Arch. Math. 112 (2019), 13–18

0003-889X/19/010013-6
published online July 21, 2018
https://doi.org/10.1007/s00013-018-1222-9

Archiv der Mathematik

CrossMark

Automorphism groups of dessins d'enfants

Rubén A. Hidalgo

Abstract. Recently, Gareth Jones observed that every finite group G can be realized as the group of automorphisms of some dessin d'enfant $\mathcal{D}$. In this paper, complementing Gareth's result, we prove that for every possible action of G as a group of orientation-preserving homeomorphisms on a closed orientable surface of genus $g \geq 2$, there is a dessin d'enfant $\mathcal{D}$ admitting G as its group of automorphisms and realizing the given topological action. In particular, this asserts that the strong symmetric genus of G is also the minimum genus action for it to act as the group of automorphisms of a dessin d'enfant of genus at least two.

Mathematics Subject Classification. Primary 30F40, 11G32, 14H57.

Keywords. Riemann surfaces, Dessins d'enfants, Automorphisms.

§ 1. Intro

Hurwitz showed (1893) that:

$\boxed{\text{every finite group } G}$ is realized as a grp. of conf. automorphisms $G < \mathrm{Aut}\, X$ for some R.S. X of genus $g \geq 2$

and that the upper bound $|G| \leq 84(g-1)$ holds.

The bound is attained $\Leftrightarrow$ $G = \langle a, b \mid a^2 = b^3 = (ab)^7 = 1 \rangle$ (as an abstract group)

Greenberg (1963) observed that

X can be chosen so that $G = \mathrm{Aut}\, X$ (full auto. grp), $g(X) \geq 2$.

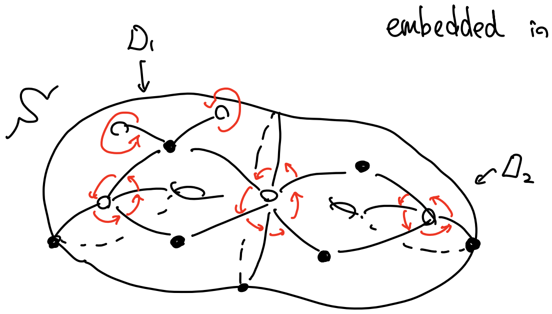
$\boxed{\text{Main theme}}$ realization of G

as $G = \mathrm{Aut}\, \mathcal{D}$ for some dessin $\mathcal{D}$

A dessin d'enfant : introduced by Grothendieck (1984)

is a pair $D = (S, G)$ $\begin{cases} S : \text{closed orientable surface} \\ G : \text{finite bipartite graph} \\ \quad \text{embedded in } S \text{ as to be filling} \end{cases}$

$(S \setminus G = \coprod (\text{top. disk}))$



identified $(\sigma_\circ, \sigma_\bullet)$: pair of permutations of edges around $\circ, \bullet$, resp.

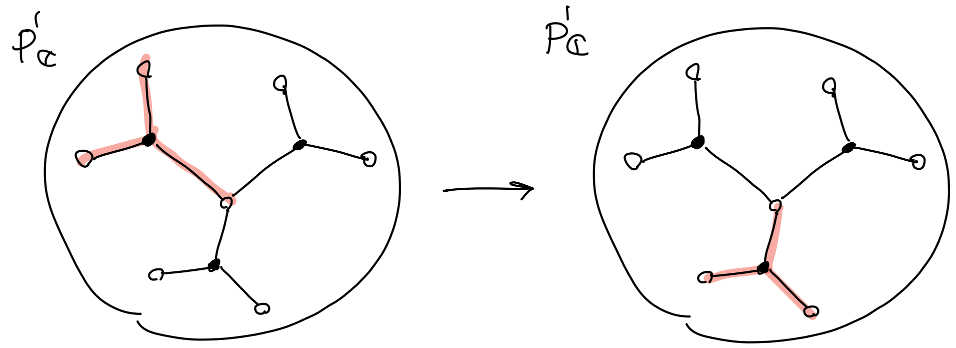
A dessin D is uniform $\Longleftrightarrow$ Each of $\left\{\begin{array}{l} (\text{black vertices}) \\ (\text{white vertices}) \\ (\text{faces}) \end{array}\right\}$ share the same order

i.e., $\begin{cases} \sigma_\bullet = \underbrace{(\cdots)}_{l}\underbrace{(\cdots)}_{l}\cdots\underbrace{(\cdots)}_{l} \\ \sigma_\circ = \underbrace{(\cdots)}_{m}\underbrace{(\cdots)}_{m}\underbrace{(\cdots)}_{m}\cdots\underbrace{(\cdots)}_{m} \\ \sigma_f = (\sigma_\circ \sigma_\bullet)^{-1} = \underbrace{(\cdots)}_{n}\cdots\underbrace{(\cdots)}_{n} \end{cases}$

Def An automorphism of a dessin $D = (S, G)$

is a graph automorphism of G (permutation of edges compatible w/ $\sigma_\circ, \sigma_\bullet$)

: that is induced from $f : S \to S$ ori. pres. homeo



A dessin is called regular $\iff$ $\mathrm{Aut}\,\mathcal{D} \curvearrowright \{\text{edges}\}$: transitive ($\Rightarrow$ uniform.)

well-known Fact grp. realizable as $\mathrm{Aut}\,\mathcal{D}$ of regular dessin $\mathcal{D}$ $\left(\begin{array}{l} \mathrm{Aut}(\text{reg. cov}) \\ = \pi_1(X^*) = \text{2-generated} \end{array} \right)$

$\iff$ 2-generated grp

For a realization as $\mathrm{Aut}\,\mathcal{D}$ of non-regular dessin $\mathcal{D}$,

Theorem 1 (Jones, 2018 pre) Every finite group is isomorphic to $\mathrm{Aut}\,\mathcal{D}$ for ${}^{\exists}\mathcal{D}$: dessin

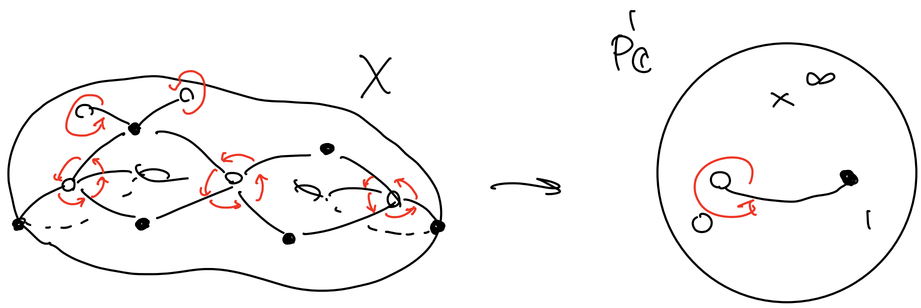
As a consequence of uniformization thm, $\Gamma < \Delta(l, m, n)$

since ${}^{\forall}$dessin is realizable as $\tilde{X} \to \underline{X = \tilde{X}/\Gamma \to \tilde{X}/\Delta(l,m,n) = \mathbb{P}^1_{\mathbb{C}}}$ w/ 3-sing,

a dessin $\mathcal{D}$ defines a Belyi covering $\boxed{\beta : X \to \mathbb{P}^1_{\mathbb{C}} : \text{branched over } 0, 1, \infty}$

where : $\beta^{-1}([0,1]) = \mathcal{G}$

$\mathrm{Aut}\,\mathcal{D} \circlearrowright G$



$$\mathrm{Aut}\,\mathcal{D} \sim \mathrm{Aut}(X, \beta) = \{ \phi : X \to X : \text{conf. auto s.t. } \beta = \beta \circ \phi \}$$

(maximal grp in the sense : G w/ β projects to X/G)

Theorem 2 (= Thm 1) Every finite group is isomorphic to $\mathrm{Aut}(X, \beta)$ for ${}^{\exists}(X, \beta)$: Belyi.

Let G_1, G_2 : finite grp of ori. pres. homeo on S_1, S_2 : cl. ori. surf.

Def We say that G_1, G_2 are topologically equivalent if

there exists $f : S_1 \to S_2$: ori. pres. homeo

s.t. $G_2 = f G_1 f^{-1}$

$$\begin{tikzcd} S_1 \arrow[r, "g_1"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & S_1 \arrow[d, "f"] \\ S_2 \arrow[r, "g_2"'] & S_2 \end{tikzcd}$$

Main result The following extension to the Jones' realization thm

THM3 $G < \mathrm{Aut}\, X$: finite grp w/ X : R.S. of $g(X) \geq 2$

Then $\exists\, (\hat{X}, \beta)$: Belyi $\begin{cases} g(\hat{X}) = g(X) \\ G \cong \mathrm{Aut}(\hat{X}, \beta) \\ G \curvearrowright X \underset{\text{Top.eq.}}{\sim} \mathrm{Aut}(\hat{X}, \beta) \curvearrowright \hat{X} \end{cases}$

THM4 (= THM3) $G < \mathrm{Aut}\, X$: finite grp w/ X : R.S. of $g(X) \geq 2$

Then there exists a dessin $\mathcal{D} = (S, \mathcal{G})$

$\begin{cases} g(S) = g(X) \\ G \cong \mathrm{Aut}\, \mathcal{D} \\ G \curvearrowright X \underset{\text{Top.eq.}}{\sim} \mathrm{Aut}\, \mathcal{D} \curvearrowright S \end{cases}$

Remark 1 For a uniform Belyi pair (X, β),

$\eta : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$; $\eta(z) = 16 z (z - \frac{3}{4})$ satisfies that

$\begin{cases} \mathrm{Aut}(X, \beta) = \mathrm{Aut}(X, \eta \circ \beta) \\ (X, \eta \circ \beta) : \text{NOT uniform} \quad (\Rightarrow \text{THM 4 is attained by non-uniform } \mathcal{D}) \end{cases}$

In particular, every 2-generated grp is realizable by both $\left\{ \begin{matrix} \text{regular} \\ \text{non-uniform} \end{matrix} \right.$ $\mathcal{D}$.

Def Suppose G : finite group $\overset{\text{realized}}{=}$ $\mathrm{Aut}\, X$, X : R.S, $g(X) \geq 2$

The minimal value $\mu(G) = \min\{ g(X) \mid G = \mathrm{Aut}\, X \}$

May, Zimmermann (2003) showed that : $\forall g \geq 2$. $\exists G$: finite grp $g = \mu(G)$.

Corollary 1 G : finite grp $\Rightarrow \exists \mathcal{D} = (X, \mathcal{G})$ $\begin{cases} \mu(G) = g(X) \\ G = \mathrm{Aut}\, X \end{cases}$

Winkelmann (2002) showed that : $\forall$ finitely generated grp G is realizable as $G = \mathrm{Aut}\, X$.

- A subtle adaption of (pf. THM3) leads the following

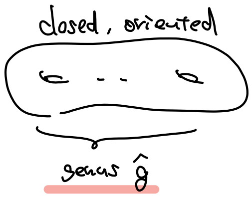
THM5 $\forall$ finitely generated G is realizable as $G = \mathrm{Aut}\,\mathcal{D}$, $D = (X, \beta)$: ~~Belyi~~

$$= \{ \phi \in \mathrm{Aut}\,X \mid \beta \circ \phi = \beta \}$$

§2 pf. of THM3

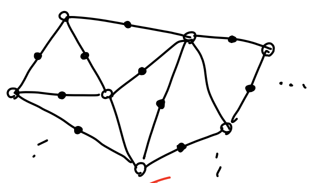
Lem 1 $\forall\, \hat{g},\ r \in \mathbb{Z}_{\geq 0}$, $\exists (\hat{X}, \delta)$: Belyi $\begin{cases} \text{non-uniform} \\ g(\hat{X}) = \hat{g} \\ \deg \delta : \text{prime} \\ \# \delta^{-1}(1) \geq r \end{cases}$

pf) Let S =



Take a clean dessin from a triangulation T w/ #edges $\geq r$.

i.e.

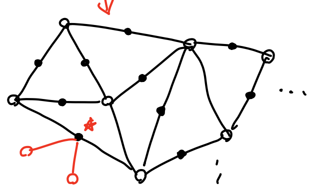


○ : vertices of T

● : center of edges of T

— : half-edges of T

We add edge(s) (at least one):



so that $\begin{cases} \text{\# edges : prime} \\ \text{added edges belong to the same face} \end{cases}$

★ : degree at least 3 $\Rightarrow$ $\begin{cases} \delta^{-1}(1) \geq \#\text{edges} + 1 > r \\ \text{non-uniform dessin}. \end{cases}$ ok

claim $G < \mathrm{Aut}\,X$: finite grp w/ X : R.S. of $g(X) \geq 2$

$$\Rightarrow \exists (\hat{X}, \beta) : \text{Belyi} \begin{cases} g(\hat{X}) = g(X) \\ G \cong \mathrm{Aut}(\hat{X}, \beta) \\ G \curvearrowright X \underset{\text{top. eq.}}{\sim} \mathrm{Aut}(\hat{X}, \beta) \curvearrowright \hat{X} \end{cases}$$

pf of THM3) Let $G < \mathrm{Aut}\, X$: finite grp w/ X : R.S. of $g(X) \geq 2$

X/G : orbifold consists of $\left\{ \begin{array}{l} \text{underlying RS. } Y \text{ of } g(Y) \geq 0 \\ \text{cone points } p_1 \cdots p_r \in Y \\ \quad \text{of order } m_1, \dots, m_r \in \mathbb{Z}_{\geq 2} \end{array} \right.$

Then : $2g(Y) - 2 + r > \sum_{j=1}^{r} \frac{1}{m_j} > 0$.

i.e. Y : hyperbolic

$\left(\begin{array}{l} \because \text{ Riem-Hurwitz.} \\ 0 < 2g(X) - 2 = |G| (2g_Y - 2) + \sum_{j=1}^{r} (m_j - 1) \\ 2g_Y - 2 > \frac{1}{|G|} \sum_{j=1}^{r} (1 - m_j) \qquad |G| \geq \max m_j \\ \qquad \geq \sum \frac{1}{m_j} - r \end{array} \right.$

By uniformization thm, $\exists\, K \curvearrowright \mathbb{H}$: Fuchsian grp

w/ std representation $K = \left\langle A_1, B_1, \dots, A_{g(Y)}, B_{g(Y)}, C_1, \dots, C_r \;\middle|\; \begin{array}{l} \prod [A_k, B_k] \prod C_j \\ = C_1^{m_1} = \dots = C_r^{m_r} = 1 \end{array} \right\rangle$

$$\begin{array}{ccccc} \mathbb{H} & \to & \mathbb{H}/\Gamma & \to & \mathbb{H}/K \\ & & \| & & \| \\ & & X & \longrightarrow & X/G \\ & & & & \| \\ & & & & Y \end{array}$$

where $K \xrightarrow{\theta} G$: sur. hom

$\Gamma = \ker\theta \trianglelefteq K$

$$\begin{array}{ccccc} x \in \mathbb{H} & \xrightarrow{\sigma} & \mathbb{H} & \ni & \sigma x \\ \gamma \downarrow & & \downarrow \gamma & & \\ \gamma x \in \mathbb{H} & \xrightarrow[\sigma_\gamma]{} & \mathbb{H} & \ni & \gamma \sigma x \end{array}$$

$G = C_{\mathrm{Aut}\,\mathbb{H}}(\Gamma)$

$K = \Gamma \cdot G$

$X \to X/G$ is induced from $\Gamma \trianglelefteq K$.

From Lem 1, choose $(\hat{X}, \beta)$: Belyi $\left\{ \begin{array}{l} g(\hat{X}) = g(Y) \\ \text{prime degree} \\ \# \beta^{-1}(1) \geq r \\ \text{non-uniform} \end{array} \right.$

We may find a qc deformation $\hat{K}$ of K,

such that $\mathbb{H}/\hat{K}$ consists of $\left\{ \begin{array}{l} \text{underlying R.S. } \hat{X} \\ \text{cone points } q_1, q_2, \dots, q_r \in \hat{\beta}^{-1}(1) \\ \quad \text{of order } m_1, m_2, \dots, m_r \in \mathbb{Z}_{\geq 2} \end{array} \right.$

Now $\exists\, f : \mathbb{H} \to \mathbb{H}$: qc

w/ $\theta_f : K \to \hat{K}$: isom of Fuchsian grp

$\gamma \mapsto f \gamma f^{-1}$; $\hat{X} = \mathbb{H}/f\Gamma f^{-1}$ $\quad \hat{\Gamma} = f \Gamma f^{-1}$

θ_f projects to $\bar{\theta}_f : G \xrightarrow{\cong} \hat{G} = f G f^{-1}$

$\Rightarrow \quad G \curvearrowright X \underset{\text{top.eq.}}{\sim} \hat{G} \curvearrowright \hat{X} = \underbrace{\mathrm{Aut}(\hat{X}, \beta = \delta \circ \pi_{\hat{G}})}_{\text{Belyi}} \curvearrowright \hat{X}$

$$\begin{tikzcd}
\hat{X} \arrow[rr, "\hat{\alpha}"] \arrow[dr, "\pi_{\hat{G}}"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & \hat{X} \arrow[dl, "\pi_{\hat{G}}"] \\
& \hat{X}/\hat{G} \arrow[d, "\delta"'] & \\
& \mathbb{P}^1_{\mathbb{C}} &
\end{tikzcd}$$

We have $\hat{\alpha} \in \hat{G} \Rightarrow \pi_{\hat{G}} \circ \hat{\alpha} = \pi_{\hat{G}}$

$\Rightarrow \beta \circ \hat{\alpha} = \beta$

$\Rightarrow \hat{G} \leq \mathrm{Aut}(\hat{X}, \beta) = \hat{G}'$

claim $\hat{G} = \hat{G}'$

(β projects to $\pi_{\hat{G}'} = \hat{X} \to \hat{X}/\mathrm{Aut}(\hat{X}, \beta)$)

Assume $\hat{G} \lneq \hat{G}'$. then δ factors through the diagram

$$\begin{tikzcd}
& \hat{X} \arrow[dl, "\pi_{\hat{G}}"'] \arrow[dr, "\pi_{\hat{G}'}"] \arrow[ddd, bend right=80, "\beta"'] & \\
\hat{X}/\hat{G} \arrow[rr, "\text{reg}"] \arrow[dr, "\delta"'] & & \hat{X}/\hat{G}' \arrow[dl, "\delta'"] \\
& \mathbb{P}^1_{\mathbb{C}} &
\end{tikzcd}$$

and $\underbrace{\deg \delta}_{\text{prime}} = \underbrace{[\hat{G}' : \hat{G}]}_{\neq 1} \deg \delta'$

$\Rightarrow \deg \delta' = 1$, β is regular

It contradicts to δ : non-uniform //

THM5 $\forall$ finitely generated G is realizable as $G = \mathrm{Aut}\, \mathcal{D}$, $\mathcal{D} = (X, \beta)$: ~~Belyi~~

pf. of THM5) Let $G = \langle g_1, \cdots, g_s \rangle$.

From Lem 1, choose (x, δ) : Belyi $\begin{cases} g(x) = s \\ \text{prime degree} \geq 3 \\ \text{non-uniform} \end{cases}$

Let $\Gamma = \langle A_1, B_1, \cdots, A_s, B_s \mid \prod [A_j, B_j] = 1 \rangle$: hyp. Fuchsian grp

s.t. $X = \mathbb{H}/\Gamma$

$\theta : \Gamma \to G$, $\begin{pmatrix} A_j \mapsto g_j \\ B_j \mapsto 1 \end{pmatrix}$ $\qquad \Gamma_\theta := \ker \theta$

$Y := \mathbb{H}/\Gamma_\theta$; $Y \xrightarrow{\pi} X$: regular cov.

$\Gamma_\theta < \Gamma$ w/ $\mathrm{Deck}\, \pi = \Gamma/\Gamma_\theta \cong G$.

Then, $(Y, \beta = \delta \circ \pi)$ has G as a $\mathrm{Aut}(Y, \beta)$-subgroup

(not necessarily finite cov.) $\qquad \| \quad G \leq G'$

$$\begin{tikzcd}
 & Y \arrow[dl, "\pi"'] \arrow[dr, "\pi_{G'}"] \arrow[ddd, bend right=60, "\beta"'] & \\
Y/G \arrow[rr, "\mathrm{reg}"] \arrow[dr, "\delta"'] & & Y/G' \arrow[dl, "\delta'"] \\
 & \mathbb{P}^1_{\mathbb{C}} & \\
 & &
\end{tikzcd}$$

possibly infinite

Belyi

$\underline{\text{claim}}$ $G = G'$

Assume $G \subsetneq G'$, then δ factors through the diagram.

← and $\deg \delta = [G' : G] \deg \delta'$

As $\deg \delta < \infty$: prime and $G \neq G'$, it follows that

$\deg \delta' = 1$, β is regular.

But it contradicts to $\beta : \underbrace{Y \xrightarrow[\text{unbranched cover}]{\pi} Y/G \xrightarrow[\text{non-uniform cover}]{\delta} \mathbb{P}^1_{\mathbb{C}}}_{\text{not regular}}$ $\#$

Reference

[Hidalgo2019] R. Hidalgo, Automorphism groups of dessin d'enfants. Archiv der Mathematik, 112, no.1 (2019) 13–18.

$$\begin{tikzcd}
 & \mathbb{H} \arrow[r, "f"] \arrow[d, "\sigma"'] \arrow[ddl, "\gamma"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{H} \arrow[d, "\hat{\sigma}"] \arrow[ddr, "\hat{\gamma}"] & \\
 & \mathbb{H} \arrow[r, "f"] \arrow[dd, "\gamma"] & \mathbb{H} \arrow[dd, "\hat{\gamma}"'] & \\
\mathbb{H} \arrow[rrr] \arrow[dr, "\sigma"'] & & & \mathbb{H} \arrow[dl, "\hat{\sigma}"] \\
 & \mathbb{H} \arrow[r, "f"] & \mathbb{H} &
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma \arrow[r, hook] \arrow[d, "f^*"'] & K \arrow[r] \arrow[d, "f^*"'] & G \arrow[d, "f^*"'] \\
\hat{\Gamma} \arrow[r, hook] & \hat{K} \arrow[r] & \hat{G}
\end{tikzcd}$$