TEICHMÜLLER GEODESICS, DELAUNAY TRIANGULATIONS, AND VEECH GROUPS

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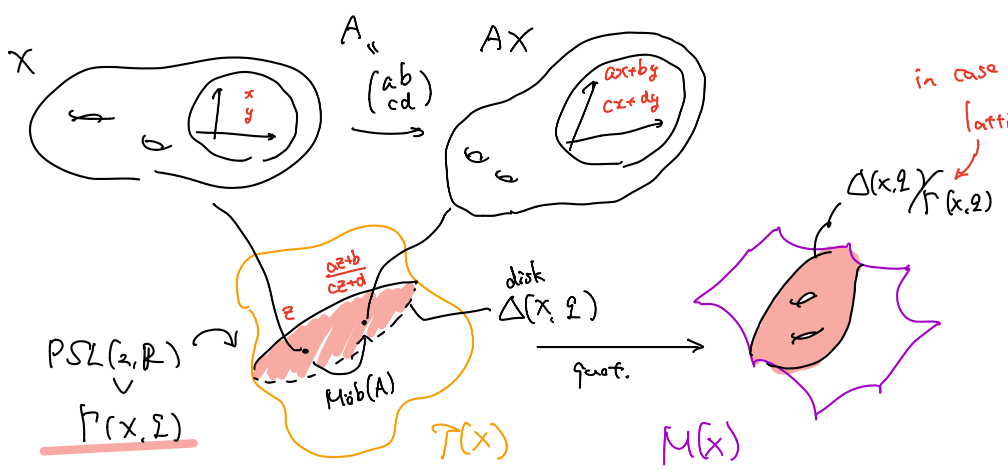
Abstract. In this article, we describe a method for computing generators of the Veech group of a flat surface (which we define as a Riemann surface with a non-zero holomorphic quadratic differential). The method employs a cell structure on the (complex) Teichmüller geodesic generated by the surface, using Delaunay triangulations, which are canonically associated to flat surfaces.

– Teich. theor. and moduli problem (2008)

§0. Intro

- Veech group $\Gamma(X, q)$ of a flat surface (X, q)

  ⋯ the stabilizer of conf. class under the affine-deform. action of $PSL(2,\mathbb{R})$.



"sufficiently large"

in case lattice VG

$\Delta(X,q)/\Gamma(X,q)$: embedded surface curve in $M(X)$ w.r.t. Teich. metric

called Teichmüller curve

in lattice case

a flat surface w/ "sufficiently large" lattice VG

is called a Veech surface

Veech's dichotomy (1989)

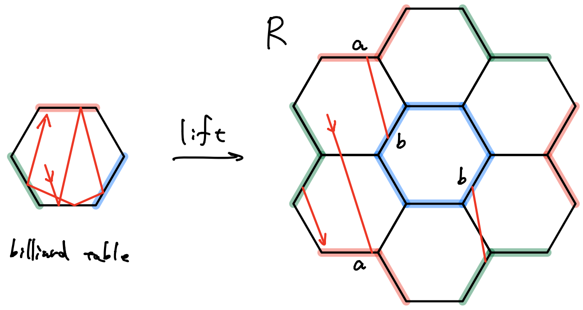
On a Veech surface, for any direction θ, one of the two following possibilities holds: (mutually exclusive)

(i) The surface decomposes in θ into metric cylinders w/ commensurable moduli.

(ii) Every trajectory in θ is densely & uniformly distributed in the surface.

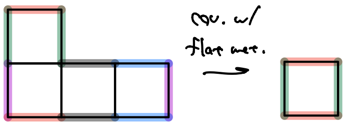
("geodesic flow is uniquely ergodic.")

○ For a rational polygon, the billiard paths could be lifted to certain flat surface, called an "overlay surface".
(Ueberlagerungsfläche)



hexagonal billiard table

○ The simplest exms of Veech surface are origami.



(Schmithüsen, 2004) algorithm for computing VG.

There have been concerted efforts

- to demonstrate the existance of Veech surfaces w/ specified prop.
- to produce exms via explicit construction.

(Kenyon, Smillie 2000) classification of lattice triangle

(Schwarz, Hooper) Java program "McBilliard", which computes periodic paths in triangle billiard table.

→ 'obtuse triangle that produces a Veech surface.' found.

(McMullen / Calta) criterion for determining when an abelian genus 2 surface is Veech.

◎ This paper is a pre-running paper to my last talk: Computing VG using Voronoi staples.

The key idea is similar to the one.

In this paper, we develop a general algorithm for computing generators of VG.

The method depends on keeping track of combinatorial data: Delaunay triangulation

dual of Voronoi decomposition

Author uses MATLAB to obtain "iso-Delaunay regions".

§1. Background.

Let X: Riem. surf. of genus ≥ 2.

Def 1.1 A holomorphic quadratic differential is a section of the $(\text{sheaf of holo. one forms})^{\otimes 2}$.

$$\xrightarrow{\text{loc.}} \quad q(z) = \underbrace{f(z)}_{\text{holo.}} dz^{\otimes 2}$$

Let q: non-trivi. h.q.d. on X, $Z := \mathrm{Zero}(q)$.

$\forall p \in X^* := X - Z$. $\boxed{p' \mapsto \int_p^{p'} \sqrt{q}}$ defines a loc. coord arr. p.

$\rightsquigarrow$ an atlas on X^* whose $\boxed{\text{any transition is of the form } z \mapsto \pm z + c}$

$\longrightarrow$ called a flat structure on X.

For a flat surf. (X, q), $|dz| = |\sqrt{q}|$ is an invariant metric on X.

$\rightsquigarrow$ $(X^*, |dz|)$: locally isometric to $(\mathbb{C}, |dz|)$

each point in Z is a cone pt : $\exists$ nbd $\overset{\text{isom}}{\simeq} (\mathbb{C}, |z^k dz|)$ (k: order.)

Def 1.4 The Teich. geodesic (= Teich. disk) is an isom. embedding $\mathbb{H} \hookrightarrow \overset{\text{Teich. sp.}}{T(X)}$

$\Delta(X,q) = \left\{ id_X : (X,q) \to (X, q_t) \;\middle|\; \begin{pmatrix} x_{q_t} \\ y_{q_t} \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{Re}\, t \\ 0 & \mathrm{Im}\, t \end{pmatrix} \begin{pmatrix} x_q \\ y_q \end{pmatrix} : t \in \mathbb{H} \right\} \hookrightarrow T(X)$: Teich. geod. generated by (X, q)

(X, q) is called Veech $\Leftrightarrow$ $\Gamma(X,q)$: lattice $\Leftrightarrow$ $\mathrm{Area}_{\mathbb{H}}(\mathbb{H}/\Gamma(X,q)) < \infty$.

$\Leftrightarrow$ the image $\mathbb{H} \hookrightarrow T(X) \to M(X)$ is an algebraic curve.

Def 1.7 The Veech group $\Gamma(X,q)$ of (X,q) is the subgrp of $PSL(2,\mathbb{R})$ acting on $\Delta(X,q)$

that corresponds to MCG of (X,q) : $\underset{\overset{\shortparallel}{(\mathrm{Aff}^+(X,q))}}{\{ f \in \mathrm{Mod}(X) : \text{stabilizes } \Delta(X,q) \}}$.

By (Thurston, 1988)'s description of $\mathrm{Mod}(X) \curvearrowright T(X)$,

we can closely relate the top. type of $g \in \Gamma(X,q)$ to the type of isometry $g \curvearrowright \mathbb{H}$ as follows:

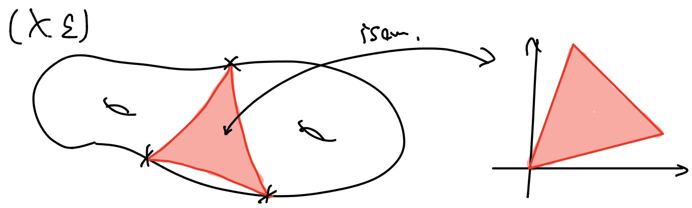
- g is *periodic* if and only if the correponding isometry of $\mathbb{H}$ is *elliptic* (fixes a point in $\mathbb{H}$); $\leftarrow$ $g^{\exists n} = id$
- g is *reducible* if and only if the corresponding isometry of $\mathbb{H}$ is *parabolic* (fixes a point on $\partial\mathbb{H}$ and the class of horocycles through this point); $\leftarrow$ $g(\exists C) = C$ — disjoint union of s.c.c.
- g is *pseudo-Anosov* if and only if the corresponding isometry of $\mathbb{H}$ is *hyperbolic* (fixes two points on $\partial\mathbb{H}$ and the Poincaré geodesic connecting them). $\longleftarrow$ $\exists$ two transverse foliation stable & unstable

§2. Delaunay triangulations.

Let (X, q) be a flat surface.

Def 2.1 A q-triangle is a 2-simplex (triangle) in X,

embedded in such a way that $\begin{cases} \text{vertices lie in } Z \\ \text{edges are geodesics w.r.t. } |\sqrt{q}| \\ \text{faces contain no pts in } Z. \end{cases}$



interior of q-triangle is purely Euclidean.

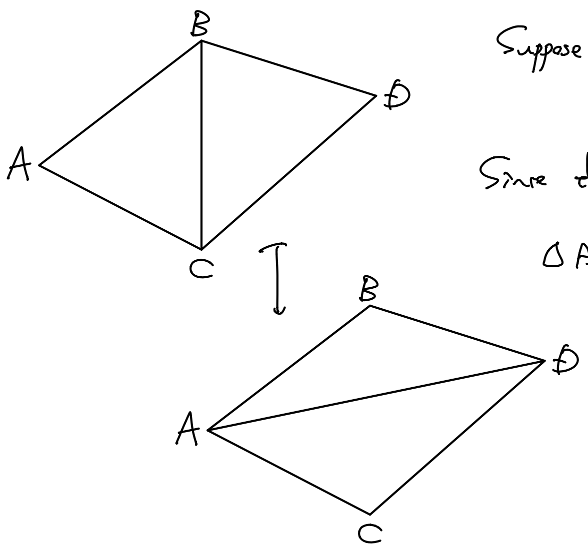
Def 2.2 A q-triangulation of X is a simplical cell str. s.t. $^\forall$2-cell is a q-triangle.

A q-triangulation of X & a q'-triangulation of X' are combinatorially equivalent

$\Leftrightarrow$ $^\exists$ bij. b/w 2-cells preserving $^\forall$edge identifications.

$\overset{\text{induce}}{\rightsquigarrow}$ natural homeo $X \to X'$: loc. affine ?? sus

The Whitehead move is a natural way to change one q-triangulation into another.



Suppose $\triangle ABC$, $\triangle BCD$: two q-Δs
(two of $A \sim D$ may coincide as a pt.)

Since the interior of $\square ABDC$ contains no crit. pts

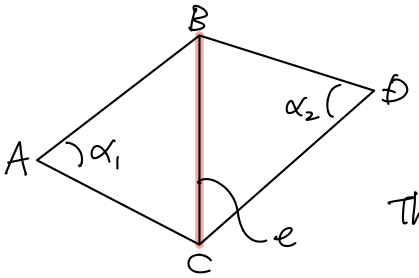
$\triangle ABD$, $\triangle ACD$: two q-Δs

Clearly (X, q) admits many q-triangulations,

We wish to describe a particular sort

that is unique for an open dense $\Delta(X, q)$-subset.

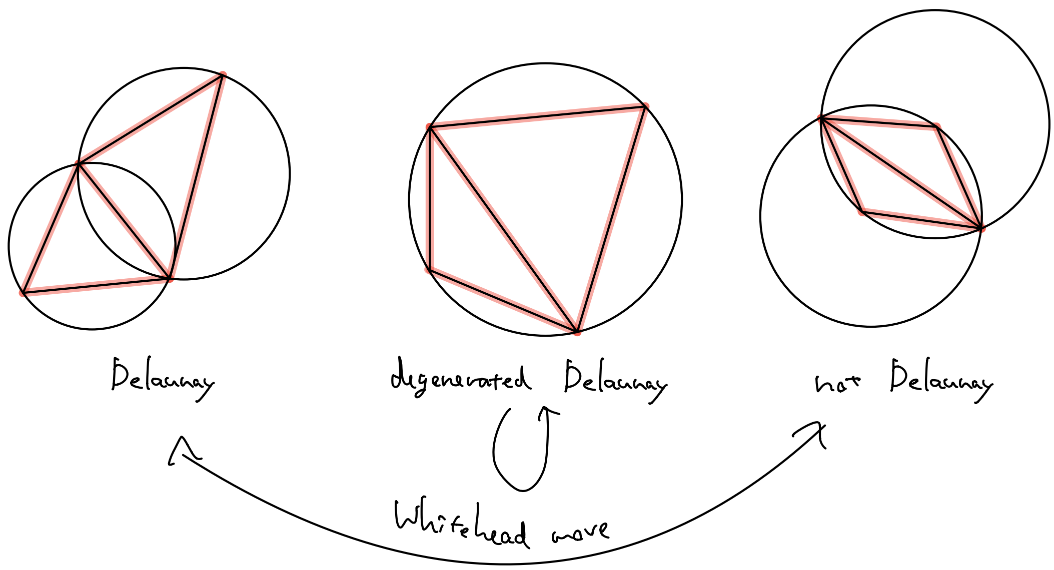
Def 2.4 For two Euclidian triangles s.t.:



The dihedral angle of e: $\alpha(e) = \alpha_1 + \alpha_2$

Def 2.5 A Σ-triangulation of X is Delaunay if $\forall e$: edge, $\alpha(e) \leq \pi$.

Delaunay triangulation is degenerated if $\exists e$: edge $\alpha(e) = \pi$.



Prop 2.6 (Rivin, 1994)

Given any Σ-triangulation on X, one obtains a Delaunay Σ-triangulation of X

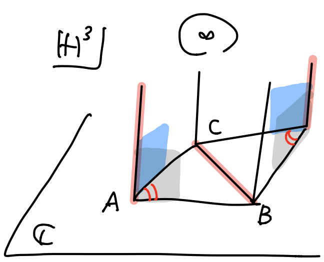
via a finite seq. of Whitehead moves.

Rem 2.8 The terminology of "dihedral angles" is closely related to the hyp. geometry.

In the hyp. 3-space model $\mathbb{H}^3$,

$\triangle ABC$ in $\partial \mathbb{H}^3 = \mathbb{C}$ induces an ideal tetrahedron $ABC\infty$.

fact opposite sides of an ideal tetrahedron have equal dihedral angles.



true "dihedral angle" at $A\infty$ is $\angle CAB$.

solid angle b/w two faces.

◎ iso-Delaunay regions in $\mathbb{H}$.

~~$\mathbb{H}$~~

Prop 2.9 Suppose a Delaunay triangulation of $(X',g') \in \Delta(X,g)$ — in the geod. (embedded)

If it is non-degenerate, then it is unique (in what sense??)

If it is degenerate, it is unique up to Whitehead moves.

(pf : from Masur, Smillie 1991 : duality b/w Delaunay $\leftrightarrow$ Voronoi

w/ the convexity of a volume fction on the triangulation.)

Prop 2.10 Let $f \in \Gamma(X,g)$ & $\mathcal{T}$: Delaunay triangulation of (X,g)

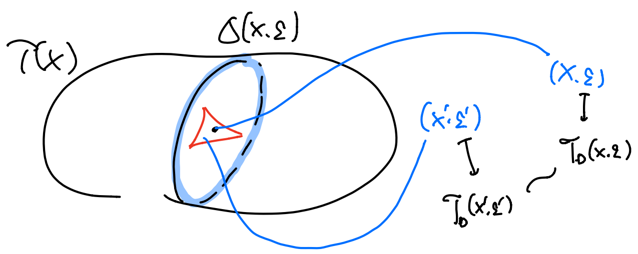
Then $f^*\mathcal{T} = \{ f^*(T) \mid T \in \mathcal{T} \}$ is a Delaunay triangulation of (f^*X, f^*g) $= (X, f^*g)$ } differ by markings

$$\begin{tikzcd}
(X,g) \supset T \arrow[r, hook] \arrow[d, "f"] & \Delta \subset \mathbb{C} \\
(f^*X, f^*g) \supset f^*T \arrow[ur, hook'] &
\end{tikzcd}$$

i.e. Delaunay triangulation is indep. of the marking.

Def 2.11 An iso-Delaunay region (IDR) is a maximal, connected open subset of $\Delta(X,g)$

s.t. $\forall (X',g')$ belonging to it has $\boxed{\text{combinatorially-equiv.}}$ Delaunay triangulations.

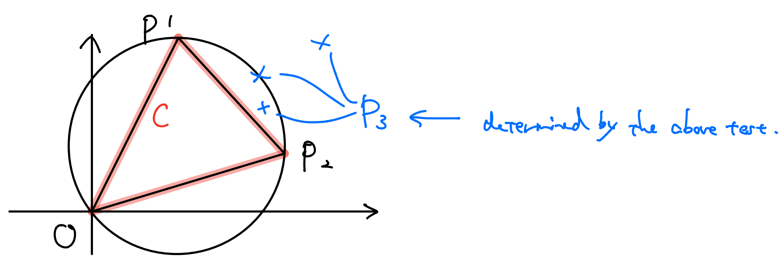


Prop 2.12 W.r.t. $d_{\mathbb{H}}$, each IDR is convex and has geodesic boundary.

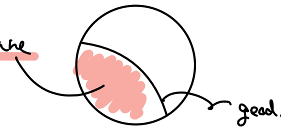
Proposition 2.13 (Incircle test). *Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3 = (x_3, y_3)$ be three points in $\mathbb{R}^2$, with O, P_1, and P_2 non-colinear. Let $\mathcal{C}$ be the circumcircle of OP_1P_2. Then*

$$(1) \qquad \text{if} \quad \det \begin{pmatrix} x_1 & y_1 & x_1^2 + y_1^2 \\ x_2 & y_2 & x_2^2 + y_2^2 \\ x_3 & y_3 & x_3^2 + y_3^2 \end{pmatrix} \begin{cases} < 0, & P_3 \text{ is in the interior of } \mathcal{C}. \\ = 0, & P_3 \text{ lies on } \mathcal{C}. \\ > 0, & P_3 \text{ is exterior to } \mathcal{C}. \end{cases}$$

} Delaunay

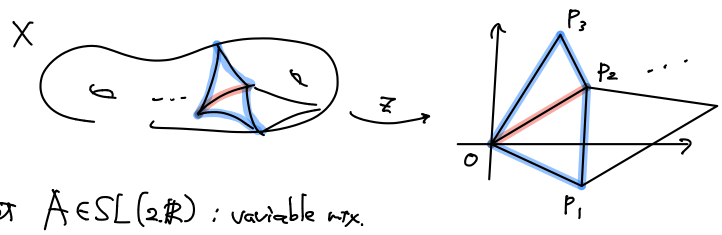


outline of pf of Prop 2.12) claim IDR is an intersection of hyp. half-plane



Suppose a Delaunay triangulation of $(X', q') \in \Delta(X,q)$.

For each edge e, choose a chart s.t.:



Let $P_j = (x_j, y_j)$ $(j=1,2,3)$

Let $A \in SL(2,\mathbb{R})$: variable mtx.

s.t. $A(X,q) \in \Delta(X,q)$ has parameter $u + iv \in \mathbb{H}$.

$\longrightarrow$ $A \sim \begin{pmatrix} 1 & u \\ 0 & v \end{pmatrix}$ mod $SO(2,\mathbb{R})$

evaluate the ineq. in Prop 2.13 w/ $\begin{matrix} x_i \\ y_i \end{matrix}$ replaced by $\begin{matrix} x_i + uy_i \\ vy_i \end{matrix}$

$$\rightsquigarrow \quad a(u^2+v^2) + 2bu + c \geq 0$$

where
$$
\begin{aligned}
a &= x_1y_2y_3(y_3 - y_2) + x_2y_1y_3(y_1 - y_3) + x_3y_1y_2(y_2 - y_1), \\
b &= x_1y_1(x_2y_3 - x_3y_2) + x_2y_2(x_3y_1 - x_1y_3) + x_3y_3(x_1y_2 - x_2y_1), \\
c &= x_1x_2y_3(x_1 - x_2) + x_2x_3y_1(x_2 - x_3) + x_1x_3y_2(x_3 - x_1).
\end{aligned}
$$

This ineq. presents a hyp. half plane bounded by a geodesic.

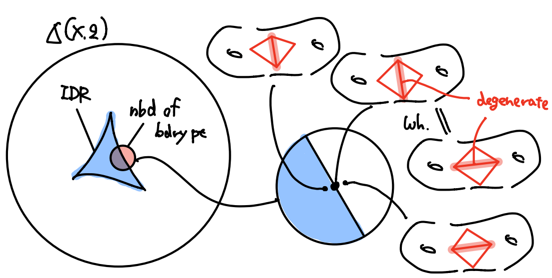
Thus $\mathcal{H}_e := \{ A(X,q) \mid A \in SL(2,\mathbb{R}) : \alpha(e) \leq 2\pi \}$ is a hyp. half-plane $\otimes$.

(Veech, 2011) The area of each IDR is finite (perhaps for Veech surfaces)

$\rightarrow$ actually IDR is a hyp. poligon

We may observe that: performing a Whitehead move on the degenerate edges

we reverse these particular ineq. arr. the pt.



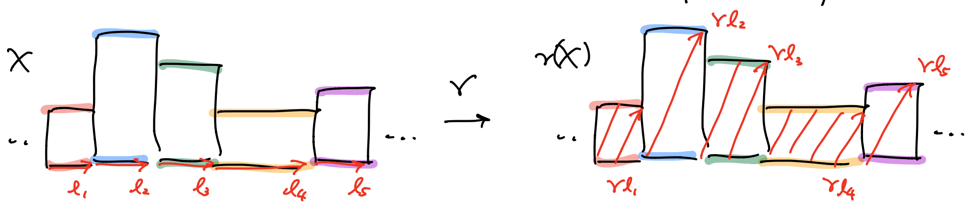
$\rightsquigarrow$ IDRs naturally partition $\Delta(X,q)$ w/ cell complex structure.

Def 2.14 This cell complex on $\Delta(X,q)$ is called the iso-Delaunay complex of (X,q).

Prop 2.10 implies that the iso-Delaunay complex of (X,q) is an invariant under $\Gamma(X,q)$-action

§3. Bouillabaisse surface

When (X, ω) has a direction in which X decomposes into cylinders of commensurable moduli;



$\Gamma(X, \omega)$ contains a parabolic element, which is locally a power of the Dehn twist on each cylinder.

Def. A bouillabaisse surface is a flat surface w/ at least two 'comm-moduli' cylinder direction.

Construction : (Thurston, 1988)

name : Hubbert's lecture, 2003.

3.3. **Relevance of bouillabaisse surface.** The following corollary to Theorem 0.1 demonstrates one measure of the importance of bouillabaisse surfaces:

Corollary 3.2. *Every Veech surface is bouillabaisse.*

Proof. Any direction containing a geodesic that connects two singular points decomposes the surface into cylinders whose moduli are commensurable. □

GuS.

§4. The algorithm

not necessarily compact.

IDR, ID-complex determined by (X,ω) partition $\Delta(X,\omega)$ into hyp. polygons.

A fundamental domain of $\Gamma(X,\omega) \curvearrowright \mathbb{H}$ can be assembled from IDRs of the surface.

To do this, we consider the "Veech equivalence" of IDRs,

the equivalence under an element in $\Gamma(X,\omega)$.

First, clearly (combinatorial eq.) $\Leftarrow$ (Veech eq.) holds by the uniqueness of Delaunay triangulations.

($\not\Rightarrow$) needs some metric structure.

Let R_1, R_2 : IDRs w/ the same combinatorial data.

& fix $(X',\omega') \in R_2$.

<u>Goal</u> to determine if $\exists (X,\omega) \in R_1$ s.t. $(X,\omega) \underset{\text{Veech}}{\sim} (X',\omega')$

◎ Tree search in $\Delta(X,\omega)$.

To assemble a f.d. of $\Gamma(X,\omega)$, we perform as follows:

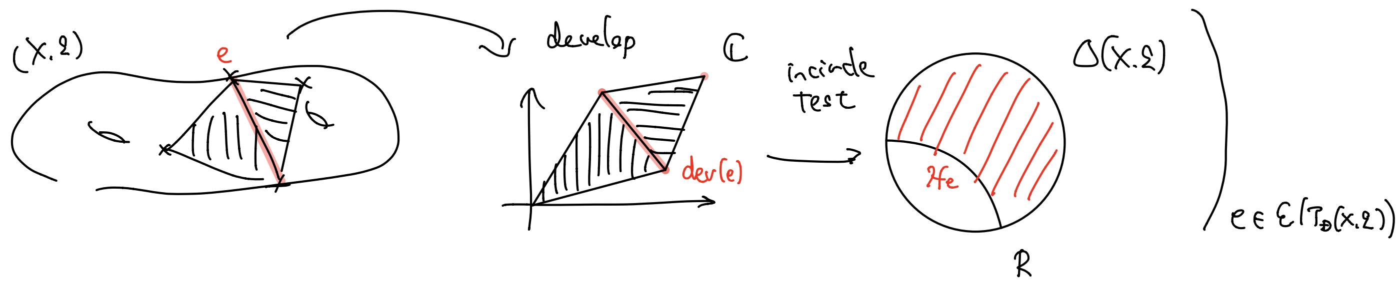
(1) Begin w/ a surface $(X_0,\omega_0) \in \Delta(X,\omega)$ w/ non-degenerating Delaunay triangulation.

and compute its IDR, as shown in the pf of Prop 2.12.

that is: develop each two neighboring Delaunay triangles

and solve the ineq. $a(u^2+v^2)+2bu+c \geq 0$ $\rightsquigarrow$ get $R_0 \subset \Delta(X,\omega)$

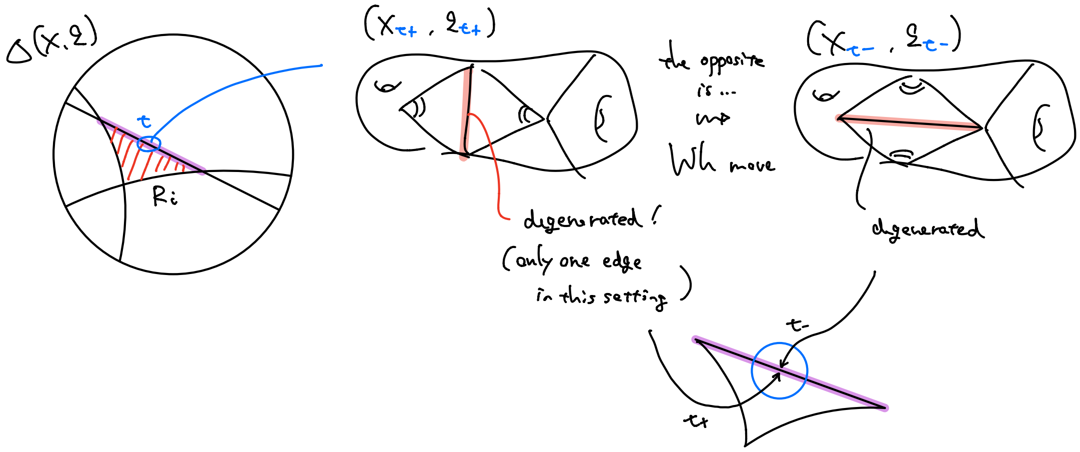
where
$$a = x_1y_2y_3(y_3-y_2) + x_2y_1y_3(y_1-y_3) + x_3y_1y_2(y_2-y_1),$$
$$b = x_1y_1(x_2y_3-x_3y_2) + x_2y_2(x_3y_1-x_1y_3) + x_3y_3(x_1y_2-x_2y_1),$$
$$c = x_1x_2y_3(x_1-x_2) + x_2x_3y_1(x_2-x_3) + x_1x_3y_2(x_3-x_1).$$



(2) Along each degenerated edge of R_i,

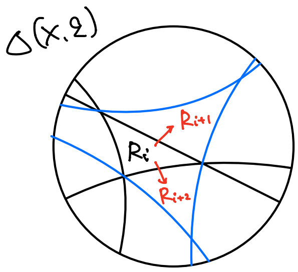
determine which edge is degenerated and compute the combinatorial type of the adjacent Delaunay triangulation

→ We "cross" the edge ($\subset \partial R_i$) into a new IDR.



(3) For each bdry segment crossed, compute the other adjacent IDR just like (1)

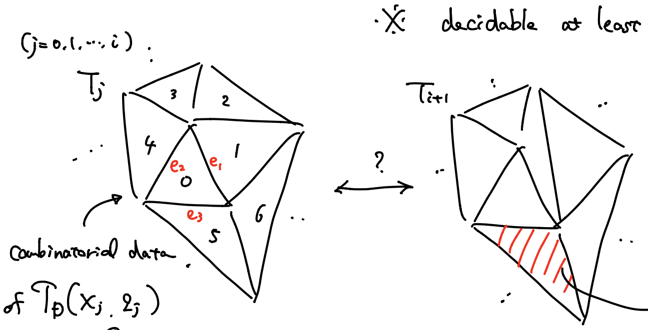
→ denote by $R_{i+1}, R_{i+2}, \cdots$



(4) Check if one of the new IDRs $R_{i+1}, R_{i+2}, \cdots$

is combinatorially equivalent to previous IDRs $R_0, R_1, \cdots R_i$

※ decidable at least $3 \# \mathcal{F}(\mathcal{T}_D(x, q))$

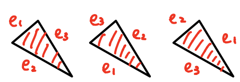


combinatorial data of $\mathcal{T}_D(X_j, q_j)$ $\cap$ R_j

data of $\mathcal{T}_D(X_{i+1}, q_{i+1})$

for each Delaunay triangle, check if $T_{i+1} = {}^{\exists}T_j$

s.t.



if ${}^{\exists} R_j \sim R_{i+1}$, find an isometry $g \in SL(2, \mathbb{R})$ that carries $R_j \mapsto R_{i+1}$

, add g to $\mathrm{Gen}(\Gamma(x, q))$: generating list

& eliminate R_{i+1} from the tree search : (renumber)

(5) if the tree is not closed, go back to (2) for the leftover IDR – bdry edges.

else : finish the algorithm.