Arithmetic Veech sublattices of $\mathrm{SL}(2,\mathbf{Z})$

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Abstract

We prove that every algebraic curve $X/\overline{\mathbf{Q}}$ is birational over $\mathbf{C}$ to a Teichmüller curve. This result is a corollary of our main theorem, which asserts that most finite index subgroups of $\mathrm{SL}(2,\mathbf{Z})$ are Veech groups.

keywords: *algebraic curve, mapping class group, Teichmüller curve, Veech group*.

MSC code: 32G15, 37D40.

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§1. Intro

$\mathcal{M}_{g,[n]}$: moduli sp of type $(g,[n])$ surfaces
unordered punctures

Teich. curve : $f: C \to \mathcal{M}_{g,[n]}$: holomorphic curve,
1-1, & local Kobayashi isometry

McMullen (2008) : $\forall$ Teich. curve has a model of
an algebraic curve defined over $\overline{\mathbb{Q}}$

THM If $X/\overline{\mathbb{Q}}$ is an algebraic curve,
then $\exists C$: Teich. curve birational to $X_{\mathbb{C}}$

In fact, this Teich. curve can be taken as an origami curve. Teich. curve arising from cov. of $S_{1,1}$

※ Not primitive Teich. : arising from covering constructions

refinement

Q. Which algebraic curves over number fields
are holomorphically isomorphic to $\exists$ Teich curves?
stronger
$\Downarrow$ (not)
merely birational occurs in Main result

Masur (1986), Veech (1989)

assert that 'Teich curve cannot be cpt.'

$\rightsquigarrow$ 'proper algebraic curves' are excluded.

further question is still open.

◎ THM1 arises from the purely grp-theoretic result: THM2 ↴

THM2 Every finite index subgrp of $\Gamma(2)$ containing $\{\pm I\}$ is a Veech grp.

↓ definition

Def $S_{g,n}$: the top. (g,n) surface

$\pi_{g,n} := \pi_1(S_{g,n})$

$\mathrm{Mod}(S_{g,n}) = \mathrm{Diff}^+(S_{g,n})/\text{isotopy}$: the MCG

$\mathrm{PMod}(S_{g,n}) = \{ [f] \in \mathrm{Mod}(S_{g,n}) \mid \bar{f}|_{\partial S_{g,n}} = \mathrm{id} \text{ where } \bar{f}: S_g \to S_g \text{ : extension} \}$

Dehn–Nielsen $\mathrm{PMod}(S_{g,n})$ is an index 2 subgrp of $\mathrm{Out}(\pi_{g,n})$, preserving these conj. classes.

$$\mathrm{Out}(\pi_{g,n}) = \mathrm{Aut}(\pi_{g,n})/\mathrm{Inn}(\pi_{g,n}), \quad \mathrm{Inn}(\pi_{g,n}) = \{\text{conj. map in } \pi_{g,n}\}$$

Df for $\Delta < \pi_{1,1}$, $[\Delta]$: conj. class in $\pi_{1,1}$

$\mathrm{Mod}(S_{1,1}) \simeq SL(2,\mathbb{Z})$ acts on the conj. classes of finite index $\pi_{1,1}$-subgrps (origamis)

by: $\mathrm{Mod}(S_{1,1}) \ni f$: $[\Delta] \mapsto [f^*\Delta]$

the stabilizer of $[\Delta]$ under this action is called the Veech grp. (Schmithüsen, 2005)

corr. quotient $\mathbb{H}/\Gamma(VG)$: Teich. curve parametrizing origami (sq-tiled surf) $\sim [\Delta]$

↑ arithmetic curve (Möller, 2005)

Teich. curves parametrizing non-sq-tiled surfs.

... more difficult. $\overset{\text{corr.}}{\sim}$ non-arithmetic triangle grps (Bouw-Möller 2010.)

THM1 ⟷ THM2

Belyi's thm can be understood as: (unbranched, algebraically 'flat')

$X/\bar{\mathbb{Q}} \overset{\text{birat. over } \mathbb{C}}{\cong}$ an finite étale cover of $\mathbb{P}^1 \setminus \{0,1,\infty\}$

$\cong \mathbb{H}/\Gamma$ where $\Gamma < \Gamma(2)$: fin. index, $\Gamma \ni \{\pm I\}$

$\mathbb{H}/\Gamma(2) \cong \mathbb{P}^1 \setminus \{0,1,\infty\}$ (birational?)

$\longrightarrow$ To prove THM 1.1. it suffices to prove :

<u>THM 1.2</u> (congruence subgroup property)

Every finite index subgroup Δ of $\Gamma(2)$ containing $\{\pm I\}$ is a Veech group.

- Classification prob. of VGs goes back to Thurston.
- THM 1.1 strengthens Möller (2005)'s thm : $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q} \overset{\text{faithful}}{\curvearrowright} \{\text{Teich. curves}\}$
- Hubbert, Lelièvre : close study of VG of genus 2-surfaces
  - $\hookrightarrow$ some are non-congruence subgps.

<u>Rem</u> Some gaps b/w grp-theor. setup & geometric pic.

Let X : Belyi curve

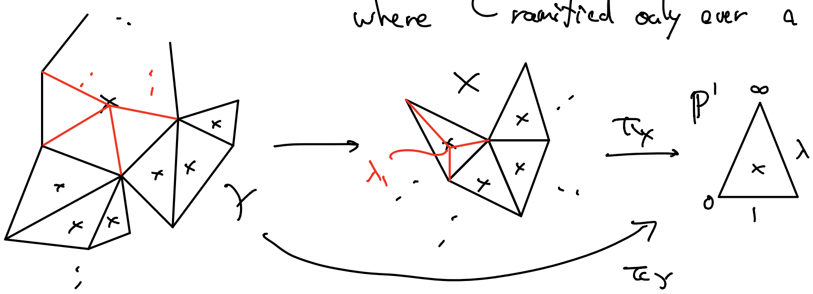
$\exists \underset{\pi_X}{\hookrightarrow} \mathbb{P}^1$: unbranched away from $\{0, 1, \infty\}$

We wish to realize X as moduli sp. of origamis ($\because$ singly branched covs. of elliptic curves)

First we chose $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$

and diagram $Y \to X \to \mathbb{P}^1$

where ramified only over a single preimages of λ.



$\rightsquigarrow$ $\pi_Y : Y \to \mathbb{P}^1$: ramified over $\{0, 1, \infty, \lambda\}$

Let H : Hurwitz space parametrizing br. cov. of $\mathbb{P}^1$ of same top. type as π_Y

(D-D-H 1989.)

$\rightarrow$ $\forall h \in H$ can be thought of remembering location of λ & choice of $\lambda_1 \in X$

$\Updownarrow$ same

or location of pt on unbr. region in X.

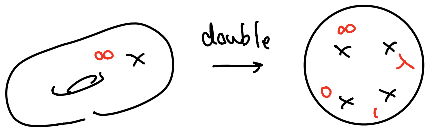
<u>Rem</u> it is delicate : Y can be chosen to make this the case.

This program succeeds in realizing H as moduli sp. of 4-br. cov. of $\mathbb{P}^1$

$\longrightarrow$ due to Diaz–Donagi–Harbater (1989) to make H from : moduli sp. of $(Y \to \mathbb{P}^1$ br/ $0, 1, \infty, \lambda)$

To : moduli sp. of $(Y' \to E$ br/ 1 pt $)_{\text{origami}}$

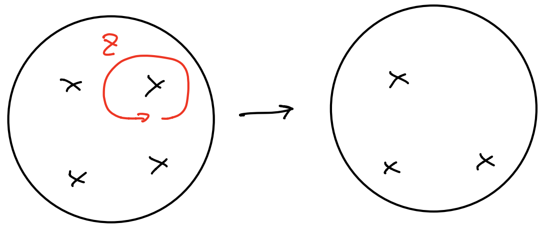
$\Leftarrow$ We let as follows :

E_λ : 

(× : branched pt)

$\pi' : Y \times_{\mathbb{P}^1} E_\lambda \to E_\lambda \to E_\lambda$

§2. Preliminaries



forgetting a puncture : $\pi_{0,4} \longrightarrow \pi_{0,3}$

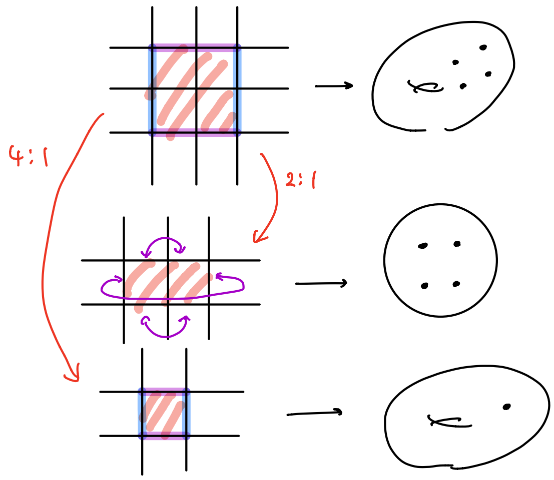
$\mathrm{Ker} = \overline{\langle z \rangle}$ (normal closure)

$[z]_{\pi_{0,4}}$: conj. class in $\pi_{0,4}$

We can think concretely : $S_{1,4} = \mathbb{R}^2 \setminus \mathbb{Z}^2 / 2\mathbb{Z}^2 =: E_2$

$S_{0,4} = E_2 / \{\pm 1\} =: E_0$

$S_{1,1} = \mathbb{R}^2 \setminus \mathbb{Z}^2 / \mathbb{Z}^2 =: E_1$



then we have natural map (from natur. cov.)

index of fin-index normal subgrp

$$\begin{tikzcd} \pi_{1,1} & & \pi_{0,4} \\ & \pi_{1,4} \arrow[ul, dash, "4"] \arrow[ur, dash, "2"'] & \end{tikzcd}$$

$\mathrm{Mod}(S_{1,1}) \cong SL(2,\mathbb{Z})$

$[f] \longleftrightarrow \exists$ (linear auto. on E_1)

$-I \in SL(2,\mathbb{Z})$: central (commutes w/ $\forall A$) $\Rightarrow$ each linear auto. on E_1 descends to the quotient $E_1/\{\pm I\}$ ($= S_{0,4}$),
yielding a homomorphism $i : \mathrm{Mod}(S_{1,1}) \longrightarrow \mathrm{Mod}(S_{0,4})$.
(that is : projection to the quotient by $\{\pm I\}$)

Birman–Hilden (1973) states that i is surjective & $\mathrm{Ker}\, i = \{\pm I\}$.

By seeing the Birman exact seq. $1 \to \pi_1(S,x) \xrightarrow{\text{Push}} \mathrm{Mod}(S,x) \xrightarrow{\text{Forget}} \mathrm{Mod}(S) \to 1$

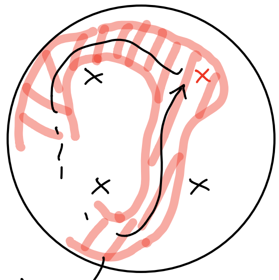
and its restriction $1 \to \pi_1(S,x) \xrightarrow{\text{Push}} \mathrm{PMod}(S,x) \xrightarrow{\text{Forget}} \mathrm{PMod}(S) \to 1$

($S = S_{0,3}$)

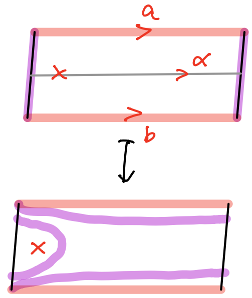
we will see that the point push map induces

an isomorphism $\text{Push} : \pi_{0,3} \xrightarrow{\cong} \mathrm{PMod}(S_{0,4})$

($\because \mathrm{PMod}(S_{0,3}) = 1$)



annular nbd



Dehn twist

$\mathrm{push}([\alpha]) = T_a T_b^{-1}$

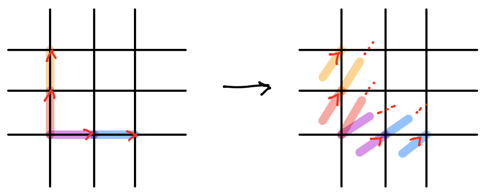
$SL(2,\mathbb{Z}) \curvearrowright S_{1,4}$ by linear auto.

this action is compatible w/ inclusions $\pi_{1,4} \subset \pi_{1,1}$, $\pi_{1,4} \subset \pi_{0,4}$

$\Gamma(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; \begin{matrix} a, d \equiv 1 \\ b, c \equiv 0 \end{matrix} \mod 2 \right\} \curvearrowright S_{1,4}$

$\downarrow$

$\Gamma < \mathrm{Mod}(S_{1,4})$: mapping classes of $\Gamma(2)$-linear actions



※ $\Gamma < \mathrm{PMod}(S_{1,4})$

let $-I \in \Gamma(2) \longmapsto \tau \in \Gamma$ ($: z \mapsto -z$ on E_2)

Lemma 2.1 $\exists$ surjection $\Gamma(2) \to \pi_{0,3}$ w/ $\mathrm{Ker} = \{\pm I\}$

s.t. the diagram

$$\begin{tikzcd} \Gamma(2) \arrow[r, two heads] \arrow[d, leftrightarrow, "\cong"'] & \pi_{0,3} \cong \mathrm{PMod}(S_{0,4}) \arrow[d, leftrightarrow, "\cong"] \\ \mathrm{Mod}(S_{1,4}) > \Gamma \arrow[r, two heads] \arrow[ur, phantom, "\circlearrowleft"] & \Gamma/\langle\tau\rangle \end{tikzcd}$$

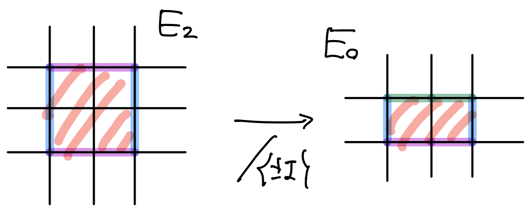
commutes.

pf) the restriction of $i : \mathrm{Mod}(S_{1,1}) \to \mathrm{Mod}(S_{0,4})$ (quotient by $\{\pm I\}$)

to $\Gamma(2) < SL(2,\mathbb{Z}) \cong \mathrm{Mod}(S_{1,1})$ is a hom. onto $\mathrm{PMod}(S_{0,4})$

w/ $\mathrm{Ker} = \{\pm I\}$

$\Gamma(2) \curvearrowright$ as $\mathrm{PMod}(S_{1,4})$



$\curvearrowleft \mathrm{PMod}(S_{0,4}) \overset{\mathrm{Push}^{-1}}{\cong} \pi_{0,3}$

($\pi_{0,3}$ - action : Point-Pushing map)

Since $\Gamma(2) \xrightarrow{\cong} \Gamma$ maps $-I \mapsto \tau$,

coming from $SL(2,\mathbb{Z}) \curvearrowright S_{1,4}$ $\qquad \pi_{0,3} \xrightarrow{\cong} \Gamma/\langle \tau \rangle$ & commutative follows. $\boxtimes$

- We briefly explain "Lem 2.1 in moduli sp. language"

$\mathcal{M}$: moduli sp. of $(C, \underbrace{P_1, P_2, P_3, P_4}_{\text{distinct 4 pts}})$ ...

smooth cpx genus 1 curve

§3. pf of main thm $\quad PMod(S_{0,4})$

Proposition 3.1. *Let H be a finite-index subgroup of $\pi_{0,4}$ satisfying the following properties:*

(a) *Let γ be an element of $\Gamma(2)/\{\pm 1\}$, considered as an outer automorphism of $\pi_{0,4}$, and α an automorphism of $\pi_{0,4}$ lying over γ. If $\alpha(H)$ is not conjugate to H, then $[H : H \cap \alpha(H)] > 2$.* $(\neq 2)$

(b) *H is not contained in $\pi_{1,4}$.*

(c) *Write H_0 for the intersection $H \cap \pi_{1,4}$. The permutations on the set $\pi_{0,4}/H$ induced by the four puncture classes represent four distinct conjugacy classes in the symmetric group on $N := [\pi_{0,4} : H] = [\pi_{1,4} : H_0]$ letters.*

Then

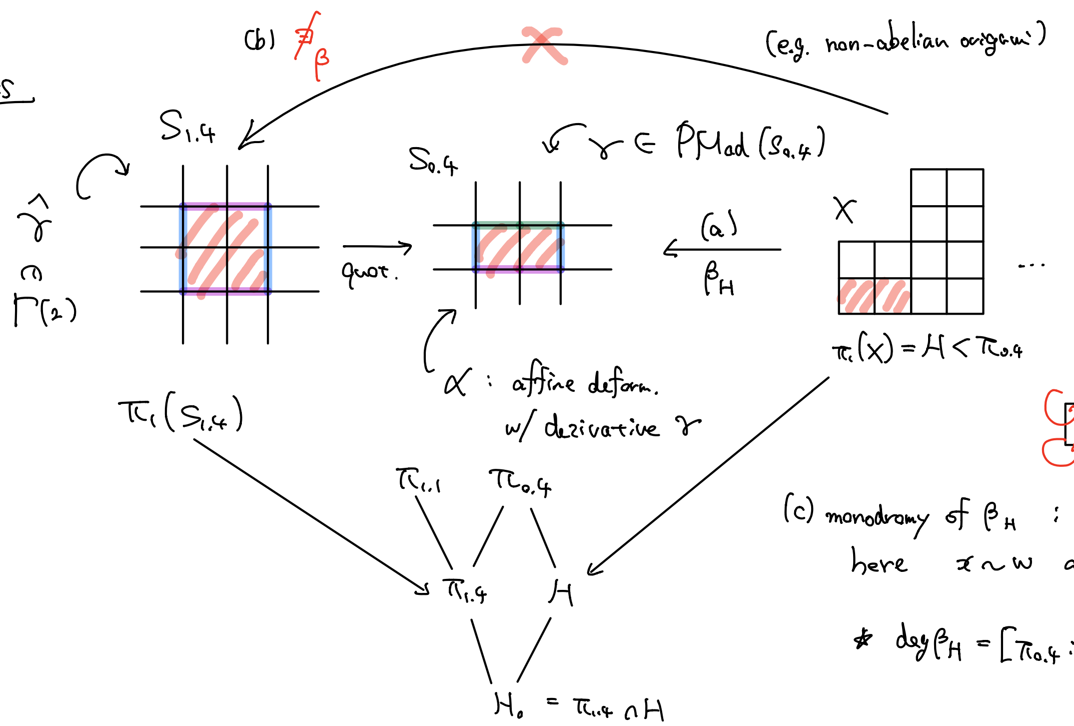
$$\mathrm{Stab}_{\mathrm{Mod}(S_{1,1})}([H_0]_{\pi_{1,1}}) = i^{-1}(\mathrm{Stab}_{\mathrm{PMod}(S_{0,4})}([H]_{\pi_{0,4}}))$$

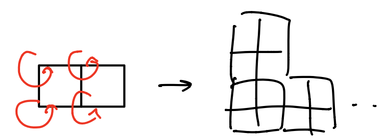
where

$$i\colon \mathrm{Mod}(S_{1,1}) \to \mathrm{Mod}(S_{0,4})$$

is the surjection defined in the previous section.

Settings





(c) monodromy of β_H : $\langle x, y, z, w \rangle < \mathfrak{S}_{[\pi_{0,4}:H]}$

here $x \sim w$ are distinct up to conj.

$\ast$ $\deg \beta_H = [\pi_{0,4} : H] = [\pi_{1,4} : H_0]$

$\Rightarrow \mathrm{Stab}_{\mathrm{Mod}(S_{1,1})}([H_0]_{\pi_{1,1}}) = i^{-1}(\mathrm{Stab}_{\mathrm{PMod}(S_{0,4})}([H]_{\pi_{0,4}}))$

where $i : \mathrm{Mod}(S_{1,1}) \to \mathrm{Mod}(S_{0,4})$: projection via $S_{1,1} \xrightarrow{\text{quot.}} S_{0,4}$

Q Let $\Delta < \Gamma(2)$: finite index, containing $\{\pm 1\}$

$\Gamma(2) < PMod(S_{1,4})$

$\Delta \xrightarrow{\text{proj.}} \Delta/\{\pm 1\}$, $\quad \Gamma(2) \to \pi_{0,3}$

$\Delta/\{\pm 1\} < \pi_{0,3} \cong PMod(S_{0,4})$: finite index

$\pi_{0,4} \xrightarrow{\text{Forget}} \pi_{0,3}$

Take preimage $G(\Delta) < \pi_{0,4}$: finite index

$G(\Delta) \supset \{c^{-1} z c \mid c \in \pi_{0,4}\}$ where z : forgotten loop-class

$$[z]_{\text{conj}_{\pi_{0,4}}} = \coprod_{j=1}^{\exists k} \{ g^{-1} z_j g \mid g \in G(\Delta) \}$$
where $\{ g^{-1} z_j g \mid g \in G(\Delta) \} = [z_j]$

$\pi_{0,4}/G(\Delta) = \{ w_1, \cdots, w_k \}$

$\pi_{0,4} = \coprod w_j \cdot G(\Delta) \qquad w_j^{-1} z w_j = z_j$

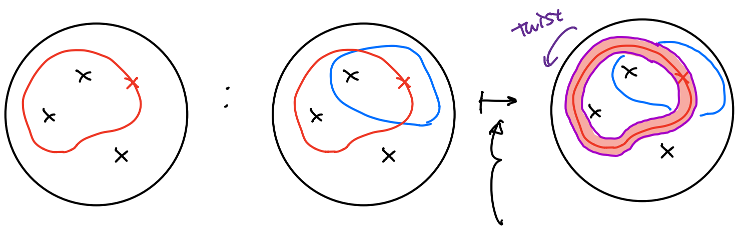
$$[z]_{\text{conj}} = \coprod_{j=1}^{k} \{ g^{-1} z g \mid g \in w_j G(\Delta) \} \quad (w_j^{-1} g = h)$$
$$= \coprod_{j=1}^{k} \{ h^{-1} z_j h \mid h \in G(\Delta) \}.$$

Lemma 3.2 $\Gamma(2) \curvearrowright \pi_{0,4}$ (as $PMod(S_{0,4})$) preserves the conj. class of $G(\Delta)$.

$\Gamma(2) \curvearrowright \{ [z_j] \mid j = 1, 2, \cdots, k \}$ is equivalent to $\Gamma(2) \curvearrowright \Gamma(2)/\Delta$: permutation action

pf) $\Gamma(2)/\{\pm I\} \cong \pi_{0,3} \curvearrowright \pi_{0,4}$ via Pushing map

$$1 \to \pi_{0,3} \xrightarrow{\text{Push}} Mod_{0,4} \xrightarrow{\text{Forget}} Mod_{0,3} \to 1$$



can be seen as a conjugation in $\pi_{0,3}$

$\longrightarrow$ this action preserves $G(\Delta)$ up to conjugacy.

Now let $F : S \to S_{0,3}$: cov. ass. to $\Delta/\{\pm I\} < \pi_{0,3}$
$\qquad\qquad\qquad\qquad \psi$
$\qquad\qquad\qquad\qquad p$: fix

$\longrightarrow$ the monodromy of F : $\pi_{0,3} \curvearrowright \pi_{0,3}/\Delta/\{\pm I\} = F^{-1}(p)$

$\qquad\qquad \cong \Gamma(2)/\Delta$: action on cosets of Δ

by assigning p to the forgotten puncture in $S_{0,4}$,

we may identify $[z_j] \subset [z]_{\text{conj}}$ w/ pt in $F^{-1}(p)$

$\qquad [z_j]$: $G(\Delta)$-conj, $\quad [z]_{\text{conj}}$: $\pi_{0,4}$-conj

& pushing action on these conj classes

w/ monodromy action on $F^{-1}(p) \sim \Gamma(2)/\Delta$ $\boxtimes$

Lemma 3.3 $\Delta < \Gamma(2)$: finite index, containing $\{\pm I\}$

$\Rightarrow \exists\, \Delta' \triangleleft \Delta$: finite index, containing $\{\pm I\}$, & $N_{\Gamma(2)}(\Delta') = \Delta$

Let Δ' as provided by this lemma.

$$\begin{tikzcd}
 & \Gamma(2) \arrow[d, "\text{proj}"'] & > & \Delta \arrow[d] & \rhd & \Delta' \arrow[d] \\
 & \pi_{0,3} & > & \Delta/\{\pm I\} & \rhd & \Delta'/\{\pm I\} \\
\text{preimage}: & \pi_{0,4} \arrow[u, "\text{forget}"] & > & G(\Delta) \arrow[u] & \rhd & G(\Delta') \arrow[u] \\
 & S_{0,4} \arrow[u, phantom, "\wr"] & & S \arrow[ll] \arrow[u, phantom, "\wr"] & & S' \arrow[ll, "\text{Normal}"'] \arrow[u, phantom, "\wr"]
\end{tikzcd}$$

$G(\Delta')$: fundamental group of a punctured surface

$\rightsquigarrow$ free group

arguments on free grps.

Let Δ' be a normal subgroup of Δ as provided by Lemma 3.3, and write G for $G(\Delta')$. Now G is the fundamental group of a punctured surface, and in particular is free. For each prime p, let

$$\phi_p\colon G \to H^1(G, \mathbf{F}_p)$$

be the canonical epimorphism. Let ℓ, p_1, p_2, p_3 be four distinct large odd primes. To be precise, let $\mathscr{P}_{bad}$ is the set of primes dividing either the index of G in $\pi_{0,4}$, or some index

$$[G : G \cap vGv^{-1}]$$

as vGv^{-1} ranges over the finite set of conjugates of G in $\pi_{0,4}$. We want p_1, p_2, p_3, ℓ to be larger than any element of $\mathscr{P}_{bad}$ and will mean by "large" precisely this. Note that the set of large primes is co-finite. For a subset A of $H^1(G, \mathbf{F}_p)$, the $\mathbf{F}_p$–span of A will be denoted by $\mathbf{F}_p[A]$.

Define

$$H_\ell = \phi_\ell^{-1}(\mathbf{F}_\ell[\phi_\ell([z_1])])$$

where $[z_1]$ is one of the conjugacy classes in $G(\Delta)$ making up $[z]_{\pi_{0,4}}$; in general it will be a finite union of conjugacy classes in G.

Let $[\gamma_1], [\gamma_2], [\gamma_3] \in \pi_{0,4}$ be the three puncture classes which do not vanish in $\pi_{0,3}$. For each $i = 1,2,3$, denote by Y_i the intersection of $[\gamma_i]$ with G; this is a finite union of conjugacy classes in G. Define

$$H_i = \phi_{p_i}^{-1}(\mathbf{F}_{p_i}[\phi_{p_i}(Y_i)]).$$

Finally, we define $H = H_\ell \cap H_1 \cap H_2 \cap H_3$. (We emphasize that the use of H_1, H_2, and H_3 is simply to ensure that the third condition of Proposition 3.1 is satisfied.)

Prop 3.4 $\mathrm{Stab}_{PMod(S_{0,4})}([H]_{\pi_{0,4}}) = \Delta/\{\pm I\}$.

Prop 3.5 H satisfies the conditions of Proposition 3.1

$\rightsquigarrow$ By Prop 3.1, $\quad \mathrm{Stab}_{\mathrm{Mod}(S_{1,1})}([H \cap \pi_{1,4}]_{\pi_{1,1}}) = i^{-1}\left(\mathrm{Stab}_{P\mathrm{Mod}(S_{0,4})}([H]_{\pi_{0,4}})\right)$

$$= i^{-1}\left(\mathcal{O}/\{\pm I\}\right)$$

$$= \Delta$$

$\Rightarrow$ Δ is the VG of origami

whose fundamental grp is $H \cap \pi_{1,4} < \pi_{1,1}$ //