Bull. Soc. Math. France
149 (2), 2021, p. 417-438

SYSTOLES IN TRANSLATION SURFACES

BY CORENTIN BOISSY & SLAVYANA GENINSKA

ABSTRACT. — For a translation surface, we define the relative systole to be the length of the shortest saddle connection. We give a characterization of the maxima of the systole function on a stratum and give a family of examples providing local but non-global maxima on each stratum of genus at least 3. We further study the relation between the (local) maxima of the systole function and the number of shortest saddle connections.

[BG2021]

§1-2 Intro & Background

$A = \{(U, \xi)\}$

A translation surface is a surface w/ a translation atlas.
($\overset{\text{def}}{\Longleftrightarrow}$ $\forall$ transition is $z \mapsto z + c$)

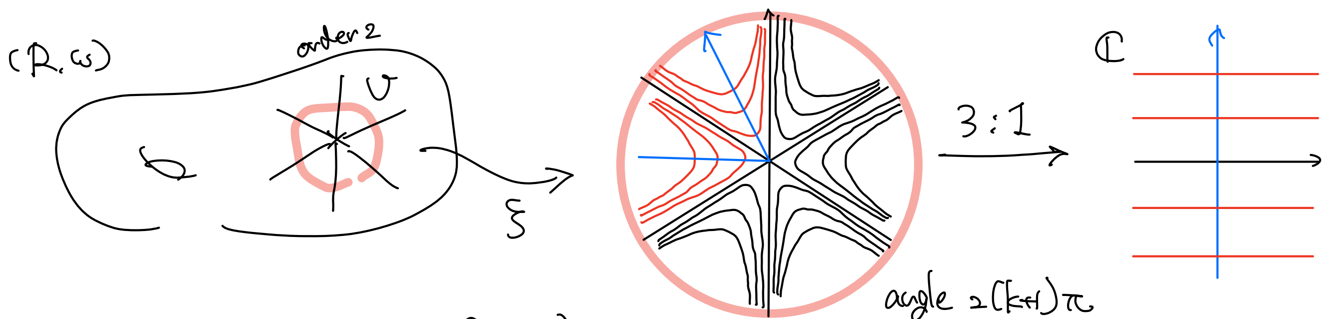
$\updownarrow$

Riemann surface w/ Abelian differential (holo. 1-form)

(R, ω) $\quad \omega \overset{\text{loc.}}{=} \omega(z)\,dz$, $\xi_{p_0} = \int_{p_0}^{p} \omega$ arr. $p_0 \in R \setminus \mathrm{Zero}(\omega)$

A point $p \in R$ is called a singularity of order $k \geq 1$ if $\mathrm{ord}_p \omega = k$.

Then, around p, R is a $(k+1)$-fold cyclic cover of $\mathbb{C}$.



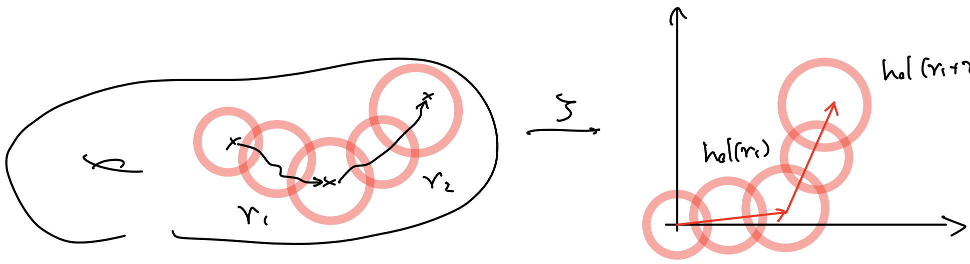
A saddle connection on (R, ω)

is a nonsingular $|\omega|$-geodesic joining two singularities.

▷ (R,ω) determines a flat metric $|\omega|$ on R, w/ conical singularities as above.

The holonomy vector of a saddle connection γ is $\int_\gamma \omega \in \mathbb{C}$.

It defines a homomorphism $hol : H^1(R, \Sigma; \mathbb{Z}) \to \mathbb{C}$.



☆ $\mathcal{H}_g = \{ (R,\omega) : \text{trans. surf.} \mid \text{genus}(R) = g,\ \int_R |\omega^2| < \infty \} / \sim$

is stratified into

$\mathcal{H}_g(k_1, \cdots, k_n) = \{ (R,\omega) \in \mathcal{H}_g$ w/ n singularities of order $k_1, \cdots, k_n \}$

such that ${}^{\forall} k_j \geq 1$, $\sum_{j=1}^{n} k_j = 2g-2$ (g is determined by $k_1 \cdots k_n$)

Masur (1982), Veech (1982, 1990):

$\mathcal{H}_g(k_1, \cdots, k_n)$ is a complex subvariety of $T^* T_{g,n}$

☆ Local coordinates of $\mathcal{H}_g(k_1, \cdots, k_n)$ are obtained by the holonomies along a basis of the relative homology:

$$\left[\begin{array}{l} \mathcal{H}_g(k_1, \cdots, k_n) \ni (R,\omega) \\ \qquad \text{w/ } (\gamma_i)_{i \in I} = \mathrm{Gen}(H_1(R, \mathrm{sing}\,\omega : \mathbb{Z})) \end{array} \right] \mapsto \left(\int_{\gamma_i} \omega \right)_{i \in I}$$

Note.

Theorem (Veech (1993))

Let α be a symbol (e.g. $(k_1, \cdots, k_n)$: including 'integrable flat surfaces')

The holonomy character map $\chi_\alpha : T_{g,n} \to K(\alpha)$ is real analytic.

$\{ \text{flat metric w/ symbol } \alpha \}$

The level sets of χ_α are the leaves of a real analytic foliation $\mathcal{F}(\alpha)$ of $T_{g,n}$ by $(2g-3+n)$-dimensional submanifolds of $T_{g,n}$.

Let $\mathcal{H} = \mathcal{H}_g(k_1, \cdots, k_n)$ be a stratum.

☆ A sequence $(R_j, \omega_j)_{j \in J} \subset \mathcal{H}$ leaves any cpt set

$\Leftrightarrow$ the length of shortest saddle-conn. $\to 0$

$\parallel$

the systole $\mathrm{Sys}(R, \omega)$ of (R, ω) (up to normalization)

Systole is maximal $\Leftrightarrow$ far from $\partial\mathcal{H}$

goal to study global & local maxima of $\mathrm{Sys}\big|_{\text{area}=1}$

Remark The concept of "true systole" (a bit different from our 'relative systole') is given by Judge, Parlier (2019)

prev. research
- char. of global maxima of Sys : known
- existence of local maxima : unknown

THEOREM. — *Let S be a genus $g \geq 1$ translation surface of area 1 and $r > 0$ singularities or marked points. Then,*

$$\mathrm{Sys}(S) \leq \left(\frac{\sqrt{3}}{2}(2g-2+r)\right)^{-\frac{1}{2}}.$$

The equality is obtained if and only if S is built with equilateral triangles whose sides are saddle connections of length $\mathrm{Sys}(S)$. Such a surface exists in any connected component of any stratum.

[BG2021]

one zero strata version : proven by Judge, Parlier (2019)

THEOREM. — *Each stratum of area 1 surfaces with genus $g = 2$ with marked points or $g \geq 3$ contains local maxima of the function* Sys *that are not global.*

[BG2021]

↑ examples obtained by considering surfaces that decompose into equilateral triangles & regular hexagons.

related Q.. the maximal number of shortest saddle-conns (realizing systole)?

A ts is called rigid $\Longleftrightarrow$ $\#\{\text{shortest saddle-conns}\}$ is local maximal in $\mathcal{H}$.

Proposition 5.2. — *Let S be a translation surface such that, when cut along its saddle connections of shortest length, it decomposes into equilateral triangles and regular hexagons. If the function* Sys *admits a local maximum at $[S] \in \mathbb{P}\mathcal{H}(k_1, \ldots, k_r)$, then S is rigid.*

[BG2021]

"equilateral triangles & regular hexagons"-surface $\not\Rightarrow$ rigid.

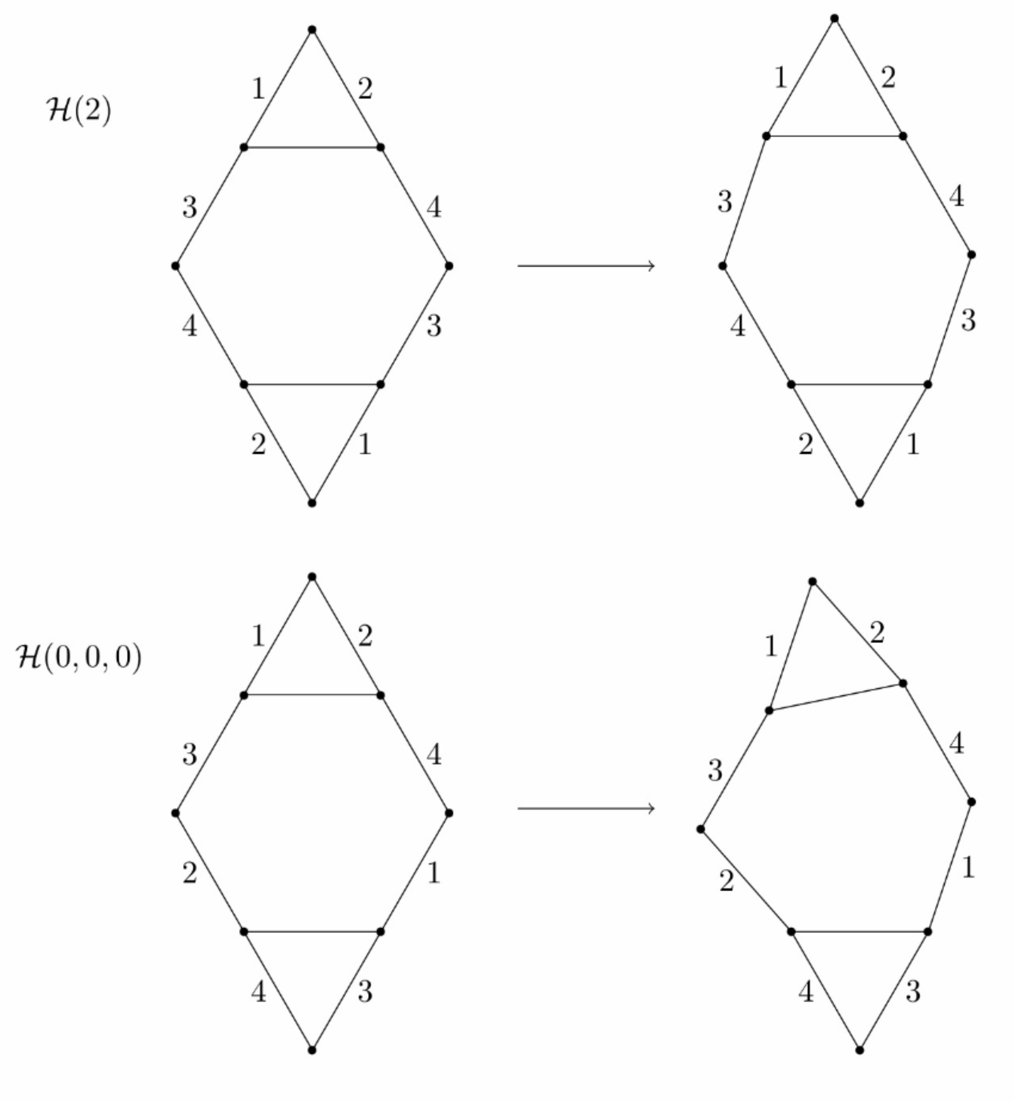


Figure 5.1. Examples of nonrigid surfaces in $\mathcal{H}(0,0,0)$ and $\mathcal{H}(2)$

[BG2021]

rigid $\not\Rightarrow$ local maxima for Sys.

Proposition 5.3. — *The translation surface given by Figure 5.2 is rigid but it is not a local maximum for the function* Sys *in $\mathbb{P}\mathcal{H}$ for $n \geq 3$.*

[BG2021]

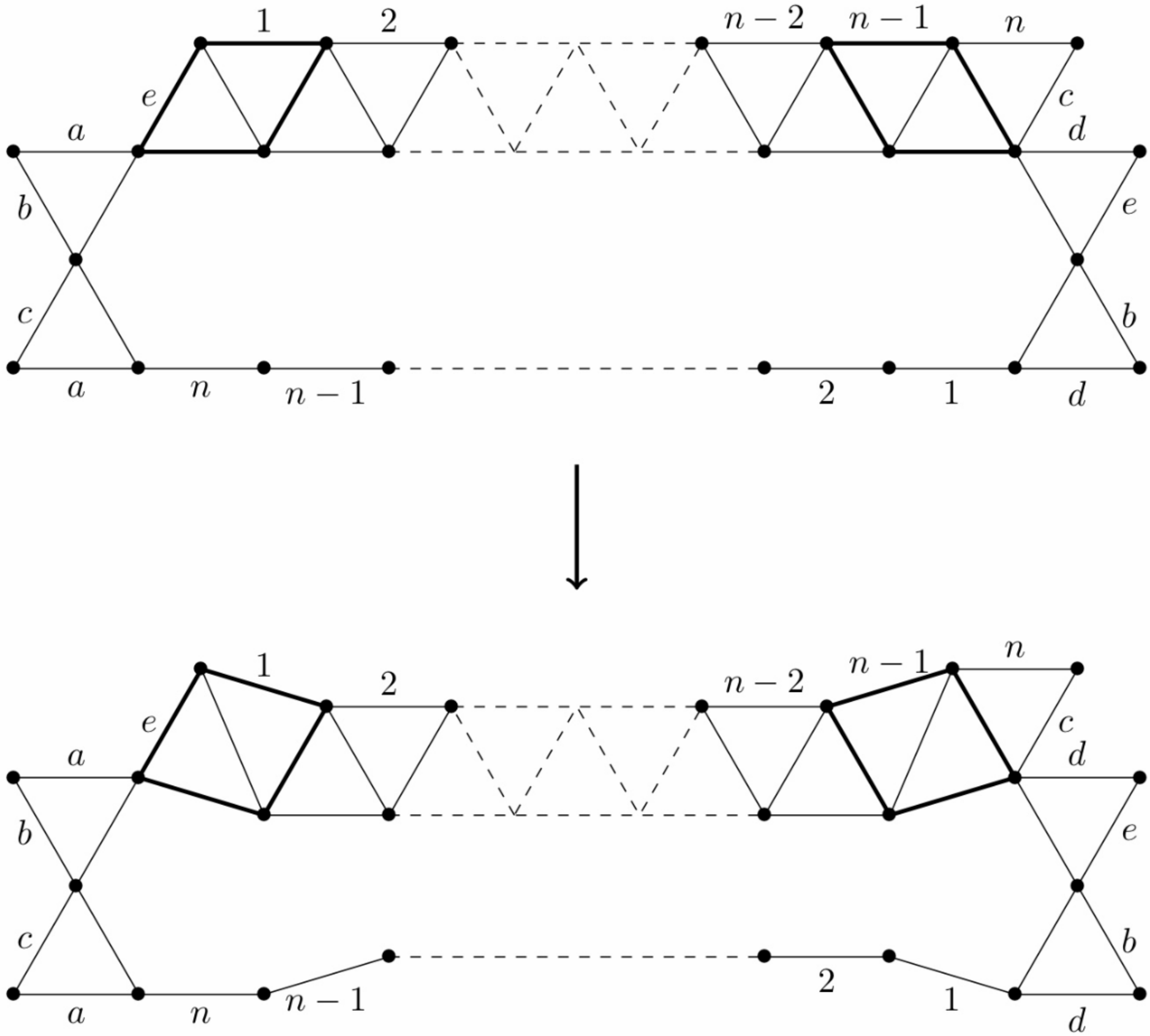


FIGURE 5.2. Example of a rigid surface that is not a local maximum

[BG2021]

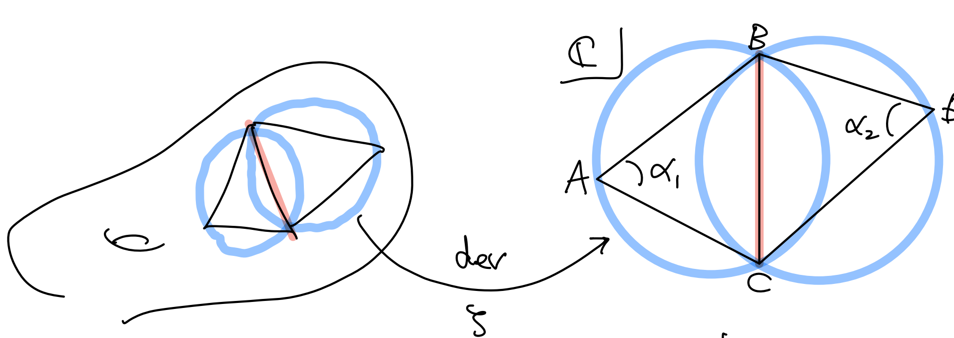
§3. Maximal systole

<u>Def</u> Let S be a translation surface.
A <u>Delaunay triangulation</u> of S is a geometric triangulation,

$$\begin{cases} \text{vertices} : \text{singularities on } S \\ \text{edges} : \text{saddle-connections on } S \end{cases}$$

such that any dihedral angle is less than π,

that is,



$\alpha_1 + \alpha_2 < \pi$

no singularity is contained in
the 'circum-circle' of each triangles

Masur & Smillie (1991)
— Delaunay triangulation exists for $\forall$ ts.

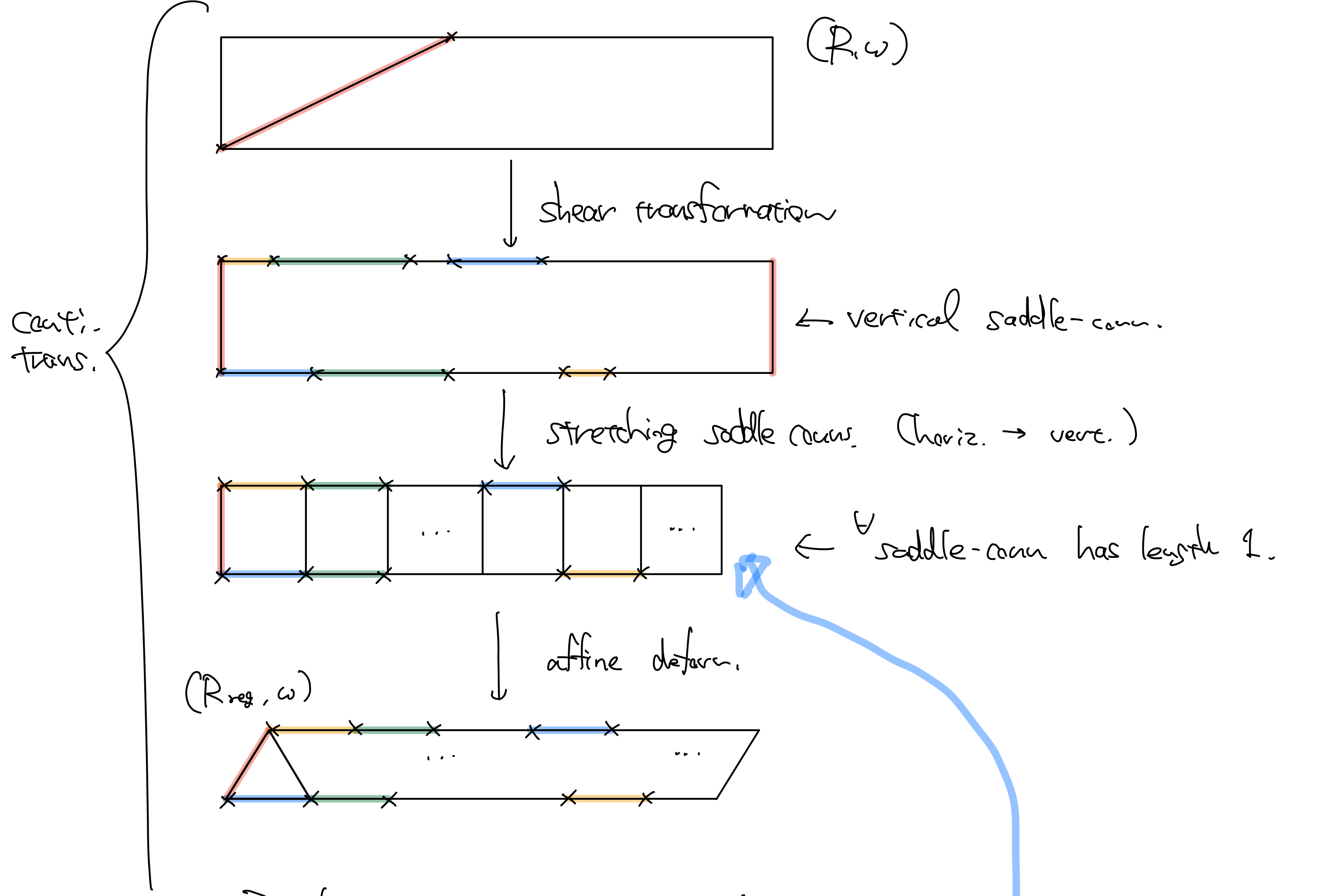
<u>Lem 3.1</u> shortest saddle-conn $\Rightarrow$ edge of Delaunay triangulation.

LEMMA 3.2. — *Let $\mathcal{C} \subset \mathcal{H}(k_1, \ldots, k_r)$ be a connected component of a stratum of abelian differentials with $k_1, \ldots, k_r \geq 0$.*

1. *There exists in $\mathcal{C}$ a surface S that decomposes into equilateral triangles whose sides are saddle connections.*
2. *Furthermore, for each i, j we can find such a surface with a side of an equilateral triangle being a saddle connection joining a singularity of degree k_i to a singularity of degree k_j, with the convention that the two singularities are different, if $i \neq j$ and equal if $i = j$.*

[BG2021]

pf) [Lem 18, Kontsevich & Zorich 2003] states that
$\forall \mathcal{C} \subset \mathcal{H}(k_1, \cdots, k_r)$; conn. comp $\exists (R, \omega) \in \mathcal{C}$
s.t. (R, ω) admits a horizontal one-cylinder decomposition.



In this way, we obtain an equilateral triangle-tiled surface in C. #(1)

Observe that each singularity appears both top & bottom sides.

Applying $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ - affine deformation to we obtain such a surface. #(2)

(Note : Not simultaneously realized for $^\forall i, j$?)

THEOREM. — *Let S be a genus $g \geq 1$ translation surface of area 1 and $r > 0$ singularities or marked points. Then,*

$$\mathrm{Sys}(S) \leq \left(\frac{\sqrt{3}}{2} (2g - 2 + r) \right)^{-\frac{1}{2}}.$$

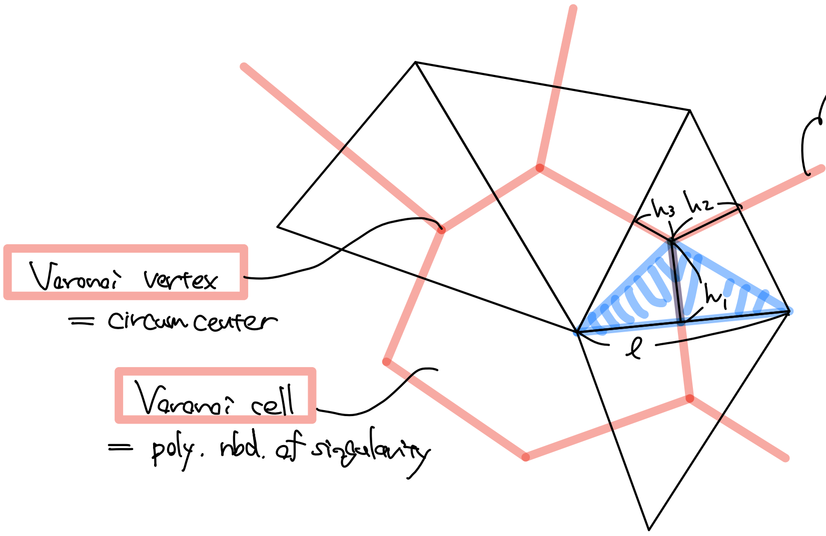
The equality is obtained if and only if S is built with equilateral triangles whose sides are saddle connections of length Sys(S). *Such a surface exists in any connected component of any stratum.*

[BG2021]

pf) Alternatively suppose to minimize Area(S) under Sys$(S) = 1$.

We consider the Voronoi diagram :

given by joining circumcenter of Delaunay triangles



$$\mathrm{Area}(S) = \sum_{\text{D.tri.}} \frac{\ell h}{2} \leq \mathrm{Sys}(S) \sum_{\text{D.tri.}} \frac{h}{2}$$

For each Delaunay triangle T,

$$\alpha(T) := h_1 + h_2 + h_3 \quad (\text{sum of circumdistances})$$

Carnot's thm : $\alpha(T) =$ inradius (T) + circumradius (T).

(in classical geometry) $\geq \frac{\sqrt{3}}{2}$ w/ equality $\Leftrightarrow$ T : equilateral

(See Lem 3.4.)

※ $\#\{\text{Triangle}\} = 2g-2+r$ ($\because$ Euler char.)

§4 Locally maximum systole

The systole function Sys is well-def on

$$\mathbb{P}\mathcal{H}_g(k_1, \cdots, k_r) := \mathcal{H}_g(k_1 \cdots k_r)/\mathbb{C}^\times \quad (\text{i.e. up to scaling \& rotation})$$

Def $S_0 \in \mathcal{H}_g(k_1, \cdots, k_r)$: base pt

Fix a basis of $H^1(S_0, \mathrm{Sing}(S_0); \mathbb{Z})$.

For S in small nbd of S_0, w/ local coordinate $v(S) = (v_1(S), \cdots, v_k(S))$

define $d(S, S_0) := \max\{|v_i(S) - v_i(S_0)| \mid i = 1, \cdots, k\}$

… distance in $\mathcal{H}_g(k_1, \cdots, k_r)$

〃 in $\mathbb{P}\mathcal{H}_g(k_1 \cdots k_r)$ by normalizing in the way that

$$\begin{cases} \text{(i)} & v_1 \in (0, +\infty) \\ \text{(ii)} & \text{shortest saddle-conn. has length } 1. \end{cases}$$

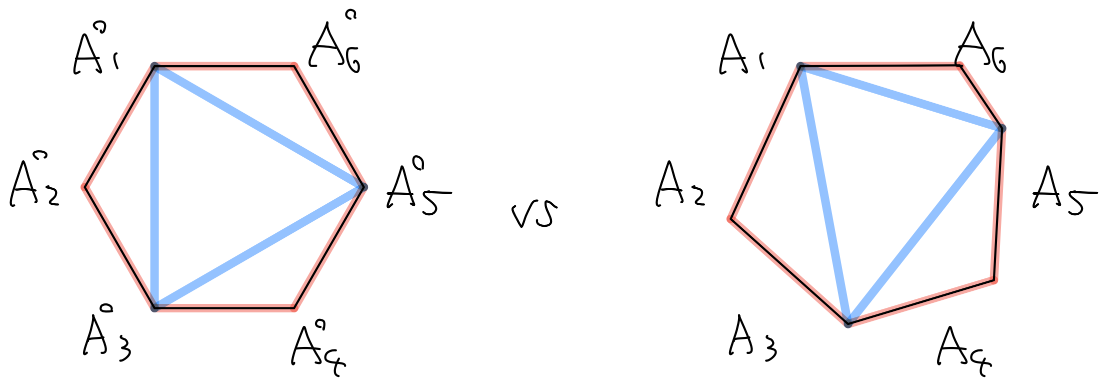
Theorem 4.1. — *Let S_{reg} be a translation surface in $\mathcal{H}_1(k_1, \ldots, k_r)$, such that when cut along its saddle connections of length $\mathrm{Sys}(S_{reg})$, it decomposes to equilateral triangles and regular hexagons so that:*

- *The set of the equilateral triangles without the vertices is connected.*
- *The boundary of each polygon is contained in the boundary of the set of triangles.*

Then, $\mathrm{Sys}(S_{reg})$ is a local maximum in $\mathcal{H}_1(k_1, \ldots, k_r)$ and even a strict local maximum in $\mathbb{P}\mathcal{H}(k_1, \ldots, k_r)$.

Lemma 4.3. — *Let H_{reg} be the regular hexagon of sides of length 1. There exists a positive constant c such that for every $\varepsilon > 0$ small enough and every convex hexagon $H = A_1A_2 \ldots A_6$ with sides of lengths in the interval $[1, 1+\varepsilon]$ and diagonals A_1A_3, A_3A_5 and A_5A_1 of lengths in the interval $[\sqrt{3}-\varepsilon, \sqrt{3}+\varepsilon]$, we have $\mathrm{Area}(H) \geq \mathrm{Area}(H_{reg}) - c\varepsilon^2$.*

[BG2021]



claim $\Delta(\quad)$ & $\Delta(\quad) < \varepsilon \Rightarrow \Delta(\mathrm{Area}) < c\varepsilon^2$.

Similar evaluation for equilateral triangles,

[BG2021]

Lemma 4.4. — *Let T_{reg} be an equilateral triangle with sides of length 1. There exists a positive constant $c \in \mathbb{R}$ such that for every $\varepsilon > 0$ small enough and every triangle T with one of its sides of length $1+\varepsilon$ and the other sides of lengths in the interval $[1, 1+\varepsilon]$, we have that $\mathrm{Area}(T) > \mathrm{Area}(T_{reg}) + c\varepsilon$.*

for positions of triangles,

Lemma 4.5. — *Let ABC be a nondegenerate triangle of sides of length $l_1 = BC$, $l_2 = AC$, and $l_3 = AB$. For ε small enough, let $A'B'C'$ be a triangle with sides of lengths l'_1, l'_2, l'_3 such that for each $i \in \{1,2,3\}$, $|l_i - l'_i| \leq \varepsilon$. We assume further that $d(A, A') \leq \varepsilon$, $d(B, B') \leq \varepsilon$, and C and C' are in the same half-plane determined by AB. Then there is a constant $J > 1$ only depending on l_1, l_2, l_3 such that $d(C, C') \leq J\varepsilon$.*

[BG2021]

outline of pf of Thm 4.1) Set the length of shortest saddle-conn by 1.

Fix a basis of $H^1(S_{reg}, \text{sing}; \mathbb{Z})$ by the shortest saddle-conns.

Consider a trans. surf. S in a small nbd of S_{reg} w.r.t. coords determined by

Let $\varepsilon > 0$: sufficiently small w/ $d(S, S_{reg}) = \varepsilon$.

$\gamma_1 \cdots \gamma_k$; the shortest saddle-conns of S_{reg}.

$l(\gamma_1) \cdots l(\gamma_k)$; deformation ratio of 'short saddle-conn' on S corr. to $\gamma_1 \cdots \gamma_k$.

$$\rho(S) := \max\{ l(\gamma_i) - 1 \}_{i=1}^{k}$$

As $\forall$ saddle-conn is a linear combination in $H^1(S_{reg}, \text{sing}; \mathbb{Z})$,

its deformation ratio is $O(\varepsilon)$.

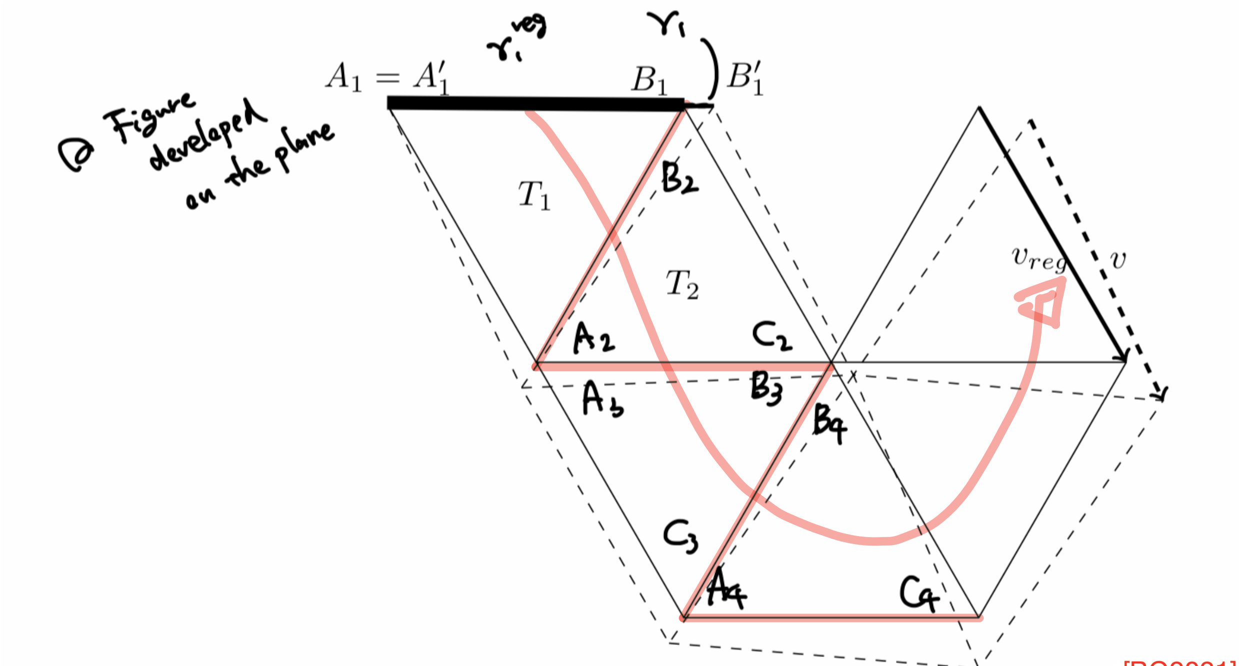
In particular, $\rho(S)$ is also $O(\varepsilon)$.

We know from Lemma 4.3 – 4.4, $\Delta(\text{Area (hexagon)}) = O(\varepsilon^2)$

$\Delta(\text{Area(equi-triangle)}) = O(\varepsilon)$.

For $\gamma \in \{\gamma_2, \cdots, \gamma_k\}$, we may take a "path" from γ_1 to γ s.t.

1. γ_1 is a side of T_1,
2. for each $i \in \{1, \ldots, l-1\}$, T_i and T_{i+1} are adjacent,
3. γ is a side of T_l.



[BG2021]

Let A_j, B_j, C_j : developed Sing. of S_{reg} w.r.t. the above path.

A_j', B_j', C_j' : 〃 S 〃

Claim $\Delta(\text{holonomy}(\gamma)) = O(\rho(S))$

We inductively see that $d_{\mathbb{C}}(C_j, C_j') = O(\rho(S))$;

(i) $d_{\mathbb{C}}(A_1, A_1') = 0$, $d_{\mathbb{C}}(B_1, B_1') = O(\rho(S))$

(ii) By Lemma 4.5, $d_{\mathbb{C}}(C_1, C_1') = O(\rho(S))$

(iii) (A_j, B_j) is either (C_{j-1}, B_{j-1}) or (A_{j-1}, C_{j-1})

(iv) Again by Lemma 4.5,

if $d_{\mathbb{C}}(A_j, A_j'), d_{\mathbb{C}}(B_j, B_j') = O(\rho(S)) \Rightarrow$ so is $d_{\mathbb{C}}(C_j, C_j')$.

Thus we observe that

$$\varepsilon = d(S_{reg}, S) = \sum \Delta(\text{hol}(\gamma)) = O(\rho(S))$$

$\Rightarrow$ if ε is small enough, it is controlled by $\rho(S)$.

$$\text{Area}(S) - \text{Area}(S_{reg}) \geq -\sum_{\text{Hexa}} c\varepsilon^2 + \sum_{\text{eq. triangle}} c'\rho(S) > 0$$

for sufficiently small $\varepsilon = d(S_{reg}, S) > 0$. $\boxtimes$.

THEOREM 4.7. — *Let $\mathcal{H}$ be a stratum of area 1 and genus $g \geq 2$ surfaces. We assume that $\mathcal{H}$ is neither $\mathcal{H}(1,1)$ nor $\mathcal{H}(2)$. Then $\mathcal{H}$ contains local maxima of the function* Sys *that are not global.*

[BG2021]

Local but non global maxima of Sys is constructed as follows:

LEMMA 4.8. — *We consider the stratum $\mathcal{H} = \mathcal{H}(m_1, \ldots, m_r, x, y)$ with $m_1, \ldots, m_r, x, y \geq 0$. We assume that there exists a surface $S_1 \in \mathcal{H}$ that satisfies the hypothesis of Theorem 4.1 and such that there is a shortest saddle connection γ_1 joining a singularity of degree x to a distinct singularity of degree y. Then:*

a) *For any $n_1, \ldots, n_k, p, q \geq 0$ with $p + q + \sum_i n_i$ even, there exists a local but nonglobal maximum of Sys in the stratum $\mathcal{H}(m_1, \ldots, m_r, p + a + 1, q + a + 1, n_1, \ldots, n_k)$.*

b) *For any $n_1, \ldots, n_k, p \geq 0$ with $p + \sum_i n_i$ even, there exists a local but nonglobal maximum of Sys in the stratum $\mathcal{H}(m_1, \ldots, m_r, p + x + y + 2, n_1, \ldots, n_k)$.*

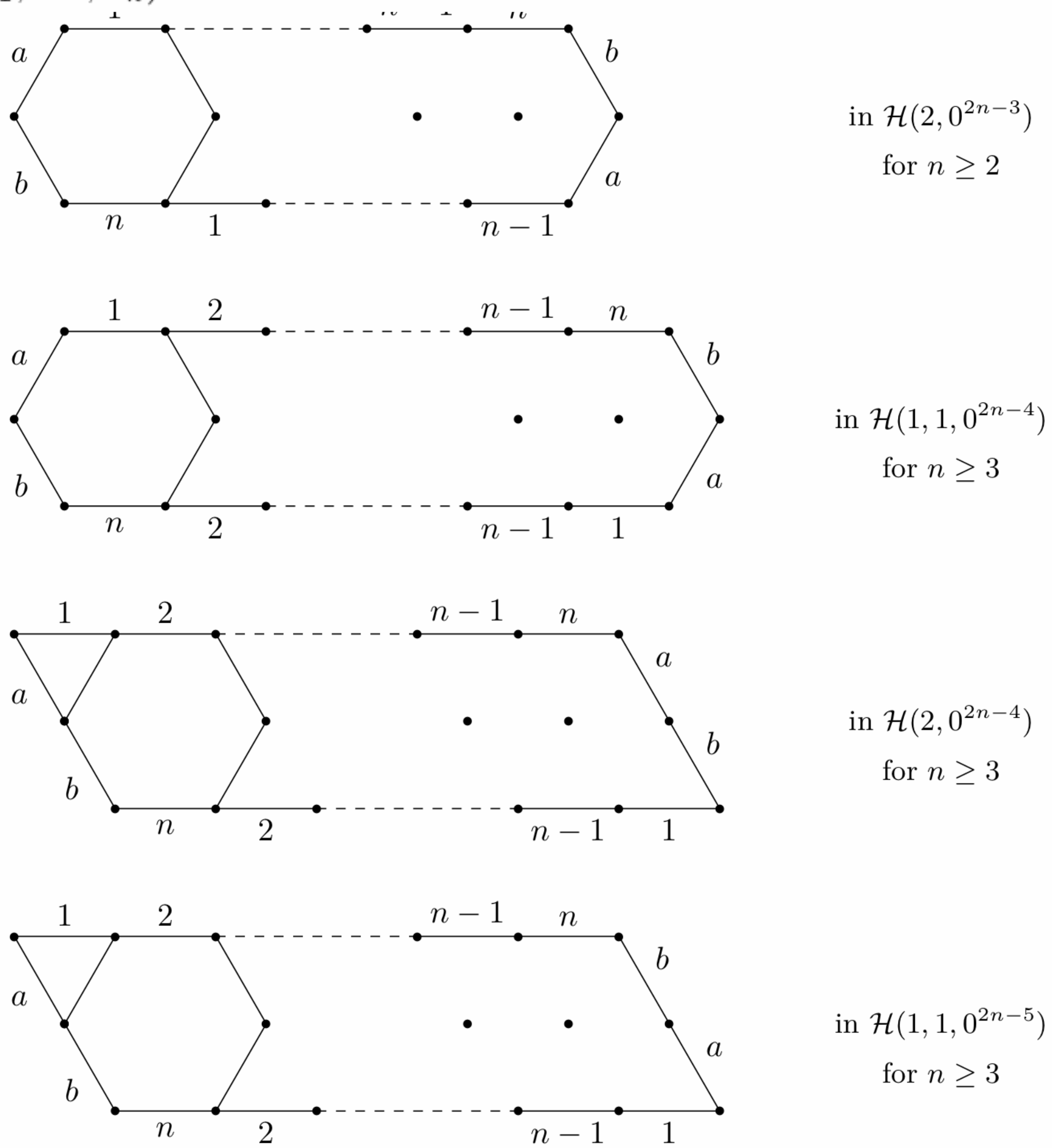


FIGURE 4.4. Examples of local but nonglobal maxima

[BG2021]

§ 5. Number of shortest saddle connections

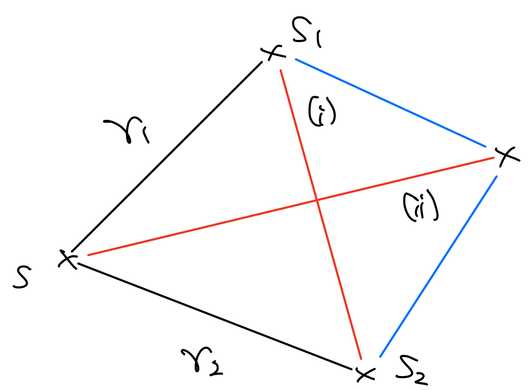
Proposition 5.1. — *The greatest number of shortest saddle connections of a surface in $\mathcal{H}(k_1, \ldots, k_r)$ is equal to $\sum_{i=1}^{r} 3(k_i+1)$, and this number is realized if and only if the surface is a global maximum for the function* Sys *in* $\mathbb{P}\mathcal{H}(k_1, \ldots, k_r)$.

[BG2021]

proof) Let $S \in \mathcal{H}(k_1, \cdots, k_r)$.

We consider two shortest saddle-conns. γ_1, γ_2 starting at the same singularity.

Assume $\text{angle}(\gamma_1, \gamma_2) < \frac{\pi}{3}$ then :



(i) $\exists$ SC $s_1 s_2$ $\Rightarrow$ $|s_1 s_2| < |\gamma_1| = |\gamma_2|$

$\Rightarrow$ contradiction

(ii) $\exists$ SC γ between γ_1, γ_2

$\Rightarrow$ As the triangulation is Delaunay,

$|\gamma| < |\gamma_1| = |\gamma_2|$

$\Rightarrow$ Contradiction

Thus $\text{angle}(\gamma_1, \gamma_2) \geq \frac{\pi}{3}$ holds.

So the maximal number of shortest saddle-connections starting at sing. of order k_i is $6(k_i+1)$

$$\#\{\text{shortest sc}\} \geq \sum_{i=1}^{r} \frac{6(k_i+1)}{2} = \sum_{i=1}^{r} 3(k_i+1).$$

equality $\Leftrightarrow$ $\forall$ Delaunay triangle is equilateral $\otimes$.

Go back to following mention in § 1-2

related Q.. the maximal number of shortest saddle-conns (realizing systole) ?

Reference

[BG2021] C. Boissy and S. Geninska, Systoles in translation surfaces. Bulletin de la Société Mathématique de France, **149**, 417–438 (2021)