

A comb of origami curves in the moduli space M_3 with three dimensional closure

Frank Herrlich · Gabriela Schmithüsen

Received: 30 July 2006 / Accepted: 5 April 2007 / Published online: 2 June 2007
© Springer Science+Business Media B.V. 2007

Abstract The first part of this paper is a survey on Teichmüller curves and Veech groups, with emphasis on the special case of origamis where much stronger tools for the investigation are available than in the general case. In the second part we study a particular configuration of origami curves in genus 3: A “base” curve is intersected by infinitely many “transversal” curves. We determine their Veech groups and the closure of their locus in M_3 , which turns out to be a three dimensional variety and the image of a certain Hurwitz space in M_3 .

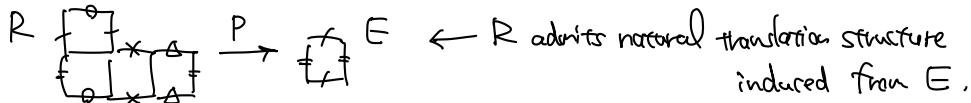
“quaternionic origami”

Keywords Teichmüller curves · Veech groups · Origamis · Hurwitz spaces

Mathematics Subject Classifications (2000) 14H10 · 14H30 · 32G15 · 53C10

§ 1.–3. Intro

• An origami is a covering $\theta = (\rho: R \rightarrow E)$ of the unit square torus $E = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ branched over 0.



• Veech group $\Gamma(\theta)$ of an origami θ

is the stabilizer of translation-isomorphism class

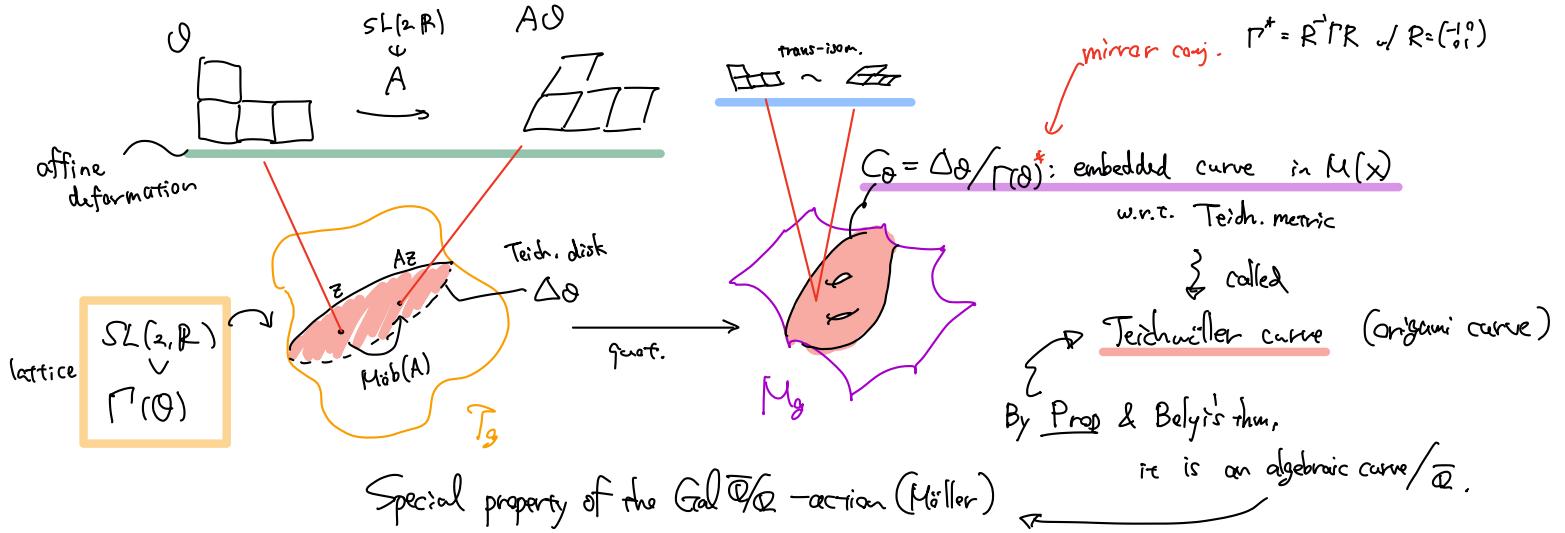
under the affine-deform. action of $SL(2, \mathbb{R})$

Proposition 5 Let $O = (p: X \rightarrow E)$ and $O' = (p': X' \rightarrow E)$ be two origamis and U (resp. U') the corresponding subgroups of F_2 . Then

- O is isomorphic to O' if and only if U is conjugate to U' in F_2 .
- $C(O) = C(O')$ if and only if there is a $\gamma \in Aut^+(F_2)$ such that $\gamma(U) = U'$.

• $\pi_1(F_2)$ is the fundamental grp $\pi_1(X) \hookrightarrow \pi_1(E) = F_2$

Prop $\Gamma(\theta)$ is a finite-index subgp of $SL(2, \mathbb{Z})$.



Theorem A[†] Let Δ be a Teichmüller disk in the Teichmüller space T_g .

- The image $\text{proj}(\Delta)$ is an algebraic curve C in M_g , if and only if the Veech group Γ is a lattice in $SL_2(\mathbb{R})$.
- In this case, \mathbb{H}/Γ^* is the normalization of the algebraic curve C .

Hence, the Veech group detects whether a Teichmüller disk leads to a Teichmüller curve and, if this happens, it determines the Teichmüller curve up to birationality.

Theorem B: (Schmithüsen)

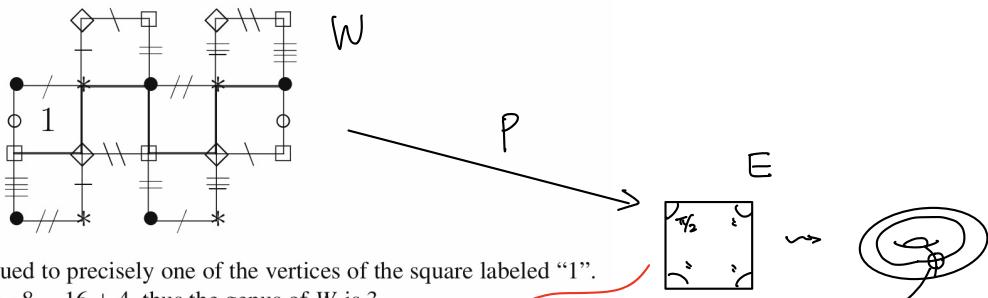
- Let U be the finite index subgroup of F_2 defined in (3) for the origami O .
- Let $\hat{\beta} : \text{Aut}(F_2) \rightarrow \text{GL}_2(\mathbb{Z}) = \text{Out}(F_2)$ be the natural projection and $\text{Aut}^+(F_2) := \hat{\beta}^{-1}(\text{SL}_2(\mathbb{Z}))$.
- Let $\text{Stab}(U) := \{\gamma \in \text{Aut}^+(F_2) \mid \gamma(U) = U\}$ be the stabilizing group of U .

Then for the Veech group $\Gamma(O)$ holds: $\Gamma(O) = \hat{\beta}(\text{Stab}(U))$.

§4. The quaternion origami.

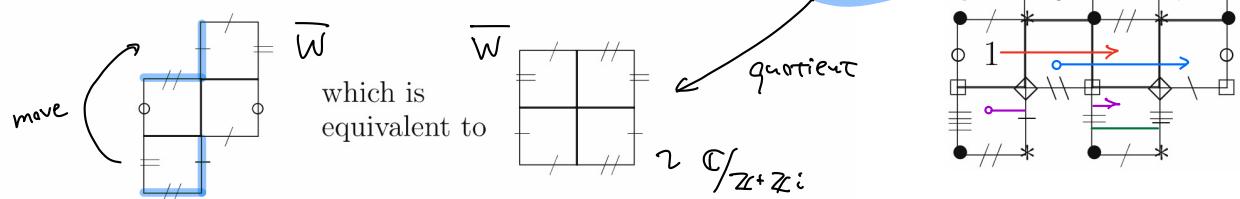
4-1. origami W

Using the combinatorial definition of origamis, W can be described by 8 squares that are glued as indicated (edges are glued if they have the same label).



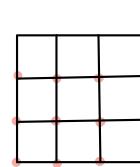
Note that every vertex is glued to precisely one of the vertices of the square labeled "1". Euler's formula gives $2 - 2g = 8 - 16 + 4$, thus the genus of W is 3.

The total angle at every vertex is 4π , so they are all ramification points of order 2 for the covering $p: W \rightarrow E$ of degree 8 to the torus E . Recall that p is obtained by mapping each of the eight squares to E . The map p can be decomposed as follows: Observe that "translation by 2 to the right" is an automorphism of W , and let $q: W \rightarrow \overline{W}$ be the quotient map for this automorphism. Then \overline{W} is the origami



We call the points in the group $\mathbb{G}/\mathbb{Z}_{12}$
of order k dividing $n \geq 2$
the n -torsion points of E .

e.g.



3-torsion points

$$E[3] \text{ w/ } |E[3]| = 3^2$$

They form a group $E[n]$ of order n^2 .

$$\cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$

$$\begin{cases} E[n] = E/\mathbb{Z} \oplus n\mathbb{Z}, \\ E \rightarrow E/E[n]: \text{origin} \end{cases}$$

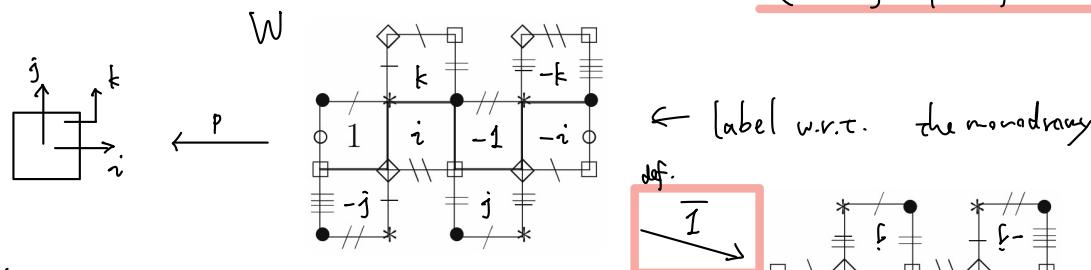
4-2. Automorphism group

We may check that $p: W \rightarrow E$ is a normal cover.

$$\text{Gal}(W/E) = \pi_1(\pi_1(E)) \cong \mathbb{Q} \quad ; \text{ the quaternion group of order 8.}$$

(monodromy group)

$$\langle 1, i, j, k \mid i^2 = j^2 = k^2, ijk = -1 \rangle$$



4-3. The automorphism group

$$\begin{aligned} G &:= \text{Aut } W = \{ f: W \rightarrow W : \text{affine, } D(f) = \pm I \}. \\ &\cong \{ r \in \text{Aut } F_2 : r(\pi_1(W)) = \pi_1(W), p(r) = \pm I \} \end{aligned}$$

$$= \frac{\{ \pm I \}}{\text{half-rotation}} \times \mathbb{Q} \quad \cdots \text{extension of } \mathbb{Q} \text{ of deg 2.}$$

↑ ↑
half-rotation translation
order 16.

which includes 6 involutions acting as rotation.

$\{ \pm I \}$	t	$+I$	c	$-I$	c
\mathbb{Q}	$ i \ j \ k \ -1 \ -i \ -j \ -k$	$ i \ j \ k \ -1 \ -i \ -j \ -k$	$ i \ j \ k \ -1 \ -i \ -j \ -k$	$ -i \ -j \ -k \ 1 \ i \ j \ k$	$ -i \ -j \ -k \ 1 \ i \ j \ k$
(notation) r	$ i \ j \ k \ -1 \ -i \ -j \ -k$	$ \bar{i} \ \bar{j} \ \bar{k} \ -1 \ -\bar{i} \ -\bar{j} \ -\bar{k}$	$ \bar{i} \ \bar{j} \ \bar{k} \ -1 \ -\bar{i} \ -\bar{j} \ -\bar{k}$	$ \bar{i} \ \bar{j} \ \bar{k} \ 1 \ -\bar{i} \ -\bar{j} \ -\bar{k}$	$ \bar{i} \ \bar{j} \ \bar{k} \ 1 \ -\bar{i} \ -\bar{j} \ -\bar{k}$
r^2	$ -1 \ -1 \ -1 \ 1 \ -1 \ -1 \ -1$		$ 1 \ 1 \ 1 \ -1 \ 1 \ 1 \ -1$		$ 1 \ 1 \ 1 \ -1 \ 1 \ 1 \ -1$

involution
(translation)

6 involutions
(rotation)

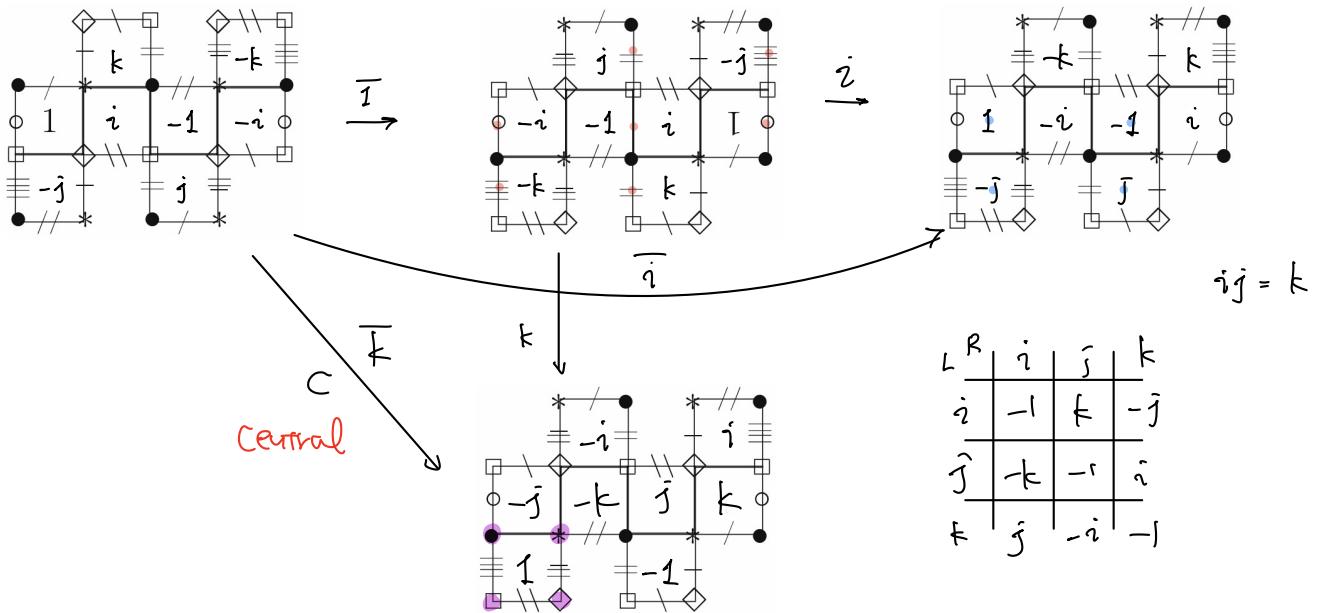
$$\begin{aligned} \bar{I} &= (ijk \ -1 \ -i \ -j \ -k) \\ \bar{-I} &= (i \ -k \ -j \ -i \ -k \ j) \\ \bar{i} &= (1 \ -j \ k \ -1 \ i \ -j \ k) \\ \bar{-i} &= (-1 \ i \ -j \ k \ 1 \ -i \ j \ -k) \\ \bar{j} &= (-k \ -j \ -i \ -1 \ k \ j \ i \ 1) \\ \bar{-j} &= (k \ j \ i \ 1 \ -k \ -j \ -i \ -1) \\ \bar{k} &= (-j \ -k \ -1 \ -i \ j \ k \ 1 \ i) \end{aligned}$$

$$(\bar{a} = a \circ \bar{I})$$

$$\therefore \langle c \rangle = C_G \text{ (center).}$$

* fixed points of G

$a \in G$ ($a \neq \pm 1$) are fixed pt-free, & $-1, \bar{a} \in G$ ($a \in \Omega$) have just 4 fixed pts.



fixed pts of other involutions are mid points of either squares or edges.

4-3. The equation of $C(W)$ \Leftrightarrow equations of surfaces in $C(W) \subset M_3$.

* $C = \overline{F} \in G$ has 4 fixed pts

Riemann-Hurwitz $W/\langle c \rangle$ has genus 0.

$W \rightarrow W/\langle c \rangle$ is a cyclic covering of degree 4 of P^1_C ,

branched over 4 pts (the vertices) normalize $0, 1, \lambda, \infty$ $\left(\frac{\lambda}{P^1_C \setminus \{0, 1, \infty\}} \right)$

Then W is given by the equation of the form

$$y^4 = x^{\varepsilon_0} (x-1)^{\varepsilon_1} (x-\lambda)^{\varepsilon_\lambda}$$

$\varepsilon_0, \varepsilon_1, \varepsilon_\lambda$ turn out to be 1 since the ramification order $\equiv 4$ around $0, 1, \lambda, \infty$.

up to Aut P^1_C w/ renormalization; λ defines the same cyc str. as $\frac{1}{\lambda}, (-\lambda), 1 - \frac{1}{\lambda}, \frac{\lambda}{\lambda-1}, \frac{1}{1-\lambda}$

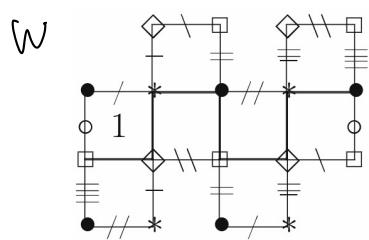
translation by -1 .

The double cover $\varrho: W \rightarrow W/\langle c \rangle = E = \mathbb{P}^1_{\mathbb{Z} + \mathbb{Z}i}$ is given by

$(x, y) \mapsto (x, y^2)$ where $E: y^2 = x(x-1)(x-\lambda)$: Legendre's normal form

Its affine deformation by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is known to be presented by $\lambda A = \lambda \frac{ax+bx}{cx+dx}$

$$\text{i.e. } E_A = \mathbb{C}/(\begin{pmatrix} a \\ c \end{pmatrix}\mathbb{Z} + \begin{pmatrix} b \\ d \end{pmatrix}\mathbb{Z}) \quad : \quad y^2 = x(x-1)(x-\lambda_A)$$



$$E = \overline{W} = W/\langle x \rangle$$

P →

which is equivalent to

$$\left(\begin{array}{l} y^4 = x(x-1)(x-\lambda) \\ \omega_W = \frac{dx}{y^2} \end{array} \right)$$

$$\left(\begin{array}{l} y^2 = x(x-1)(x-\lambda) \\ \omega_E = \frac{dx}{y} \end{array} \right)$$

Res: $E_1: (2y)^2 = 4x(x-1)(x+1)$

... $\begin{pmatrix} 1 & a \\ 0 & 1/2 \end{pmatrix}$ - deformation of E

4-4 VG of W

It can be checked that $\Gamma(W) = SL(2\mathbb{Z})$. (cf. Thm B).

As a consequence, $C(W)$ is an affine line, embedded in M_3 . (cf. Thm A.)

4-5 Remarkable properties of W

(Herrlich, Schmithüsen, 2008) The Jacobian of each of $W_\lambda \quad \lambda \in \mathbb{P}_{\mathbb{C}}^1 \setminus \{0, 1, \infty\}$
splits (up to isogeny) into a product of three elliptic curves.

$$\dots \underline{\text{Jac}(W_\lambda)} \underset{\text{isogeny}}{\sim} E_\lambda \times E_{-1} \times E_{-1}.$$

↑
The quotient $\mathbb{C}^g/L(W)$ of is the period lattice $L(W) := \{ (\int_{r\omega_1}, \dots, \int_{r\omega_g}) \in \mathbb{C}^g \mid r \in H^1(W, \mathbb{Z}) \}$
where $\omega_1, \dots, \omega_g$: normalized basis of hol. 1-forms on W.

(Möller, 2005pre - 2011) By the fact that $\text{Jac}(W_\lambda)$ has a constant part of codim 1

shows that $C(W)$ is a Shimura curve. (furthermore is the only one case

Teich. curve of trans. surf & Shimura curve
at the same time.)

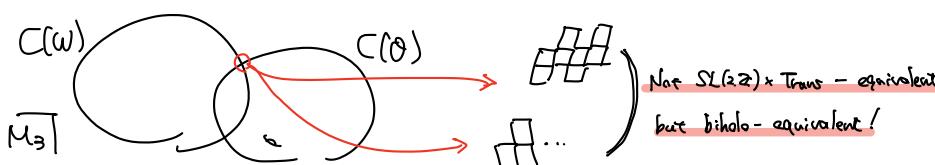
The most interesting is:

(Herrlich, Schmithüsen, 2008) Theorem C $C(W)$ intersects infinitely many other origami curves.

$$\text{Note: } C(\theta) = \{ A\theta \mid A \in SL(2\mathbb{Z}) \} / \text{trans-iso.} \cong \mathbb{H}/\Gamma(\theta) \subset M_g$$

So if $\theta' \neq \text{trans. } A\theta$ for $A \in SL(2\mathbb{Z})$ then $C(\theta) \neq C(\theta')$.

"Intersections of origami curves" imply points in M_g which represent such $\theta, A\theta'$ at the same time.



(Outline) We have seen that $G = \text{Aut}(W_\lambda)$ contains 7 involutions

and all of them have 4 fixed pts $\Rightarrow \forall g \in G, W_\lambda/\langle g \rangle$ has genus 1.

$-1 \in G$ induces $W_\lambda \rightarrow E$

$\forall g \in G$: involution, c descends via $W_\lambda \rightarrow W_\lambda/\langle g \rangle$ to $\bar{c} \in \text{Aut}(W_\lambda/\langle g \rangle)$ of order 4.
 ↳ central automorphism w/ 2 fixed pts.

$W_\lambda/\langle g \rangle$ must be E_{-1} .

(The only elliptic curve w/ such an automorphism)

Let $K = K_\lambda : W_\lambda \rightarrow E_{-1}$

↙ ramification occurs at the 4 fixed pts of g .

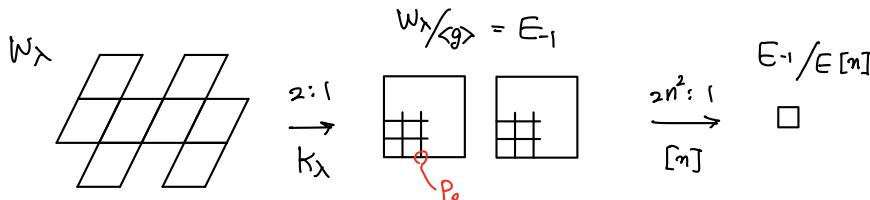
$$E_{-1} \xrightarrow{\bar{c}} \begin{matrix} \square \\ a \\ \circ \end{matrix} \xrightarrow{\pi_2} \begin{matrix} \square \\ \sigma \\ \circ \end{matrix} \xrightarrow{\bar{c}} E_{-1}$$

Their images, $P_0(\lambda) \cdots P_3(\lambda) \in E_{-1}$, form an orbit of \bar{c} .

Choose one fixed pt \mathcal{O} of \bar{c} as the origin of E_{-1} .

Then: if $P_0(\lambda) \in E_{-1}$ is an n -torsion \Rightarrow same holds for $P_1(\lambda) \sim P_3(\lambda) \in E_{-1}$.

$\Rightarrow [n] \circ K$ is normalized over \mathcal{O} , so defines an origami $[n] \circ K : W_\lambda \rightarrow E_{-1}[n]$ of order $2n^2$.



Authors showed that: for $\forall n \geq 3$ and $\forall P \in E_{-1}$: n -torsion,

there exists $\lambda \in P_C^1 \setminus \{0, 1, \infty\}$ s.t. $P_0(\lambda) = P$.

and its origami curve intersects $C(W)$. $\underline{\underline{}}$

§5. The Hurwitz space H

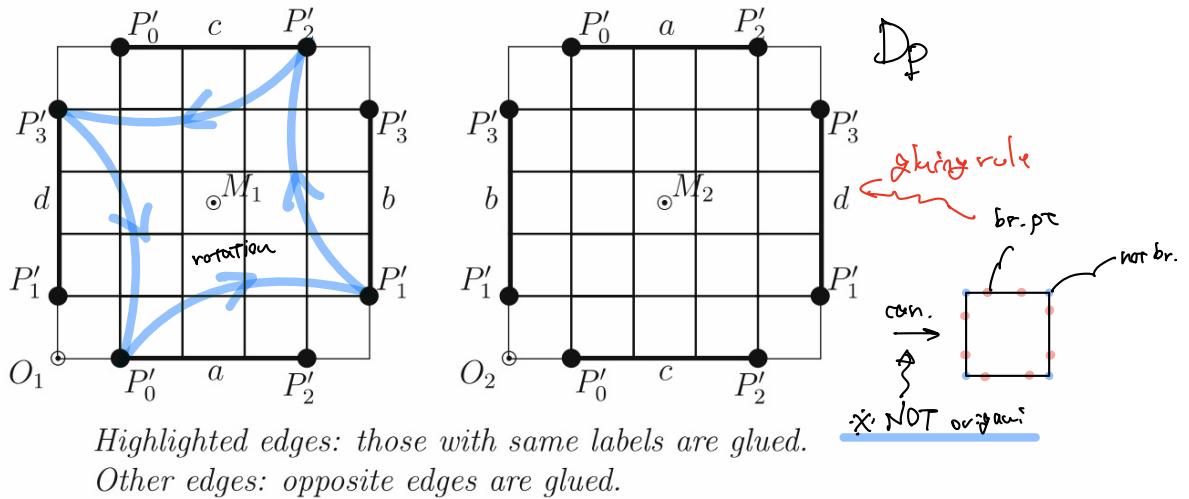
5.1 origamis D_p

recall $\forall n \geq 3$, every n -torsion pt. P on E_{-1} induces an origami D_P of degree $2n^2$.

where $C(D_P)$ intersects $C(W)$.

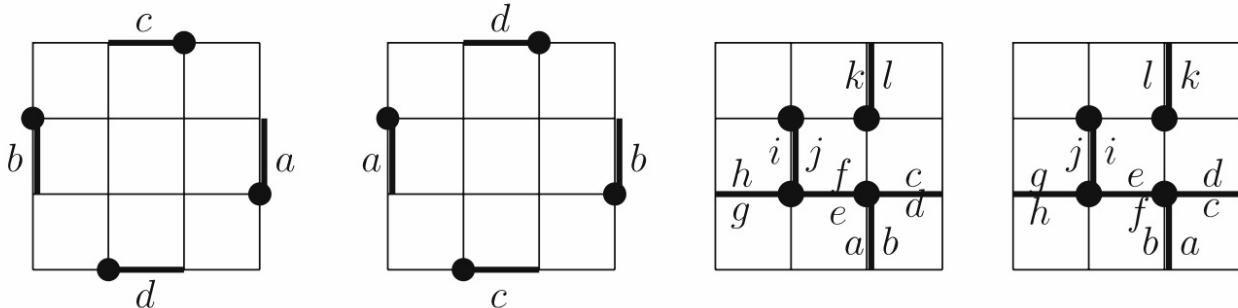
D_P consists of 2 large squares made of n^2 squares.

squares each; the point $P = P_0$ corresponds to a (primitive) vertex of one of the small squares. P_1, P_2 and P_3 are obtained from P_0 by rotation by an angle of $\frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$, resp., around the center of the large square. The two large squares are glued in such a way that the canonical map (of degree 2) from the resulting surface X_P to the torus corresponding to the large square is ramified exactly over P_0, P_1, P_2 and P_3 . For the precise description of the glueing we refer to [6]; here we confine ourselves to an example with $n = 5$:



Here, P'_0, P'_1, P'_2 , and P'_3 are the preimages of the ramification points P_0, P_1, P_2 and P_3 , M_1 and M_2 the preimages of M and O_1 , O_2 the preimages of O , respectively. As in Sect. 4.5, O and M denote the fixed points of the automorphism \bar{c} of order 4 on the torus E .

For $n = 3$ there are only two different possibilities for the orbit P_0, P_1, P_2, P_3 :



[Let $D_P = (d_P : X \rightarrow E)$: the degree $2n^2$ ords.]

$$C(D_P) = \{X_A \mid A \in SL(2, \mathbb{Z})\} /_{\text{isom}}$$

\Downarrow
 X_I is an intersection point of $C(D_P)$ & $C(W)$.

c : the central automorphism of W_I has derivative $\lambda^r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

and does not belong to $\text{Aut } D_P$.

c defines an affine diffeomorphism $C_A \in \text{Aff}^+(W_{\lambda_A})$ w/ derivative $A \circ A^{-1}$ (not hole in general)

on the other hand $c^2 = I$ defines a hole auto. of derivative $-I$.

5.2 Coverings w/ given ramification data.

The orbifolds D_p : defined by coverings w/ the same ramification behavior.

It is classically known that such coverings are classified by an algebraic variety : Hurwitz space.

More precisely : Hurwitz spaces classify $p: X \rightarrow Y$ w/ the following data fixed :

- degree d , genus $g(Y)$, $\#Br(p)$, $\{\text{ord}_x p \mid x \in p^{-1}(Br(p))\}$

It is also possible to specify a certain $\begin{cases} \text{geometric configuration of } Br(p) : \text{branched pts} \\ (\mu : \text{monodromies}) \end{cases}$
(To define subspaces?)

Note: once Y , p & ramification orders of $Br(p)$ are fixed,

$\begin{cases} (\text{points, lines}) \text{ s.t.} \\ \#\text{lines crossing a pt} = \text{const} \\ \#\text{pts located on a line} = \text{const} \end{cases}$

there are finitely many possibilities of

$$\text{monodromy homomorphism } \mu : \pi_1(Y - Br(p)) \rightarrow G_{\text{deg}}.$$

Restricting the monodromy typically leads to an irreducible component of the Hurwitz space.

Our case degree 2 covers of an elliptic curve, that are ramified over 4 pts P, P', Q, Q'

taking suitable base point, we may assume $P' = -P, Q' = -Q$.

* " P_0, P_1, P_2, P_3 form \mathbb{C} -orbit" cannot be formulated in an algebraic way ($\overline{G_A}$: not a g.s. automorphism)

We denote by \widehat{H} the Hurwitz space of such coverings.

$(X, p) : X : \text{opt R.S. } p : X \rightarrow E : \text{degree 2 cover}$
w/ 4 crit. pt. $P, Q, P', Q' \in X \quad (P' = -P, Q' = -Q)$

$p: X \rightarrow E$ & $p': X' \rightarrow E'$: considered isomorphic iff $\begin{array}{ccc} X & \xrightarrow{P} & E \\ \exists \varphi \downarrow & & \downarrow \exists \bar{\varphi} \\ X' & \xrightarrow{P'} & E' \end{array} : \text{isomorphism}$

By Riem-Hurwitz, $\#(X, p) \in \widehat{H} \quad g(X) = 3$.

\widehat{H} admits natural morphism $\pi : \widehat{H} \rightarrow \mathbb{M}_3$
 \downarrow
 $(X, p) \mapsto X$

- $(X, p) \in \widehat{H}$ is completely determined by its deck transformation σ ,

$\rightarrow \widehat{H} \sim \{(X, \sigma) \mid \sigma \text{ of Aut } X : \text{involution w/ 4 fixed points}$
whose images on $E = X/\langle \sigma \rangle : \stackrel{\exists}{=} \text{pt-symmetric}\}$

$$(X, \sigma) \sim (X', \sigma') \Leftrightarrow \stackrel{\exists}{=} \varphi : X \xrightarrow{\text{isom}} X' \text{ s.t. } \sigma' = \varphi \circ \sigma \circ \varphi^{-1}$$

Prop Let $(x, p) \in \hat{H}$. Then $\text{Aut}(x)$ contains a subgroup $\cong V_4$: Klein's four group $V_4 = \{1, i, j, k\}$

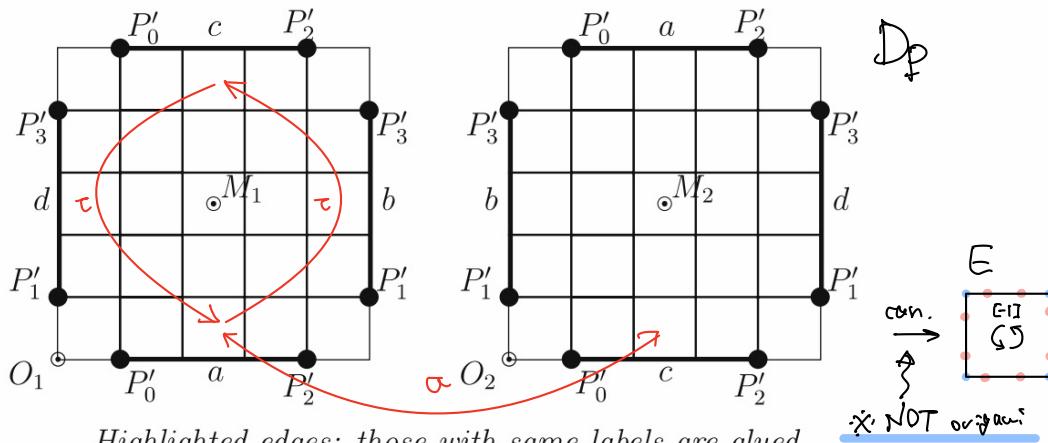
pf) Every covering of degree 2 is normal.

$\Rightarrow \exists \sigma \in \text{Aut}X$ that interchanges two inverse pts under p (crit. pts P, P', Q, Q' fixed)

i	j	k
j	k	i
k	i	1
1	1	1

Next consider a lift τ of $[E]$ on E (multiplication by -1)

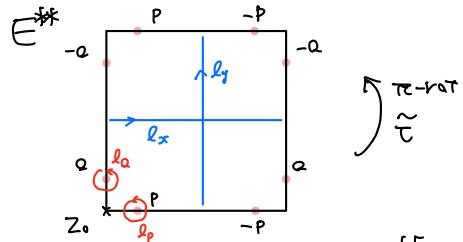
τ is an automorphism that commutes w/ σ .



Highlighted edges: those with same labels are glued.
Other edges: opposite edges are glued.

$[E]$ also is an acta on $E^{**} = E \setminus \{\pm P, \pm Q\} \xrightarrow{\text{induce}} \tilde{\tau} \in \text{Aut}(E^{**})$. ($\tilde{\tau} = [E] \mid_{E^{**}}$)

From general theor. of topology, $[E]$ lifts to X iff $\tilde{\tau}$ preserves $U = \pi_1(X^{**}) < \pi_1(E^{**})$



$$\text{We see: } \tilde{\tau}(l_x) = -l_x \quad \tilde{\tau}(l_y) = -l_y$$

$$\tilde{\tau}(l_p) = l_{-p} \quad \tilde{\tau}(l_q) = l_{-q}.$$

$U = \pi_1(X^{**}) < \pi_1(E^{**})$ is the kernel of monodromy hom $\mu: \pi_1(E^{**}) \rightarrow S_2$.

Since P is ramified over $\pm P, \pm Q$, $\mu(l_{\pm P}) = \mu(l_{\pm Q}) = (12)$.

For the other two generators, $\mu(l_x), \mu(l_y)$ are freely determined.

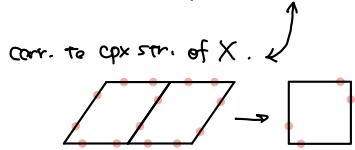
$$\mu(-l_x), \mu(-l_y)$$

Thus lifting elements determined by $\underline{\hspace{2cm}}$, we have four automorphisms on X

which form V_4 . \otimes

5.3 origami covers in H

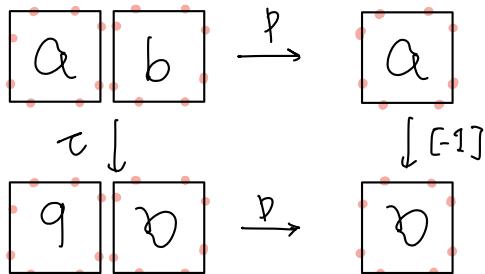
Data $(E, (\pm p, \pm q), \text{degree } 2)$ given \Rightarrow there are 4 different coverings.



(monodromy $\mu: \pi_1(E^{\text{reg}}) \rightarrow S_2$ is fixed except for the choice of $\mu(l_x), \mu(l_y): l_x, l_y \in \pi_1(E)$)

Let $T \in \text{Aut}(X)$ be the lift of $[E:\mathbb{F}] \in \text{Aut} E$ fixing $\{\pm p, \pm q\}$

order 2, commutes w/ α .



The surface is hyperelliptic \Leftrightarrow involution w/ 4 fixed pts
(quotient: $P^1_{\mathbb{C}}$)

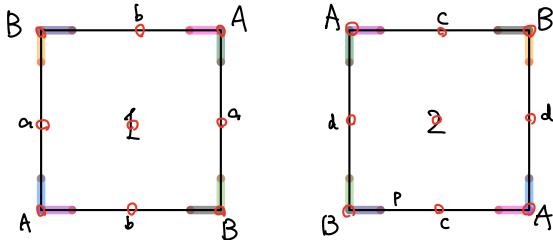
general pt. $(X, \alpha) \in \widetilde{H}$ has 3 involutions
 $\alpha, T, \text{ and } \alpha T$.

By def of \widetilde{H} : α has 4 fixed pts

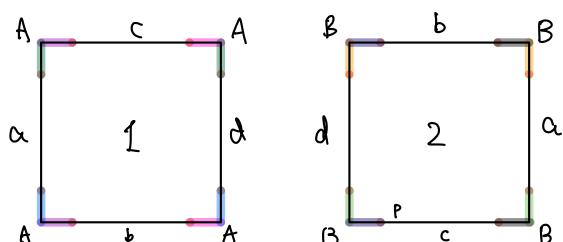
Possible fixed pts of $\tau, \alpha T$: inverse images of 4 fixed pts of $[E:\mathbb{F}]$ on E .

More precise possibilities $\begin{cases} (\alpha) & \text{one fixes 4 pts \& other fixes the next 4 pts} \\ (\beta) & \text{one fixes 8 pts \& other is fixed pt-free} \end{cases}$

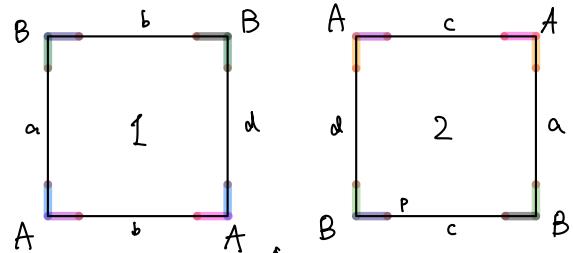
(i) $x=y=1 \dots (\beta)$ halfpt of edges
center of faces are fixed pts.
vertices



(ii) $x=(12) \dots (\alpha)$



(iii) $x=(12) \dots (\alpha)$



(iv) $x=1 \quad y=(12) \dots (\alpha)$

$\frac{\pi}{2}$ -rotation
(anti)

Def 9 $H := \{(X, \alpha) \in \widetilde{H} \mid \tau \text{ has exactly 4 fixed pts in } X\}$

For $(X, \alpha) \in H$, $X/\tau, X/\alpha, X/\alpha T$ all have genus 1.

Later we shall see that for generic $(X, \alpha) \in H$, $\text{Aut}(X) = \{1, \alpha, \tau, \alpha \tau\}$.

w.r.t. algebraic parametrization of H

→ We could characterize $H = \{(x, \alpha) \in \widehat{H} \mid X \text{ is not hyperelliptic}\}$

also
alg. curve

- $p \in E_1$: n -torsion point, the origin: D_p defines an alg. curve $C(D_p) \hookrightarrow C_p \hookrightarrow H \subset \widehat{H} \rightarrow M_3$

In the same way, origin: W defines an alg. curve

$$C(W) \hookleftarrow C_W \hookrightarrow H \subset \widetilde{H} \rightarrow M_3.$$

Let $\pi: \widetilde{H} \rightarrow M_3$ be a finite morphism of varieties.

$$(W_x \xrightarrow{P} W_x/\langle g \rangle = E_1)$$

* here n -torsion not necessary

Thm 1 $\bigcup_{\substack{n \geq 3 \\ p \in E_1: n\text{-torsion}}} C_p \subset H$: dense.

pf) prove in cpx top. of Hurwitz sp : parametrized by branched pts $(\Rightarrow$ Zariski top. also holds.)

Let $(x, \alpha) \in H$: non hyperelliptic

↑ determine

$$(E, (\pm P, \pm Q), \mu)$$

Clearly we can approximate P, Q by torsion points. ↳ Let (P_n, Q_n) : pair of n -torsions near to (P, Q) .

For a suitable choices of monodromies, we may define $(X_n, \alpha_n) \in H$ close to (x, α) .

Note that $X_n \rightarrow E_1 \rightarrow E_1/E[n]$ defines an origin,

and so (X_n, α_n) lies on $C = \pi^*(\underline{C(X_n \rightarrow E_1/E[n])})$: alg. curve.
origin curve

Next we show that $\exists p: n\text{-torsion pt s.t. } (X_n, \alpha_n) \in C_p$

in fact we show that the stronger, claim $C_p = C$: for n : prime.

Schmitz's theory states that two origin curves C, C_p are equal iff

the corresponding subgroups $U, U_p \subset F_2$ are Aut F_2 -equivalent:

fundamental grps

\cong

Aff $^+(H)$: univ. affine grp.

If $\gamma \in \text{Aut}^+(F_2)$ joining $U \& U_p$,

induced by

$f: H \rightarrow H$: affine diffeo morphism

proj.

$\tilde{f}: X \rightarrow X_p$: s.t. $\tilde{f}^* \pi_p(x) = \pi_p(x_p)$

proj.

$\tilde{f}: E \rightarrow E_1$: transforming $\begin{pmatrix} \pm P_n & \pm P \\ \pm Q_n & \pm C(P) \end{pmatrix}$

& respects monodromy.

$E \quad E_1$ (for some P)

\cong

To find such \tilde{f} , consider $(x', \alpha') \in C$ through (X_n, α_n) which lies over E_1 .

To (x', α') there corresponds an aff. diffeo. $A \in SL(2\mathbb{Z}) \begin{cases} E \rightarrow E_1, \\ P_n, Q_n \rightarrow n\text{-torsion pts on } E_1. \end{cases}$

$$E \rightarrow \{x \in E_1 : n\text{-torsion}\} \cong (\mathbb{Z}/n\mathbb{Z})^2$$

\Downarrow

$$\begin{matrix} P_n, Q_n \\ \xrightarrow{\text{send}} \end{matrix} \begin{matrix} P_n, Q_n \end{matrix}$$

$$@ C = \mathbb{G}_p \text{ for } p \in (\mathbb{Z}/n\mathbb{Z})^2$$

$$\Leftrightarrow {}^z B \in SL_2(\mathbb{Z}/n\mathbb{Z}) \text{ s.t. } \begin{pmatrix} B(P_n) \\ B(Q_n) \end{pmatrix} = \begin{pmatrix} P \\ \overline{C}(P) \end{pmatrix} \quad \square$$

Since \overline{C} acts as $\frac{\pi}{2}$ -rotation, it acts on $(\mathbb{Z}/n\mathbb{Z})^2$ through $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Using linear algebra over $\mathbb{Z}/n\mathbb{Z} \cong F_n$ (n : prime),

we may construct P s.t. \square concretely. \square

5.4 Affine coordinates for H

Proposition 11 *For any point $(X, \sigma) \in H$, X can be represented by a plane quartic with equation*

$$x^4 + y^4 + z^4 + 2ax^2y^2 + 2bx^2z^2 + 2cy^2z^2 = 0 \quad (11)$$

for some complex numbers a, b and c .

Proof Since X is not hyperelliptic, the canonical map on X gives an embedding into the projective plane as a smooth quartic. Moreover, every automorphism of X is induced by a projective automorphism of \mathbb{P}^2 . Thus we can choose coordinates on \mathbb{P}^2 such that σ acts by $(x : y : z) \mapsto (-x : y : z)$ and τ by $(x : y : z) \mapsto (x : -y : z)$. Then $\sigma\tau$ acts by $(x : y : z) \mapsto (-x : -y : z) = (x : y : -z)$. A quartic that is invariant under σ and τ must therefore be a polynomial in x^2, y^2 and z^2 . We can still multiply x, y and z by suitable constants and thus obtain that the coefficients of x^4, y^4 and z^4 are 1, which gives us an equation of the form (11). \square

The Eq. (11) describes a family of plane projective curves of degree 4 over the affine parameter space $\mathbb{A}^3(\mathbb{C}) = \mathbb{C}^3$. The non-singular curves in this family are determined by Prop 12 ,

Def $U := \mathbb{A}^3 \setminus \{\text{critical parameters}\}$ w.r.t prop [2].

Corollary 14 *The kernel of the homomorphism $\rho : \text{Aut}(\mathcal{C}) \rightarrow \text{Aut}(U)$ is G_0 . In particular, for general $(a, b, c) \in U$, $\text{Aut}(C_{abc}) = G_0$.*

Proof As observed in the proof of Proposition 11, any automorphism of a curve in \mathcal{C} is induced by an automorphism of $\mathbb{P}^2(\mathbb{C})$. Thus any $\varphi \in \ker(\rho)$ is a linear change of the homogeneous coordinates x, y, z that preserves the terms ax^2y^2, bx^2z^2 and cy^2z^2 for all $(a, b, c) \in U$. This is only possible if $\varphi \in G_0$. Alternatively, the result can be checked by comparison with the list of automorphism groups in genus 3 in [9]. \square

5.5 Maps to moduli space

In this subsection we study the relations between the spaces U , H and M_3 . To do so, we shall factor the morphism $m : U \rightarrow M_3$, $(a, b, c) \mapsto [C_{abc}]$, in two ways.

We saw in the proof of Proposition 13 that the subgroup L of $\text{Aut}(U)$ is contained in the image of the natural homomorphism $\rho : \text{Aut}(\mathcal{C}) \rightarrow \text{Aut}(U)$. Therefore m factors through U/L .

On the other hand we know that, for every $(a, b, c) \in U$, C_{abc} admits the automorphism $\alpha : (x : y : z) \mapsto (-x : y : z)$. We find:

Proposition 15 $(a, b, c) \mapsto (C_{abc}, \alpha)$ is a surjective morphism $h : U \rightarrow H$.

So far we have found a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & H \\ \tilde{q} \downarrow & \searrow m & \downarrow \pi \\ U/L & \xrightarrow{q} & M_3 \end{array}$$

The final goal in this section is to show

Proposition 16 q is birational.

Note: q is not an isomorphism, nor is H isomorphic to U/L_H .

This can be seen e. g., by looking at the Fermat curve C_{000} : it is mapped isomorphically onto $C_{0,3,0}$ by the transformation $(x : y : z) \mapsto (x + z : \sqrt[4]{8}y : x - z)$, which does not extend to an automorphism of \mathcal{C} since we have seen in Proposition 13 d) that they all fix C_{000} .

References

- [Herrlich, Schmithüsen, 2007] Herrlich, F., Schmithüsen, G.: A comb of origami curves in M_3 . *Geom. Dedicata*, **124**, 69–94 (2007)