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FINITE TRANSLATION SURFACES WITH MAXIMAL NUMBER OF TRANSLATIONS

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ABSTRACT

The natural automorphism group of a translation surface is its group of translations. For finite translation surfaces of genus $g \geq 2$ the order of this group is naturally bounded in terms of g due to a Riemann–Hurwitz formula argument. In analogy with classical Hurwitz surfaces, we call surfaces which achieve the maximal bound Hurwitz translation surfaces. We study for which g there exist Hurwitz translation surfaces of genus g.

0. Intro

A finite translation surface (X, μ) w/ μ $\left\{ \begin{array}{l} \forall \text{ coord. changes are translations } z \mapsto z+c \\ \text{defined on } X \text{ up to finitely many cone singularities} \end{array} \right.$

surface

max. atlas

naturally shows up when studying Teichmüller spaces $T(X)$

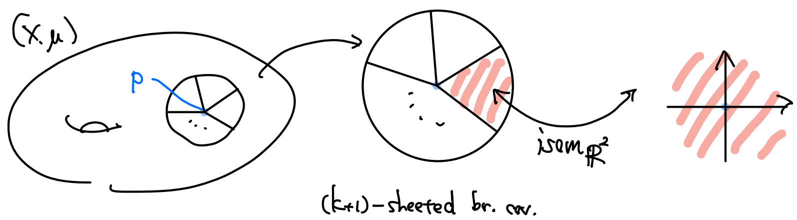
Since it comes from (X, ω) by $\mu = \{(U, p \mapsto \int^p \omega) \mid p \in U \subset X\}$

non-zero holo. 1-form

where (X, ω^2) $\begin{cases} \text{forms pt in cot bundle of } T(X). \\ \text{determines Teichmüller disk } \Delta \hookrightarrow T(X) \end{cases}$

cf. Earle, Gardiner '97, Veech '89 ...

In general, translation surfaces are distinguished by their singularities.



$\Longleftrightarrow$: $p \in (X,\mu)$ is a singularity of order k

(k+1)-sheeted br. cov. of the Euclidian plane

w/ cone angle of $2(k+1)\pi$

Def We denote the space of translation surfaces w/ n singularities of order $k_1, k_2, \cdots, k_n$ by $H(k_1, k_2, \cdots, k_n)$

rem Euler-characteristic calc. states that the genus g of (X,μ) is determined by $k_1 + k_2 + \cdots + k_n = 2g-2$

Rem genus$(X) \geq 1$. for trans. surf. X

Q What is the maximal number of automorphism a surface in genus g can have?

— Hurwitz automorphism theorem (Hurwitz, 1890's)

For any Riemann surface in genus g, $|\mathrm{Aut}(X)| \leq 84(g-1)$

There has been a vivid study to describe R.S. achieving this upper bound.

$\longrightarrow$ called a Hurwitz surface (Hs)

Conder, 1990

Accola, 1968

Maclachlan, 1969 — genera in which no Hs occurs

Larsen, 2001 — genera in which Hs are characterized as rare cases.

Schlage-Puchta, Wolfart — good genera there have to be a lot Hs.

- natural automorphism on trans. surf.

  $=$ translation $z \mapsto z + c$.

THEOREM 1 (proven in Section 2): *Let $g \geq 2$.*

(i) *A finite translation surface (X, μ) of genus g has at most $4g - 4$ translations. It has precisely $4g - 4$ translations if and only if (X, μ) is a normal origami in the stratum $H(1, \ldots, 1)$.*

(ii) *A finite group G is the automorphism group of a Hurwitz translation surface if and only if it can be generated by two elements a and b such that their commutator $[a, b]$ has order 2.*

We then study in which genus there exist Hurwitz translation surfaces and obtain the answer to this question in Theorem 2.

THEOREM 2 (proven in Section 3): *There exists a Hurwitz translation surface of genus g if and only if g is odd or $g - 1$ is divisible by 3.*

1. Basics.

Definition 1: An **origami** O is equivalently given by one of the three following objects:

(1) A translation surface (X, μ) obtained by taking finitely many copies of the Euclidean unit squares and gluing their edges via translations, up to equivalence. Two such surfaces (X_1, μ_1) and (X_2, μ_2) are equivalent, if there exists a homeomorphism $f : X_1 \to X_2$ which is a translation with respect to the charts in μ_1 and μ_2.

(2) A finite degree cover $p : X \to E$ of the torus E which is ramified at most over one point, up to equivalence. Two such covers $p_1 : X_1 \to E_1$ and $p_2 : X_2 \to E_2$ are equivalent, if there exists a homeomorphism $h : X_1 \to X_2$ with $p_2 \circ h = p_1$.

(3) A pair of permutations (σ_a, σ_b) in the symmetric group S_d with some $d \in \mathbb{N}$ up to equivalence by simultaneous conjugation.

The equivalence of these definitions is, e.g., described in [Sch06, Section 1].

A special class of origami : normal origami

$\Longleftrightarrow$ in above Def, $(p : X \to E)$: normal cover (a)

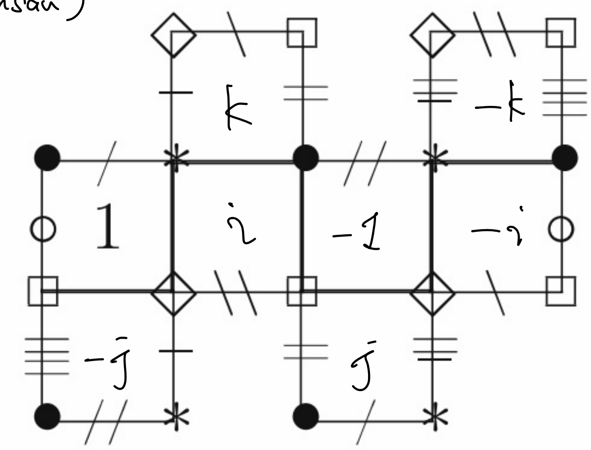
$\overset{(b)}{\parallel}$

$(q : X \to X/G)$: quot. by $G = \mathrm{Trans}(X) = \mathrm{Aut}(X, \mu)$

$S_d > \langle \sigma_a, \sigma_b \rangle \underset{(c)}{\cong} \pi_1(X^*) \underset{(d)}{\triangleleft} \pi_1(E^*)$. ((a)~(d) : equiv.)

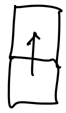
Exm Quarternion origami (Eierlegende Wollmilchsau)

X: obtained by gluing $\longrightarrow$



$G = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$ ... the quarternion grp

acting on X by $\begin{cases} i: \text{ translation by} \\ j: \text{ translation by} \end{cases}$



horiz. monodromy : $\sigma_a = (1\ i\ -1\ -i)(j\ -k\ -j\ -k)$
vert. monodromy : $\sigma_b = (1\ j\ -1\ -j)(i\ k\ -i\ -k)$

$G \cong \langle \sigma_a, \sigma_b \rangle \triangleleft S_8$, $\quad p: X \to E \cong X/G$. Normal origami.

2. Hurwitz trans. surfaces

In this section we show that a translation surface of genus g can have at most $4g-4$ automorphisms. We call a translation surface a **Hurwitz translation surface**, if it achieves this upper bound. We prove the criterion in Theorem 1 for translation surfaces to be Hurwitz translation surface and give first examples for Hurwitz translation surfaces.

LEMMA 4: *Let X be a finite translation surface of genus $g \geq 2$; X has at most $c(g) = 4g-4$ translations. If X has $4g-4$ translations, then it lies in the stratum $H(1, \ldots, 1)$ and is an origami.*

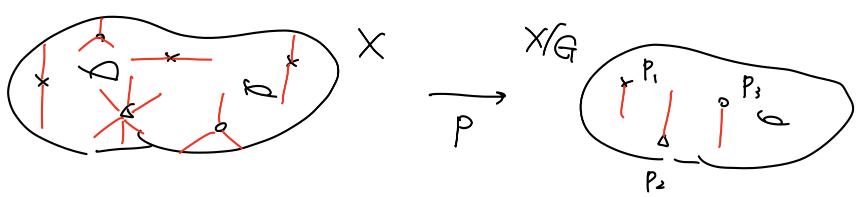
pf) We may consider a normal translation cover $p: X \to X/G$ where $G = \mathrm{Trans}(X, \mu)$

Now both X & X/G are finite translation surface $\Rightarrow g(X/G) \geq 1$.

Let $P_1 \cdots P_t \in X/G$ be branched pts of p.

As p: normal cov., $^\forall$pts in $p^{-1}(P_i)$ have the same multiplicity m_i for $^\forall i$.

($=$ ord g where $g \sim$ the branched locus)



We have $d = |p^{-1}(P_i)| \times m_i$ and $|p^{-1}(P_i)| \leq \frac{d}{2}$. ($m_i \geq 2$)

By Riemann-Hurwitz formula, it holds that

$$2g(X)-2 = d\,(2g(X/G)-2) + \sum_{i=1}^{k} \underbrace{|p^{-1}(P_i)|\,(m_i}_{d}-1)$$

$$= d\,(2g(X/G)-2) + dk - \sum_{i=1}^{k} |p^{-1}(P_i)|$$

$$\geq d\,(2g(X/G)-2) + dk - \frac{dk}{2}$$

$$= d\left(2g(X/G)-2+\frac{k}{2}\right)$$

Since $g(X/G) \geq 1$, we have $2g(X/G)-2+\frac{k}{2}=0 \Leftrightarrow g(X/G)=1$ & $k=0$.

$\Rightarrow g(X)=1$: contradiction

So we have $2g(X/G)-2+\frac{k}{2} \geq \frac{1}{2}$,

$$2g(X)-2 \geq d\,(\quad) \geq d\cdot\frac{1}{2}$$

hence $d \leq 4g(X)-4$ //.

Equality holds $\Leftrightarrow$ equalities of ___ & ___ hold

$\Leftrightarrow$ $|p^{-1}(P_i)| = \frac{d}{2}$ $\forall i$ & $g(X/G)=1$, $k=1$

$\Leftrightarrow$ normal origami in $H(1,\dots,1)$ $\boxtimes$

<u>Def 5</u> We say that (X,μ) is a Hurwitz translation surface (Hts) iff $|\mathrm{Trans}(X,\mu)| = 4g-4$.

Example 7: The following origamis are Hurwitz translation surfaces:

(1) The Eierlegende Wollmilchsau from Example 2 is a Hurwitz translation surface. (quarternion origami)

(2) The Escalator with 8 squares (see Figure 4) defined by the permutations

$$\sigma_a = (1,2)(3,4)(5,6)(7,8), \quad \sigma_b = (2,3)(4,5)(6,7)(8,1).$$

The automorphism group is the dihedral group

$$D_4 = \langle \tau_1, \tau_2; \tau_1^2, \tau_2^2, (\tau_1\tau_2)^4 \rangle$$

of 8 elements. Going to the right corresponds to multiplication by τ_1, going up corresponds to multiplication by τ_2. The origami has four singularities of total angle 4π and is thus of genus 3.

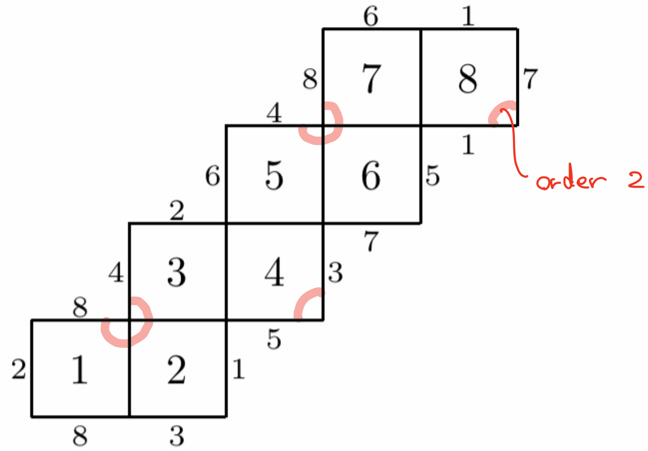


Figure 4. The Escalator: a normal origami of genus 3 with 8 translations.

(3) The origami given by the following two permutations (see Figure 5):

$$\sigma_a = (1,5,7)(2,4,8)(11,12,10)(3,6,9),$$
$$\sigma_b = (1,4)(2,6)(3,5)(7,11)(8,10)(9,12).$$

The origami is of genus 4. Its automorphism group is the alternating group A_4. The squares correspond to the elements in A_4 as follows:

$1 \leftrightarrow \mathrm{id}, 2 \leftrightarrow (1,4,3), 3 \leftrightarrow (1,3,4), 4 \leftrightarrow (1,2)(3,4),$

$5 \leftrightarrow (1,2,3), 6 \leftrightarrow (1,2,4), 7 \leftrightarrow (1,3,2), 8 \leftrightarrow (2,4,3),$

$9 \leftrightarrow (1,4)(2,3), 10 \leftrightarrow (1,4,2), 11 \leftrightarrow (2,3,4), 12 \leftrightarrow (1,3)(2,4).$

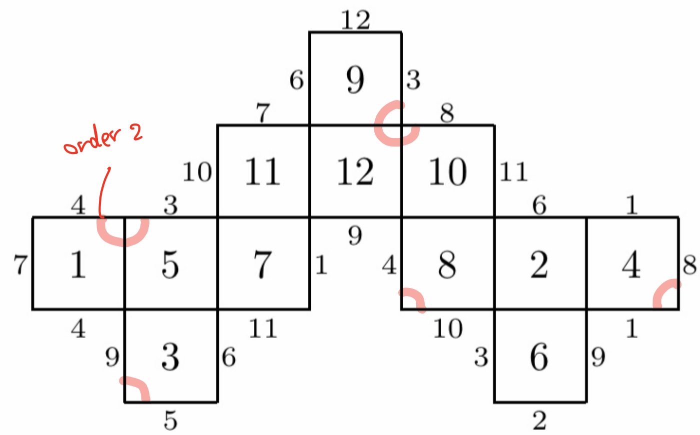


Figure 5. A normal origami in genus 4 whose deck group is the alternating group A_4.

LEMMA 8: *Let G be a finite group of order d which is generated by two elements a and b. Suppose that the commutator $[a,b]$ has order 2; G acts on itself by multiplication from the right. Identify the elements of G with the numbers $1,\ldots,d$. Then each element g of G defines a permutation σ_g in S_d. Let O be* *the origami defined by the pair of permutations (σ_a, σ_b). Then O is a Hurwitz translation surface and we obtain any Hurwitz translation surface in this way.*

pf) $G = \langle a, b \rangle$ acts on G itself by right multiplication.

As G is finite, doubly-generated & the action is transitive,

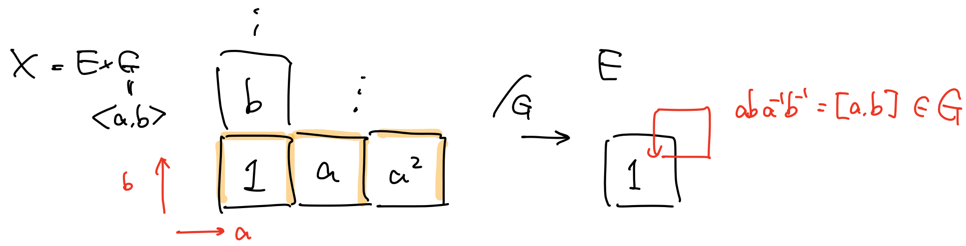
it defines an origami whose cells labelled in G.

$\rightsquigarrow$ We may construct an origami equivalent to $X \to X/G$; normal cover.

in the following way :

$O_G : X = E \times G/\sim \ \to E$; $(p, g) \mapsto p$ $\quad \left(\binom{0}{y}, g\cdot a\right) \sim \left(\binom{1}{y}, g\right)$

$\quad \left(\binom{x}{0}, g\cdot b\right) \sim \left(\binom{x}{1}, g\right)$



Here we have multiplication indices arr. every vertices

by (the orders of translation by $[a, b]$) $= 2$.

$\Rightarrow O_G \in H(1, 1, \cdots, 1)$.

Converse is clear. $\#$

<u>Rem</u> There is no Hts of genus 2.

($\because$ there is no grp $\langle a, b \rangle$ of order $4 \cdot 2 - 4 = 4$ w/ ord $[a, b] = 2$)

3. Translation Hurwitz numbers

Exm 7 shows : Hts in genus 3, 4

Rem 9 shows : no Hts in genus 2

Lem 8 gives an answer to : Which group occurs as Trans$(\cdot)$ of $\exists$ Hts ?

Q. In what genus does Hts exist ?

Def 10 A finite group G is called translation Hurwitz (tH)

$\Leftrightarrow$ $G = \langle a, b \rangle$ w/ $|[a, b]| = 2$.

An integer n is called translation Hurwitz (tH)

Euler char. calc.

$\Leftrightarrow$ $\exists G$: tH group s.t. $|G| = n$ $\left(\Leftrightarrow g(\mathcal{O}_G) = \frac{n}{4} + 1 \right)$

We'll observe that :

$\leadsto$ n shall be divisible by 4 !

(cf. Lem 14)

Prop 11 An integer n is tH $\Leftrightarrow$ n is divisible by 8 or 12.

As a direct consequence, it follows that :

thm 2 There exists Hts of genus g $\Leftrightarrow$ $g-1$ is divisible by 2 or 3.

One directions of Prop 11 follows from the following exms.

$a = 4$ C_4

Prop 12 ("$\Leftarrow$" of prop 11)

(1) $\forall a \geq 3$ 2^a is tH.

(2) $\forall b \geq 1$ $4 \cdot 3^b$ is tH.

(3) n is tH & $(n, m) = 1$ $\Rightarrow$ nm is tH.

0 1 2 3 4 5 6 7

(0 1 2 3 4 5 6 7)

$\downarrow$

(0 5 2 7 4 1 6 3)

pf) (1) consider $G = C_{2^{a-1}} \rtimes_\varphi C_2$ where $\varphi : C_2 \to \mathrm{Aut}(C_{2^{a-1}})$

$1 \mapsto (1 \mapsto 1 + 2^{a-2}) = \begin{pmatrix} 2i-1 \mapsto 2^{a-2} + 2i - 1 \\ 2i \mapsto 2i \end{pmatrix}$

$(\xi_1, \varepsilon_1) \cdot (\xi_2, \varepsilon_2) = (\xi_1 \varphi_{\varepsilon_1}(\xi_2), \varepsilon_1 \varepsilon_2)$ ($\because$ abelian)

We choose a pair of generators $x = (1, 0)$ $y = (0, 1)$ ($\leadsto$ Normal origami)

$\leadsto$ $[x, y] = (1, 0)(0, 1)(-1, 0)(0, 1)$

$= (1 + \cancel{\varphi_0(0)}, 0 + 1)(-1, 0)(0, 1)$

$= (1 + \varphi_1(-1), 1 + 0)(0, 1)$

$= (\cancel{1} + 2^{a-2} \cancel{-1}, 1)(0, 1) = (2^{a-2}, 0)$: order 2 $\Rightarrow$ G is tH. $\lrcorner$

(2) Use the alternating grp $A_4 = \langle x_1, y_1 \rangle$ w/ $x_1 = (1\ 2\ 3)$ $y_1 = (1\ 2)(3\ 4)$

As $[x_1, y_1]$ has order 2 & $|A_4| = 4 \cdot 3^1$, claim holds for $b = 1$.

For the higher b, $G = A_4 \times C_{3^{b-1}} = \langle x_b, y_b \rangle$ w/ $x_b = (x_1, 0)$ $y = (y_1, 1)$

give a suitable Hts. $\longrightarrow$

(3) Let $G = \langle x, y\rangle$: cH w/ $|G| = n$, $[x, y]$: order 2

Consider $G_m = G \times C_m$ & $x_m = (x, 0)$ $y_m = (y, 1)$

Since $(n, m) = 1$, $\exists u, v \in \mathbb{Z}$ s.t. $un + vm = 1$,

we have : $y_m^{un} = ((y^n)^u, 1 - vm)$ $x_m^{un} = (1, 0)$
$= (1, 1)$

$\Rightarrow$ $G_m = \langle x_m, y_m\rangle$ & $[x_m, y_m]$: order 2

$\Rightarrow$ nm : cH $\longrightarrow$ $\boxtimes$

For the converse,

- Thompson's classification ('68) of minimal finite simple groups

3. Statement of main theorem and corollaries.

Main Theorem. *Each nonsolvable N-group is isomorphic to a group $\mathfrak{G}$ such that $I(\mathfrak{S}) \subseteq \mathfrak{G} \subseteq \mathrm{Aut}(\mathfrak{S})$, where $\mathfrak{S}$ is one of the following N-groups*:
(a) $L_2(q)$, $q > 3$.
(b) $Sz(q)$, $q = 2^{2n+1}$, $n \geqq 1$.
(c) $L_3(3)$.
(d) M_{11}.
(e) A_7.
(f) $U_3(3)$.

Corollary 1. *Every minimal simple group is isomorphic to one of the following minimal simple groups*:
(a) $L_2(2^p)$, p *any prime.*
(b) $L_2(3^p)$, p *any odd prime.*
(c) $L_2(p)$, p *any prime exceeding* 3 *such that* $p^2 + 1 \equiv 0 \pmod 5$.
(d) $Sz(2^p)$, p *any odd prime.*
(e) $L_3(3)$.

Corollary 2. *A finite group is solvable if and only if every pair of its elements generates a solvable group.*

Def G is a simple group

$\Leftrightarrow$ nontrivial grp w/ no nontrivial normal subgrp.

Def G is a solvable group

$\Leftrightarrow$ $\exists$ seq. $1 \triangleleft G_1 \triangleleft \cdots \triangleleft G_k = G$

s.t. $\forall$ G_j / G_{j-1} : commutative (abelian)

Def G is a minimal simple grp

$\Leftrightarrow$ non-abelian simple grp. & $\forall$ proper subgrp is solvable

Lemma 13 (Hall): *Let G be a finite solvable group, π a set of prime divisors of $|G|$. Write $|G| = nm$, where all prime divisors of n are in π, and all prime divisors of m are not in π. Then there exists a subgroup U of G with $|U| = n$, and all subgroups of this form are conjugate.*

We call a subgroup of this form a π-Hall group. If π consists of a single prime p, we write p in place of $\{p\}$, and for a set π we denote by π' the complement of π. In particular, $2'$ denotes the set of all odd primes.

LEMMA 15: *Let π be a set of primes, G a solvable group, U a π-Hall group. Let n be the number of conjugates of U. Then n divides the π'-part of $|G|$, and there exist non-negative integers a_p, $p \in \pi$, such that*

$$1 + \sum_{p \in \pi} a_p p = n.$$

COROLLARY 16: *Suppose that $|G|$ is divisible by 4, but not by 8 or 12. Then G has a normal subgroup of index 4.*

pf of prop 11) Suppose G : tH grp.

$g = \frac{|G|}{4} + 1$

Consider the genus of the normal origami: → $|G|$ is divisible by 4.

By the fact that every group of order 4 is commutative & Cor 16,

we see that : $|G|$ is not divisible by 8, 12

$\Rightarrow$ $G \rhd^{\exists} U$: index 4. & $U \geq \mathrm{comm}(G)$.

then $[G : \mathrm{comm}(G)]$ is divisible by 4

$\Rightarrow$ $|\mathrm{comm}(G)|$: odd,

$\nexists\, (x, y) \in G$ s.t. $[x, y]$: even order.

$\Rightarrow$ Not tH $\boxtimes$

Summary

THEOREM 1 (proven in Section 2): *Let $g \geq 2$.*

(i) *A finite translation surface (X, μ) of genus g has at most $4g - 4$ translations. It has precisely $4g - 4$ translations if and only if (X, μ) is a normal origami in the stratum $H(1, \ldots, 1)$.*

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