

Periodic points on the regular and double n -gon surfaces

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Abstract

Using the transfer principle, we classify the periodic points on the regular n -gon and double n -gon translation surfaces and deduce consequences for the finite blocking problem on rational triangles that unfold to these surfaces.

[ASC20]

§ I. Intro

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$$\begin{aligned} \Omega M_g &= \{(X, \omega) \mid X: \text{cpt. R.S. of genus } g, \omega: \text{Abelian differential on } X\} \\ &\stackrel{\text{stratified}}{=} \bigsqcup_{\mu} \Omega M_g(\mu) \quad \text{where } \mu = (m_1, m_2, \dots) : \text{order of zeros } \omega / \sum m_i = 2g-2 \\ \Omega M_g &\xrightarrow{\text{GL}_2^+(\mathbb{R})} \text{affine action, preserving strata} \\ \text{for } (X, \omega), &\quad \text{generated by rotation, dilation, \& Teichmüller flow} \\ &\quad \omega \mapsto c\omega \quad \omega \mapsto \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} Re \omega \\ Im \omega \end{pmatrix} \\ \text{Orb}_{\text{GL}_2^+(\mathbb{R})}(X, \omega) &\text{ projects to } \mathbb{H}/\text{SL}(X, \omega) \\ &\text{where } \text{SL}(X, \omega) := \text{Stab}_{\text{Aff}^+}(X, \omega) \subset \text{SL}_2(\mathbb{R}) : \text{Veech group} \\ (X, \omega) : \text{Veech surface} &\iff \text{SL}(X, \omega) : \text{affine} \\ &\iff \text{vol}(\mathbb{H}/\text{SL}(X, \omega)) < \infty \end{aligned}$$

In this paper, we deal with periodic pts on trans. surf.

Def 1.1 A point p in a Veech surface (X, ω) is called periodic.

$$\iff p \notin \text{Zero}(\omega) \quad \& \quad |\text{Orb}_{\text{Aff}^+(X, \omega)}(p)| < \infty$$

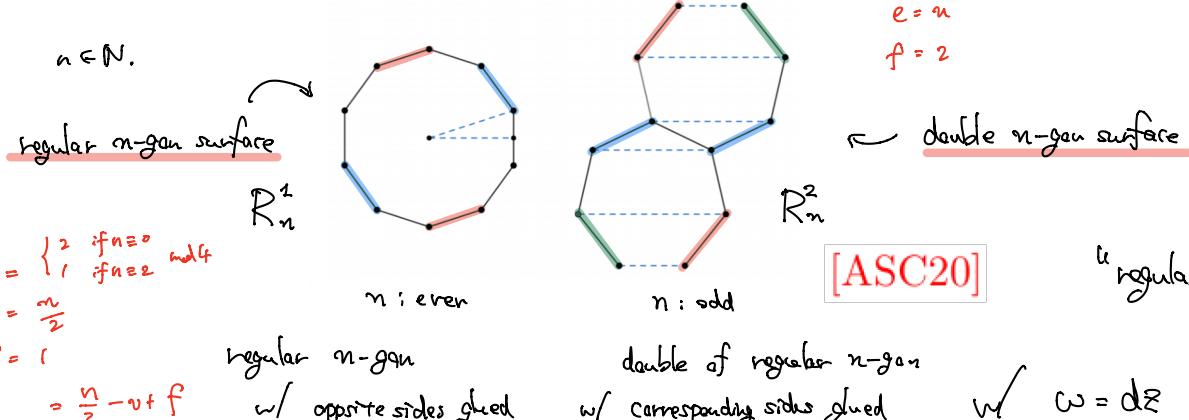
↪ grp of affine diff.

cf. Lem (Müller, '06) $p \in (X, \omega)$ is periodic or in $\text{Zero}(\omega)$

\iff \exists (algebraic) section of some fibred surface $f: \hat{X} \rightarrow \mathbb{C}$
associated to the Teich. curve, which passes through p on the fibre X .

Consider the following translation surfaces:

$$\begin{aligned} v &= \begin{cases} 1 & \text{if } n \equiv 0 \pmod{4} \\ 2 & \text{if } n \equiv 2 \pmod{4} \end{cases} \\ e &= n \\ f &= 2 \end{aligned}$$



[ASC20]

"regular n -gon
translation surface"

regular n -gon
w/ opposite sides glued

w/ corresponding sides glued

w/ $\omega = dz$: natural Abelian diff.

Both are hyperelliptic w/ involutions : affine diff. w/ der. = -I

$\stackrel{\text{def}}{\Leftrightarrow}$ hyperelliptic \Leftrightarrow \exists hyp. involution
that is : $i: R \rightarrow R$: auto.
 $\text{st. } i^2 = \text{id},$
 $| \text{Stab } i | = 2g(R) + 2$

i.e. $\begin{cases} \text{in } R_n^1: \pi\text{-rotation around the center of face} \\ \text{in } R_n^2: \pi\text{-rotation around the center of edges} \end{cases}$

fixed pts of hyperelliptic involutions are called Weierstrass points.

Main result : classification of periodic pts on $R_n^1 \& R_n^2$

Thm 1.3 When $n \geq 5$ & $n \neq 6$,

the periodic pts on $R_n^1 \& R_n^2$ are exactly Weierstrass pts that are not singularities.

$\nwarrow n=5, 8, 10$: Möller, '06.

when $n = 3, 4, 6$, surfaces are tori & have infinitely many periodic pts.

Thm 1.3 apply 'the blocking problem'

Def 1.5 two pts P, Q on a translation surface M are finitely blocked.

$\Leftrightarrow \exists S \subset M \setminus \{P, Q\}$: finite set every straight line joining P, Q pass through a point in S .

\therefore for billiard table & its trajectory, defined similarly.

Cor 1.6 When $n \geq 5$ & $n \neq 6$,

the pairs of finitely blocked pts on $R = R_n^1$ or R_n^2 consist precisely of

pairs $(P, i(P))$ where $\forall P \in R \setminus \text{Sing}(w)$ $\& \forall i$: hyperelliptic involution

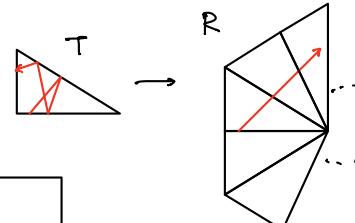
Via the unfolding construction of Katok-Zemlyakov ('75),

Cor 1.6 yields similar results for rational right triangle billiards.

Cor 1.7 When $n \geq 5$ & $n \neq 6$,

The $(\frac{\pi}{2}, \frac{\pi}{n}, \pi - \frac{\pi}{2} - \frac{\pi}{n})$ -triangle admits finitely blocked pts iff n is even.

Furthermore the only such a pair is the $\frac{\pi}{n}$ -vertex & itself.



Note. Classification of periodic pts in strata of Abelian differentials : Apisa '20
quadratic differentials : Apisa-Wright '17.

Classification of periodic pts on every surf. in ΩM_2 : Möller '06 & Apisa '17 (arXiv)

\leftarrow McMullen's ('07) classification of $\text{SL}_2^+(\mathbb{R})$ -orbit closure of genus two strata.

§ 2. Preliminaries

Denote by Γ_n the VG of the regular n -gon translation surface.

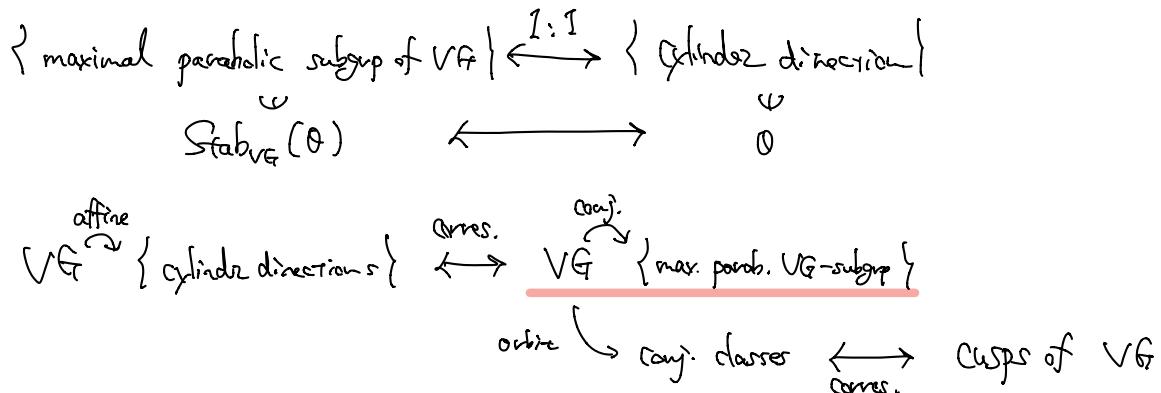
$$r_n = \begin{pmatrix} \cos \frac{\pi}{n} & -\sin \frac{\pi}{n} \\ \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{pmatrix}, \quad s_n = \begin{pmatrix} 1 & 2 \cot \frac{\pi}{n} \\ 0 & 1 \end{pmatrix}.$$

Theorem 2.1 ([Vee89] (Definition 5.7, Theorem 5.8); see also [MT02] (Theorem 5.4)). When n is even, Γ_n is generated by $\{r_n^2, s_n, r_n s_n r_n^{-1}\}$ and is isomorphic to the $(n/2, \infty, \infty)$ triangle group. In particular, \mathbb{H}/Γ_n has two cusps.

When n is odd, Γ_n is generated by $\{r_n, s_n\}$ and is isomorphic to the $(2, n, \infty)$ triangle group. In particular, \mathbb{H}/Γ_n has one cusp.

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From \mathbb{Z}^+ it is known that for Veech surfaces



Thus, $\text{Orb}_{\text{VG}}(\text{cyl. dir}) \xleftrightarrow{\text{Orbs.}} \text{cusps of VG}$

→ every cylinder direction is one of prescribed directions up to VG-action (below)

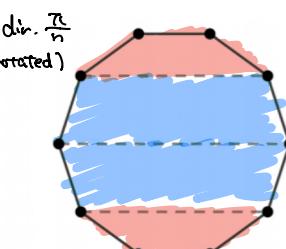
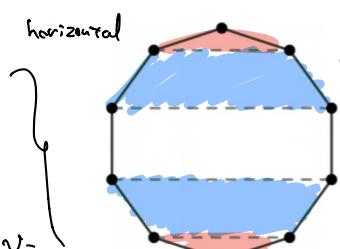
by Thm 2.1, on \mathbb{R}_n^2 (double n -gon), any cyl. dir. can be sent to any other by VG.
↪ exactly one cusp $\rightarrow \#(\text{cylinder}) = \frac{n-1}{2}$

[ASC20] on \mathbb{R}^n (regular n -gon), there are exactly two VG-orbit of cyl. dir.

$n = \text{even}$

$$\#(\text{cyl}) = \lceil \frac{n}{4} \rceil$$

$$\text{Vertices : } i \exp \frac{2\pi j i}{n} =: v_j \quad (j=0, \dots, n-1)$$



$$\#(\text{cyl.}) = \lceil \frac{n-2}{4} \rceil = \lfloor \frac{n}{4} \rfloor \quad (\because n = \text{even})$$

these two belong to different VG-orbits
by the difference of $\#(\text{cyl.})$

$$\hookrightarrow \text{cylinder heights : } h_j = \text{Im}(v_{j+1} - v_j) = \dots$$

$$= 2 \sin \frac{\pi}{n} \sin \frac{(2j+1)\pi}{n} \quad (j=0, 1, \dots, \#(\text{cyl})-1).$$

$$\text{vertices : } i \exp \frac{(2j+1)\pi i}{n} =: v_j$$

$$\text{cylinder heights : } h'_j = \text{Im}(v_{j+1} - v_j) = \dots$$

$n = \text{odd}$

$$\text{Similarly, } h_j = 2 \sin \frac{\pi}{n} \sin \frac{k\pi}{n} \quad (k=0, 1, \dots, \#(\text{cyl})-1).$$

$$= 2 \sin \frac{\pi}{n} \sin \frac{(2j+2)\pi}{n} \quad (j=0, 1, \dots, \#(\text{cyl})-1)$$

Figure 1: Two cylinder directions for the regular decagon.

Len 2.4 (a part of McMullen '96)

For rational number $0 < \alpha < \beta \leq \frac{1}{2}$,

$$\frac{\sin \pi \alpha}{\sin \pi \beta} \text{ is rational} \Leftrightarrow (\alpha, \beta) = \left(\frac{1}{6}, \frac{1}{2}\right)$$

Len 2.5 On regular n -gon translation surfaces, at least one cylinder direction has the property:

(*) heights (circumferences) of cylinders in this direction has irrational ratio.

If $n \not\equiv 0, 6 \pmod{12}$ then every cylinder direction has property (*).

Moreover, any two neighbouring parallel cylinders, their heights (circumferences) has irrational ratios.

∴ By above Remark, any cyl. dir. vector in horizontal or $\frac{\pi}{n}$ (n : even) -direction.

⇒ ratio of heights : of the form $\frac{h_j}{h_k}$ or $\frac{h_k}{h_j}$ and hence $\frac{\sin \pi \alpha}{\sin \pi \beta}$ ($0 < \alpha < \beta \leq \frac{1}{2}$) up to inverting.

by Len 2.5 it is rational iff $(\alpha, \beta) = \left(\frac{1}{6}, \frac{1}{2}\right)$

... it never occurs when n : odd.

Suppose n : even. We have $\frac{h_j}{h_k} = \frac{\sin((2j+1)\pi/n)}{\sin((2k+1)\pi/n)}$: assuming $j < k$, it is rational iff

$$\frac{2j+1}{n} = \frac{1}{6} \cdot \frac{2k+1}{n} = \frac{1}{2} \rightarrow n = [2j+6, k=3j+1]$$

In the same way for $\frac{h_k}{h_j}$; assuming $j < k$, it is rational iff

$$\frac{2j+2}{n} = \frac{1}{6} \cdot \frac{2k+2}{n} = \frac{1}{2} \rightarrow n = [2j+12, k=3j+2]$$

} thus the result follows \square

Len 2.7 regular n -gon translation surfaces (R, ω) are primitive.

i.e. there is no covering $f: (R, \omega) \rightarrow (S, \eta)$ s.t. $g(R) > g(S)$ & $f^* \eta = \omega$.

(pf) Assume $\exists f$ as  (translation cover of (R, ω))

by Len 2.5 & Ren , \exists cyl. dir. w/ irrational height ratio
 & # (cyl.) ≥ 2

$$|Z(\omega)| \leq 2 \Rightarrow |Z(\eta)| \leq 2$$

$$\left. \begin{array}{l} g(S) = g(R) - 1 \quad \& |Z(\eta)| = 2 \\ \uparrow \\ \text{R.H. formula} \end{array} \right\} g(R) \leq 3.$$

again by Len 2.5, both case $n = 10, 14$

$$n = 10 \text{ or } 14$$

we have no cyl. dir. w/ rational height ratio

$$\Rightarrow g(S) = g(R) \quad , \text{ contradiction } \otimes$$

Cor 2.8 For regular n -gon translation surfaces (R, ω) ,

$$SL(R, \omega) \cong \text{Aff}^*(R, \omega).$$

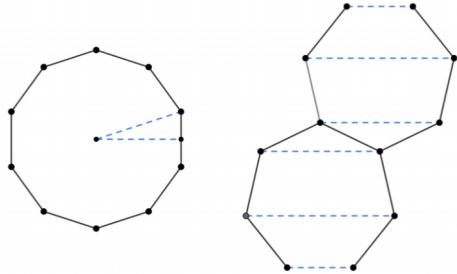
§ 3. pf of Main results

Def 3.1 R_n : regular n -gon trans. surf. Γ_n : VG of R_n .

denote by $P_n \in R_n : \left\{ \begin{array}{l} (n: \text{even}) \text{ the center of } n\text{-gon.} \\ (n: \text{odd}) \text{ the unique cone pt.} \end{array} \right\} \times \text{fixed by VG.}$

Prop 3.2 $\forall p \in R_n : \text{periodic pt. } \text{Orb}_{\Gamma_n}(p) \text{ contains a point on } \left\{ \begin{array}{l} F_0(P_n) \\ \text{or } F_{\frac{n}{2}}(P_n) \quad (n: \text{even only}) \end{array} \right\}$
where F_0 is the directional foliation on R_n .

[ASC20]



These segments are called conjugate line segments

Figure 2: After applying Proposition 3.2 any periodic point can be assumed to lie on one of the dashed lines or its image under the hyperelliptic involution.

(Outline)

The transfer principle $X : \text{top. sp. } G/X \curvearrowright H : \text{top. group acting continuously}$

\Rightarrow there is a 1-1 correspondence among closed dense orbits:

$$\begin{array}{ccc} \text{closed (resp. dense)} & \leftrightarrow & \text{closed (resp. dense)} \\ G\text{-orbits on } X/H & & G \times H \text{-orbits on } X \\ \Downarrow & & \Downarrow \\ \text{Orb}_G(xH) & \leftrightarrow & \text{Orb}_{G \times H}(x) \end{array} \quad \begin{array}{ccc} \text{closed (resp. dense)} & \leftrightarrow & \text{closed (resp. dense)} \\ G \times H \text{-orbits on } X & & H \text{-orbits on } G/X \\ \Downarrow & & \Downarrow \\ \text{Orb}_{G \times H}(x) & \leftrightarrow & \text{Orb}_H(Gx) \end{array}$$

In this pf $\left\{ \begin{array}{l} G \text{ is the VG } \Gamma_n \\ H \text{ is the unipotent subgroup } U = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\} \subset SL(2, \mathbb{R}) \end{array} \right.$

$$\begin{array}{c} \text{Rm } SL(2, \mathbb{R})/U \cong \mathbb{R}^2 \setminus \{0\} \\ \Downarrow \qquad \Downarrow \\ g \leftrightarrow g(\cdot) \end{array}$$

Thm (Parabolic)

the only U -orbits of $\Gamma_n \backslash SL(2, \mathbb{R})$ are closed or dense.
(std. linear action)

($n: \text{even}$)

using Transferprinciple.

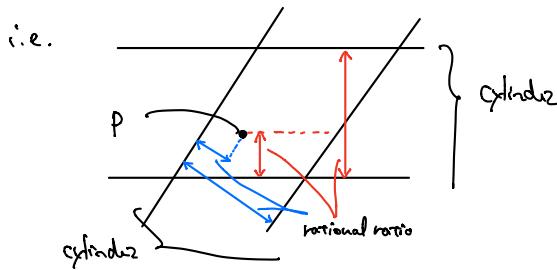
$\xrightarrow{\text{only vectors w/ closed orbit are parallel to } \Gamma_n \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \text{ or } \Gamma_n \left(\begin{pmatrix} \cos \frac{\pi}{n} & \sin \frac{\pi}{n} \\ -\sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)}$
& then 2.1 (description of Γ_n)

Since $\Gamma_n \cdot p$ is finite, it should contain a pt in horizontal leaf

or $\frac{\pi}{n}$ -directional leaf ($n: \text{even}$). \square

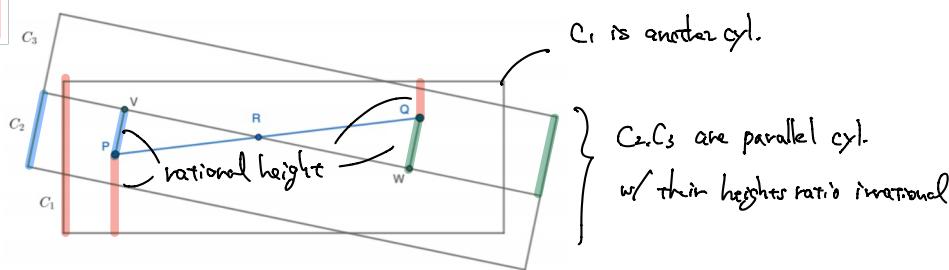
Lem 3.5 (rational height lemma, Apisa 20)

A periodic pt P on a Veech surface has rational height in any cylinder containing it.



Lem 3.6 (rough statement) Let C_1, C_2, C_3 be cylinders as follows:

[ASC20]



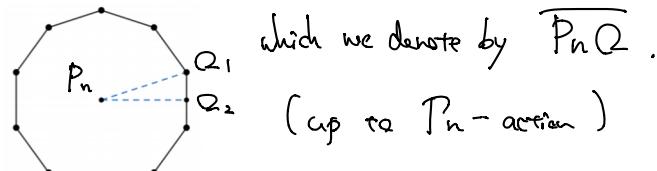
Take P, Q so that (i) \overline{PQ} is neither \perp to $\text{core}(C_i) \sim \text{core}(C_3)$
(ii) $\overline{PQ} \subset C_1, \overline{PQ} \subset C_2 \cup C_3$: properly contained
(iii) $\overline{PQ} \cap (\partial C_2 \cup \partial C_3) = \{R\}$ w/ $R \neq P, Q$.
(iv) P (resp. Q) has rational height in C_1, C_2 (resp. C_1, C_3)
 \Rightarrow no interior pts in $\left\{ \begin{array}{c} \overline{PR} \\ \overline{RQ} \end{array} \right\}$ has rational height in $\left\{ \begin{array}{c} C_1, C_2 \\ C_1, C_3 \end{array} \right\}$

Theorem 1.3 When $n \geq 5$ & $n \neq 6$,

the periodic pts on R_n^1 & R_n^2 are exactly Weierstrass pts that are not singularities.

pf of Thm 1.3) Let P be a periodic pt on R_n .

(I) n: even by prop 3.2, We may assume P lies on a candidate segment,

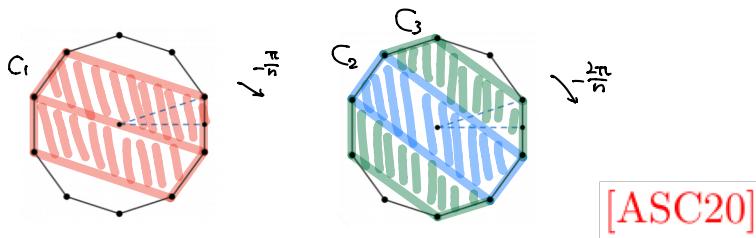


which we denote by $\overline{P_n Q_1}$.

(up to P_n -action)

\rightarrow Each edge (P_n, Q) is either $\left\{ \begin{array}{c} \text{Weierstrass pt or} \\ \text{singularity.} \end{array} \right\}$

For Lem 3.5, we take cyl. $C_1 \sim C_3$ as follows:



by Len 2.5 (neighboring cyl.) G, G has irrational height ratio.

Since δ (candidate segmets) are either $\begin{cases} \text{periodic pt or} \\ \text{singularity} \end{cases}$,
they have rational height in any

cylinder containing \rightarrow by Len 2.5

By Lemma 2.6, $\#$ pt in \overline{PQ}^i has irrational height in one of G, G_2, G_3 .

\rightarrow By Lemma 2.5, such pt cannot be periodic.

I) n : odd ... Though more precise modification is needed

it follows almost similar strategy.



Cor 1.6 When $n \geq 5$ & $n \neq 6$,

the pairs of finitely blocked pts on $R = R_n^1$ or R_n^2 consist precisely of

pairs $(P, i(P))$ where $\#P \in R \setminus \text{Sing}(\omega)$ & i : hyperelliptic involution

Outline) $\overset{\wedge}{\tau}$ trans, surf. (X, ω) that is not an origami, we may take :

$$\begin{array}{ll} (\text{M\"oller '06}) & (\text{Apra, Wright '17}) \\ (X, \omega) \xrightarrow{\quad T_{\min} \quad} (X_{\min}, \omega_{\min}) \xrightarrow{\quad \pi \quad} (Q_{\min}, q_{\min}) \\ \text{minimal translation} & \text{minimal half-translation} \end{array}$$

(Apra & Wright '17) states that $\overset{\wedge}{\tau}(p, q)$: finitely blocked pair one of the following occurs :

(i) p, q are periodic pts on zeros of ω
& the blocking set consists of all other periodic pts.

(ii) Neither p nor q are periodic pts on zeros of ω

but proj. image of p, q in Q_{\min} coincide $\rightarrow \bar{p} \in Q_{\min}$

& the blocking set consists of all periodic pts b $\pi_{\min}^{-1}(\bar{\pi}(\bar{p}))$.

Cor 1.7 When $n \geq 5$ & $n \neq 6$,

The $(\frac{\pi}{2}, \frac{\pi}{n}, \pi - \frac{\pi}{2} - \frac{\pi}{n})$ -triangle admits finitely blocked ps iff n is even.

Furthermore the only such a pair is the $\frac{\pi}{n}$ -vertex & itself.

References

- [ASC20] Paul Apisa, Rafael M. Saavedra, Christopher Zhang: Periodic points on the regular and double n-gon surfaces. arXiv:2011.02668 (2020)