CYLINDER CURVES IN FINITE HOLONOMY FLAT METRICS

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ABSTRACT. For an orientable surface of finite type equipped with a flat metric with holonomy of finite order q, the set of maximal embedded cylinders can be empty, non-empty, finite, or infinite. The case when $q \leq 2$ is well-studied as such surfaces are (half-)translation surfaces. Not only is the set always infinite, the core curves form an infinite diameter subset of the curve complex. In this paper we focus on the case $q \geq 3$ and construct examples illustrating a range of behaviors for the embedded cylinder curves. We prove that if $q \geq 3$ and the surface is *fully punctured*, then the embedded cylinder curves form a finite diameter subset of the curve complex. The proof is then used to characterize the flat metrics for which the embedded cylinder curves have infinite diameter.

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exm: $q = 6$

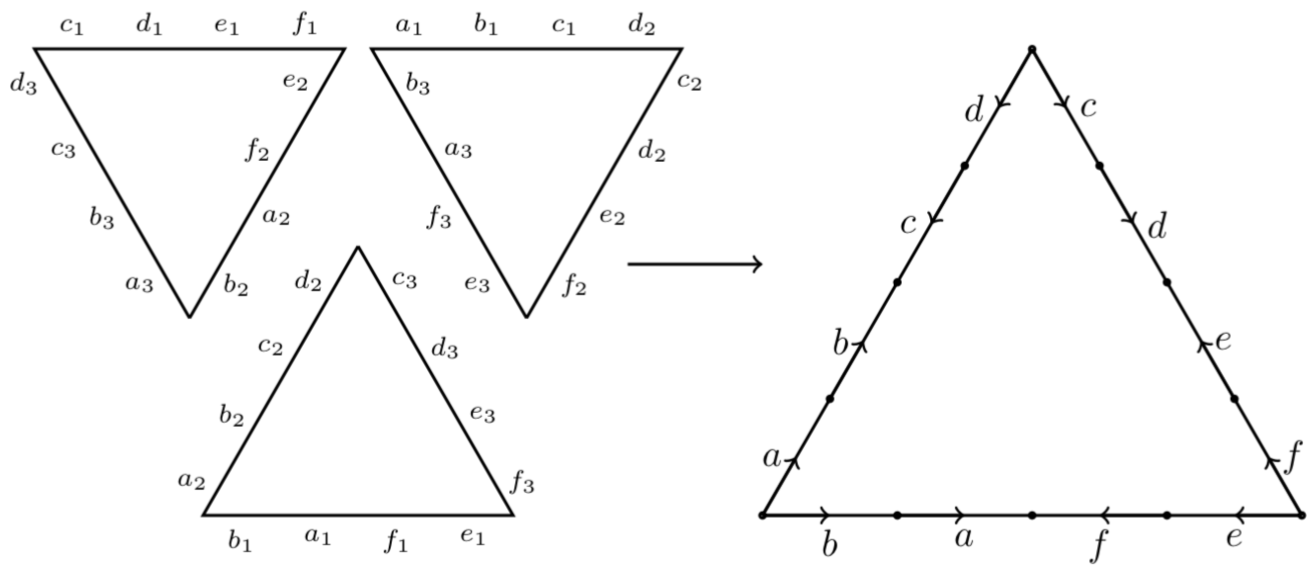
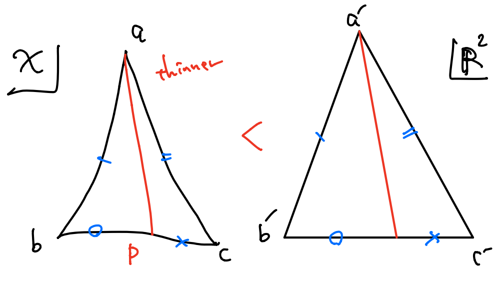


FIGURE 1. A flat metric with holonomy of order 6 and the (degree 3) holonomy almost trivializing cover.

∘ CAT(0) - space (cf. www.math.keio.ac.jp/~kei/GDS/2nd/catK.html)

Consider a metric sp. (X, d) on which $d(x,y) = \min\{ \ell(r) \mid r \subset X : \text{path joining } x, y \}$ for $^\forall x, y \in X$.
for $a, b, c \in X$, take $a', b', c' \in \mathbb{R}^2$ s.t. $\| a' - b' \| = d(a,b)$ $\| b' - c' \| = d(b,c)$ $\| c' - a' \| = d(c,a)$
and call $\triangle a'b'c' \subset \mathbb{R}^2$ a comparison triangle

Def X is a CAT(0)-sp $\Leftrightarrow$ $^\forall p \in \widetilde{bc}$ (geodesic).
for $p' \in \overline{b'c'}$ w/ $\| b' - p' \| = d(b,p)$
$\boxed{d(a,p) \leq \| a' - p' \|}$ holds.



§2. Notation

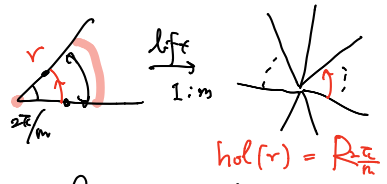
A Euclidean cone metric φ on S (surf.) is: a metric which is locally isometric to Euc. plane away from finite numbers of pts. (except for ?)

at these pts, the metric has a cone singularity.

Rem the metric is non-positively curved (locally CAT(0)) if $\forall$ cone angle $> 2\pi$

※ Now we assume that the metric completion of S is a surface w/ finite area.

$\longrightarrow$ we call such a metric a flat metric.



- φ : flat metric on S.

$\varphi^\circ := \varphi|_{S^\circ = S \setminus \mathrm{Cone}(\varphi)}$: associated fully punctured metric

※ special orthogonal grp.

the holonomy homomorphism $\pi_1(S^\circ) \to SO(2)$※ is:

$[r] \longmapsto hol(r) :=$ (rotation of lift of r via the loc-isometries to $\mathbb{R}^2$)

$\hookrightarrow$ When $\mathrm{ord}(hol) = 1$ or 2. S comes from trans. surf. or half-trans. surf.

in this paper we focus on the case $\mathrm{ord}(hol) \geq 3$.

Def If $\mathrm{ord}(hol)$ divides d (≥ 1), we say that the metric is a d-flat metric.

Given $d \geq 1$ and a mero d-differential on a cl. R.S. w/ poles of order $\leq d-1$.

$\longrightarrow$ it determines a d-flat metric φ. (by puncturing at all poles.)

$\longrightarrow$ $z_{\varphi, p_0}(p) = \int_{p_0}^{p} \sqrt[d]{\varphi}$ defines an atlas $\forall p, p_0 \in S^\circ$.

whose transition maps are $z \mapsto e^{\frac{2\pi}{d}i} z + c$.

$\longleftarrow$ We call a preferred coordinate.

- We can also describe d-flat metrics by gluing sides of finite numbers of Euclidean polygons.

e.g.

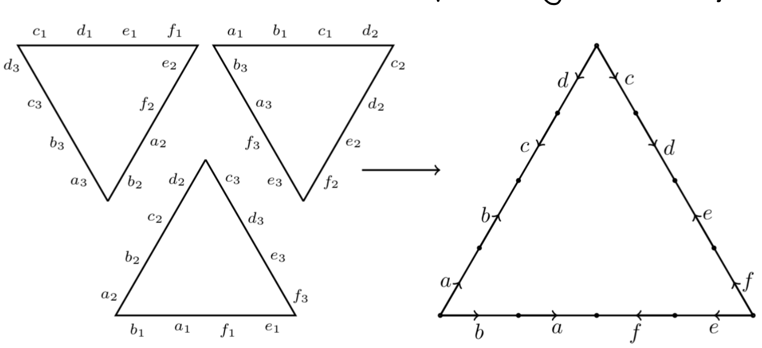


FIGURE 1. A flat metric with holonomy of order 6 and the (degree 3) holonomy almost trivializing cover.

· Given a fully punctured flat metric φ on S, w/ $\mathrm{ord}(\mathrm{hol}) = q$,

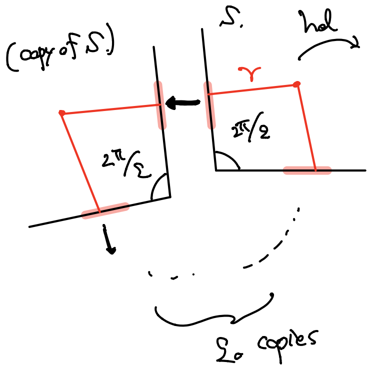
the holonomy almost trivializing cover $p : (S_0, \varphi_0) \to (S, \varphi)$ is : ✻ minimum cover which is translation or half-trans. surface.

the loc-isom cover $\overset{\text{corres.}}{\sim}$ $\{\pm I\} < \mathrm{hol}(\pi_1(S^\circ)) < SO(2)$

hol $\uparrow$ $\uparrow$ hol

$\boxed{H = \mathrm{hol}^{-1}(\{\pm I\}) < \pi_1(S^\circ)}$ ← take a cover from this inclusion.

The cover p is a cyclic cover of degree $q_0 = \begin{cases} q & \text{if } q : \text{odd} \\ q/2 & \text{if } q : \text{even.} \end{cases}$



⇝ by taking q_0-copies of S, and gluing them,

we will obtain S_0, $p : S_0 \to S$, and $\varphi_0 = p^*\varphi$

for which holonomy-image is $\{\pm I\}$.

Def If a (simple ?) closed curve γ has a geod. rep. containing no cone pts,

then γ is called a cylinder curve. : which lies inside a unique maximal immersed Euclidean cylinder.

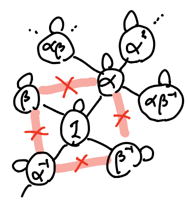
… maximal cylinder

Combining Masur's result of hol. al. triv. cover, it follows that

" q-flat metric has infinitely many homot. classes of cyl. curves. "

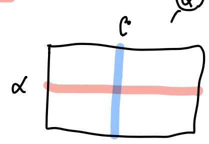
We focus on $EC(\varphi) := \{ [\gamma] :$ $\overset{\text{free}}{\text{homot.}}$ class of embedded cyl. curve $\gamma \subset S \}$.

$ESC(\varphi) := \{ [\delta] :$ $\overset{\text{fixed endpt}}{\text{homot.}}$ class of saddle conn. $\delta \subset S \}$



$C(S)$: curve graph : vertices are isotopy classes of ess. s.c.c.

$A(S)$: arc graph : vertices are isotopy classes of arcs.



in each case, vertices are joined by an edge

if the isotopy classes admit disjoint representatives. (except endpts: $A(S)$)

◎ We view $EC(\varphi) \subset C(S)$, $ESC(\varphi) \subset A(S)$ (as set of vertices.)

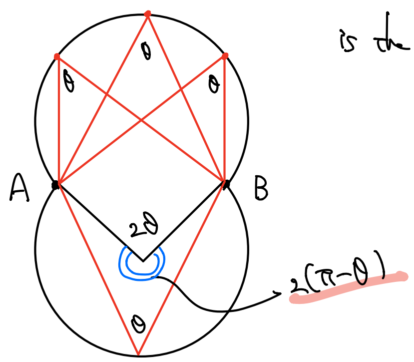
and refer to the diameter of them as $\boxed{\sup\{\text{min. numbers of edges of paths joining 2 vertices}\}}$ (edge-path distance.

Theorem 1. *Let φ be a fully punctured flat metric on a surface S with finite holonomy of order at least 3. Then $\mathrm{EC}(\varphi) \subset \mathcal{C}(\mathrm{S})$ has finite diameter.*

§3. Exms

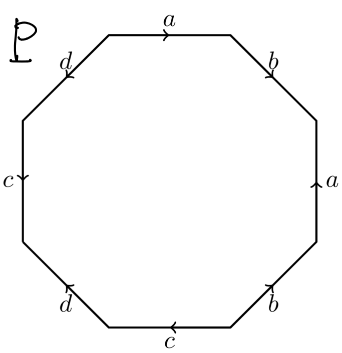
<u>Prop 1</u> $A, B \in \mathbb{R}^2$: distinct pts. for $0 < \theta < \pi$.

$$I_{AB}(\theta) = \{ C \in \mathbb{R}^2 \mid \angle ACB = \theta \}$$



is the two circular arcs of central angle $2(\pi - \theta)$ joining A and B.

3.1.1. Octagon



Starting w/ a regular octagon P, we identify sides by isometry as indicated in left Figure.

$\rightsquigarrow$ Resulting surf. S is a genus 2 surf. w/ 4-flat metric φ

φ has a single cone pt w/ angle 6π.

(S, φ) has 3 embedded cylinders :

In fact, these are the only maximal cylinders. ←

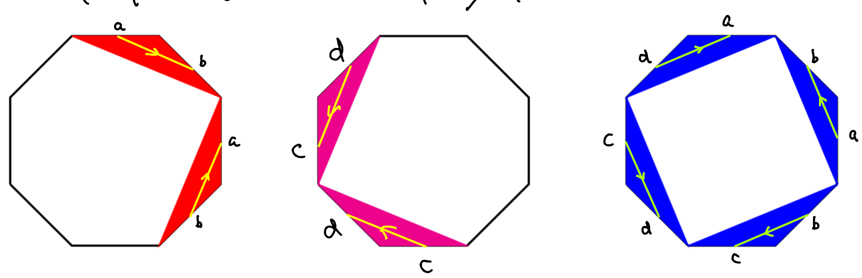


FIGURE 3. Cylinders in the octagon example.

i.e.

<u>Prop 2</u> $\mathrm{EC}(\varphi)$ consists of exactly three curves.

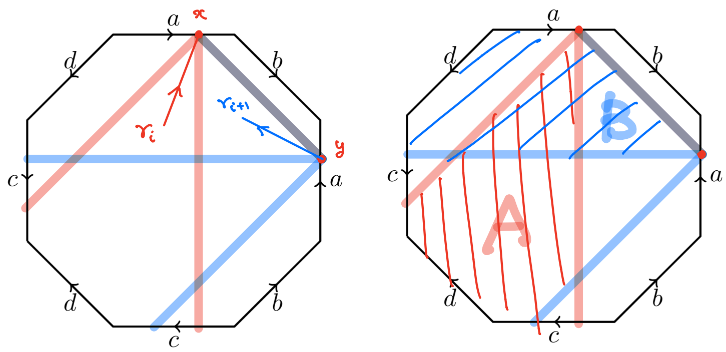
pf) Suppose γ : core curve of some emb. cylinder in S.

i.e. closed geod. containing no cone pts.

<u>claim</u> γ can be embedded only if it is one of the three.

Let γ cuts through P in $\gamma_1, \gamma_2, \cdots, \gamma_k$: segments in P, joining sides

Step 1 if $\exists \gamma_i$ meets a side making angle $\theta \in [\frac{\pi}{4}, \frac{\pi}{2}]$,
then γ will have a self-intersection.



if γ_i lies in $\begin{pmatrix} A \\ C \end{pmatrix}$, then γ_{i+1} will lie in $\begin{pmatrix} B \\ D \end{pmatrix}$ resp.

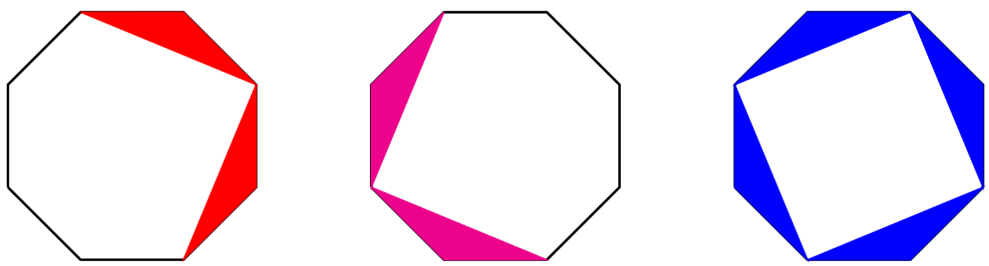
$\longrightarrow$ γ should have self-intersection. //

($\leadsto$ γ is NOT embedded !!)

Step 2. any line segment that connects non-adjacent sides of P
makes angle $\geq \frac{\pi}{4}$.

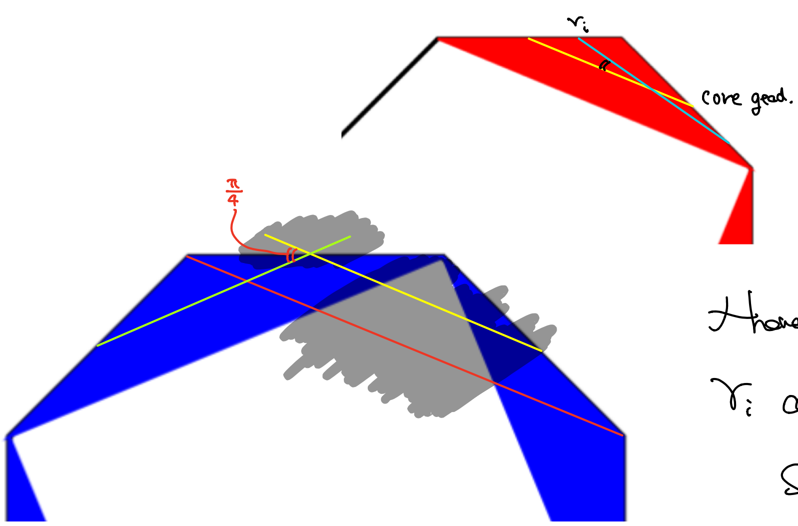
So, if γ is embedded $\Rightarrow$ each γ_i should join adjacent edges.

Each pair of adjacent sides of P determines following triangles.



Step 3 each γ_i is embedded in one of these triangles

$\longrightarrow$ γ_i makes angle $\leq \frac{\pi}{8}$ w/ the core geodesic.



if γ is entirely contained in one of the three $\longrightarrow$ OK.

otherwise $\longrightarrow$ contained in (at least) two of the three.

$\leadsto$ switch b/w intersecting cylinders happens.

However, core curves of intersecting cylinders meet at angle $\frac{\pi}{4}$.

γ_i cannot make angle $\leq \frac{\pi}{8}$ w/ both of core geodesics.

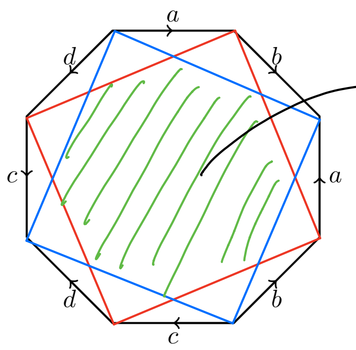
So "otherwise" never happens. $\boxtimes$

3.1.2 cylinder-free region.

Here we construct "building blocks" for 'cylinder-free regions'.

↳ subsets which are disjoint from emb. cylinders.

exm,



cylinder-free.

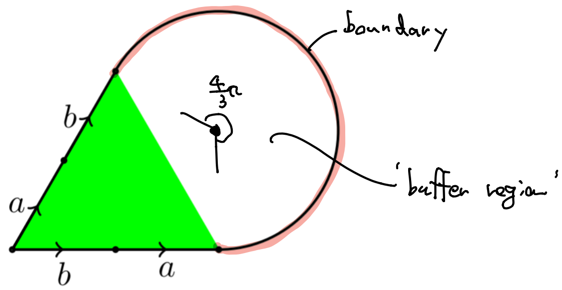
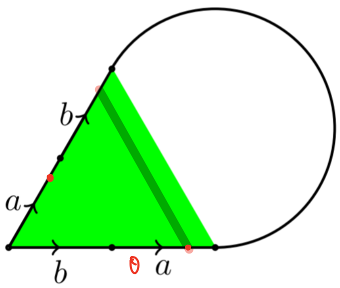


FIGURE 5. The building block $\mathcal{B}$ with a triangular cylinder-free region.

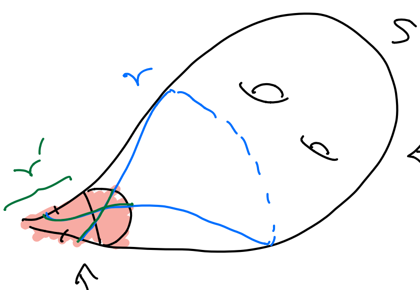


Prop 3 Suppose the building block $\mathcal{B}$ is loc. isom.ly immersed in (S,φ).
Then the image of triangle in $\mathcal{B}$ is cylinder-free.

pf) Suppose γ: closed non-singular geod. in (S,φ) that intersects $\mathcal{B}\subset S$.

claim γ has a self-intersection.

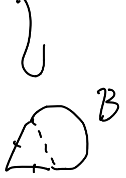
Consider a maximal arc γ' of γ that lifts to $\mathcal{B}$ meeting the triangle.



(a) γ' enters/exits $\mathcal{B}$ in $\partial\mathcal{B}$ (circle)

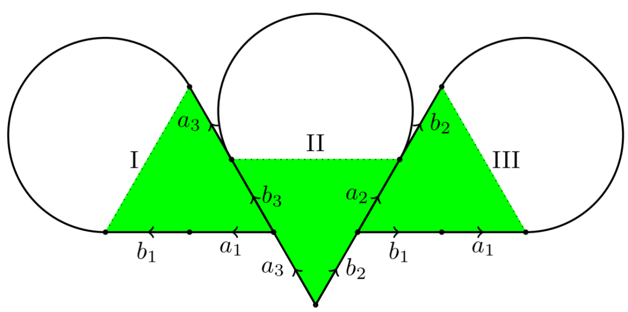
or

(b) $\gamma' = \gamma$



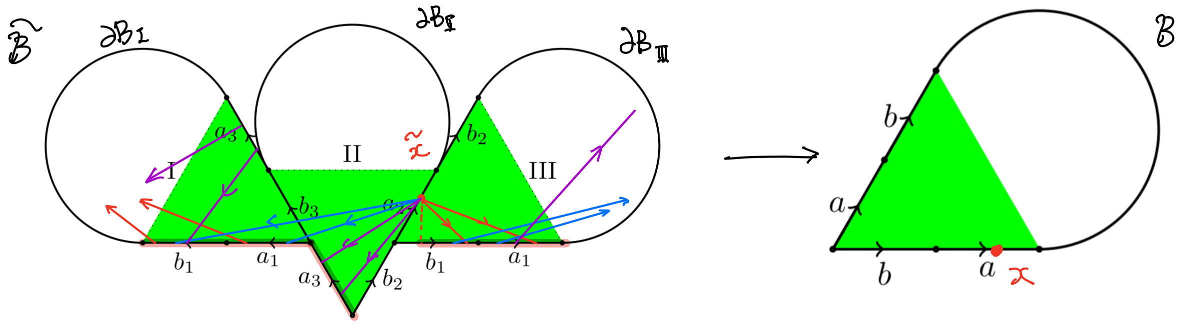
Let γ' cuts through $\mathcal{B}\setminus(a\cup b)$ in $\gamma_1\gamma_2\cdots\gamma_k$.

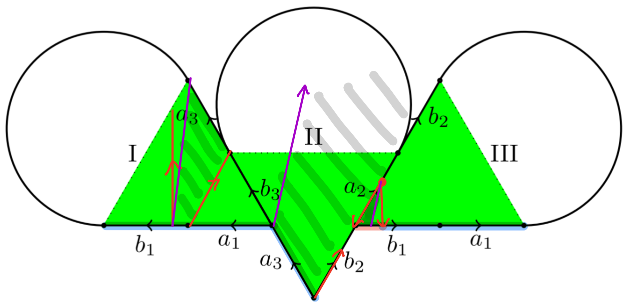
We consider further lift of γ' via the hol. alm. triv. cover $\tilde{\mathcal{B}}\to\mathcal{B}$.



claim I γ' must have endpts in ∂B and $k \le 5$ $(\gamma = \gamma_1\gamma_2 \cdots \gamma_k)$.

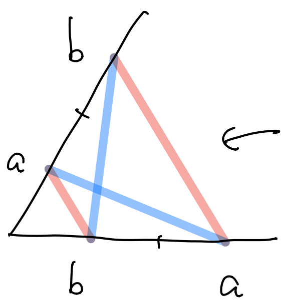
Consider any intersection pt x of γ w/ $a \cup b$.
we may assume $x \in \gamma_1 \cap a$!



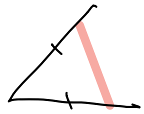


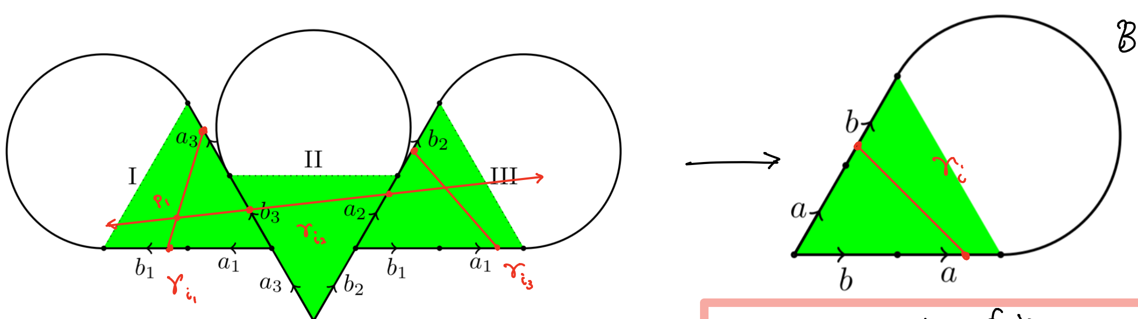
from the pictures, for any direction γ' exits ∂B after intersecting w/ $a \cup b$!
also we may see $k \le 5$ from the pictures.

Next, for each $1 < i < k$, γ_i is contained in the triangle and joins two of labeled segments.
(of course, neither nor)



homot. type of γ_i is one of them. (4 cases)

1st, suppose $\gamma_i \sim$  . lifting γ_i we see :



$\gamma_{i_1} \sim \gamma_{i_3}$ are preimages of γ_i.

maximal arc of γ_{i_2} intersects γ_{i_1} & γ_{i_3}.

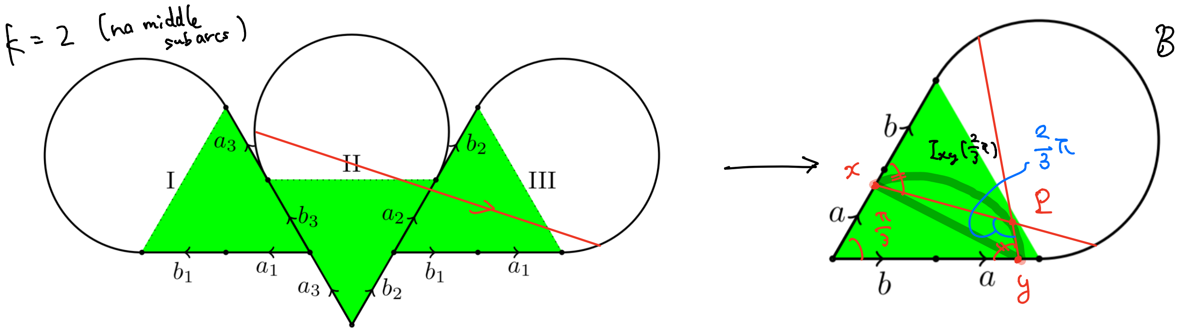
taking two lifts $\gamma_{i_1}, \gamma_{i_2} \in \widetilde{B}$ of γ_i.

we'll see that γ' has self-intersection in this case.

2nd, suppose other cases.

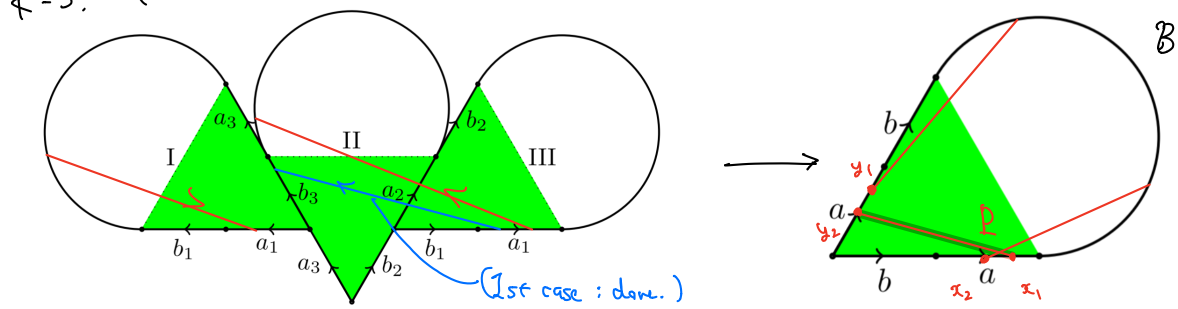
there are 3 possible cases.

$k = 2$ (no middle subarcs)



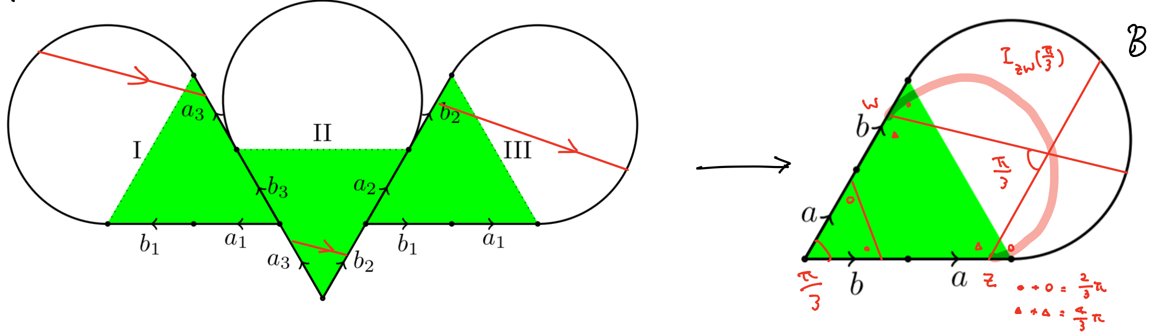
intersection pt P lies in $I_{xy}(\frac{2}{3}\pi) \subset \mathcal{B}$ ✓

$k=3$. (a–a or b–b)



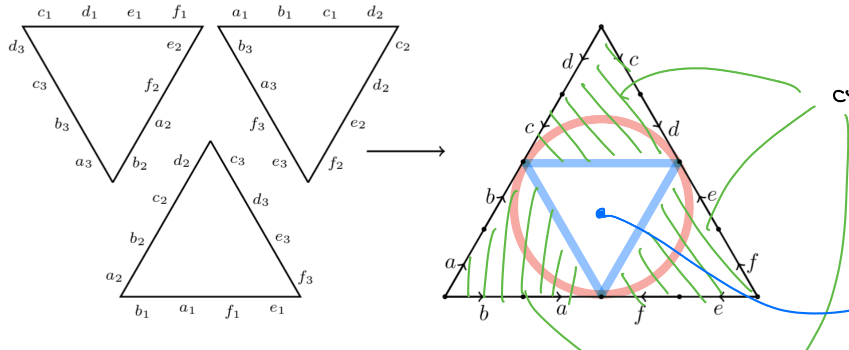
intersection pt P lies in the triangle.

$k=3$. (adjacent a–b)



exm.

intersection pt P lies in $I_{zw}(\frac{\pi}{3}) \subset \mathcal{B}$ ✓ ☒



cylinder-free! (from building block)

the rest.
this region clearly contain no cylinders.

totally cylinder-free!

FIGURE 1. A flat metric with holonomy of order 6 and the (degree 3) holonomy almost trivializing cover.

3.2. Large diameter sets of cylinder curves

__Goal__ to construct (S, φ) w/ finite holonomy. for which $EC(\varphi)$ has arbitrary large diameter.

Let $\alpha, \beta \subset S$: s.c.c. that fill S & ... (intersection number ?)

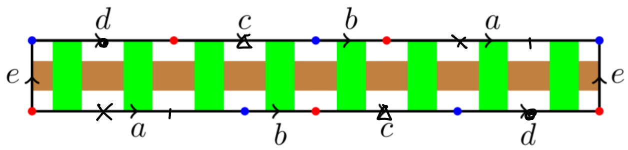
like Strebel's existance thm.

$\rightsquigarrow$ $\exists \varphi$: square-tiled translation structure

s.t. α, β : horiz / vert. foliation resp.

$\#(\text{cone pts}) \leq 2g-2$ by Poincaré Hopf.

__This is :__



(1 horiz. cyl. & 1 vert. cyl.)

FIGURE 8. Square-tiled flat metric with a single horizontal cylinder and a single vertical cylinder.

$\delta = i(\alpha, \beta)$

$n = \#(\text{vertices})$

$\#(\text{faces}) = \delta$, $\#(\text{edges}) = 2\delta$

$\chi = n - \delta$ $\quad g = 1 + \frac{\delta - n}{2}$

$\boxed{\delta = 8 \quad n = 6 \quad g = 2.}$

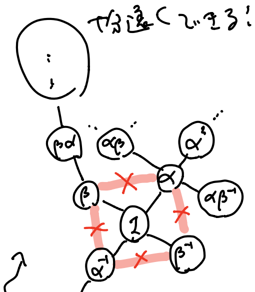
__rem__ α, β : curves on S^0.

$\forall d \geq 3$, we can find (α, β) so that $d_{C(S^0)}(\alpha, \beta) \geq d$

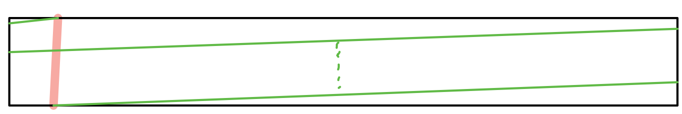
... by replacing β to a deformation of β

by sufficiently large power of $f \in \mathrm{Aff}^+(S, \varphi)$

十分遠くにいる3!



e.g.



十分多く巻く.

$\longrightarrow$ So we can assume $d(\alpha, \beta) \geq d$.

Now we deform $\varphi^0 \longmapsto \hat{\varphi}^0$ on S^0

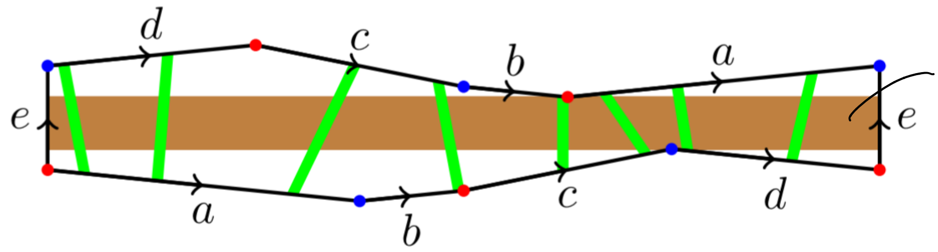
so that α, β are still cylinder curves

but they $\mathrm{ord}(\mathrm{hol}) \geq 3$.

$\downarrow$ To do this.

deform the rectangle to a polygon whose angles b/w horizon are (rational)×π.

now we keep all segments w/ same label to have same length.



still cylinder

FIGURE 9. Deformed flat metric along with the cylinders.

To ensure that β is still a cylinder curve, we need more care.

Let $\delta_1 \sim \delta_k$: φ-horizontal saddle connections. (before deformation)

$j = 1 \sim k$, $n_j := i(\beta, \delta_j)$

we choose $\psi_1 \sim \psi_j \in \mathbb{Q}\pi$: small so that $\sum n_j \psi_j = 0$.

rotate δ_j^+ by ψ_j & δ_j^- by $-\psi_j$.

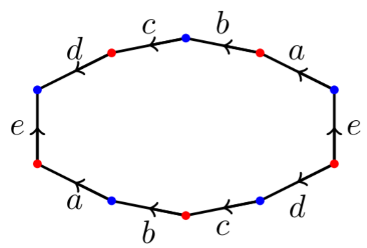
⇝ we will have ______.

Therefore, we can construct a fully punctured flat metric w/ finite holonomy on S° containing α, β which have arbitrary large distance in $C(S^\circ)$

3.3 Infinite diameter set of cylinder curves

Goal construct non-fully-punctured flat metric w/ infinite diameter (w/ ord(hol) = 4)

Consider →



• • } angle 4π

choose a direction θ and $\varepsilon > 0$: small

FIGURE 10. Translation surface of genus 2 with 2 cone points.

Starting at a cone point, we cut 4 slits of length ε to dir. θ.

⇝ subsurface $\Sigma \subset S$

we glue Σ w/ a square of length 2ε ⇝ 4 cone pts

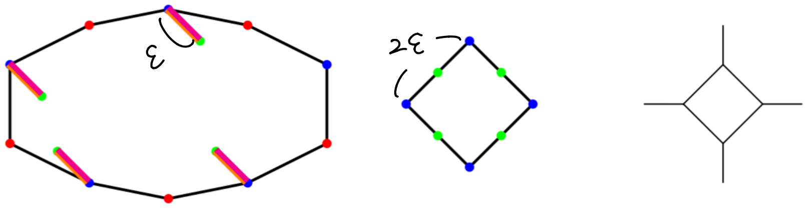


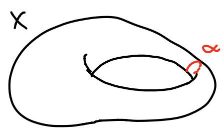
FIGURE 11. Slits and gluing in a square without changing the surface type. The far right shows the picture near the glued in square.

additional assumption : φ has no saddle conn. to direc. θ.

$\rightarrow$ foriation $\mathcal{F}_\theta$ is minimal & arational ?

Next. rotate so that $\theta \mapsto$ vertical. and apply the Teichmüller deformation $\varphi \mapsto \varphi_t$ $(t \in \mathbb{D})$

We additionally assume that θ is chosen so that φ_t recurs to some fixed thick part of Teichmüller space ? at a sequence $\{t_n\}$
くり返す.



Def a curve α on hyp. R.S. X is a systole
$\Leftrightarrow$ $\ell_X(\alpha)$ is minimal among all non-triv. loops on X

$\forall \varepsilon > 0$. ε-thick part of $\mathcal{M}_{g,n}$ is
$\mathcal{M}_{g,n}^{\varepsilon} := \{X \in \mathcal{M}_{g,n} \mid \ell(X) \geq \varepsilon\}$
where $\ell(X)$ is the length of systole.

(i.e, the convergence to a nodal surface does not happen.)

Prop 4 For the flat metric $\hat{\varphi}$ on the closed surface of genus 2 just defined, $EC(\varphi) \subset \mathcal{C}(S)$ has infinite diameter.

... consequence of Masur's work. (in 1986)

4. Main Theorem

Theorem 1 *Let φ be a fully punctured flat metric on a surface S with finite holonomy of order at least 3. Then* $\mathrm{EC}(\varphi)$ *has finite diameter in* $\mathcal{C}(S)$.

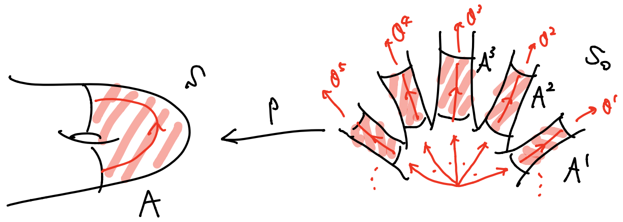
Suppose φ : fully punctured q-flat metric on S w/ $q \geq 3$

and $p : (S_0, \varphi_0) \to (S, \varphi)$: the hol. alm. triv. cover of degree q_0
h.a.t. cover

· φ : full-punctured $\Rightarrow$ φ_0 so is.

Consider : $p^{-1}(A)$ for a maximal embedded cylinder $A \subset S$.

$\rightsquigarrow$ $p^{-1}(A)$ is the union of pairwise disjoint open cylinders in equally spaced directions $\theta^1, \cdots, \theta^{q_0} \in \mathbb{RP}^1$



✩ directions of $A^1, A^2, \cdots, A^{q_0} \subset S_0$ are pairwise distinct

$\to$ $\overline{A^1}, \overline{A^2}, \cdots, \overline{A^{q_0}}$ are pairwise disjoint

but note that each $\overline{A^j}$ may self-intersect by a saddle conn.

Lemma 1. *Let φ be a fully punctured flat metric on a surface S with holonomy of order $q \geq 3$ and $p \colon (S_0, \varphi_0) \to (S, \varphi)$ the holonomy almost trivializing cover of degree q_0. Given a sequence $\{A_n\}$ of distinct, embedded maximal cylinders in (S, φ), let $\{A_n^1, \ldots, A_n^{q_0}\}$ be the sequence of components of the preimage in S_0.*

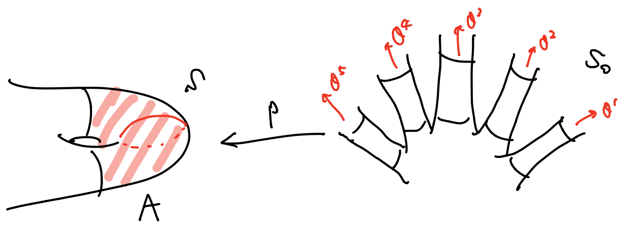
Then, there are pairwise disjoint closed subsurfaces $Z^1, \ldots, Z^{q_0}$ of S_0, foliated in equally spaced directions $\theta^1, \ldots, \theta^{q_0}$ in $\mathbb{RP}^1$, and a subsequence of $\{A_n\}$ so that for each j, (any choice of) a core geodesic $\{a_n^j \subset A_n^j\}$ Hausdorff converges to Z^j. Moreover, for sufficiently large n, a_n^j is contained in Z^j.

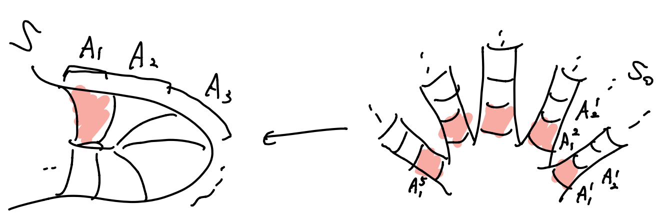
Rem Hausdorff converge $\iff$ conv. w.r.t. the Hausdorff (pseudo) distance on S

$d_H : \mathcal{P}^{\times}(S) \times \mathcal{P}^{\times}(S) \to \mathbb{R}_{\geq 0}$ (non empty)

$d_H(X, Y) = \max\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\}$

$= \inf\{\varepsilon > 0 \mid X \subset N_\varepsilon(Y),\ Y \subset N_\varepsilon(X)\}$





CYLINDER CURVES IN FINITE HOLONOMY FLAT METRICS

SER-WEI FU AND CHRISTOPHER LEININGER

arXiv: 1809.13760 (2018)

- Masur & Minsky (1999), Bowditch (2006) : connection b/w flat metrics and $C(S)$.
  - arising naturally in cpx anal. via extremal probs.

- Leininger, Duchin & Rafi (2010) : rigidity result w.r.t. the Σ-length-spectrum for a set of metrics (Q. Is a metric determined by lengths of curves ?)
  - w/ this work $EC(\varphi)$ is an important tool in identifying the metric / $\mathrm{Aff}^+(S,\varphi)$

- embedded cylinders $\hookrightarrow$ ori. surf. S equiped w/ a flat metric φ { finite type, finite area, finite holonomy

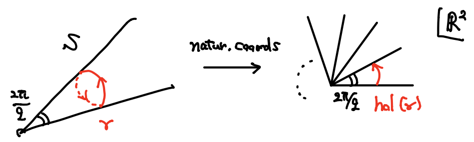
$\rightsquigarrow$ The set $EC(\varphi)$ of embedded cylinder curve
  - homot. classes of core curves of immersed Euclidian cylinders.

$\bigcap$

$C(S)$: curve graph w/ distance (related to intersections of curves)

The holonomy $\mathrm{hol} : \pi_1(S^\circ) \xrightarrow{\text{hom}} SO(2)$

$[\gamma] \mapsto$ rotation in $\mathbb{R}^2$ w.r.t. a lift of γ.



$\rightsquigarrow$ transition arr. cone pt is : $z \mapsto e^{\frac{2\pi}{2}i} z + c$

so, e.g. $\mathrm{ord}(\mathrm{hol}) = 1$ or 2 $\Rightarrow$ φ comes from half-translation structure

- Masur's work (1986) implies that : $EC(\varphi) \subset C(S)$ has infinite diameter.

In this paper we focus on the cases '$\mathrm{ord}(\mathrm{hol}) \geq 3$', for which $EC(\varphi)$ exhibits a variety of behaviors.

$\longrightarrow$ In §3 we construct exms where $EC(\varphi)$ is { empty, non-empty { finite, infinite

especially in these cases : $\hookrightarrow$ We find some for which $EC(\varphi)$ has infinite diameter in $C(S)$

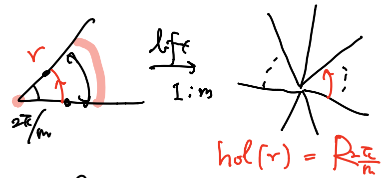
Despite this fact, the authors showes that

Theorem 1. *Let φ be a fully punctured flat metric on a surface S with finite holonomy of order at least 3. Then $\mathrm{EC}(\varphi) \subset \mathcal{C}(S)$ has finite diameter.*

§2. Notation

A Euclidean cone metric φ on S (surf.) is: a metric which is locally isometric to Euc. plane away from finite numbers of pts. (except for ?)

at these pts, the metric has a cone singularity.



Rem the metric is non-positively curved (locally CAT(0)) if $\forall$ cone angle $> 2\pi$

☆ Now we assume that the metric completion of S is a surface w/ finite area.

$\rightarrow$ we call such a metric a flat metric.

- φ: flat metric on S.

$\varphi^\circ := \varphi|_{S^\circ = S \setminus \mathrm{Cone}(\varphi)}$: associated fully punctured metric ※ special orthogonal grp.

the holonomy homomorphism $\pi_1(S^\circ) \to SO(2)^{※}$ is :

$[r] \mapsto \mathrm{hol}(r) := $ (rotation of lift of r via the loc-isomorphisms to $\mathbb{R}^2$)

$\hookrightarrow$ When $\mathrm{ord}(\mathrm{hol}) = 1$ or 2. S comes from trans. surf. or half-trans. surf.

in this paper we focus on the case $\mathrm{ord}(\mathrm{hol}) \geq 3$.

Def If $\mathrm{ord}(\mathrm{hol})$ devides k (≥ 1), we say that the metric is a k-flat metric.

Given $k \geq 1$ and a mero k-differential on a cl. R.S. w/ poles of order $\leq k-1$.

$\rightarrow$ it determines a k-flat metric φ. (by puncturing at all poles.)

$\rightarrow$ $z_{\varphi, p_0}(p) = \int_{p_0}^{p} \sqrt[k]{\varphi}$ defines an atlas $\forall p, p_0 \in S^\circ$.

whose transition maps are $z \mapsto e^{\frac{2\pi}{k} i} z + c$.

$\leftarrow$ We call a preferred coordinate.

- We can also describe k-flat metrics by gluing sides of finite numbers of Euclidean polygons.

e.g.

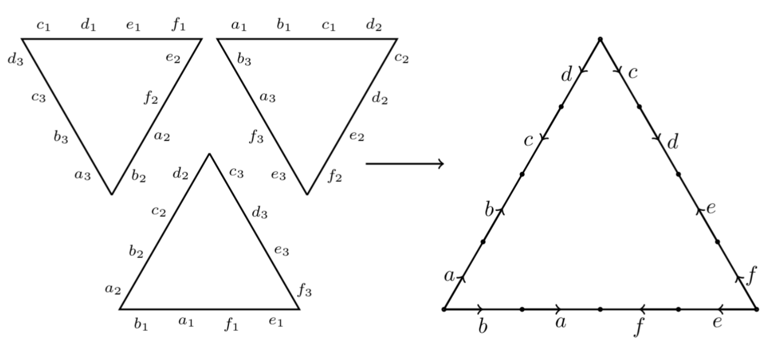


FIGURE 1. A flat metric with holonomy of order 6 and the (degree 3) holonomy almost trivializing cover.

· Given a fully punctured flat metric φ on S. of $\mathrm{ord}(hol) = q$.

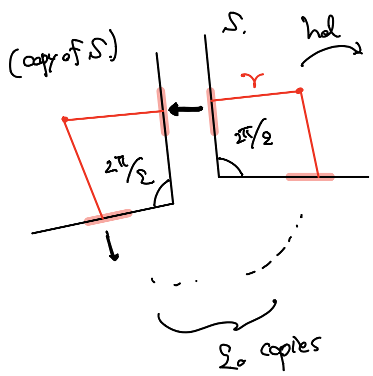
the holonomy almost trivializing cover $p:(S_0,\varphi_0)\to(S,\varphi)$ is : minimum cover which is translation or half-trans. surface.

the lee-ison cover $\overset{\text{corres.}}{\sim}$ $\{\pm I\} < hol(\pi_1(S^\circ)) < SO(2)$

$$\begin{array}{ccc} hol \uparrow & & \uparrow hol \\ \boxed{H = hol^{-1}(\{\pm I\}) < \pi_1(S^\circ)} & & \end{array}$$

← take a cover from this inclusion.

The cover p is a cyclic cover of degree $q_0 = \begin{cases} q & \text{if } q : \text{odd} \\ q/2 & \text{if } q : \text{even.} \end{cases}$



$\frac{2\pi}{q}$ - rotation

⇝ by taking q_0-copies of S, and gluing them.

we will obtain S_0, $p: S_0 \to S$, and $\varphi_0 = p^*\varphi$

for which holonomy-image is $\{\pm I\}$.

Def If a (simple?) closed curve γ has a geod. rep. containing no cone pts.

then γ is called a cylinder curve. : which lies inside a unique maximal. immersed Euclidian cylinder.

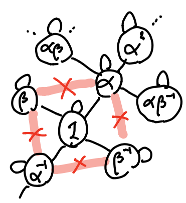
… maximal cylinder.

Combining Masur's result of hol. al. triv. cover, it follows that

" q-flat metric has infinitely many homot. classes of cyl. curves. "

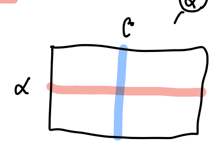
We focus on $EC(\varphi) := \{[\gamma] :$ free homot. class of embedded cyl. curve $\gamma \subset S\}$.

$ESC(\varphi) := \{[\delta] ;$ fixed endpt homot. class of saddle conn. $\delta \subset S\}$



$C(S)$; curve graph : vertices are isotopy classes of ess. s.c.c.

$A(S)$: arc graph : vertices are isotopy classes of arcs.



in each case, vertices are joined by an edge

if the isotopy classes admit disjoint representatives. (except endpts; $A(S)$)

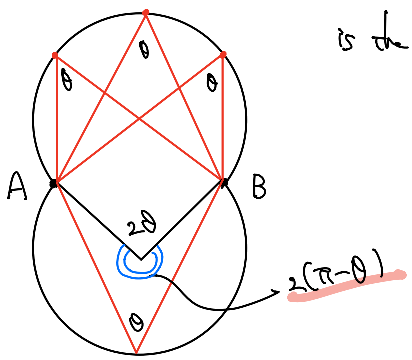
◎ We view $EC(\varphi) \subset C(S)$, $ESC(\varphi) \subset A(S)$ (as set of vertices.)

and refer to the diameter of them as $\boxed{\sup\{\text{min. numbers of edges of paths joining 2 vertices}\}}$ edge-path distance.

§3. Exms

Prop 1 $A, B \in \mathbb{R}^2$: distinct pts. for $0 < \theta < \pi$.

$$I_{AB}(\theta) = \{ C \in \mathbb{R}^2 \mid \angle ACB = \theta \}$$



is the two circular arcs of central angle $2(\pi-\theta)$ joining A and B.

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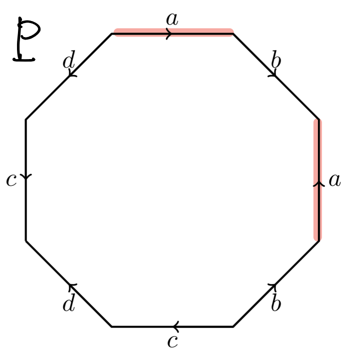
3.1.1. Octagon



Starting w/ a regular octagon P, we identify sides by isometry as indicated in left figure.

$\rightsquigarrow$ Resulting surf. S is a genus 2 surf. w/ 4-flat metric φ

φ has a single cone pt w/ angle 6π.

(S, φ) has 3 embedded cylinders:

In fact, these are the only maximal cylinders. ←

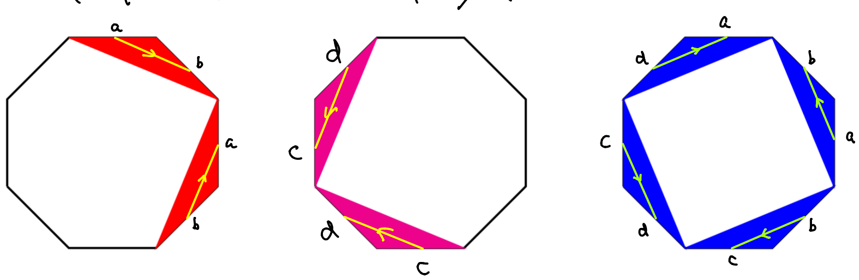


FIGURE 3. Cylinders in the octagon example.

i.e.

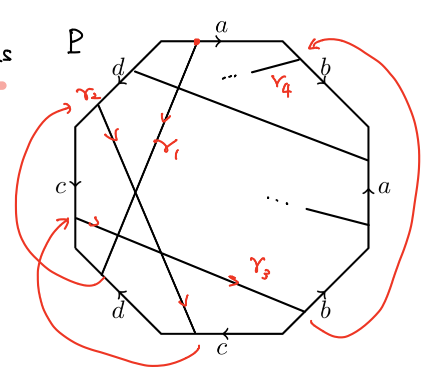
Prop 2 $EC(\varphi)$ consists of exactly three curves.

pf) Suppose γ : core curve of some emb. cylinder in S.

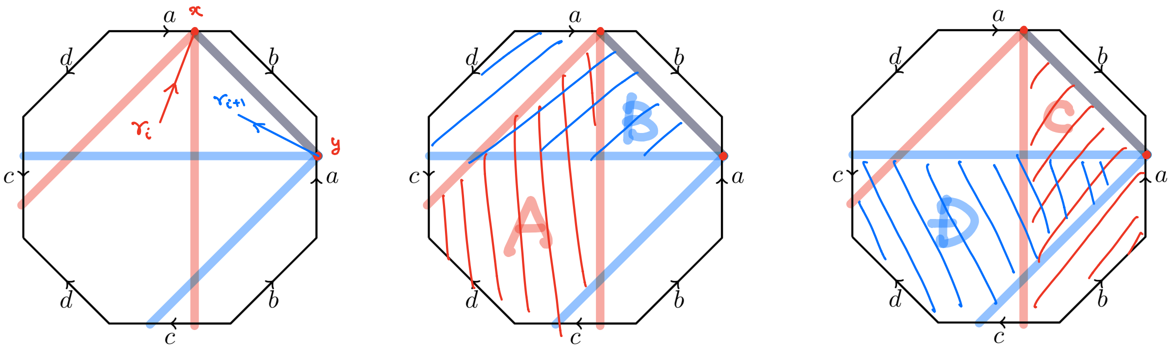
i.e. closed geod. containing no cone pts.

claim γ can be embedded only if it is one of the three.

Let γ cuts through P in $\gamma_1, \gamma_2, \cdots, \gamma_k$: segments in P, joining sides



Step 1 if $\exists \gamma_i$ meets a side making angle $\theta \in [\frac{\pi}{4}, \frac{\pi}{2}]$, then γ will have a self-intersection.



if γ_i lies in A (C), then γ_{i+1} will lie in B (D, resp.)

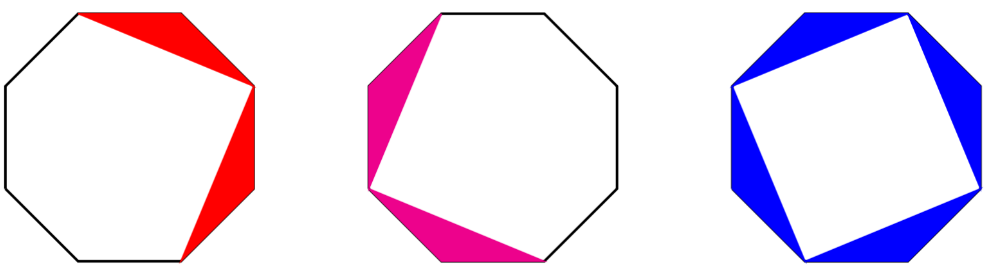
$\longrightarrow$ γ should have self-intersection. //

($\rightsquigarrow$ γ is NOT embedded!!)

Step 2. any line segment that connects non-adjacent sides of P makes angle $\geq \frac{\pi}{4}$.

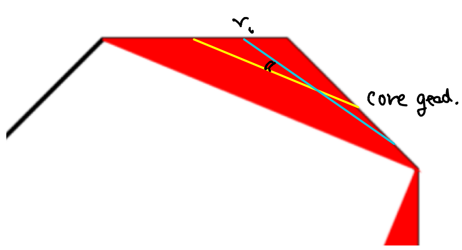
So, if γ is embedded $\Rightarrow$ each γ_i should join adjacent edges.

Each pair of adjacent sides of P determines following triangles.



Step 3 each γ_i is embedded in one of these triangles

$\longrightarrow$ γ_i makes angle $\leq \frac{\pi}{8}$ w/ the core geodesic.



if γ is entirely contained in one of the three $\longrightarrow$ OK.

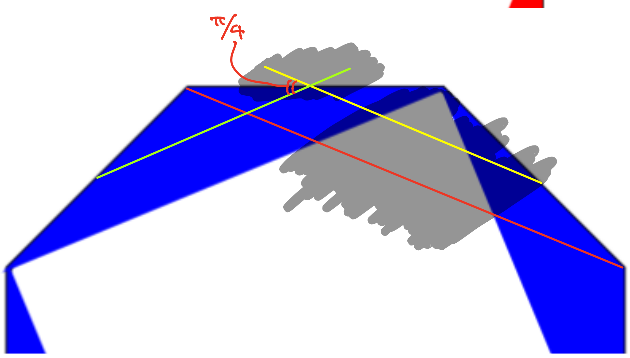
otherwise $\longrightarrow$ contained in (at least) two of the three.

$\rightsquigarrow$ switch b/w intersecting cylinders happens.

However, core curves of intersecting cylinders meet at angle $\frac{\pi}{4}$.

cannot make angle $\leq \frac{\pi}{8}$ w/ both of core geodesics.

So "otherwise" never happens. $\boxtimes$

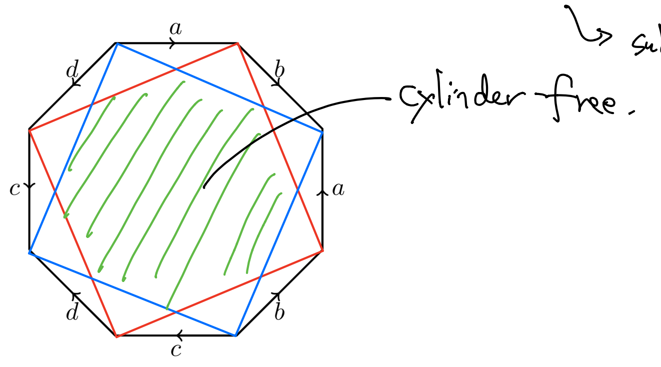


3.1.2 cylinder-free region.

Here we construct "building blocks" for 'cylinder-free regions'.

↳ subsets which are disjoint from emb. cylinders.

ex,



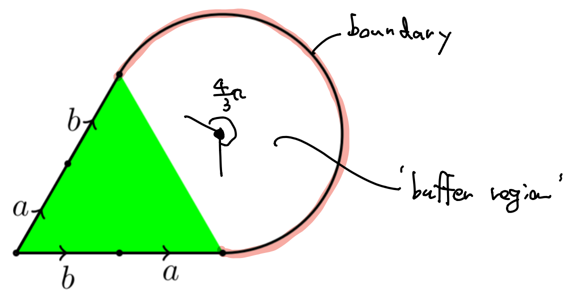
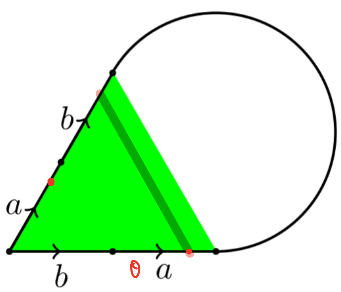


FIGURE 5. The building block $\mathcal{B}$ with a triangular cylinder-free region.

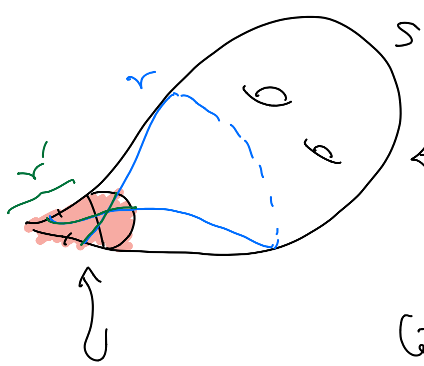


<u>Prop 3</u> Suppose the building block $\mathcal{B}$ is loc. isom.ly immersed in (S,φ). then the image of triangle in $\mathcal{B}$ is cylinder-free.

pf) Suppose γ: closed non-singular geod. in (S,φ) that intersects $\mathcal{B}\subset S$

<u>claim</u> γ has a self-intersection.

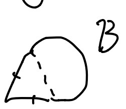
Consider a maximal arc γ' of γ that lifts to $\mathcal{B}$ meeting the triangle.



(a) γ' enters/exits $\mathcal{B}$ in $\partial\mathcal{B}$ (circle)

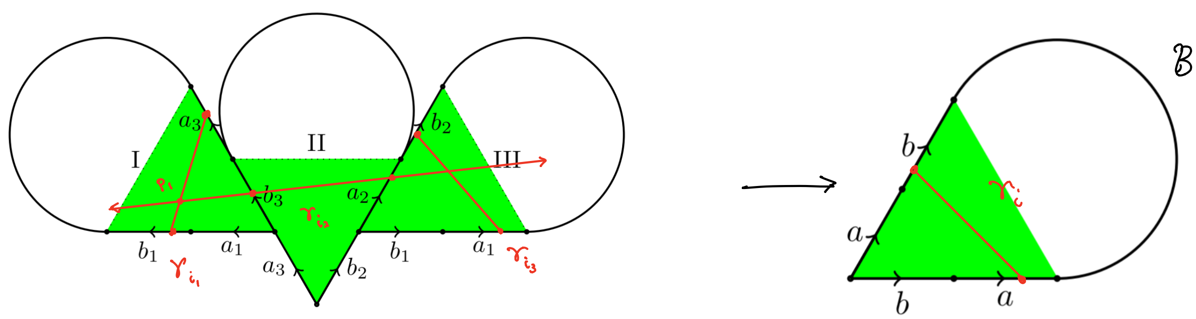
or

(b) $\gamma' = \gamma$

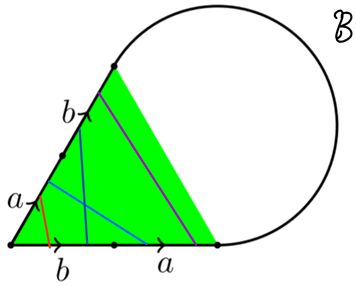


Let γ' cuts through $\mathcal{B}\subset(a,b)$ in $\gamma_1\gamma_2\cdots\gamma_k$.

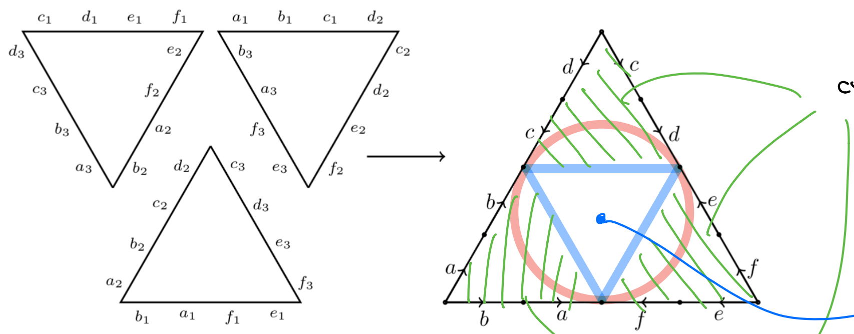
We consider further lift of γ' via the hol. alm. triv. cover $\tilde{\mathcal{B}}\to\mathcal{B}$.



possible patterns of γ_i :



by lifting via $\tilde{\mathcal{B}} \to \mathcal{B}$, we will see that γ self-intersects.



cylinder-free? (from building block)

the rest.
this region clearly contain no cylinders.

FIGURE 1. A flat metric with holonomy of order 6 and the (degree 3) holonomy almost trivializing cover.

$$EC(\varphi) = \phi.$$

3.2. Large diameter sets of cylinder curves

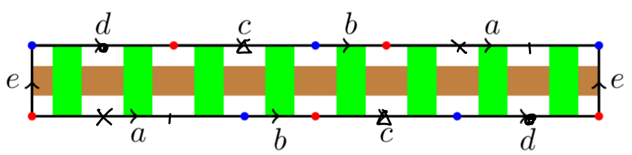
Goal: to construct (S,φ) w/ finite holonomy, for which $EC(\varphi)$ has arbitrary large diameter.

Let $\alpha, \beta \subset S$: s.c.c. that fill S & ... (intersection number ?)
like Strebel's existence thm.

$\rightsquigarrow$ $\exists \varphi$: square-tiled translation structure
s.t. α, β : horiz / vert. foliation resp.

$\#$(cone pts) $\leq 2g-2$ by Poincaré Hopf.

This is :



(1 horiz. cyl. & 1 vert. cyl.)

FIGURE 8. Square-tiled flat metric with a single horizontal cylinder and a single vertical cylinder.

$\delta = i(\alpha, \beta)$ $\#(\text{faces}) = \delta$, $\#(\text{edges}) = 2\delta$

$n = \#(\text{vertices})$ $\chi = n - \delta$ $g = 1 + \frac{\delta - n}{2}$

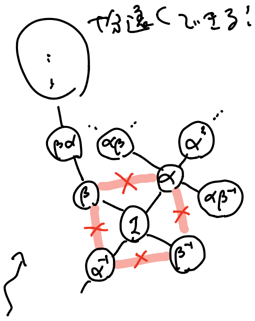
$\boxed{\delta = 8 \quad n = 6 \quad g = 2.}$

rem α, β : curves on S^0.

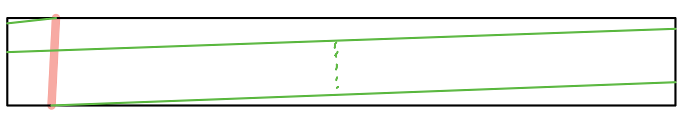
$\forall d \geq 3$, we can find (α, β) so that $d_{C(S^0)}(\alpha, \beta) \geq d$

... by replacing β to a deformation of β by sufficiently large power of $f \in \mathrm{Aff}^+(S, \varphi)$

十分遠くできる!



e.g.



十分多く巻く.

$\longrightarrow$ So we can assume $d(\alpha, \beta) \geq d$.

Now we deform $\varphi^0 \longmapsto \hat{\varphi}^0$ on S^0

so that α, β are still cylinder curves

but they $\mathrm{ord}(\mathrm{hol}) \geq 3$.

$\downarrow$ To do this.

deform the rectangle to a polygon whose angles b/w horizon are (rational) $\times\pi$.

now we keep all segments w/ same label to have same length.

still cylinder

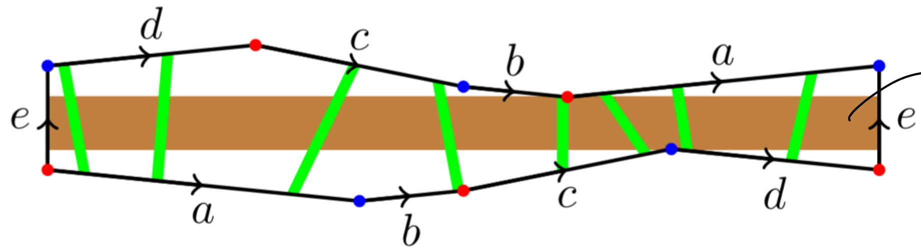


FIGURE 9. Deformed flat metric along with the cylinders.

To ensure that β is still a cylinder curve, we need more care.

let $\delta_1 \sim \delta_k$: φ – horizontal saddle connections. (before deformation)

$j = 1 \sim k$, $n_j := i(\beta, \delta_j)$

we choose $\psi_1 \sim \psi_j \in \mathbb{Q}\pi$; small so that $\sum n_j \psi_j = 0$.

rotate δ_j^+ by ψ_j & δ_j^- by $-\psi_j$.

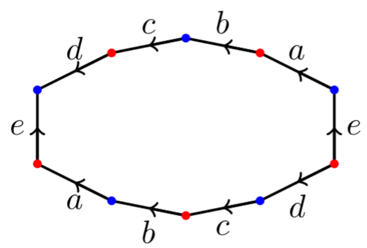
$\rightsquigarrow$ we will have ________.

Therefore, we can construct a fully punctured flat metric w/ finite holonomy on S° containing α, β which have arbitrary large distance in $C(S^\circ)$

3.3 Infinite diameter set of cylinder curves

Goal construct non-fully-punctured flat metric w/ infinite diameter (w/ ord(hol) = 4)

Consider $\longrightarrow$



: } angle 4π

choose a direction θ and $\varepsilon > 0$: small

FIGURE 10. Translation surface of genus 2 with 2 cone points.

starting at a cone point, we cut 4 slits of length ε to dir. θ.

$\rightsquigarrow$ subsurface $\Sigma \subset S$

we glue Σ w/ a square of length 2ε $\rightsquigarrow$ 4 cone pts

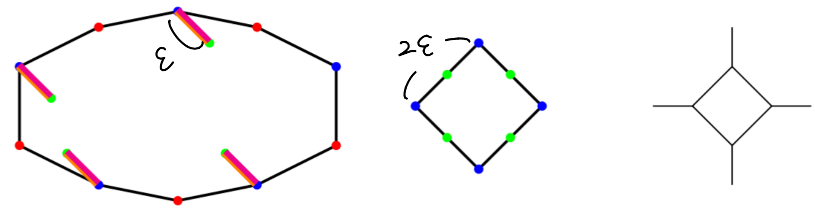


FIGURE 11. Slits and gluing in a square without changing the surface type. The far right shows the picture near the glued in square.

additional assumption : φ has no saddle conn. to direc. θ.

$\rightarrow$ foliation $\mathcal{F}_\theta$ is minimal & arational ?

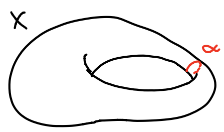
Next. rotate so that $\theta \mapsto$ vertical. and apply the Teichmüller deformation $\varphi \mapsto \varphi_t$ $(t \in \mathbb{D})$

We additionally assume that θ is chosen so that φ_t recurs to some fixed thick part of Teichmüller space ?
くり返す.

at a sequence $\{t_n\}$

(i.e, the convergence to a nodal surface does not happen.)

X



Def a curve α on hyp. R.S. X is a systole $\Leftrightarrow$ $l_X(\alpha)$ is minimal among all non-triv. loops on X

$\forall \varepsilon > 0$. ε-thick part of $\mathcal{M}_{g,n}$ is
$\mathcal{M}_{g,n}^{\varepsilon} := \{ X \in \mathcal{M}_{g,n} \mid \ell(X) \geq \varepsilon \}$
where $\ell(X)$ is the length of systole.

Prop 4 For the flat metric $\hat{\varphi}$ on the closed surface of genus 2 just defined, $EC(\varphi) \subset C(S)$ has infinite diameter.

$\because$ consequence of Masur's work. (in 1986)

4. Main Theorem

Theorem 1 *Let φ be a fully punctured flat metric on a surface S with finite holonomy of order at least 3. Then* $\mathrm{EC}(\varphi)$ *has finite diameter in* $\mathcal{C}(S)$.

Suppose φ : fully punctured q-flat metric on S w/ $q \geq 3$

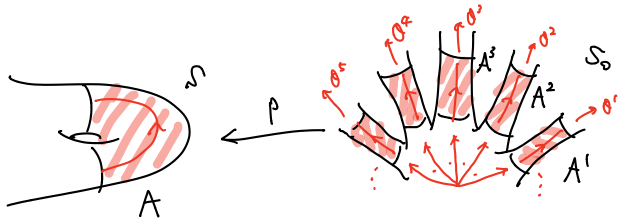
and $p : (S_0, \varphi_0) \to (S, \varphi)$: the hol. alm. triv. cover of degree q_0

h.a.t. cover

· φ : full-punctured $\Rightarrow$ φ_0 so is.

Consider : $p^{-1}(A)$ for a maximal embedded cylinder $A \subset S$.

$\rightsquigarrow$ $p^{-1}(A)$ is the union of pairwise disjoint open cylinders in equally spaced directions $\theta^1, \cdots, \theta^{q_0} \in \mathbb{RP}^1$



$\not\in$ directions of $A^1, A^2, \cdots, A^{q_0} \subset S_0$ are pairwise distinct

$\rightarrow$ $\overline{A^1}, \overline{A^2}, \cdots, \overline{A^{q_0}}$ are pairwise disjoint

but note that each $\overline{A^j}$ may self-intersect by a saddle conn.

Lemma 1. *Let φ be a fully punctured flat metric on a surface S with holonomy of order $q \geq 3$ and $p \colon (S_0, \varphi_0) \to (S, \varphi)$ the holonomy almost trivializing cover of degree q_0. Given a sequence $\{A_n\}$ of distinct, embedded maximal cylinders in (S, φ), let $\{A_n^1, \ldots, A_n^{q_0}\}$ be the sequence of components of the preimage in S_0.*

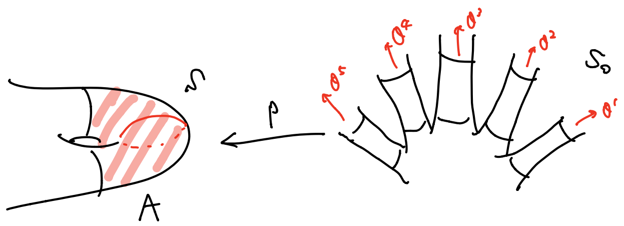
Then, there are pairwise disjoint closed subsurfaces $Z^1, \ldots, Z^{q_0}$ of S_0, foliated in equally spaced directions $\theta^1, \ldots, \theta^{q_0}$ in $\mathbb{RP}^1$, and a subsequence of $\{A_n\}$ so that for each j, (any choice of) a core geodesic $\{a_n^j \subset A_n^j\}$ Hausdorff converges to Z^j. Moreover, for sufficiently large n, a_n^j is contained in Z^j.

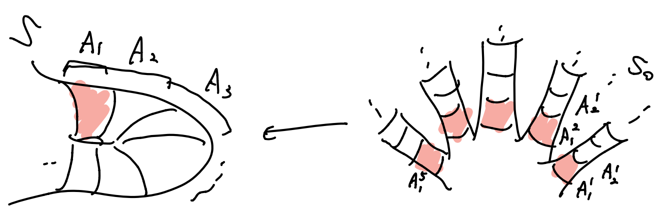
<u>Rem</u> Hausdorff converge $\Longleftrightarrow$ Conv. w.r.t. the Hausdorff pseudo distance on S

$d_H : P^\times(S) \times P^\times(S) \to \mathbb{R}_{\geq 0}$ non empty

$d_H(X, Y) = \max\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\}$

$= \inf\{\varepsilon > 0 \mid X \subset N_\varepsilon(Y), Y \subset N_\varepsilon(X)\}$





for each $A_n \subset S$. $\xleftarrow{p}$ $A_n^1, A_n^2 \cdots A_n^{2q}$ w/ direction $\theta_n^1, \theta_n^2, \cdots, \theta_n^{2q}$ $\in \mathbb{RP}^1$

by passing to subseq, we may assume $\theta_n^j \to {}^\exists\theta^j \in \mathbb{RP}^1$

$$\overrightarrow{A_n^j} \xrightarrow[d_H]{} Z^j \subset S_0 .$$

$$\begin{array}{ccc} x_n^j & \longrightarrow & x^j \\ \cap & & \cap \\ \text{core geod.} & \longrightarrow & \text{geod.} \\ \text{to direc. } \theta_n^j & & \text{to direc. } \theta^j \end{array}$$

Theorem 2. *Suppose φ is a flat metric on a closed surface with holonomy of finite order $q \geq 3$ and that $\{A_n\}$ is a collection of maximal cylinders for which the set of core curves $\{a_n\}$ has infinite diameter in $\mathcal{C}(S)$. Then there is a subsurface $Z \subset S$ with a singular foliation by geodesics so that the complement is a collection of disks.*

In this theorem, the surface Z may not be embedded, but only fails to be so at a finite number of cone points. That is, there is a surface with boundary and map $Z \to S$ so that Z is injective except possibly at a finite number of points in ∂Z which all map to cone points.

Duchin, Leininger, Rafi (2010) より

Σ: finite-type surface

$C(\Sigma) = \{$ homot. classes of curves on $\Sigma \}$

$S(\Sigma) = \{$ homot. classes of simple closed curves on $\Sigma \}$

for ρ: isotopy class of metrics on Σ, $\alpha \in C(\Sigma)$,

$$l_\rho(\alpha) = \inf_{\gamma \in \alpha} L_\rho(\gamma)$$

for ρ: isotopy class of metrics on Σ, $\Gamma \subset C(\Sigma)$,

$$\lambda_\Gamma(\rho) := (l_\rho(\alpha))_{\alpha \in \Gamma} \in \mathbb{R}^\Gamma \ : \ \Gamma\text{-length spectrum}$$

Q. Let $\mathcal{G}(\Sigma) = \{$ isot. classes of metrics on $\Sigma \}$.

Is the map $\mathcal{G} \to \mathbb{R}^\Sigma$: $\rho \mapsto \lambda_\Gamma(\rho)$ an injection?

(If so, we say that Γ is spectrally rigid over $\mathcal{G}$).

e.g. $\Gamma = S(\Sigma)$, $\mathcal{G} = \mathcal{T}(\Sigma)$: Teich. sp. of complete, finite-area, hyperbolic metrics on Σ.

classical fact: $\mathcal{T}(\Sigma) \to \mathbb{R}^{S(\Sigma)}$ is injective (Fricke).

$S(\Sigma)$ is spectrally rigid over $\mathcal{T}(\Sigma)$.

another family : $\mathrm{Flat}(\Sigma) = \{ \rho_q$: the q-metric $\mid q$; q.d. on Σ, $\|q\| = 1 \}$

$= Q^1(\Sigma) / q \sim e^{i\theta} q$.

※ (Σ, ρ_q) is loc. flat away from finite cone pts. of angle $k\pi$

Theorem 1 (D., L., R.)

For any finite-type surface Σ, $S(\Sigma)$ is spectrally rigid over $\mathrm{Flat}(\Sigma)$.

Theorem 2 (D., L., R.)

Thurston's space of proj. meas. foliations

let $3g-3+n \geq 2$. $S' \subset S(\Sigma) \subset \mathcal{PMF}$ is spec. rigid over $\mathrm{Flat}(S)$

$\Leftrightarrow$ $S' \subset \mathcal{PMF}$: dense

$$\mathcal{MF} = \{ k\, i_\alpha : \mathbb{R}_+ \times \mathbb{R}^S \mid i_\alpha(\underset{\in S}{\beta}) = \min i(\underset{\in \alpha}{a}, \underset{\in \beta}{b}) \}$$