ORIGAMI EDGE-PATHS IN THE CURVE GRAPH

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Abstract. An *origami* (or flat structure) on a closed oriented surface, S_g, of genus $g \geq 2$ is obtained from a finite collection of unit Euclidean squares by gluing each right edge to a left one and each top edge to a bottom one. The main objects of study in this note are *origami pairs of curves*—filling pairs of simple closed curves, (α, β), in S_g such that their minimal intersection is equal to their algebraic intersection—they are *coherent*. An origami pair of curves is naturally associated with an origami on S_g. Our main result establishes that for any origami pair of curves there exists an *origami edge-path*, a sequence of curves, $\alpha = \alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n = \beta$, such that: α_i intersects α_{i+1} at exactly once; any pair (α_i, α_j) is coherent; and thus, any filling pair, (α_i, α_j), is also an origami. With their existence established, we offer shortest origami edge-paths as an area of investigation.

bounds no disks / annuli

arXiv: 2008.09179 (2020. Aug)

non-trivial / non-peripheral

§1 Intro (basically, $g \geq 2$)

- Harvey's (1978) Curve graph $\mathcal{C}^1(S_g)$ ~ study of MCG
  - We will focus on the non-separating curve graph $NS(S_g) \subset \mathcal{C}^1(S_g)$

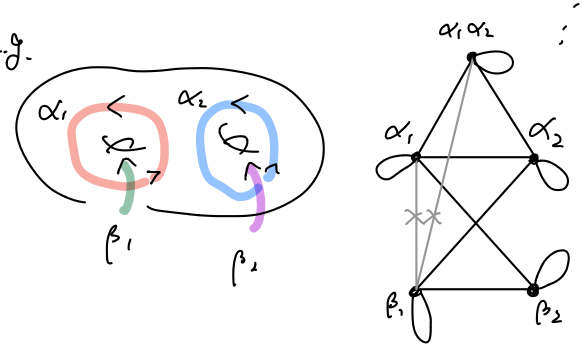
$NS(S_g)$ — vertices : (free homot.) isot. classes of non-separating, s.c.c. on S_g

s.c.c. $\alpha \subset S_g$: = $\Leftrightarrow$ $S_g \setminus \alpha$ is connected.

— edges : signs that the joining vertices have disjoint rep.

equipped w/ the edge-path distance $\rightarrow$ metric space

e.g.



$NS(S_2)$

$d(\alpha_1, \beta_1) = 2$

fact $NS(S_g)$ $(g \geq 2)$ has an infinite diameter :

$$d_{NS}(u,v) \leq 2\, i(u,v) + 1 \quad \text{(Masur–Minsky, 1999)}$$

Def Let $\alpha,\beta \subset S_g$ represent $u,v \in V(\mathcal{NC}(S))$ resp.

$|\alpha\cap\beta|$: assumed to be minimal over all rep.

\# intersection

→ We often $|u\cap v| := |\alpha\cap\beta|$.

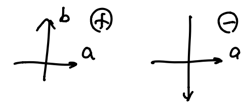
(i) We say (α,β) : filling pair ($\Leftrightarrow$ (u,v) is)

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall$ components of $S\setminus(\alpha\cup\beta)$ are top. disks.

(ii) an assignment of orientation to α,β allows us to calculate

the algebraic intersection number $\langle\alpha,\beta\rangle := \sum^{\text{local}}_{a\subset\alpha,\, b\subset\beta\,:\,\text{subarc}} \varepsilon(a,b)\, i(a,b)$

sign of intersection :



We say (α,β) : coherent pair ($\Leftrightarrow$ (u,v) is)

$\overset{\text{def}}{\Longleftrightarrow}$ $|\text{the alg. intersec number}| = |\alpha\cap\beta|$

i.e. all intersections share the same sign.

(iii) an origami for a closed surface S_g ($g\geq 2$)

: a separation of S_g into finite number of Euclidian square, w/ $\boxed{\text{left side} \leftrightarrow \text{right side, upper side} \leftrightarrow \text{bottom side by translation producing connected surf.}}$

origami rule

$\boxed{\text{Theorem 1.1}}$ A coherent filling pair of curves naturally corresponds to an origami.

→ We call a coherent filling pair an origami pair.

Def 1.2 $\mathcal{E} = \{v_0,\dots,v_n\} \subset V(\mathcal{NS}(S_g))$ is an origami edge-path if

(i) $|v_i\cap v_{i+1}| = 1$ ($i=0,1,\dots,n-1$)

(ii) any pair (v_i,v_j) ($i\neq j$) is a coherent pair.

(iii) and thus, any filling pair (v_i,v_j) is an origami pair.

We denote the length $n =: |\mathcal{E}|$

✱ existence of filling pairs?

Main results

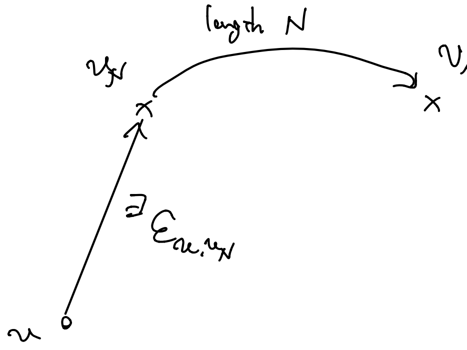
Thm 1.3 For any origami pair (u,v), there exists an origami edge-path joining u & v.

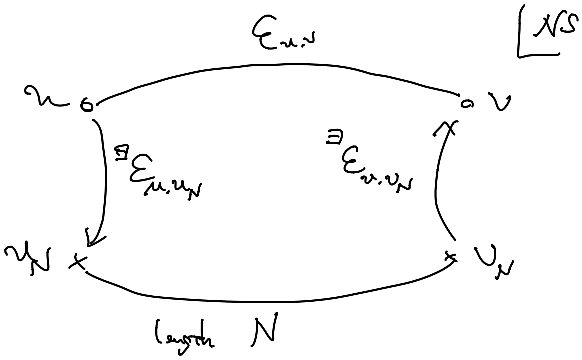
i.e. $\exists\, \mathcal{E} = \{u=v_0, v_1, \dots, v_n = v\}$.

Thm 1.4 $\forall N>0,\ \exists (u,v)$: origami pair s.t. $d_{NS}(u,v) > N$.

in particular, $\operatorname{diam}(\{v : \text{endpt of an origami edge-path starting from } u\}) = \infty$ $\forall u$.

(ε in Thm 1.3 can be arbitrary large)





§2. Thm 1.1

claim coherent filling pair $\longleftrightarrow$ origami

Let (α,β) : 〃 on S_g.

$\hookrightarrow$ $\alpha\cup\beta$ induces a cellular decomposition on S_g ($\because$ filling)

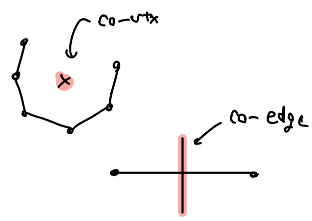
$$\begin{cases} \text{0-cells (vertices)}: & \alpha\cap\beta \\ \text{1-cells (edges)}: & (\alpha\cup\beta)\setminus(\alpha\cap\beta) \\ \text{2-cells (faces)}: & S_g\setminus(\alpha\cup\beta). \end{cases}$$

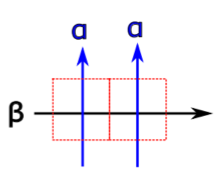
lem since α,β : s.c.c. (there is no self intersection) faces should be $2m$-gon. the coherence gives that m should be even. $\to$ $4n$-gon.

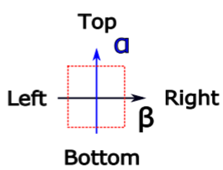
This cellular decomposition admits a dual decomposition:

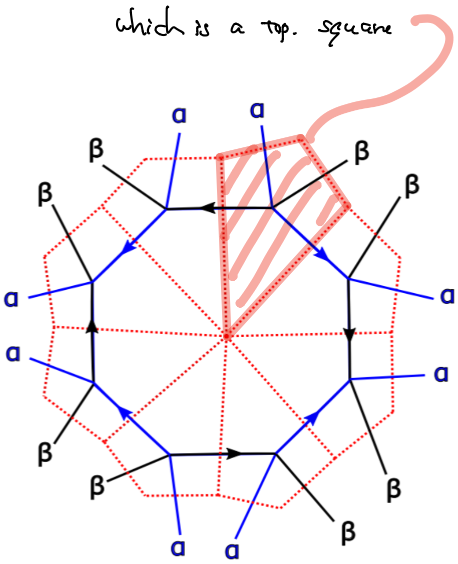
$$\begin{cases} \text{co-vertices}: & \text{centers of } 4n\text{-gons} \\ \text{co-edges}: & \text{edges (1:1 crossing)} \\ \text{co-faces}: & \text{centered at vertices} \end{cases}$$

which is a top. square









$\longrightarrow$ as a result, S_g is covered by squares. $\#\{\text{square}\} = |\alpha \cap \beta|$. $\longrightarrow$ connected tilling.

since α intersects β at the center of each co-face,

& half-edges along $\alpha, \beta, \alpha^{-1}, \beta^{-1}$ from the center will arrive at co-edges individually,

we can regard bdries of squares as (right, upper, left, bottom) sides w.r.t. $(\alpha, \beta, \alpha^{-1}, \beta^{-1})$.

The coherence of (α, β) implies that

these bdries of squares are glued respecting the origami rule. $\boxtimes$

rem α, β are cores of horizontal, vertical annuli respectively.

§3. Bicorn curves & thm 1.3

(according to Przytycki & Sisto)

Def 3.1 $\alpha, \beta \subset S_g$: s.c.c. intersecting minimaly (up to isot.)

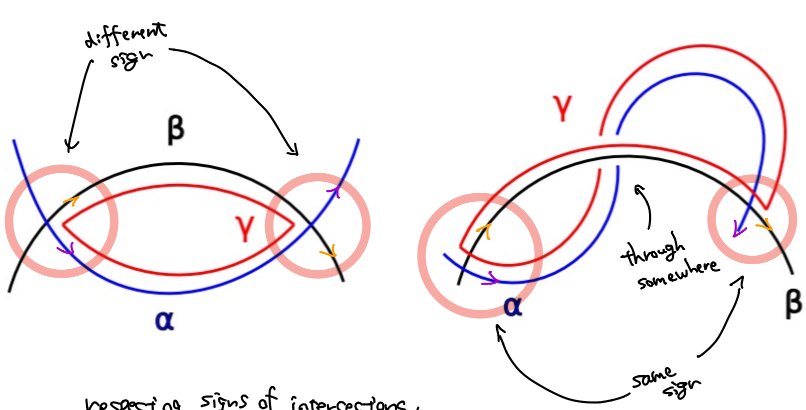
A s.c.c. $\gamma \subset S_g$ is a bicorn curve between α & β

$\Longleftrightarrow$ γ is represented by $\alpha' \cup \beta'$ where $\underset{(\alpha\text{-arc})}{\alpha' \subset \alpha}$, $\underset{(\beta\text{-arc})}{\beta' \subset \beta}$: arcs

intersecting at only their endpoints.

including the cases $\gamma = \alpha$ (β' : empty) and $\gamma = \beta$ (α' : empty).

Rem



respecting signs of intersections,

FIGURE 2. Two configurations of bicorn curves γ between α and β. Only the right configuration occurs for coherent pairs.

(left doesn't occur)

Prop 3.2 Let (α, β) : coherent pair on S_g.

Then there exists a seq. $(\alpha_j)_{j=0}^{n} \subset \mathcal{L}(\mathcal{NS}(S_g))$

s.t. $\alpha_0 = \alpha$, $\alpha_n = \beta$ & $|\alpha_i \cap \alpha_{i+1}| = 1$.

Moreover, ${}^{\forall}(\alpha_i, \alpha_j)$ is coherent.

pf) Let (α, β) : coherent pair on S_g. (w/ same orientation)

Take a minimal subarc $b_1 \subset \beta$ intersecting α at only endpts.

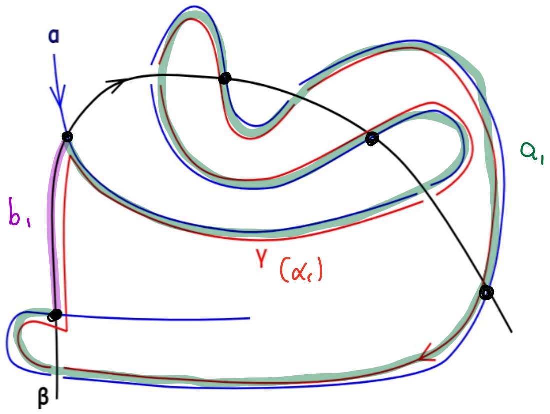


FIGURE 3. Bicorn curve γ between α and β with induced orientation from α and β.

denote $\alpha_1 = a_1 \cup b_1$ where $a_1 \subset \alpha_1$: subarc s.t. $\partial a_1 = \partial b_1$

the minimality of b_1 implies that $a_1 \cap b_1 = \partial a_1 = \partial b_1$ (same endpts),

and that $\alpha_1 = a_1 \cup b_1$ is a bicorn curve.

Note Rem w/ Fig. 2 and so $|\alpha_0 \cap \alpha_1| = 1$ holds (right configuration).

Next we extend b_1 to the next pt $\in a_1 \cap \beta$ following the orientation of β.

$\longrightarrow$ denote by b_2.

take $a_2 \subset \alpha$ w/ $\partial a_2 = \partial b_2$

$\Rightarrow$ $\alpha_2 = a_2 \cup b_2$: bicorn curve

& $i(\alpha_1, \alpha_2) = 1$. (by Rem)

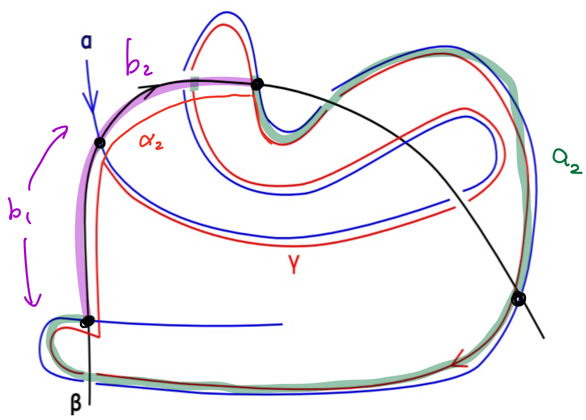


FIGURE 3. Bicorn curve γ between α and β with induced orientation from α and β.

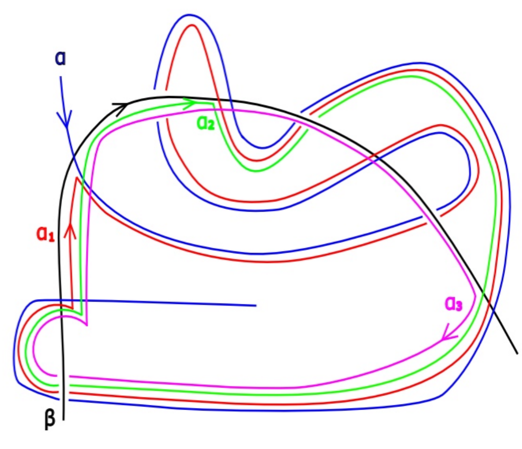


FIGURE 4. An example that extends bicorn curve α_1 along β to obtain bicorn curves α_2 and α_3 such that $|\alpha_2 \cap \alpha_3| = 1$ and $|\alpha_1 \cap \alpha_3| = 3$.

repeat this (take $\alpha_3, \alpha_4, \cdots$)

We can see that
- $\forall \alpha_j$ is a bicorn curve.
- $|\alpha_{j-1} \cap \alpha_j| = 1$,
- and thus (α_{j-1}, α_j) is coherent.
- finally $\exists n$ s.t. $\alpha_n = \beta$ ($\because |\alpha \cap \beta| < \infty$)

By induction $\forall \alpha_j$: non-separating.

Furthermore $\forall (\alpha_i, \alpha_j)$ is coherent because their intersections w/ α or β share the same orientation by the coherence of $\forall$ adjacent pair. ⊠

The sequence of bicorn curves constructed above is called bicorn path.

(rem Not necesarily actual path : adjacent curves are not disjoint.
It is an actual path in Rasmussen's [22] N.S. curvegraph.)

Prop 3.2 actually shows the following

Thm 3.3 For any origami pair (u, v), there exists an origami edge-path joining u & v.
i.e. $\exists\, \mathcal{E} = \{ u = v_0, v_1, \cdots, v_n = v \}$. □

[12] Hensel, Przytycki, Webb (2015) : introducing unicorn paths (specialization of bicorn paths) to show the uniform hyperbolicity of $\mathcal{A}(S_g)$

[21] Przytycki, Sisto (2017) : 1. $B_2(\{ e \in \mathcal{E}(\mathcal{NS}(S_g)) : \text{bicorn path joining } \alpha, \beta \})$ is connected

2. $\{ e \in \mathcal{E}(\mathcal{NS}(S_g)) : \text{unicorn path joining } \alpha, \beta \} \subset B_6(\gamma : \text{geod. joining } \alpha, \beta)$

Prop 3.4 is a bicorn ver. of this result

Lemma 3.3. *Let $x_0, \cdots, x_m$ be a sequence of curves in $\mathcal{C}^1(S_g)$ with $2^{n-1} < m \leq 2^n$ for some positive integer n, then for any bicorn curve $c \in \mathcal{B} \in B(x_0, x_m)$, there exists a curve $c^* \in \mathcal{B}^* \in B(x_i, x_{i+1})$ such that $d(c, c^*) \leq 2n = 2\lceil log_2(m) \rceil$, where $\lceil * \rceil$ denotes the least integer that is larger than or equal to $*$.*

Lem 3.3 follows from a result in [21].
using this lemma, the bicorn analog of the method in [12] shows the following.

Prop 3.4 Let Γ : geodesic joining paths α, β in $\mathcal{C}(S_g)$

$\Rightarrow \{ e \in \mathcal{E}(\mathcal{NS}(S_g)) : \text{bicorn path joining } \alpha, \beta \} \subset B_{14}(\Gamma)$ □

§4. example

- A well known example of filling pair

$\longrightarrow$ Hempel's example ; genus 2, distance 4

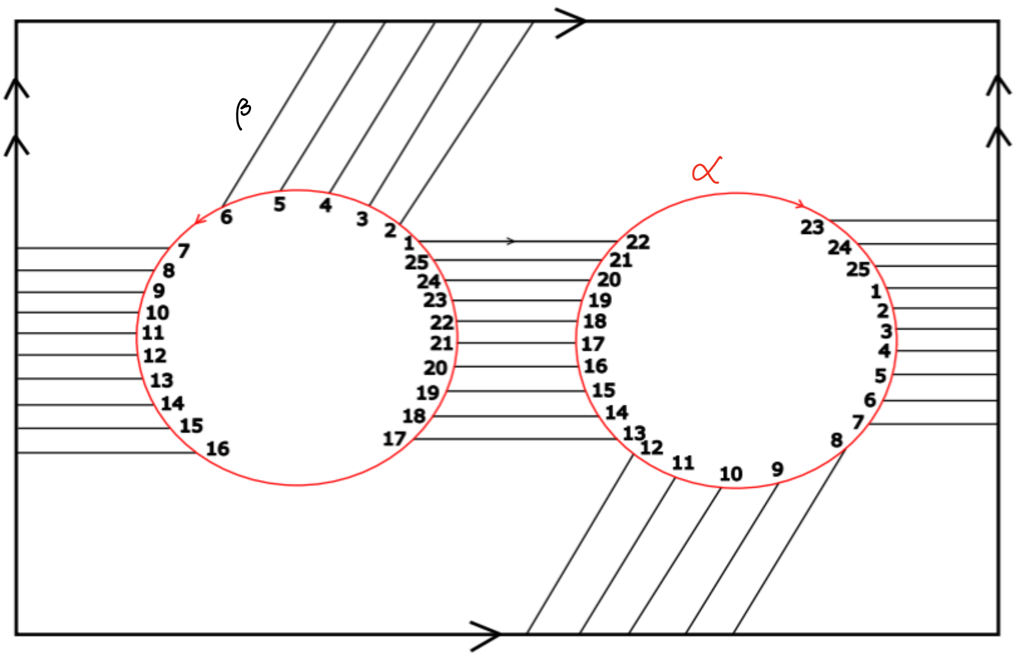


FIGURE 6. Hempel's example is a coherent filling pair.

Lem 5.3 Given a closed oriented surf S_g $(g \geq 2)$,

there exists a coherent filling pair on it.

$\leftarrow$ show by using Hempel's example,

w/ Aougab–Menasco–Nieland's construction [in prepar.]

Prop 5.4 $^{\forall}\alpha \subset S_g$: non-separating curve, $^{\exists}\beta$: another curve intersecting coherently to α,

and $d_{c'}(\alpha, \beta)$ in the curve graph $C'(S_g)$ can be arbitrary large.

Fact $C'(S_g)$ & $NS(S_g)$ are quasi-isometric.

i.e. $^{\exists}A > 1, {}^{\exists}B > 0$ s.t. $\frac{d_{c'}(u,v)}{A} - B \leq d_{NS}(u,v) \leq A\, d_{c'}(u,v) + B$

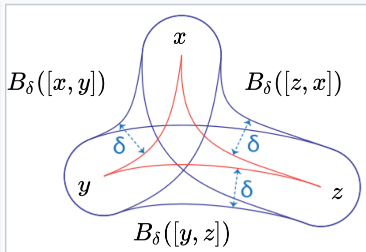
$\longrightarrow$ Thm 1.4 holds.

§ ex. Backgrounds & Future Works

$\mathcal{C}(S_g)$: the curve complex of S_g is introduced by Harvey (1978),

to study the $MCG(S_g)$.

Masur & Minsky (1999) : $\mathcal{C}(S_g)$ is δ-hyperbolic



The δ-slim triangle condition

$\forall$ geod. triangle is δ-thin

i.e, one side is contained in

δ-nbd of the other two sides.

Rasmussen (2020) : $NS(S_g)$ is conn. & uniformly δ-hyperbolic.

w/ infinite diameter.

↳ shown by the 'bicorn technology'

Bell & Webb (2015-16) : Poly-time algorithm for computing $d_{\mathcal{C}}(u,v)$

using bicorn tech.

Glen & Menasco (2014.17) : calculation for short distances based upon

"efficient geodesics".

Now, such distance calculations are still difficult.

Authors are aiming to use origami edge-paths

to approach this problem.

For an origami pair (u,v),

$E(u,v) := \min\{\#\text{edges in } E \mid E : \text{origami edge-path joining } u \,\&\, v\}$

... the origami length

Prop. 3.4 (using bicorn construction)

says that an origami edge-path lies in 14-nbd of a geodesic

→ Authors conjecture origami edge-paths realizing $E(u,v)$ are well-behaved w.r.t the distance.

Conjecture 1.7 Shortest origami edge-paths are quasi-geodesics.

Authors also presented a test for origami edge-paths being a quasi-geod. as Conj. 1.8.

Furthermore, Authors aim to construct comes from.

a greedy algorithm for finding a shortest origami edge-path using Conj. 1.9

basic method for optimization

$\square$