Periodic points on the regular and double n-gon surfaces

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November 6, 2020

Abstract

Using the transfer principle, we classify the periodic points on the regular n-gon and double n-gon translation surfaces and deduce consequences for the finite blocking problem on rational triangles that unfold to these surfaces.

§I. Intro

arXiv 2011.02668v1 (Nov. 2020)

$\Omega\mathcal{M}_g = \{ (X,\omega) \mid X:$ cpt. R.S. of genus g, ω : Abelian differential on $X \}$

stratified

$= \coprod_{\mu} \Omega\mathcal{M}_g(\mu)$ where $\mu = (m_1, m_2, \cdots)$: order of zeros w/ $\sum m_j = 2g-2$

$\Omega\mathcal{M}_g \curvearrowleft GL_2^+(\mathbb{R})$: affine action, preserving strata

generated by rotation, dilation, & Teichmüller flow

$\omega \mapsto c\omega$ $\qquad$ $\omega \mapsto \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,\omega \\ \mathrm{Im}\,\omega \end{pmatrix}$

for $(X,\omega) \in \Omega\mathcal{M}_g$,

$\mathrm{Orb}_{GL_2^+(\mathbb{R})}(X,\omega)$ projects to $\mathbb{H}/SL(X,\omega)$

where $SL(X,\omega) := \mathrm{Stab}_{GL^+}(X,\omega) \subset SL_2(\mathbb{R})$: Veech group

(X,ω) : Veech surface $\Leftrightarrow$ $SL(X,\omega)$: lattice

$\Leftrightarrow$ $\mathrm{vol}(\mathbb{H}/SL(X,\omega)) < \infty$

In this paper, we deal with periodic pts on trans. surfs.

Def 1.1 A point p in a Veech surface (X,ω) is called periodic

$\Leftrightarrow$ $p \notin \mathrm{Zero}(\omega)$ & $|\mathrm{Orb}_{\mathrm{Aff}^+(X,\omega)}(p)| < \infty$

grp of affine diffeos.

cf. Lem (Möller, '06) $\quad p \in (X,\omega)$ is periodic or in $\mathrm{Zero}(\omega)$

$\Leftrightarrow$ $\exists$ (algebraic) section of some fibred surface $f : \hat{X} \to C$ associated to the Teich. curve, which passes through p on the fibre X.

Consider the following translation surfaces :

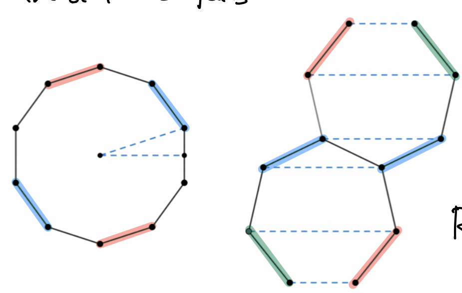
$n \in \mathbb{N}$.

$v = \begin{cases} 1 & \text{if } n \equiv \\ 2 & \text{if } n \equiv \end{cases}$ mod 4

$e = n$

$f = 2$

regular n-gon surface



R_n^1 $\qquad$ R_n^2

← double n-gon surface

$v = \begin{cases} 2 & \text{if } n \equiv 0 \\ 1 & \text{if } n \equiv 2 \end{cases}$ mod 4

$e = \frac{n}{2}$

$f = 1$

$2g \quad = \frac{n}{2} - v + f$

n : even $\qquad$ n : odd

regular n-gon w/ opposite sides glued

double of regular n-gon w/ corresponding sides glued

"regular n-gon translation surface" w/ $\omega = dz$: natural Abelian diff.

Both are hyperelliptic w/ involutions : affine diffeo. w/ der. $= -I$

i.e. in R_n^1 : π-rotation around the center of face

in R_n^2 : π-rotation around the center of edges

def

R : hyperelliptic $\overset{\text{def}}{\iff}$ $\exists$ hyp. ell. involution

that is : $i : R \to R$: auto. s.t. $i^2 = \mathrm{id}$, $|\mathrm{Stab}\, i| = 2g(R)+2$

fixed pts of hyperelliptic involutions are called Weierstrass points.

Main result : classification of periodic pts on R_n^1 & R_n^2

> **Thm 1.3** When $n \geq 5$ & $n \neq 6$,
>
> the periodic pts on R_n^1 & R_n^2 are exactly Weierstrass pts that are not singularities.

$n = 5, 8, 10$: Möller, '06.

when $n = 3, 4, 6$, surfaces are tori & have infinitely many periodic pts.

thm 1.3 $\underset{\text{apply}}{\rightsquigarrow}$ 'the blocking problem'

Def 1.5 Two pts P, Q on a translation surface M are finitely blocked

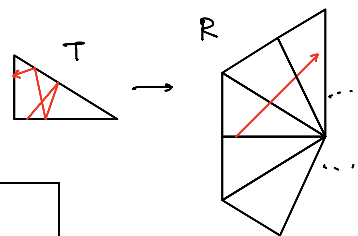
$\iff$ $\exists S \subset M \setminus \{P, Q\}$: finite s.t. every straight line joining P, Q pass through a point in S.

※ for billiard table & its trajectory, defined similarly.

> **Cor 1.6** When $n \geq 5$ & $n \neq 6$,
>
> the pairs of finitely blocked pts on $R = R_n^1$ or R_n^2 consist precisely of
>
> pairs $(P, i(P))$ where $\forall P \in R \setminus \mathrm{Sing}(\omega)$ & $\forall i$: hyperelliptic involution

Via the unfolding construction of Katok-Zemlyakov ('75),

Cor 1.6 yields similar results for rational right triangle billiards.



> **Cor 1.7** When $n \geq 5$ & $n \neq 6$,
>
> The $\left(\frac{\pi}{2}, \frac{\pi}{n}, \pi - \frac{\pi}{2} - \frac{\pi}{n}\right)$-triangle admits finitely blocked pts iff n is even.
>
> Furthermore the only such a pair is the $\frac{\pi}{n}$-vertex & itself.

Note. Classification of periodic pts in strata of Abelian differentials : Apisa '20
quadratic differentials : Apisa-Wright '17

Classification of periodic pts on every surf. in ΩM_2 : Möller '06 & Apisa '17 (arXiv)

$\xleftarrow{\text{using}}$ McMullen's ('07) classification of $GL_2^+(\mathbb{R})$-orbit closure of genus two strata.

§ 2. Preliminaries

Denote by Γ_n the VG of the regular n-gon translation surface.

$$r_n = \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix}, \qquad s_n = \begin{pmatrix} 1 & 2\cot\frac{\pi}{n} \\ 0 & 1 \end{pmatrix}.$$

Theorem 2.1 ([Vee89] (Definition 5.7, Theorem 5.8); see also [MT02] (Theorem 5.4)). *When n is even, Γ_n is generated by $\{r_n^2, s_n, r_n s_n r_n^{-1}\}$ and is isomorphic to the $(n/2, \infty, \infty)$ triangle group. In particular, $\mathbb{H}/\Gamma_n$ has two cusps.*

When n is odd, Γ_n is generated by $\{r_n, s_n\}$ and is isomorphic to the $(2, n, \infty)$ triangle group. In particular, $\mathbb{H}/\Gamma_n$ has one cusp.

~~Rem~~ It is known that for Veech surfaces

$$\{\text{maximal parabolic subgrp of VG}\} \overset{1:1}{\longleftrightarrow} \{\text{cylinder direction}\}$$
$$\text{Stab}_{VG}(\theta) \longleftrightarrow \theta$$

VG $\overset{\text{affine}}{\curvearrowright}$ {cylinder directions} $\overset{\text{corres.}}{\longleftrightarrow}$ VG $\overset{\text{conj.}}{\curvearrowright}$ {max. parab. VG-subgrp}

orbit ↳ conj. classes $\underset{\text{corres.}}{\longleftrightarrow}$ cusps of VG

Thus, Orb_{VG} (cyl. dir) $\overset{\text{corres.}}{\longleftrightarrow}$ cusps of VG

→ every cylinder direction is one of prescribed (below) directions up to VG-action

by thm 2.1, on R_n^2 (double n-gon), any cyl. dir. can be sent to any other by VG.
↳ exactly one cusp → $\#(\text{cylinder}) = \frac{n-1}{2}$

on R_n^1 (regular n-gon), there are exactly two VG-orbit of cyl. dir.

$\boxed{n = \text{even}}$

$\#(\text{cyl}) = \lceil \frac{n}{4} \rceil$

horizontal

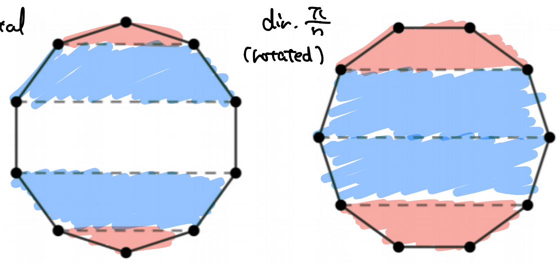


Figure 1: Two cylinder directions for the regular decagon.

dir. $\frac{\pi}{n}$ (rotated)

$\#(\text{cyl.}) = \lceil \frac{n-2}{4} \rceil = \lfloor \frac{n}{4} \rfloor$ ($\because n$: even)

these two belong to different VG-orbits by the difference of $\#(\text{cyl})$

vertices : $i\exp\frac{2j\pi i}{n} =: v_j$ $(j = 0, 1, \cdots, n-1)$

↳ cylinder heights : $h_j = \text{Im}(v_{j+1} - v_j) = \cdots$

$= 2\sin\frac{\pi}{n}\sin\frac{(2j+1)\pi}{n}$ $(j = 0, 1, \cdots \#(\text{cyl}) - 1)$.

vertices : $i\exp\frac{(2j+1)\pi i}{n} =: v_j'$

cylinder heights : $h_j' = \text{Im}(v_{j+1}' - v_j') = \cdots$

$= 2\sin\frac{\pi}{n}\sin\frac{(2j+2)\pi}{n}$ $(j = 0, 1, \cdots \#(\text{cyl}) - 1)$

$\boxed{n : \text{odd}}$

Similarly, $h_j = 2\sin\frac{\pi}{n}\sin\frac{k\pi}{n}$ $(k = 0, 1, \cdots \#(\text{cyl}) - 1)$.

Lem 2.4 (a part of McMullen '06)

For rational number $0 < \alpha < \beta \leq \frac{1}{2}$,

$$\frac{\sin \pi\alpha}{\sin \pi\beta} \text{ is rational} \iff (\alpha, \beta) = \left(\frac{1}{6}, \frac{1}{2}\right)$$

Lem 2.5 On regular 2n-gon translation surfaces, at least one cylinder direction has the property:

(*) heights (circumferences) of cylinders in this direction has irrational ratio.

If $n \not\equiv 0, 6 \mod 12$ then every cylinder direction has property (*).

Moreover, any two neighboring parallel cylinders, their heights (circumferences) has irrational ratio.

(∵) By above Remark, any cyl. dir. results in horizontal or $\frac{\pi}{n}$ (n: even) - direction.

$\Rightarrow$ ratio of heights : of the form $\frac{h_j}{h_k}$ or $\frac{h_j'}{h_k'}$ and hence $\frac{\sin \pi\alpha}{\sin \pi\beta}$ $(0 < \alpha < \beta \leq \frac{1}{2})$ up to inverting.

by Lem 2.5 it is rational iff $(\alpha, \beta) = \left(\frac{1}{6}, \frac{1}{2}\right)$

$\cdots$ it never occurs when n : odd.

Suppose n : even. We have $\frac{h_j}{h_k} = \frac{\sin (2j+1)\pi/n}{\sin (2k+1)\pi/n}$: assuming $j < k$, it is rational iff

$\frac{2j+1}{n} = \frac{1}{6}, \frac{2k+1}{n} = \frac{1}{2} \rightsquigarrow n = 12j+6, k = 3j+1$

In the same way for $\frac{h_j'}{h_k'}$; assuming $j < k$, it is rational iff

$\frac{2j+2}{n} = \frac{1}{6}, \frac{2k+2}{n} = \frac{1}{2} \rightsquigarrow n = 12j+12, k = 3j+2$

thus the result follows ☒

Lem 2.7 regular n-gon translation surfaces (R, ω) are primitive.

i.e. there is no covering $f : (R, \omega) \to (S, \eta)$ s.t. $g(R) > g(S)$ & $f^*\eta = \omega$.

(translation cover of (R, ω))

(pf) Assume $\exists f$ as .

by Lem 2.5 & Rem, $\exists$ cyl. dir. w/ irrational height ratio & $\#(\text{cyl.}) \geq g$

$|Z(\omega)| \leq 2 \Rightarrow |Z(\eta)| \leq 2$

$\xrightarrow{\text{R.H. formula}}$ $g(S) = g(R) - 1$ & $|Z(\eta)| = 2$

$g(R) \leq 3.$

$n = 10$ or 14

again by Lem 2.5, both case $n = 10, 14$

we have no cyl. dir. w/ rational height ratio

$\Rightarrow g(S) = g(R)$, contradiction ☒

Cor 2.8 For regular n-gon translation surfaces (R,ω),

$$SL(R.\omega) \cong \mathrm{Aff}^{+}(R.\omega).$$

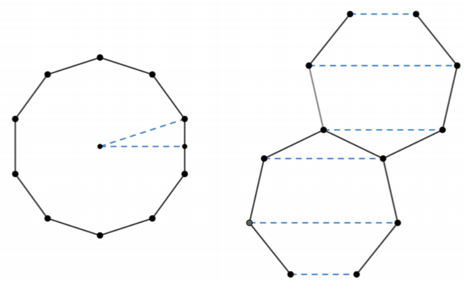
§3. pf of Main results

Def 3.1 R_n : regular n-gon trans surf. Γ_n : VG of R_n.

denote by $P_n \in R_n$: $\begin{cases} (n: \text{even}) & \text{the center of n-gon.} \\ (n: \text{odd}) & \text{the unique cone pt.} \end{cases}$ ※ fixed by VG.

Prop 3.2 $^{\forall} p \in R_n$: periodic pt, $\mathrm{Orb}_{\Gamma_n}(p)$ contains a point on $\begin{cases} F_0(P_n) \\ \text{or } F_{\frac{\pi}{n}}(P_n) \ (n: \text{even only}) \end{cases}$

where F_θ is the directional foliation on R_n.



these segments are called candidate line segments

Figure 2: After applying Proposition 3.2, any periodic point can be assumed to lie on one of the dashed lines or its image under the hyperelliptic involution.

Outline)

The transfer principle X : top. sp. $G \curvearrowright X \curvearrowleft H$: top. grps acting continuously

$\Rightarrow$ there is a 1-1 correspondence among $\begin{matrix}\text{closed} \\ \text{dense}\end{matrix}$ orbits :

closed (resp. dense) G-orbits on X/H $\leftrightarrow$ closed (resp. dense) $G\times H$-orbits on X $\leftrightarrow$ closed (resp. dense) H-orbits on $G\backslash X$

$$\begin{matrix} \Psi & & \Psi & & \Psi \\ \mathrm{Orb}_G(xH) & \longleftrightarrow & \mathrm{Orb}_{G\times H}(x) & \longleftrightarrow & \mathrm{Orb}_H(Gx) \end{matrix}$$

In this pf $\begin{cases} G \text{ is the VG } \Gamma_n \\ H \text{ is the unipotent subgrp } U = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \middle| s \in \mathbb{R} \right\} < SL(2,\mathbb{R}) \end{cases}$

Rem $SL(2.\mathbb{R})/U \cong \mathbb{R}^2 \setminus \{0\}$

$$\begin{matrix} \Psi & & \Psi \\ g & \longleftrightarrow & g\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{matrix}$$

Thm (Dani '78)

the only U-orbits of $\Gamma_n \backslash SL(2\mathbb{R})$ are closed or dense.

(Std. linear action)

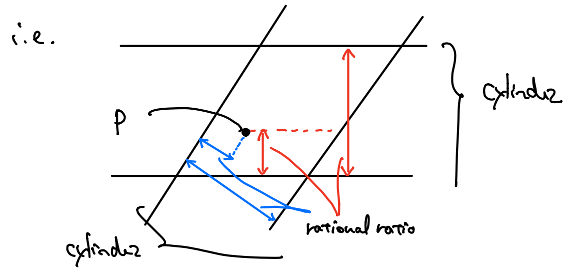
using Transfer principle $\xrightarrow{\quad}$ ... $\xrightarrow{\quad}$ & thm 2.1 (description of Γ_n) only vectors w/ closed orbit are parallel to $\Gamma_n\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\Gamma_n\begin{pmatrix} \cos \pi/n \\ \sin \pi/n \end{pmatrix}$ (n: even)

Since $\Gamma_n \cdot p$ is finite, it should contain a pt in horizontal leaf or $\frac{\pi}{n}$-directional leaf (n: even). ☒

Lem 3.5 (rational height lemma, Apisa '20)

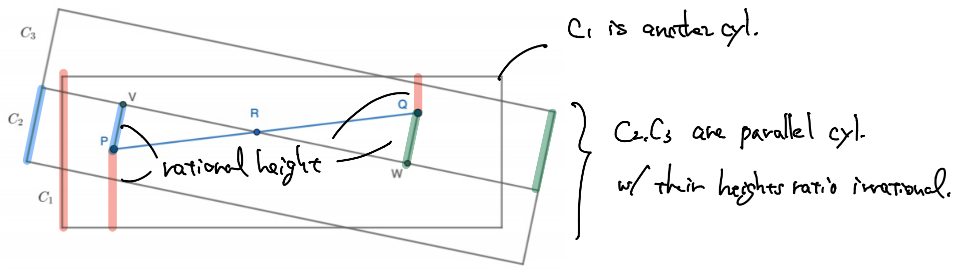
A periodic pt P on a Veech surface has rational height in any cylinder containing it.

i.e.



Lemma 3.6 (rough statement)

Let C_1, C_2, C_3 be cylinders as follows:



Take P, Q so that (i) $\overline{PQ}$ is neither $\left\{ \begin{matrix} /\!/ \\ \perp \end{matrix} \right\}$ to core(C_1) & core(C_3)

(ii) $\overline{PQ} \subset C_1$, $\overline{PQ} \subset C_2 \cup C_3$; properly contained

(iii) $\overline{PQ} \cap (\partial C_2 \cup \partial C_3) = \{R\}$ w/ $R \neq P, Q$.

(iv) P (resp. Q) has rational height in C_1, C_2 (resp. C_1, C_3)

$\Rightarrow$ no interior pts in $\left\{ \begin{matrix} \overline{PR} \\ \overline{RQ} \end{matrix} \right\}$ has rational height in $\left\{ \begin{matrix} C_1, C_2 \\ C_1, C_3 \end{matrix} \right\}$

Thm 1.3 When $n \geq 5$ & $n \neq 6$,

the periodic pts on R_n^1 & R_n^2 are exactly Weierstrass pts that are not singularities.

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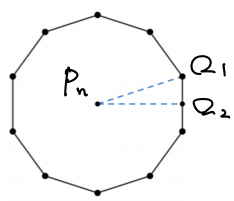
pf of thm 1.3) Let P be a periodic pt on R_n.

(I) n : even by prop 3.2, We may assume P lies on a candidate segment,

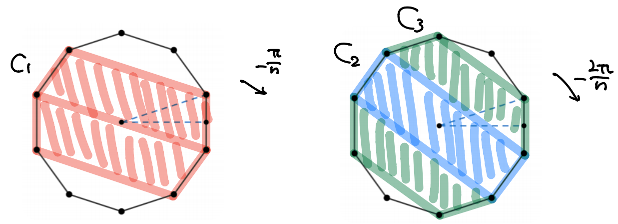


which we denote by $\overline{P_n Q}$.

(up to T_n-action)

$\to$ Each endpt (P_n, Q) is either $\left\{ \begin{matrix} \text{Weierstraß pt or} \\ \text{singularity.} \end{matrix} \right.$

For Lem 3.5, we take cyl. C_1 & C_3 as follows:



by Lem 2.5 (neighboring cyl.) C_2, C_3 has irrational height ratio.

Since ∂(candidate segments) are either $\left\{ \begin{array}{l} \text{periodic pt or} \\ \text{singularity} \end{array} \right\}$,
they have rational height in any cylinder containing it by Lem 3.5

By Lemma 3.6, $\forall$ pt in $\overline{PQ}^{i}$ has irrational height in one of C_1, C_2, C_3.

$\longrightarrow$ By Lemma 3.5, such a pt cannot be periodic.

II) n : odd ... Though more precise modification is needed it follows almost similar strategy. $\otimes$

Cor 1.6 When $n \geq 5$ & $n \neq 6$,

the pairs of finitely blocked pts on $R = R_n^1$ or R_n^2 consist precisely of pairs $(P, i(P))$ where $^\forall P \in R \setminus \mathrm{Sing}(\omega)$ & $^\forall i$: hyperelliptic involution

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Outline) $^\forall$ trans. surf. (X, ω) that is not an origami, we may take :

$$\underset{(\text{Möller '06})}{} \qquad \underset{(\text{Apisa, Wright '17})}{}$$
$$(X, \omega) \xrightarrow[\pi_{min}]{} (X_{min}, \omega_{min}) \xrightarrow[\pi]{} (Q_{min}, q_{min})$$
minimal translation   minimal half-translation

(Apisa & Wright '17) states that $^\forall (p, q)$: finitely blocked pair one of the following occurs :

(i) p, q are periodic pts or zeros of ω

& the blocking set consists of all other periodic pts.

(ii) Neither p nor q are periodic pts or zeros of ω

but proj. image of p, q in Q_{min} coincide $\rightarrow \overline{p} \in Q_{min}$

& the blocking set consists of all periodic pts & $\pi_{min}^{-1}(\pi^{-1}(\overline{p}))$.

Cor 1.7 When $n \geq 5$ & $n \neq 6$,

The $\left(\frac{\pi}{2}, \frac{\pi}{n}, \pi - \frac{\pi}{2} - \frac{\pi}{n}\right)$ - triangle admits finitely blocked pts iff n is even.

Furthermore the only such a pair is the $\frac{\pi}{n}$ - vertex & itself.