

TEICHMÜLLER GEODESICS, DELAUNAY TRIANGULATIONS, AND VEECH GROUPS

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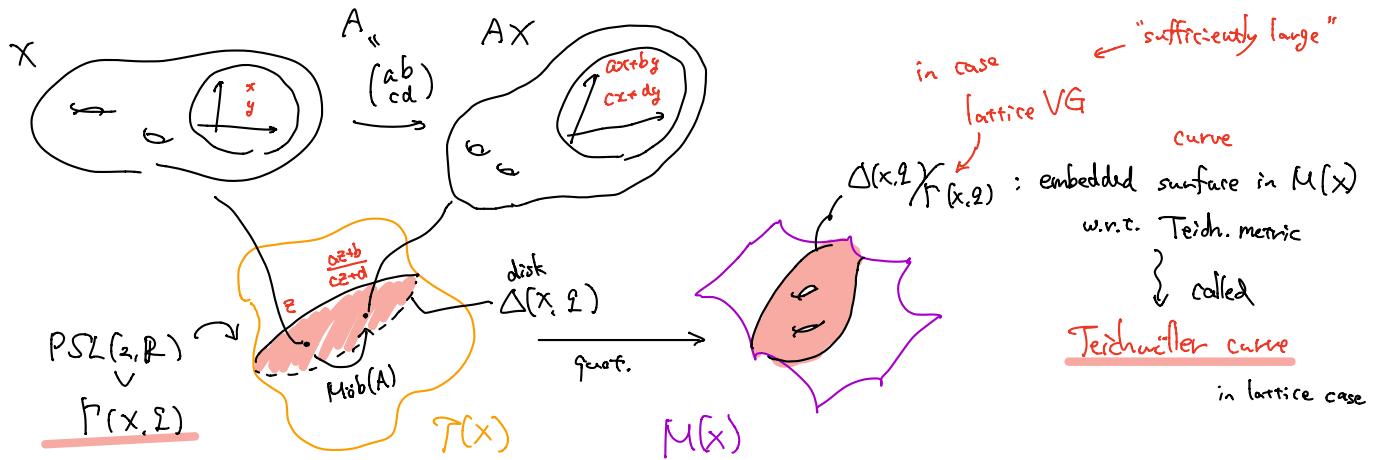
ABSTRACT. In this article, we describe a method for computing generators of the Veech group of a flat surface (which we define as a Riemann surface with a non-zero holomorphic quadratic differential). The method employs a cell structure on the (complex) Teichmüller geodesic generated by the surface, using Delaunay triangulations, which are canonically associated to flat surfaces.

- Teich. theor. and moduli problem (2008)

↳ 0. Intro

• Veech group $\Gamma(X, \mathbb{Q})$ of a flat surface (X, \mathbb{Q})

... the stabilizer of conf. class under the affine-deform. action of $PSL(2, \mathbb{R})$.



a flat surface w/ "sufficiently large" lattice VG

is called a Veech surface.

Veech's dichotomy (1988)

(mutually exclusive)

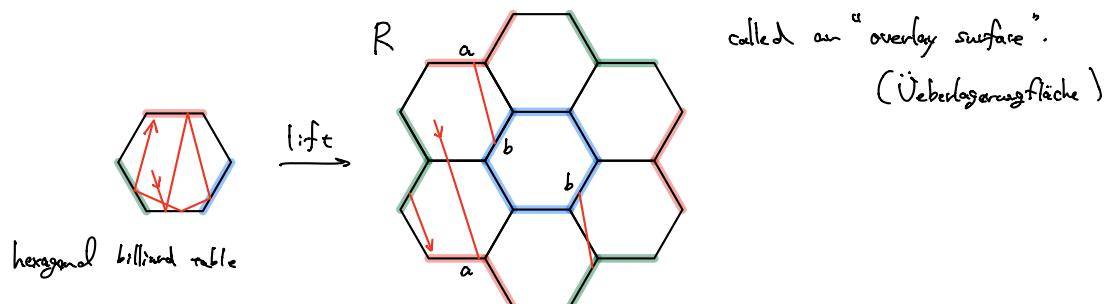
[On a Veech surface, for any direction θ , one of the two following possibilities holds :

(i) The surface decomposes in θ into metric cylinders w/ commensurable moduli.

(ii) Every trajectory in θ is densely & uniformly distributed in the surface.

("geodesic flow is uniquely ergodic")

For a rational polygon, the billiard paths could be lifted to certain flat surface,



The simplest examples of Veech surfaces are origami.



(Schwartz, 2007) algorithm for computing VG.

There have been concerted efforts

- to demonstrate the existence of Veech surfaces w/ specified prop.
- to produce exns via explicit construction.

(Kollar, Smillie 2008) classification of (lattice triangle

(Schwarz, Hooper) Java program "McBilliard", which computes periodic paths in triangle billiard table.

→ 'obtuse triangle that produces a Veech surface.' found.

(McMullen / Calta) criterion for determining when an abelian genus 2 surface is Veech.

② This paper is a pre-warning paper to my last talk: Computing VG using Voronoi samples.

The key idea is similar to the one.

In this paper, we develop a general algorithm for computing generators of VG.

The method depends on keeping track of combinatorial data: Polygonal triangulation

Author uses MATLAB to obtain "iso-Pelennay regions". { dual of Voronoi decomposition

§ 1. Background.

Let X : Riem. surf. of genus ≥ 2 .

Def. 1.1 A holomorphic quadratic differential is a section of the (sheaf of hol. one forms)^{⊗2}.

$$\xrightarrow{\text{loc.}} q(z) = \underbrace{f(z)}_{\text{holo.}} dz^{\otimes 2}$$

Let g : non-neg. h.g.d. on X , $Z := \text{Zero}(g)$.

$\forall p \in X^* := X - Z$. $p' \mapsto \int_p^{p'} \sqrt{g}$ defines a loc. coord. on p .

→ an atlas on X^* whose any transition is of the form $z \mapsto \pm z + c$

→ called a flat structure on X .

For a flat surf. (X, \mathfrak{L}) , $|dz| = |\sqrt{g}|$ is an invariant metric on X .

→ $(X^*, |dz|)$: locally isometric to $(\mathbb{C}, |dz|)$

each point in Z is a cone pt.: symbol $\sim (\mathbb{C}, (z^k dz))$ (k : ord.).

Def. 1.4 The Teich. geodesic (= Teich. disk) is an isom. embedding $H \hookrightarrow T(X)$

$$\Delta(X, \mathfrak{L}) = \left\{ \text{id}_X : (X, \mathfrak{L}) \rightarrow (X, \mathfrak{L}_t) \mid \begin{pmatrix} x_{z_0} \\ y_{z_0} \end{pmatrix} = \begin{pmatrix} 1 & \text{Re } t \\ 0 & \text{Im } t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} : t \in \mathbb{H} \right\} \hookrightarrow T(X) : \text{Teich. geod. generated by } (X, \mathfrak{L})$$

(X, \mathfrak{L}) is called Veech $\Leftrightarrow \Gamma(X, \mathfrak{L})$: lattice $\Leftrightarrow \text{Area}_{\mathbb{H}/\Gamma(X, \mathfrak{L})} < \infty$.

\Leftrightarrow the image $H \hookrightarrow T(X) \rightarrow M(X)$ is an algebraic curve.

Def. 1.7 The Veech group $\Gamma(X, \mathfrak{L})$ of (X, \mathfrak{L}) is the subgroup of $\text{PSL}(2, \mathbb{R})$ acting on $\Delta(X, \mathfrak{L})$

($\text{Aff}^+(X, \mathfrak{L})$)

that corresponds to MCG of (X, \mathfrak{L}) : $\{ f \in \text{Mod}(X) : \text{stabilizer } \Delta(X, \mathfrak{L}) \}$.

By (Thurston, 1988)'s description of $\text{Mod}(X) \curvearrowright T(X)$,

we can closely relate the top. type of $g \in \Gamma(X, \mathfrak{L})$ to the type of isometry $g \curvearrowright H$ as follows:

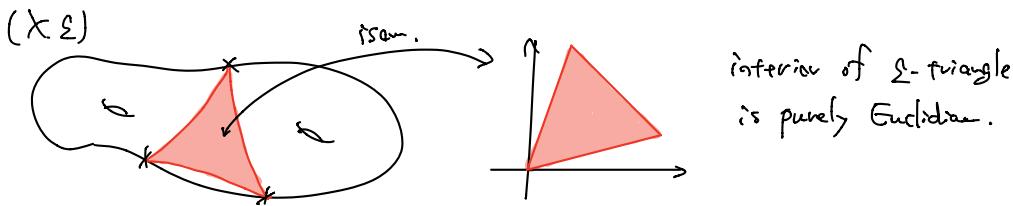
- g is *periodic* if and only if the corresponding isometry of \mathbb{H} is *elliptic* (fixes a point in \mathbb{H}); $\leftarrow g^n = \text{id}$
- g is *reducible* if and only if the corresponding isometry of \mathbb{H} is *parabolic* (fixes a point on $\partial\mathbb{H}$ and the class of horocycles through this point); $\leftarrow g(\mathcal{C}) = \mathcal{C}$
disjoint union of s.c.c.
- g is *pseudo-Anosov* if and only if the corresponding isometry of \mathbb{H} is *hyperbolic* (fixes two points on $\partial\mathbb{H}$ and the Poincaré geodesic connecting them). \leftarrow two transverse foliations
stable & unstable

§ 2. Delaunay triangulations.

Let (X, Σ) be a flat surface.

Def 2.1 A 2-triangle is a 2-simplex (triangle) in X ,

embedded in such a way that $\begin{cases} \text{vertices lie in } \Sigma \\ \text{edges are geodesics w.r.t. } |\Sigma| \\ \text{faces contain no pts in } \Sigma. \end{cases}$



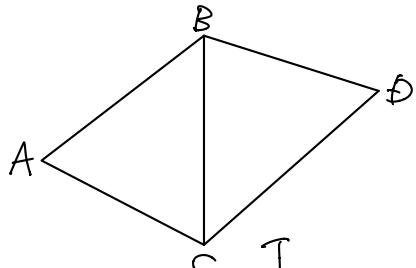
Def 2.2 A 2-triangulation of X is a simplicial cell str. s.t. ^2-cell is a 2-triangle.

A 2-triangulation of X & a 2'-triangulation of X' are combinatorially equivalent.

$\Leftrightarrow \exists$ bij. b/w 2-cells preserving $\overset{\vee}{\text{edge identifications.}}$

$\xrightarrow{\text{induce}}$ natural homeo $X \rightarrow X'$: loc. affine ??
sus

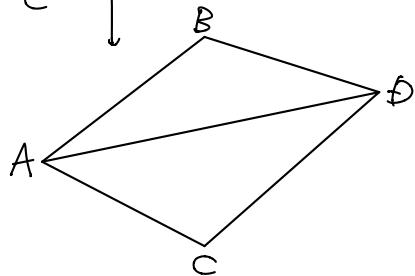
The Whitehead move is a natural way to change one 2-triangulation into another.



Suppose $\triangle ABC, \triangle BCD$: two 2- Δ s
(two of A and D may coincide as a pt.)

Since the interior of $\square ABCD$ contains no crit. pts

$\triangle ABD, \triangle ACD$: two 2- Δ s

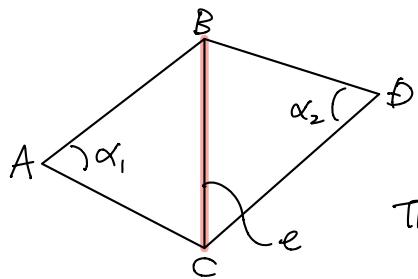


Clearly (X, Σ) admits many 2-triangulations,

We wish to describe a particular one

that is unique for an open dense $\Delta(X, \Sigma)$ -subset.

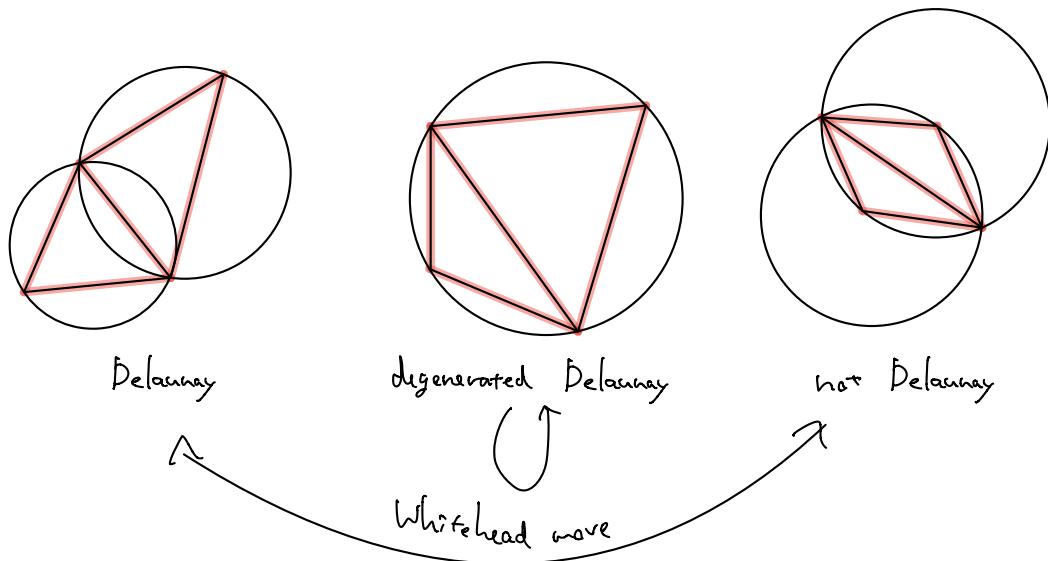
Def 2.4 For two Euclidean triangles s.t.:



The dihedral angle of e : $\alpha(e) = \alpha_1 + \alpha_2$

Def 2.5 A 2-triangulation of X is Delaunay if $\forall e$: edge, $\alpha(e) < \pi$.

Delaunay triangulation is degenerated if $\exists e$: edge $\alpha(e) = \pi$.



Prop 2.6 (Rivin, 1994)

Given any 2-triangulation on X , one obtains a Delaunay 2-triangulation of X

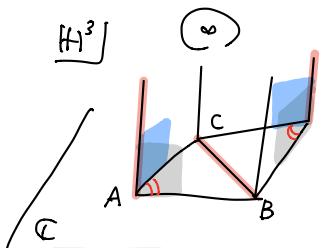
via a finite seq. of Whitehead moves.

Rec 2.8 The terminology of "dihedral angles" is closely related to the hyp. geometry.

In the hyp. 3-space model H^3 ,

$\triangle ABC$ in $\partial H^3 = \mathbb{C}$ induces an ideal tetrahedron $\triangle ABC\infty$.

fact opposite sides of an ideal tetrahedron have equal dihedral angles.



true "dihedral angle" at $A\infty$ is $\angle CAB$.
 { solid angle b/w two faces.

② iso-Delaunay regions in H1.

~~H1~~

Prop 2.9 Suppose a Delaunay triangulation of $(x, \varrho) \in \Delta(x, \varrho)$

If it is non-degenerate, then it is unique (in what sense??) in the good.
(embedded)

If it is degenerate, it is unique up to Whitehead moves.

(pf : from Masař, Šmilová 1991 : duality b/w Delaunay \leftrightarrow Voronoi w/ the convexity of a volume function on the triangulation.)

Prop 2.10 Let $f \in \Gamma(x, \varrho)$ & T : Delaunay triangulation of (x, ϱ)

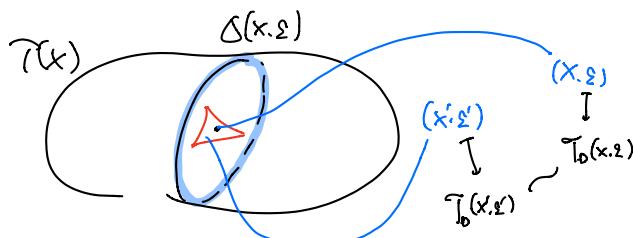
Then $f^* T = \{f^*(T) \mid T \in T\}$ is a Delaunay triangulation of $(f^* x, f^* \varrho) \rightarrow$ differ by markings
 $= (x, f^* \varrho)$

$(x, \varrho) \supset T \leftarrow \Delta \subset \mathbb{C}$
 $\downarrow f \quad \supset Q$
 $(f^* x, f^* \varrho) \supset f^* T$

i.e. Delaunay triangulation is indep. of the marking.

Def 2.11 An iso-Delaunay region (IDR) is a maximal, connected open subset of $\Delta(x, \varrho)$

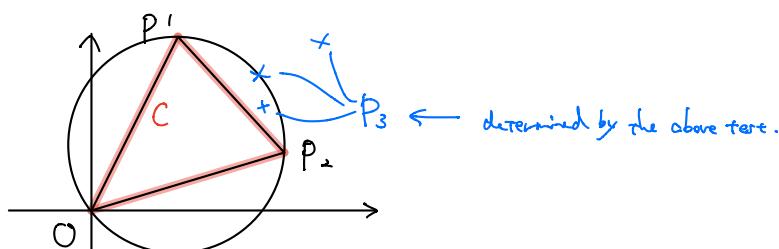
s.t. $\forall (x', \varrho') \text{ belonging to it has combinatorially-equiv. Delaunay triangulations.}$

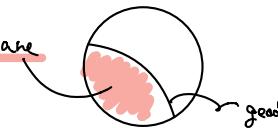


Prop 2.12 W.r.t. d_H, each IDR is convex and has geodesic boundary.

Proposition 2.13 (Incircle test). Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3 = (x_3, y_3)$ be three points in \mathbb{R}^2 , with O , P_1 , and P_2 non-collinear. Let \mathcal{C} be the circumcircle of OP_1P_2 . Then

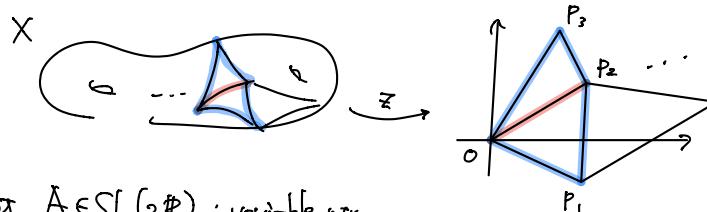
$$(1) \quad \text{if } \det \begin{pmatrix} x_1 & y_1 & x_1^2 + y_1^2 \\ x_2 & y_2 & x_2^2 + y_2^2 \\ x_3 & y_3 & x_3^2 + y_3^2 \end{pmatrix} \quad \begin{cases} < 0, & P_3 \text{ is in the interior of } \mathcal{C}. \\ = 0, & P_3 \text{ lies on } \mathcal{C}. \\ > 0, & P_3 \text{ is exterior to } \mathcal{C}. \end{cases} \quad \} \text{Delaunay}$$



outline of pf of Prop 2.12) claim IDR is an intersection of hyp. half-plane  geod.

Suppose a Delaunay triangulation of $(X, \mathcal{L}) \in \Delta(X, \mathcal{L})$.

For each edge e , choose a chart c.c:



$$\text{Let } P_j = (x_j, y_j) \quad (j=1, 2, 3)$$

Let $A \in SL(2, \mathbb{R})$: variable mix.

s.t. $A(X, \mathcal{L}) \in \Delta(X, \mathcal{L})$ has parameter $u, v \in \mathbb{H}$.

$$\rightarrow A \sim \begin{pmatrix} u & \\ & v \end{pmatrix} \text{ mod } SO(2, \mathbb{R})$$

evaluate the ineq. in Prop 2.13 w/ $\frac{x_i}{y_i}$ replaced by $\frac{x_i + uy_i}{vy_i}$:

$$\rightarrow a(u^2 + v^2) + 2bu + c \geq 0$$

where $a = x_1 y_2 y_3 (y_3 - y_2) + x_2 y_1 y_3 (y_1 - y_3) + x_3 y_1 y_2 (y_2 - y_1),$
 $b = x_1 y_1 (x_2 y_3 - x_3 y_2) + x_2 y_2 (x_3 y_1 - x_1 y_3) + x_3 y_3 (x_1 y_2 - x_2 y_1),$
 $c = x_1 x_2 y_3 (x_1 - x_2) + x_2 x_3 y_1 (x_2 - x_3) + x_1 x_3 y_2 (x_3 - x_1).$

This ineq. presents a hyp. half plane bounded by a geodesic.

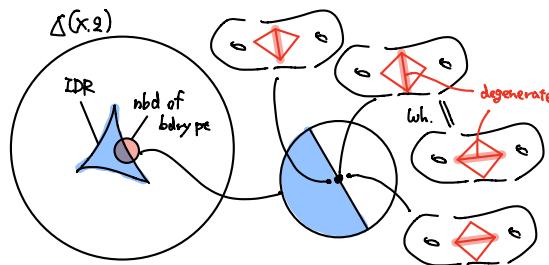
Thus $\mathcal{H}_e := \{A(X, \mathcal{L}) \mid A \in SL(2, \mathbb{R}); \alpha(e) \leq \pi\}$ is a hyp. half-plane

(Veech, 2011) The area of each IDR is finite (perhaps for Veech surfaces)

\rightarrow actually IDR is a hyp. polygon

We may observe that: performing a Whitehead move on the degenerate edges

we reverse these particular ineq. arr. the pt.



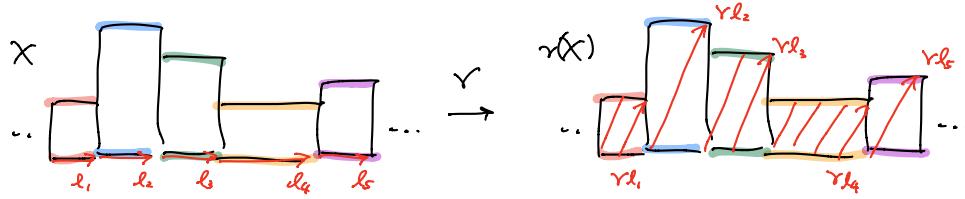
\rightarrow IDRs naturally partition $\Delta(X, \mathcal{L})$
w/ cell complex structure.

Def 2.14 This cell complex on $\Delta(X, \mathcal{L})$ is called the iso-Delaunay complex of (X, \mathcal{L}) .

Prop 2.10 implies that the iso-Delaunay complex of (X, \mathcal{L}) is an invariant under $\mathcal{P}(X, \mathcal{L})$ -action

§ 3. Bouillabaisse surface

When (X, ω) has a direction in which X decomposes into cylinders of commensurable moduli;



$r(X, \omega)$ contains a parabolic element, which is locally a power of the Dehn twist on each cylinder.

Def. A bouillabaisse surface is a flat surface w/ at least two 'comm-moduli' cylinder directions.

Construction : (Thurston, 1982)

name : Hubbard's lecture, 2003.

3.3. Relevance of bouillabaisse surface. The following corollary to Theorem 0.1 demonstrates one measure of the importance of bouillabaisse surfaces:

Corollary 3.2. Every Veech surface is bouillabaisse.

Proof. Any direction containing a geodesic that connects two singular points decomposes the surface into cylinders whose moduli are commensurable. \square

SWS,

§4. The algorithm

IDR, ID-complex determined by (X, Σ) partition $\Delta(X, \Sigma)$ into hyp. polygons.

not necessarily compact.

A fundamental domain of $\Gamma(X, \Sigma)^{\text{orb}}$ can be assembled from IDR's of the surface.

To do this, we consider the "Veech equivalence" of IDR's,

the equivalence under an element in $\Gamma(X, \Sigma)$.

First, clearly (Combinatorial eq.) \Leftarrow (Veech eq.) holds by the uniqueness of Delaunay triangulations.
(X) needs some metric structure.

Let R_1, R_2 : IDR's w/ the same combinatorial data.

\Leftarrow $\text{fix } (X, \Sigma) \in R_2$.

Goal → determine if $\exists (X, \Sigma) \in R_1$, s.t. $(X, \Sigma) \sim_{\text{Veech}} (X', \Sigma')$

② Tree search in $\Delta(X, \Sigma)$.

To assemble a f.d. of $\Gamma(X, \Sigma)$, we perform as follows:

(i) Begin w/ a surface $(x_0, \Sigma_0) \in \Delta(X, \Sigma)$ w/ non-degenerating Delaunay triangulation.

and compute its IDR, as shown in the pf of Prop 2.12.

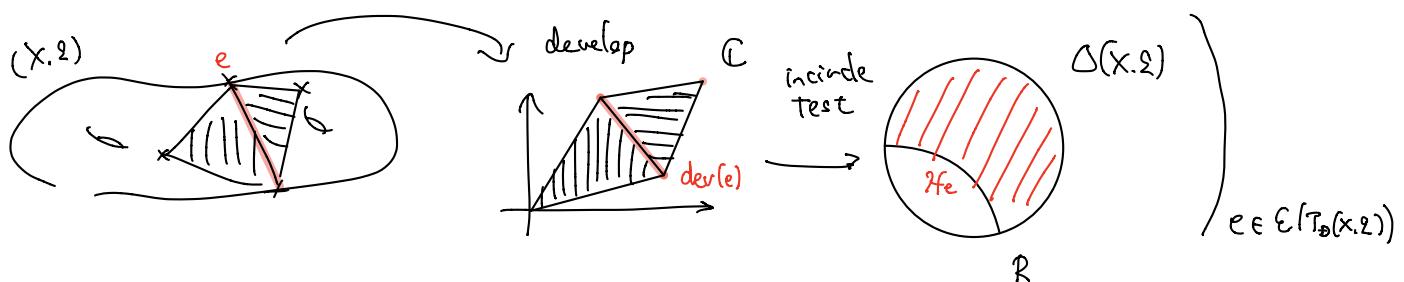
that is: develop each two neighboring Delaunay triangles

and solve the ineq. $a(u^2 + v^2) + 2bu + c \geq 0 \rightarrow \text{get } R_0 \subset \Delta(X, \Sigma)$

where $a = x_1y_2y_3(y_3 - y_2) + x_2y_1y_3(y_1 - y_3) + x_3y_1y_2(y_2 - y_1)$,

$b = x_1y_1(x_2y_3 - x_3y_2) + x_2y_2(x_3y_1 - x_1y_3) + x_3y_3(x_1y_2 - x_2y_1)$,

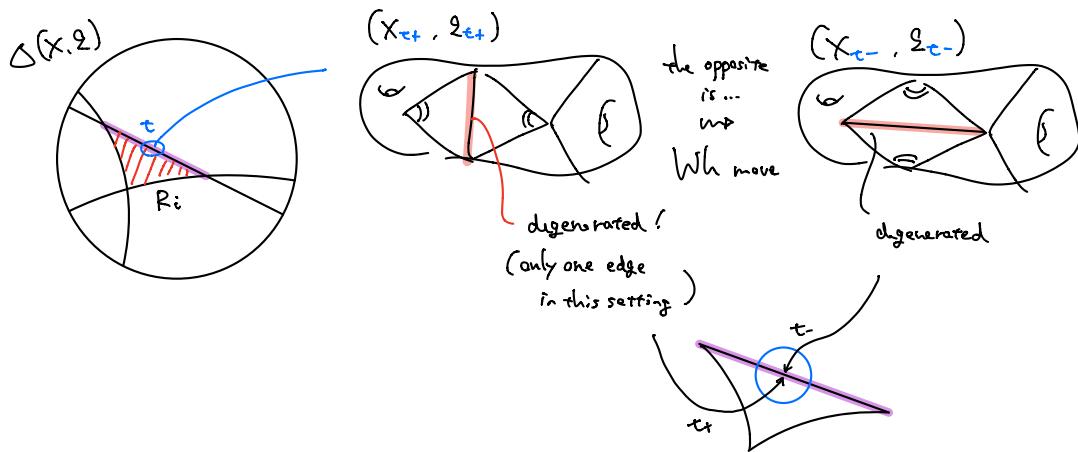
$c = x_1x_2y_3(x_1 - x_2) + x_2x_3y_1(x_2 - x_3) + x_1x_3y_2(x_3 - x_1)$.



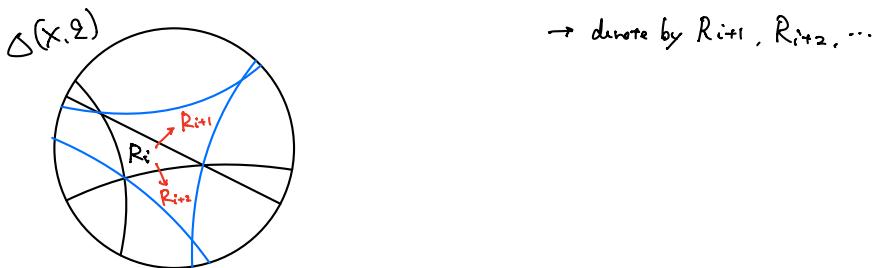
(2) Along each degenerated edge of R_i ,

determine which edge is degenerated and compute the combinatorial type of the adjacent Delaunay triangulation

→ We "cross" the edge $(c \partial R_i)$ into a new IDR.

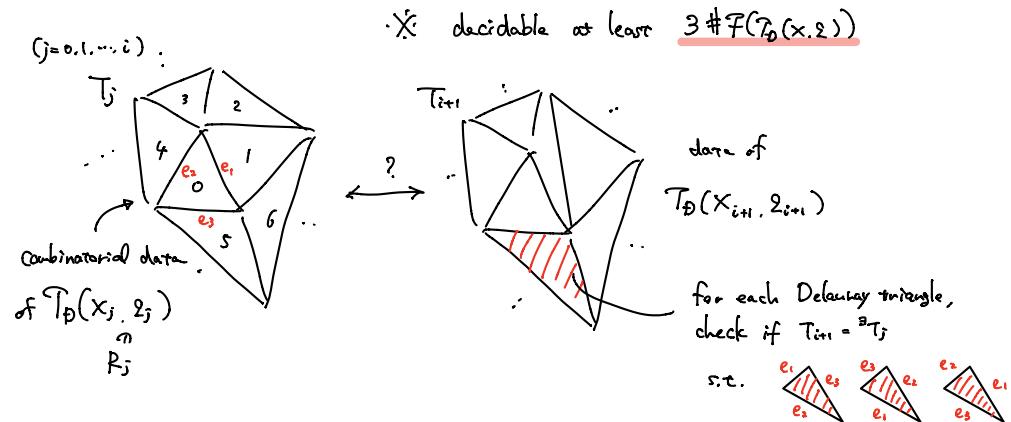


(3) For each bdry segment crossed, compute the other adjacent IDR just like (1)



(4) Check if one of the new IDRs R_{i+1}, R_{i+2}, \dots

is combinatorially equivalent to previous IDRs R_0, R_1, \dots, R_i



if ${}^3 R_j \sim R_{i+1}$, find an isometry $g \in \text{SL}(2, \mathbb{R})$ that carries $R_j \mapsto R_{i+1}$

, add g to $\text{Gen}(\Gamma(x, \omega))$: generating list

& eliminate R_{i+1} from the tree search:
(remember)

(5) if the tree is not closed, go back to (2) for the leftover IDR - bdry edges.

else: finish the algorithm.

References

- [ASC20] Paul Apisa, Rafael M. Saavedra, Christopher Zhang: Periodic points on the regular and double n-gon surfaces. arXiv:2011.02668 (2020)