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ORIGINAL PAPER

A comb of origami curves in the moduli space M_3 with three dimensional closure

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Abstract The first part of this paper is a survey on Teichmüller curves and Veech groups, with emphasis on the special case of origamis where much stronger tools for the investigation are available than in the general case. In the second part we study a particular configuration of origami curves in genus 3: A "base" curve is intersected by infinitely many "transversal" curves. We determine their Veech groups and the closure of their locus in M_3, which turns out to be a three dimensional variety and the image of a certain Hurwitz space in M_3.

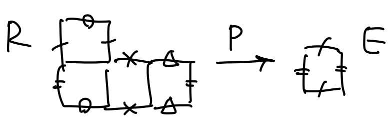
"quarternion origami"

Keywords Teichmüller curves · Veech groups · Origamis · Hurwitz spaces

Mathematics Subject Classifications (2000) 14H10 · 14H30 · 32G15 · 53C10

§1.-3. Intro

- An origami is a covering $\mathcal{O}=(p:R\to E)$ of the unit square torus $E=\mathbb{C}/\mathbb{Z}+i\mathbb{Z}$ branched over 0.



← R admits natural translation structure induced from E.

- Veech group $\Gamma(\mathcal{O})$ of an origami $\mathcal{O}$

  is the stabilizer of translation-isomorphism class

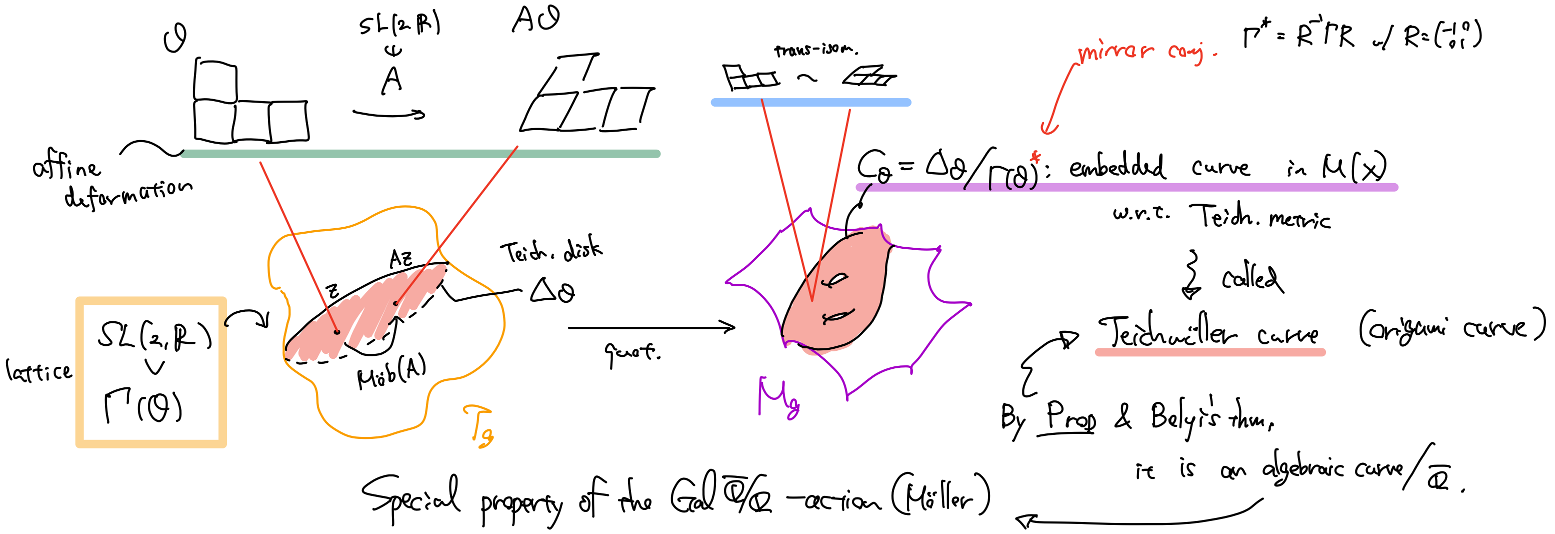
  under the affine-deform. action of $SL(2,\mathbb{R})$

Proposition 5 *Let $O = (p : X \to E)$ and $O' = (p' : X' \to E)$ be two origamis and U (resp. U') the corresponding subgroups of F_2. Then*

(a) *O is isomorphic to O' if and only if U is conjugate to U' in F_2.*
(b) *$C(O) = C(O')$ if and only if there is a $\gamma \in \mathrm{Aut}^+(F_2)$ such that $\gamma(U) = U'$.*

※ $U < F_2$ is the fundamental grp $\pi_1(X^*) \hookrightarrow \pi_1(E^*) = F_2$

Prop $\Gamma(\mathcal{O})$ is a finite-index subgrp of $SL(2,\mathbb{Z})$.



Theorem A† *Let Δ be a Teichmüller disk in the Teichmüller space T_g.*

- *The image* $\mathrm{proj}(\Delta)$ *is an algebraic curve C in M_g, if and only if the Veech group Γ is a lattice in* $\mathrm{SL}_2(\mathbb{R})$.
- *In this case, $\mathbb{H}/\Gamma^*$ is the normalization of the algebraic curve C.*

Hence, the Veech group detects, whether a Teichmüller disk leads to a Teichmüller curve and, if this happens, it determines the Teichmüller curve up to birationality.

Theorem B: (Schmithüsen)

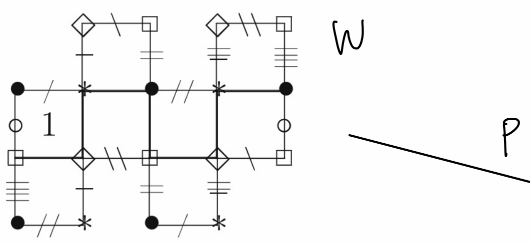
- *Let U be the finite index subgroup of F_2 defined in (3) for the origami O.*
- *Let $\hat{\beta} : \mathrm{Aut}(F_2) \to \mathrm{GL}_2(\mathbb{Z}) = \mathrm{Out}(F_2)$ be the natural projection and $\mathrm{Aut}^+(F_2) := \hat{\beta}^{-1}(\mathrm{SL}_2(\mathbb{Z}))$.*
- *Let $\mathrm{Stab}(U) := \{\gamma \in \mathrm{Aut}^+(F_2) | \gamma(U) = U\}$ be the stabilizing group of U.*

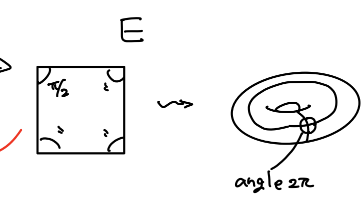
Then for the Veech group $\Gamma(O)$ holds: $\Gamma(O) = \hat{\beta}(\mathrm{Stab}(U))$.

§4. The quarternion origami.

4-1. origami W

Using the combinatorial definition of origamis, W can be described by 8 squares that are glued as indicated (edges are glued if they have the same label).

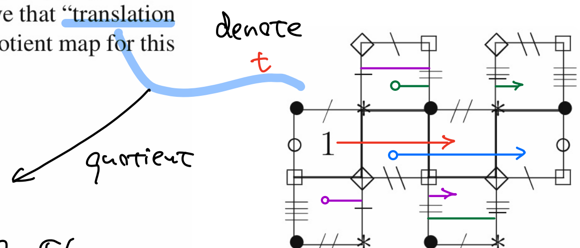


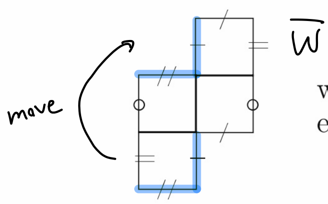


Note that every vertex is glued to precisely one of the vertices of the square labeled "1". Euler's formula gives $2 - 2g = 8 - 16 + 4$, thus the genus of W is 3.

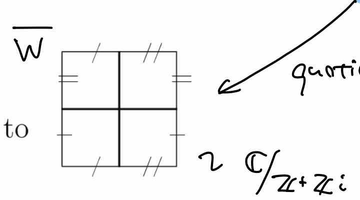
The total angle at every vertex is 4π, so they are all ramification points of order 2 for the covering $p : W \to E$ of degree 8 to the torus E. Recall that p is obtained by mapping each of the eight squares to E. The map p can be decomposed as follows: Observe that "translation by 2 to the right" is an automorphism of W, and let $q : W \to \overline{W}$ be the quotient map for this automorphism. Then $\overline{W}$ is the origami

denote t



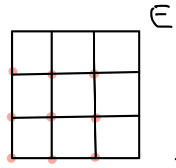


which is equivalent to



We call the points in the group $\mathbb{C}/\mathbb{Z}+\mathbb{Z}i$ of order k dividing $n\geq 2$: the n-torsion points of E.

e.g.



3-torsion points $E[3]$ w/ $|E[3]| = 3^2$

$\rightsquigarrow$ They form a group $E[n]$ of order n^2,

$\cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

※ $\left(\begin{array}{l} E[n] = E/_{\mathbb{Z}+n\mathbb{Z}}, \\ E \to E/E[n] : \text{origami} \end{array} \right.$

4-2. Automorphism group

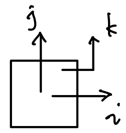
We may check that $p : W \to E$ is a normal cover.

$\mathrm{Gal}(W/E) = m_p(\pi_1(E)) \cong Q$: the quarternion group of order 8.

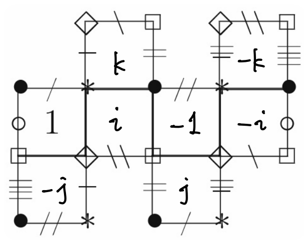
(monodromy grp)

$= \langle 1, i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$

W

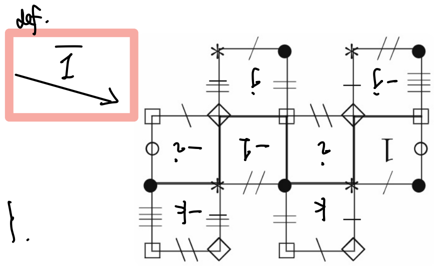


$\xleftarrow{p}$



$\leftarrow$ label w.r.t. the monodromy

def.



4-3. The automorphism group

$G := \mathrm{Aut}\,W = \{ f : W \to W : \text{affine},\ D(f) = \pm I \}$.

(Schmithüsen)

$\cong \{ \gamma \in \mathrm{Aut}^+ F_2 : \gamma(\pi_1(W)) = \pi_1(W),\ \rho(\gamma) = \pm I \}$

$= \{\pm I\} \times Q$ $\cdots$ extension of Q of deg 2.

half-rotation $\uparrow$ $\uparrow$ translation

order 16.

which includs 6 involutions acting as rotation.

					t							c				c^{-1}
$\{\pm I\}$	$+I$								$-I$							
Q	1	i	j	k	-1	$-i$	$-j$	$-k$	1	i	j	k	-1	$-i$	$-j$	$-k$
(notation) γ	1	i	j	k	-1	$-i$	$-j$	$-k$	$\overline{1}$	$\overline{i}$	$\overline{j}$	$\overline{k}$	$\overline{-1}$	$\overline{-i}$	$\overline{-j}$	$\overline{-k}$
γ^2	1	-1	-1	-1	1	-1	-1	-1	1	1	1	-1	1	1	1	-1

$\uparrow$ involution (translation)

6 involutions (rotation)

$$\overline{1} = \frac{\begin{matrix} 1 & i & j & k & -1 & -i & -j & -k \end{matrix}}{\begin{matrix} -i & -1 & k & j & i & 1 & -k & -j \end{matrix}}$$

$\overline{-1} = i \quad 1 \quad -k \quad -j \quad -i \quad -1 \quad k \quad j$

$\overline{i} = 1 \quad -i \quad j \quad -k \quad -1 \quad i \quad -j \quad k$

$\overline{-i} = -1 \quad i \quad -j \quad k \quad 1 \quad -i \quad j \quad -k$

$\overline{j} = -k \quad -j \quad -i \quad -1 \quad k \quad j \quad i \quad 1$

$\overline{-j} = k \quad j \quad i \quad 1 \quad -k \quad -j \quad -i \quad -1$

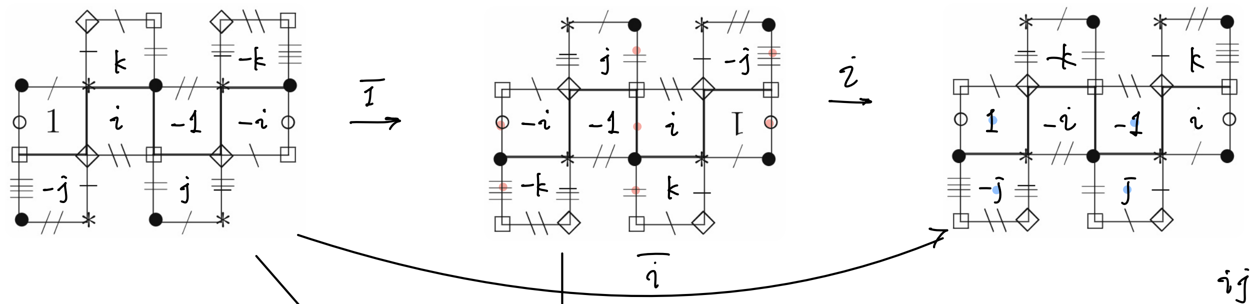
$\overline{k} = -j \quad -k \quad -1 \quad -i \quad j \quad k \quad 1 \quad i$

$(\overline{a} = a \circ \overline{1})$

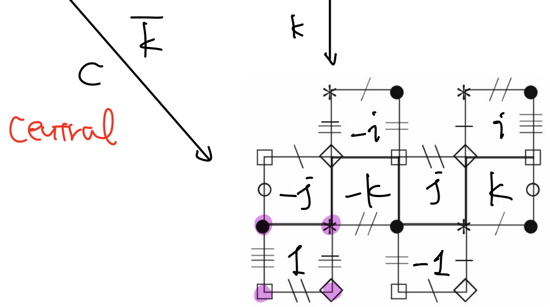
※ $\langle c \rangle = C_G$ (center).

$\ast$ fixed points of G

$a \in G$ ($a \neq \pm 1$) are fixed pt.-free, & $-1, \bar{a} \in G$ ($a \in Q$) have just 4 fixed pts.



$ij = k$



$L \backslash R$	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

fixed pts of other involutions are mid points of either squares or edges.

4-3. The equation of $C(W)$

※ equations of surfaces in $C(W) \subset M_3$.

$\ast$ $c = \bar{k} \in G$ has 4 fixed pts

$\xrightarrow{\text{Rien.-Hurwitz}}$ $W/\langle c \rangle$ has genus 0.

$W \to W/\langle c \rangle$ is a cyclic covering of degree 4 of $P^1_{\mathbb{C}}$,

branched over 4 pts (the vertices) $\xrightarrow{\text{normalize}}$ $0, 1, \lambda, \infty$ $\left(\lambda \in P^1_{\mathbb{C}} \setminus \{0,1,\infty\} \right)$

Then W is given by the equation of the form

$$y^4 = x^{\varepsilon_0}(x-1)^{\varepsilon_1}(x-\lambda)^{\varepsilon_\lambda}$$

$\varepsilon_0, \varepsilon_1, \varepsilon_\lambda$ turn out to be 1 since the ramification orders $= 4$ around $0, 1, \lambda, \infty$.

up to Aut $P^1_{\mathbb{C}}$ w/ renormalization ; λ defines the same cpx str. as $\frac{1}{\lambda}, 1-\lambda, 1-\frac{1}{\lambda}, \frac{\lambda}{\lambda-1}, \frac{1}{1-\lambda}$

The double cover $g : W \to W/\langle t \rangle = E = \mathbb{C}/\mathbb{Z}+\mathbb{Z}i$ is given by
translation by -1.

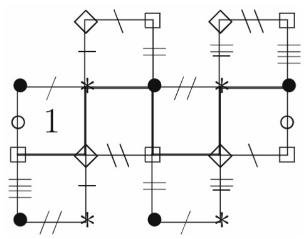
$(x, y) \mapsto (x, y^2)$ where $E : y^2 = x(x-1)(x-\lambda)$: Legendre's normal form

Its affine deformation by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is known to be presented by $\lambda_A = \lambda \frac{ai+b}{ci+d}$

?

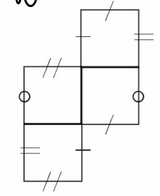
i.e. $E_A = \mathbb{C}/\binom{a}{c}\mathbb{Z} + \binom{b}{d}\mathbb{Z}$; $y^2 = x(x-1)(x-\lambda_A)$

W

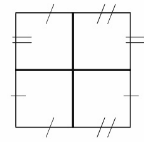


$\xrightarrow{p}$

$E = \overline{W} = W/\langle t \rangle$



which is equivalent to



$$\begin{cases} y^4 = x(x-1)(x-\lambda) \\ \omega_W = \dfrac{dx}{y^2} \end{cases} \qquad \begin{cases} y^2 = x(x-1)(x-\lambda) \\ \omega_E = \dfrac{dx}{y} \end{cases}$$

Rem $\boxed{E_{-1} : (2y)^2 = 4x(x-1)(x+1) \quad \cdots \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\text{-deformation of } E}$

4-4 VG of W

It can be checked that $\Gamma(W) = SL(2,\mathbb{Z})$. (cf. Thm B).

As a consequence, $C(W)$ is an affine line, embedded in M_3. (cf. Thm A.)

4-5 Remarkable properties of W

(Herrlich, Schmithüsen, 2008) The Jacobian of each of W_λ $\lambda \in P^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ splits (upto isogeny) into a product of three elliptic curves.

$\cdots \quad Jac(W_\lambda) \underset{isog.}{\sim} E_\lambda \times E_{-1} \times E_{-1}$.

The quotient $\mathbb{C}^g / L(W)$ of is the period lattice $L(W) := \{ (\int_\gamma \omega_1, \cdots \int_\gamma \omega_g) \in \mathbb{C}^g \mid \gamma \in H_1(W, \mathbb{Z}) \}$

where $\omega_1, \cdots, \omega_g$: normalized basis of holo. 1-forms on W.

(Möller, 2005_{pre} - 2011) By the fact that $Jac(W_\lambda)$ has a constant part of codim 1

shows that $C(W)$ is a Shimura curve. (furthermore is the only one case Teich. curve of trans. surf & Shimura curve at the same time.)

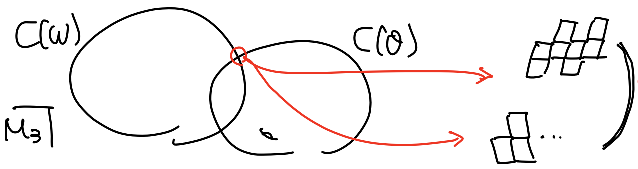
The most interesting is :

(Herrlich, Schmithüsen, 2008) <u>Theorem C</u> $C(W)$ intersects infinitely many other origami curves.

Note : $C(\mathcal{O}) = \{ A\mathcal{O} \mid A \in SL(2,\mathbb{Z}) \} / \text{trans-isom.} \cong \mathbb{H}/\Gamma(\mathcal{O}) \hookrightarrow M_g$

So if $\mathcal{O}' \underset{\text{Trans.}}{\not\cong} A\mathcal{O}$ for ${}^\forall A \in SL(2,\mathbb{Z})$ then $C(\mathcal{O}) \neq C(\mathcal{O}')$.

"Intersections of origami curves" imply points in M_g which represent such $\mathcal{O}$, $A\mathcal{O}'$ at the same time.



Not $SL(2,\mathbb{Z}) \times$ Trans – equivalent

but biholo - equivalent!

(Outline) We have seen that $G = \mathrm{Aut}(W_\lambda)$ contains 7 involutions

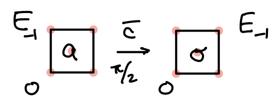
and all of them have 4 fixed pts $\rightsquigarrow$ $\forall g \in G$, $W_\lambda/\langle g\rangle$ has genus 1.
involution

$-1 \in G$ induces $W_\lambda \to E$

$\forall g \in G$: involution, c descends via $W_\lambda \to W_\lambda/\langle g\rangle$ to $\bar{c} \in \mathrm{Aut}\, W_\lambda/\langle g\rangle$ of order 4,
central automorphism w/ 2 fixed pts.

$W_\lambda/\langle g\rangle$ must be E_{-1}.
(The only elliptic curve w/ such an automorphism)

Let $K = K_\lambda : W_\lambda \to E_{-1}$
ramification occurs at the 4 fixed pts of g.



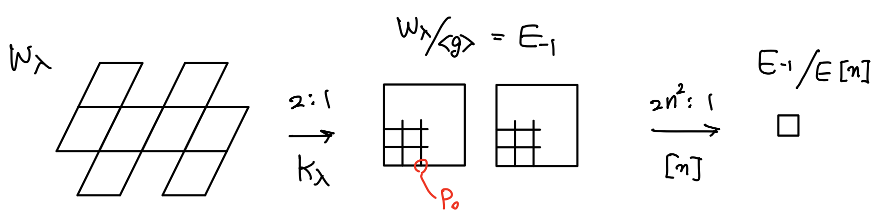
Their images, $P_0(\lambda) \cdots P_3(\lambda) \in E_{-1}$, form an orbit of $\bar{c}$.

Choose one fixed pt O of $\bar{c}$ as the origin of E_{-1}.

Then: if $P_0(\lambda) \in E_{-1}$ is an $\exists n$-torsion $\Rightarrow$ same holds for $P_1(\lambda) \sim P_3(\lambda) \in E_{-1}$.

$\Longrightarrow$ $[n]\circ K$ is ramified over O, so defines an origami $[n]\circ K : W_\lambda \to E_{-1}[n]$ of order $2n^2$

quotient by $\mathbb{Z} \mapsto n\mathbb{Z}$.



Authors showed that : for $\forall n \geq 3$ and $\forall P \in E_{-1}$: n-torsion,

there exists $\lambda \in \mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$ s.t. $P_0(\lambda) = P$.

and its origami curve intersects $C(W)$. //

§5. The Hurwitz space H

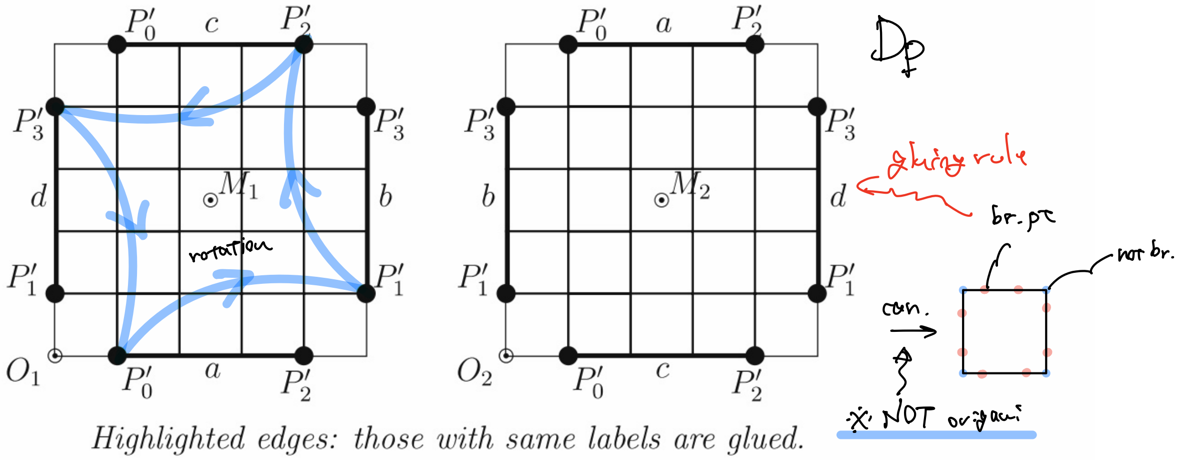
5.1 origamis D_P

<u>recall</u> $\forall n \geq 3$, every n-torsion pt. P on E_{-1} induces an origami D_P of degree $2n^2$,

where $C(D_P)$ intersects $C(W)$.

D_P consists of 2 large squares made of n^2 squares.

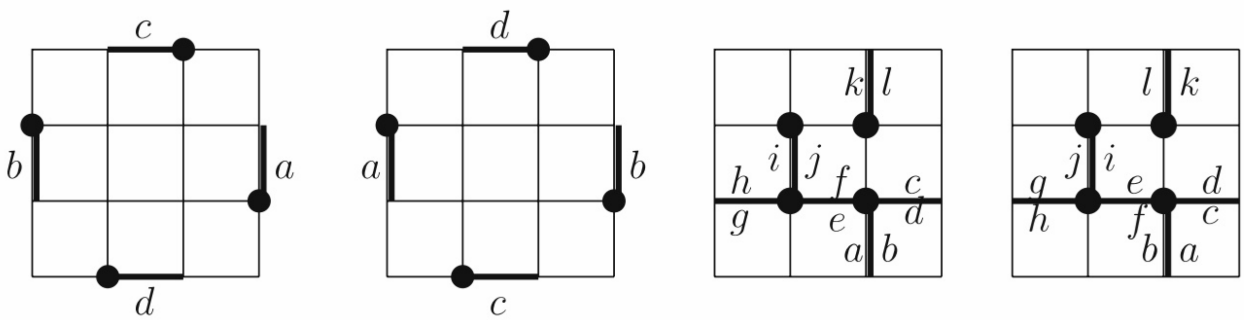
squares each; the point $P = P_0$ corresponds to a (primitive) vertex of one of the small squares. P_1, P_2 and P_3 are obtained from P_0 by rotation by an angle of $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$, resp., around the center of the large square. The two large squares are glued in such a way that the canonical map (of degree 2) from the resulting surface X_P to the torus corresponding to the large square is ramified exactly over P_0, P_1, P_2 and P_3. For the precise description of the glueing we refer to [6]; here we confine ourselves to an example with $n = 5$:



Highlighted edges: those with same labels are glued.
Other edges: opposite edges are glued.

Here, P_0', P_1', P_2', and P_3' are the preimages of the ramification points P_0, P_1, P_2 and P_3, M_1 and M_2 the preimages of M and O_1, O_2 the preimages of O, respectively. As in Sect. 4.5, O and M denote the fixed points of the automorphism $\bar{c}$ of order 4 on the torus E.

For $n = 3$ there are only two different possibilities for the orbit P_0, P_1, P_2, P_3:



Let $D_P = (d_P : X \to E)$: the degree $2n^2$ origami.

$C(D_P) = \{X_A \mid A \in SL(2,\mathbb{Z})\} / \text{isom}$

Ψ

X_I is an intersection point of $C(D_P)$ & $C(W)$.

c : the central automorphism of W_I has derivative $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

and does not belong to Aut D_P.

c defines an affine diffeomorphism $c_A \in \mathrm{Aff}^{+}(W_{X_A})$ w/ derivative ASA^{-1} (not holo in general)

on the other hand, $c^2 = \tau$ defines a holo. auto. of derivative $-I$.

5.2 coverings w/ given ramification data.

The origamis D_p : defined by coverings w/ the same ramification behavior.

It is classically known that such coverings are classified by an algebraic variety : Hurwitz space

More precisely : Hurwitz spaces classify $p : X \to Y$ w/ the following data fixed :

degree d, genus $g(Y)$, $\# Br(p)$, $\{ \mathrm{ord}_x p \mid x \in p^{-1}(Br(p)) \}$

※ also possible to specify a certain $\begin{cases} \text{geometric configuration of } Br(p) : \text{branched pts} \\ \mu \quad : \text{monodromies} \end{cases}$

(to define subspaces ?)

(points , lines) s.t.
$\begin{cases} \# \text{ lines crossing a pt} = \text{const} \\ \# \text{ pts locating on a line} = \text{const} \end{cases}$

Note : once Y, p & ramification orders of $Br(p)$ are fixed,

there are finitely many possibilities of

monodromy homomorphism $\mu : \pi_1(Y - Br(p)) \to G_{\deg p}$.

Restricting the monodromy typically leads to an irreducible component of the Hurwitz space.

Our case degree 2 covers of an elliptic curve, that are ramified over 4 pts P, P', Q, Q'

taking suitable base point, we may assume $P' = -P$, $Q' = -Q$.

※ "P_0, P_1, P_2, P_3 form $\bar{C}$-orbit" cannot be formulated in an algebraic way ($\overline{C_A}$: not alg. automorphism)

We denote by $\widetilde{H}$ the Hurwitz space of such coverings.

$\rotatebox{90}{$\in$}$

$$\boxed{(X, p) : \quad X : \text{cpt R.S.} \quad p : X \to E : \text{degree 2 cover} \quad \text{w/ 4 crit. pt } P, Q, P', Q' \in X \quad (P' = -P, Q' = -Q)}$$

$p : X \to E$ & $p' : X' \to E'$: considered isomorphic iff
$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\exists\varphi"'] & E \arrow[d, "\exists\bar{\varphi}"] \\ X' \arrow[r, "p'"'] & E' \end{tikzcd} \quad : \text{isomorphism}$$

By Riem.-Hurwitz, $\forall (X, p) \in \widetilde{H}$ $g(X) = 3$.

$\widetilde{H}$ admits natural morphism $\pi : \widetilde{H} \to M_3$
$\qquad (X, p) \mapsto X$

- $(X, p) \in \widetilde{H}$ is completely determined by its deck transformation σ,

$\rightsquigarrow \widetilde{H} \sim \{ (X, \sigma) \mid \sigma \in \mathrm{Aut}\, X : \text{involution w/ 4 fixed points whose images on } E = X/\langle\sigma\rangle : \exists\text{pe-symmetric} \}$

$(X, \sigma) \sim (X', \sigma') \Leftrightarrow \exists \varphi : X \xrightarrow{\text{isom}} X'$ s.t. $\sigma' = \varphi \circ \sigma \circ \varphi^{-1}$

Prop 8 Let $(X, p) \in \hat{H}$. Then $\mathrm{Aut}(X)$ contains a subgrp $\cong V_4$: Klein's four grp $V_4 = \{1, i, j, k\}$

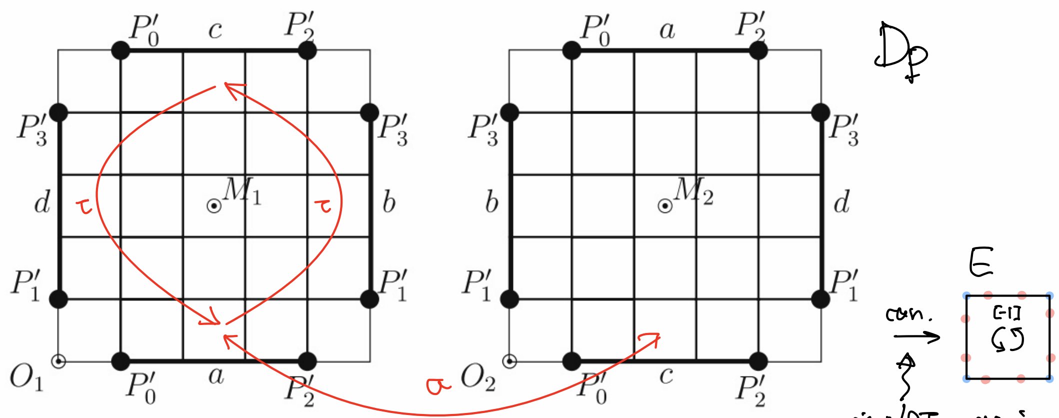
	i	j	k
i	1	k	j
j	k	1	i
k	j	i	1

pf) Every covering of degree 2 is normal.

$\Rightarrow$ $\exists \sigma \in \mathrm{Aut}\,X$ that interchanges two inverse pts under p (crit. pts P, P', Q, Q' : fixed)

Next consider a lift τ of $[-1]$ on E (multiplication by -1)

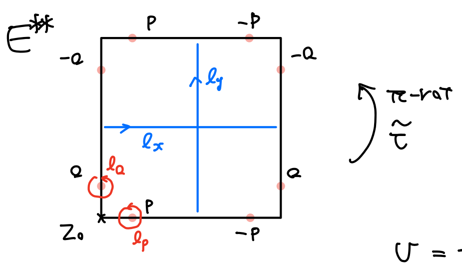
τ is an automorphism that commutes w/ σ.



Highlighted edges: those with same labels are glued.
Other edges: opposite edges are glued.

$[-1]$ also is an auto on $E^{**} = E \setminus \{\pm P, \pm Q\}$ $\overset{\text{induce}}{\rightsquigarrow}$ $\hat{\tau} \in \mathrm{Aut}(E^{**})$. $(\hat{\tau} = [-1]|_{E^{**}})$

From general theor. of topology, $[-1]$ lifts to X iff $\hat{\tau}$ preserves $U = \pi_1(X^{**}) < \pi_1(E^{**})$



We see : $\hat{\tau}(l_x) = -l_x$ $\quad \hat{\tau}(l_y) = -l_y$

$\hat{\tau}(l_P) = l_{-P}$ $\quad \hat{\tau}(l_Q) = l_{-Q}$.

$U = \pi_1(X^{**}) < \pi_1(E^{**})$ is the kernel of monodromy hom $\mu : \pi_1(E^{**}) \to S_2$.

Since p is ramified over $\pm P, \pm Q$, $\mu(l_{\pm P}) = \mu(l_{\pm Q}) = (1\,2)$.

For the other two generators, $\mu(l_x), \mu(l_y)$ are freely determined.

$\mu(l_x) = \mu(-l_x)$, $\mu(l_y) = \mu(-l_y)$

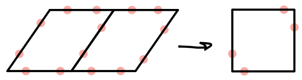
Thus lifting elements determined by ▬▬, we have four automorphism on X

which form V_4. ⊠

5.3 origami curves in H

Data $(E, \{\pm P, \pm Q\}, \text{degree } 2)$ given $\Rightarrow$ there are 4 different coverings.

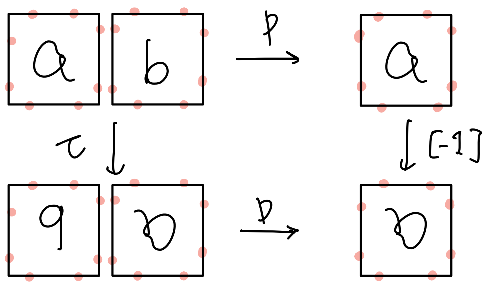
corr. to cpx str. of X.



(monodromy $\mu : \pi_1(E^{*4}) \to S_2$ is fixed except for the choice of $\mu(l_x), \mu(l_y) : l_x, l_y \in \pi_1(E)$)

Let $\tau \in \mathrm{Aut}(X)$ be the lift of $[-1] \in \mathrm{Aut}\, E$ fixing $\{\pm P, \pm Q\}$

order 2, commute w/ σ.



The surface is hyperelliptic $\Leftrightarrow$ $\exists$ involution w/ 6 fixed pts (quotient : $\mathbb{P}^1_{\mathbb{C}}$)

general pt. $(X, \sigma) \in \tilde{H}$ has 3 involutions

$\sigma, \tau,$ and $\sigma\tau$.

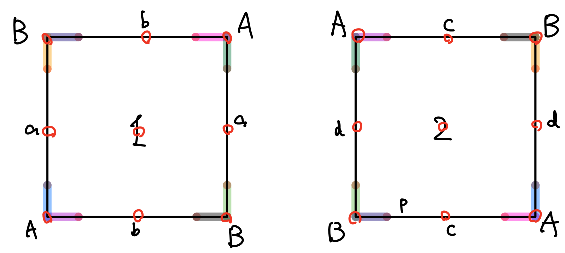
By def of $\tilde{H}$: σ has 4 fixed pts

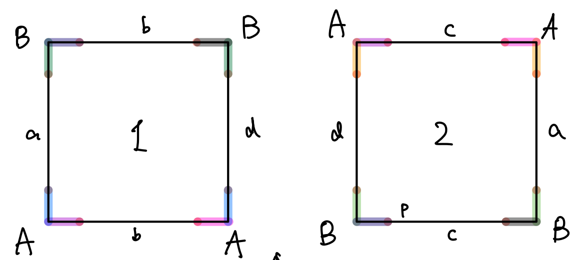
Possible fixed pts of $\tau, \sigma\tau$: inverse images of 4 fixed pts of $[-1]$ on E.

More precise possibilities $\begin{cases} (\alpha) & \text{one fixes 4 pts \& other fixes the rest 4 pts} \\ (\beta) & \text{one fixes 8 pts \& other is fixed pt-free} \end{cases}$

(i) $x = y = 1$ ... (β) half pt of edges, center of faces, vertices are fixed pts.

(ii) $x = (12)$ $y = 1$ ... (α)

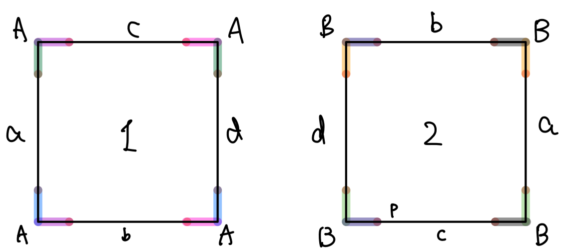




(iii) $x = 1$ $y = (12)$ ... (α) $\frac{\pi}{2}$-rotation

(omit)

(iv) $x = y = (12)$ ... (α)



Def 9 $H := \{(X, \sigma) \in \tilde{H} \mid \tau \text{ has exactly 4 fixed pts in } X\}$

For $(X, \sigma) \in H$, X/τ, X/σ, $X/\sigma\tau$ all have genus 1.

Later we shall see that for generic $(X, \sigma) \in H$, $\mathrm{Aut}(X) = \{1, \sigma, \tau, \sigma\tau\}$.

w.r.t. algebraic parametrization of H

$\longrightarrow$ We could characterize $H = \overline{\{(X,\omega)\in\widetilde{H} \mid X \text{ is not hyperelliptic}\}}$

also alg. curve

- $^{\forall} p \in E_{-1}$: n-torsion point, the origami D_p defines an alg. curve $C(D_p) \leftarrow C_p \hookrightarrow H < \widetilde{H} \to M_3$

In the same way, origami W defines an alg. curve $C(W) \leftarrow C_W \hookrightarrow H < \widetilde{H} \to M_3$

$\begin{matrix} \uparrow & & \uparrow \\ (W_\lambda & \xrightarrow{p} & W_\lambda/\langle g\rangle = E_{-1}) \end{matrix}$

※ here n-torsion not necessary

Let $\pi : \widetilde{H} \longrightarrow M_3$ be a finite morphism of varieties.

Thm 1 $\bigcup^{n\geq 3}_{P\in E_{-1}: n\text{-torsion}} C_p \subset H$: dense.

pf) prove in cpx top. of Hurwitz sp : parametrized by branched pts ($\Rightarrow$ Zariski top. also holds.)

Let $(X,\omega)\in H$: non hyperelliptic

$\updownarrow$ determine

$(E, (\pm P, \pm Q), \mu)$

Clealy we can approximate P, Q by torsion points. $\rightsquigarrow$ Let (P_n, Q_n) ; pair of n-torsions near to (P, Q).

For a suitable choices of monodromies, we may define $(X_n, \omega_n)\in H$ close to (X, ω).

Note that $X_n \to E_{-1} \to E_{-1}/E[n]$ defines an origami,

and so (X_n, ω_n) lies on $C = \pi^{-1}(\underbrace{C(X_n \to E_{-1}[n])}_{\text{origami curve}})$: alg. curve.

Next we show that $^{\exists}p$: n-torsion pt s.t. $(X_n, \omega_n)\in C_p$

in fact we show that the stronger, claim $C_p = C$: for n : prime.

Schmithüsen's theory states that two origami curves C, C_p are equall iff

the corresponding subgroups $U, U_p < F_2$ are $\mathrm{Aut}^+ F_2$-equivalent.

U, U_p : fundamental grps

$\mathrm{Aut}^+F_2 \cong \mathrm{Aff}^+(\mathbb{H})$: univ. affine grp.

If $^{\exists}\gamma \in \mathrm{Aut}^+(F_2)$ joining U & U_p,

$\wr$ induced by

$^{\exists} f : \mathbb{H} \longrightarrow \mathbb{H}$: affine diffeomorphism

proj. $\wr$

$\tilde{f} : X \to X_p$: " s.t. $\tilde{f}^* \pi_1(X) = \pi_1(X_p)$

proj. $\wr$

$\bar{f} : E \to E_{-1}$: " transforming $\begin{pmatrix} \pm P_n & \longmapsto & \pm P \\ \pm Q_n & & \pm c(P) \\ \cap & & \cap \\ E & & E_{-1} \end{pmatrix}$ (for some P)

& respects monodromy.

To find such $\bar{f}$, consider $(X', \omega')\in C$ through (X_n, ω_n) which lies over E_{-1}.

To (X', ω') there corresponds an aff. diffeo. $A \in SL(2,\mathbb{Z})$ $\begin{cases} E \to E_{-1}. \\ P_n, Q_n \to n\text{-torsion pts on } E_{-1}. \end{cases}$

$$\begin{array}{ccc} E & \longrightarrow & \{x \in E_1 ; n\text{-torsion}\} \cong (\mathbb{Z}/n\mathbb{Z})^2 \\ \Psi & & \Psi \\ P_n, Q_n & \xrightarrow{\text{send}} & P_n, Q_n \end{array}$$

@ $C = G_p$ for $p \in (\mathbb{Z}/n\mathbb{Z})^2$

$\Leftrightarrow \exists B \in SL_2(\mathbb{Z}/n\mathbb{Z})$ s.t. $\begin{pmatrix} B(P_n) \\ B(Q_n) \end{pmatrix} = \begin{pmatrix} P \\ \bar{C}(P) \end{pmatrix}$ — ※

Since $\bar{C}$ acts as $\frac{\pi}{2}$-rotation, it acts on $(\mathbb{Z}/n\mathbb{Z})^2$ through $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Using linear algebra over $\mathbb{Z}/n\mathbb{Z} \sim \mathbb{F}_n$ (n : prime),

we may construct P s.t. ※ concretely. $\boxtimes$

5.4 Affine coordinates for H

Proposition 11 *For any point $(X, \sigma) \in H$, X can be represented by a plane quartic with equation*

$$x^4 + y^4 + z^4 + 2ax^2y^2 + 2bx^2z^2 + 2cy^2z^2 = 0 \tag{11}$$

for some complex numbers a, b and c.

Proof Since X is not hyperelliptic, the canonical map on X gives an embedding into the projective plane as a smooth quartic. Moreover, every automorphism of X is induced by a projective automorphism of $\mathbb{P}^2$. Thus we can choose coordinates on $\mathbb{P}^2$ such that σ acts by $(x : y : z) \mapsto (-x : y : z)$ and τ by $(x : y : z) \mapsto (x : -y : z)$. Then $\sigma\tau$ acts by $(x : y : z) \mapsto (-x : -y : z) = (x : y : -z)$. A quartic that is invariant under σ and τ must therefore be a polynomial in x^2, y^2 and z^2. We can still multiply x, y and z by suitable constants and thus obtain that the coefficients of x^4, y^4 and z^4 are 1, which gives us an equation of the form (11). □

The Eq. (11) describes a family of plane projective curves of degree 4 over the affine parameter space $\mathbb{A}^3(\mathbb{C}) = \mathbb{C}^3$. The non-singular curves in this family are determined by Prop 12.

Def $U := \mathbb{A}^3_{\mathbb{C}} \setminus \{\text{critical parameters}\}$ w.r.t. prop 12.

Corollary 14 *The kernel of the homomorphism $\rho : \mathrm{Aut}(\mathcal{C}) \to \mathrm{Aut}(U)$ is G_0. In particular, for general $(a, b, c) \in U$, $\mathrm{Aut}(C_{abc}) = G_0$.*

Proof As observed in the proof of Proposition 11, any automorphism of a curve in $\mathcal{C}$ is induced by an automorphism of $\mathbb{P}^2(\mathbb{C})$. Thus any $\varphi \in \ker(\rho)$ is a linear change of the homogeneous coordinates x, y, z that preserves the terms ax^2y^2, bx^2z^2 and cy^2z^2 for all $(a, b, c) \in U$. This is only possible if $\varphi \in G_0$. Alternatively, the result can be checked by comparison with the list of automorphism groups in genus 3 in [9]. □

5.5 Maps to moduli space

In this subsection we study the relations between the spaces U, H and M_3. To do so, we shall factor the morphism $m : U \to M_3$, $(a, b, c) \mapsto [C_{abc}]$, in two ways.

We saw in the proof of Proposition 13 that the subgroup L of $\mathrm{Aut}(U)$ is contained in the image of the natural homomorphism $\rho\colon \mathrm{Aut}(\mathcal{C}) \to \mathrm{Aut}(U)$. Therefore m factors through U/L.

On the other hand we know that , for every $(a, b, c) \in U$, C_{abc} admits the automorphism $\alpha\colon (x : y : z) \mapsto (-x : y : z)$. We find:

Proposition 15 *$(a, b, c) \mapsto (C_{abc}, \alpha)$ is a surjective morphism $h\colon U \to H$.*

So far we have found a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "\tilde{q}"'] \arrow[dr, "m"] & H \arrow[d, "\pi"] \\ U/L \arrow[r, "q"'] & M_3 \end{tikzcd}$$

The final goal in this section is to show

Proposition 16 *q is birational.*

Note: q is not an isomorphism, nor is H isomorphic to U/L_H.
This can be seen e. g., by looking at the Fermat curve C_{000}: it is mapped isomorphically onto $C_{0,3,0}$ by the transformation $(x : y : z) \mapsto (x + z : \sqrt[4]{8}\, y : x - z)$, which does not extend to an automorphism of $\mathcal{C}$ since we have seen in Proposition 13 d) that they all fix C_{000}.