A GENUS 4 ORIGAMI WITH MINIMAL HITTING TIME AND AN INTERSECTION PROPERTY

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ABSTRACT. In a minimal flow, the hitting time is the exponent of the power law, as r goes to zero, for the time needed by orbits to become r-dense. We show that on the so-called *Ornithorynque* origami the hitting time of the flow in an irrational slope equals the diophantine type of the slope. We give a general criterion for such equality. In general, for genus at least two, hitting time is strictly bigger than diophantine type.

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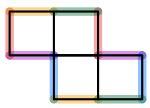
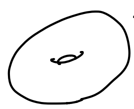
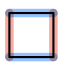
1. Intro

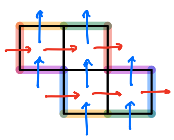
origami (square-tiled surface)

(exm)  glue =  X torus tiled by 4 squares

cov. $p_X \downarrow$

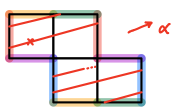
 ... std torus $\mathbb{R}^2/\mathbb{Z}^2$

p_X corresponds to its monodromy $h, v \in G_{\deg p_X}$



h : horizontal
v : vertical } monodromy (permutation of squares)

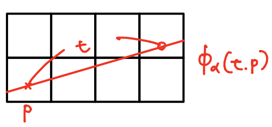
On a given origami X, $\alpha \in \mathbb{R}$ defines a linear flow in slope α

X 

$$\phi_\alpha : \mathbb{R} \times X \to X$$

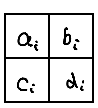
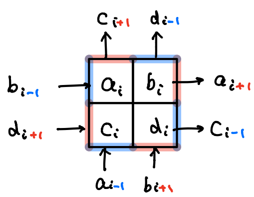
w/ $p_X(\phi_\alpha(t, p)) = p_X(p) + t e_\alpha \mod \mathbb{Z}^2$, $\quad e_\alpha = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$

dev $\downarrow$ (p_X & lift to $\mathbb{R}^2$)



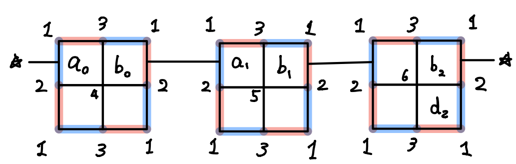
※ flow stops for singular leaves : trajectories hitting $0 \in \mathbb{R}^2/\mathbb{Z}^2$

In this paper we consider the Ornithorynque origami X_O (degree 12)

tile T_i $(i = 0, 1, 2)$  $\times 3$ w/ gluing rule 

red slit : to next copy w/ same row (column)
blue slit : to prev. copy w/ same row (column)

$\rightsquigarrow$ monodromy : $\begin{cases} h = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \mapsto \begin{pmatrix} b_i & a_{i+1} \\ d_i & c_{i-1} \end{pmatrix} \\ v = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \mapsto \begin{pmatrix} c_{i+1} & d_{i-1} \\ a_i & b_i \end{pmatrix} \end{cases}$



6 vertices $\rightsquigarrow \chi = 6 - 24 + 12 = -6$

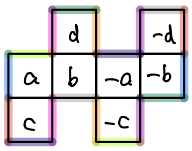
$g = 4$

X_0 is discovered by Forni & Matheus (2008)

as a rare exam of flat surf. w/ special property of the Teichmüller geodesic flow
("totally degenerate Lyapunov spectrum")

only other exam : 'Eierlegende Wollmilchsau' X_ε (Quaternion origami)

X_ε



$$\begin{cases} h = (ab-a-b)(cd-c-d) \\ v = (ca-c-a)(bd-b-d) \end{cases}$$

… degree 8 , genus 3

. main statement

Def ∘ diophantine type of $\alpha \in \mathbb{R}$

p, q : coprime

$$w(\alpha) := \sup\left\{ w>0 : \left|\alpha - p/q\right| < \frac{1}{q^{w+1}} \text{ for infinitely many } p/q \in \mathbb{Q} \right\} \geq 1 \quad = 1 \text{ for a.e. } \alpha$$

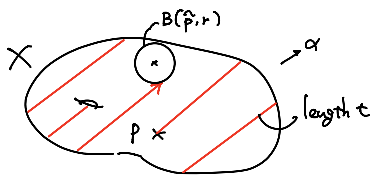
order of diophantine approximation

Dirichlet's thm

known

∘ Fix an origami X , $\alpha \in \mathbb{R}$.
For any $p \in X$ and $r>0$, the time needed by the positive ϕ_α-orbit of p to be r-dense is :

$$T(X, \alpha, p, r) := \sup_{\tilde{p} \in X} \inf\left\{ t>r \mid d(\tilde{p}, \phi_\alpha(t,p)) < r \right\}$$



⇝ positive ϕ_α-orbit of p of length $H(X,\alpha,p,r)$ intersects $B(\tilde{p}, r)$ for $\forall \tilde{p} \in X$ and ↑ is minimal.

defined outside singular leaves or periodic leaves by Veech's dictionary (1987)
probably

∘ The scaling law of $T(X, \alpha, p, r)$ as $r \to +0$: irregular in general
but can be bounded by a power r^{-H}

⇝ $H = H(X, \alpha, p)$: hitting time

$$H(X, \alpha, p) := \limsup_{r \to +0} -\log_r T(X, \alpha, p, r)$$

Theorem 1.1 Let X_0 be the Ornithorynque origami.

Then $\forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\forall p \in X_0$: outside singular leaves $H(X, \alpha, p) = w(\alpha)$ holds.

no periodic leaves

We will show thm 1.1 in a more general setting.

∘ nontrivial part : $H(X, \alpha, p) \leq w(\alpha)$ … "minimal hitting time" ← also true for X_ε.

In the contrast, we state as follows :

Lem 1.2 For any origami X, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $p \in X$ outside singular leaves,

$$H(X, \alpha p) \geq w(\alpha) \text{ holds.}$$

Kim & Marchese (2018) observe that :

- For any origami X and $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $p \mapsto H(X, \alpha, p)$: invariant under ϕ_α.
  (in the same trajectory)

- Let X be an origami w/ genus 2. & one cone pt of order 2

  $\forall \lambda \in [1, 2]$, $\exists \alpha \in \mathbb{R}$ s.t. $H(X, \alpha, p) = w(\alpha)^\lambda$ for a.e. $p \in X$

- For any origami X & irrational α,

$$\liminf_{r \to +0} -\log_r (\inf\{t<r \mid d(\phi_\alpha(t,p), \tilde{p}) < r) = 1 \text{ for a.e. } p, \tilde{p} \in X.$$

("hitting time from p to $\tilde{p}$")

Combining thm1.1 & above result & recalling $w(\alpha) = 1$ generically, we get :

for a.e. α & a.e. $p \in X_0$, there exists a limit

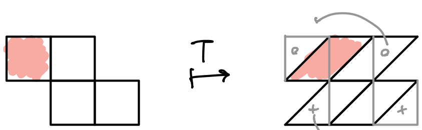
$$\liminf_{r \to +0} -\log_r (\inf\{t<r \mid d(\phi_\alpha(t,p), \tilde{p}) < r) = 1 \quad \ldots ?$$

§2. background

$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $V := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - bc = 1 \right\} = \langle T, R \rangle$ $\qquad V = TRT$

$SL(2, \mathbb{Z})$ acts on an origami by affine deformation :



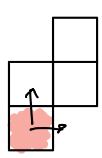
$\leftarrow$ another origami

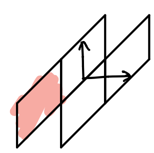
where slopes of geods. change by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \alpha = \dfrac{a\alpha + b}{c\alpha + d}$

For each origami, $SL(2, \mathbb{Z})$ - action on the monodromy (h, v) is :

$T(h, v) = (h, vh^{-1})$

$R(h, v) = (v^{-1}, h)$





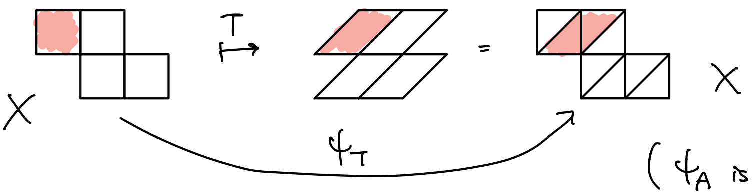
by simple calculation, we get the following :

Prop 2.1 $A \cdot X_0 = X_0$ for $\forall A \in SL(2, \mathbb{Z})$

- Each $SL(2,\mathbb{Z})$-action defines a self-diffeo. ϕ_A on X_Θ

  which is locally affine w.r.t. the natural charts induced from $\mathbb{R}^2/\mathbb{Z}^2$.

  ϕ_A maps $\{$conical pt on $X_\Theta\}$ itself.



(ϕ_A is also defined for any origami X and $A \in SL(2,\mathbb{Z})$ satisfying $A \cdot X = X$)

$A =: D\phi_A$: linear part of ϕ_A

§3. The intersection property of X_Θ

Let X : any origami , $\{p_1 \cdots p_n\}$: set of conical pts on X

Def A (straight) segment in X : a path $S : (a,b) \to X \setminus \{p_1, \cdots, p_n\}$ s.t. $\exists v \in \mathbb{R}^2$: vector

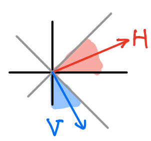
w/ $\frac{d}{dt} \rho_X(S(t)) = v$ $\forall t \in (a,b)$

We define $\mathrm{Slope}(S) := \frac{y}{x}$ where $v = \binom{x}{y}$.

$|S| := |b-a| \cdot \|v\|_{\mathbb{R}^2}$.

A saddle connection in X : a straight segment joining two conical points.

Goal Prop 3.1 Let X_Θ : Ornithorinque origami

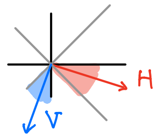


H, V : segments in X w/ $\mathrm{slope}(H) \in (0,1)$ $\mathrm{slope}(V) < -1$

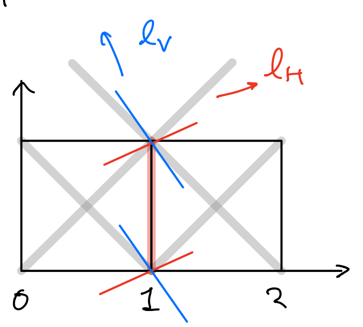
R-action

$\Rightarrow$ if $|H|, |V| \geq \sqrt{288}$ then $H \cap V \neq \phi$.

Cor 3.2 the same holds for



$-\frac{\pi}{2}$-rotation



3.1. **Preliminary Lemmas.** The proof of Lemma 3.3 below is left to the reader.

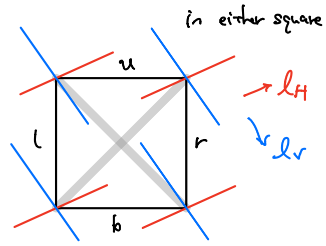
Lemma 3.3. *Let $Q_1 := [0,1]^2$ and $Q_2 := [1,2] \times [0,1]$. Let ℓ_H and ℓ_V be two lines with $0 < \mathrm{Slope}(\ell_V) < 1$ and $\mathrm{Slope}(\ell_H) < -1$ and set $P := \ell_H \cap \ell_V$. If both ℓ_V and ℓ_H intersect $\{1\} \times [0,1]$, then either $P \in Q_1$ or $P \in Q_2$.*

ℓ_H, ℓ_V intersect in either square

Lemma 3.4. *Let X be any origami labelled by a finite set $\mathcal{Q}$. Fix a square Q_j with $j \in \mathcal{Q}$ and let H and V be segments in X with $0 < \mathrm{Slope}(V) < 1$ and $\mathrm{Slope}(H) < -1$, such that both H, V have endpoints in $\bigcup_{l \neq j} \partial Q_l$. If both $H \cap Q_k \neq \emptyset$ and $V \cap Q_k \neq \emptyset$ then $H \cap V \neq \emptyset$.*

in any case of $3 \times 3 = 9$ patterns, we may conclude intersection or use Lem 3.3.

ℓ_H intersects (l,u) or (l,r) or (r,b)
ℓ_V intersects (l,b) or (u,b) or (r,u)



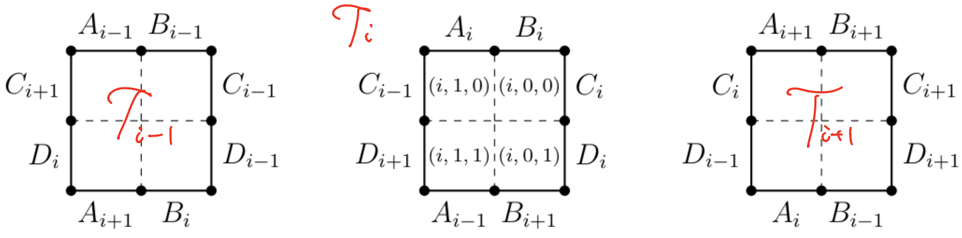


FIGURE 1. The Ornithorynque origami $X_{\mathcal{O}}$.

Df Let $\mathcal{A} = \{A_i, B_i, C_i, D_i \mid i = 0,1,2\}$; 12 letters alphabet

$\gamma \in \mathcal{A}$: saddle connection (geometrically)

consider : $\gamma_1, \cdots, \gamma_n \in \mathcal{A} \rightsquigarrow (\gamma_1, \cdots, \gamma_n)$: words arising cutting sequences.

For a straight segment $S : (0,1) \to X$,

(nonsingular)

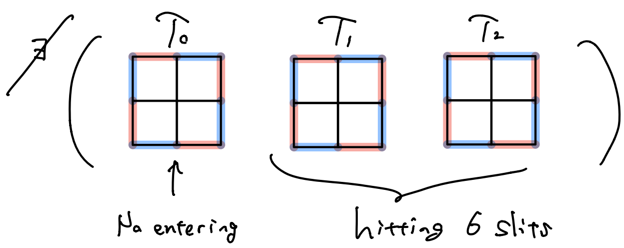
take a seq $0 \le t_1 < t_2 < \cdots < t_n \le 1$ so that $S(t) \in^{\exists !} \gamma \in \mathcal{A}$ iff $t \in \{t_k\}$.

and let $[S] := (\gamma_1, \gamma_2, \cdots, \gamma_n)$ where $S(t_k) \in \gamma_k \in \mathcal{A}$

Lem 3.5 (3.6) Let V : segment w/ $\mathrm{slope}(V) \in (0,1)$ or $\mathrm{slope}(V) < -1$ (R-action)

If $[V]$ contains at least 6 letters then V intersects every tile $T_0 \sim T_2$.

that is : $\nexists$



Lem 3.7 Let H, V : segments w/ $\mathrm{slope}(H) < -1$ $\mathrm{slope}(V) \in (0,1)$

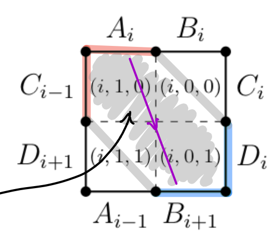
$[H] = (h_1, \cdots, h_n)$ $[V] = (v_1, \cdots, v_m)$

$n \ge 4$. $m \ge 8$.

If $\exists\, i, j$ s.t. $h_j \in \{C_{i-1}, A_i\}$ and $h_{j+1} \in \{B_{i+1}, D_i\}$

then H & V intersect.

$\to$ $\exists$ segment of H in this region



Cor 3.7' $\mathrm{slope}(H) \in (0,1)$ & $\mathrm{slope}(V) < -1$

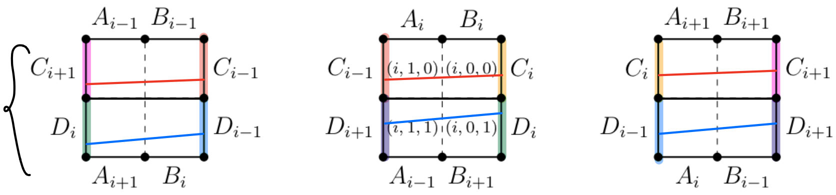
Lem 3.8 Let H, V : segments w/ $\mathrm{slope}(H) \in (0,1)$ $\mathrm{slope}(V) < -1$

$$[H] = (h_1, \cdots, h_n) \quad [V] = (v_1, \cdots, v_m)$$
$$n \geq 3 \qquad\qquad m \geq 7$$

If $\exists\, i, j$ s.t. $(h_j, h_{j+1}, h_{j+2}) = (C_{i-1}, C_i, C_{i+1})$ or (D_{i+1}, D_i, D_{i-1})

then H & V intersect.

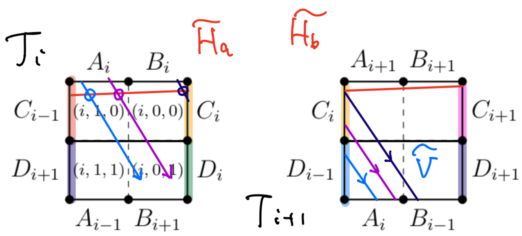
$\exists$ either type of segment



pf) We may assume $(h_j, h_{j+1}, h_{j+2}) = (C_{i-1}, C_i, C_{i+1})$ w/o l.o.g.

Let $\widetilde{H}_a, \widetilde{H}_b \subset H$: minimal subseg. w/ $[\widetilde{H}_a] = (C_{i-1}, C_i)$ $[\widetilde{H}_b] = (C_i, C_{i+1})$

Let $\widetilde{V} \subset V$: $\quad$ " $\quad$ w/ $[\widetilde{V}] = (v_1, \cdots, v_{m-1})$



$\widetilde{H}_a \subset T_i$

$\widetilde{H}_b \subset T_{i+1}$.

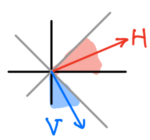
By Lem 3.5 $\widetilde{V}$ intersects T_{i+1}.

If $\widetilde{V}$ does not intersect $\widetilde{H}_b$, $\exists (v_k, v_{k+1}) = (D_{i-1}, A_i)$ or (C_i, A_i) or (C_i, B_{i-1})

$\rightarrow$ anyway V intersects $\widetilde{H}_a$.

$-I \cdot X_0 = X_0$ shows the claim in other case immediately. $\otimes$.

Prop 3.1 Let X_0 : Ornithorinque origami

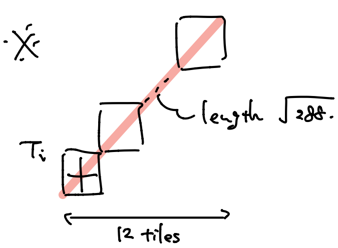


H, V : segments in X w/ $\mathrm{slope}(H) \in (0,1)$ $\mathrm{slope}(V) < -1$

pf) Let $[H] = (h_1, \cdots, h_n)$, $[V] = (v_1, \cdots, v_m)$

Since $|H|, |V| \geq \sqrt{288}$, then we have $n, m \geq 12$. $(\sqrt{288} = \sqrt{2} \cdot 12)$

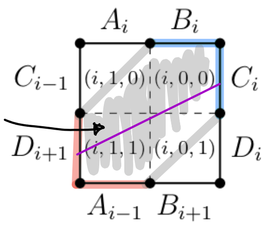
Assume that H does not satisfy the hypothesis of Lem 3.8,



then we must have $\frac{1}{6} < \mathrm{Slope}(H) < 1$.

Since $n \geq 12$, we may observe that the hypothesis of Lem 3.7' holds.

$\exists$ segment in this region



§4 The general criterion

Theorem 4.1. *Let X be an origami and assume that there exists a constant $\mathcal{K} > 0$ such that for any origami $Y \in \mathrm{SL}(2,\mathbb{Z}) \cdot X$ and any pair of segments $H, V \subset Y$ we have $H \cap V \neq \emptyset$ whenever they have length $|H|, |V| \geq \mathcal{K}$ and satisfy*

- *either* $\mathrm{Slope}(H) < -1$ *and* $0 < \mathrm{Slope}(V) < 1$
- *or* $-1 < \mathrm{Slope}(H) < 0$ *and* $\mathrm{Slope}(V) > 1$.

intersection property

Then $H(X, \alpha, p) = w(\alpha)$ for any α irrational and any p outside (X, α)-singular leaves.

4.1. **Proof of Main Theorem 1.1.** Recall that $\mathrm{SL}(2,\mathbb{Z}) \cdot X_{\mathcal{O}} = X_{\mathcal{O}}$ by Proposition 2.1. Theorem 1.1 follows combining Theorem 4.1 with Proposition 3.1 and Corollary 3.2.

hypothesis of thm 4.1 w/ $X = X_{\mathcal{O}}$, $\mathcal{K} = \sqrt{288}$.

- continued fraction

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

<u>Def</u> for $a_1, \cdots, a_n \in \mathbb{Z}_{>0}$.

$$g(a_1, \cdots, a_n) := \begin{cases} V^{a_1} \circ T^{a_2} \circ \cdots \circ V^{a_{n-1}} \circ T^{a_n} & \text{if } n : \text{even} \\ V^{a_1} \circ T^{a_2} \circ \cdots \circ T^{a_{n-1}} \circ V^{a_n} & \text{if } n : \text{odd} \end{cases} \in SL(2,\mathbb{Z})$$

for $\alpha \in \mathbb{R}$, dfn $[\alpha], \{\alpha\}$ by

$\underset{\mathbb{Z}}{\underset{\cap}{}}$ $\alpha = [\alpha] + \{\alpha\}$ ... integer part & fractional part.

$$[\alpha] \in \mathbb{Z}, \quad \{\alpha\} \in [0,1)$$

the Gauss map : $G : [0,1) \to [0,1)$
$$\alpha \mapsto \{1/\alpha\}$$

- Any $\alpha \in [0,1) \setminus \mathbb{Q}$ admits an unique continued fraction expansion

$$\alpha = [0; a_1, a_2 \cdots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}} \quad \text{where} \quad \begin{cases} a_n = [1/\alpha_{n-1}] \\ \alpha_n = G(\alpha_{n-1}) \\ \alpha_0 = \alpha \end{cases}$$

The n-th convergent : $p_n/q_n := [0; a_1, a_2 \cdots a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots\, a_{n-1} + \cfrac{1}{a_n}}}}$

Then we have
$$\begin{cases} g(a_1, \cdots, a_{2n-1}) = \begin{pmatrix} p_{2n-1} & p_{2n-2} \\ q_{2n-1} & q_{2n-2} \end{pmatrix} \\ g(a_1, \cdots, a_{2n}) = \begin{pmatrix} p_{2n-1} & p_{2n} \\ q_{2n-1} & q_{2n} \end{pmatrix} \end{cases}$$

from $q_n = a_n q_{n-1} + q_{n-2}$ & $p_n = a_n p_{n-1} + p_{n-2}$.

Therefore we have

Möbius action

$$p_n/q_n = \begin{cases} g(a_1, \cdots, a_n)\cdot 0 & \text{if } n: \text{even} \\ g(a_1, \cdots, a_n)\cdot \infty & \quad : \text{odd} \end{cases}$$

We have $\alpha_n^{-1} = a_{n+1} + \alpha_{n+1} \Leftrightarrow \alpha_n = V^{a_{n+1}} \cdot \alpha_{n+1}^{-1} \Leftrightarrow \alpha_n^{-1} = T^{a_{n+1}} \cdot \alpha_{n+1}$. Hence

$$\alpha = g(a_1, \ldots, a_{2k}) \cdot \alpha_{2k} = g(a_1, \ldots, a_{2k}, a_{2k+1}) \cdot \frac{1}{\alpha_{2k+1}} \quad \text{for any} \quad k \in \mathbb{N}. \tag{4.5}$$

Theorem 4.1. *Let X be an origami and assume that there exists a constant $\mathcal{K} > 0$ such that for any origami $Y \in \mathrm{SL}(2,\mathbb{Z}) \cdot X$ and any pair of segments $H, V \subset Y$ we have $H \cap V \neq \emptyset$ whenever they have length $|H|, |V| \geq \mathcal{K}$ and satisfy*

- *either* $\mathrm{Slope}(H) < -1$ *and* $0 < \mathrm{Slope}(V) < 1$
- *or* $-1 < \mathrm{Slope}(H) < 0$ *and* $\mathrm{Slope}(V) > 1$.

Then $H(X, \alpha, p) = w(\alpha)$ for any α irrational and any p outside (X, α)-singular leaves.

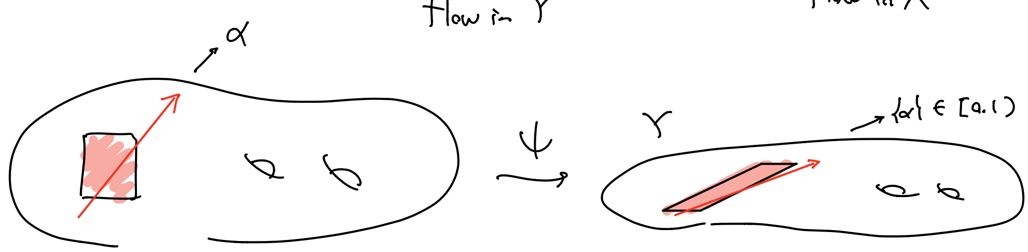
4.1. **Proof of Main Theorem 1.1.** Recall that $\mathrm{SL}(2,\mathbb{Z}) \cdot X_{\mathcal{O}} = X_{\mathcal{O}}$ by Proposition 2.1. Theorem 1.1 follows combining Theorem 4.1 with Proposition 3.1 and Corollary 3.2.

pf of thm 4.1) Let X, α be as above.

Set $Y = T^{-[\alpha]} \cdot X$ and $\psi : X \to Y$: natur. affine diffeo w/ $D\psi = T^{-[\alpha]}$.

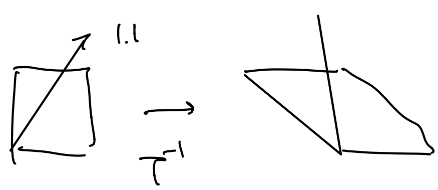
We have $^{\exists} K > 0$ w/ $\phi_{\{\alpha\}}(t, \psi(p)) = \psi(\phi_\alpha(Kt, p))$

flow in Y flow in X



Thus $H(Y, \{\alpha\}, \psi(p)) = H(X, \alpha, p)$

$p/q \mapsto \frac{p+q}{q}$



$\begin{pmatrix}1\\0\end{pmatrix} \mapsto$

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x+y\\ y\end{pmatrix}$$

Proposition 4.2. *Let X and $\mathcal{K} > 0$ be an origami and a constant as in Theorem 4.1. Fix a slope $\alpha = [a_1, a_2, \dots] \in (0,1)$. For any $p \in X$ outside singular leaves and $n \in \mathbb{N}$ we have*

$$T(X_{\mathcal{O}}, \alpha, p, r_n) \leq 4\mathcal{K} \cdot q_n \quad \textit{where} \quad r_n := \frac{2(\mathcal{K}+1)}{q_n}$$

Set $w := w(\alpha)$, so that $q_n \leq K \cdot q_{n-1}^w$ for some K and all n. Fix $p \in X$ outside singular leaves. For any $r > 0$ small enough consider n with $r_{n-1} \leq r < r_n$. Proposition 4.2 gives

$$\begin{aligned}\frac{\log T(X_{\mathcal{O}}, \alpha, p, r)}{|\log r|} &\leq \frac{\log T(X_{\mathcal{O}}, \alpha, p, r_n)}{|\log r_{n-1}|} \leq \frac{\log 4\mathcal{K} + \log q_n}{\log q_{n-1} - \log 2(\mathcal{K}+1)} \\ &\leq \frac{\log 4\mathcal{K} + \log K + w \cdot \log q_{n-1}}{\log q_{n-1} - \log 2(\mathcal{K}+1)} \to w \quad \text{for} \quad n \to +\infty.\end{aligned}$$

Hence $H(X, \alpha, p) \leq w$. Lemma 1.2 gives the other inequality. Theorem 1.1 is proved. $\square$