ORIGAMI-SCHOTTKY GROUPS

RUBÉN A. HIDALGO

ABSTRACT. A Kleinian group K, with region of discontinuity Ω, is an origami-Schottky group if (i) it contains a Schottky group Γ as a finite index subgroup and (ii) Ω/K is an orbifold of genus one with exactly one conical point. In this paper, we provide a geometrical structural picture of origami-Schottky groups in terms of the Klein-Maskit combination theorems. Examples of Hurwitz translation surfaces in terms of Schottky groups are provided.

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§1. intro.

'square-tiling'

origami : $(X, \mathcal{G})$: surf. X w/ embedded, finite graph $\mathcal{G}$
- $X \setminus \mathcal{G}$ consists of squares
- edges : bicoloured (horiz./vert.)
- each face is bdd by h^+, v^+, h^-, v^-,

$\Updownarrow$ eq.

$\eta : X \to S^1 \times S^1$: top. cov. of a torus. : branched over 1 pt. $p \in S^1 \times S^1$

put a str.

$(\eta : S) \to E$: cov. of cl. R.S.

origami pair

very interesting case

combinatorial object which is centered in study of origamis.

origami is regular if η : regular br. cov. ($\Leftrightarrow$ Gal $\eta \cong G = \pi_1(S)$: monodromy grp) $\triangleleft\ \pi_1(E) \cong F_2$.

when $g \geq 2$.

Riemann-Hurwitz formula asserts $|G| \leq 4(g-1)$,

when "=" holds, such an origami is called a Hurwitz translation surface.

Q in terms of Fuchsian groups.

for (S, η) : genus $g \geq 2$.

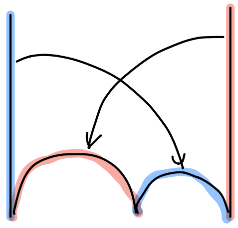
$E^* = E \setminus \{p\}$ is uniformed by $\mathbb{H}$ w/ Fuchsian grp

$F[r, \alpha] = \left\langle A(z) = \frac{r(z+1)}{z+\alpha},\ B(z) = \frac{r-z}{z+1-r-\alpha} \right\rangle$ $\cong \pi_1(E)$ $\quad A = \begin{pmatrix} r & r \\ 1 & \alpha \end{pmatrix}$ $\quad B = \begin{pmatrix} -1 & r \\ 1 & 1-r-\alpha \end{pmatrix}$

$(r > 0,\ \alpha > 1)$

$(r > 1 ?)$

which type?



← just of this form…

(so A.B should be of hyp type?)

(?)

at least. Fuchsian grp of E belongs to some sort of 2-parametered family : $\mathcal{F}[r,\alpha]$. (denote)

$E \cong \mathbb{H}/\mathcal{F}[r,\alpha] \rightsquigarrow S \cong \mathbb{H}/H$ for some $H < \mathcal{F}[r,\alpha]$: subgrp of index deg η.

η : regular $\Leftrightarrow$ $H \triangleleft \mathcal{F}[r,\alpha]$

The lowest regular, planar coverings of closed R.S. of genus g are provided by Schottky grps of rank g. (Maskit, 1988)

Thm $\eta : \hat{S} \to S$: reg. cov (planar)
$\mathrm{Gal}\,\eta$ is a lowest planar regular cover of S
$\Leftrightarrow$ $\mathrm{Gal}\,\eta$ is a Schottky grp.

→ Q. description of origami pairs in terms of Schottky grps?

Recall a Kleinian group is a discrete subgrp $K < PSL(2,\mathbb{C})$
has the Möbius-trans. action on $\hat{\mathbb{C}}$, which is naturally extended to a one on $\mathbb{H}^3$.
$K_o = \{\gamma(o) \mid \gamma \in K\}$ ramificates in $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$. the set $\Lambda = \{\text{ramif. pt of } K_o\} \subset \hat{\mathbb{C}}$: Limit set of K
$\Omega := \hat{\mathbb{C}} \setminus \Lambda$: region of discontinuity of K

- $\Lambda \subset \hat{\mathbb{C}}$: closed set
- $\Lambda = \{$fixed pts of $\gamma \in K$: loxodromic / parabolic (order ∞)$\}$.
- Λ, Ω is K-invariant
- K acts prop. discont.ly on Ω
- Λ is minimum K-inv. & Ω is maximum w/ discont. action in $\hat{\mathbb{C}}$.

- a Schottky grp is a Kleinian grp $\Gamma = \langle \gamma_1 \cdots \gamma_g \rangle$ where $g \geq 2$, γ_j is a loxidromic Möbius trans. which maps $\exists C_{2j-1}$ & its interior $\to$ $\exists C_{2j}$ & its exterior where $C_1 \sim C_{2g}$: disjoint circle in $\hat{\mathbb{C}}$. (or Jordan curve)
- a Schottky grp is purely loxodromic, a free grp of rank g
- $\Lambda(\Gamma)$ is a Cantor set
- finitely generated, purely loxodromic, free Kleinian grp is a Schottky grp. (← Maskit 1967.)

Def an origami pair (S,η) is of Schottky type if $\exists$ (K : Kleinian grp $> \Gamma$: Schottky grp : finite index) such that $S \cong \Omega/\Gamma$, $E \cong \Omega/K$, η is given by $K > \Gamma$. where Ω is the region of discontinuity of K. (also of Γ.)

K is an origami-Schottky grp of type n if $\exists (S,\eta)$: origami of degree n s.t. Schottky type where K is as in above def.

Hidalgo (1994) & Reni, Zimmerman (1995) $\xrightarrow{\text{follows}}$ origami-Schottky groups of any type exist.

Rem "origami-Schottky" does NOT imply being Schottky.

• Main result.

Theorem 1. *Let $n \geq 2$ be an integer.*

$PSL(2, \mathbb{C})$

(1) *If K is an origami-Schottky group of type n, then (up to conjugation by a suitable Möbius transformation) one of the following holds.*

(a) *The group K is constructed from the Klein-Maskit combination theorems as an HNN-extension of the dihedral group*

$$D_n = \langle A(z) = e^{2\pi/n}z, B(z) = 1/z \rangle$$

by a cyclic group $\langle T \rangle$, where T is a loxodromic transformation conjugating B into AB.

(b) *$n = 2$ and K can be constructed from the Klein-Maskit combination theorems as an HNN-extension of the alternating group*

$$\mathcal{A}_4 = \langle A(z) = i(1-z)/(z+1), B(z) = -z \rangle$$

by a cyclic group $\langle T \rangle$, where T is a loxodromic transformation commuting with A.

(2) *Every Kleinian group constructed as in (1) above is an origami-Schottky group of type n.*

Remark 2. If K is as in (a) of part (1) in Theorem 1, then (since $A = [T, B] = TBT^{-1}B$) we have the presentation $K = \langle B, T : B^2 = [T, B]^n = ([T, B]B)^2 = 1 \rangle$.

def $G = \langle \overset{\text{gen system}}{S} \mid \overset{\text{relation}}{R} \rangle$, $\alpha : H \to K$: isomorphism b/w $H, K < G$.

(G. Higmann, B. Neumann, H. Neumann, 1949) $G *_\alpha := \langle S, t \mid R, tht^{-1} = \alpha(h) \ \forall h \in H \rangle$ ($t \notin S$: new symbol refering in G)

is called the HNN-extension of G relative to α.

⇝ should be viewed in parallel w/ amalgamated products.

Thm (Klein, 1883) G_1, G_2 : Kleinian grps w/ certain conditions

$\Rightarrow$ Kleinian grp. $\langle G_1, G_2 \rangle \cong G_1 * G_2$

Thm (Maskit, 1965) G_1, G_2 : Kleinian grps H : common subgrp of G_1, G_2 $\psi_i : H \hookrightarrow G_i$: embed.

$\boxed{\text{conditions for fund. sets}}$ $\Rightarrow$ $\langle G_1, G_2 \rangle \overset{\text{claim}}{=} G_1 * G_2 / H$: Kleinian grp

free product w/ the relation $\psi_1(h)\psi_2(h^{-1}) = 1_{G_1 * G_2}$ $\forall h \in H$

amalgamated free product version of Klein's thm

Maskit gave HNN-extension version (1994), too.

Q HNN-extension of D_n (A_4) by $\langle T \rangle$?

$$= \langle D_n, T \rangle \underset{\text{Maskit}}{=} D_n *_{\underset{\text{conjugation isomorphism}}{T}}$$

(Maskit, 1994)

Theorem II (the second combination theorem). *Let J_1 and J_2 be geometrically finite subgroups of the Kleinian group, G_0. Assume the following.*

(A) *For $m = 1, 2$, there is a J_m-invariant closed topological disc B_m, with boundary loop W_m; there is a set of rimpoints Θ_m given on W_m; and there is a Möbius transformation, f, mapping the exterior of B_1 onto the interior of B_2, so that (B_1, B_2) is jointly f-simple (i.e., (B_m, Θ_m) is a (J_m, G_0)-simple disc; if there is an $x \in B_1 \cap g(B_2)$, for some $g \in G_0$, then either $x \in \Lambda(J_1) \cap g(\Lambda(J_2))$, or $x \in \Theta_1 \cap g(\Theta_2)$; f conjugates J_1 onto J_2; and $f(\Theta_1) = \Theta_2$.*

(B) *Every cyclic stabilizer is parabolic.*

(C) *$A = \widehat{\mathbb{C}} - (B_1 \cup B_2) \neq \varnothing$.*

Let D_0 be a coordinated fundamental set for G_0; let D be the corresponding adjusted set; let A_0 be the complement of the union of the G_0-translates of $(B_1 \cup B_2)$; and let $G = \langle G_0, f\rangle$. Then

(i) *$G = G_0 *_f$ (i.e., G is the HNN-extension of G_0 by the element f conjugating the subgroup J_1 onto the subgroup J_2).*

(ii) *G is discrete.*

(iii) *Every element of G that is not a conjugate of an element of G_0, and is not a conjugate of a cyclic stabilizer, is loxodromic.*

(iv) *W_1 is precisely embedded, and (W_1, Θ_1) is a (J_1, G)-swirl; it is strong if and only if B_1 and B_2 are both strong simple discs.*

(viii) *D is a fundamental set for G.*

(ix) *A_0 is precisely invariant under G_0. Let $A_0^* = \overline{A}_0 \cap \Omega(G_0)$; then $\Omega/G = A_0^*/G_0$, where the two possibly disconnected and possibly empty boundaries, $(W_1 \cap \Omega(G))/J_1$ and $(W_2 \cap \Omega(G))/J_2$ are identified by f.*

(x) *G is geometrically finite if and only if G_0 is geometrically finite.*

(xi) *Assume that G_0 is geometrically finite, and that $W_1 \cap \Omega(J_1)$ is smooth. Then there is a spanning disc Q_m for W_m, where (Q_1, Q_2) is precisely invariant under (J_1, J_2), and $f(Q_1) = Q_2$. Further, $\mathbb{H}^3/G$ can be described as follows. Let A_0^3 be the region in $\mathbb{H}^3$, bounded by the translates of $Q_1 \cup Q_2$, whose Euclidean boundary is A_0. Then $\mathbb{H}^3/G$ is A_0^3/G_0, where the two boundaries, Q_1/J_1 and Q_2/J_2, are identified by f.*

(xii) *G is analytically finite if and only if G_0 is analytically finite.*

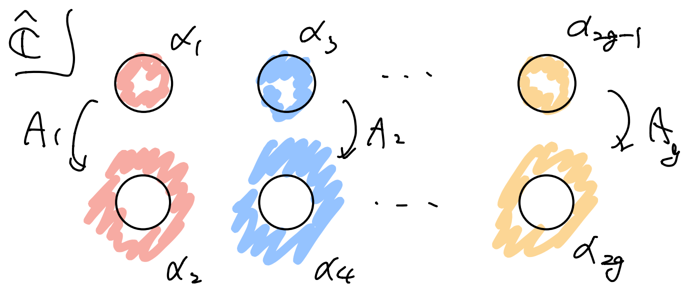
(xiii) *If G is analytically finite, then*

$$\operatorname{area}(G) = \operatorname{area}(G_0) - \operatorname{area}(J_1) = \operatorname{area}(G_0) - \operatorname{area}(J_2).$$

$G_0 = D_n(A_4)$, $f = T^*$: $\langle A\rangle \mapsto \langle AB\rangle$

§2. Preliminaries

- Kleinian grp : discrete subgrp of $PSL(2,\mathbb{C})$: Möbius trans. grp. : denote K
- region of discontinuity : maximal set in $\hat{\mathbb{C}}$ on which given grp acts properly discontinuously : denote $\Omega = \Omega(K)$
  - if $\Gamma < K$: finite index inclusion b/w Kleinian grps
    $\Rightarrow$ both has the same region of discontinuity.
- A function grp is a pair (K, Δ) where K is a finitely generated Kleinian grp (finite type (finite area)) and $\Delta \subset \Omega$ is a K-inv component
- a Schottky group of rank $g \geq 1$ is a Kleinian grp : denote Γ
  generated by loxodromic trans. $A_1 \sim A_g$ where



☆ $\Gamma \cong F_g$: free grp

It is well known that

(i) $\Omega(\Gamma)$ is connected $\overset{\text{imp.}}{\rightsquigarrow}$ (Γ, Ω) is a fct. grp.

(ii) $S = \Omega/\Gamma$ (the surface uniformized by Γ) is a closed R.S. of genus g.

<u>Thm</u> (retrosection thm, Köebe 1910?, Bers 1975)

every closed Riemann surface of genus g is uniformized by a Schottky grp of rank g.

- Maskit's decomposition thm states that

every function group can be constructed from
- elementary groups (Kleinian grps w $\#\Lambda < \infty$) acting on finitely punctured sphere
- quasi-Fuchsian grps (function grps w/ Λ = s.c.c.)
- degenerate grps (function grps w/ Ω is conn. & simply conn.)

→ by a finite numbers of applications of Klein-Maskit combination theorem

amalgams are produced along cyclic groups in construction

◎ as a consequence of decomposition results :

(Maskit) Schottky grps are purely loxodromic, geom.-finite fct. groups w/ totally disconn. limit set.

in particular → K in Thm 1 are exms of geom.-finite fct. groups w/ totally disconn. limit set

so, → $\forall \Gamma < K$: finite index, torsion free is a Schottky grp.

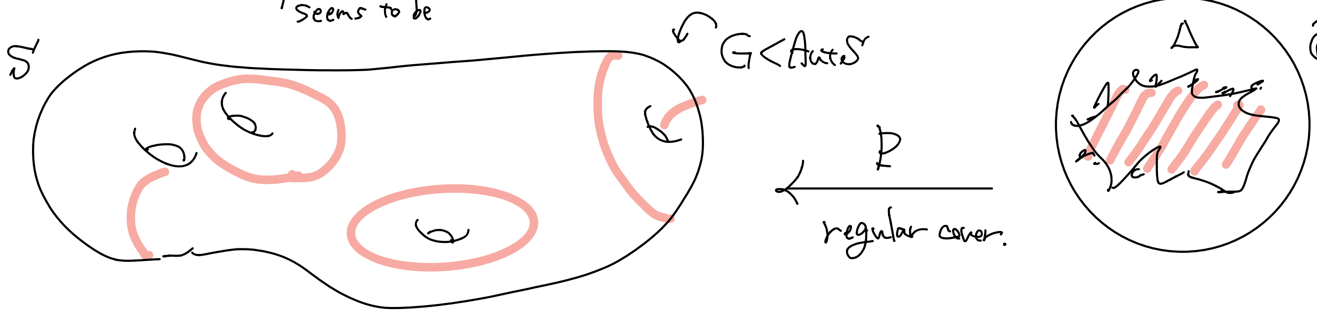
◎ MAIN TOOL. (Hidalgo, 2005 ; conseq. of the Equivariant Loop Theorem (Meeks, Yau, 1980))

Theorem 5. [6] *Let (Γ, Δ) be a torsion free function group uniformizing a closed Riemann surface S of genus $g \geq 2$, that is, there is a regular covering $P : \Delta \to S$ with Γ as covering group. If G is a group of automorphism of S, then it lifts with respect to P if and only if there is a collection $\mathcal{F}$ of pairwise disjoint simple loops on S such that:*

(i) ? *$\mathcal{F}$ defines the regular planar covering $P : \Delta \to S$; and*
(ii) *$\mathcal{F}$ is invariant under the action of G.*

setwise ? pathwise? pointwise?

↑ seems to be



Rem 6. (1) In cases of Schottky grps, $S \setminus \mathcal{F}$ consists of planar surfaces

(2) by uniformization thm S has a natural hyp. str. from $\mathbb{H}$.

$\mathcal{F}$ can be assumed to be formed by s.c. geods.

Rem 4 There is a p-adic version of origamis

$\eta : S \to E$: cov. of "Mumford curve" $/\mathbb{C}_p$

(S) ≀ p-adic Schottky grp. (E) ≀ p-adic Kleinian grp.

(Kramer, 2016) p-adic, regular origami ver. of thm 1 is proved.

(this paper) complex, not necessarily regular origami ver. of Kramer's work using Thm 5.

§3. Pf of Thm 1.

3.1 part (1) K: origami-Schottky grp: Type $n \geq 2$.

by def, $E = \Omega/K$: orbifold of genus 1 w/ 1 cone pt of ord n.

$\exists\, \Gamma < K$ (finite index): Schottky grp

∘ $\Omega = \Omega(K) = \Omega(\Gamma)$.

first we may assume $\Gamma \triangleleft K$ (?)

$\Rightarrow G = \mathrm{Gal}\,\eta = K/\Gamma$.

Consider reg. covs.

$$\begin{array}{ccc} K \circlearrowright \Omega \circlearrowleft \Gamma & & \\ (\mathrm{br})\ \pi \downarrow & \searrow P\ (\text{no br}) & \\ \Omega/K & \Omega\ \Omega/\Gamma & \\ \| & & \| \\ E & \xleftarrow[Q\ (\mathrm{br})]{} & S \circlearrowleft G \end{array}$$

G (NOT Abelian grp)

$\nexists\ \pi = Q \circ P$.

by properties of Deck trans grps, we can take $\Phi : K \to G$: hom s.t. ...

$\forall \omega \in \Omega\ \forall k \in K$.

$$\begin{array}{ccc} \pi(\omega) & = & \pi(k\omega) \\ \| & & \| \\ Q(P\omega) & = & Q(Pk\omega) \end{array}$$

$\therefore\ \exists g_k \in G$ s.t. $gP\omega = Pk\omega$

$\Phi : K \to G$, $k \mapsto g_k$ defines a hom w/ $P \circ k = \Phi(k) \circ P$.

∘ As Γ has no parab. element & K is a finite extension of Γ, neither K does.

($\forall k \in K$ is elliptic or parabolic)

∘ As Γ: geom. finite & of finite index in K

$\Rightarrow$ So K is.

→ consequence of (Hidalgo, 1994).

Lemma 7. *Let $k \in K$ be an elliptic transformation different from the identity. Then either (i) both of its fixed points belong to Ω or (ii) there is a loxodromic transformation in K commuting with it.*

As

$$\begin{array}{ccc} \Omega \circlearrowleft & & K \\ P \downarrow & & \uparrow \text{lift} \\ \Omega/\Gamma = S & \circlearrowleft & G < \mathrm{Aut}\, S \end{array}$$

($G < \mathrm{Aut}\, S$ lifts via P into K.)

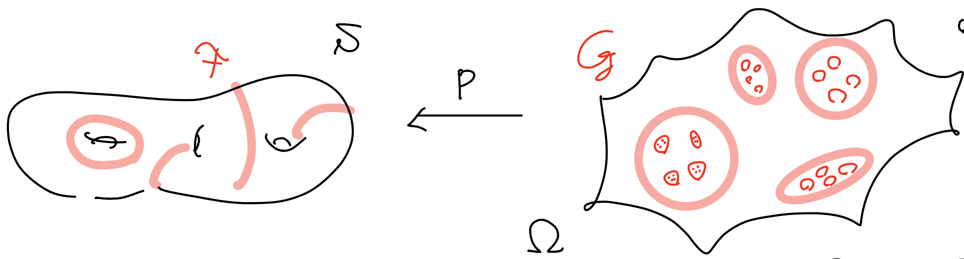
thm 5 asserts $\exists \mathcal{F}$: G-invariant collection of s.c.g. in S

such that (i) $\mathcal{F}$ devides S into planar surfs.

(ii) $\forall \gamma \in \mathcal{F}$ lifts via P to s.c.g. on Ω.

We may take $\mathcal{F}$ to be minimal in the sense of

(a) G-invariance & (b) planarity of complemental components



exm of $\mathcal{G}$ may be produced by Γ- orbits of the loop family of definition of Schottky grp.

@ why $\mathcal{G}$: collection of loops ?

(can be arcs...?)

character of Schottky gp ?

(たぶん)

We lift S via P

to $\mathcal{G}$: structure loops.

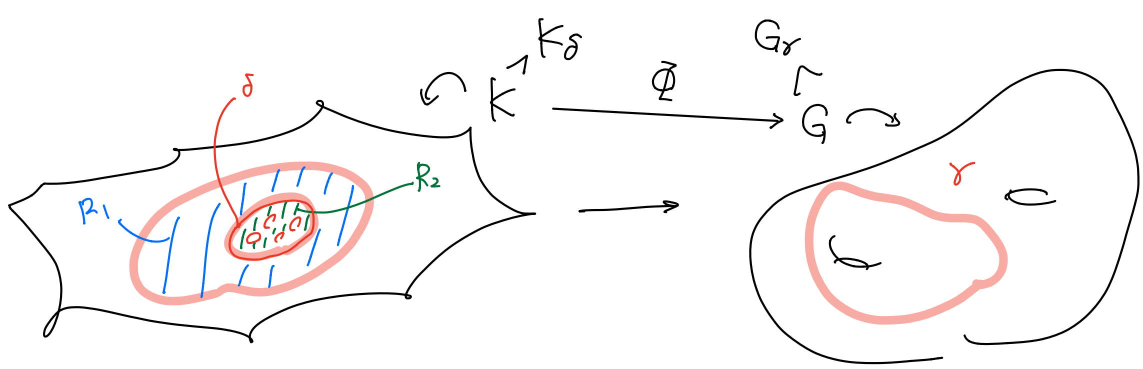
components of $\Omega \setminus \mathcal{G}$: structure regions.

We may observe that :

$$\begin{cases} (1) \ \forall \delta \in \mathcal{G}, \ \gamma = P(\delta) \in \mathcal{F} \ \& \ P|_\delta : \delta \to \gamma \text{ is homeo.} \\ (2) \ \forall F : \text{str. region.} \quad P|_F : F \to P(F) \text{ is homeo.} \end{cases}$$

Lemma 8. *If $\delta \in \mathcal{G}$, then its K-stabilizer is either trivial or a cyclic group generated by an elliptic transformation keeping invariant each of the two structure regions sharing δ in the boundary.*

pf)



finite

$\Rightarrow K_\delta$: finite grp of Möb. trans

we can see that : $\Phi|_{K_\delta} : K_\delta \to G_\gamma$

: isomorphism.

$\Rightarrow$ possibility of K_δ :

$\{id\}$, $\langle T \rangle$, D_n, A_4, A_5, G_4

triv. cyclic dihedral alternating sym

by existance of invariant s.c.c., possibility :

trivial, cyclic, or dihedral.

Let R_1, R_2 : str. region in Ω sharing δ at bdry . $k \in K_\delta \setminus \{id\}$

$\Rightarrow$ possibility for k : (i) alterating $R_1 \leftrightarrow R_2$: ellipeic element of order 2

(ii) leaving R_1, R_2 invariant : some elliptic element .

In case (i) ; $K_\delta = \langle k \rangle = \{id, k\}$ and $\{\text{fixedpts of } k\} \subset \delta$

or $D = \langle k, l \rangle$ ...

$\rightarrow$ any way, existance of $k \in K_\delta$ alternating $R_1 \leftrightarrow R_2$

E must have two cone pt : contradiction!

So, K_δ = trivial or $\langle k \rangle$ where k : ell. element keeping R_1, R_2 invariant. $\otimes$

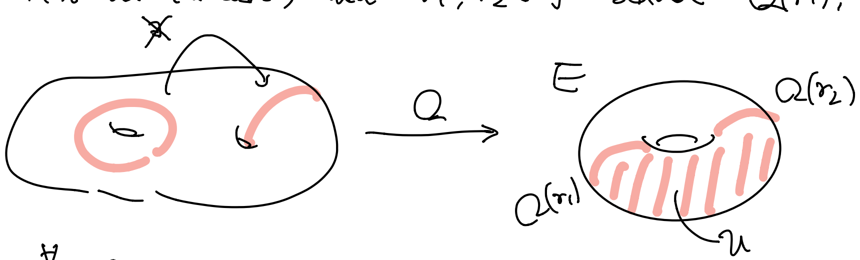
<u>Lemma 9.</u> (1) $\forall$ loops in $\mathcal{F}$ are G-equivalent ($\Leftrightarrow$ $\forall$ loops in $\mathcal{G}$ are K-equivalent) ..

(2) $\forall$ structure regions are G-equivalent

mapped transitively by K

pf) show by contradiction :

if it is not the case, then $\exists \gamma_1, \gamma_2 \in \mathcal{F}$ sothat $Q(\gamma_1), Q(\gamma_2)$ bounds a cylinder.



$\because$ $\forall g \in G$ $\gamma_1 \neq g(\gamma_2)$

since $g(\gamma_2) \in \mathcal{F}$, $\underline{\gamma_1 \cap g(\gamma_2) = \phi}$

$\downarrow Q$

$Q(\gamma_1)$, $Q(\gamma_2)$ are disjoint loops in E $\Rightarrow$ bounds a cyl. $\exists u \subset E$!

as $Q(\gamma_1)$ & $Q(\gamma_2)$ are homot. in $E \setminus \{p\}$,

G_{γ_1} & G_{γ_2} : both triv

or

both isomorphic to the same cyclic grp of order m.

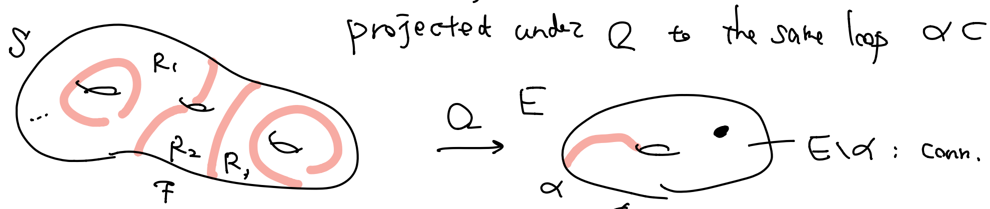
$\Rightarrow$ any component of $Q^{-1}(u)$ must be a cylinder (why?)

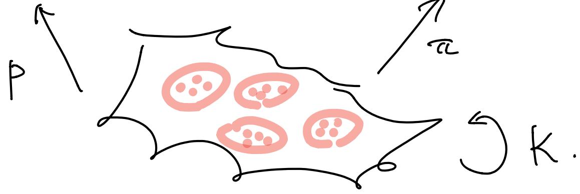
this contradicts to the fact that no elements in $\mathcal{F}$ are homotopic $\Rightarrow$ (1) holds.

(they are s.c.g.)

(2) (1) asserts that $\forall \gamma \in \mathcal{F}$ are

projected under Q to the same loop $\alpha \subset E$. (which no conept lies on)





$\Rightarrow$ every str. region is mapped under π to the same region $E \setminus \alpha$

$\Rightarrow$ K-equivalent $\otimes$

- fix str. region $R \subset \Omega$.

  let ∂R consists of str. loops $\delta_1, \delta_2, \cdots, \delta_r \in \mathcal{G} \xrightarrow{\pi} \alpha \subset E$

  K_R : stabilizer of R, K_{δ_j} : stabilizer of δ_j. ($=$ trivial or cyclic by Lemma 8)

  by Lemma 9, $^{\forall}K_{\delta_j}$ are conjugated by K. ($\uparrow$ is a common property)

Lem 10 K-stabiliser of any str. loop is a nontrivial cyclic grp. w/ the same isom class.

pf) Assume $^{\forall}K_{\delta_j}$ are trivial in above setting.

$\Rightarrow$ $^{\forall}G$-stabiliser G_γ of any $\gamma \in \mathcal{F}$ is trivial.

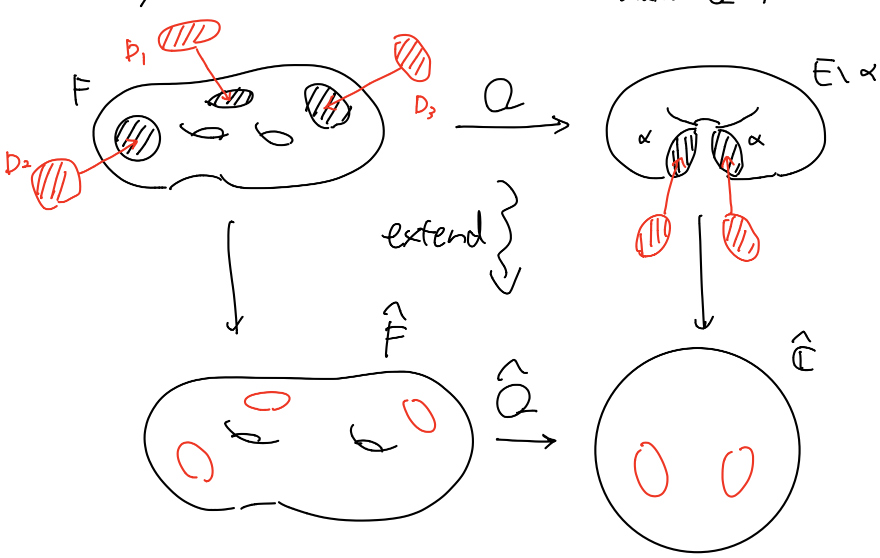
Let F be a component of $Q^{-1}(E \setminus \alpha)$

then $F = {}^{\exists}\hat{F} \setminus \sqcup D_\lambda$ ← disks in $\hat{F}$

$\hat{F}$: closed surf.

We extend $Q|_F : F \to E \setminus \alpha$: cov. continuously

by adding disks at bdries $\rightsquigarrow$ we obtain $\hat{Q} : \hat{F} \to \hat{\mathbb{C}}$



however, it is not possible by Euler characteristics. $\otimes$

consequence of Lem 10 → $K_{\delta_j} = \langle k_j \rangle$ ($k_j \in K \setminus \{id\}$).

as $K_R \ni {}^{\exists}k_j$. ($\partial R \supset \delta_j$), K_R : non-trivial.

Lemma 11 K_R cannot be a cyclic grp.

Assume $K_R = \langle v \rangle$. ($v \in K$: elliptic)

As R/K_R has one cone pt, of order n, it follows that

- ord $v = n$.
- only one fixed pt of v belongs to R. (the other is far.)

$\Rightarrow$ there is str. loop $\delta_{j_R} \in \mathcal{G}$ on ∂R w/ $K_R = \langle k_{j_R} \rangle$ ($= \langle v \rangle$)

$\Rightarrow$ any other str. loop on ∂R should be stabilized by k_{j_R}.

conj. in K to δ_{j_R}.

$\Rightarrow$ $\langle k_j \rangle \triangleleft K$ or ∂R consists of exactly one loop

? impossible

$\Rightarrow$ Contradiction $\otimes$

Lemma 12 Possibility for K_R : D_n or A_4.

By Lem 11. neither K_R : triv nor cyclic.

In the pf of Lemma 8 we observed that

only possibilities we need to rule out are : A_5 & G_4.

Let $K_R = A_5 = \langle a, b \mid a^5 = b^3 = (ab)^2 = 1 \rangle$

as R/K_R has 1 cone pt, $\partial R = \delta_1 \sqcup \delta_2$ ($\exists \delta_1, \delta_2 \in \mathcal{G}$)

each one has invariant order

either 2, 3, or 5, but of different order $\Rightarrow$ contradiction to Lem 10

The argument for G_4 is similar $\otimes$

" $\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^2 = (s_2s_3)^3 = (s_3s_1)^3 = 1 \rangle$

NOT doubly-generated group

Q For $K_R = D_n$ or A_4.

(1) $K_R = D_n = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle$

$\Rightarrow$ We may assume $\partial R = \alpha \sqcup \beta$ where $k_\alpha \sim b$, $k_\beta \sim ab$, $K_\alpha \overset{\text{conj}}{\sim} \langle b \rangle$, $K_\beta \overset{\text{conj}}{\sim} \langle ab \rangle$

$\# \mathcal{G} = r = 2n$,

and we may assume $a \sim A$ $b \sim B$ in thm 1 upto conj.

(2) $K_R = A_4 = \langle a, b \mid a^3 = b^2 = (ab)^3 = 1 \rangle$

$\Rightarrow$ We may assume $\partial R = \alpha \sqcup \beta$ where $\begin{array}{c} k_\alpha \sim a \\ K_\alpha \overset{\text{conj}}{\sim} \langle a \rangle \end{array}$ $\begin{array}{c} k_\beta \sim ab \\ K_\beta \overset{\text{conj}}{\sim} \langle ab \rangle \end{array}$

$\# \mathcal{G} = r = 8$,

and we may assume $a \sim A$ $b \sim B$ in Thm 1 up to conj.

Q Which kinds of Kleinian group can uniformize a torus & dominate a Schottky-origami covering?