

(contd.)

Q Spec : spectrum functor (comm. rings) \rightarrow (affine schemes)

$\forall R$: comm. ring, $\text{Spec}(R) = \{ I \subset R : \text{prime ideal} \}$

w/ a sys. of closed nbhd : $\{ V_I = \{ I' \subset R : \text{prim. ideal containing } I \} \mid I \in \text{Spec } R \}$

($\{ D_I := V_I^c \} : \text{open base}$) \leftarrow Zariski topology.

For given R : comm. ring,

$\forall I \in \text{Spec}(R)$, $f(D_I, \mathcal{O}_X) := I^c R$: localization

$$\{ (i, r) \in I \times R \mid [(i, r) \sim (j, r_j)] \mid j \in I \}$$

This defines a structure sheaf \mathcal{O}_X

$$\text{s.t. } f(D, \mathcal{O}_X) = (\bigcap_{p \in D} p^c)^c R \quad \forall D \subset \text{Spec}(R) : \text{gen.}$$

Df Let K : field

A K -scheme is a pair (S, p)

S : scheme $p : S \rightarrow \text{Spec}(K)$: morphism	locally m.sch. of Spec structure $f : A \rightarrow B$: hom. $\text{Spec } A \rightarrow \text{Spec } B : p \mapsto f(p)$ $\mathcal{O}_{\text{Spec } A} \rightarrow \mathcal{O}_{\text{Spec } B} : D^c A \mapsto f(D)^c B$
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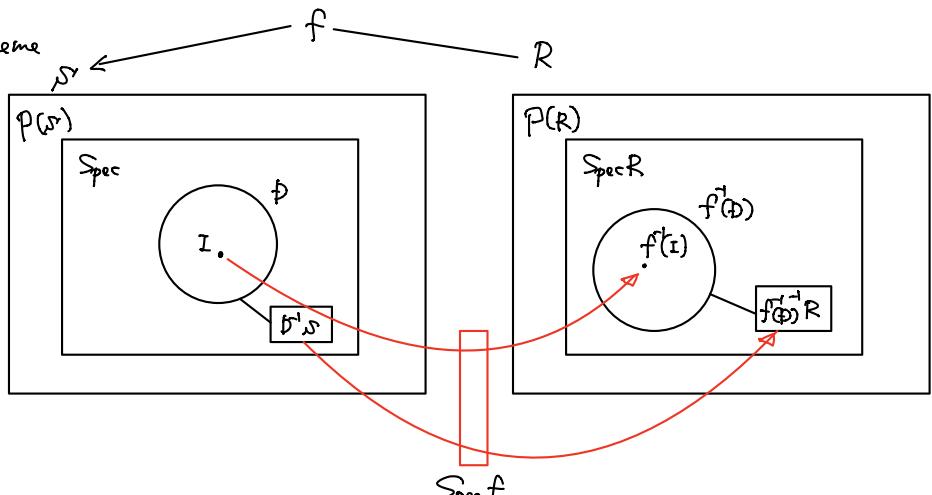
A K -variety is a K -scheme (S, p) s.t. S : reduced & p : separated mor. of finite type.

Df $\forall \alpha \in \text{Aut } K$, $\alpha(S, p) : K\text{-scheme}$

$$S \xrightarrow{p} \text{Spec}(K) \xrightarrow{\text{Spec}(\alpha)} \text{Spec}(K)$$

α

$$(S, p^\alpha) := (S, \text{Spec}(\alpha) \circ p)$$



This defines $\text{Aut } K \curvearrowright (\text{Schemes over } K)$
 $\curvearrowright (\text{K-Varieties})$

Df 3.9 Let $(S, p) : K\text{-scheme (var.)}$

sub field $k \subset K$ is called a field of definition of (S, p)

if : $\exists (S', p') : k\text{-scheme (var.)}$

s.t.

$$\begin{array}{ccc} \exists & S \xrightarrow{\alpha} S' \\ p \downarrow & \alpha \downarrow p' \\ \text{Spec}(K) \rightarrow \text{Spec}(k) & & \end{array}$$

$\text{Spec } i$

$(i : k \hookrightarrow K)$

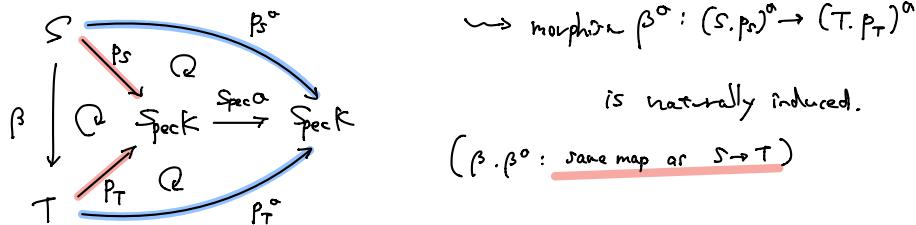
we say $\Leftrightarrow (S, p)$ is defined over k .

We define $\mathcal{U}(S, p) \subseteq \text{Aut}(k)$ by

$$\mathcal{U}(S, p) := \{ \alpha \in \text{Aut}(k) \mid (S, p)^\alpha \cong (S, p) \}$$

The moduli field $M(S, p) := \text{Stab}_k \mathcal{U}(S, p)$ (fixed field)

Def 3.10 $(S, p_S), (T, p_T)$: k -scheme (var.) $\beta: (S, p_S) \rightarrow (T, p_T)$: morphism



Let $\beta': (S', p_{S'}) \rightarrow (T', p_{T'})$: another k -scheme morph.

We write $\underline{\beta \cong \beta'}$ if $\exists \phi: S \xrightarrow{\cong} S' \exists \psi: T \xrightarrow{\cong} T'$

$$s.t. \quad \psi \circ \beta = \beta' \circ \phi$$

Specifically, $\beta \cong \beta^o$ if:

$$\begin{array}{ccc} S & \xrightarrow{\alpha \circ \phi} & S \\ \beta^o \downarrow & \square & \downarrow \beta \\ T & \xrightarrow{\beta^o \circ \phi} & T \\ p \downarrow & \alpha & \downarrow p \\ \text{Spec } k & \xrightarrow{\text{Spec } \alpha} & \text{Spec } k \end{array}$$

Def 3.11 A morph. $S \xrightarrow{\beta} T$ is called defined over $k \subset K$

$$\text{if: } \exists S' \xrightarrow{\beta'} T' \quad \text{morph s.t.} \quad \begin{array}{ccc} S & \xrightarrow{\beta} & T \\ \beta \downarrow & \square & \downarrow \beta' \\ T & \xrightarrow{\beta \circ \phi} & T' \\ p \downarrow & \alpha & \downarrow p' \\ \text{Spec } k & \xrightarrow{\text{Spec } \alpha} & \text{Spec } k \end{array}$$

denoted $\beta \times_{\text{Spec } k} \text{Spec } k$.

We define $\mathcal{U}(\beta) \subseteq \text{Aut}(k)$ by

$$\mathcal{U}(\beta) := \{ \alpha \in \text{Aut}(k) \mid \beta^\alpha \cong \beta \}$$

The moduli field $M(\beta) := \text{Stab}_k \mathcal{U}(\beta)$ (fixed field)

Def 3.12 Let $\beta: X \rightarrow P_C : \text{Belyi morph.}$

Let $U_\beta^\beta \subset \text{Aut } C$: group of $\alpha \in \text{Aut } C$ such that

$\exists f_\alpha: X^\alpha \rightarrow X : C\text{-isomorphism}$

$$\begin{array}{ccc} \text{c.t.} & X^\alpha \xrightarrow{f_\alpha} X \\ & \beta^\circ \downarrow \quad \downarrow \beta & \text{where } \text{proj}_\alpha: P_C \rightarrow P_C \\ & P_C^\alpha \xrightarrow{\text{proj}_\alpha} P_\alpha & [x_i, x_j] \mapsto [\alpha(x_i), \alpha(x_j)] \end{array}$$

The a. we call $U_\beta^\beta := \text{Fix}_{P_C} U_\beta^\beta$ (fixed field)

the fixed field of the divisor $\sim \beta$.

Reur Difference to Def 3.11 (b)

In Def 3.11, $U_\beta = \{ \alpha \in \text{Aut } C \mid \beta \leq \beta^\alpha \}$

$$\begin{array}{ccc} \text{"}\psi\text{" admits covering} & \leftarrow & \begin{array}{ccc} \textcircled{1} & \xrightarrow{\psi} & \textcircled{2} \\ \beta & \downarrow & \downarrow \beta^\circ \\ \text{Proj}_\alpha \text{ or } \psi \in \text{Aut } P_C & \xrightarrow{\text{Proj}_\alpha} & \textcircled{3} \end{array} \\ (\psi = \psi' \circ \text{proj}_\alpha) & & \end{array}$$

i.e. U_β^β is the " ~~$\psi \rightarrow \psi = \text{proj}_\alpha$~~ " ver of U_β .

$\hookrightarrow U_\beta^\beta \subset U_\beta \text{ & } M_\beta^\beta \supset M_\beta$.

Prop 3.13 K -field. $\beta: S \rightarrow T$: morphism of K -schemes (or var.)

a) $M(S)$ depends only on the K -isom type of S
 $M(\beta) \cong \beta : \text{Belyi}.$

c) Etale field of definition of S_β contains $M(S)$
 $M(\beta).$

d) $M(S) \subseteq M(\beta)$

e) for $\beta: \text{Belyi} \vdash M(\beta) \subset M^\beta(\beta)$
f) and $M^\beta(\beta)$ is a number field
 $M_\beta \& M_S$... finite extension of \mathbb{Q} .

Thm 1 (Belyi) A smooth proj. curve C is definable over a number field iff it admits a Belyi morph. $\beta: C \rightarrow P_C$.

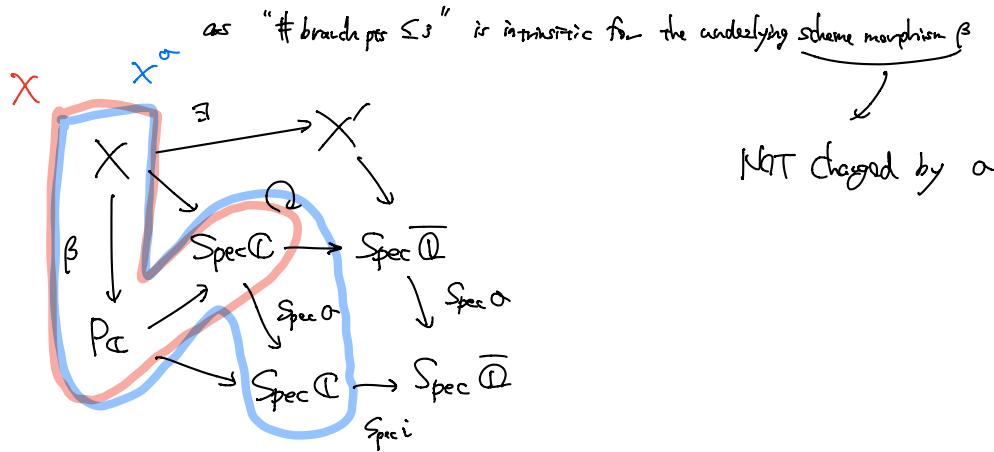
Thm 2 (Hanner, Herrlich 2003) X : curve over a field K

$\Rightarrow X$ is defined over a finite extension of $M(X)$.

3.2 $G_{\overline{Q}} \curvearrowright (\text{dessins})$

Given a Belyi morphism $\beta: X \rightarrow P_C$ & $\alpha \in \text{Aut } C$,

$\beta^\alpha: X^\alpha \rightarrow P_C$ is again a Belyi morphism.



So $\text{Aut } C$ acts on set of Belyi morphs, equivalently on the set of dessins.

By Belyi thm, this action factors through $\text{Aut } \overline{Q} = G_{\overline{Q}}$.

Theorem 3. For every $g \in \mathbb{N}$,

$G_{\overline{Q}} \curvearrowright (\text{dessins of genus } g)$ is faithful.
 ↳ $(\text{clean dessins of genus } g)$ is still faithful.
 (unicellular) { break vertices are "—"

In genus 1, this can be seen as the action is compatible w/ the $\tilde{\gamma}$ -invariant

$$\text{i.e. } \alpha(\tilde{\gamma}(E)) = \tilde{\gamma}(E^\alpha) \quad (E: \text{ell. curve})$$

($\tilde{\gamma}$ as $\tilde{\gamma}$ being a hol. map $H/\mathcal{O}_{\mathbb{Z}[2, 2]} \rightarrow \mathbb{C}$)

(Araknechtz, 2001) This generalizes to higher genera,

by even restricting to hyperelliptic curves only. $\rightarrow g \geq 2$. ok.

(Schnepp, 1994) Said pf for $g=0$

Re- $G_{\overline{Q}}$ -action-faithfulness does not break

when restricting to unicellular / clean dessins (cf. (Nisbach 2011))

THE GALOIS ACTION ON M-ORIGAMIS AND THEIR TEICHMÜLLER CURVES

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ABSTRACT. We consider a rather special class of translation surfaces (called *M-Origamis* in this work) that are obtained from dessins by a construction introduced by Möller in [Mölo5]. We give a new proof with a more combinatorial flavour of Möller's theorem that $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the Teichmüller curves of M-Origamis and extend his result by investigating the Galois action in greater detail.

We determine the Strebel directions and corresponding cylinder decompositions of an M-Origami, as well as its Veech group, which contains the modular group $\Gamma(2)$ and is closely connected to a certain group of symmetries of the underlying dessin. Finally, our calculations allow us to give explicit examples of Galois orbits of M-Origamis and their Teichmüller curves.

found at arXiv : 1408.6769v1

Similar contents are found in the author's PhD thesis.

Intro : Back ground

(correspondence)

(Complex)

- Serre's GAGA (1958) states a close relationship b/w Algebraic Geometry and Analytic Geometry.

$$\left(\begin{array}{ccc} \text{e.g. "projective curve"} & \leftrightarrow & \text{"cpx Riemann surf."} \\ P_C & \leftrightarrow & \hat{\mathbb{P}} \\ \text{elliptic curve } y^2 = x^3 - x & \leftrightarrow & \text{torus } \mathbb{C}/\mathbb{Z}^2 \\ & & \text{etc...} \end{array} \right)$$

Belyi cover	origami cover	\leftarrow study of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ - action
$\beta : X \rightarrow P_C$	$\hat{\beta} : X \rightarrow \hat{\mathbb{P}}$	$(= \text{Aut } \hat{\mathbb{P}})$ field auto

covering of $P_C = \hat{\mathbb{P}}$, ramified over $\{0, 1, \infty\}$	Covering of $E = \mathbb{C}/\mathbb{Z}^2$, ramified over $\{0 + \mathbb{Z}^2\}$
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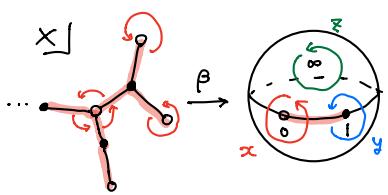
Graph structure induced from β	Graph structure induced from $\hat{\beta}$

R.g.

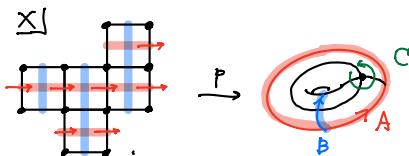
$$\left(\begin{array}{l} \sigma \in \text{Gal } \overline{\mathbb{Q}}/\mathbb{Q} \\ \downarrow \\ \{ \text{alg. proj. curve} \} \\ \downarrow \\ y = a_0 + a_1 x + a_2 x^2 + \dots \\ \downarrow \sigma \\ y = \sigma(a_0) + \sigma(a_1)x + \sigma(a_2)x^2 + \dots \end{array} \right)$$

↔ "dessin"

↔ "origami"



monodromy group : $G = \langle P_x, P_y \rangle < G_d$
 $(P_z = P_x^{-1} P_y^{-1})$



monodromy group : $G = \langle P_A, P_B \rangle < G_d$
 $(P_C = P_A P_B P_A^{-1} P_B^{-1})$

local ramification behavior

is an invariant

under the $\text{Gal } \overline{\mathbb{Q}}/\mathbb{Q}$ -action
 $(= \text{Aut } \overline{\mathbb{Q}}$
field auto.)

(Müller)

$\text{Gal } \overline{\mathbb{Q}}/\mathbb{Q}$ -action on origami's
is induced from the one on
dessins of origami curves.

Thm 1 (Belyi, 1979)

A smooth proj. curve C is definable over a number field iff it admits a Belyi morph. $\beta: C \rightarrow P_C$.

Thm 2 (Harmer, Herrlich 2003) X : curve over a field K

$\Rightarrow X$ is defined over a finite extension of $M(X)$.

Thm 3. For every $g \in \mathbb{N}$,

$\text{Gal } \overline{\mathbb{Q}}/\mathbb{Q}$ (dessins of genus g) is faithful.
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In genus 1, this can be seen as the action is compatible w/ the \tilde{J} -invariant

$$\text{i.e. } \sigma(\tilde{J}(E)) = \tilde{J}(E^\sigma) \quad (E: \text{ell. curve})$$

(\tilde{J} as \tilde{J} being a hol. map $H/\text{SL}(2, \mathbb{Z}) \rightarrow \mathbb{C}$)

(Arinkin, 2001) This generalizes to higher genera,

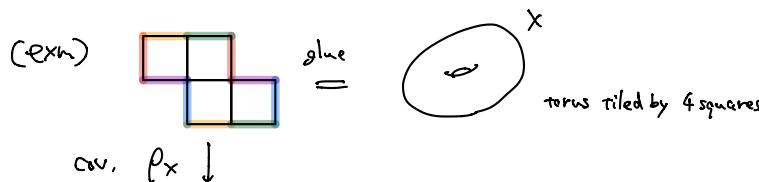
by even restricting to hyperelliptic curves only. $\rightarrow g \geq 2$. cf.

(Schnepp, 1984) See pf for $g=0$

4. Origami & their Teichmüller curves

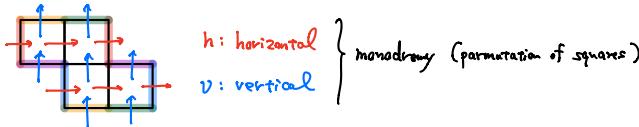
(Short Summary)

origami (square-tiled surface)



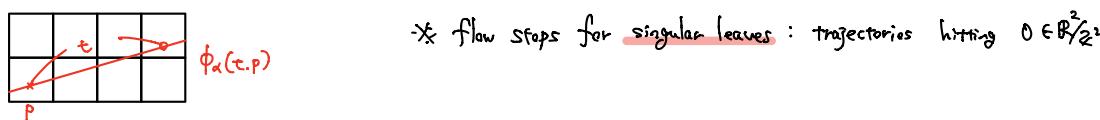
 ... std torus $\mathbb{R}^2/\mathbb{Z}^2 \rightarrow$ Natural translation structure is induced.

ρ_X corresponds to its monodromy $h, v \in \mathcal{G}_{\deg \rho_X}$



On a given origami X , $\alpha \in \mathbb{R}$ defines a linear flow in slope α

$$X \quad \begin{array}{c} \text{origami } X \\ \text{der} \int (\rho_X \text{ is lift to } \mathbb{R}^2) \end{array} \quad \phi_\alpha : \mathbb{R} \times X \rightarrow X \quad (\cos \alpha) \quad \text{and} \quad \text{w/ } \rho_X(\phi_\alpha(t.p)) = \rho_X(p) + t e_\alpha \mod \mathbb{Z}^2$$



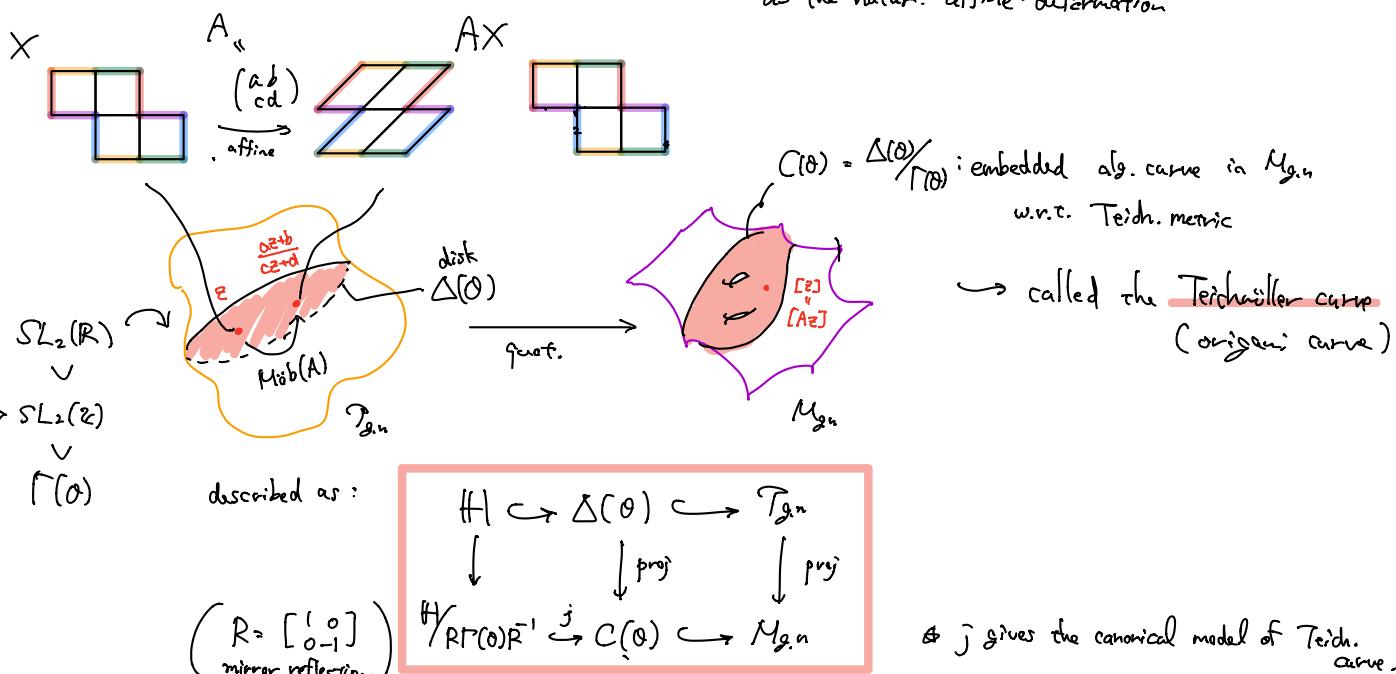
$\Gamma(\theta)$ on origami θ

• Veech group $\Gamma(X)$ of a translation surface X

is the conjugacy class stabilizer of X in $SL(2, \mathbb{R})$ acting on $\Omega T(X)$

↓
stratum of translation surfaces whose analytic type
= one of X .

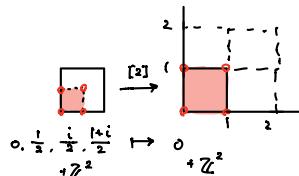
as the natur. affine-deformation



5. Galois action on M -origami

E : the unit square torus \mathbb{C}/\mathbb{Z}^2 : Belyi curve ($y^2 = x^3 - x$, $\beta(x) = x^2$)

$[2]: E \rightarrow E$ multiplication by 2
(4: 1-map)



$h: E \rightarrow P_C$: quotient map under

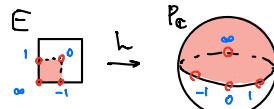
$$\text{the involution } z + \mathbb{Z}^2 \mapsto -z + \mathbb{Z}^2$$

taken s.t. $h(\infty) = \infty$ & the other three crit. values are $0, 1, \lambda$ (-1?)

i.e. h is of the form (P_C -act.) \circ β -fraction of lattice \mathbb{Z}^2

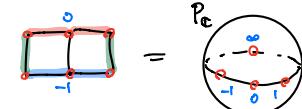
Through h , we name the 4 Weierstrass points on E

$$\text{by } 0, \frac{1}{2}, \frac{i}{2}, \frac{1+i}{2} =: \infty, \lambda, 1, 0$$



Let $r: Y \rightarrow P_C$: pillow case covering

that is, Y : non-singular, conn., proj. curve
 r : non-const morphism w/ $\text{Crit } r \subset \{0, 1, \infty, \lambda\}$

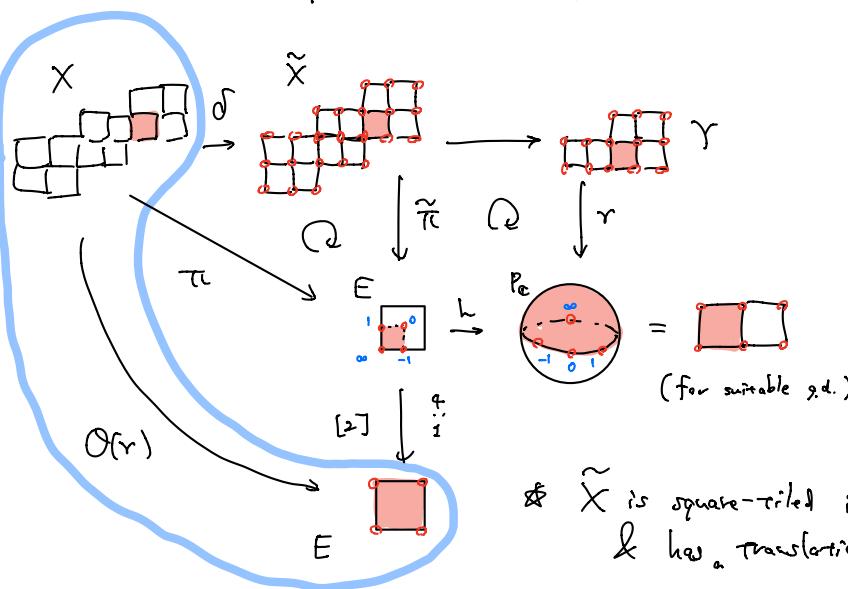


Let $\tilde{\pi}: \tilde{X} = E \times_{P_C} Y \rightarrow E$: pullback

$$\{(e, y) \in E \times Y \mid h(e) = r(y)\}$$

$\delta: X \rightarrow \tilde{X}$: desingularization of \tilde{X}
 $\pi: X \rightarrow E$:

$$\begin{array}{c} E \times Y \supset E \times_{P_C} Y \xrightarrow{\text{proj.}} Y \\ \downarrow \text{proj} \quad \downarrow r \\ E \xrightarrow{h} P_C \end{array}$$



for each non-singular pt $y \in Y$, there are two pts $\pm e \in E$ s.t. $h(\pm e) = r(y)$

So $Y \times_{P_C} E$ is a sort of double of Y .

& \tilde{X} is square-tiled in non-abelian way.
& has a translation str. naturally induced from E .

We consider the map $[2] \circ \pi: X \rightarrow E$.

By the construction, $\tilde{\pi} \circ \pi$ is ramified at most over $\{0, 1, \infty, \lambda\}$.

They are mapped to $\infty \in E$ by $[2]$, thus this defines an origami.

Def 5.1 (a) We call $\mathcal{O}(\gamma) := [2] \circ \pi : X \rightarrow E$ the pillow case origami:

associated to the pillow case covering $\gamma : Y \rightarrow P_C$.

(b) If furthermore $\beta := \gamma$ is unramified over λ i.e. Belyi, we call $\mathcal{O}(\beta)$ the M -origami associated to γ .

$(Y \times_{P_C} E)$

Q In a topological view point: Replacing algebraic-geometric fibre product by topological one,

We obtain some important results : $\left\{ \begin{array}{l} X^+ \cong E^+ \times_{P_C^+} Y^+ \text{ as coverings} \\ \text{description of monodromy of } \mathcal{O}(\beta). \text{ etc.} \end{array} \right.$

* The monodromy of M -origami.

Proposition 2.2. Let X be a coverable space, $f : A \rightarrow X$, $g : B \rightarrow X$ coverings of degree d and d' , respectively, with given monodromy maps m_f resp. m_g . Then, we have for the fibre product $A \times_X B$:

a) For each path-wise connected component $A_i \subseteq A$, the restriction

$$p_A|_{p_A^{-1}(A_i)} : p_A^{-1}(A_i) = A_i \times_X B \rightarrow A_i \\ m_g \circ (f|_{A_i})_*.$$

$\swarrow \deg f \times \deg g$

b) The map $f \circ p_A = g \circ p_B : A \times_X B \rightarrow X$ is a covering of degree dd' with monodromy

$$m_f \circ m_g : \pi_1(X, x_0) \rightarrow S_d \times S_{d'} \subseteq S_{dd'}, \gamma \mapsto ((k, l) \mapsto (m_f(k), m_g(l))),$$

where $(k, l) \in \{1, \dots, d\} \times \{1, \dots, d'\}$.

Proposition 2.3. In the situation described above, let γ_i , $i = 1, \dots, d$, be right coset representatives of Γ_1 in Γ , with $\gamma_1 = 1$, such that $e_{y_i}^f(\gamma_i) = y_1$. So, we have $\Gamma = \bigcup \Gamma_1 \cdot \gamma_i$.

Then, we have:

$$m_{f \circ g}(\gamma)(i, j) = (m_f(\gamma)(i), m_g(c_i(\gamma))(j))$$

Here, we denote $c_i(\gamma) := (f_*)^{-1}(\gamma_k \gamma \gamma_i^{-1})$, $k := m_f(\gamma)(i)$

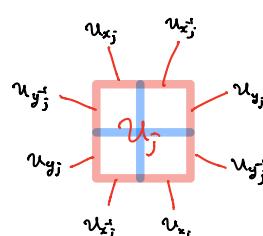
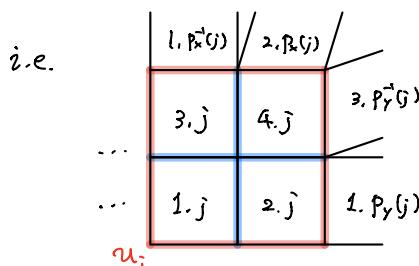
Theorem 5. Let $\beta : Y \rightarrow P_C$: Belyi cover of degree d . ($: \text{pillow case}$)

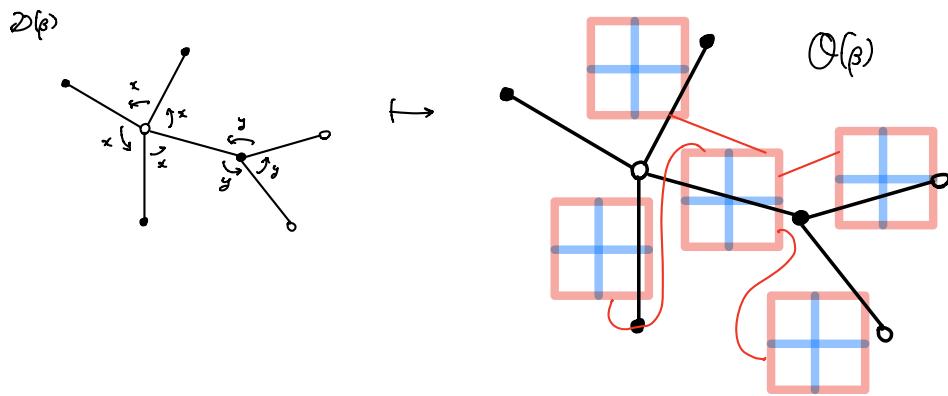
$p_x := m_p(x)$, $p_y := m_p(y)$... monodromy generators.

Then the monodromy of the M -origami $\mathcal{O}(\beta) = [2] \circ \pi : X \rightarrow E$

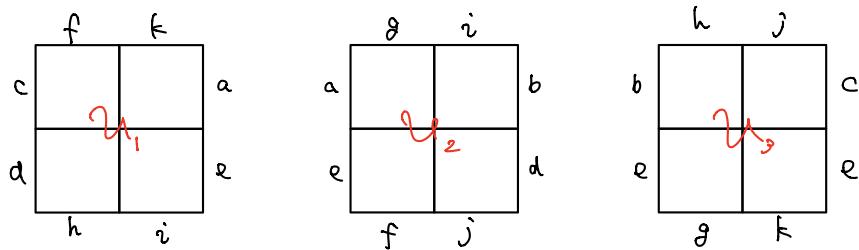
is given by $m_{\mathcal{O}(\beta)} : \pi_1(E^+) \rightarrow S_{4d}$

where $m_{\mathcal{O}(\beta)}(A) = \begin{cases} (1, j) \mapsto (2, j) & m_{\mathcal{O}(\beta)}(B) = \begin{cases} (1, j) \mapsto (3, j) \\ (2, j) \mapsto (4, j) \\ (3, j) \mapsto (1, p_x^{-1}(j)) \\ (4, j) \mapsto (2, p_x^{-1}(j)) \end{cases} \\ (2, j) \mapsto (1, p_y(j)) \\ (3, j) \mapsto (4, j) \\ (4, j) \mapsto (3, p_y^{-1}(j)) \end{cases}$



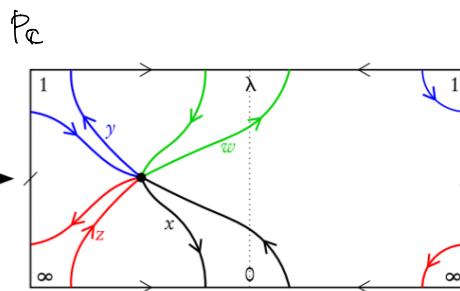
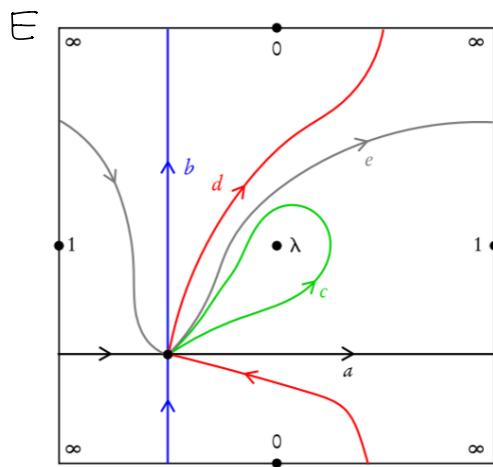
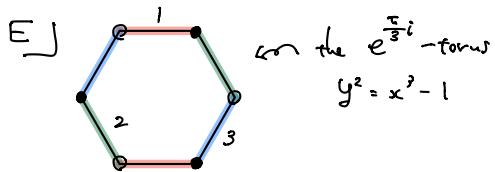


exm the Demiraynque origami



is the M -origami associated to the drawing

$$\langle x, y \rangle = \langle (123), (123) \rangle \subset G_3$$



referring
this figure

@ monodromy around the Weierstrass points.

Lemma 5.5. If we choose the following simple loops around the Weierstrass points

$$x' := l_0 := db^{-1}$$

$$z' := l_\infty := bed^{-1}a^{-1}$$

$$y' := l_1 := ac^{-1}e^{-1}$$

$$w' := l_\lambda := c,$$

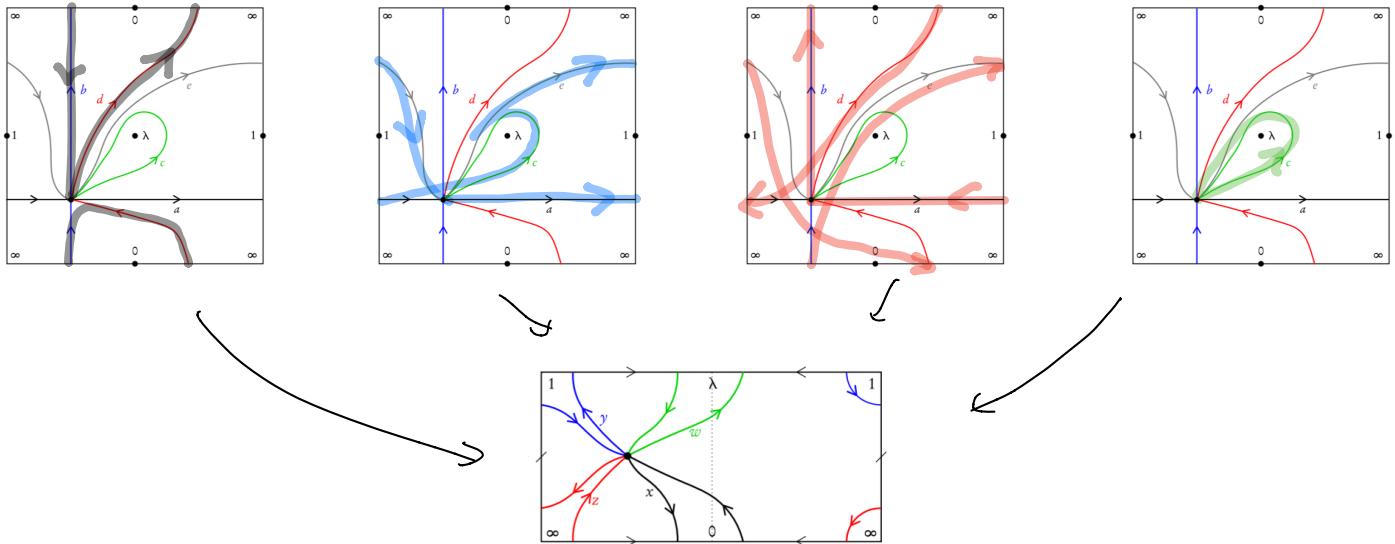
then we have

$$h_*(x') = x^2$$

$$h_*(z') = z^2$$

$$h_*(y') = y^2$$

$$h_*(w') = w^2.$$



④ analytic type of M-orbifolds.

Prop 5.6 Let $\beta: \Sigma \rightarrow P_C$: Belyi mrv. of degree d

For $z = 0, 1, \infty$, let $m_z :=$ the monodromy arr. $z \in G_{\text{fd}}$
 $g_z := \#(\text{cycles of even length in } m_z)$

Then we have (a) $g(x) = g(\gamma) + d - \frac{1}{2} \sum g_z$

$$(b) \quad g(\gamma) + \lceil \frac{d}{4} \rceil \leq g(x) \leq g(\gamma) + d$$

(pf: from Riemann-Hurwitz for β & π .)

Rem 5.9 a) Let $n(p) := \#\pi^{-1}(p)$ for $p \in W := \{0, 1, \lambda, \infty\}$: Weierstrass p

$$\left\{ \begin{array}{l} \text{then } n(p) = \begin{cases} \#(\text{cycles in } m_p) + g_p & (p \neq \lambda) \\ \deg \pi = \deg \beta & \end{cases} \\ \deg \pi = \deg \beta \end{array} \right.$$

$$b) \text{ For an M-orbifold } \Theta(\beta), \quad \#\beta^{-1}(\infty) = \sum_{p \in W} n(p).$$

⑤ Veech group of M-orbifolds

Prop 5.10 β : Belyi morphism. of degree d \sim desir. $\langle p_x, p_y \rangle \subset G_d$

$$\Theta_\beta = \Theta(p_x, p_y) := \Theta(\beta): \text{associated M-orbifold}$$

$$\left\{ \begin{array}{l} \bullet \quad \sigma \cdot \Theta_\beta = \Theta(p_y, p_x) : \text{M-orbifold} \\ \bullet \quad T \cdot \Theta_\beta = \Theta(p_x, p_y) : \quad \approx \quad \text{where } p_x = p_x^{-1} p_y^{-1} \\ \bullet \quad -I \cdot \Theta_\beta = \Theta_\beta. \end{array} \right.$$

It will turn out in the next theorem that the VG of M-origami's often is $\Gamma(2)$.

Theorem 9 Let $\beta: \text{Belyi} \sim \langle p_x, p_y \rangle$: dessin.

(a) $\Gamma(2) \subset \Gamma(\mathcal{O}_\beta)$ in particular, $\Gamma(\mathcal{O}_\beta)$: congruence level ≥ 2

(b) $SL(2, \mathbb{Z}) \cdot \mathcal{O}_\beta = \{ \mathcal{O}(p_x^*, p_y^*): \text{M-origami} \mid \langle p_x, p_y \rangle \xrightarrow{\text{weak}} \sim \langle p_x^*, p_y^* \rangle \}$

where $\langle p_x, p_y \rangle \xrightarrow{\text{weak}} \sim \langle p_x^*, p_y^* \rangle \stackrel{\text{def}}{\iff} \begin{array}{c} X \xrightarrow{\beta_\#} X' \\ \downarrow P \quad \downarrow P' \\ P_C \xrightarrow{\text{act}} P'_C \end{array} \leftarrow \text{i.e. biholomorphically equivalent up to permutations of } \{0, 1, \infty\} \right. \\ \left. \text{permutes } \{0, 1, \infty\} \right\}$

(c) If $\Gamma(2) = \Gamma(\mathcal{O}_\beta)$, then

β has no non-trivial weak isomorphism.

$$\text{as } \left. \begin{array}{c|ccccc} \text{id} & 1 & \infty & 1 & \infty & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & \infty & \infty \\ \infty & \infty & 0 & 0 & 1 & 1 \end{array} \right\} 6 \text{ patterns.}$$

In the case β : filtry (\iff neither p_x^*, p_y^* , nor $p_z^* = 1$)

the converse is true.

(pf) (a) is proved by the fact that $\alpha \in \{I, S, T, ST, TS, TST\}$ maps \mathcal{O}_β in the following way:

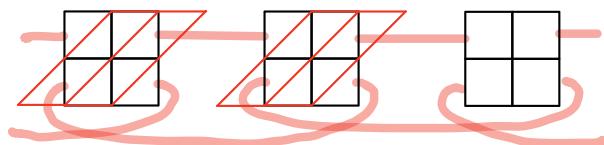
I	S	T	ST	TS	TST
p_x	p_y	p_x	p_y	p_x	p_x
p_y	p_x	p_y	p_x	p_x	p_x

← images are the M-origami ass. to those dessins.
... that is permutation.

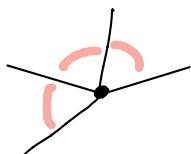
thus we may see $T^2 \mathcal{O}(p_x, p_y) = ST^{2^{-1}} \mathcal{O}(p_x, p_y) = \mathcal{O}(p_x, p_y)$

→ w/ prop 5.19, we see $\Gamma(\mathcal{O}) \supset \langle T^2, ST^{2^{-1}}, -I \rangle = \Gamma(2)$.

(b) might be too strong?
 $SL(2, \mathbb{Z}) \cdot \mathcal{O}_\beta \not\propto \{ \text{weakly-isom} \}$?
if (b) holds: $[SL(2, \mathbb{Z}) : \Gamma(\mathcal{O}_\beta)] \leq 6$: too strong?



each cylinder on $\mathcal{O}_\beta \sim$ cycle in p_x, p_y



$S, T \in SL(2, \mathbb{Z})$ acts on $\{ \text{M-origami} \}$
as a modification of cylinders on dessins.

Prop 5.12 $W = \{ \psi \in \text{Aut } P_{\mathbb{C}} \mid \psi(\{0, 1, \infty\}) = \{0, 1, \infty\} \}$

acts on the set of origami w/ VG-congruence level ≥ 2)

via the group action: $\psi : W \rightarrow \frac{\text{SL}(2, \mathbb{Z})}{\Gamma_0(2)} \quad \begin{cases} (z \mapsto 1-z) \mapsto S \Gamma_0(2) \\ (z \mapsto \frac{1}{z}) \mapsto T \Gamma_0(2) \end{cases}$

Furthermore, under this action,

the M-origami map $\Omega : (\text{dessins}) \xrightarrow{\psi} (\text{M-origami})$ is W -equivariant.
 $\beta \mapsto \Omega(\beta)$

i.e. $\Omega(-\beta) = S \Omega(\beta)$
 $\Omega(\frac{1}{\beta}) = T \Omega(\beta)$.

⑤ Cylinder decomposition

By thm 9 ($\Gamma(\beta) \supset \Gamma_0(2)$),

an M-origami curve $C(O_\beta)$ is covered by $\mathbb{H}/\Gamma_0(2) \cong P_{\mathbb{C}} \setminus \{0, 1, \infty\}$: three cusps

In particular, it has at most three cusps

at most three non $\Gamma_0(2)$ -equivalent cylinder directions.

... namely $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$

By calculating the cylinder decomposition in these directions,

We may observe the situation of cylinders lying on M-origamis.

from information of dessins!

Theorem 10. Let O_β be an M-Origami associated to a dessin β which is given by a pair of permutations (p_x, p_y) . Then we have:

- a) If (p_x, p_y) does not define one of the dessins listed in Lemma 5.13, then, in the Strebel direction $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$, O_β has:
 - for each fixed point of p_y one maximal cylinder of type $(2, 2)$,
 - for each cycle of length 2 of p_y one maximal cylinder of type $(4, 2)$,
 - for each cycle of length $l > 2$ of p_y two maximal cylinders of type $(2l, 1)$.
- b) In particular, we have in this case:

$$\# \text{max. horizontal cylinders} = 2 \cdot \# \text{cycles in } p_y - \# \text{fixed points of } p_y^2.$$

- c) We get the maximal cylinders of O_β in the Strebel directions $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ and $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ if we replace in a) the pair (p_x, p_y) by (p_y, p_x) and (p_y, p_z) , respectively.

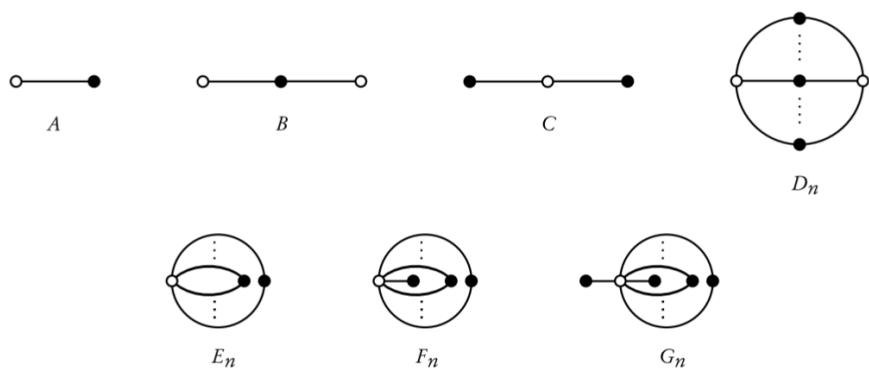


FIGURE 3.

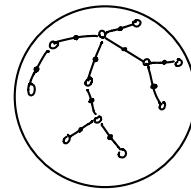
- Remark 5.14.**
- a) The M-Origami coming from D_n has two maximal horizontal cylinders of type $(2n, 2)$ for $n \geq 3$ (and else one of type $(2n, 4)$).
 - b) The M-Origami coming from E_n has one maximal horizontal cylinder of type $(4n, 2)$.
 - c) The M-Origami coming from F_n has one maximal horizontal cylinder of type $(4n - 2, 2)$.
 - d) The M-Origami coming from G_n has one maximal horizontal cylinder of type $(4n - 4, 2)$.

② Main part : Möller's theorem & variations.

We may reprove Möller's work in 2005 in an almost purely topological way.

Thm 1 (reform ver. of Möller's)

(a) Let $\alpha \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, β : Belyi map ass. to clean tree dessin.



i.e. a dessin of genus 0, totally ramified over ∞ ,

s.t. $p^{-1}(1)$ consists of order 2 ramif. pts.

and assume $\beta^\alpha \neq \beta$.

Then, we also have for the origin curve that $C(\alpha_\beta) \neq C(\beta_\beta)^\alpha$. (in $M_{g,n}$)

(b) In particular, the Galois action on the set of origin curves is faithful.

Lemma 5.15 Let $\beta \sim (p_x, p_y)$, $\beta' \sim (p'_x, p'_y)$ be two dessins.

If $\beta \not\cong \beta'$, then we also have $\pi \not\cong \pi'$. (in Def of M-origami)
(outline)

Assuming $\deg \beta = \deg \beta'$, $\pi \cong \pi' \Rightarrow \beta \cong \beta'$ will be shown in the same strategy as Thm 8
(dessin \leftrightarrow monodromy of M-origami)
post-composing [2];

Lemma 5.16 $\pi: X \rightarrow E$, $\pi': X' \rightarrow E$: two coverings

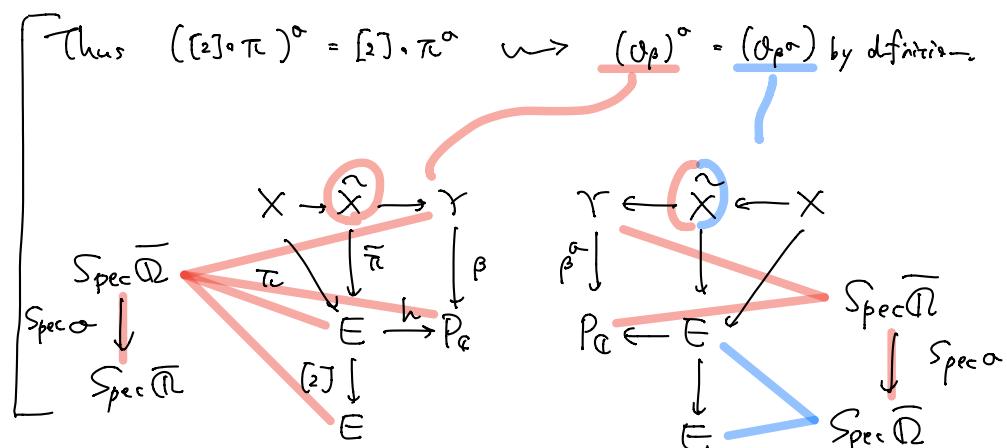
$$\text{Then } [2] \circ \pi \cong [2] \circ \pi' \Rightarrow {}^2\varphi \in \text{Deck}[2] \quad \varphi \circ \pi \cong \pi'$$

Prop 5.17 Assume we have a dessin $\beta \sim (p_x, p_y)$, and $\alpha \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ s.t. $\beta^\alpha \neq \beta$.

If furthermore the 4-tuple $(p_x^2, p_y^2, p_e^2, 1) \in G_d^2$ contains a permutation whose cycle str. is distinct from others

then $\alpha_\beta \neq \alpha_{\beta^\alpha}$.

pf) We have $E = \mathbb{G}_{\mathbb{Z}^2}$ & $[2]: E \rightarrow E$: defined over \mathbb{Q} , so $E^\alpha \cong E$ & $[2]^\alpha \cong [2]$.



Assume $\mathcal{O}_\beta \cong \mathcal{O}_{\beta^\alpha}$.

then ${}^3\varphi \in \text{Deck}[\mathbb{Z}]$ $\varphi_* \pi \cong \pi'$.

Here $\text{Deck}[\mathbb{Z}] \curvearrowright E$ is a translation action, so it has no fixed pts.

Recall that Lem 5.15 says that

$(p_x^2, p_y^2, p_z^2, 1)$ describes the ramification of π around the 4 Weierstrass points,

and it is preserved by σ .

(Local ramification behavior is Galois invariant.)

So by the assumption (4) on this tuple, it follows that φ should act trivially.

$$\Rightarrow \pi \cong \pi' \xrightarrow{\text{Lem 5.15}} \beta \cong \beta^\alpha \quad \square,$$

Sheaf \mathcal{F} of top. sp. X

