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ABSTRACT. We present an algorithm, based on work of the first named author, for computing the Veech group, $\text{SL}(X, \omega)$, of a translation surface (X, ω) . This presentation is informed by an implementation of the algorithm by the second named author.

For each stratum of the space of translation surfaces, we introduce an infinite translation surface containing in an appropriate manner a copy of every translation surface of the stratum. We show that a matrix is in $\text{SL}(X, \omega)$ if and only if an associated affine automorphism of the infinite surface sends each of a finite set, the “marked” Voronoi staples, arising from orientation-paired segments appropriately perpendicular to Voronoi 1-cells, to another pair of orientation-paired “marked” segments. (“Segments” are also known as “saddle connections”.)

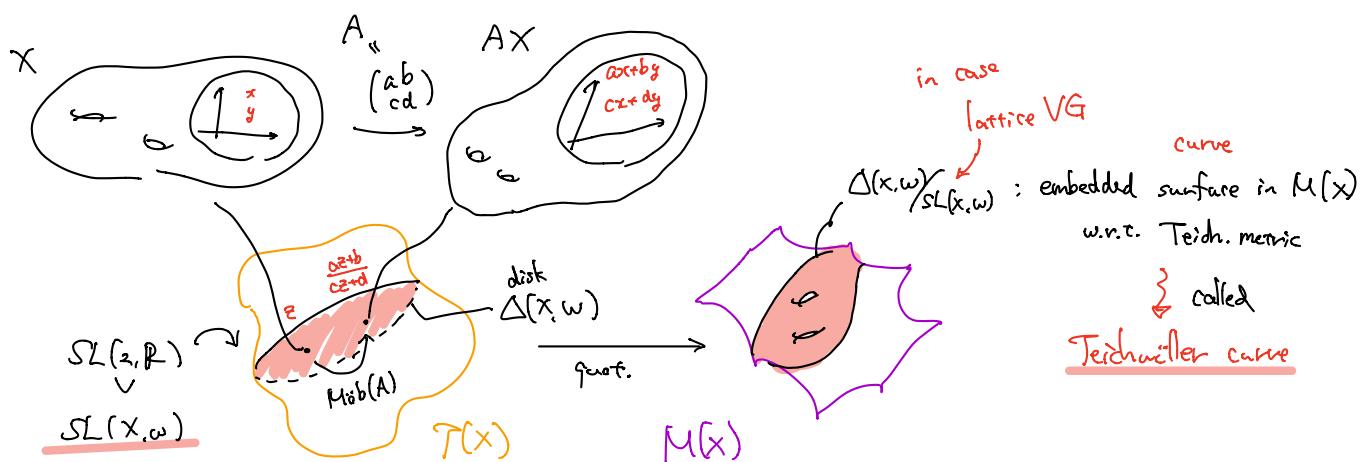
We prove a result that is of independent interest. For each real $a \geq \sqrt{2}$ there is an explicit hyperbolic ball such that for any Fuchsian group trivially stabilizing i , the Dirichlet domain centered at i of the group already agrees within the ball with the intersection of the hyperbolic half-planes determined by the group elements whose Frobenius norm is at most a . When $\text{SL}(X, \omega)$ is a lattice we use this to give a condition guaranteeing that the full group $\text{SL}(X, \omega)$ has been computed.

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1. Intro

- Veech group $\text{SL}(X, \omega)$ of a translation surface (X, ω)

- ... the stabilizer of isom.class under the affine-deform. action of $\text{SL}(2, \mathbb{R})$.



(Veech (1982)) [“] Veech group is a non-cyclic Fuchsian group.

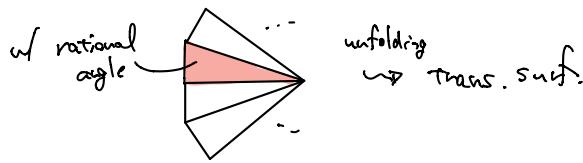
Certain triangle grp arises as VG.

(Bowen-Möller 2010) up to finite index, every non-cyclic Fuchsian triangle grp arises as VG,

On the other hand, (Möller 2009) for a.e. trans. surf. $VG = \{I\}$.

(Kenyon, Smillie 2000 / Puchta 2001)

the only 3 non-isoscele acute triangles unfold to induce Teich. curves.



Open Prob. which non-cocomp Fuchsian groups can be VG?

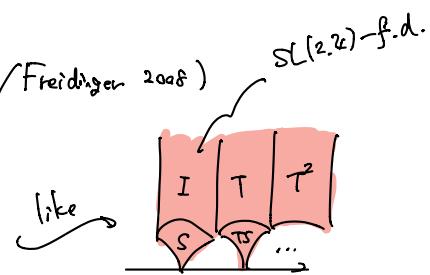
(Kenyon, Smillie 2000) certain equality of trace field for VG containing a hyp. element.

In this paper, we give a new algorithm for computing VGs.

[for lattice VG, our algorithm completely calculates VG.

[approach different from { (Bawn, 2010)
special case algorithms (Scharlaußen 2004 / Freidinger 2008)

As w/ most of others, our algorithm { determines VG elements
builds a fundamental domain in H .



To determine elements, we view a trans. surf.

as being \coprod (Voronoi 2-cells) w/ shared edges identified.

We introduce model surface for each stratum

into which the Voronoi 2-cells inject.

§ 2. background

translative surface $\left\{ \begin{array}{l} \text{i) Surface w/ atlas outside finite set } \Sigma \\ \text{whose transitions are } z \mapsto z + c \\ \text{ii) Surface of an abelian differential } \omega. \\ \text{iii) collection of Euclidian polygons whose edges are glued by translations.} \end{array} \right.$

$$\begin{aligned} Z_\omega(p) &= \int_p^\infty \omega \\ \Sigma &= \text{Crit}(\omega) \end{aligned}$$

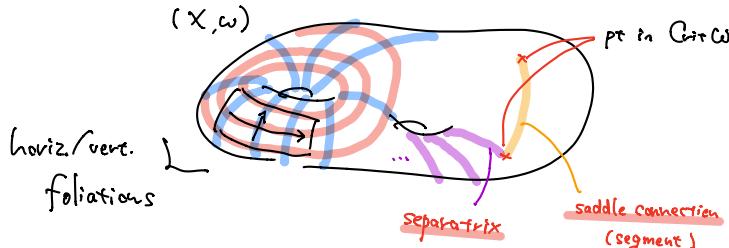
affine diffeomorphism : $f: (X, \omega) \xrightarrow{\text{homeo}} (X, \omega)$: loc. affine w.r.t. $Z\omega$

Jacobian derivative of — : constant matrix in $SL(2, \mathbb{Z})$.

$$SL(X, \omega) = \{ \text{der}(f) \mid f: \text{affine diff. on } (X, \omega) \}$$

... Veech group

- The horizontal foliation on \mathbb{R}^2 induces a foliation on $X \setminus \Sigma$.



for a saddle conn. r , $\text{hol } r = \int_r \omega$: holonomy vector

- Stratum $\mathcal{H}(d_1, d_2, \dots, d_g) := \{ (X, \omega) : \text{trans. surf. } \left/ \int_{\text{ord}_p \omega} \omega \mid p \in \text{Crit}(\omega) \right. \} = \{ d_1, \dots, d_g \} \}$

→ by Riemann-Roch's or Gauss-Bonnet's,

$(X, \omega) \in \mathcal{H}(d)$ share the same genus. \uparrow sometimes denoted by $\Omega M_g(d)$.

- Voronoi decomposition

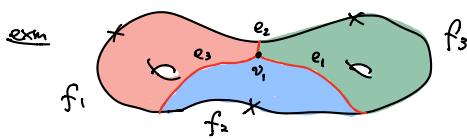
↪ pull back of Euclidian metric on \mathbb{R}^2 via $Z\omega$.

Let $(X, \omega) \in \mathcal{H}(d)$. We take the natural flat metric on X .

Voronoi 2-cell f_p corresponding to $p \in \text{Crit}(\omega)$ is defined by

$$f_p = \left\{ q \in X \mid d(p, q) \leq d(p, q') \quad \forall p' \in \text{Crit}(\omega) \right\}^{\text{int}}$$

Bdries of Voronoi 2-cells are unions of



Voronoi 1-cells : open geodesics
Voronoi 0-cells : bdry pts of 2-cells

§3. A model surface, Voronoi strakes, and a VG membership criterion.

Given $\mathcal{H}(d_1, d_2, \dots, d_s)$, for each $i = 1, 2, \dots, s$. ($d_1 \leq d_2 \leq \dots \leq d_s$)

Let $\Omega_i := (\mathbb{R}^2, z^{d_i} dz)$ & $\Omega = \bigsqcup_{i=1}^s \Omega_i$.

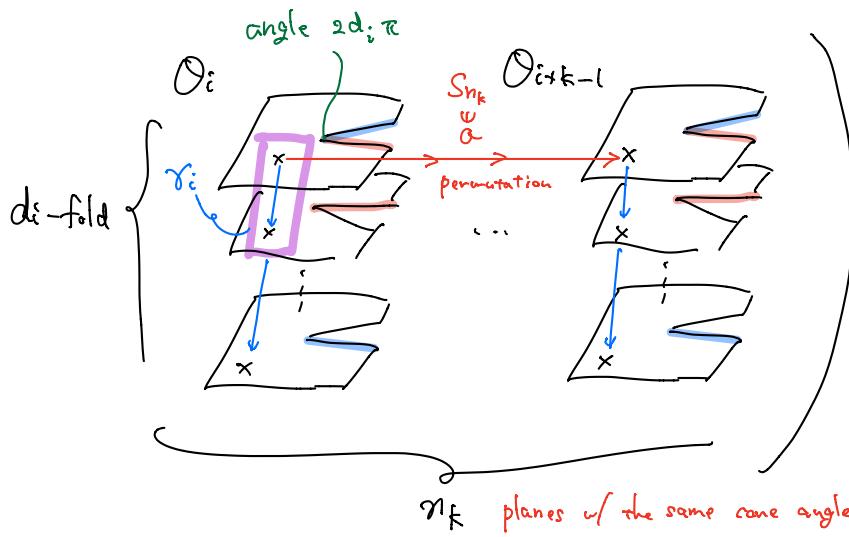
We partition $\{1, \dots, s\}$ w/ n_1, n_2, \dots, n_t ($\sum n_k = s$)

where $d_1 = d_2 = \dots = d_{n_1}, d_{n_1+1} = \dots = d_{n_1+n_2}, \dots$, and so on. (corr. to # repeated values.)

Lemma 1 $\text{Trans}(d_i) = \langle r_i \rangle$ where $r_i^{d_i+1} = 1$.

Ker(den L_Ω)

$\text{Trans}(\Omega) \cong \langle S_{n_1}, S_{n_2}, \dots, S_{n_t}, r_1, r_2, \dots, r_s \rangle$



$\text{Trans}(\Omega)$ is generated by those elements.

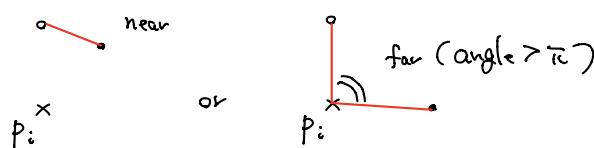
Len singularities Ω_{Ω_i} should be fixed.

Lemma 2 $\forall A \in GL_2(\mathbb{R})$, $\exists f_A \in \text{Aff}^+(\Omega)$ s.t. $\text{der } f_A = A$.

pf) $\exists \mapsto A \exists \left(\int_0^P z^{d_i} dz : \text{natur. coord.} \right)$ belongs to $\text{Aff}(\Omega_i)$ for $\forall i$. \otimes

Lemma 3 \forall two distinct pts on Ω_i $\exists!$ geod. connecting them

which is :

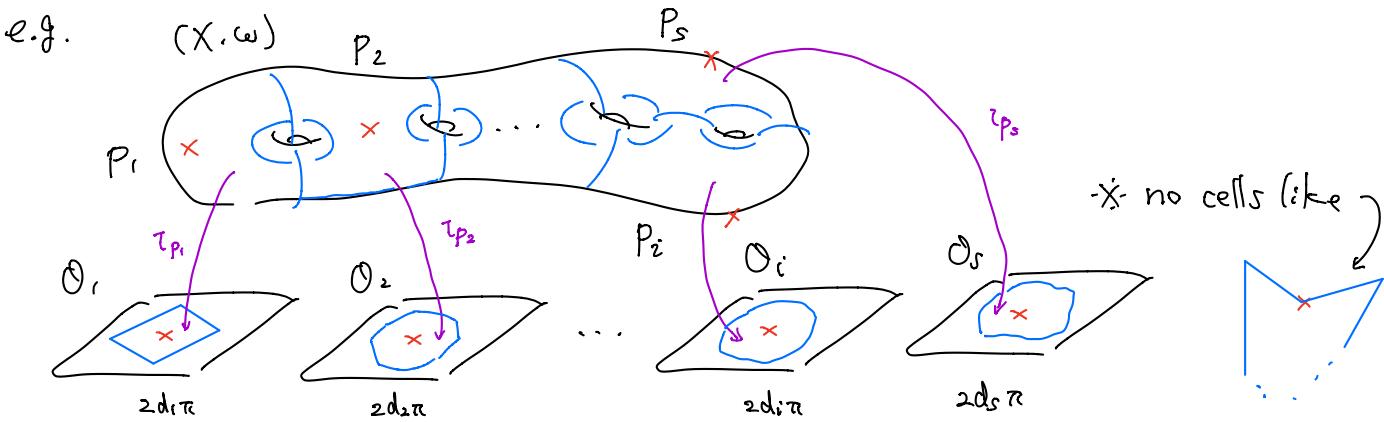


cf) Strelbel, 1984

Lemma 4 Let $(x, \omega) \in \mathcal{H}(d_1, d_2, \dots, d_s)$

Then there is an isometric injection

$\bigsqcup_i (\text{Voronoi 2-cells } \sim d_i) \hookrightarrow \Omega = \bigsqcup_i \Omega_i$ where $f_i \hookrightarrow \Omega_i$.



Def 5 We will assume through out that an isometric embedding \underline{L} as above,

For each $p \in \text{Crit}(\omega) \cap \text{order } d_i$,

We denote by $\left\{ \begin{array}{l} O_p : \Theta_i, \\ C_p : \text{the Voronoi 2-cell centred at } p, \text{ and} \\ \tau_p : C_p \hookrightarrow O_p : \text{the restriction of } \tau. \end{array} \right.$

Note that

$(X, \omega) \underset{i \in \text{some}}{\cong} \bigsqcup_{p \in \text{Carw}} \overline{C}_p / \sim$ for some gluing rule \sim of Voronoi 2-cells.

From the data of $\bar{C}_{p_1}, \dots, \bar{C}_{p_s}$ in X ,

$\iota(C_1), \dots, \iota(C_s)$ have a collection of segments which are glued to recover (X, w) .

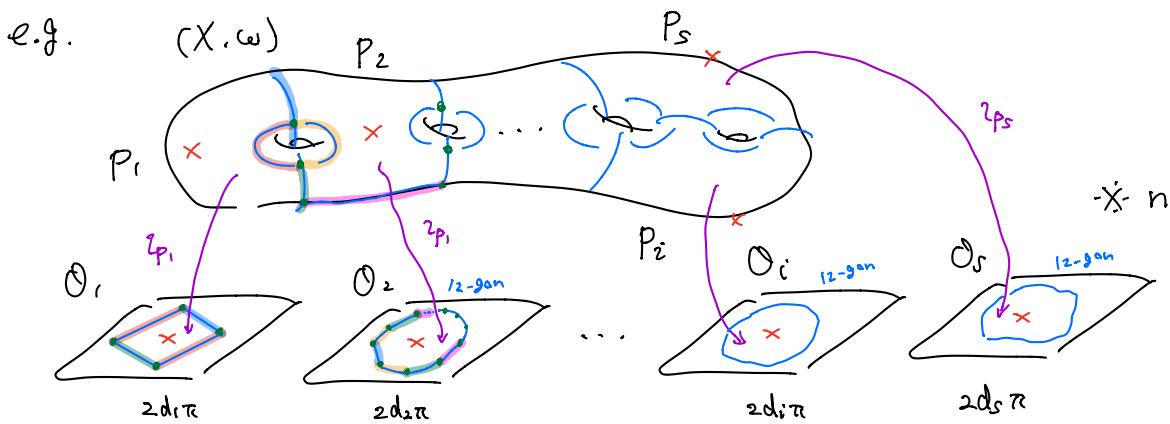
as in the following :

Prop 6 Let $\gamma := \bigcup_{p \in \text{Gr}(\omega)} \overline{\tau(p)}$.

$\forall p \in \text{Crit } \omega, \quad \forall r \in \dot{\tau}(c_p) : \text{bdry segment},$

$$\exists^I_p \in C_{n+1} \quad \exists^I_r \subset \delta_I(C_p) :$$

s.t. the relation ($r \sim r'$ for $\forall r$ as above) makes a trans. scrf. \mathcal{Y}/\sim isom. to (X, w) .



Def 7. Let $j : \gamma = \sqcup \mathcal{O}_p / \sim \rightarrow X$ be the isometry constructed from $(\tau_p^{-1})_{p \in \text{crit. co.}}$.

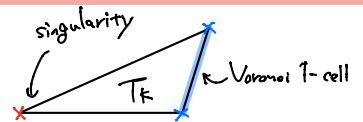
$$\hat{z} := j^{-1}(z), \hat{e} := j^{-1}(e) \quad \text{for } z \in R: \text{point} \& e \in R: \text{edge/e (segment)}$$

For $z \in \mathcal{O}_p \setminus \{\text{pt}\}$, let $D(z) := B(z; d(0, z)) \subset \mathcal{O}_p$: open disk.

Prop 8. Let $e \in \bar{\mathcal{C}}_p$: edge & $z \in e$: point.

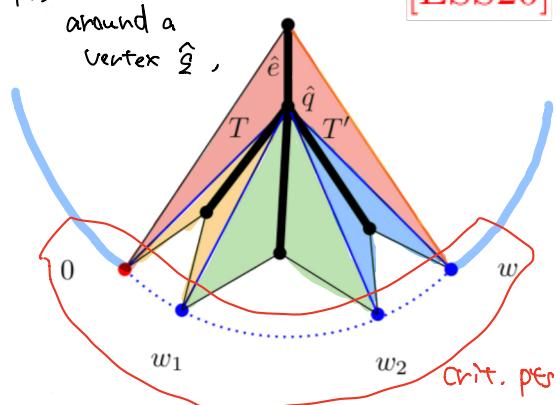
Then $D(\hat{z})$ is contained in $\bigcup_{k=1}^m T_k$: union of triangles s.t.

say: "Voronoi triangle"



In particular, (as boundary behavior) j maps $D(\hat{z})$ to an open set of $X \setminus \mathcal{L}$.

pf)



[ESS20]

by Prop 6 & def of Voronoi decomposition,

the reflection T' of a Voronoi triangle T

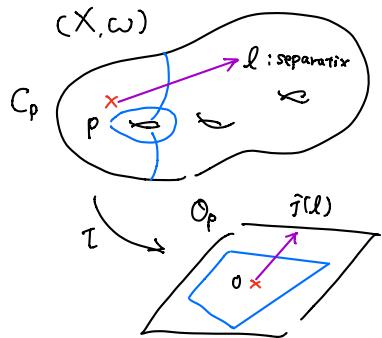
is again ____.

Thus $D(\hat{z}) \subset \bigcup \{T \cup T' \mid T: \text{Voronoi triangle neighboring } \hat{z}\}$.
!!
 $P(\hat{z})$

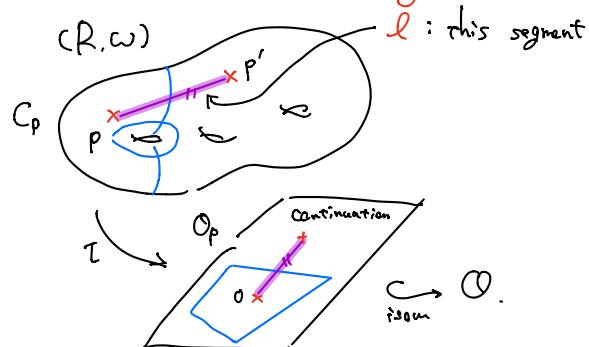
& for $\forall \hat{z}_1, \hat{z}_2 \in \hat{\mathcal{L}}$: edge $D(\hat{z}) \subset P(\hat{z}_1) \cup P(\hat{z}_2)$ holds. \square

3.2. Closed Voronoi 2-cells as convex bodies on \mathcal{O} .

Def 9

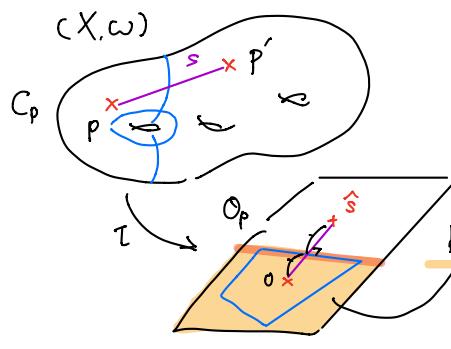


if l meets w some singularity,



We denote the embedded image of \hat{l} by $\hat{l}(e)$.

Prop 11 $\forall s$: saddle conn. of (X, ω) ,



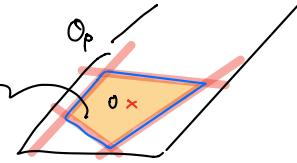
$H_c(s) := \{z \in \mathcal{O}_p \mid d(0, z) \leq d(s, z)\}$... the half space of s

Then, $\partial H_c(s)$ is the perpendicular bisector of s .

Prop 13 For $p \in \text{Crit}(\omega)$, we define the convex body

$$\Omega_p := \bigcap_{s \in \text{sep}: s.c.} H_s(s)$$

Then, $\Omega_p = \overline{\mathcal{C}(C_p)}$.



3.3. Voronoi staples determine (X, ω)

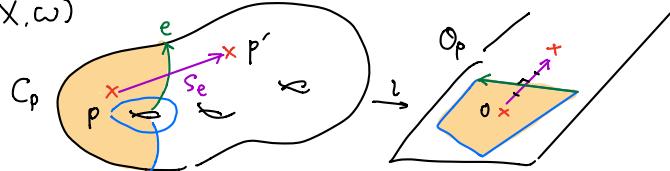
Def 14 For each saddle-conn. s , we fix an orientation

$$\text{Let } i : s \rightarrow s \leftrightarrow s' \leftarrow s' \quad \text{inversion}$$

$$\mathcal{M}(X, \omega) := \{ (s, s') \mid s \text{ saddle-conn. on } (X, \omega) \}$$

$$\widehat{\mathcal{M}}(X, \omega) := \{ (\widehat{s}(s), \widehat{s}(s')) \mid \dots \}$$

Consider: (X, ω)



Every edge e of C_p is associated to a saddle-conn. s_e

which is a perpendicular bisector in Ω_p .

$$\text{Let } \mathcal{E}_p^\perp = \{ s_e : \text{saddle conn.} \mid e : \text{edge of } C_p \} \subset \text{SC}(X, \omega)$$

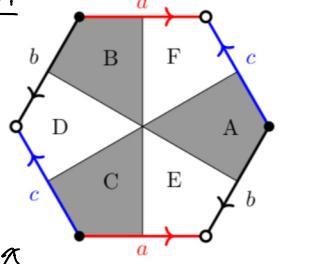
$$\mathcal{S}(X, \omega) := \{ (s, s') \in \mathcal{M}(X, \omega) \mid s \in \mathcal{E}_p^\perp, p \in \text{Crit}(\omega) \}$$

$$\widehat{\mathcal{S}}(X, \omega) := \{ (\widehat{s}(s), \widehat{s}(s')) \in \widehat{\mathcal{M}}(X, \omega) \mid \dots \}$$

set of marked Voronoi staples

[ESS20]

exam



regular hexagon tiling.

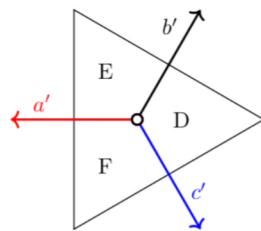


image in \mathcal{O}_1

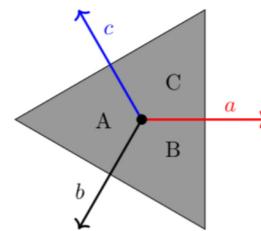
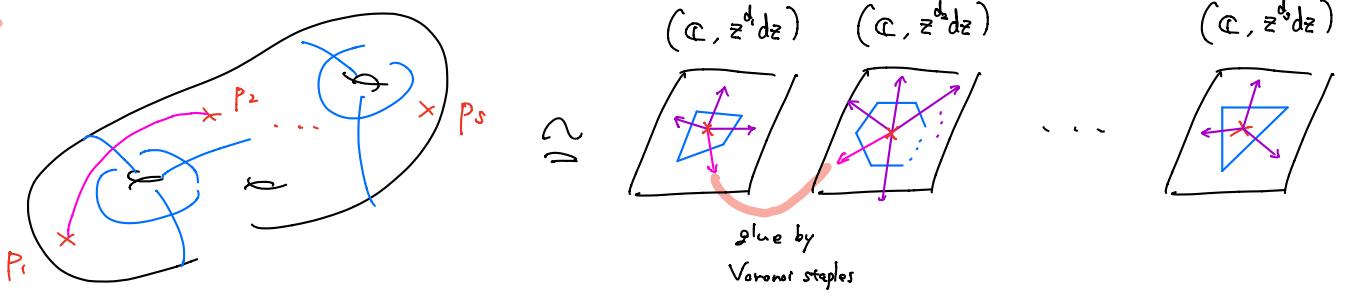


image in \mathcal{O}_2

there are two Voronoi 2-cells and three Voronoi staples.

We obtain (X, ω) by gluing Voronoi 2 cells by 'staples.'

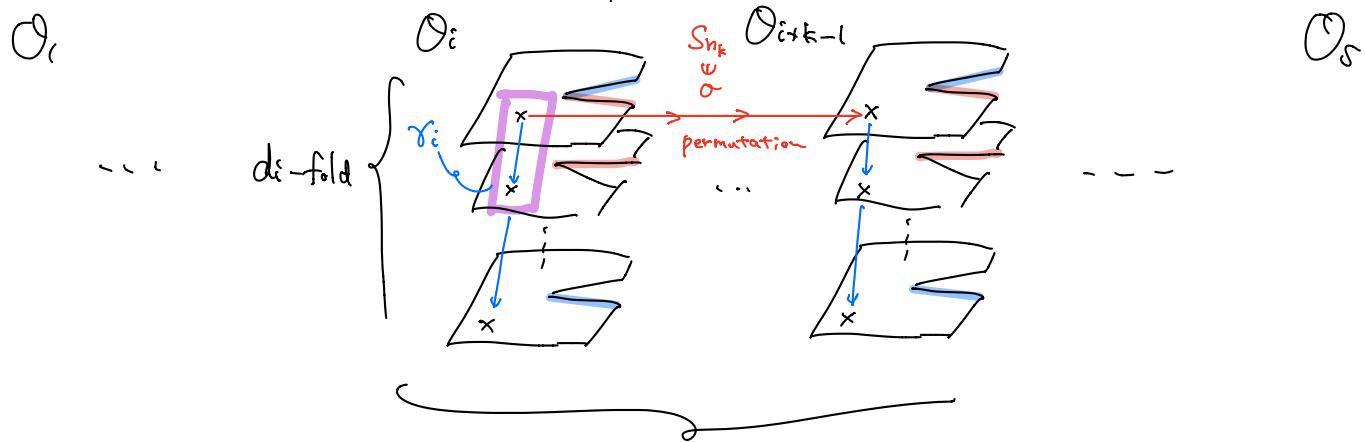
Summary



[ESS20]

Theorem 15. Suppose that both $(X_1, \omega_1), (X_2, \omega_2)$ belong to $\mathcal{H}(d_1, \dots, d_s)$ and choose maps ι_1, ι_2 to \mathcal{O} . Then $(X_1, \omega_1), (X_2, \omega_2)$ are equivalent translation surfaces if and only if $\widehat{\mathcal{S}}(X_1, \omega_1)$ and $\widehat{\mathcal{S}}(X_2, \omega_2)$ are in the same $\text{Trans}(\mathcal{O})$ -orbit.

recall $\text{Trans}(\mathcal{O}) \cong \langle S_{n_1}, S_{n_2}, \dots, S_{n_r}, \gamma_1, \gamma_2, \dots, \gamma_s \rangle$



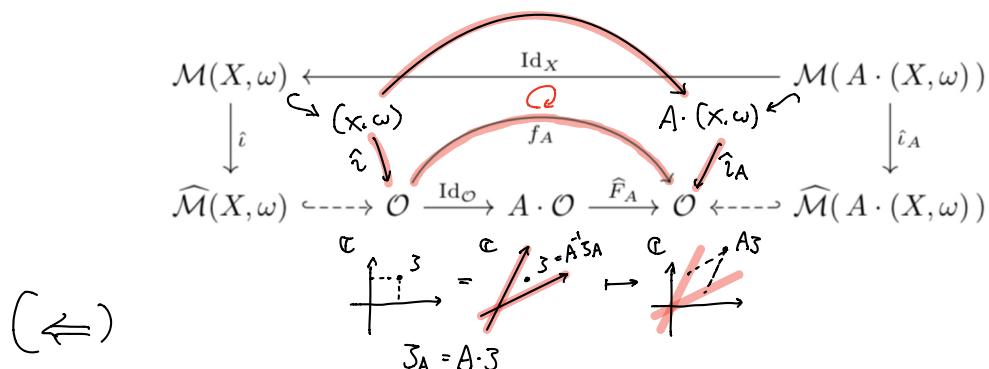
3.4 Membership criterion

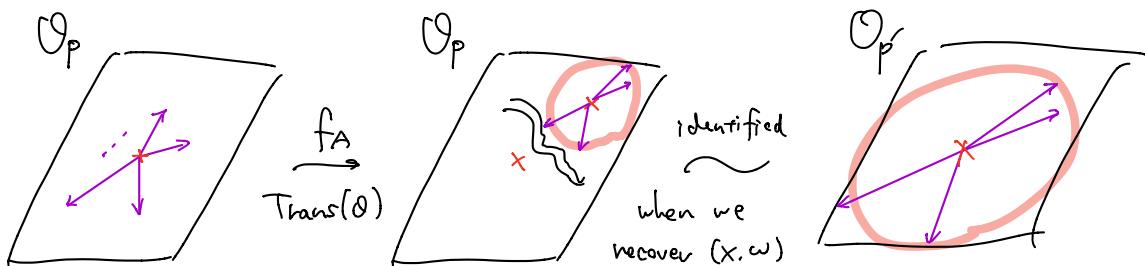
" $SL(2, \mathbb{R}) \ni A$ membership in $SL(X, \omega)$ "

recall $f_A : \mathbb{C} \rightarrow \mathbb{C}$: the natural affine deformation
 $\downarrow \quad \downarrow$
 $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Proposition 16 (Membership Criterion, initial form). Fix (X, ω) and suppose $A \in SL_2 \mathbb{R}$. Then $A \in SL(X, \omega)$ if and only if f_A sends $\widehat{\mathcal{S}}(X, \omega)$ to a subset of $\widehat{\mathcal{M}}(X, \omega)$, up to some element of $\text{Trans}(\mathcal{O})$.

[ESS20]





for each $p \in \text{Crit}(\omega)$, f_A sends each origin O_p & staples emanating from O_p to a point & segments emanating from it which will be identified w/ $O_{p'}$ & staples emanating from $O_{p'}$ for some $p' \in \text{Crit}(\omega)$.

So, if $\alpha = (f_A(\alpha), f_A(\alpha')) \in \widehat{\mathcal{M}}(x, \omega)$ for $(\alpha, \alpha') \in \widehat{\mathcal{S}}(x, \omega)$
it implies that $\alpha \in \widehat{\mathcal{S}}(x, \omega)$.

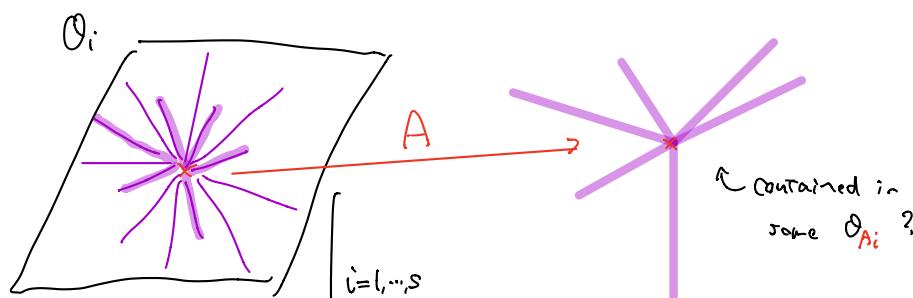
Thus " $f_A(\widehat{\mathcal{S}}(x, \omega)) \subset \widehat{\mathcal{M}}(x, \omega)$ up to $\text{Trans } \emptyset$ "

implies that f_A keep the same Voronoi staples picture up to permutation.

Since Voronoi staple set $\widehat{\mathcal{S}}(x, \omega)$ has enough data to recover (x, ω) ,
we have the conclusion \otimes

We may apply this criterion

by observing the planar geometry on $\Omega = \bigsqcup O_i$
concerning $\widehat{\mathcal{S}}(x, \omega)$ & $\widehat{\mathcal{M}}(x, \omega)$



$$A : \{1, 2, \dots, s\} \rightarrow \{1, 2, \dots, s\}$$

$$\downarrow \quad \quad \quad \downarrow$$

$$i \mapsto A_i$$

rotation & permutation?

(i) build $\widehat{\mathcal{M}}(x, \omega)$

w/ $\widehat{\mathcal{S}}(x, \omega)$

(ii) check whether

$$A \cdot \widehat{\mathcal{S}}(x, \omega) \bigr|_{O_i} \subset \widehat{\mathcal{M}}(x, \omega)$$

(iii) check whether

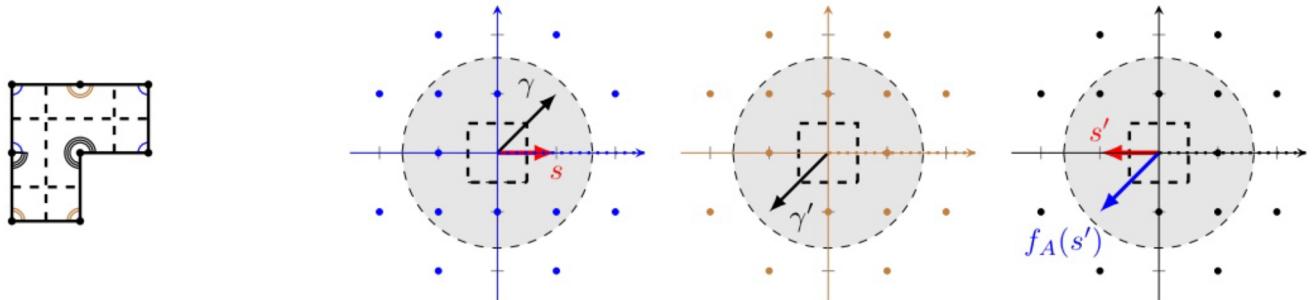
A acts on $\widehat{\mathcal{S}}(x, \omega)$
as $\text{Trans } \emptyset$,

We may use some geometric data

to improve the membership criterion.

- bounds of lengths of $\widehat{S}(x, \omega)$

[ESS20]



Corollary 19 Let $\ell = \max\{\text{len}(s) \mid (s, s') \in S(x, \omega)\}$

Suppose $A \in \text{SL}(2, \mathbb{R})$ has maximum singular value $\leq \nu \in [1, \infty)$
(eigenvalue??)

$\Rightarrow A \in S(x, \omega)$ iff f_A maps $\widehat{S}(x, \omega)$ into $\widehat{\mu}(x, \omega)$ w/ max length $\leq \nu \ell$
up to $\text{Trans}(0)$.

- Frobenius norm of $A \in \text{SL}(2, \mathbb{R})$

$$\dots \|A\| = \sqrt{\text{Tr}(A^* A)}$$

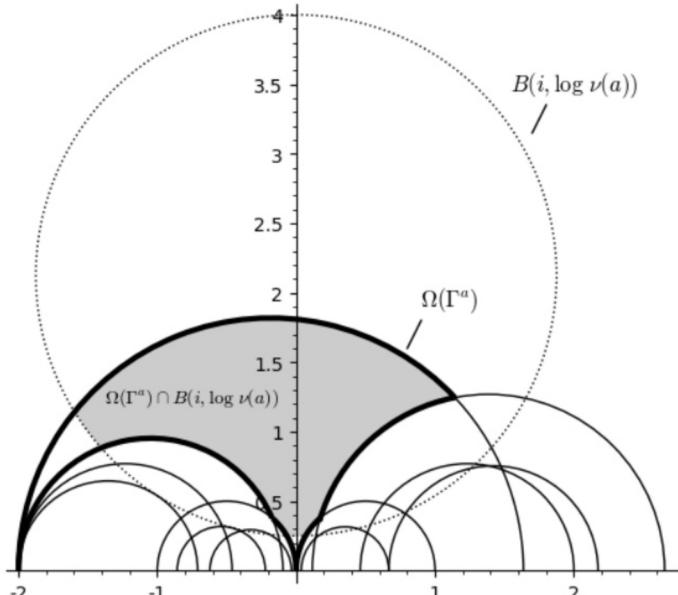
For $\Gamma < \text{SL}(2, \mathbb{Z})$, $\Gamma^a := \{A \in \Gamma \mid \|A\| \leq a\}$

hyp. ball

Theorem 25. Suppose that $a \geq \sqrt{2}$ is such that

$$\mu_{\mathbb{H}}(\Omega(\Gamma^a)) < 2\mu_{\mathbb{H}}(\Omega(\Gamma^a) \cap B(i, \log \nu(a))). \quad \nu(a) = \sqrt{\frac{a^2 + \sqrt{a^2 - 4}}{2}}$$

Then the subset Γ^a generates Γ . In particular, Γ is a lattice.



[ESS20]

shaded region shows where the computed intersection of half-planes of small Frobenius norm must agree with the Dirichlet group, as guaranteed by Proposition 24. Here, $\Gamma = \text{SL}(M \cdot X, \omega)$ is given in Example 20, see also Figure 6, and $M = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}$ in a surface whose Veech group has trivial stabilizer of $z = i/4$.

- Basic steps of algorithm

(I) Given a polygonal presentation (X, ω) ,

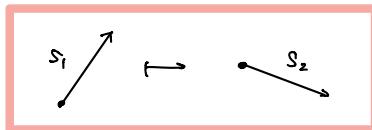
one can calculate the Voronoi decomposition.

And also $\widehat{S}(X, \omega)$ & l : maximal length are knowable.

We test membership using Cor 19.

(II) We may take candidate matrices

by choosing two distinct saddle-combs.



(III) Beginning w/ $a = \sqrt{2}$.

$$P := \left\{ s : \text{saddle-comb w/ length } \leq v(a)l \right\} : \text{finite}$$

We then check whether

f_A maps $\widehat{S}(X, \omega)$ into P modulo Trans \mathcal{O} .

This test is sufficient to obtain generating set of $SL(X, \omega)$



References

[ESS20] Edwards, B., Sanderson, T., Schmidt, T.: Computing Veech groups. arXiv.2012.12444 (2020)