forall i = 0, 1, \infty\right)$

exm



$\leftarrow [6 ; 2, 4 ; 2, 2, 2]$ - dessin

2. Galois action on dessins

Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Thm (Belyi, 1979)

A complete, nonsingular algebraic curve $/\mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ if and only if it admits a Belyi morphism (dessin).

$G_{\mathbb{Q}} \curvearrowright$ (dessin) induced from $G_{\mathbb{Q}} \curvearrowright$ (alg. curve $/\overline{\mathbb{Q}}$ & rational fct on it) defined by the transformations of coeffs cients.

$$\left\{ \begin{array}{l} \{\sum a_{ij} X^i Y^j\} \xrightarrow{\sigma} \{\sum \sigma(a_{ij}) X^i Y^j\} \\ R(X,Y) = \sum b_{ij} X^i Y^j \xrightarrow{\sigma} R^{\sigma}(X,Y) = \sum \sigma(b_{ij}) X^i Y^j \end{array} \right.$$

Goal
- to obtain the defining equation of (C, β) from D_β
- to obtain the $G_{\mathbb{Q}}$-orbit of a dessin

To do these, first we consider a criterion for determining the Galois-stability of dessin as detailed as possible.

For exa, sometimes we may observe that a dessin shares some Galois invariants w/ no other possible patterns.

Then we see that such a dessin has no Galois conjugate, and hence defined over $\mathbb{Q}$,

cf. Beckman (1989): $^\forall$prime number not dividing $|\mathrm{Mon}(D)|$ is unramified in the defining field of D.

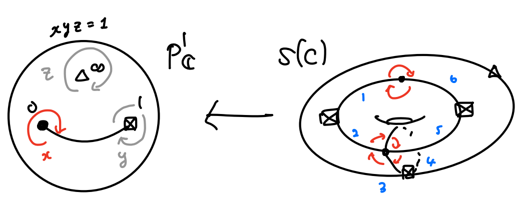
3. Basic Galois invariant.

The following are known as basic Galois invariants of a dessin.

- valency list $\leftarrow$ algebraic info.
- degree $\leftarrow$ Σ. valency
- genus $\leftarrow$ Riemann-Hurwitz
- monodromy grp. $\mathrm{Mon}(D) < G_d$
- Nielsen class $NC(D)$
- Cartographic grp. $\mathrm{Car}(D) < G_{2d}$
- automorphism grp.

Def 3.2 monodromy grp

$\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}) \cong F_2$ acts on $\mathcal{E}(D)$ by the monodromy:



$x = (16)(2354)$

$y = (12)(34)(56)$

$(z = (xy)^{-1})$

The monodromy group: $\mathrm{Mon}(D) = \langle x, y, z \mid xyz = 1 \rangle < G_{\deg \beta}$ (x, y, z : monodromy arround $0, 1, \infty$)

Q the isomorphism type of $\mathrm{Mon}(D)$ as an abstract grp is a Galois invariant.

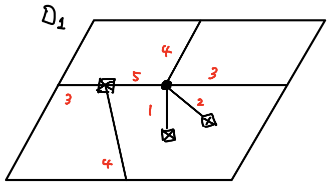
Each of x, y, z has the cycle str. corresponding to the valency list.

Def 3.3 Nielsen class

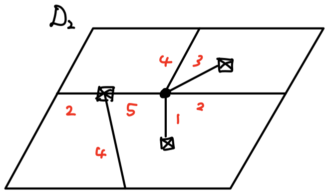
The Nielsen class $NC(D)$ of a dessin D is the list of $Mon(D)$-conjugacy classes of x, y, z.

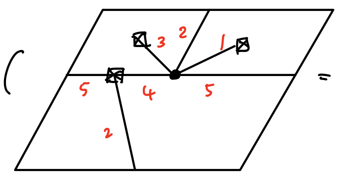
Exm 3.3.1 $[5;5;3,1,1]$-dessin : there are 2 patterns

D_1



D_2



( $= D_2$)

$x = (12345)$
$y = (345)$
$z = (15342)$
$\downarrow y^*$
(13452)
$\downarrow y^*$
(14532), ...

x, z belong to different $Mon(D)^*$-classes

$\to NC(D_1) = (5A, 5B, 3A)$

$x = (12345)$
$y = (245)$
$z = (15324)$
$\downarrow y^*$
(12345) $=$
$\downarrow y^*$
(14352)

$\rightsquigarrow NC(D_2) = (5A, 5A, 3A)$

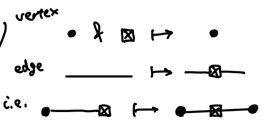
(A, B : distinguishing symbol)

as $NC(D_1) \neq NC(D_2)$,
D_1, D_2 are NOT $G_{\mathbb{Q}}$-conjugate!

☆ There are some exms in which val. list, $Mon(D)$, etc. do not distinguish Galois orbits but $NC(D)$ does.

Def 3.4 Cartographic grp

The cartographic grp $Car(D)$ of a dessin is the monodromy grp of the 'clean' dessin D_2 given by $\beta \mapsto 4\beta(1-\beta)$ which maps



D & D_2 are in a Galois-invariant relation.

In addition to $\beta \mapsto 4\beta(1-\beta)$, Belyi-extending map : $\tau : \mathbb{P}^1_{\mathbb{C}} \xrightarrow{\text{Belyi}} \mathbb{P}^1_{\mathbb{C}}$ w/ $\tau(\{0,1,\infty\}) \subset \{0,1,\infty\}$
$D \mapsto D_2$ is studied as a tool to produce new Galois invariants. (Wood 2006)

Rem 3.5 The graph-automorphism grp of a dessin is a Galois invariant.

Rem 3.6 Only one case of possible decisions of Galois conjugacy - mirror images

As $i \mapsto -i \in G_{\mathbb{Q}}$ acts as the mirror reflection ι, we can decide the following :

(A) $\iota(D) = D \Rightarrow D$ is defined over $\mathbb{R} \cap \overline{\mathbb{Q}}$
(B) $\iota(D) \neq D \Rightarrow D$ is not defined over $\mathbb{R}$, in particular NOT over $\mathbb{Q}$.

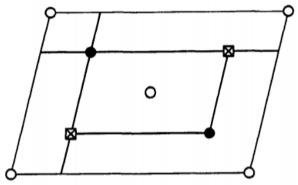
4. enumerating dessins & determining Galois orbits

Rem 4.1. To enumerate dessins w/ given val. list, (just drawing pictures)
it is better to take grp-theoritic exhaustive search than combinatorial geometric method.

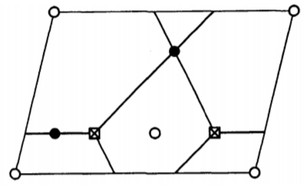
e.g. using GAP-software

$\to$ val. list ~ cycle type of x, y, z $\leftarrow$ enumerate conj. classes of such x, y, z satisfying $xyz = 1$.

exm 4.1.1 $[4,2\,;\,4,2\,;\,3,3]$ – dessin w/ genus = 1 : there are just two patterns.



$x = (1\,2\,3\,4)(5\,6)$
$y = (1\,5\,3)(2\,6\,4)$
$\mathrm{Mon}(D) = {}_6T_{10}, \# = 36$



$x = (1\,2\,3\,4)(5\,6)$
$y = (1\,2\,5)(3\,4\,6)$
$\mathrm{Mon}(D) = {}_6T_7, \# = 24$

※ ${}_nT_i$: Butler – McKay's classification of transitive permutation grp (1983)

As their Mon(D) differ, both are defined over $\mathbb{Q}$.
It is hard to see the non-existance of other patterns by drawing pictures...

exm 4.1.2 Suppose $[4,2\,;\,3,3\,;\,3,3]$ – dessin D w/ genus 1.

Riem.-Hurwitz : $\chi(D) = 6 \cdot \chi(\mathbb{P}^1_{\mathbb{C}}) - \Sigma(\nu_p - 1)$

$$2 - 2\cdot 1 = 6(2 - 2\cdot 0) - (3+1+2+2+2+2)$$
$$0 = 12 - 12 \quad \text{success to hold.}$$

However, GAP-calculation shows that there are no pattern.

Rem 4.2 After enumeration, if there is a dessin sharing Galois invariants w/ no other, it is defined $/\mathbb{Q}$.

a dessin —— " —— w/ only the mirror reflection, they are defined over an imaginary quadratic field $\mathbb{Q}(\sqrt{-k})$.

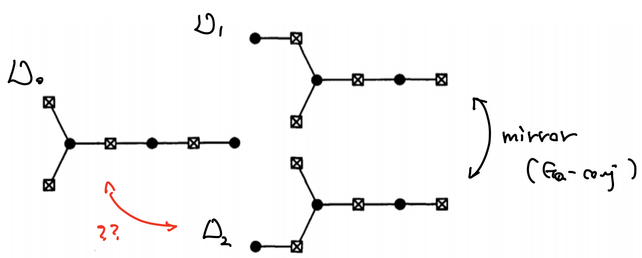
Conversely, what invariant is perfect (i.e. dessins sharing it are Galois conjugate) is an open problem.

We often need to calculate concretely, as in the following section.

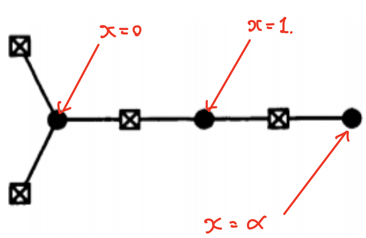
5. calculation exms of genus 0

Since $C = \mathbb{P}^1_{\mathbb{C}}$ is unique, we only have to specify β.

exm. 5.0.1 Consider $[6\,;\,3,2,1\,;\,2,2,1,1]$ – dessin of genus 0.
There are three patterns.



Normalize :



Let $z = k\beta = x^3(x-1)^2(x-\alpha)$. ($k$: const)

Crit. values of z are $0, k, \infty$.
Since $z - k$ has double zero $\times 2$ & simple zero $\times 2$,

$$z - k = x^3(x-1)^2(x-\alpha) - k = (x^2 - b_1 x + b_2)^2 (x^2 - c_1 x + c_2)$$

By comparing coefficients we will obtain the desired equation,

but there is a nice method called "differentiation trick".

Differentiate both sides :

$$x^2(x-1)(6x^2-(5a+4)x+3a)$$
$$=(x^2-b_1x+b_2)(6x^3-(4b_1+5c_1)x^2+(2b_2+3c_1b_1+4c_2)x-(b_2c_1+2c_2b_1))$$

As $(x^2-b_1x+b_2)$ & $x^2(x-1)$ are mutually coprime,

$$\begin{cases} 6x^2-(5a+4)x+3a=6(x^2-b_1x+b_2) \\ 6x^2(x-1)=6x^3-(4b_1+5c_1)x^2+(2b_2+3c_1b_1+4c_2)x-(b_2c_1+2c_2b_1) \end{cases}$$

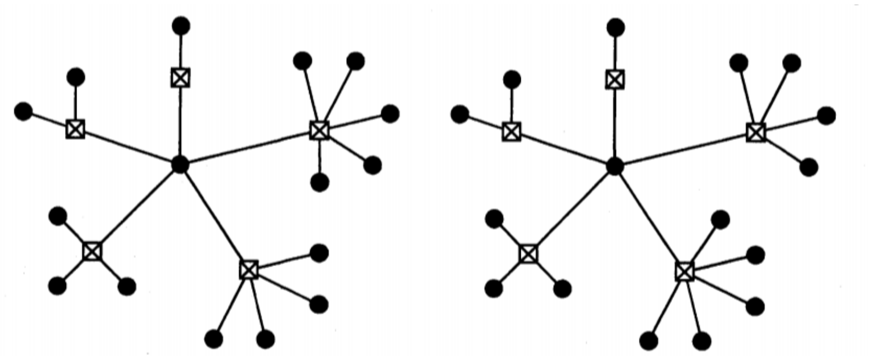
By comparing coefficients, $25a^3-12a^2-24a-16=0$ holds.

LHS is an irreducible poly. of degree 3, So $\mathcal{D}$ is defined over its rupture field. $\boxtimes$

exm 5.2.2 (Leila's flower)

Suppose $[20; 5,1^{15}; 6.5.4.3.2]$ - dessin.

There are $4! = 24$ patterns and they are not distinguished by the basic invariants above.



However, a calculation shows that they are separated to two Galois orbits of length 12.

This indicates that the basic invariants are not perfect.

They are distinguished by the 'spin-structure' invariant (Zapponi, 2000)

6. calculation of genus 1

Unlike the case $g=0$, we need to specify both C & β.

Rather than setting $C: y^2 =$ (cubic poly. of x),

it is better to choose x, y according to poles & zeros of β.

6.1 : prev. work for $g>0$

- There are several works discussing some special exms : $\left\{ \begin{array}{ll} \text{Shabat, Voevodsky} & 1990 \\ \text{Zapponi} & 1997 \\ \text{Hoshino, Nakamura} & 2010 \end{array} \right.$

few works exhausting equations of Belyi pairs

$\longrightarrow$ Birch 1994 $\left[\begin{array}{l} \text{all of degree} \leq 5 \\ \text{some easy exms of degree} = 6 \end{array} \right.$

6.2 Main result

- re-calculation of degree ≤ 5 dessins
- calculation of all of $[6; *; *]$-dessins
- calculation of some of other degree $= 6$ dessins
- calculation of all of $[7; 7; *]$-dessins

6.3 Observations

As far as in the results in this paper, we couldn't find any separated orbits w/ the same basic invariants.
(sufficient to see $\mathrm{Mon}(D)$, $NC(D)$, $\mathrm{Car}(D)$)

This is thought to be because the degree is still low.

7. Exms.

Lem 7.1 Let (C, β) be a Belyi pair of degree d & genus g.

Then $\#\beta^{-1}(\{0, 1, \infty\}) = d - (2g-2)$

in particular, $g = 1 \Rightarrow \#\beta^{-1}(\{0, 1, \infty\}) = d$.

($\because$ Riem.-Hurwitz)

7.2: Case $d = 5$

By Lem 7.1 we have $\#\beta^{-1}(\{0, 1, \infty\}) = 5$, so at least one crit. pt. has valency 5.

Let ∞ be a crit. pt. of valency 5 & $P = (0, 0) \in \beta^{-1}(0)$ have the next greatest valency.

Possible val. list : $\overset{\beta_1}{[5; 5; 3,1,1]}$, $[5; 5; 2,2,1]$, $[5; 4,1; 4,1]$, $[5; 4,1; 3,2]$, $[5; 3,2; 3,2]$

memo. **Recall** For a divisor δ on X : R.S., $L(\delta) := \{ f \in \mathbb{C}(X) \mid \mathrm{ord}_p f + \delta(p) \geq 0 \ \forall p \in X \}$: vector sp.

$\ell(\delta) = \dim_{\mathbb{C}} L(\delta)$

$\deg(\delta) = \sum_{p \in X} \delta(p)$

Riem.-Roch Let $K = K_\omega : p \mapsto \mathrm{mult}_p \omega$ for some ω : mero 1-form on X

e.g. $K_{dz} = 0$ on a torus.

Then for $\forall \delta$: divisor on X, $\ell(\delta) - \ell(K - \delta) = \deg \delta + 1 - g(X)$

by $\delta = 0$ & K, we obtain $\deg K = 2g - 2$.

When $\deg \delta \geq 2g-1$ we have $\ell(K - D) = 0$

and $\ell(D) = \deg \delta + 1 - g$.

We may take x, y s.t. $L(2\infty - P) = \langle x \rangle$ & $L(3\infty - 2P) = \langle y \rangle$

$$u := x^2 - y \in L(4\infty - 2P), \quad v := y^2 - x^3 \in L(6\infty - 3P)$$

cancel $(\xi - p)^2$ cancel ξ^{-6}

By replacing x, y w/ some const. multiple, we may adjust so that $u \in L(4\infty - 3p)$, $v \in L(5\infty - 3p)$

For a Belyi morphism β, let $z = k\beta$ (k: const) whose crit. values are $0, k, \infty$.

Now we have :

	1	x	y	x^2	u	xy	v	z
ord_∞	0	-2	-3	-4	-4	-5	≥ -5	-5
ord_P	0	1	2	2	≥ 3	3	3	≥ 3

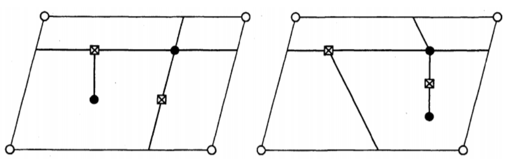
Since $v, z \in L(5\infty - 3p) = \langle xy, u \rangle$: $v = {}^{\exists}a\,xy + {}^{\exists}b\,u$ $\quad z = xy + {}^{\exists}c\,u$ (for suitable k)

By solving $\left\{ \begin{array}{l} u = x^2 - y \\ v = y^2 - x^3 \\ v = a xy + bu \\ z = xy + cu \end{array} \right\}$ $\xrightarrow{\text{cancel } y}$ $f(x,z) := x^5-(2c-b+c^2+ac)x^4+c(c+ac-b)x^3+z(2c+a)x^2-z(b+ac)x-z(z-bc)=0$

adjust a, b, c so that f has desired roots at $z = 0, k$.

Rem $c = 0$, $z = xy \Leftrightarrow \mathrm{ord}_p z = 3$: $[5; 3,2; 3,2]$ - only.

exm 7.2.1 $\quad$ $[5; 4,1; 3,2]$ - dessin : there are 2 patterns



Both are mirror symmetric $\Leftrightarrow$ defined over $\mathbb{R}$

and so either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{k})$

However, $\mathrm{Mon}(\mathcal{D})$, $NS(\mathcal{D})$, $\mathrm{Car}(\mathcal{D})$ do not distinguish them.

$[5; 4,1; 3,2]$ - dessin (remark that $c \neq 0$)

$\longrightarrow$ $f(x,z) := x^5-(2c-b+c^2+ac)x^4+c(c+ac-b)x^3+z(2c+a)x^2-z(b+ac)x-z(z-bc)=0$

has quadruple root & simple root at $z = 0$ $\rightsquigarrow$ $f(x, 0) = x^4 (x - \tau)^1$

has triple root & double root at $z = k$ $\longrightarrow$ $f(x, k) = (x - k_1)^3 (x - k_2)^2$

By solving above equations, we obtain : $c^2 - 54c - 135 = 0 \rightsquigarrow c = 27 \pm \sqrt{6}$

We conclude that these two dessins are Galois conjuate, defined over $\mathbb{Q}(\sqrt{6})$. $\boxtimes$

Result of $d \leq 5$: any other is easier to check.

val. list	#	定義体	Mon	Nielsen 類	j	ID	備考
[3,3,3]	1	Q	C_3	(3A, 3A, 3A)	0	27A3	
[4,4,31]	1	Q	$\mathfrak{S}_4$	(4A, 4A, 3A)	$2^1 3^{-8} 47^2$	48A6	
[4,4,22]	1	Q	C_4	(4A, 4A, 2B)	$2^6 3^3$	32A2	
[5,5,311]	1	Q	$\mathfrak{A}_5$	(5A, 5A, 3A)	$-2^{-10} 3^{-5} 269^3$	150A3	
	1	Q	$\mathfrak{A}_5$	(5A, 5B, 3A)	$2^{12} 3^{-5} 5^1$	75C1	
[5,5,221]	1	Q	$\mathfrak{A}_5$	(5A, 5B, 2A)	$2^{-15} 5^1 211^3$	50B2	
[5,41,41]	1	Q	$\mathfrak{S}_5$	(5A, 4A, 4A)	$2^1 3^3 5^1$	400H1	
	2	$Q(\sqrt{-1})$	F_{20}	(5A, 4A, 4B)	$-2^{-1} 5^2$	400C1	#
[5,41,32]	2	$Q(\sqrt{6})$	$\mathfrak{S}_5$	(5A, 4A, 6A)			
[5,32,32]	1	Q	$\mathfrak{S}_5$	(5A, 6A, 6A)	$-2^{13} 3^{-3} 5^1$	900D1	

) distinguished by NC (cf. exm 3.3.1)

← related by mirror asymmetry

← exm 7.2.1

7.4 degree 6, no crit pt. of order 6

possible patterns: [51, 51, 51], [51, 51, 42], [51, 51, 33], [51, 42, 42], [51, 42, 33], [51, 33, 33],
[42, 42, 42], [42, 42, 33], [42, 33, 33], [33, 33, 33]

The only successful cases are when Belyi-extending maps are valid.

e.g. $z \mapsto 4z(1-z)$, $\quad z \mapsto \frac{(z+\omega)^3}{(z+\omega)^3-(z-\omega)^3}$ etc..

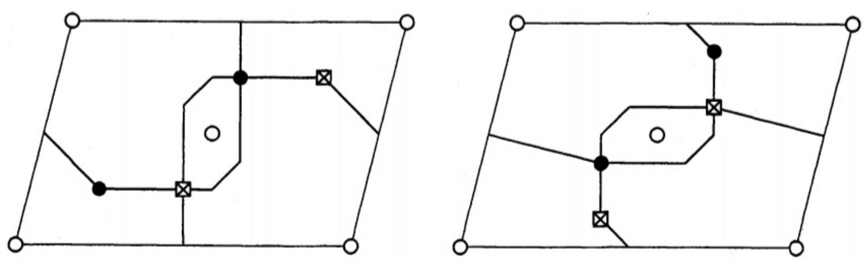
exm 7.4.1 There are four $[5.1;4.2;4.2]$-dessins.

Two of them are mirror-symmetric ($/\mathbb{R}$) & rest two form a mirror conjugate pair. ($/\mathbb{Q}(\sqrt{-15})$)

- $/\mathbb{Q}(\sqrt{-15})$-pair is distinguished from both $/\mathbb{R}$, by $\mathrm{Car}(D)$.
- Whether $/\mathbb{R} \times 2$ are Galois conjugate is NOT detected. (common basic invariants)

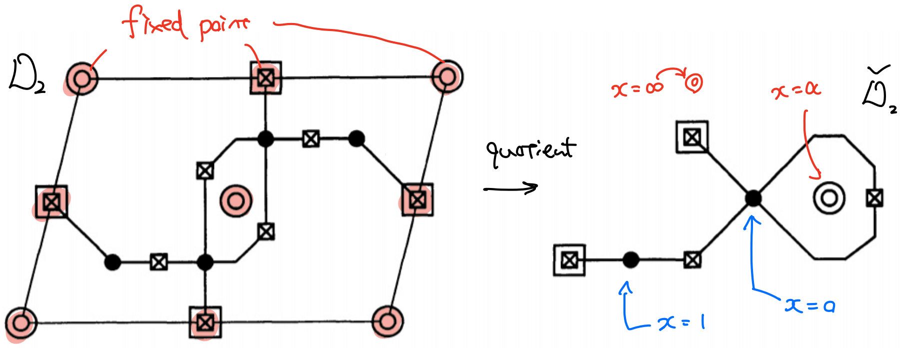
... Few clues, difficult to conclude.

$/\mathbb{Q}(\sqrt{-15})$-pair :



Each of them is invariant under $\bullet \leftrightarrow \boxtimes$ $(\beta \mapsto 1-\beta)$.

So the barycentric division 'D_2' has a graph-automorphism of order 2.



The quotient $\check{D}_2$ is $[5.1;4.2;2.2.1.1]$-dessin of genus 0.

Letting $[5.1;4.2;2.2.1.1]$ ($x=\infty$, α, 0, 1), $\quad \check{\beta}(x) = \frac{x^4(x-1)^2}{k(x-\alpha)}$, $\quad \check{\beta}(x)-1 = \frac{g(x)^2 h(x)}{k(x-\alpha)}$ $\quad$ (g, h : degree 2 monic)

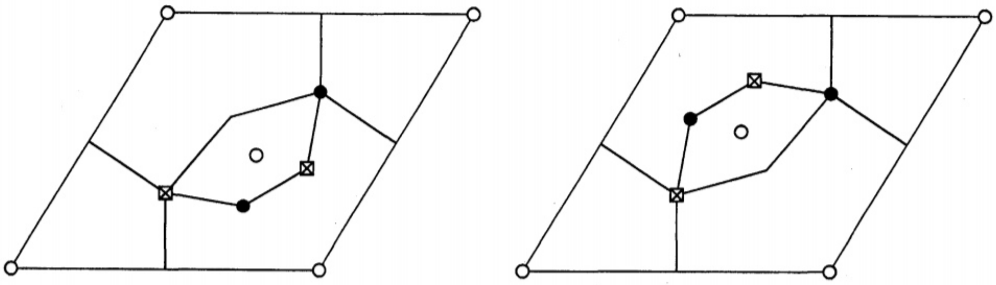
Calculation $\rightsquigarrow$ defined over $\mathbb{Q}(\sqrt{-15})$.

$$\begin{tikzcd} E \arrow[r] & \mathbb{P}^1_{\mathbb{C}} \\ D_2 \arrow[u] \arrow[r] & \check{D}_2 \arrow[u] \end{tikzcd}$$

is given by the Belyi-extending map $\beta \mapsto 4\beta(1-\beta)$.

Finally we obtain that D is also defined over $\mathbb{Q}(\sqrt{-15})$. $\boxtimes$

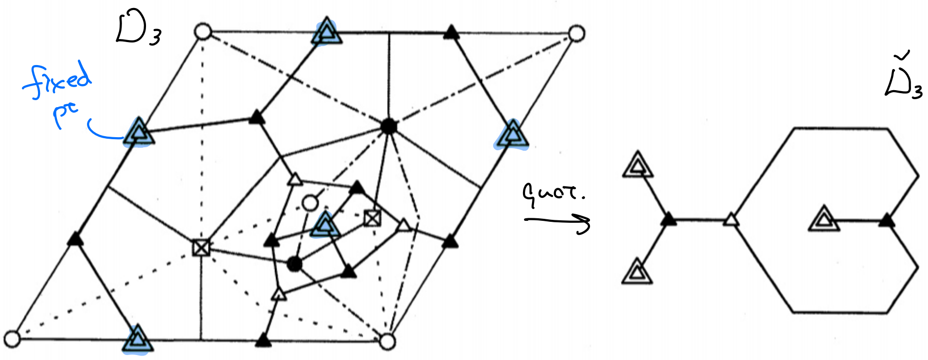
Exm 7.4.2 There are two $[4,2;4,2;4,2]$-dessins, which form a mirror conjugate pair $(/\mathbb{Q}(\sqrt{-k}))$



- Each of them is invariant under $\bullet \leftrightarrow \boxtimes$ $(\beta \mapsto 1-\beta)$.
- Each of them is invariant under $\bullet \to \boxtimes \to \circ$ $\left(\beta \mapsto \frac{1}{1-\beta} \mapsto \frac{\beta-1}{\beta} \mapsto\right)$

So by applying the Belyi-extending map $t: \beta \longmapsto \dfrac{(\beta+\omega)^3}{(\beta+\omega)^3-(\beta-\omega)^3}$: $\begin{pmatrix} 0,1,\infty \mapsto \infty \\ -\omega \mapsto 0 \\ \omega \mapsto 1 \end{pmatrix}$, degree 3

we obtain a new dessin $D_3 \sim (E, t\circ\beta)$ w/ automorphism of order 3 :



The quotient $\check{D}_3$ is a $[4,2;3,3;3,1,1,1]$-dessin of genus 0.

Calculating $\check{D}_3$ & passing through t, we finally obtain that D is defined over $\mathbb{Q}(\sqrt{-3})$. $\boxtimes$