COMPUTING PERIODIC POINTS ON VEECH SURFACES

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ABSTRACT. A non-square-tiled Veech surface has finitely many periodic points, i.e. points with finite orbit under the affine automorphism group. We present an algorithm that inputs a non-square-tiled Veech surface and outputs its set of periodic points. Applying our algorithm to Prym eigenforms in the minimal stratum in genus 3, we obtain experimental evidence that these surfaces do not have periodic points, except for the fixed points of the Prym involution.

arXiv:2112.02698v1 [math.DS] 5 Dec 2021 [CEFL21]

[CEFL21]

§1 Intro

A translation surface (TS) (X,ω) : R.S. X w/ nonzero holo 1-form ω.

$\mathrm{Aff}^+(X,\omega)$: grp. of ori. pres affine auto. on (X,ω)

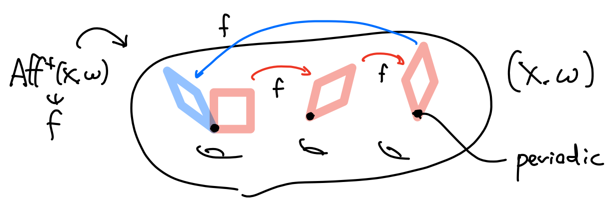
We consider the "periodic points" of $\mathrm{Aff}^+(X,\omega) \curvearrowright (X,\omega)$

For simplicity, we assume : (X,ω) is a Veech surface (VS).

$\overset{\text{def}}{\iff}$ $\mathrm{der}(\mathrm{Aff}^+(X,\omega)) < SL_2\mathbb{R}$: lattice

Then, $p \in (X,\omega)$ is called periodic $\Leftrightarrow$ has finite $\mathrm{Aff}^+(X,\omega)$-orbit

exm.



background periodic points of VS plays a role in :

- finite-blocking problem (for $p, Q \in X$, search for $S \subset X$ finite s.t. $\forall$ good P–Q passes through S)
- illumination problem (study of the illumination of rooms w/ mirrored walls)
- higher-rank orbit closures

Gutkin–Hubert–Schmidt '03 : $\forall$ non sq-tiled VS has finitely-many periodic points

Shinomiya '16 : explicit upper bounds on the number of ↑

explicit classification of periodic points in some cases :

Möller '06 : genus 2 case

Apisa - Saavedra - Zhang '20 : reg n-gon & double reg n-gon ($n=5, \geq 7$)

Wright '21 : Bouw - Möller exms

open determining the periodic pts on an arbitrary VS

Theorem 1.1. *There is an algorithm that, given a non-square-tiled Veech surface as input, outputs the periodic points on that translation surface.*

[CEFL21]

Str. of the algorithm :

I. we use the fact that periodic points must lie at rational height in cylinders.

II. from two cyl-directions, we produce a finite collection of line segments containing $\forall$ periodic pts.

III. applying well-chosen $f \in \mathrm{Aff}^+(X, \omega)$ to those line segment yields a new ____ ,

that intersects the first ____ in a finite set

IV. we determine which pts $\in$ ____ are periodic.

The algorithm is implemented in SageMath. (available at [CEFL21] in GitHub)

A novelty of the implementation : using Delaunay triangulations to decompose a VS into cylinders in $\forall$ sc-direction

Using this implementation, we investigated periodic points on Weierstrass Prym eigenforms in the minimal stratum in genus 3 discovered by McMullen '06.

Theorem 1.2. *For Weierstrass Prym eigenforms in genus 3 of nonsquare discriminant D at most 104, the periodic points are the fixed points of the Prym involution.*

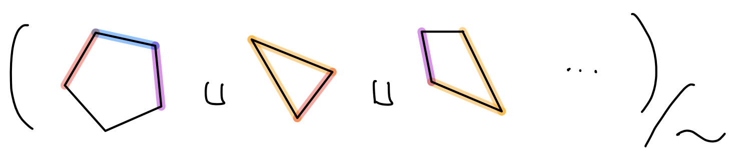
[CEFL21]

§2 Background

A translation surface (X, ω) : R.S. X w/ nonzero holo 1-form ω.

= a collection of polygons in $\mathbb{C}$ whose parallel sides are identified by translation / cut & paste equiv.

exm.

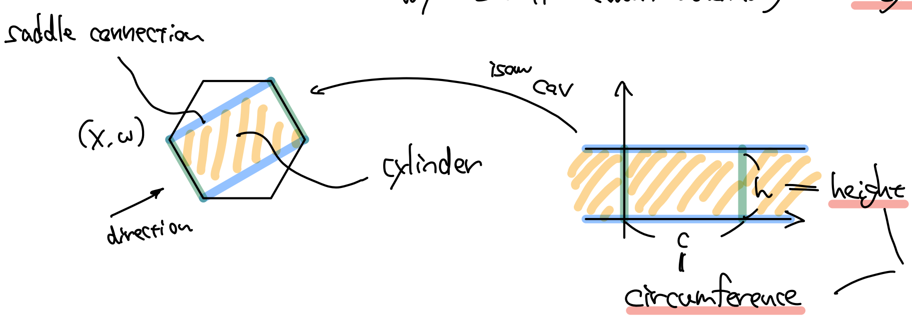


↑ (X,ω) admits a flat metric whose geod. { is a loc. segment, has well-def direction

pts $\in$ Zero(ω) : singularities or cone pts.

nonsingular line segments joining two singularities : saddle connections.

isometric image of an open right-angled flat Euclidian cylinder w/ saddle conn. boundary : cylinder



ratio $m = h/c$: modulus

We say (X,ω) is periodic in the direction θ

$\Leftrightarrow$ (X,ω) can be decomposed as a union of cylinders in the direction θ.

$SL(X,\omega) := \mathrm{Stab}_{SL(2,\mathbb{R})}(X,\omega) = D(\mathrm{Aff}^+(X,\omega))$

grp of affine self-homeos on (X,ω)

proj. image of $SL(X,\omega)$ in $PSL(2,\mathbb{R})$: the Veech group

If $SL(X,\omega) < SL(2,\mathbb{R})$ is a lattice, (X,ω) is called a Veech surface.

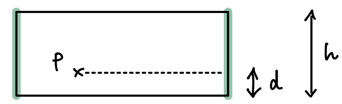
On a Veech surface, knowing that a surface is horizontally periodic tells us that the Veech group $\mathrm{SL}(X,\omega)$ contains a particular parabolic element: [CEFL21]

Proposition 2.1. *Let (X,ω) be a horizontally periodic Veech surface, decomposed into horizontal cylinders $\{C_i\}$. Then all moduli m_{C_i} have rational ratios, and $\mathrm{SL}(X,\omega)$ contains the matrix* $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ *where* $t := \mathrm{lcm}(m_{C_1}^{-1}, \dots, m_{C_n}^{-1})$. [McM '03, Lem 9.7]

The multiplicity of the cylinder C_i : $k_{C_i} := t/m_{C_i}$.

We say that a point p on a cylinder C has rational height

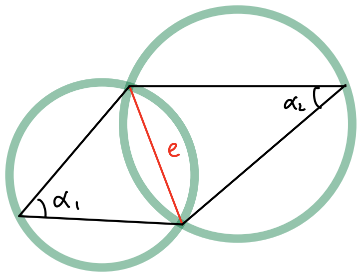
$\iff$ $d(p, \partial C)/h_C$ is rational.



Lemma 2.3 (Rational Height Lemma). *Let $\mathcal{C}$ be an equivalence class of cylinders so that any two have a rational ratio of moduli. If a periodic point belongs to the interior of a cylinder in $\mathcal{C}$ then it lies at rational height.*

Proof. For a proof see e.g. Lemma 5.4 in [Api20] or Lemma 2.3 in [Wri21]. □

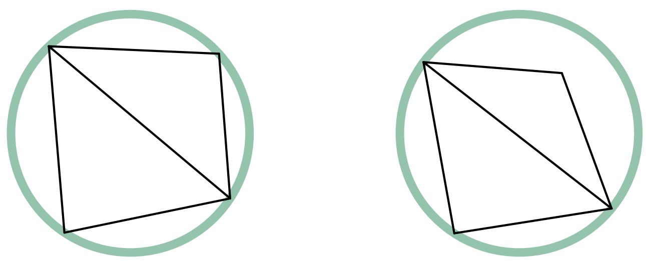
A Dleaunay Triangulation of (X, ω) is a geometric triangulation τ w/ the following Dleaunay condition:



no vertex lies inside the circumcircle of any triangle.

$\iff \alpha(e) = \alpha_1 + \alpha_2 \leq \pi$ in the left fig.

if the equality holds for some edge e, τ is called degenerate.



degenerate   not Delaunay

Masur and Smillie [9] : existence of D-Triangulation for $\forall (X, \omega)$

uniqueness for generic (X, ω)

Given $g \in SL(2, \mathbb{R})$ & τ : triangulation of (X, ω),

$g \cdot \tau = \{ g \cdot T \mid T \in \tau \}$ is a triangulation of (X, ω).

The Iso-Delaunay Region (IDR) for τ is the maximal open subset $\Omega \subset \mathbb{H} = SO(2,\mathbb{R}) \backslash SL(2, \mathbb{R})$

consisting of $[g]$ s.t. $g(\tau)$ is a Delaunay triangulation of $g(X, \omega)$.

$\longrightarrow$ Denote by (Ω, τ)

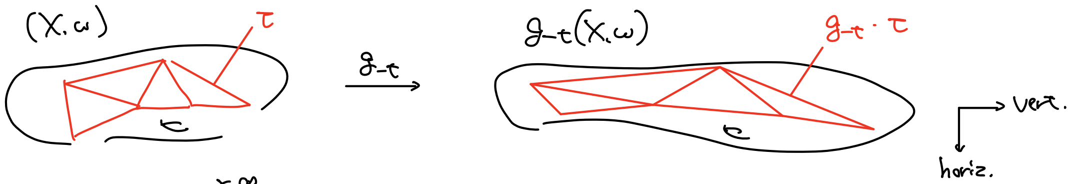
Bowman '08 : each IDR is a finite-area, geodesically convex hyperbolic polygon.
(possible w/ some vertices at ∞ : called cusps)

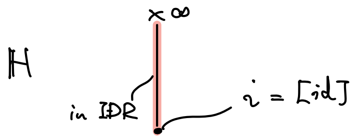
Consider the family of triangulations $\{g_{-t}\cdot\tau\}$ (vertical contraction), where

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$ is the Teichmüller contraction flow.

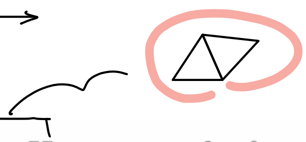
◦ τ is Delaunay for (X,ω) w/ IDR (S,τ) has a cusp at ∞

$\Rightarrow$ $g_{-t}\cdot\tau$ is Delaunay for $g_{-t}(X,\omega)$, $\forall t \geq 0$.





$$\frac{e^t\cdot i}{e^{-t}} = e^{2t}\, i$$



(Veech, '11)

Definition 2.5. *We say a hinge H composed of two triangles in a Delaunay triangulation τ is eternally Delaunay if, for all $t \geq 0$ the transformed hinge $g_{-t}\cdot H$ satisfies the local Delaunay property. If every hinge H in a Delaunay triangulation τ is eternally Delaunay, then we say the triangulation τ is eternally Delaunay.*

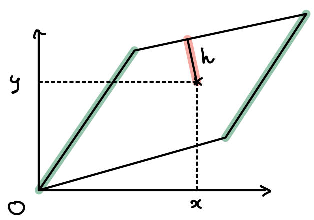
[CEFL21]

§3 Algorithm for finding periodic points

§§ 3.1. Rational height lemma (RHL)

Fix a coord. sys. for a cylinder s.t. some corner $\mapsto 0$.

$\Rightarrow$



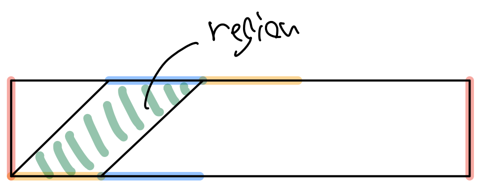
height $= ax + by + c \in K[x,y]$

"field of definition of (X,ω)"

Lem 2.3 gives the 'constraint'

(x,y) : periodic pt $\Rightarrow$ $ax+by+c \in \mathbb{Q}$.

<u>Df</u> A region is a conn. comp. of the intersection of two cylinders.

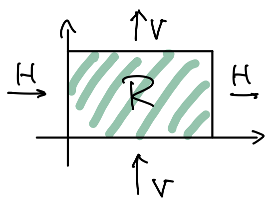


w.l.g., we may assume that our TS is horiz-vert periodic.

(up to rotation & shear)

Then, $\forall$ region R is $H \cap V$ for some horiz. / vert. cylinder H / V.

We embed R into $\mathbb{C}$ as



$\rightsquigarrow$ coord. defined

For $(x,y) \in R$: periodic,

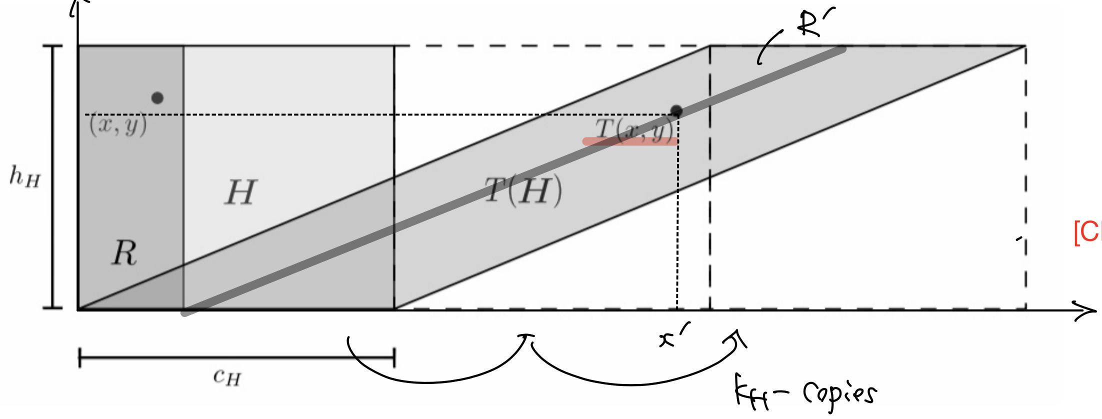
applying RHL, we obtain two constraints :

$$Q_0 = \frac{y}{h_H} \in \mathbb{Q} \qquad Q_1 = \frac{x}{h_V} \in \mathbb{Q} .$$

We will apply RHL again to obtain the 3rd constant.

By prop 2.1, we have $T = \begin{pmatrix} 1 & k_H/m_H \\ 0 & 1 \end{pmatrix} \in SL(X,\omega)$

In our embedding we may draw as follows :



[CEFL21]

T- action on the periodic pt ; $\mathbb{C} \to \mathbb{C}$: $(x,y) \mapsto (\underbrace{x + \frac{k_H}{m_H} y}_{x'}, y)$

Let $R' = H' \cap V'$ be the region containing $T(x,y) \in (X,\omega)$

and the left side of R' is d apart from ___ of R.

RHL on the horizontal cyl. H' gives the constraint

$$\frac{1}{h_{V'}}(x' - d) = \frac{1}{h_{V'}}\left(x + \frac{k_H}{m_H} y - d\right) \in \mathbb{Q}$$

In this way, we obtain three constraints.

§§ 3.2 The Constraint Reduction Lemma

Each constraint produces a meas-zero set of possible periodic pts, yet infinite & dense in cyl..

The following CRL resolve this.

Lemma 3.2 (Constraint Reduction Lemma). *Let K be a finite extension of $\mathbb{Q}$. Take three linear polynomials*

$$Q_i(x, y) = a_i x + b_i y + c_i \in K[x, y], \ i = 1, 2, 3$$

where K is a nontrivial extension of $\mathbb{Q}$. Then there exists an algorithm which takes Q_i as input, and outputs a polynomial

$$Q(x, y) = ax + by + c \in K[x, y]$$

such that for any $(x, y) \in K^2$, the constraints $Q_i(x, y) \in \mathbb{Q}$ imply $Q(x, y) = 0$. Moreover, if Q is a constant polynomial, then there exist $d_i \in \mathbb{Q}$ such that $\sum a_i d_i = \sum b_i d_i = 0$.

[CEFL21]

possible periodic pts in $V.S'$ (X,ω) ?

$$\text{§§3.1} \downarrow \text{RHL}$$

three constraints "$Q_i \in \mathbb{Q}$", $i = 1,2,3$

$$\text{§§3.2} \downarrow \text{CRL}$$

(non-const) linear eq. $Q = 0$ producing a line.

§§ 3.3 Producing a Finite set of Line Segments for Each Region

Lemma 3.3. *Consider a Veech surface (X, ω). Let R and R' be two regions in the same horizontal cylinder H, and on vertical cylinders V and V' respectively. Let T be the horizontal shear in $\mathrm{SL}(X, \omega)$ defined by $\boxed{\text{Proposition 2.1}}$, c_H the circumference of cylinder H and h_V the height of cylinder V. Then either $c_H/h_V \in \mathbb{Q}$, or all periodic points $p = (x, y) \in R$ such that $T(p) \in R'$ lie on some line segment in R.* [CEFL21]

Only the case we have to exclude

Lemma 3.4. *Consider a non-square-tiled Veech surface (X, ω). For each region R on the surface, there exists a set of line segments L such that all periodic points in R lie on some line segment $\ell \in L$.* [CEFL21]

Gutkin, Judge '00 : the "trace field" of (X, ω) cannot be $\mathbb{Q}$

Hubert, Lanneau '06 : ______ implies that $\frac{r c_H}{h_H} \cdot \frac{s c_V}{h_V} \notin \mathbb{Q}$,

where $P_H = \begin{pmatrix} 1 & r c_H/h_H \\ 0 & 1 \end{pmatrix}$ $P_V = \begin{pmatrix} 1 & 0 \\ s c_V/h_V & 1 \end{pmatrix}$

are the horiz, vert parabolic elements in $SL(X, \omega)$

§§ 3.4 From Segments to Points

We now have a finite set $\mathcal{S}$ of line segments on which periodic pts must lie.

$\forall$ Element in $SL(X, \omega)$ maps periodic pt $\leftrightarrow$ periodic pt.

$\longrightarrow$ If we pick $g \in SL(X, \omega)$ s.t. $g \cdot \mathcal{S} \cap \mathcal{S}$ is finite, we will have reduced our candidate line segments to points.

Finding such $g \in SL(X, \omega)$:

Lemma 3.5. *Given a finite set of line segments S on a translation surface (X, ω), there exists a hyperbolic element in the Veech group $g_0 \in \mathrm{SL}(X, \omega)$ such that none of the line segments in S are parallel to eigenvectors of g_0.* [CEFL21]

Construction of g_0 : $\underbrace{M_H}_{\text{Horiz.}}{}^{a} \underbrace{M_V}_{\text{Vert.}}{}^{a}$, $a \in \mathbb{N}$: always hyperbolic

Horiz. Vert. shears w.r.t. cylinders H, V

Explicit calc. shows that its eigenvectors are $\frac{1}{2}\left(a\tau_H \pm \sqrt{a^2\tau_H^2 + 4\,\tau_H/\tau_V}\right)$ $\square$

Lemma 3.6. *Consider a hyperbolic $h \in \mathrm{SL}(X,\omega)$, and a set S of line segments none of which are parallel to an eigenvector of h. Then there exists n such that $h^n \cdot S \cap S$ is a finite set of points.*

[CEFL21]

§§ 3.5 Reducing Candidate Pts to Periodic Pts

Lemma 3.7. *Consider a translation surface (X,ω) and a finite set S which contains all of its periodic points. There exists an algorithm which takes the generators of $\mathrm{SL}(X,\omega)$ as input and outputs the set of periodic points of the surface.*

[CEFL21]

... how to get an input ?

§§ 3.6 Pf of THM 1.1

Theorem 3.8 ($\boxed{\text{Theorem 1.1}}$). *Suppose (X,ω) is a non-square-tiled Veech surface. There is an algorithm that takes (X,ω) as input and outputs the periodic points on the surface.*

Proof. The algorithm has the following steps, whose correctness can be verified by the lemmas in the previous sections:

(1) Obtain the cylinder decomposition of the surface in two distinct periodic directions. Without loss of generality, by applying shears to the surface we may assume these two periodic directions are the horizontal and vertical directions. These cylinders partition the surface into connected components of intersections of a horizontal cylinder with a vertical cylinder. Call these regions $R_1, R_2, \ldots R_k$. In practice, Proposition 4.1 proves that these regions can be obtained through a computationally tractable triangulation, and subsection 4.2 describes the algorithm to obtain a cylinder decomposition producing such regions.

(2) For each region R_i, obtain the set of line segments L_i output by Lemma 3.4. We know that all periodic points on R_i lie on some line segment $\ell \in L_i$. Take $S = \bigcup_{i=1}^k L_i$. Then any periodic point on (X,ω) lies on some segment in S.

(3) By Lemma 3.6, find some $g \in \mathrm{SL}(X,\omega)$ such that $g \cdot S \cap S = P$ is a finite set of points. Any periodic point in S also lies on some segment in $g \cdot S$. Thus P is a finite set of points that contains all the periodic points of (X,ω).

(4) Apply the algorithm described in Lemma 3.7 on the set P, to obtain the set of points periodic under the entire Veech group.

$\square$

[CEFL21]

additional arguments are used to get a "nice" Delaunay triangulation.

§4 Results on Delaunay triangulations of TS.

To apply THM 1.1, we need a rep. of TS.

In this §, we detail results making a Delaunay triangulation of TS be computationally tractable.

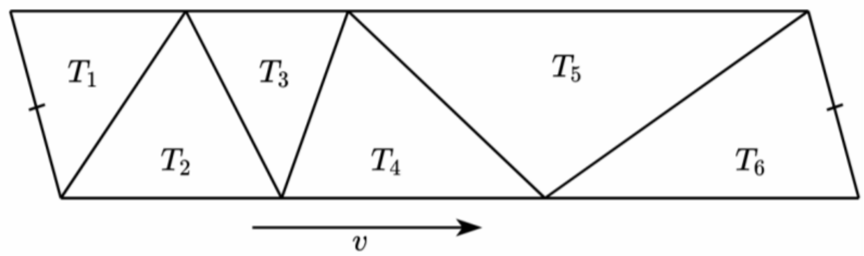


FIGURE 4.1. A refinement of a cylinder by a collection of triangles with edges parallel to direction v.

4.1. **Cylinder Refinement.** Call a collection $\{T_i\}$ of triangles in a triangulation of translation surface (X, ω) a *refinement of a cylinder C*, if $C = \bigsqcup T_i$ (see Figure 4.1 for an example). In addition, we say a triangulation τ of a translation surface *comes from a Delaunay triangulation $\bar{\tau}$* of a different translation surface, if $\tau = M \cdot \bar{\tau}$ for some $M \in SL(2, \mathbb{R})$.

We begin by stating the following "cylinder refinement proposition," that is used in a number of steps of our implementation of the algorithm given in Theorem 1.1, as in determining intersection regions of horizontal and vertical cylinders of the surface.

Proposition 4.1 (Cylinder Refinement). *Let (X, ω) be a Veech surface, periodic in direction v. Then there exists a triangulation τ of (X, ω), coming from a Delaunay triangulation, such that every triangle $T_i \in \tau$ has an edge parallel to v.*

Corollary 4.2. *Let (X, ω) be a Veech surface, periodic in direction v with cylinder decomposition $\{C_i\}$ in direction v. Then there exists a triangulation τ of (X, ω), coming from a Delaunay triangulation, such that for each cylinder C_i, there is a collection of triangles in τ that form a refinement of C_i.*

[CEFL21]

τ can be not Delaunay but Delaunay up to $SL(2,\mathbb{R})$-action

In §4.2, we present an algorithm:

(Delaunay triangulation & periodic dir.) $\mapsto$ (triangulation coming from Delaunay that forms cyl. refinement.)

§5 Experimental result

§§5.1. Eigenforms in $\mathcal{H}(2)$

Eigenforms in $\mathcal{H}(2)$: constructed by McMullen '03 & Calta '04

come from L-shaped polygons

Möller '06 : the periodic pts of such TS are fixed pts of hyperel. involution

← Checked via the algorithm as:

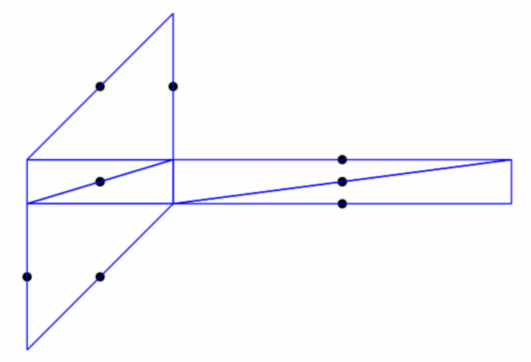


FIGURE 5.1. The dots are the computed periodic points on L table with parameter $D = 44$. Note that the same point may appear twice as opposite parallel sides are identified.

§§5.2 Prym eigenforms in genus 3

Prym involution : McMullen '06 's extension of the concept of hyperelliptic involution

loci of Prym eigenforms are closed $SL(2,\mathbb{R})$-invariant.

Each integer $D \geq 8$ w/ $D \equiv 0, 1, 4 \bmod 8$ determines two S-shaped surfaces:

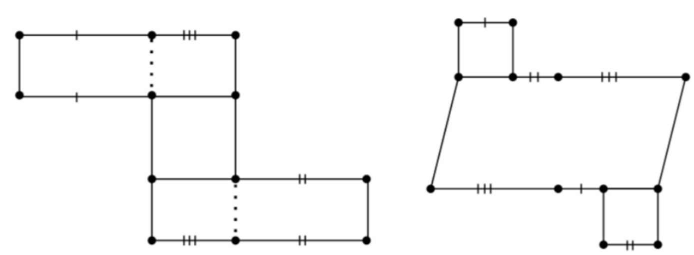


FIGURE 5.2. Example form of Model $A+$ surface left, and Model $A-$ surface right.

and known to admit Prym invol. w/ 3 fixed pts out of $Z(\omega)$.

The algorithm on $D = 104$ shows that $\forall$ periodic pts are the fixed pts of the Prym involution

[CEFL21] Z. Chowdhury, S. Everett, S. Freedman, D. Lee, Computing Periodic Points on Veech Surfaces. arXiv:2112.02698