NON-PLANARITY OF SL$(2,\mathbb{Z})$-ORBITS OF ORIGAMIS IN $\mathcal{H}(2)$

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ABSTRACT. We consider the SL$(2,\mathbb{Z})$-orbits of primitive n-squared origamis in the stratum $\mathcal{H}(2)$. In particular, we consider the 4-valent graphs obtained from the action of SL$(2,\mathbb{Z})$ with respect to a generating set of size two. We prove that, apart from the orbit for $n = 3$ and one of the orbits for $n = 5$, all of the obtained graphs are non-planar. Specifically, in each of the graphs we exhibit a $K_{3,3}$ minor, where $K_{3,3}$ is the complete bipartite graph on two sets of three vertices.

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§ 1-2 Intro & Preliminaries

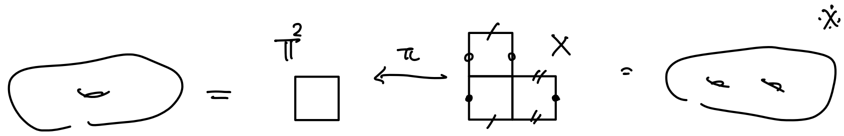
Origami : oriented, connected surface X obtained from finite copies of unit squares by translation $\begin{cases} \text{left edge} \leftrightarrow \text{right edge} \\ \text{upper edge} \leftrightarrow \text{lower edge} \end{cases}$

$=$ finite cover $\pi : X \to \mathbb{T}^2 = \mathbb{C}/\mathbb{Z}+i\mathbb{Z}$ branched only at $0 \in \mathbb{T}^2$

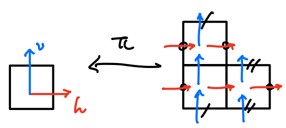
↳ natural abelian differential $\omega = \pi^* dz$

↳ special case of translation surface (surface equipped w/ abelian differential)

※ abelian differential gives translation coordinates defined by local integrals : $z \mapsto \int_0^z \omega$



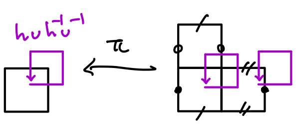
monodromy :



$m : F_2 \to \mathrm{Sym}(n)$

$\mathrm{Mon}(O) = \langle h, v \rangle$

Commutator $[h,v] = hvh^{-1}v^{-1}$ corresponds to the monodromy arraound $\infty \in \mathbb{T}^2$: branch pt



⇝ cycle lengths of $[h,v]$ ~ orders of singularities

$\mathcal{H}_g := \{ (X,\omega) : \text{translation surface} \} / \sim$: moduli space of translation surfaces

By Riemann-Roch's $\sum_{p \in Z(\omega)} \mathrm{ord}_p \omega = 2g-2$ and $\mathcal{H}_g$ is naturally stratified as

$\mathcal{H}_g = \bigsqcup \mathcal{H}(k_1, k_2, \cdots, k_s)$ where $\begin{matrix} k_i \geq 1. \\ \sum k_i = 2g-2 \end{matrix}$ ⟵ assigning $\{ \mathrm{ord}_p \omega \mid p \in Z(\omega) \}$.

In terms of monodromy, we have :

Prop $O = (h,v) \in \mathcal{H}(k_1, k_2, \cdots, k_s) \Leftrightarrow [h,v]$ has cycle lengths $k_1+1, k_2+1, \cdots, k_s+1$

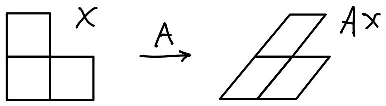
- Each stratum $\mathcal{H}(k_1 \cdots k_s)$ is a complex orbifold of dimension $2g+s-1$

  parametrized by the positions of conical singularities

  so that origamis are thought to be integer lattice pts.

  $\xrightarrow{\text{due to}}$ Eskin–Okounkov (2001), Zorich (2002)

- Each stratum $\mathcal{H}$ admits a natural $SL(2,\mathbb{R})$-action by affine-deformations.



$SL(2,\mathbb{Z}) < SL(2,\mathbb{R})$ is thought to be acting on $\mathcal{H}_{\mathbb{Z}} \subset \mathcal{H}$ (origamis)

Note that $SL(2,\mathbb{Z}) = \langle T, S \rangle$ where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

It can be checked that
$$\left.\begin{aligned} T\cdot(h,v) &= (h, vh^{-1}) \\ S\cdot(h,v) &= (hv^{-1}, v) \end{aligned}\right\} (☆)$$

Remark: The number of squares, the stratum, the primitivity, the monodromy grp are $SL(2,\mathbb{Z})$-invariant.

↑ (☆) ↑ affine deform. (stated later) ↑ (☆) ↑ as an abstract grp

- In this paper, we consider $SL(2,\mathbb{Z})$-orbits of primitive, n-sheeted origamis in $\mathcal{H}(2)$.

Df $\mathcal{O} = (h,v)$: primitive origami $\overset{df}{\iff}$ No nontrivial partition of $\{1, \cdots, n\}$ is $\langle h,v \rangle$-invariant.

(trivial $\iff$ $\{1 \cdots n\}$ (no partition) or $\{1|2|\cdots|n\}$ (into singletons))

These orbits is well classified in the works on McMullen (2005) & Hubbert–Lelièvre (2006)

Theorem 2.1 (I) $n=3$ or $n \geq 4$ even : single orbit w/ $\mathrm{Mon}(\mathcal{O}) = \mathrm{Sym}(n)$

(II) $n \geq 5$ odd : two orbits $\begin{cases} \mathrm{Mon}(\mathcal{O}) = \mathrm{Sym}(n) \leftarrow \text{A-orbit} \\ \mathrm{Mon}(\mathcal{O}) = \mathrm{Alt}(n) \leftarrow \text{B-orbit} \end{cases}$

In particular, the monodromy groups of primitive origamis in $\mathcal{H}(2)$ is a strong $SL(2,\mathbb{Z})$-invariant.

Here, each $SL(2,\mathbb{Z})$-orbit can be realised as a 4-valent graph (valency of $\forall$vertex : 4)

— denote by $\mathcal{G}_n$ (single orbit case) or $\mathcal{G}_n^A, \mathcal{G}_n^B$ (two orbits case)

Conjecture (McMullen) $\mathcal{G}_n, \mathcal{G}_n^A, \mathcal{G}_n^B$ form an expander family.

Def Let $\Gamma = (V, E, \xi)$ be a graph : $\xi : E \to V \times V / \text{alt.}$: endpoint - assignment
$(v \neq \text{singleton})$

- The rate of expansion of Γ is :

$$h(\Gamma) := \min \left\{ \frac{|\partial V'|}{|V|} \;\middle|\; V' \subset V,\ 0 < |V'| \le \frac{|V|}{2} \right\} \quad \text{where } \partial V' = \{ e \in E \mid \xi(e) \in V' \times V'^{c} \}$$
V'^{c} : compl.

- Γ is an ε-expander $\iff h(\Gamma) \ge \varepsilon$

- Graph family $\{ \Gamma_i = (V_i, E_i, \xi_i) \mid i = 1, 2, \cdots \}$ is an expander family

$\iff$
(i) $|V_i| \to \infty$ as $i \to \infty$ — ascending $\# V_i$
(ii) $^{\exists} r > 0$ s.t. $^{\forall}\Gamma_i$ (valency of $^{\forall} v \in V_i$) $\le r$ — uniformly bounded valency
(iii) $^{\exists} \varepsilon > 0$ s.t. $^{\forall}\Gamma_i$: ε-expander — common expanding rate

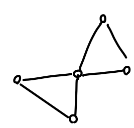
Expander constructions have found extensive aplications in computer sciences
constructing algorithm of { error correcting codes, random walks, sorting networks } in polynomial time ?

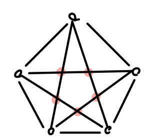
→ cf. Badaoui, Bretto, Ellison et.al (2018) §1

Nevertheless, constructing expander families is not easy.

Theorem 1.1 $\mathcal{G}_n, \mathcal{G}_n^A, \mathcal{G}_n^B$ are non-planar graphs w/ the exception of $\mathcal{G}_3, \mathcal{G}_5^B$.

(planar $\iff$ realizable on the plane so that no edge crosses any other) →


planar


non planar

Theorem 1.1 provides indirect evidence for McMullen's conjecture.

Indeed, separator theorem by Lipton–Tarjan (1979)

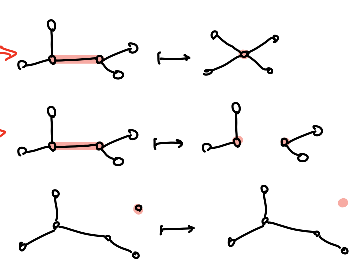
states that planar graphs cannot form an expander family.

cf. Cerbu–Ireland (2021) : The Markoff graph mod p is non planar for $p = 4m+1$: prime
(conjectured to form an expander family.
related to " $x^2 + y^2 + z^2 = xyz$ on $\mathbb{F}_p$ "

Key tool : Wagner (1937), Kuratowski (1930) — characterization of planar graph in terms of forbidden minor.

Def Let G, H : graph. H is realized as a minor inside G

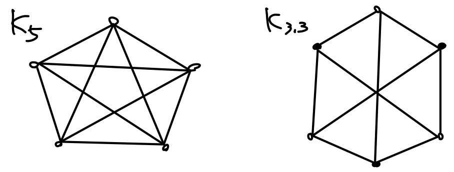
$\iff$ $^{\exists}$ transformation $G \mapsto H$ consisting of { edge-contraction, edge-deletion, deletion of isolated vertex }



Thm (Wagner (1937), Kuratowski (1930))

Graph G is nonplanar $\Leftrightarrow$ G contains K_5 or $K_{3,3}$ minor.

where K_5 : complete graph. $K_{3,3}$: complete bipartite graph



We prove Thm 1.1 by realizing $K_{3,3}$ minor in each case.

§3 Construction the minor

3.1. **Even squared orbits.** Let $n \geq 4$ be even and define the origamis $O_1, \ldots, O_6$ as follows: $[h, v]$

$O_1 = ((n-1,n),(1,2,\ldots,n-1))$, —— $(n, n-1, n-2)$
$O_2 = ((1,2,\ldots,n),(1,n-1,n-3,\ldots,3,n,n-2,\ldots,2))$, —— $(1,3,2)$
$O_3 = ((1,2,\ldots,n),(n-1,n))$, —— $(n-2, n-1, n)$
$O_4 = ((1,2,\ldots,n),(2,3,\ldots,n))$, —— $(1, n, n-1)$
$O_5 = ((1,2,\ldots,n),(2,n,n-1,\ldots,3))$, and —— $(n, 2, 1)$
$O_6 = ((2,3,\ldots,n),(1,2,n,n-1,\ldots,3))$. —— $(1, 2, 3)$

in $\mathcal{H}(2)$

Rem For the cycle representation of origamis, Sym-conjugations are described naturally :

e.g. $\sigma^{*}O_1 = ((\sigma(n-1), \sigma(n)), (\sigma(1), \sigma(2), \cdots, \sigma(n-1)))$

In particular, $O_1 \sim O_6$ are mutually distinct origamis.

Prop 3.1 $n \geq 4$: even.
Then there exists a $K_{3,3}$ minor in G_n w/ partition $\{O_1, O_2, O_3 \mid O_4, O_5, O_6\}$

pf) Recall $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

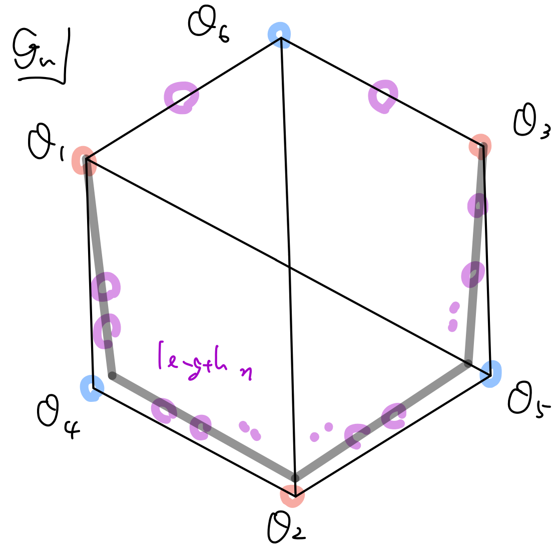
$$\left.\begin{aligned} T\cdot(h,v) &= (h, vh^{-1}) \\ S\cdot(h,v) &= (hv^{-1}, v) \end{aligned}\right\} (\ast)$$

Using this, we observe the following gives paths in $K_{3,3}$ minor.

$\overline{14} = S^{-1}$, $\overline{15} = S$, $\overline{16} = ST$

$\overline{24} = T^{n-3}$, $\overline{25} = T^{-1}$, $\overline{26} = T^{-(n-4)}S^{-1}$

$\overline{34} = T^{-1}$, $\overline{35} = T$, $\overline{36} = TS$ where $\overline{ij}$ connects O_i and O_j.



← We may observe that this is moved to $K_{3,3}$

via $\left\{\begin{array}{l}\text{edge-contraction}\\ \text{edge-delation}\\ \text{deletion of isolated vertex}\end{array}\right.$.

Q Similar argument successes for $\mathcal{G}_n^A$ & $\mathcal{G}_n^B$.

exception

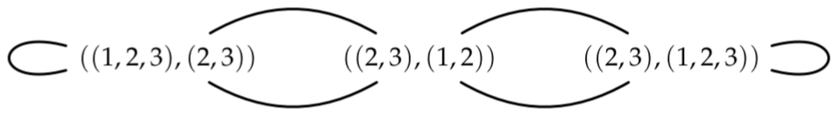


FIGURE 3.1. The graph $\mathcal{G}_3$.

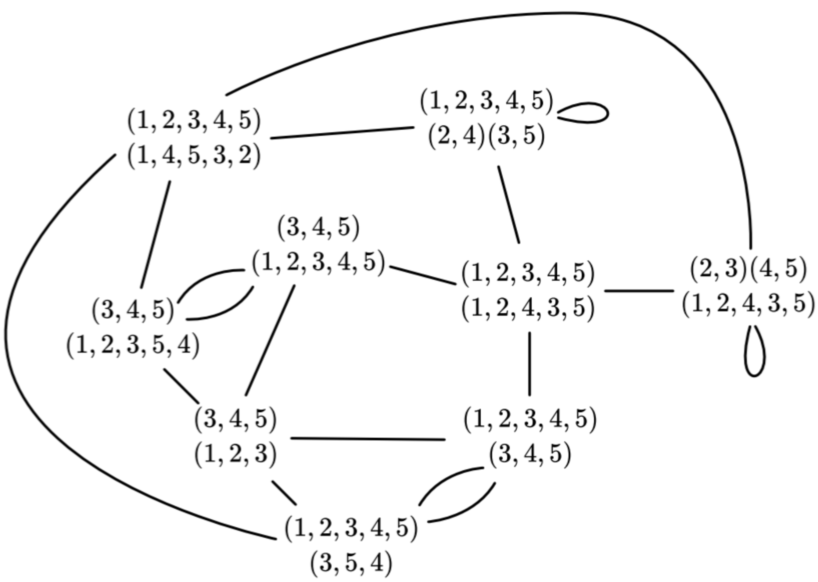


FIGURE 3.2. The graph $\mathcal{G}_5^B$.

4. Further Questions

We finish with two natural questions. Firstly:

Question 4.1. *Can one find generalisations of these structures that exist in the* $SL(2,\mathbb{Z})$*-orbits of primitive origamis in different strata?*

We remark that the classification of these $\mathrm{SL}(2,\mathbb{Z})$-orbits is open in general.

Secondly, the work of de Courcy-Ireland for Markoff graphs modulo p gave, for certain primes p, a construction of a $K_{3,3}$ minor whose path lengths did not depend on the prime p. They called such a construction a 'local' construction. In our case, this would correspond to finding a $K_{3,3}$ minor in $\mathcal{G}_n$ whose paths have lengths that do not depend on n. As such, we ask the following.

Question 4.2. *Does there exist a 'local' construction of a* $K_{3,3}$ *(or* K_5*) minor in the* $SL(2,\mathbb{Z})$*-orbits of primitive origamis in* $\mathcal{H}(2)$*?*