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INFINITE TRANSLATION SURFACES WITH INFINITELY GENERATED VEECH GROUPS

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(Communicated by Anton Zorich)

ABSTRACT. We study infinite translation surfaces which are $\mathbb{Z}$-covers of finite square-tiled surfaces obtained by a certain *two-slit cut and paste construction*. We show that if the finite translation surface has a one-cylinder decomposition in some direction, then the Veech group of the infinite translation surface is either a lattice or an infinitely generated group of the first kind. The square-tiled surfaces of genus two with one zero provide examples for finite translation surfaces that fulfill the prerequisites of the theorem.

1. Intro

After [Veech, 1989], a lot effort has been put into understanding Veech groups (VGs) of trans. surfs. w/ $\left\{\begin{array}{l}\text{finite area}\\ \text{finite genus}\end{array}\right\}$ → infinitely-generated grps were discovered

— Hubbert, Schmitz 2004
— McMullen 2003.

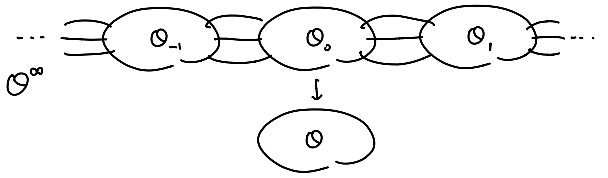
※ VG : Stabilizer of $\mathrm{MCG} \curvearrowright \mathbb{D} \xhookrightarrow[\text{isom.}]{} \mathcal{T}_{g,n}$ $\left\{\begin{array}{l}\text{Möbius on } \mathbb{D}\\ \text{Affine on the reference surface}\end{array}\right.$

Recently trans. surfs. of infinite genus have attracted more attention.

We work w/ origamis : coverings of "□" – unit square torus

- VG of such a surface : $SL_2(\mathbb{Z})$ – subgrp

In this paper : we study $\mathbb{Z}$-cover of finite origami



Goal Construct a $\mathbb{Z}$-cover $\mathcal{O}^\infty \to \mathcal{O}$ so that the following holds.

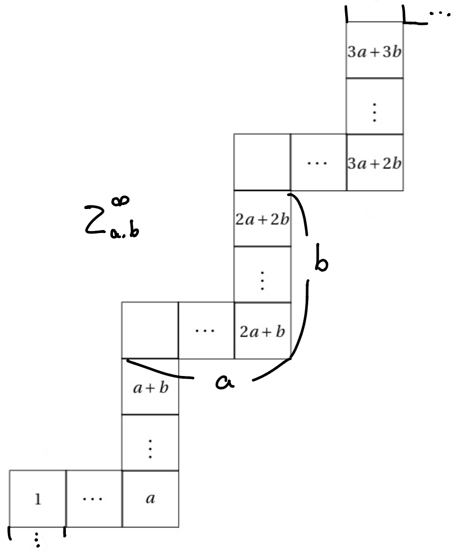
THEOREM 1. *If there is a one-cylinder direction on $\mathcal{O}$, then the Veech group of $\mathcal{O}^\infty$ is either a lattice or infinitely generated. Its limit set is equal to $\mathbb{P}^1(\mathbb{R})$.*

We mainly consider a special family

$Z^{\infty}_{a,b}$ ($a \geq 2$, $b \geq 0$) : infinite staircase origami

& as a simple generalization of the observations

we obtain Theorem 1.

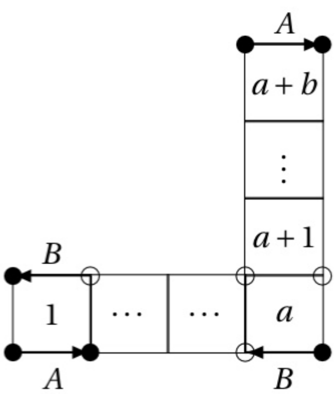


PROPOSITION 7. *The countable family of infinite area translation surfaces $Z^{\infty}_{a,b}$ with $a \geq 2$, $b \geq 0$, a or b even, and $(a,b) \neq (2,0)$, which all are $\mathbb{Z}$-covers of genus 2 origamis, have Veech groups that are infinitely generated subgroups of* $\mathrm{SL}_2(\mathbb{Z})$. *The Veech groups are Fuchsian groups of the first kind.*

§3 an infinite family of infinite trans. surfs.

DEFINITION 1. Let $Z_{a,b} = Z^{1}_{a,b}$ ($a \geq 2$, $b \geq 0$) be the origami drawn in Figure 1. More precisely, the corresponding covering map to the torus is described by the two permutations

$$\sigma_h = (1 \quad 2 \quad \ldots \quad a) \quad \text{and} \quad \sigma_v = (1 \quad a \quad a+1 \quad \ldots \quad a+b).$$



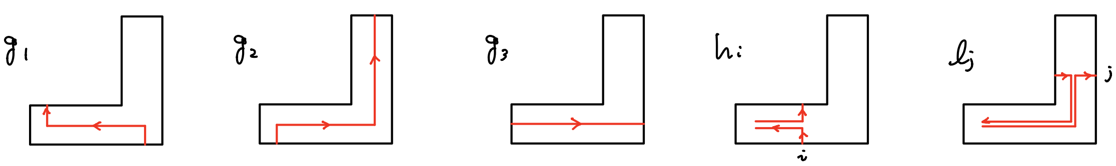
admits trivial one cyl. direction $v = \binom{0}{1}$

FIGURE 1. The origami $Z_{a,b}$. Edges labeled by the same letter and unlabeled opposite edges are identified.

Let $Z^{*}_{a,b}$ be the punctured surface obtained from $Z_{a,b}$ by removing all the vertices of the squares. Its fundamental group $U_{a,b} = \pi_1(Z^{*}_{a,b})$ is

$$U_{a,b} = \langle g_1, g_2, g_3, h_i, l_j \mid i \in \{1, \ldots, a-2\},\ j \in \{1, \ldots, b\} \rangle$$

where we define $g_1 = yx^{-(a-1)}$, $g_2 = x^{a-1}y^{b+1}$, $g_3 = x^{a}$, $h_i = x^{i}yx^{-i}$, and $l_j = x^{a-1}y^{j}xy^{-j}x^{-(a-1)}$. We have chosen the base point of $\pi_1(Z^{*}_{a,b})$ in the square labeled by 1.



GLUING RULES.

- *Crossing the A-edge of the copy labeled by l in the direction bottom to top leads to the copy labeled by $l+1$.*
- *Crossing the B-edge in the direction bottom to top leads to the copy labeled by $l-1$.*

Observation g: genus n: zero (singularity) angle: cone angle of singularity

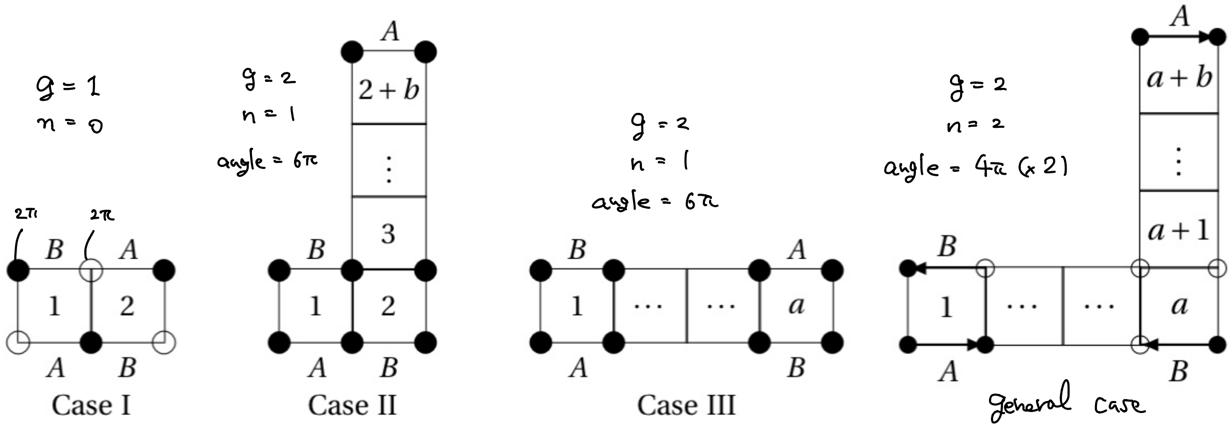
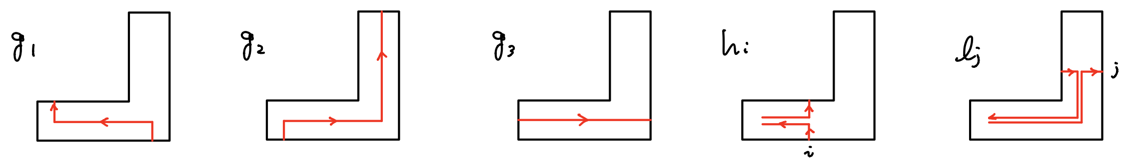


FIGURE 2. The three special cases for $Z_{a,b}$: Case I: $a = 2$, $b = 0$, Case II: $a = 2$, $b \geq 1$, Case III: $a \geq 3$, $b = 0$.

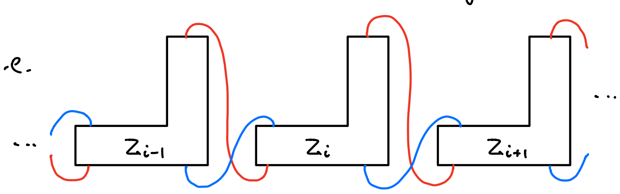
recall $U_{a.b} = \pi_1(Z^*_{a.b}) = \langle g_1, g_2, g_3, h_i, l_j \rangle$



Def 2 Let $p^k = p^k_{a.b} : Z^k_{a.b} \to Z_{a.b}$ be the k-fold cover given by the monodromy:

$$m^k = m^k_{a.b} : U_{a.b} \to \mathrm{Sym}\,\mathbb{Z}/k\mathbb{Z} \quad \begin{cases} g_1 \mapsto (z \mapsto z-1) \\ g_2 \mapsto (z \mapsto z+1) \\ \text{other generator} \mapsto \mathrm{id} \end{cases}$$

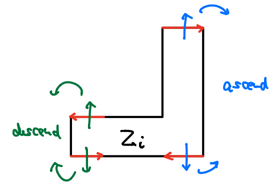
i.e.



Def 3 Let $p^\infty = p^\infty_{a.b} : Z^\infty_{a.b} \to Z_{ab}$ be the $\mathbb{Z}$-cover defined by the similar monodromy as Def 2.

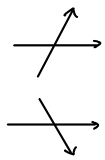
Denote by $Z^{\infty *}_{ab}$ the punctured surface $Z^\infty_{ab} \setminus \{\text{vertices}\}$

§4. Monodromy of the infinite origami



Let $c \in U_{a,b}$. $\overset{\text{lift}}{\leadsto}$ $\tilde{c}$: path on $Z^\infty_{a,b}$

by the gluing rule of $Z_{a,b}$, $\tilde{c}$ ascends when it crosses A-slit or B-slit in a 'positive-crossing'
(i-th → (i+1)-th)
descends when it crosses A-slit or B-slit in a 'negative-crossing'
(i-th → (i−1)-th)



Lemma 4 Let $c = w(g_i, h_j, l_k) \in U_{a,b}$
(word)

(i) $m^\infty(c) = \#(\text{positive crossing}) - \#(\text{negative crossing}) \in \mathbb{Z} \hookrightarrow \mathrm{Sym}\,\mathbb{Z}$
$\tau \mapsto (z \mapsto z+\tau)$

(ii) $m^\infty(c) = -\#_{g_1}(w) + \#_{g_2}(w) \in \mathbb{Z}$

recall cusps : pts in $Z^k_{a,b} \setminus Z^{k*}_{a,b}$ or $Z^\infty_{a,b} \setminus Z^{\infty *}_{a,b}$. we obtain their type as follows :

COROLLARY 5. *For a cusp P let l_P be a small positively oriented loop around P. We distinguish the four different cases described after Definition 1 and shown in Figure 1 and Figure 2:*

General case: *If $P = \bullet$, the monodromy of l_P is 1; if $P = \circ$, it is -1.*

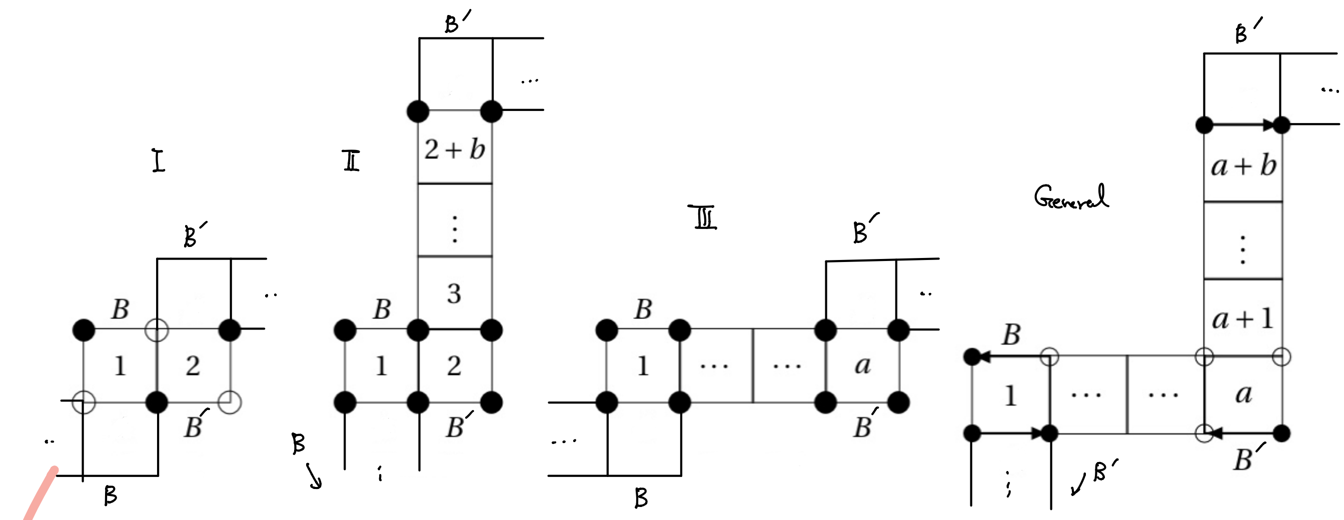
Thus on $Z^k_{a,b}$ both cusps have precisely one preimage, which is a zero of angle $2k\pi$. Hence the genus of $Z^k_{a,b}$ is $2k$. On $Z^\infty_{a,b}$ the singularities have each precisely one preimage, which is an infinite angle singularity. The genus of $Z^\infty_{a,b}$ is infinite.

Case I: *If $P = \circ$, the monodromy of l_P is 2; if $P = \bullet$, it is -2.*

Thus each cusp has one preimage on $Z^k_{a,b}$ if k is odd and two preimages if k is even. The genus of $Z^k_{a,b}$ is k if k is odd and $k-1$ if k is even. On $Z^\infty_{a,b}$ each cusp of $Z_{a,b}$ has two preimages, which are infinite angle singularities. $Z^\infty_{a,b}$ has infinite genus.

Cases II and III: *For $P = \bullet$, the monodromy of l_P is 0.*

Hence the maps $p^k_{a,b}$ and $p^\infty_{a,b}$ are unramified even above the cusps. In particular, the cusp $\bullet$ on $Z_{a,b}$ has k preimages on $Z^k_{a,b}$ and infinitely many preimages on $Z^\infty_{a,b}$, each of angle 6π. The genus of $Z^k_{a,b}$ is $k+1$, and $Z^\infty_{a,b}$ is again an infinite genus surface.



§ 5. The Veech group   ($\mathbb{Z}^2$: the $\mathbb{Z}$-module spanned by dev. vectors of saddle connections)

Rem Since the VG-action preserves all singularities we don't have to distinguish $\dot{Z}_{ab}$ & $\dot{Z}_{ab}^*$.

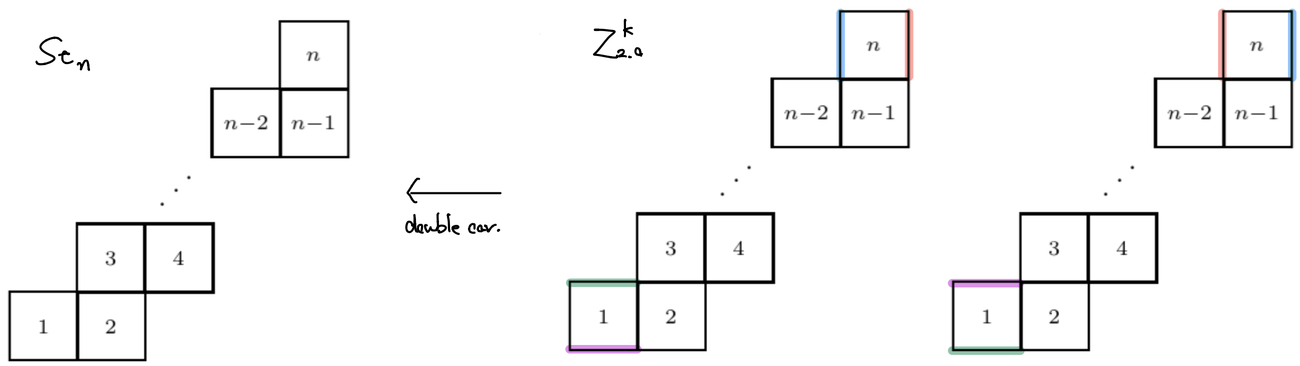
$(\ \Gamma(\dot{Z}_{ab}) = \Gamma(\dot{Z}_{ab}^*)\ )$

Surfaces $Z^k_{2.0}$ $(k:\text{odd})$ appear in [Herrlich, 2006]. (#square $= 2k$)

They occur as the smallest normal cover of the "staircase origami" St_k by e.g. [Gutkin, Judge 2000],

$\Gamma(Z^k_{2.0}) > \Gamma(St_k)$

4.3. Odd stairs. Let $n \geq 3$ be odd and St_n the origami



where opposite edges are glued in horizontal and in vertical direction.

In [Schmithüsen, 2006] their VGs were calculated.

$$\Gamma(St_k) = \begin{cases} \Gamma(2) & \text{if } k: \text{even} \\ \Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle|\ a+c \ \& \ b+d : \text{odd} \right\} \end{cases}$$

We will see in Prop 6. that $\Gamma(Z^k_{2.0}) = \Gamma$ for $^\forall k$.

In §4 in [Lelièvre, Silhol 2007], the algebraic eq. for these families have been examined.

Prop 6. Let $k \geq 3$ or $k = \infty$. $\Gamma(Z^k_{2.0}) = \Gamma$ holds.

pf) $Z^k_{2.0}$ decomposes into cylinders of length, in both horizontal & vertical directions.

Thus VG $\Gamma(Z^k_{2.0})$ contains $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ & $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; these two generate $\Gamma(2)$.

$\to \Gamma(Z^k_{2.0}) \supset \Gamma(2)$.

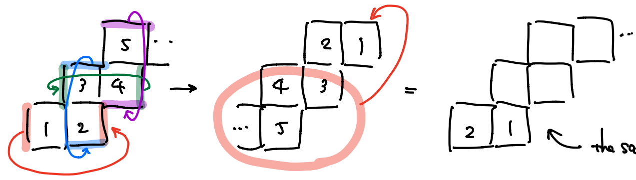
Next we may observe $\gamma_S : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x^{-1} \end{pmatrix} \in \mathrm{Aut}^+ F_2$ stabilizes N_k

and thus $\Gamma(Z^k_{2.0}) \supset \Gamma = \langle \Gamma(2), S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$. ([Schmithüsen, 2006])

Finally, $\Gamma \overset{\text{index } 3}{<\Gamma(Z^k_{2.0})<} SL(2,\mathbb{Z}) \Rightarrow \Gamma(Z^k_{2.0}) = \Gamma$ or $SL(2,\mathbb{Z})$

Here we can see that $N_k := \pi_1(Z^k_{2,0})$ is a normal subgrp of F_2 (i.e. $Z^k_{2,0}$: normal origami)

(e.g. by the observation :



$x^* N_k = N_k$ //)

the same structure w/ base cell "2".

Thus we have $N_k = \begin{cases} \langle\langle x^2, y^2, (xy)^k \rangle\rangle & (k < \infty) \\ \langle\langle x^2, y^2 \rangle\rangle & (k = \infty) \end{cases}$ where $\langle\langle \cdot \rangle\rangle$ denotes the normal closure.

Consider $\gamma_T : \binom{x}{y} \mapsto \binom{x}{xy} \in \mathrm{Aut}^+ F_2$ and observe $\gamma_T(y^2) \notin N_k$.

By $N_k \lhd F_2$ we may conclude $\gamma_T(N_k) \neq N_k$ and $\Gamma(Z^k_{2,0}) \neq SL(2,\mathbb{Z})$. ([Schmithüsen, 2006])

This finishes the pf. ⊗

It follows from this Prop that $\Gamma(Z^\infty_{2,0})$ is a lattice.

But the following Prop states that this is not general.

From now on we assume $(a,b) \neq (2,0)$

Prop 7. Let $\Gamma^\infty_{ab} = \Gamma(Z^\infty_{ab})$ w/ $(a,b) \neq (2,0)$.

If a or b : even, then Γ^∞_{ab} is infinitely generated.

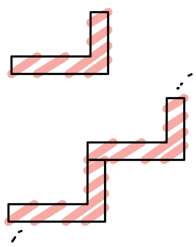
pf) In Lemma 8 we show $[SL(2,\mathbb{Z}) : \Gamma^\infty_{ab}] = \infty$.

It remains to show that the limit set $\Lambda(\Gamma^\infty_{ab}) = \overline{\mathbb{R}}$. To prove this,

(I) We show in Corollary 10 that

[if $Z_{a,b}$ admits one-cylinder direction v

then Z^∞_{ab} decompose in v into cylinders $\underset{\text{isom}}{\sim}$ the one cylinder in $Z_{a,b}$]



$\Rightarrow$ Γ^∞_{ab} contains parabolic elements A_v w/ eigenvector v.

$\Rightarrow$ $z_v = p_v/q_v \in \mathbb{R} \subset \overline{\mathbb{H}}$ is a cusp of Γ^∞_{ab}

e.g. $v = \binom{0}{1}$ is a one-cyl. dir.
$\Rightarrow$ $0 \in \mathbb{R}$ is a cusp.

($\because$ let $A_v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{p}{q} = \lambda \binom{p}{q} \Leftrightarrow \dfrac{a\,p/q + b}{c\,p/q + d} = p/q$, p/q : parab. fixed pt.)

(II) We show in Lemma 11 that

$\left[\begin{array}{l} a : \text{even} \Rightarrow \binom{2}{1} : \text{one-cyl. dir. on } Z_{a,b} \\ b : \text{even} \Rightarrow \binom{1}{2} : \quad〃 \end{array}\right\}$ denote by v.

It follows that $\gamma(v)$ is also one-cyl. dir. on $Z_{a,b}$ for $\forall \gamma \in \Gamma(Z_{a,b})$

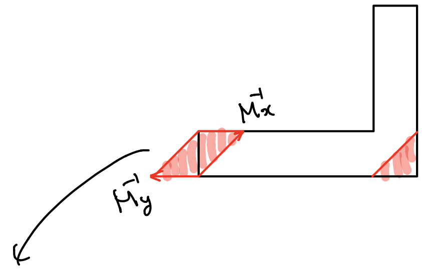
In particular all pts $Z_{\gamma(\omega)}$ are cusps of $\Gamma^{\infty}_{a,b}$

Since $\Gamma(Z_{a,b}) < SL(2,\mathbb{R})$ is a lattice, they lie densely in $\overline{\mathbb{R}}$.

LEMMA 8. *For any $a \geq 2$, $b \geq 0$ with $(a,b) \neq (2,0)$ the Veech group $\Gamma^{\infty}_{a,b}$ of the translation surface $Z^{\infty}_{a,b}$ has infinite index in* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. We proceed as follows: we consider the translation surface

$$Y^{\infty}_{a,b} = M \cdot Z^{\infty}_{a,b} \qquad \text{where} \quad M = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$



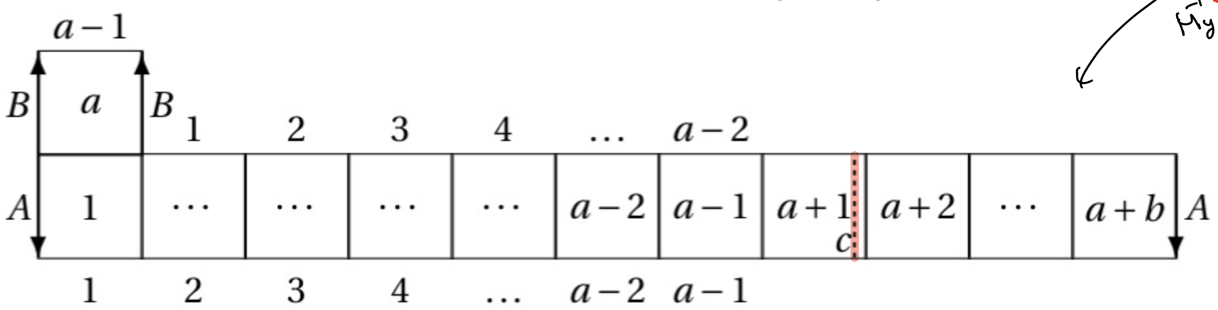


FIGURE 6. The origami $Y_{a,b} = M \cdot Z_{a,b}$. Edges labeled with the same number or letter and unlabeled opposite edges are identified.

We show $\Gamma(Y^{\infty}_{a,b})$ does not contain any power of $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Let $c \subset Y_{a,b}^{*}$: closed curve as in Fig. 6.

— not crossing any cutting slit $\Rightarrow$ $\forall$ lift $\hat{c} \subset Y^{\infty *}_{a,b}$: closed curve w/ $\mathrm{dev}(\hat{c}) = \binom{0}{1}$

Suppose $\exists\, \hat{f} \in \mathrm{Aff}^{+}(Y^{\infty}_{a,b})$ w/ $d\hat{f} = T^{n} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for some $n \in \mathbb{N}$.

— We may assume $n \geq a+b$ by passing to a power if necessary.

$Y^{\infty}_{a,b} \supset \hat{C}_2 = \hat{f}(\hat{c}_1) \xmapsto{\mathrm{dev}} T^{n}\binom{0}{1} = \binom{n}{1}$

$M^{*}p^{\infty}_{a,b}$

$Y_{a,b} \supset C_2 \xmapsto{\mathrm{dev}} \binom{n}{1}$, furthermore its monodromy is 0.

Here, we may see that any closed curve w/ dev $= \binom{n}{1}$ intersects a horizontal saddle conn.

$\longrightarrow$ we may assume C_2 starts from a point on bdries of two horizontal cylinders...

since dev $C_2 = \binom{n}{1}$ again, C_2 hits at most one of the slits A, B

and its crossing-direction never changes. $\rightsquigarrow$ Contradiction

Lem 4

$\boxtimes$

※ for the case $b=0$, we take c_1 w/ $\mathrm{dev}(c_1) = \binom{1}{1}$ instead $\binom{0}{1}$.

LEMMA 9. *In the following we fix a direction $v = \binom{p}{q}$. Let k be the number of cylinders of $Z^*_{a,b}$ in direction v, and let $c_1, \ldots, c_k$ be geodesics in direction v which are core curves of the cylinders. We then have*

$$\sum_{i=1}^{k} m(c_i) = 0,$$

where $m = m^\infty_{a,b}$ is the monodromy of the cover $p^\infty_{a,b}\colon {Z^\infty_{a,b}}^* \to {Z_{a,b}}^*$.

From Lemma 9 we immediately obtain the following corollary.

COROLLARY 10. *If $v = \binom{p}{q}$ is a one-cylinder direction on $Z_{a,b}$ and c is a closed geodesic in direction v, then we have:*

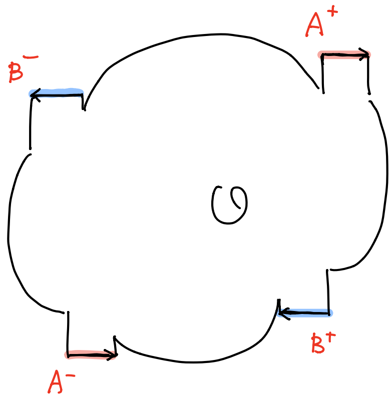
i) *The monodromy $m(c)$ is* 0.
ii) *$Z^\infty_{a,b}$ decomposes in direction v into cylinders isometric to the one cylinder on $Z_{a,b}$.*
iii) *The Veech group $\Gamma(Z^\infty_{a,b})$ contains a parabolic element in $\mathrm{SL}_2(\mathbb{Z})$ with eigenvector $v = \binom{p}{q}$.*
iv) *p/q is a cusp of $\Gamma(Z^\infty_{a,b})$.*

Observe that, so far, we have not used the prerequisite that a or b is even. We will need this now in the last step, where we find a one-cylinder direction on $Z_{a,b}$.

LEMMA 11. *Let $v = \binom{1}{2}$ and $v' = \binom{2}{1}$. If b is even, then v is a one-cylinder direction on $Z_{a,b}$. If a is even, then v' is a one-cylinder direction.*

§6 Generalization

Let $\mathcal{O}$ be a finite origami.



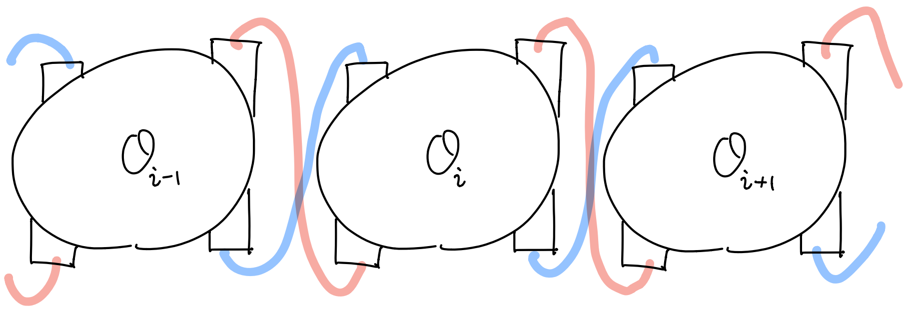
A, B : horizontal saddle-conn. of length 1,

oriented in opposite directions, not homologous.

We cut $\mathcal{O}$ along A, B to obtain a trans. surf. w/ bdries.

We name 4 bdry components A^+, A^-, B^+, B^-

where the signs respect the centres of squares.



We construct an infinite origami $\mathcal{O}^\infty$

which is a $\mathbb{Z}$- cover of $\mathcal{O}$:

by gluing A^+, B^+ in i-th copy

w/ A^-, B^- in $(i+1)$-th copy.

Now we can observe that the monodromy m^∞ w.r.t $\mathcal{O}^\infty \to \mathcal{O}$ is just same as in Lemma 4.

$\forall c \in \pi_1(\mathcal{O})$, $m^\infty(c) = \#(\text{positive crossing}) - \#(\text{negative crossing}) \in \mathbb{Z} \hookrightarrow \mathrm{Sym}\,\mathbb{Z}$

We may enjoy the same arguments as in the previous § and we obtain;

THEOREM 1. *If there is a one-cylinder direction on $\mathcal{O}$, then the Veech group of $\mathcal{O}^\infty$ is either a lattice or infinitely generated. Its limit set is equal to $\mathbb{P}^1(\mathbb{R})$.*

OPEN QUESTIONS.

1. There are origamis without one-cylinder decompositions. For instance, the Veech group of $Z_{3,1}^\infty$ has two cusps. Each of them corresponds to directions in which the surface is decomposed into two cylinders. We don't know whether the limit set of the Veech group of $Z_{3,1}^\infty$ is $\mathbb{P}^1(\mathbb{R})$ or a Cantor set.
2. When Theorem 1 holds, it seems difficult to give a general criterion to decide whether the group is a lattice or infinitely generated.