(contra.)

@ Spec : spectrum functor (comm. rings) $\to$ (affine schemes)

$\quad {}^{\forall}R$: comm. ring, $\mathrm{Spec}(R) = \{ I \subset R : \text{prime ideal} \}$

w/ a sys. of closed nbd : $\{ V_I = \{ I' \subset R : \text{prim ideal containing } I \} \mid I \in \mathrm{Spec}\, R \}$

$(\{ D_I := V_I^c \}$: open base $)$ $\leftarrow$ Zariski Topology

For given R : comm. ring,

$\quad {}^{\forall} I \in \mathrm{Spec}(R)$, $\Gamma(D_I, \mathcal{O}_X) := I^{-1}R$: localization

$$I^{-1}R = \{ (i, r) \in I \times R \} / [(i, r) \sim (ij, rj) \mid j \in I]$$

This defines a structure sheaf $\mathcal{O}_X$

s.t. $\Gamma(D, \mathcal{O}_X) = \left(\bigcap_{p \in D} p^c \right)^{-1} R \quad {}^{\forall} D \subset \mathrm{Spec}(R)$: open

Df Let K : field

A K-scheme is a pair (S, p) $\left\{ \begin{array}{l} S : \text{scheme} \\ p : S \to \mathrm{Spec}(K) \ \text{: structure morphism} \end{array} \right.$

locally morph. of Spec

$\left\{ \begin{array}{l} f : A \to B \text{ : hom} \\ \mathrm{Spec}\, A \to \mathrm{Spec}\, B : p \mapsto f(p) \\ \mathcal{O}_{\mathrm{Spec}A} \to \mathcal{O}_{\mathrm{Spec}B} : D^{-1}R \mapsto f(D)^{-1}S \end{array} \right.$

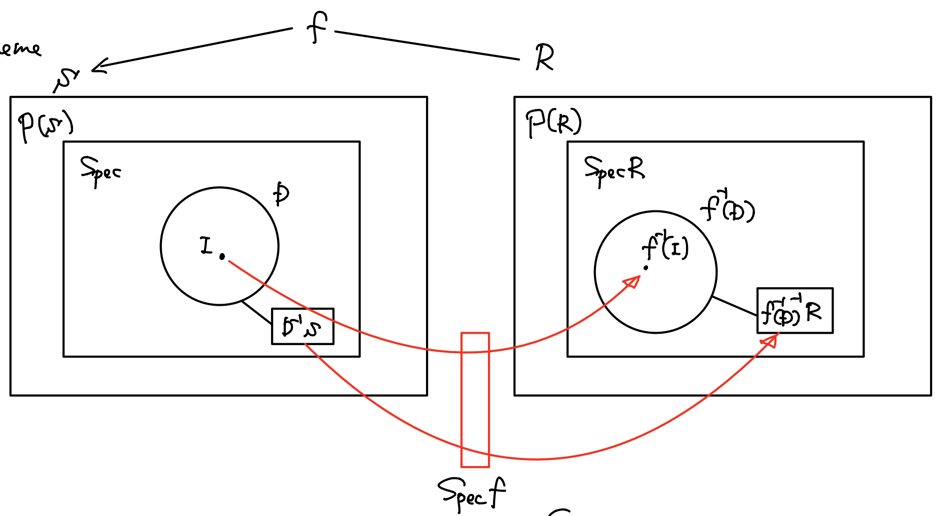
A K-variety is a K-scheme (S, p) s.t. S : reduced & p : separated mor. of finite type.

Def ${}^{\forall}\sigma \in \mathrm{Aut}\, K$, ${}^{\forall}(S, p)$: K-scheme

$$\begin{tikzcd} S \arrow[r, "p"] \arrow[rr, bend right, "p^\sigma"'] & \mathrm{Spec}(K) \arrow[r, "\mathrm{Spec}(\sigma)"] & \mathrm{Spec}(K) \end{tikzcd} \quad \circlearrowleft$$

$(S, p)^\sigma := (S, p^\sigma) = (S, \mathrm{Spec}(\sigma) \circ p)$

This defines $\boxed{\begin{array}{l} \mathrm{Aut}\, K \curvearrowright (\text{Schemes over } K) \\ \qquad\qquad\qquad \vee \\ \qquad \curvearrowright (K\text{-Varieties}) \end{array}}$



Spec is a contravariant functor

Df 3.9 Let (S, p) : K-scheme (var)

sub field $k < K$ is called a field of definition of (S, p)

$\left\{ \begin{array}{l} \text{if : } {}^{\exists}(S', p') : k\text{-scheme (var.) s.t. } \boxed{\begin{tikzcd} S \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & S' \arrow[d, "p'"] \\ \mathrm{Spec}(K) \arrow[r, "\mathrm{Spec}\, i"'] & \mathrm{Spec}(k) \end{tikzcd}} \quad (i : k \overset{\text{incl.}}{\hookrightarrow} K) \\ \text{we say} \\ \Longleftrightarrow (S, p) \text{ is defined over } k. \end{array} \right.$

We define $\mathcal{U}(S,p) \le \mathrm{Aut}(K)$ by

$$\mathcal{U}(S,p) := \{\sigma \in \mathrm{Aut}(K) \mid (S,p)^\sigma \cong (S,p)\}$$

The moduli field $\mathcal{M}(S,p) := \mathrm{Stab}_K\, \mathcal{U}(S,p)$ (fixed field)

Def 3.10 $(S,p_S),(T,p_T)$: K-scheme (var.) $\beta : (S,p_S) \to (T,p_T)$: morphism

$$\begin{tikzcd}
S \arrow[dr, "p_S"] \arrow[drr, bend left, "p_S^\sigma"] \arrow[dd, "\beta"'] & & \\
& \mathrm{Spec}\,K \arrow[r, "\mathrm{Spec}\,\sigma"] & \mathrm{Spec}\,K \\
T \arrow[ur, "p_T"'] \arrow[urr, bend right, "p_T^\sigma"'] & &
\end{tikzcd}$$

$\rightsquigarrow$ morphism $\beta^\sigma : (S,p_S)^\sigma \to (T,p_T)^\sigma$ is naturally induced.

$(\beta, \beta^\sigma$: same map as $S \to T)$

Let $\beta' : (S',p_{S'}) \to (T',p_{T'})$: another K-scheme morph.

We write $\beta \cong \beta'$ if $\exists \phi : S \xrightarrow{\cong} S'$ $\exists \psi : T \xrightarrow{\cong} T'$

s.t. $\psi \circ \beta = \beta \circ \phi$

Specially, $\beta \cong \beta^\sigma$ if :

$$\begin{tikzcd}
S^\sigma \arrow[r, "\phi"] \arrow[d, "\beta^\sigma"'] & S \arrow[d, "\beta"] \\
T^\sigma \arrow[r, "\psi"] \arrow[d, "p"'] & T \arrow[d, "p"] \\
\mathrm{Spec}\,K \arrow[r, "\mathrm{Spec}\,\sigma"'] & \mathrm{Spec}\,K
\end{tikzcd}$$

Def 3.11 A morph. $S \xrightarrow{\beta} T$ is called defined over $k < K$

if : $\exists S' \xrightarrow{\beta'} T'$ morph s.t.

$$\begin{tikzcd}
S \arrow[r, "\exists\phi"] \arrow[d, "\beta"'] & S' \arrow[d, "\beta'"] & \\
T \arrow[r, "\exists\psi"] \arrow[d, "p"'] & T' \arrow[d] \arrow[dr, "p'"] & \\
\mathrm{Spec}\,K \arrow[r] & \mathrm{Spec}\,K & \mathrm{Spec}\,k \arrow[l, "\mathrm{Spec}\,i"]
\end{tikzcd}$$

denoted $\beta' \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,K$.

We define $\mathcal{U}(\beta) \le \mathrm{Aut}(K)$ by

$$\mathcal{U}(\beta) := \{\sigma \in \mathrm{Aut}(K) \mid \beta^\sigma \cong \beta\}$$

The moduli field $\mathcal{M}(\beta) := \mathrm{Stab}_K\, \mathcal{U}(\beta)$ (fixed field)

Def 3.12 Let $\beta: X \to \mathbb{P}_{\mathbb{C}}$: Belyi morph.

Let $\mathcal{U}^{\mathbb{P}}_{\beta} < \mathrm{Aut}\,\mathbb{C}$: group of $\sigma \in \mathrm{Aut}\,\mathbb{C}$ such that

$\exists f_\sigma : X^\sigma \to X$: $\mathbb{C}$-isomorphism

s.t.

$$\begin{tikzcd} X^\sigma \arrow[r, "f_\sigma"] \arrow[d, "\beta^\sigma"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\beta"] \\ \mathbb{P}_{\mathbb{C}}^\sigma \arrow[r, "\mathrm{proj}\,\sigma"'] & \mathbb{P}_{\mathbb{C}} \end{tikzcd}$$

where $\mathrm{proj}\,\sigma : \mathbb{P}_{\mathbb{C}} \to \mathbb{P}_{\mathbb{C}}$, $[x_0, x_1] \mapsto [\sigma(x_0), \sigma(x_1)]$

Then we call $\mathcal{M}^{\mathbb{P}}_{\beta} := \mathrm{Stab}_{\mathbb{C}}\,\mathcal{U}^{\mathbb{P}}_{\beta}$ (fixed field)

the moduli field of the dessin $\sim \beta$.

Rem Difference to Def 3.11 (b)

In Def 3.11, $\mathcal{U}_\beta = \{\sigma \in \mathrm{Aut}\,\mathbb{C} \mid \beta \cong \beta^\sigma\}$

$\Updownarrow$

"ϕ" admits composing $\leftarrow$

$$\begin{tikzcd} \circ \arrow[r, "\exists\phi"] \arrow[d, "\beta"'] & \circ \arrow[d, "\beta^\sigma"] \\ \circ \arrow[r, "\exists\psi"'] & \circ \end{tikzcd}$$

$\mathrm{Proj}\,\sigma$ w/ $\psi' \in \mathrm{Aut}\,\mathbb{P}^1_{\mathbb{C}}$

$(\psi = \psi' \circ \mathrm{proj}\,\sigma)$

i.e. $\mathcal{U}^{\mathbb{P}}_{\beta}$ is the "$\cancel{\exists\psi} \to \psi = \mathrm{proj}\,\sigma$" ver of $\mathcal{U}_\beta$.

$\rightsquigarrow \mathcal{U}^{\mathbb{P}}_{\beta} < \mathcal{U}_\beta$ & $\mathcal{M}^{\mathbb{P}}_{\beta} > \mathcal{M}_\beta$.

Prop 3.13 K: field. $\beta: S \to T$: morphism of K-schemes (or var.)

a) $\mathcal{M}(S)$ / $\mathcal{M}(\beta)$ depends only on the K-isom type of S / β.

b) $\mathcal{M}^{\mathbb{P}}(\beta)$ 〃 β : Belyi.

c) Every field of definition of S / β contains $\mathcal{M}(S)$ / $\mathcal{M}(\beta)$.

d) $\mathcal{M}(S) \subseteq \mathcal{M}(\beta)$

e) for β: Belyi $\mathcal{M}(\beta) \subset \mathcal{M}^{\mathbb{P}}(\beta)$

f) and $\mathcal{M}^{\mathbb{P}}(\beta)$ is a number field

$\Downarrow$

$\mathcal{M}_\beta$ & $\mathcal{M}_S$. ... finite extension of $\mathbb{Q}$.

Thm 1 (Belyi) A smooth proj. curve C is definable over a number field iff it admits a Belyi morph. $\beta: C \to \mathbb{P}_{\mathbb{C}}$.

Thm 2 (Hammer, Herrlich 2003) X: curve over a field K

$\Rightarrow X$ is defined over a finite extension of $\mathcal{M}(X)$.

3.? $G_{\bar{\mathbb{Q}}} \curvearrowright$ (dessins)

Given a Belyi morphism $\beta : X \to P_{\mathbb{C}}$ & $\sigma \in \mathrm{Aut}\,\mathbb{C}$,

$\beta^\sigma : X^\sigma \to P_{\mathbb{C}}$ is again a Belyi morphism.

as "# branch pts ≤ 3" is intrinsic for the underlying scheme morphism β $\rightarrow$ NOT changed by σ

X $\quad$ X^σ

$$\begin{tikzcd}
X \arrow[rr, "\exists"] \arrow[d, "\beta"'] \arrow[r] & \mathrm{Spec}\,\mathbb{C} \arrow[r] \arrow[d, "\mathrm{Spec}\,\sigma"] & X' \arrow[d] \\
P_{\mathbb{C}} \arrow[ur] \arrow[r] & \mathrm{Spec}\,\mathbb{C} \arrow[r, "\mathrm{Spec}\,i"'] & \mathrm{Spec}\,\bar{\mathbb{Q}}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Spec}\,\mathbb{C} \arrow[r] & \mathrm{Spec}\,\bar{\mathbb{Q}} \arrow[d, "\mathrm{Spec}\,\sigma"] \\
\mathrm{Spec}\,\mathbb{C} \arrow[r, "\mathrm{Spec}\,i"'] & \mathrm{Spec}\,\bar{\mathbb{Q}}
\end{tikzcd}$$

So $\mathrm{Aut}\,\mathbb{C}$ acts on set of Belyi morphs, equivalently on the set of dessins.

By Belyi thm, this action factors through $\mathrm{Aut}\,\bar{\mathbb{Q}} = G_{\bar{\mathbb{Q}}}$.

Thm 3. For every $g \in \mathbb{N}$,

$G_{\bar{\mathbb{Q}}} \curvearrowright$ (dessins of genus g) is faithful.

$\hookrightarrow$ (clean dessins of genus g) is still faithful.

(unicellular —) { black vertices are "—•—"

In genus 1, this can be seen as the action is compatible w/ the j-invariant

i.e. $\sigma(j(E)) = j(E^\sigma)$ (E: ell. curve)

(& as j being a hol. fct $\mathbb{H}/SL(2,\mathbb{Z}) \to \mathbb{C}$)

(Armknecht, 2001) This generalizes to higher genera,

by even restricting to hyperelliptic curves only. $\to g \geq 2$. ok.

(Schneps. 1994) Sed pf for $g=0$

Rem: $G_{\bar{\mathbb{Q}}}$-action-faithfulness does not break

when restricting to unicellular / clean dessins (cf. [Nisbach 2011])

THE GALOIS ACTION ON M-ORIGAMIS AND THEIR TEICHMÜLLER CURVES

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September 10, 2018

[N11]

Abstract. We consider a rather special class of translation surfaces (called *M-Origamis* in this work) that are obtained from dessins by a construction introduced by Möller in [Möl05]. We give a new proof with a more combinatorial flavour of Möller's theorem that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the Teichmüller curves of M-Origamis and extend his result by investigating the Galois action in greater detail.

We determine the Strebel directions and corresponding cylinder decompositions of an M-Origami, as well as its Veech group, which contains the modular group $\Gamma(2)$ and is closely connected to a certain group of symmetries of the underlying dessin. Finally, our calculations allow us to give explicit examples of Galois orbits of M-Origamis and their Teichmüller curves.

found at arXiv: 1408.6769v1

Similar contents are found as the author's PhD thesis.

Intro: Back ground

- Serre's GAGA (1956) states a close relationship (correspondence) b/w Algebraic Geometry and (Complex) Analytic Geometry.

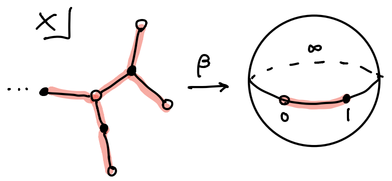
e.g. "projective curve" $\longleftrightarrow$ "cpt Riemann surf."

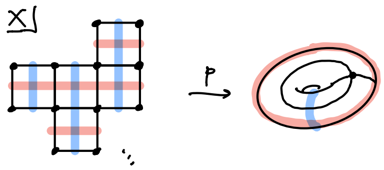
$P^1_{\mathbb{C}} \longleftrightarrow \hat{\mathbb{C}}$

elliptic curve $y^2 = x^3 - x \longleftrightarrow$ torus $\mathbb{C}/\mathbb{Z}^2$

etc...

Belyi cover	origami cover
$\beta : X \to P_{\mathbb{C}}$	$p : X \to E$
covering of $P_{\mathbb{C}} = \hat{\mathbb{C}}$. ramified over $\{0, 1, \infty\}$	Covering of $E = \mathbb{C}/\mathbb{Z}^2$, ramified over $\{0 + \mathbb{Z}^2\}$
Graph structure induced from β	Graph structure induced from p





← study of $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ - action
(= Aut $\overline{\mathbb{Q}}$)
field auto

$\sigma \in \mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$

↻ {alg. proj. curve}

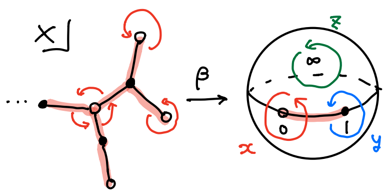
e.g.

$y = a_0 + a_1 x + a_2 x^2 + \cdots$

$\downarrow \sigma$

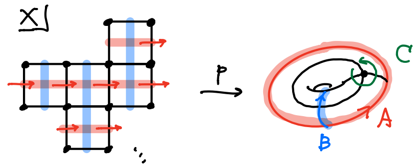
$y = \sigma(a_0) + \sigma(a_1) x + \sigma(a_2) x^2 + \cdots$

⇝ "dessin" | ⇝ "origami"



monodromy group : $G = \langle p_x, p_y \rangle < G_d$

$(p_z = p_x^{-1} p_y^{-1})$



monodromy group : $G = \langle p_A, p_B \rangle < G_d$

$(p_C = p_A p_B p_A^{-1} p_B^{-1})$

local ramification behavior is an invariant under the Gal $\bar{\mathbb{Q}}/\mathbb{Q}$-action ($=$ Aut $\bar{\mathbb{Q}}$)
field auto.

(Möller) Gal $\bar{\mathbb{Q}}/\mathbb{Q}$-action on origamis is induced from the one on dessins of origami curves.

Thm 1 (Belyi, 1979)

A smooth proj. curve C is definable over a number field iff it admits a Belyi morph. $\beta : C \to \mathbb{P}_{\mathbb{C}}$.

Thm 2 (Hammer, Herrlich 2003) X : curve over a field K

$\Rightarrow$ X is defined over a finite extension of $M(X)$.

Thm 3. For every $g \in \mathbb{N}$,

$G_{\mathbb{Q}} \curvearrowright$ (dessins of genus g) is faithful.

$\hookrightarrow$ (clean dessins of genus g) is still faithful.

$\subset$ (unicellular —) { $\forall$ black vertices are "—•—" }

In genus 1, this can be seen as the action is compatible w/ the j-invariant

i.e. $\sigma(j(E)) = j(E^{\sigma})$ (E : ell. curve)

(& as j being a hol. fct $\mathbb{H}/SL(2,\mathbb{Z}) \to \mathbb{C}$)

(Armknecht, 2001) This generalizes to higher genera,

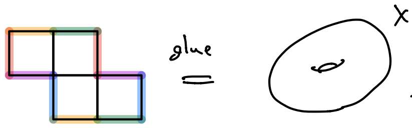
by even restricting to hyperelliptic curves only. $\to g \geq 2$. cf.

(Schneps, 1994) Std pf for $g = 0$

4. Origamis & their Teichmüller curves

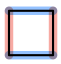
(Short Summary)

origami (square-tiled surface)

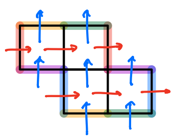
(exm) 

torus tiled by 4 squares

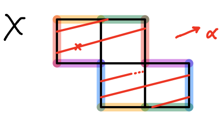
cov. $\rho_X \downarrow$

 … std torus $\mathbb{R}^2/\mathbb{Z}^2$ $\rightarrow$ Natural translation structure is induced.

ρ_X corresponds to its monodromy $h, v \in G_{\deg \rho_X}$


h : horizontal
v : vertical
} monodromy (parmutation of squares)

On a given origami X, $\alpha \in \mathbb{R}$ defines a linear flow in slope α

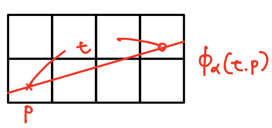


$\phi_\alpha : \mathbb{R} \times X \to X$

$e_\alpha = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$

w/ $\rho_X(\phi_\alpha(t, p)) = \rho_X(p) + t e_\alpha \mod \mathbb{Z}^2$

dev $\downarrow$ (ρ_X & lift to $\mathbb{R}^2$)



※ flow stops for singular leaves : trajectories hitting $0 \in \mathbb{R}^2/\mathbb{Z}^2$

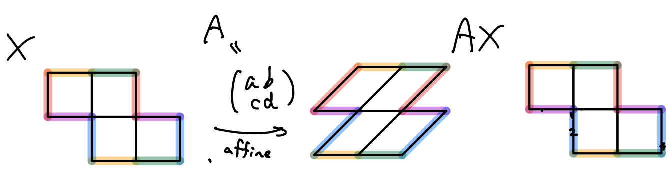
$\Gamma(\mathcal{O})$ an origami $\mathcal{O}$

∘ Veech group $\Gamma(X)$ of a translation surface X

is the isom-class stabilizer of $\overset{\mathcal{O}}{X}$ in $SL(2,\mathbb{Z})$ acting on $\Omega T(X)$

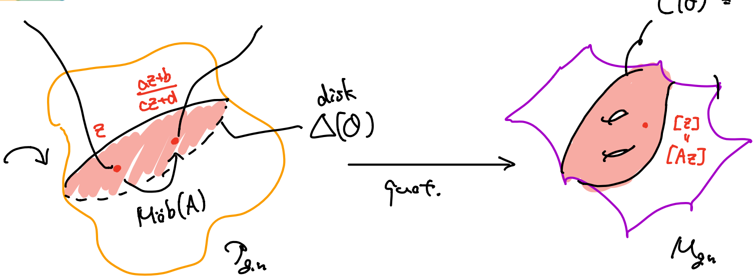
stratum of translation surfaces whose analytic type = one of X.

as the natur. affine-deformation



$C(\mathcal{O}) = \Delta(\mathcal{O})/\Gamma(\mathcal{O})$: embedded alg. curve in $M_{g,n}$

w.r.t. Teich. metric



$\rightsquigarrow$ called the Teichmüller curve (origami curve)

$SL_2(\mathbb{R})$
∨
$SL_2(\mathbb{Z})$
∨
$\Gamma(\mathcal{O})$

"square-tiled"-preserving

described as :

$$\boxed{\begin{array}{ccccc} \mathbb{H} & \hookrightarrow & \Delta(\mathcal{O}) & \hookrightarrow & T_{g,n} \\ \downarrow & & \downarrow \text{proj} & & \downarrow \text{proj} \\ \mathbb{H}/R\Gamma(\mathcal{O})R^{-1} & \overset{j}{\xrightarrow{\sim}} & C(\mathcal{O}) & \hookrightarrow & M_{g,n} \end{array}}$$

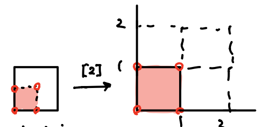
$\left(R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$ mirror reflection

※ j gives the canonical model of Teich. curve.

5. Galois action on $\mathcal{M}$-origamis

E : the unit square torus $\mathbb{C}/\mathbb{Z}^2$: Belyi curve ($y^2 = x^3 - x$, $\beta(x) = x^2$)

$[2] : E \to E$ multiplication by 2 (4 : 1-map)



$0, \frac{1}{2}, \frac{i}{2}, \frac{1+i}{2} + \mathbb{Z}^2 \mapsto 0 + \mathbb{Z}^2$

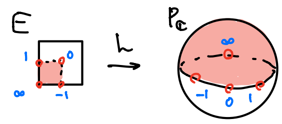
$h : E \to \mathbb{P}_{\mathbb{C}}$: quotient map under the involution $z + \mathbb{Z}^2 \mapsto -z + \mathbb{Z}^2$

taken s.t. $h(\infty) = \infty$ & the other three crit. values are $0, 1, \lambda$ $(-1?)$

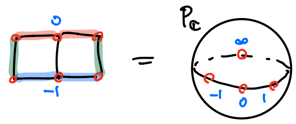
i.e. h is of the form $(\mathbb{P}_{\mathbb{C}}\text{-auto.}) \circ \wp$-function of lattice $\mathbb{Z}^2$

Through h, we name the 4 Weierstrass points on E

by $0, \frac{1}{2}, \frac{i}{2}, \frac{1+i}{2} + \mathbb{Z}^2 =: \infty, \lambda, 1, 0$



Let $\gamma : Y \to \mathbb{P}_{\mathbb{C}}$: pillow case covering



that is, Y : non-singular, conn., proj. curve

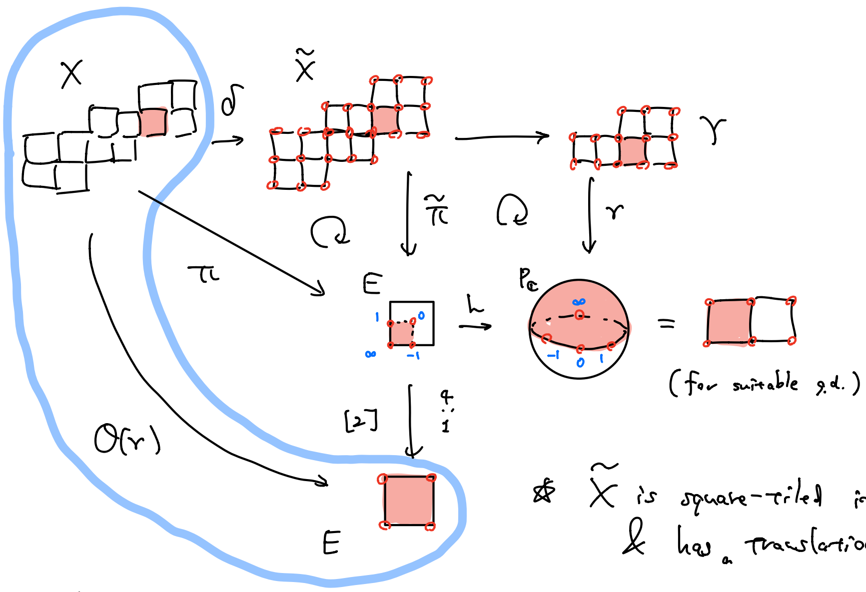
γ : non-const morphism w/ $\mathrm{Crit}\,\gamma \supset \{0, 1, \infty, \lambda\}$

Let $\tilde{\pi} : \tilde{X} = E \times_{\mathbb{P}_{\mathbb{C}}} Y \to E$: pullback

$\{(e, y) \in E \times Y \mid h(e) = \gamma(y)\}$

$$\begin{tikzcd} E \times Y \supset E \times_{\mathbb{P}_{\mathbb{C}}} Y \arrow[r, "\text{proj.}"] \arrow[d, "\text{proj}"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "\gamma"] \\ E \arrow[r, "h"'] & \mathbb{P}_{\mathbb{C}} \end{tikzcd}$$

$\delta : X \to \tilde{X}$; $\pi : X \to E$: desingularization of $\tilde{X}$, $\tilde{\pi} : \tilde{X} \to E$



for each non-singular pt $y \in Y$, there are just two pts $\pm e \in E$ s.t. $h(\pm e) = \gamma(y)$

So $Y \times_{\mathbb{P}_{\mathbb{C}}} E$ is a sort of double of Y.

* $\tilde{X}$ is square-tiled in non-abelian way.

& has a translation str. naturally induced from E.

We consider the map $[2] \circ \pi : X \to E$.

By the construction, $\tilde{\pi} \circ \pi$ is ramified at most over $\{0, 1, \infty, \lambda\}$.

they are mapped to $\infty \in E$ by $[2]$, thus this defines an origami.

Def 5.1 (a) We call $\mathcal{O}(\gamma) := [2] \circ \pi : X \to E$ the pillow case origami associated to the pillow case covering $\gamma : Y \to P_{\mathbb{C}}$.

(b) If furthermore $\beta := \gamma$ is unramified over λ i.e. Belyi, we call $\mathcal{O}(\beta)$ the $\mathcal{M}$-origami associated to γ.

@ In a topological view point: Replacing algebro-geometric fibre product $(Y \times_{P_{\mathbb{C}}} E)$ by topological one, we obtain some important results : $\left\{ \begin{array}{l} X^* \cong E^* \times_{P_{\mathbb{C}}^*} Y^* \text{ as coverings} \\ \text{description of monodromy of } \mathcal{O}(\beta). \end{array} \right.$ etc.

- The monodromy of $\mathcal{M}$-origami.

Proposition 2.2. Let X be a coverable space, $f : A \to X$, $g : B \to X$ coverings of degree d and d', respectively, with given monodromy maps m_f resp. m_g. Then, we have for the fibre product $A \times_X B$: [N11]

a) For each path-wise connected component $A_i \subseteq A$, the restriction

$$p_{A|p_A^{-1}(A_i)} : p_A^{-1}(A_i) = A_i \times_X B \to A_i$$

$$m_g \circ (f_{|A_i})_*.$$

b) The map $f \circ p_A = g \circ p_B : A \times_X B \to X$ is a covering of degree $\underset{\deg f \times \deg g}{dd'}$ with monodromy

$$m_f \times m_g : \pi_1(X, x_0) \to S_d \times S_{d'} \subseteq S_{dd'}, \gamma \mapsto \Big((k, l) \mapsto (m_f(k), m_g(l))\Big),$$

where $(k, l) \in \{1, \ldots, d\} \times \{1, \ldots, d'\}$.

Proposition 2.3. In the situation described above, let γ_i, $i = 1, \ldots, d$, be right coset representatives of Γ_1 in Γ, with $\gamma_1 = 1$, such that $e^f_{y_i}(\gamma_i) = y_1$. So, we have $\Gamma = \dot\bigcup \Gamma_1 \cdot \gamma_i$.
Then, we have:

$$m_{f \circ g}(\gamma)(i, j) = \Big(m_f(\gamma)(i), m_g(c_i(\gamma))(j)\Big)$$

Here, we denote $c_i(\gamma) := (f_*)^{-1}(\gamma_k \gamma \gamma_i^{-1})$, $k := m_f(\gamma)(i)$

Theorem 8 Let $\beta : Y \to P_{\mathbb{C}}$: Belyi cover of degree d. ($P_{\mathbb{C}}$: pillow case)

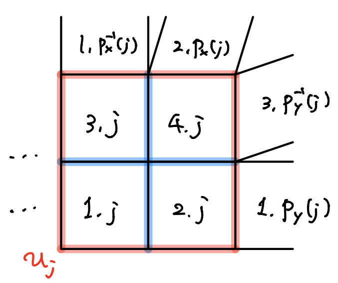
$p_x := m_\beta(x)$, $p_y := m_\beta(y)$ ... monodromy generators.

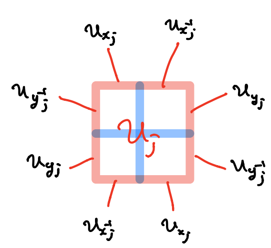
Then the monodromy of the $\mathcal{M}$-origami $\mathcal{O}(\beta) = [2] \circ \pi : X \to E$ is given by $m_{\mathcal{O}(\beta)} : \pi_1(E^*) \to S_{4d}$

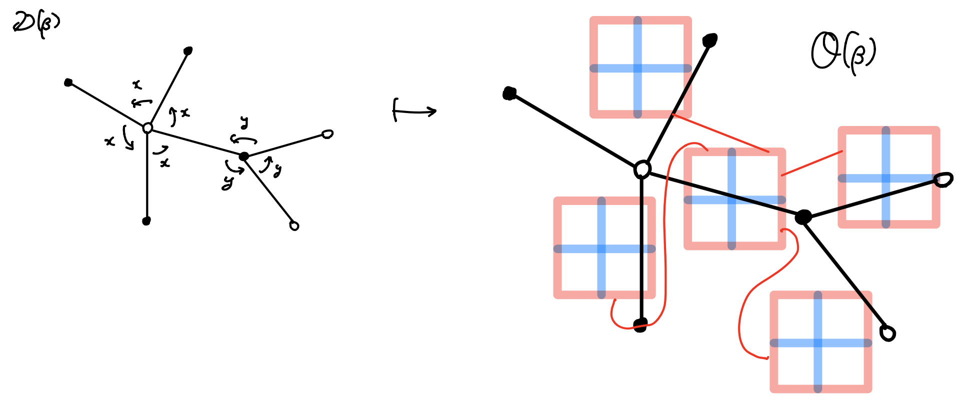
where

$$m_{\mathcal{O}(\beta)}(A) = \left\{ \begin{array}{l} (1,j) \mapsto (2,j) \\ (2,j) \mapsto (1, p_y(j)) \\ (3,j) \mapsto (4,j) \\ (4,j) \mapsto (3, p_y^{-1}(j)) \end{array} \right. \qquad m_{\mathcal{O}(\beta)}(B) = \left\{ \begin{array}{l} (1,j) \mapsto (3,j) \\ (2,j) \mapsto (4,j) \\ (3,j) \mapsto (1, p_x^{-1}(j)) \\ (4,j) \mapsto (2, p_x(j)) \end{array} \right.$$

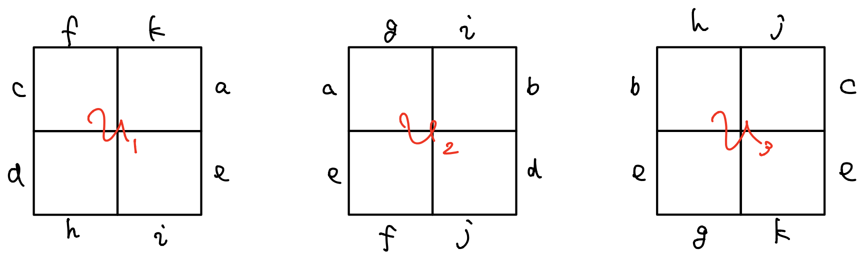
i.e.







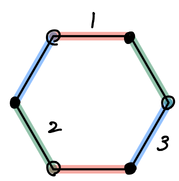
ex the Ornithorynque origami



is the M-origami associated to the dessin

$\langle x, y \rangle = \langle (1\,2\,3), (1\,2\,3) \rangle < G_3$

E]

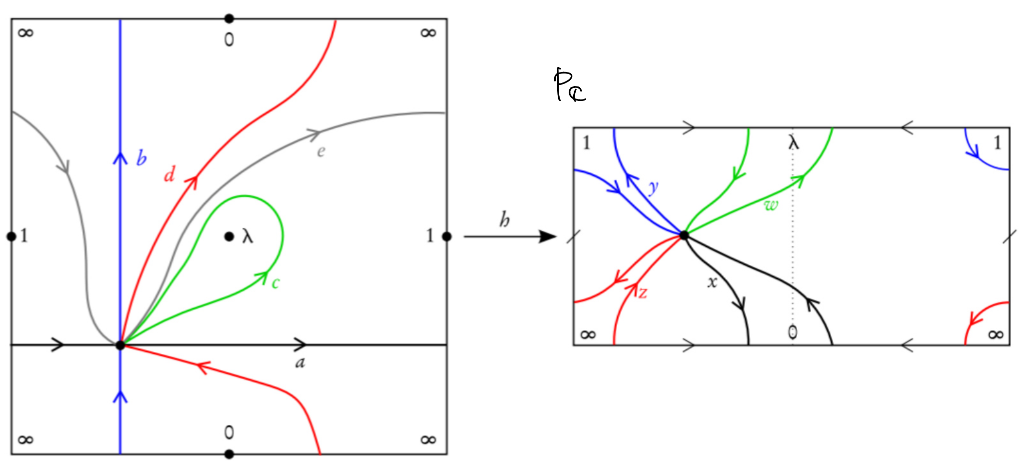


on the $e^{\frac{\pi}{3}i}$-torus

$y^2 = x^3 - 1$

E

Pc



[N11]

reffering this figure

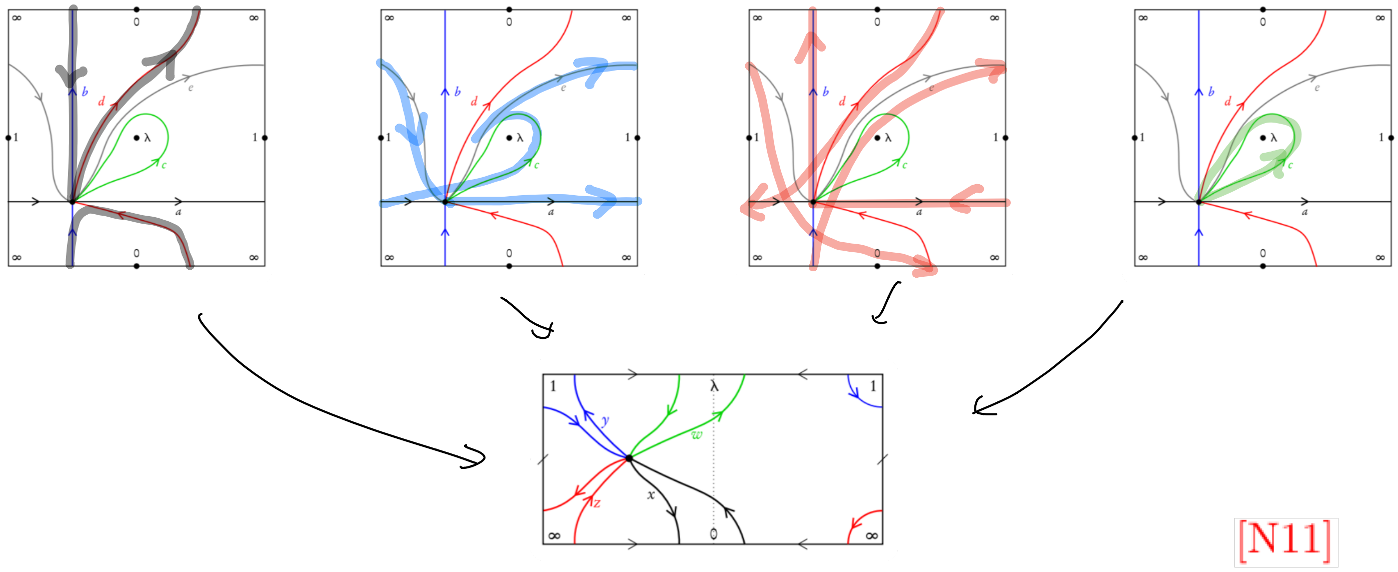
@ monodromy around the Weierstrass points.

Lemma 5.5. If we choose the following simple loops around the Weierstraß points

$$x' := l_0 := db^{-1} \qquad y' := l_1 := ac^{-1}e^{-1}$$
$$z' := l_\infty := bed^{-1}a^{-1} \qquad w' := l_\lambda := c,$$

then we have

$$h_*(x') = x^2 \qquad h_*(y') = y^2$$
$$h_*(z') = z^2 \qquad h_*(w') = w^2.$$



◎ analytic type of M-origami's.

__Prop 5.6__ Let $\beta : Y \to \mathbb{P}_{\mathbb{C}}$: Belyi cov. of degree d

For $z = 0, 1, \infty$, let $m_z :=$ the monodromy arr. $z \in G_{4d}$

$g_z := \#($cycles of even length in $m_z)$

Then we have (a) $g(X) = g(Y) + d - \frac{1}{2}\sum g_z$

(b) $g(Y) + \lceil \frac{d}{4} \rceil \leq g(X) \leq g(Y) + d$

(pf : from Riemann-Hurwitz for β & π.)

__Rem 5.9__ a) Let $n(p) := \#\pi^{-1}(p)$ for $p \in W := \{0, 1, \lambda, \infty\}$: Weierstraß pt

Then $n(p) = \begin{cases} \#(\text{cycles in } m_p) + g_p & (p \neq \lambda) \\ \deg \pi = \deg \beta \end{cases}$

b) For an M-origami $O(\beta)$, $\# p^{-1}(\infty) = \sum_{p \in W} n(p)$.

◎ Veech group of M-origami's

__Prop 5.10__ β : Belyi morphism. of degree d ~ dessin $\langle p_x, p_y \rangle < G_d$

$O_\beta = O(p_x, p_y) := O(\beta)$: associated M-origami

Then, $\begin{cases} \bullet\ S \cdot O_\beta = O(p_y, p_x) \ : \text{M-origami} \\ \bullet\ T \cdot O_\beta = O(p_z, p_y) \ : \ \text{''} \qquad \text{where } p_z = p_x^{-1} p_y^{-1} \\ \bullet\ -I \cdot O_\beta = O_\beta . \end{cases}$

It will turn out in the next thm that the VG of M-origamis often is $\Gamma(2)$.

Thm 9 Let β : Belyi $\sim \langle p_x, p_y \rangle$: dessin.

(a) $\Gamma(2) \subset \Gamma(\mathcal{O}_\beta)$ in particular, $\Gamma(\mathcal{O}_\beta)$: congruence level ≥ 2

(b) $SL(2,\mathbb{Z}) \cdot \mathcal{O}_\beta = \left\{ \mathcal{O}(p_x', p_y') : M\text{-origami} \;\middle|\; \langle p_x, p_y \rangle \overset{\text{weak}}{\sim} \langle p_x', p_y' \rangle \right\}$

where $\langle p_x, p_y \rangle \overset{\text{weak}}{\sim} \langle p_x', p_y' \rangle \overset{\text{def}}{\Longleftrightarrow}$
$$\begin{array}{ccc} X & \xrightarrow[\text{biholo}]{\exists \varphi} & X' \\ \beta \downarrow & \circlearrowleft & \downarrow \beta' \\ \mathbb{P}_\mathbb{C} & \xrightarrow[\exists \psi]{\text{auto}} & \mathbb{P}_\mathbb{C} \end{array}$$
← i.e. biholomorphically equivalent up to permutations of $\{0,1,\infty\}$

$\wr$ $\beta : X \to \mathbb{P}_\mathbb{C}$ $\qquad$ $\wr$ $\beta' : X' \to \mathbb{P}_\mathbb{C}$

ψ permutates $\{0,1,\infty\}$ as

id					
0	1	∞	1	∞	0
1	0	1	∞	0	∞
∞	∞	0	0	1	1

} 6 patterns.

(c) If $\Gamma(2) = \Gamma(\mathcal{O}_\beta)$, then

β has no non-trivial weak isomorphism.

In the case β : filthy ($\Leftrightarrow$ neither p_x^2, p_y^2, nor $p_z^2 = 1$)

the converse is true.

pf) (a) is proved by the fact that $\alpha \in \{I, S, T, ST, TS, TST\}$ maps $\mathcal{O}_\beta$ in the following way:

I	S	T	ST	TS	TST
p_x	p_y	p_z	p_y	p_z	p_x
p_y	p_x	p_y	p_z	p_x	p_z

← images are the M-Origamis ass. to those dessins.

... that is permutation.

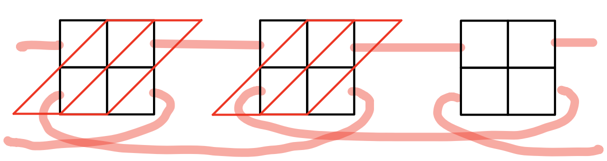
thus we may see $T^2 \mathcal{O}(p_x, p_y) = S T^{-2} S^{-1} \mathcal{O}(p_x, p_y) = \mathcal{O}(p_x, p_y)$

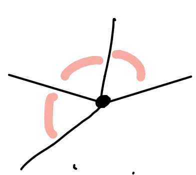
⇝ w/ prop 5.19, we see $\Gamma(\mathcal{O}) \supset \langle T^2, S T^{-2} S, -I \rangle = \Gamma(2)$.

((b) might be too strong?

$SL(2,\mathbb{Z}) \cdot \mathcal{O}_\beta \not\simeq \{$ weakly-isom $\}$

if (b) holds: $[SL(2,\mathbb{Z}) : \Gamma(\mathcal{O}_\beta)] \leq 6$: too strong?) ?





each cylinder on $\mathcal{O}_\beta \sim$ cycle in p_x, p_y

$S, T \in SL(2,\mathbb{Z})$ acts on $\{M\text{-Origamis}\}$

as a modification of cylinders $\sim$ dessins.

Prop 5.12 $W = \{ \varphi \in \mathrm{Aut}\, \mathbb{P}^1_{\mathbb{C}} \mid \varphi(\{0,1,\infty\}) = \{0,1,\infty\} \}$

acts on the (set of origamis w/ VG-congruence level ≥ 2)

via the grp. isom. $\boxed{\phi : W \to SL(2,\mathbb{Z})/\Gamma(2) \quad \begin{cases} (z \mapsto 1-z) \mapsto S\Gamma(2) \\ (z \mapsto 1/z) \mapsto T\Gamma(2) \end{cases}}$

Furthermore, under this action,

the M-origami map $\mathcal{O} : (\text{dessins}) \to (\text{M-origamis})$ is W-equivariant.

$\beta \mapsto \mathcal{O}(\beta)$

i.e. $\mathcal{O}(1-\beta) = S\,\mathcal{O}(\beta)$

$\mathcal{O}(1/\beta) = T\,\mathcal{O}(\beta)$.

◦ Cylinder decomposition

By thm 9 $(\Gamma(\mathcal{O}_\beta) \supset \Gamma(2))$,

an M-origami curve $C(\mathcal{O}_\beta)$ is covered by $\mathbb{H}/\Gamma(2) \cong \mathbb{P}_{\mathbb{C}} \setminus \{0,1,\infty\}$: three cusps

In particular, it has at most three cusps

$\sim$ at most three non $\Gamma(\mathcal{O})$-equivalent cylinder directions.

... namely $\binom{1}{0}, \binom{0}{1}, \binom{1}{1}$

By calculating the cylinder decomposition in these directions,

we may observe the situation of cylinders lying on M-origamis.

from information of dessins!

Theorem 10. Let O_β be an M-Origami associated to a dessin β which is given by a pair of permutations (p_x, p_y). Then we have:

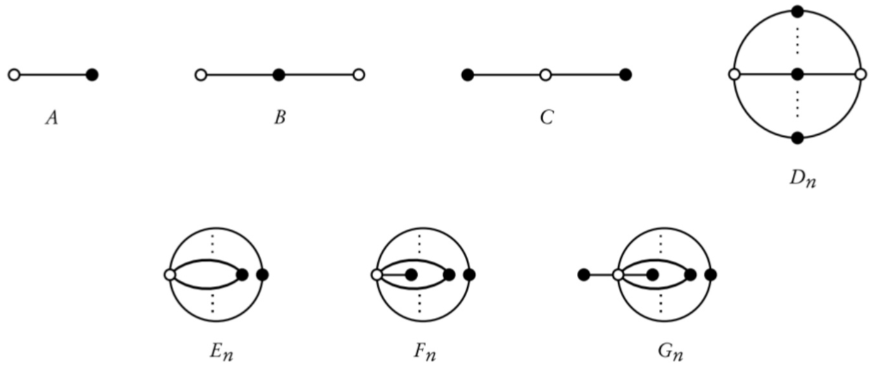
a) If (p_x, p_y) does not define one of the dessins listed in Lemma 5.13, then, in the Strebel direction $\binom{1}{0}$, O_β has:
   - for each fixed point of p_y one maximal cylinder of type $(2,2)$,
   - for each cycle of length 2 of p_y one maximal cylinder of type $(4,2)$,
   - for each cycle of length $l > 2$ of p_y two maximal cylinders of type $(2l, 1)$.

b) In particular, we have in this case:

$$\#\,\text{max. horizontal cylinders} = 2 \cdot \#\,\text{cycles in } p_y - \#\,\text{fixed points of } p_y^2.$$

c) We get the maximal cylinders of O_β in the Strebel directions $\binom{0}{1}$ and $\binom{1}{1}$ if we replace in a) the pair (p_x, p_y) by (p_y, p_x) and (p_y, p_z), respectively.

[N11]



[N11]

FIGURE 3.

Remark 5.14. a) The M-Origami coming from D_n has two maximal horizontal cylinders of type $(2n, 2)$ for $n \geq 3$ (and else one of type $(2n, 4)$).

b) The M-Origami coming from E_n has one maximal horizontal cylinder of type $(4n, 2)$.

c) The M-Origami coming from F_n has one maximal horizontal cylinder of type $(4n - 2, 2)$.

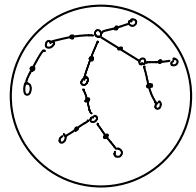
d) The M-Origami coming from G_n has one maximal horizontal cylinder of type $(4n - 4, 2)$.

@ Main part : Möller's theorem & variations.

We may reprove Möller's work in 2005 in an almost purely topological way.

<u>Thm 11</u> (reform ver. of Möller's)

(a) Let $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, β : Belyi map ass. to clean tree dessin.



i.e. a dessin of genus 0, totally ramified over ∞,

s.t. $\beta^{-1}(1)$ consists of order 2 ramif. pts.

and assume $\beta^\sigma \neq \beta$.

Then, we also have for the origami curve that $C(\mathcal{O}_\beta) \neq C(\mathcal{O}_\beta)^\sigma$. (in $M_{g,[n]}$)

(b) In particular, the Galois action on the set of origami curves is faithful.

<u>Lemma 5.15</u> Let $\beta \sim (p_x, p_y)$, $\beta' \sim (p_x', p_y')$ be two dessins.

If $\beta \not\cong \beta'$, then we also have $\pi \not\cong \pi'$. (in Def of M-origami)

(outline)

Assuming $\deg\beta = \deg\beta'$, $\pi \cong \pi' \Rightarrow \beta \cong \beta'$ will be shown in the same strategy as Thm 8

(dessin $\mapsto$ monodromy of M-origami)

post composing [2] ;

<u>Lemma 5.16</u> $\pi : X \to E$, $\pi' : X' \to E$: two coverings

Then $[2]\circ\pi \cong [2]\circ\pi' \Rightarrow \exists\varphi \in \mathrm{Deck}[2]$ $\varphi\circ\pi \cong \pi'$

<u>Prop 5.17</u> Assume we have a dessin $\beta \sim (p_x, p_y)$, and $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ s.t. $\beta^\sigma \not\cong \beta$.

If furthermore the 4-tuple $(p_x^2, p_y^2, p_z^2, 1) \in G_d^4$ contains a permutation whose cycle str. is distinct from others

(*)

then $\mathcal{O}_\beta \not\cong \mathcal{O}_\beta^\sigma$.

pf) We chose $E = \mathbb{C}/\mathbb{Z}^2$ & $[2] : E \to E$: defined over $\mathbb{Q}$, so $E^\sigma \cong E$ & $[2]^\sigma \cong [2]$.

Thus $([2]\circ\pi)^\sigma = [2]\circ\pi^\sigma$ $\rightsquigarrow$ $(\mathcal{O}_\beta)^\sigma = (\mathcal{O}_{\beta^\sigma})$ by definition

$$\begin{tikzcd} & X \arrow[r] \arrow[dr, "\pi"'] & \tilde{X} \arrow[r] \arrow[d, "\tilde{\pi}"] & Y \arrow[d, "\beta"] \\ \mathrm{Spec}\,\overline{\mathbb{Q}} \arrow[d, "\mathrm{Spec}\,\sigma"'] & & E \arrow[r, "h"] \arrow[d, "{[2]}"] & \mathbb{P}_{\mathbb{C}} \\ \mathrm{Spec}\,\overline{\mathbb{Q}} & & E & \end{tikzcd}
\qquad
\begin{tikzcd} Y \arrow[d, "\beta^\sigma"'] & \tilde{X} \arrow[l] \arrow[d] & X \arrow[l] \arrow[dl] & \\ \mathbb{P}_{\mathbb{C}} & E \arrow[l] \arrow[d] & & \mathrm{Spec}\,\overline{\mathbb{Q}} \arrow[d, "\mathrm{Spec}\,\sigma"] \\ & E & & \mathrm{Spec}\,\overline{\mathbb{Q}} \end{tikzcd}$$

Assume $\mathcal{O}_\beta \cong \mathcal{O}_{\beta^\sigma}$.

then $\exists \varphi \in \mathrm{Deck}[2]$ $\varphi \circ \pi \cong \pi'$.

Here $\mathrm{Deck}[2] \curvearrowright E$ is a translation action, so it has no fixed pts.

Recall that Lem 5.5 says that

$(p_x^2, p_y^2, p_z^2, 1)$ describes the ramification of π around the 4 Weierstrass points,

and it is preserved by σ.

(local ramification behavior is Galois invariant)

So by the assumption (*) on this tuple, it follows that φ should act trivially.

$$\Rightarrow \quad \pi \cong \pi' \quad \overset{\text{Lem 5.15}}{\Longrightarrow} \quad \beta \cong \beta^\sigma \quad \boxtimes.$$

References

[N11] Nisbach, F.: The Galois action on Origami curves and a special class of Origamis. KIT, PhD. thesis (2011) url:http://digbib.ubka.uni-karlsruhe.de/volltexte/1000025252

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