ZAPPONI-ORIENTABLE DESSINS D'ENFANTS

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ABSTRACT. Almost two decades ago Zapponi introduced a notion of orientability of a clean dessin d'enfant, based on an orientation of the embedded bipartite graph. We extend this concept, which we call Z-orientability to distinguish it from the traditional topological definition, to the wider context of all dessins, and we use it to define a concept of twist orientability, which also takes account of the Z-orientability properties of those dessins obtained by permuting the roles of white and black vertices and face-centres. We observe that these properties are Galois-invariant, and we study the extent to which they are determined by the standard invariants such as the passport and the monodromy and automorphism groups. We find that in general they are independent of these invariants, but in the case of regular dessins they are determined by the monodromy group.

— Revista Matemática Iberoamericana (EMS), 36 (2020)

§ 1-2 Intro

'GAGA' : cpt R.S. $\xleftrightarrow{\text{identify}}$ cpx proj. curve

(bi)holomorphic map. $\longleftrightarrow$ (bi)rational map.

Belyi (1979) : cpx. proj. curve is defined over $\overline{\mathbb{Q}}$: field of alg. numbers

$\Longleftrightarrow$ cpt. R.S. admits a Belyi fct

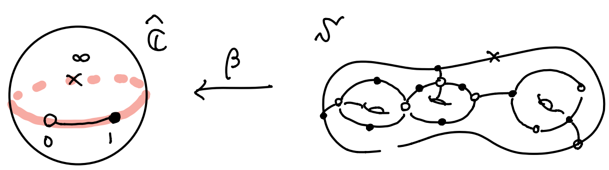
hol. fct. branched over at most three points $0, 1, \infty$ (up to $\mathrm{Aut}\,\hat{\mathbb{C}}$)

such a fct $\beta : \mathcal{S} \to \hat{\mathbb{C}}$ is called a Belyi fct.

a cpt R.S. $\mathcal{S}$ is called a Belyi surface (curve)

a pair $(\mathcal{S}, \beta)$ is called a Belyi pair

A Belyi fct. induces a bipartite graph $\mathcal{G} = \beta^{-1}([0,1]) \hookrightarrow \mathcal{S}$.



- vertices : $\beta^{-1}(0) \cup \beta^{-1}(1)$: bicoloured
- edges : $\beta^{-1}((0,1))$ … there are $\deg\beta$ edges.
- faces : $\beta^{-1}(\hat{\mathbb{C}} \setminus [0,1])$ … there are $\#\beta^{-1}(\infty)$ faces.

$\mathcal{G}$ is equivalently determined by

- a transitive permutation grp $M = \langle x, y \rangle < \mathfrak{S}_{\deg\beta}$ (= monodromy grp $\langle m_\beta(x), m_\beta(y) \rangle$)
- a fundamental grp $H = \pi_1(\mathcal{S} \setminus \mathrm{Crit}\,\beta) \hookrightarrow \pi_1(\hat{\mathbb{C}} \setminus \{0,1,\infty\}) \cong F_2$ w/ index $\deg\beta$ (= automorphism grp $\mathrm{Aut}\,\mathcal{S}$)

etc. upto certain equivalences. (see §1.)

$\boxed{\text{this talk } (x, y, z) \xleftrightarrow[\text{meaning}]{\text{same}} \text{the paper } (\sigma_0, \sigma_1, \sigma_\infty)}$

Rem $\mathrm{Core}(M) \cong H$

$\longrightarrow$ This combinatorial object is called a 'dessin d'enfant'.

Remark that a dessin determines an isomorphism class of Belyi surface (i.e. its cpx str.)

One of major problems in this area is that of determining Galois orbits,

orbits of the action $G_{\overline{\mathbb{Q}}} = \mathrm{Aut}_f \overline{\mathbb{Q}} \curvearrowright \{ \text{alg. curve} / \overline{\mathbb{Q}} \}$ as $\sigma \in G_{\overline{\mathbb{Q}}} : \sum a_{ij} X^i Y^j = 0 \mapsto \sum \sigma(a_{ij}) X^i Y^j = 0$

$\qquad \{\text{Belyi surface}\}$ $\qquad$ action on the coefficient of defining equations

$\qquad \{\text{dessin}\}$ $\qquad$ (rational fct $\rightsquigarrow G_{\overline{\mathbb{Q}}} \curvearrowright \{\text{Belyi fct.}\}$)

Key ingredient : Galois invariant

… If dessins differ in some Galois invariant property then we know that they are in different Galois orbits.

classically known Galois invariants :
- genus of Belyi surf.
- type of Belyi surf.
- class of monodromy grp
- class of automorphism grp.
- passport
- regularity

def The passport of a dessin is the list $(a_1, \cdots, a_l ; b_1, \cdots, b_m ; c_1, \cdots c_n)$ of valencies at points in $\beta^{-1}(0) ; \beta^{-1}(1) ; \beta^{-1}(\infty)$ respectively, arranged in ascending order.

$\|$ $\qquad$ i.e. $l = \#\beta^{-1}(0)$, $m = \#\beta^{-1}(1)$, $n = \#\beta^{-1}(\infty)$.

orders of $x, y, z = (xy)^{-1}$ in monodromy grp G.

A dessin is called regular $\iff$ β is a regular cover
$\iff$ $M \cong H = \mathrm{Centr}(M)$
$\iff$ $\mathrm{Stab}_M 1 = \{id\}$

A dessin is called uniform $\iff$ $a_i \equiv a$, $b_j \equiv b$, $c_k \equiv c$ $\forall i, j, k$ $\quad (\Leftarrow \text{regular})$

◎ The roles played by the three branched pts $0, 1, \infty$ can be converted to produce an another dessin.

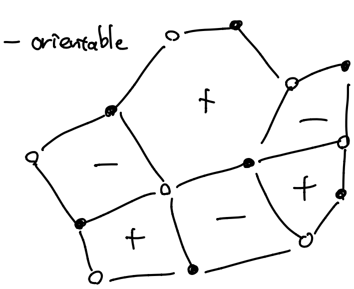
$\overset{\text{same}}{\longleftrightarrow}$ consider $\langle \beta \mapsto 1/\beta , \beta \mapsto 1-\beta \rangle \curvearrowright \{\text{dessin}\}$: $\langle \circ \leftrightarrow \times , \circ \leftrightarrow \bullet \rangle$

We say dessins obtained in this way are twist-related.

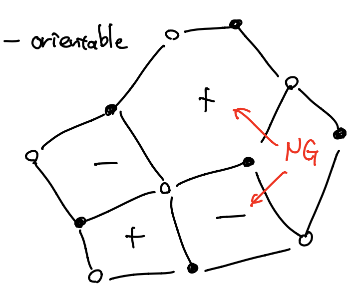
◎ Main result

Def A dessin is Zapponi-orientable (Z-orientable) if its faces can be assigned labels + or − so that each edge is incident w/ faces having different labels.

Z − orientable

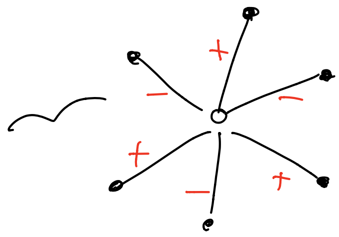


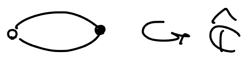
not Z − orientable



Theorem 1. *Each of the following properties of a dessin $\mathcal{D}$ is equivalent to its Z-orientability:*

1. *The function $\beta(1-\beta)$ is the square of a meromorphic function on S.*
2. *The edges of $\mathcal{D}$ can be labelled $+$ or $-$ so that successive edges around each vertex have different signs.*
3. *M acts imprimitively on the edges, with two blocks transposed by σ_0 and σ_1.*
4. *M has a subgroup of index 2 containing σ_∞ and the stabiliser of each edge.*
5. *$\mathcal{D}$ covers the unique dessin of degree 2 and type $(2,2,1)$.*





Next, twisting dessin produces a further refinement of Z-orientability as a useful Galois invariant.

Def The twist-invariant orientability type tot(D) of dessin D

is the triple of Z-orientabilities of dessins D, D', D'' obtained by twisting D.

Theorem 2. *The Galois invariant* tot *of a dessin is independent of its monodromy group, point stabilisers and automorphism group (as abstract groups) and of its passport.*

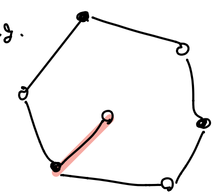
However, in Section 5 we will prove the following (see Corollary 19):

Theorem 3. *For regular dessins the invariant* tot *is determined by the monodromy group (as an abstract group).*

§3. Z-orientable dessin d'enfants

Note that Z-orientability imposes the obvious necessary conditions:

- every ∘ & • have even degree. e.g.
- there is no monofacial edges.



Theorem 3.1 Let D be a dessin w/ associated Belyi pair (S, β).

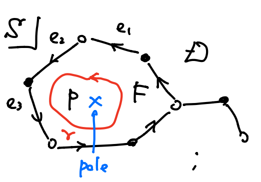
Then D is Z-orientable $\Leftrightarrow$ $\beta(1-\beta)$ is a square of mero. fn. on S.

outline of pf) Consider $\Phi = \dfrac{d\beta^{\otimes 2}}{\beta(1-\beta)}$: mero. quadratic differential

Then RHS $\Longleftrightarrow$ Φ is a square of mero 1-form η : $\Phi = \eta^{\otimes 2}$.

($\Leftarrow$)

Suppose :



;

face F, centred at pole p of order m.

edges $e_1, e_2, \cdots e_{2m}$ forming ∂F, oriented according to F.

γ : small loop around p

as $\mathrm{ord}_p \varphi$ is m, we know $\mathrm{Res}_p \eta$ is either im or $-im$.

Now we can associate this residue to the integral

$$\frac{1}{2\pi i}\int_\gamma \eta = \frac{1}{2\pi i}\sum_{k=1}^{2m}\int_{e_k}\eta = \frac{1}{2\pi i}\sum_k \int_0^1 \frac{dx}{\pm\sqrt{x(1-x)}} = \frac{1}{2i}\sum_k \varepsilon_k = \pm im$$

↑ Computable

where $\varepsilon_k \in \{\pm 1\}$

and we know that ε_k takes the same value for each face F.

↳ sign of F determined.

This gives the $\mathbb{Z}$-orientability

(⇒) We show that a choice of $\sqrt{\beta(1-\beta)}$ can be made around each pt so that $\frac{d\beta}{\sqrt{\beta(1-\beta)}}$ yields a globally defined 1-form.

Fix a sign for each face $\mathcal{F}$ of $\mathcal{D}$.

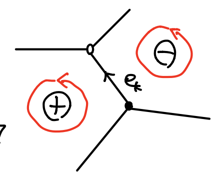
For each face $\mathcal{F}$ of $\mathcal{D}$, we choose the one of $\pm\sqrt{\beta(1-\beta)}$ so that its residue is $\operatorname{sign}(\mathcal{F}) \times im$.

⇝ Denote the resulting 1-form on $U_{\mathcal{F}}$ by η.

For each edge e_k of $\mathcal{D}$, as e is placed b/w two faces w/ distinct signs,

we may choose local branch $\eta_k \in \{\pm\sqrt{\beta(1-\beta)}\}$ so that $\int_{e_k}\eta_k = \pi$

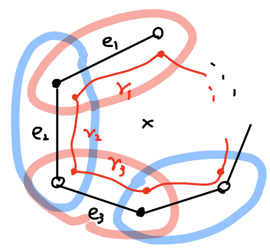
where e_k is oriented according to ⇝



It remains to show $\eta = \eta_k$ on $U_k := (\text{nbd of } \eta_k) \cap \mathcal{F}$.

This holds because $\int_\gamma \eta = \operatorname{sign}\mathcal{F} \cdot 2\pi i m = \sum_{k=1}^{2m}\int_{\gamma_k}\eta_k$

where $\gamma \subset \bigcup_{k=1}^{2m} U_k$ & $\gamma = \sum \gamma_k$ w/ $\gamma_k \subset U_k$: fixed split



Corollary 3.2 $\mathbb{Z}$-orientability is a Galois invariant.

(∵) $G_{\bar{\mathbb{Q}}}$ acts on Belyi morphisms by transformations of coeffs. of the rational representation

$\beta(1-\beta) = \phi^2 \Rightarrow$ as $\phi \in \mathcal{F}(S) = \mathbb{C}[f_1, f_2]$

is described by $\exists R \in \mathbb{C}[X_1, X_2]$: rational w/ $\phi = R(f_1, f_2)$

$\beta^\sigma(1-\beta^\sigma) = \sigma(\phi^2) = \sigma(R(f_1,f_2)^2) = R^\sigma(f_1,f_2)^2$, again rational ~ meromorphic

Q faithfulness : $\forall \sigma \in G_{\bar{\mathbb{Q}}}\ \exists ?\ \mathcal{D}$ s.t. $\mathcal{D} \not\sim \mathcal{D}^\sigma$

—— of $G_{\bar{\mathbb{Q}}} \curvearrowright$ { dessins } : well-known

—— of $G_{\bar{\mathbb{Q}}} \curvearrowright$ { Belyi surface admitting regular dessin } : González-Diez, Jaikin-Zapirain (2015)

—— of $G_{\bar{\mathbb{Q}}} \curvearrowright$ { $\mathbb{Z}$-orientable dessins } : seen by the following

(∵ $\forall \sigma \in G_{\bar{\mathbb{Q}}}$, take $\mathcal{D}$ s.t. $\mathcal{D} \not\sim \mathcal{D}^\sigma$ and apply the next prop.)

Prop. 3.3 Every Belyi surface admits $\mathbb{Z}$-orientable dessin

∵) Use the "multiplicative inverse of the Klein j-function"

$$t(x) = 27x^2(x-1)^2/4(x^2-x+1)^3$$

$\forall (S, \beta)$: Belyi pair,

we have $\begin{cases} \tau \ \& \ \tau\circ\beta : \text{Belyi} \\ \tau\circ\beta\,(1-\tau\circ\beta) = \left(\dfrac{3\sqrt{3}\,(\beta-2)(\beta-1)\beta(\beta+1)(2\beta-1)}{4(\beta^2-\beta+1)^3}\right)^2 \end{cases}$

$\Rightarrow$ $\tau\circ\beta$ induces a $\mathbb{Z}$-orientable dessin on S ⊠

Prop 3.4 (Zapponi's criterion)

A dessin D: $M = \langle x, y\rangle$ is $\mathbb{Z}$-orientable

$$\Longleftrightarrow \quad \begin{array}{ccc} \rho : M & \xrightarrow{\text{hom}} & \{\pm 1\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x, y & \mapsto & -1 \end{array} \quad : \text{well-def} \ \& \ \forall \tau \in \mathrm{Stab}_M 1 \quad \rho(\tau) = +1.$$

$\Longleftrightarrow$ $\underline{\forall \tau \in M,\ \tau(1) = 1 \ \Rightarrow\ \#_x\tau + \#_y\tau \ : \text{even}}$

$$\left(\Longleftrightarrow \quad \begin{array}{ccc} \rho : M & \to & \{\pm 1\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x, y & \mapsto & -1 \end{array} \quad \text{defines an epimorphism} \left(= \begin{array}{c}\text{homomorphism} \\ \text{onto}\end{array} \text{ in this case}\right) \text{ s.t. } \mathrm{Stab}_M 1 < \ker\rho. \right.$$

$$※ \quad \forall \left(\begin{array}{c|cc} & +1 & -1 \\ \hline f & A & B \\ g & C & D \end{array}\right) \rightsquigarrow f\circ\rho = g\circ\rho \Rightarrow f = g$$

is sufficiently satisfied by $\rho(x^2) = 1$ & $\rho(x) = -1$.)

Cor 3.7 A monodromy grp M of $\mathbb{Z}$-orientable dessin D

needs to contain an index 2 subgroup

∵) as $x^2, y^2, xy, yx \in \ker\rho$, $\begin{cases} M/\ker\rho = \{1, [x]\} & \text{if } M/M^2 \cong \mathbb{Z}_2 \quad (x \equiv y \bmod \ker\rho \\ M/\langle\ker\rho, y\rangle = \{1, [x]\} & \text{if } M/M^2 \cong \mathbb{Z}_2\times\mathbb{Z}_2 \quad \text{otherwise} \end{cases}$

Cor 3.8 A monodromy grp M of $\mathbb{Z}$-orientable dessin D

needs to satisfy $M^2 = \langle x^2, y^2\rangle \neq M$.

◎ An interesting property of $\mathbb{Z}$-orientability is that it can be checked w/ a simple glance at the dessin unlike the monodromy grp & other sophisticated invariants.

Exm 3.9 Z-orientability is independent of the passport & isom type of monodromy grps.

Below two dessins have the same passport $(4^2;4^2;4^2)$ & isom monodromy grps.

checked by GAP algorithm.

four coloured edges break Z-orientability.

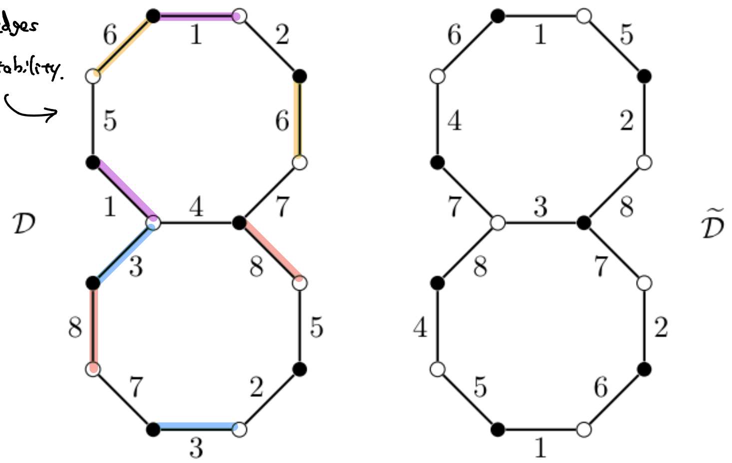


FIGURE 1. A non-Z-orientable dessin $\mathcal{D}$ and a Z-orientable dessin $\widetilde{\mathcal{D}}$ with the same passport and the same monodromy group.

$$\begin{aligned} x &= (1234)(5678) & \tilde{x} &= (1546)(2837) = z \\ y &= (1526)(3748) & \tilde{y} &= (1526)(3748) = y \\ z &= (1546)(2837) & \tilde{z} &= (1234)(5678) = x \end{aligned}$$

That is, $\widetilde{\mathcal{D}}$ is obtained by applying $\begin{pmatrix} x \leftrightarrow z \\ \beta \leftrightarrow 1/\beta \end{pmatrix}$ to $\mathcal{D}$.

So, $\widetilde{\mathcal{D}} \sim (S, 1/\beta)$ where $\mathcal{D} \sim (S, \beta)$.

§5. The twist-invariant orientability type.

Def 5.1 The twist-invariant orientability type $\mathrm{tot}(\mathcal{D})$ of a dessin $\mathcal{D} \sim \underset{\langle x, y\rangle}{(S, \beta)}$ is defined as follows:

$$\mathrm{tot}(\mathcal{D}) := \#\{\widetilde{\mathcal{D}} \underset{\text{twist}}{\sim} \mathcal{D} : \text{Z-orientable}\} \in \{0,1,2,3\}.$$

where $\widetilde{\mathcal{D}} \underset{\text{twist}}{\sim} \mathcal{D} \Leftrightarrow \widetilde{\mathcal{D}} \sim (S, \tilde{\beta})$ w/ $\tilde{\beta} \in \{\beta, 1-\frac{1}{\beta}, \frac{1}{1-\beta}\}$

$\Leftrightarrow \widetilde{\mathcal{D}} \sim \langle \tau(x), \tau(y) \rangle$ w/ $\tau \in \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} z \\ x \\ y \end{pmatrix}, \begin{pmatrix} y \\ z \\ x \end{pmatrix} \right\}$

Prop 5.2 $\mathrm{tot}(\mathcal{D})$ is a Galois invariant, and $\mathrm{tot}(\mathcal{D}) \neq 2$ for $\forall \mathcal{D}$.

pf) Obviously Thm 3.1 shows that $\mathrm{tot}(\mathcal{D})$ is a Galois invariant.

By thm 3.1, $\mathrm{tot}(\mathcal{D}) \geq 2$ implies that two of $\beta, 1-\frac{1}{\beta}, \frac{1}{1-\beta}$ pass the square test.

Simple calculation shows $\beta \times \frac{\beta-1}{\beta} \times \frac{1}{1-\beta} = -1$ & $(1-\beta) \times \left(1-\frac{\beta-1}{\beta}\right) \times \left(1-\frac{1}{1-\beta}\right) = -1$.

So the rest one also passes the square test & $\mathrm{tot}(\mathcal{D}) = 3$ by thm 3.1 again. ∎

Rem 5.3 • $\mathcal{D}$ has a face of odd degree $\Rightarrow$ $\mathrm{tot}(\mathcal{D}) \leq 1$: the same as Z-orientability

(odd number belongs to the third entry in the passport)

• $\mathrm{tot}(\mathcal{D}) = 0 \overset{\text{Cor 3.8}}{\Longleftrightarrow} M = \langle x, y\rangle = \langle y, z\rangle = \langle z, x\rangle$ contains no index-two-subgroup.

• $\mathrm{tot}(\mathcal{D}) = 3 \overset{\text{pf of prop 5.2}}{\Longleftrightarrow}$ two of β, $1-\frac{1}{\beta}$, $\frac{1}{1-\beta}$ are squares of mero. fcts.

Theorem 5.4. *Let $\mathcal{D}$ be a dessin with monodromy group $M = \langle \sigma_0, \sigma_1 \rangle$ and let H be the stabiliser in M of a point.*

(1) *If $M = M^2$, then $\mathcal{D}$ is not Z-orientable.*

(2) *If $M/M^2 \cong \mathbb{Z}_2$, then exactly one of the following holds:*

$$\begin{cases} \mathcal{D} \text{ is Z-orientable, which occurs when } H \leq M^2 \text{ and } \sigma_\infty \in M^2, \\ \mathcal{D}' \text{ is Z-orientable, which occurs when } H \leq M^2 \text{ and } \sigma_0 \in M^2, \\ \mathcal{D}'' \text{ is Z-orientable, which occurs when } H \leq M^2 \text{ and } \sigma_1 \in M^2, \\ \mathcal{D}, \mathcal{D}', \mathcal{D}'' \text{ are all non-Z-orientable, which occurs when } H \not\leq M^2. \end{cases}$$

(3) *If $M/M^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then*

$$\begin{cases} \mathcal{D} \text{ is Z-orientable if and only if } H \leq \langle M^2, \sigma_\infty \rangle, \\ \mathcal{D}' \text{ is Z-orientable if and only if } H \leq \langle M^2, \sigma_0 \rangle, \\ \mathcal{D}'' \text{ is Z-orientable if and only if } H \leq \langle M^2, \sigma_1 \rangle. \end{cases}$$

Cor 5.6 If we add an assumption $H \leq M^2$ to above thm, then

$$\mathrm{tot}(\mathcal{D}) = \begin{cases} 0 & \text{iff} \quad M = M^2 \\ 1 & \text{iff} \quad M/M^2 \cong \mathbb{Z}^2 \\ 3 & \text{iff} \quad M/M^2 \cong \mathbb{Z}^2 \times \mathbb{Z}^2 \end{cases}$$

In particular, for regular dessins ($H = 1$) tot invariant is completely determined by its monodromy grp - fundamental grp.

Exm 5.5 Exm 3.9 shows the exm

s.t. $\mathcal{D}$: not Z-orientable & $\mathcal{D}'$: Z-orientable

So by Prop 5.2 $\mathcal{D}''$ should be not Z-orientable.

Rem 5.7 Thm 5.4 & Cor 5.6 imply that the property

'the stablizer H being a subgrp of M^2' is a Galois invariant.

$\because$ (i) $M = M^2$: always true

(ii) $M/M^2 \cong \mathbb{Z}_2$: $H \leq M^2 \Leftrightarrow \mathrm{tot}(\mathcal{D}) = 1$

(iii) $M/M^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$: $H \leq M^2 \Leftrightarrow \mathrm{tot}(\mathcal{D}) = 3$

($\Leftarrow$: by Thm 5.4, $H \leq \langle M^2, x\rangle \cap \langle M^2, y\rangle \cap \langle M^2, z\rangle = M$)

Next, we consider 5-tuples $(M, H, m_0, m_1, m_\infty)$ where $M = \langle m_0, m_1, m_\infty \mid m_0 m_1 m_\infty = 1 \rangle$: finite group
$H < M$ s.t. $\bigcap_{m \in M} m^{-1} H m = 1$.

Take $\varphi : M \xrightarrow{\text{hom}} \mathrm{Sym}\, M/H$; $m \mapsto (m'H \mapsto mm'H)$

$\overset{\text{corres.}}{\rightsquigarrow}$ dessin associated to the monodromy grp $\varphi(M)$ & point-stabilizer $\varphi(H)$.
(converse correspondence also exists.)

Cor 5.8 For a dessin $\mathcal{D} \sim (M, H, m_0, m_1, m_\infty)$ w/ $H \not\subset M^2$.

we take a new dessin $\mathcal{D}_0 \sim (M, H_0 = H \cap M^2, m_0, m_1, m_\infty)$

Rem $\mathcal{D}, \mathcal{D}_0$ have the same abstract monodromy grp.

Then, (as $\varphi(H_0) < \varphi(M)^2$ by construction,)

(1) If $M/M^2 \cong \mathbb{Z}^2$: $\mathrm{tot}(\mathcal{D}_0) = 1$, $\mathrm{tot}(\mathcal{D}) = 0$ & $\mathcal{D}_0 \to \mathcal{D}$: 2-fold cover

(2) If $M/M^2 \cong \mathbb{Z}^2 \times \mathbb{Z}^2$: $\mathrm{tot}(\mathcal{D}_0) = 3$, $\mathrm{tot}(\mathcal{D}) = 0$ or 1 & $\mathcal{D}_0 \to \mathcal{D}$: 2-fold or $\mathbb{Z}_2 \times \mathbb{Z}_2$-fold cover.

Exm 5.9 For case (1), we may consider the dihedral grp

$$M = \langle a, b, c \mid a^2 = b^2 = c^n = abc = 1 \rangle \quad (n \geq 3 \text{ ; odd})$$

$\longrightarrow$ $H = \langle a \rangle$, $M^2 = \langle c \rangle$: index two subgrp not containing H.

$\bullet \leftrightarrow \circ$

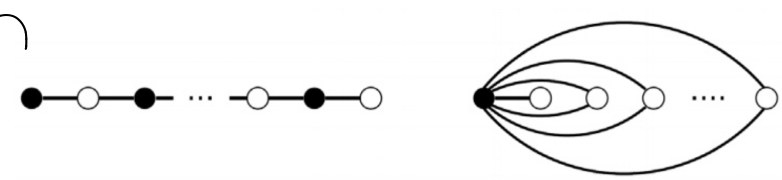


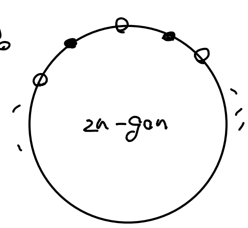
FIGURE 3. The dessins $\mathcal{D}$ and $\mathcal{D}'$.

As none of them are Z-orientable, $\mathrm{tot}(\mathcal{D}) = 0$.

We have $H_0 = H \cap M^2 = \langle a \rangle \cap \langle c \rangle = 1$

so $\mathcal{D}_0$ is a regular dessin of type $(2, 2, n)$.

$\mathcal{D}_0$



← Clearly Z-orientable

By construction, the other two twists have vertex of odd valency,

we have $\mathrm{tot}(\mathcal{D}_0) = 1$

& $M/M^2 \cong \mathbb{Z}^2$, $\mathcal{D}_0 \to \mathcal{D}$: 2-fold cov. //