VEECH GROUPS OF FLAT SURFACES WITH POLES

GUILLAUME TAHAR

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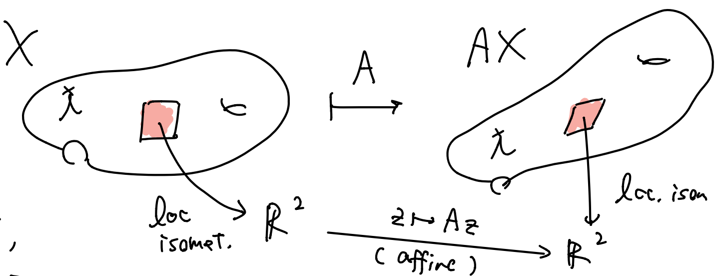
ABSTRACT. Flat surfaces that correspond to meromorphic 1-forms with poles or to meromorphic quadratic differentials containing poles of order two and higher have infinite flat area. We classify groups that appear as Veech groups of translation surfaces with poles. We characterize those surfaces such that their $GL^+(2,\mathbb{R})$-orbit or their $SL(2,\mathbb{R})$-orbit is closed. Finally, we provide a way to determine the Veech group for a typical infinite surface in any given chamber of a stratum.

§1. Intro.

○ moduli sp. of trans. surf. $\mathcal{A}$ $\curvearrowleft^{\text{act}}$ $GL^+(2,\mathbb{R})$
(sp. of Abelian differentials)

ex :



for $X \in \mathcal{A}$,

Veech grp : $\Gamma(X) = \mathrm{Stab}_{GL^+(2,\mathbb{R})}(X)$.

Characterization of Veech grps of Abelian differentials : open prob.

In this paper, we show that : → infinite area

(☆) 'flat surfaces w/ h.a.p.' (corres. mero. quadratic diff.)
are rigid enough to allow a full answer to this problem.

2012. Valdez : VG of some families of flat surf. of area ∞ ass. to irrational billiards

2010. Hubbert and Schmithüsen : exms. of sq-tiled surf. w/ infinite tiles
w/ VG : inf. generated $< SL(2,\mathbb{R})$

This paper :

$\underbrace{}_{\text{finite genus}}$ $\underbrace{}_{\text{finite sing.s w/ finite holonomy order}}$

※ q.d. w/ poles of higher order (ord ≥ 2) $\longmapsto$ flat surf. of infinite area.

$$\left(\mathrm{Area}(X, q) = \sum \iint_{loc} |q| = \sum \iint_{D} |q(z)|\,dxdy = \sum \iint_{D} (re^{i\theta})^{ord} \times r\,dr\,d\theta \right)$$

We cannot normalize the area ∵ the $GL^+(2,\mathbb{R})$-action : NOT reduces to $SL^+(2,\mathbb{R})$

Note. zeros & poles of q.d. $\xleftrightarrow{\text{corres.}}$ conical sings. of flat metric : naturally induced.

the stratum $Q(a_1, a_2, \cdots, a_n, -b_1, -b_2, \cdots, -b_p)$ of mero q.d. w/ zeros / poles (including simple poles) of order $a_1 \cdots a_n$ / $b_1 \cdots b_p$

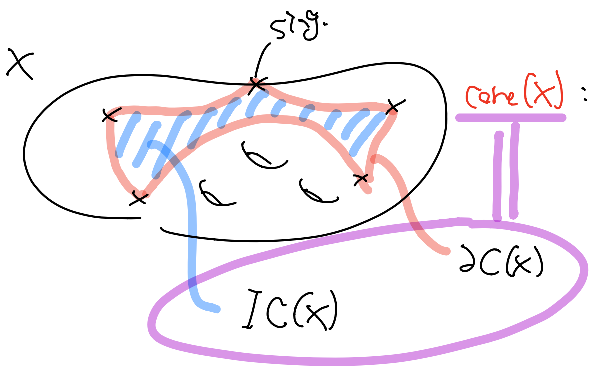
dimension $2g+n+p-2$

$\sim$ conical sing. of degrees $a_1 \sim a_n \in \{-1\} \cup \mathbb{N}_+$
higher poles of 〃 $b_1 \sim b_p \in \mathbb{Z}_{\le -2}$

we have $\sum a_i + \sum b_j = 4g-4$ (g = genus(X))

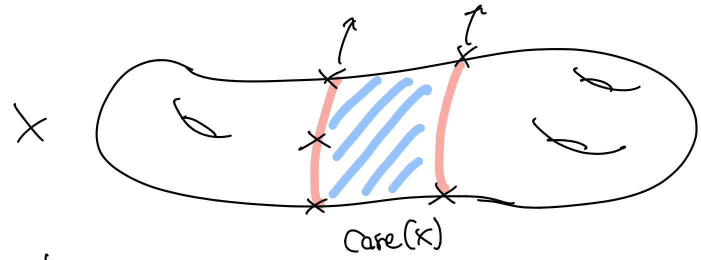
basically we assume $\left\{ \begin{array}{l} \text{flat surf. w/ poles of higher order} \\ n>0 \ \& \ p>0. \\ n+p>3 \text{ if } g=0 \\ \text{no marked pts} \end{array} \right.$

§2. MAIN Results



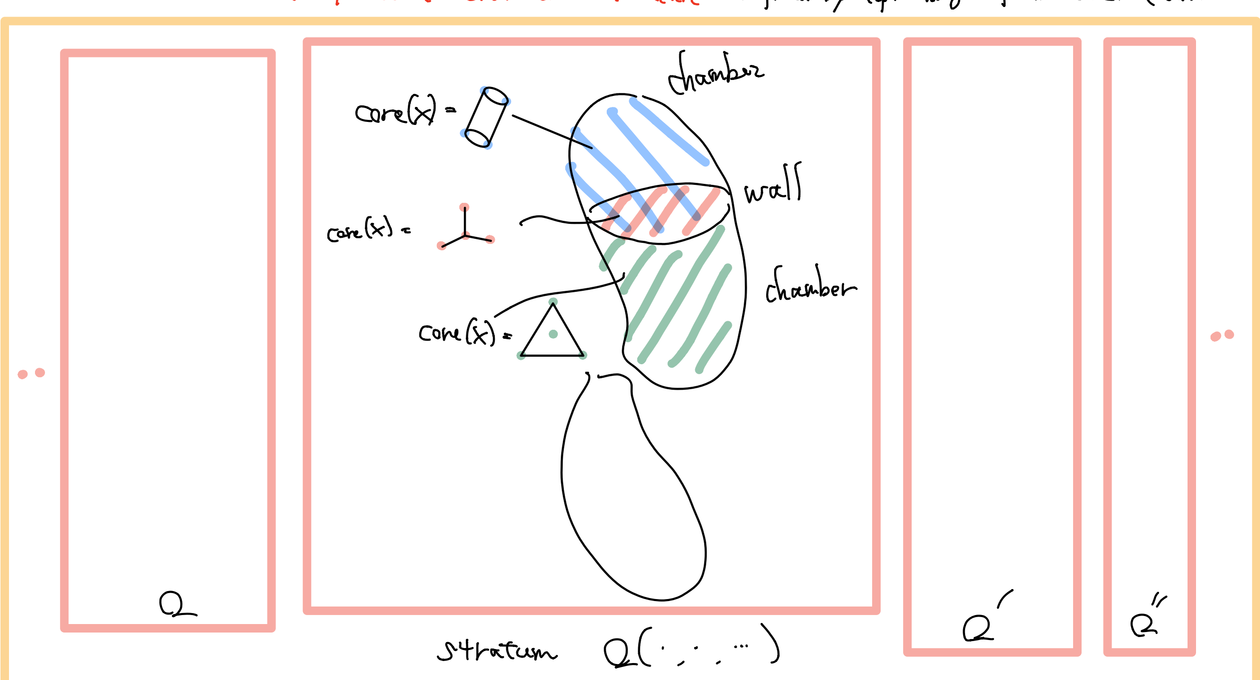
core(X) : the cvx hull of sing(X).

[a polygon bounded by saddle connections
invariant under $GL^+(2,\mathbb{R})$-action



☆ strata of flat surfaces of higher order poles have

a walls-and-chambers structure defined by top. changes of the core. (unlike ones of trans. surf.!)



ambient space QD(X)

like this roughly...

Theorem 2.1. *The Veech group of a flat surface with poles of higher order belongs to one of the following three types of subgroups of $GL^+(2,\mathbb{R})/\{\pm Id\}$:*
- Finite type: conjugated to a finite rotation group ;
- Cyclic parabolic type: conjugated to $\left\{\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix} \mid k \in \mathbb{Z}\right\}$ *;*
- Continuous type: conjugated to $\left\{\begin{pmatrix}1 & b\\ 0 & a\end{pmatrix} \mid a \in \mathbb{R}_+^*, b \in \mathbb{R}\right\}$.

It should be noted that flat cones that belong to strata $\mathcal{Q}(a, -a-4)$ with $a \in \{-1\} \cup \mathbb{N}^*$ have the whole $GL^+(2,\mathbb{R})/\{\pm Id\}$ as Veech group. We exclude these trivial cases by assuming $n + p \geq 3$.

• for classical translation surfaces, (Veech, 1989)

analog. $GL^+(2,\mathbb{R})$-orbits : closed $\Leftrightarrow$ $VG < SL(2,\mathbb{R})$: lattice

Theorem 2.2. *The following statements are equivalent for a flat surface with poles of higher order (X, q):*
(i) The Veech group of (X, q) is of continuous type.
(ii) All saddle connections of (X, q) share the same direction.
(iii) The $GL^+(2,\mathbb{R})$-orbit of (X, q) is closed in the ambient stratum.

$Q(\cdot,\cdot,\cdots,\cdot)$ which q belongs to.

Q $GL^+(2,\mathbb{R})$-action $\neq$ $SL^+(2,\mathbb{R})$-action.

Surface whose $SL^+(2,\mathbb{R})$-orbit is closed is very special.

Theorem 2.3. *The $SL(2,\mathbb{R})$-orbit of a flat surface with poles of higher order (X, q) is closed in the ambient stratum in these two cases:*
(i) Not all saddle connections of $\partial\mathcal{C}(X)$ share the same direction.
(ii) All saddle connections of $\partial\mathcal{C}(X)$ share the same direction and core(X) *decomposes into a (maybe empty) family of cylinders of commensurable moduli.*
Otherwise, the $SL(2,\mathbb{R})$-orbit of (X, q) is not closed in the ambient stratum.

Definition (Möller) a flat surf $(X,q) \in Q(\cdots)$ is generic

$\Leftrightarrow$ (X,q) lies in $Q(\cdots) \setminus \bigcup_{n\in\mathbb{N}} (Q'_n \subset Q \mid \mathrm{codim}_Q Q'_n = 1)$?

· Two saddle conns. are parallel $\Leftrightarrow$ their relative homology classes are linearly indep.

* VG of Generic flat surface of finite area, of genus $g \geq 2$

is either $\mathbb{Z}/2$ (hyperelliptic) or trivial (otherwise). [Möller 2009]

similar result for 'poles of higher order' :

Theorem 2.4. *In strata of dimension one, the Veech group of every surface is of continuous type.*
In strata of dimension at least three, the Veech group of a generic surface is trivial.
In strata of dimension two, we distinguish several cases:
(i) The Veech group of every surface is of cyclic parabolic type in chambers where the core is a cylinder in the following strata:
- $\mathcal{Q}(a, -a)$ *with* $a \geq 2$,
- $\mathcal{Q}(a, -1^2, -2-a)$ *with* $a \geq 1$,
- $\mathcal{Q}(a, b, -a-2, -b-2)$ *with* $a, b \geq 1$.

(ii) In chambers where the core is a triangle, the Veech group of every surface is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. *Such chambers exist in the following strata:*
- $\mathcal{Q}(3b-4, -b, -b, -b)$ *with* $b \geq 2$,
- $\mathcal{Q}(a, a, a, -3a-4)$ *with* $a \geq 1$,

(iii) In the chamber of each stratum $\mathcal{Q}(3a, -3a)$ *with* $a \geq 1$ *where the core is a triangle and the three elementary loops have the same topological index, the Veech group of every surface is isomorphic to* $\mathbb{Z}/3\mathbb{Z}$.
(iv) The Veech group of every surface is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ *in the the open* $GL^+(2,\mathbb{R})$*-orbits of the following flat surfaces:*
- *Flat surface whose core is degenerated and where all angles are congruent angles in* $\mathcal{Q}(4k, -4k)$ *with* $k \geq 1$,
- *Flat surface whose core is degenerated and whose four angles are* $(2+a)\pi$, $(2+a)\pi$, $\frac{(2k+1)\pi}{2}$ *and* $\frac{(2k+1)\pi}{2}$ *in strata* $\mathcal{Q}(2k-1, a, a, -2a-2k-3)$ *with* $k \geq 1$ *and* $a \in \{-1\} \cup \mathbb{N}^*$,
- *Flat surface whose core is degenerated and whose four angles are* $(b-1)\pi$, $(b-1)\pi$, $\frac{(2k+1)\pi}{2}$ *and* $\frac{(2k+1)\pi}{2}$ *in* $\mathcal{Q}(2b+2k-3, -b, -b, -2k-1)$ *with* $b \geq 2$ *and* $k \geq 1$,
- *Flat surface whose core is formed by two saddle connections between two distinct conical singularities and whose four angles are* $\frac{(1+2k)\pi}{2}$, $\frac{(1+2k)\pi}{2}$, $\frac{(1+2l)\pi}{2}$ *and* $\frac{(1+2l)\pi}{2}$ *in* $\mathcal{Q}(k+l-1, k+l-1, -2k-1, -2l-1)$ *with* $k, l \geq 1$,
- *Flat surface whose core is formed by two saddle connections between two distinct conical singularities and whose four angles are* $\frac{(1+2k)\pi}{2}$, $\frac{(1+2k)\pi}{2}$, $\frac{(1+2l)\pi}{2}$ *and* $\frac{(1+2l)\pi}{2}$ *in* $\mathcal{Q}(2k-1, 2l-1, -k-l-1, -k-l-1)$ *with* $k, l \geq 1$.

(v) Outside these chambers or orbits, the Veech group of a generic surface is trivial.

conj. to $\left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \middle| \begin{array}{l} a > 0 \\ b \in \mathbb{R} \end{array} \right\}$
$(\mathrm{Stab}_{GL^+_2\mathbb{R}} F_x)$

$\boxed{\begin{array}{lll} \dim \mathcal{Q} = 1 & : & \text{conti. type} \\ \dim \mathcal{Q} = 2 & : & \text{several cases} \\ \dim \mathcal{Q} \geq 3 & : & \text{trivial.} \end{array}}$

$n = 1 \quad p = 3 \quad \dim \mathcal{Q} = 2$
$\mathcal{Q}(2, -2, -2, -2) \quad g = 0$

§3. Defs & Tools

Notation (X, q) : flat surf
cpt. R.S. (X), mero q.d. (q)
equipped w/ the natur. flat metric
from the q-coords : $p \in X \setminus (\Lambda \cup \Delta) \underset{\text{loc}}{\longmapsto} \int_{p_0}^{p} \sqrt{q} \in \mathbb{C}$

$\Lambda, \Delta \subset X$
cone pts (Λ), h.o. poles (Δ)

Def 3.1 A saddle conn is a geod. whose $\begin{cases} \text{end pts } \subset \Lambda \\ \text{interior pts } \not\subset \Lambda \end{cases}$

Thm 3.2 [Tahar. 2016] for $(X, q) \in \mathcal{Q}(a_1 \cdots a_n, -b_1 \cdots -b_p)$
$\underbrace{|SC|}_{\#(\text{saddle conns})} \leq 2g - 2 + n + p$

@ Moduli sp. of mero. q.d. $= \{(X, q)\} / \text{biholo pres' q.d.}$
$\because (X, q) \sim (X', q') \Leftrightarrow \exists f : X \xrightarrow{\text{bihol}} X'$ s.t. $q = f_* q'$.
$\hookrightarrow$ stratified into strata $\{ \underbrace{\mathcal{Q}(a_1 \cdots a_n | -b_1 \cdots -b_p)}_{\text{dimension } 2g-2+n+p} \mid \sum a_i + \sum b_j = 4g(X) - 4 \}$

@ canonical double cover : the double cov. $\pi : (\tilde{X}, \omega) \to (X, q)$ where ω : Abelian differential

obtained by the continuation of $\sqrt{q}$: loc. Abelian differential

s.t. $\pi_* q = \omega^2$

if $(X, q) \in Q(\underbrace{2c_1, \cdots, 2c_s}_{\text{even}}, \underbrace{2d_1+1, \cdots, 2d_t+1}_{\text{odd}})$ $(c_\cdot, d_\cdot \in \mathbb{Z}^*)$

then $(\hat{X}, \omega) \in \mathcal{H}(c_1, c_1, \cdots, c_s, c_s, 2d_1+2, \cdots, 2d_t+2)$

stratum of Abelian differentials

if $(X, q) \in Q(2c_1, \cdots, 2c_s)$ i.e. $q = \omega^2$: global square

then $(\hat{X}, \omega)$ is disjoint double of X & $(X, \omega) \in \mathcal{H}(c_1, \cdots, c_s)$.

@ Period coordinates.

The can. double $(\hat{X}, \omega)$ of (X, q) admits an involution τ

$\xrightarrow{\text{induce}}$ an involution τ^* on $H_1(X \setminus \Delta, \Delta)$

Theorem 2.1. *Let $(X, \xi, z_1, \ldots, z_n) \in \Omega^k\mathcal{M}_g(m_1, \ldots, m_n)$ be a (possibly meromorphic) k-differential on a smooth curve X. Then the tangent space of the stratum at ξ can be identified as*

$$T_{(X,\xi,z_1,\ldots,z_n)}\Omega^k\mathcal{M}_g(m_1, \ldots, m_n) = H^1(X, \mathcal{C}^\bullet(\mathcal{L}_\xi)).$$

Moreover, the connected component of $\Omega^k\mathcal{M}_g(m_1, \ldots, m_n)$ containing the point $(X, \xi, z_1, \ldots, z_n)$ is a smooth orbifold of dimension $2g - 1 + n$ if ξ is the k-th power of a holomorphic abelian differential and of dimension $2g - 2 + n$ otherwise.

$\pi : \hat{X} \to X$: the canonical double $\hat{P} = \pi^{-1}(P) \subset \hat{X}$, $\hat{Z} = \pi^{-1}(Z) \subset \hat{X}$

Corollary 2.3. *Let V_ζ be the eigenspace of $H_1(\widehat{X} \setminus \widehat{P}, \widehat{Z}; \mathbb{C})$ associated to the eigenvalue ζ. Locally at the primitive k-differential ξ, the stratum $\Omega^k\mathcal{M}_g(m_1, \ldots, m_n)$ has coordinates given by the periods $\int_{\gamma_i} \omega$ where $\{\gamma_i\}$ is a basis of V_ζ and $\pi^*\xi = \omega^k$ for an abelian differential ω on $\widehat{X}$.*

We remark that Theorem 2.1 is a generalization of [Möl08, Theorem 2.3] and [Mon17, Proposition 3.1] from the case of abelian differentials to k-differentials. Recently Mondello [Mon16] and Schmitt [Sch16] have also independently obtained this result. Theorem 2.2 generalizes the known case of quadratic differentials, originally studied by Hubbard-Masur [HM79] and Veech [Vee86].

homology classes $\longleftrightarrow$ saddle conn.s

$$1 \to C_\bullet(\tilde\Lambda) \hookrightarrow C_\bullet(\tilde X \setminus \tilde\Delta) \to C_\bullet(\tilde X \setminus \tilde\Delta)/C_\bullet(\tilde\Lambda) \to 1$$

$$H_1(\tilde X \setminus \tilde\Delta, \tilde\Lambda) = H_1\left(C_\bullet(\tilde X \setminus \tilde\Delta)/C_\bullet(\tilde\Lambda)\right)$$

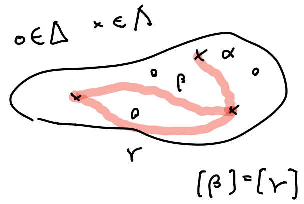
$$H_1(C_\bullet(\square)) = \mathrm{Ker}\,\partial_1/\mathrm{Im}\,\partial_2 \quad \text{in} \quad C_2(\square) \xrightarrow{\partial_2} C_1(\square) \xrightarrow{\partial_1} C_0(\square)$$

$$\mathrm{Ker}\,\partial_1 = \left\{ \gamma \in C_1(\tilde X \setminus \tilde\Delta)/\underset{\phi}{\cancel{C_1(\tilde\Lambda)}} \;\middle|\; \partial\gamma \in C_0(\tilde\Lambda) \right\}$$

$$= \{ \gamma \subset \tilde X \setminus \tilde\Delta : \text{1-chain cpx} \mid \partial\gamma \subset \tilde\Lambda \}$$

$$\mathrm{Im}\,\partial_1 = \{ \partial F \subset \tilde X \setminus \tilde\Delta \mid F \subset \tilde X \setminus \tilde\Delta : \text{2-chain cpx} \}$$

$$H_1(\tilde X \setminus \tilde\Delta, \tilde\Lambda) = \{ \gamma \subset \tilde X \setminus \tilde\Delta : \text{1-chain cpx} \mid \partial\gamma \subset \tilde\Lambda \} \big/ \partial\{\text{2-chain cpx in } \tilde X \setminus \tilde\Delta\}$$

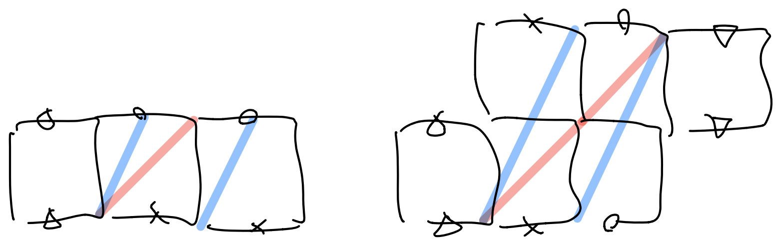


$$= SC(\tilde X \setminus \tilde\Delta) \big/ \partial\{SC\text{-polygon in } \tilde X \setminus \tilde\Delta\}$$

※oriented.

for $\gamma \in SC(X \setminus \Delta)$ ※oriented

$\xleftarrow{\pi} \gamma_1, \gamma_2 \in SC(\tilde X \setminus \tilde\Delta)$: preimage.



∘ Index of a loop.

Let $\gamma \subset (X, q)$: s.c.c. (of finite length?) parametrized by arc length $t \in [0, T]$

$\pi \uparrow$ $\eta \subset (\tilde{X}, \omega)$: (the ??) lifting

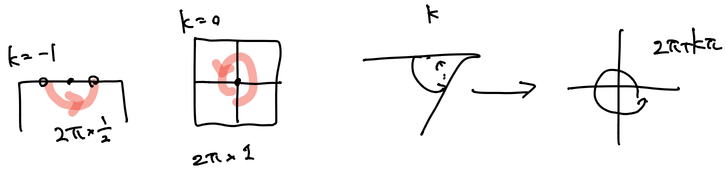
then $\eta'(t) = e^{i\theta(t)}$

$\Delta\theta = \int_0^T \theta'(t)\, dt \in \pi\mathbb{Z}$ (holonomy angle of flat surf.)

<u>def</u> $\mathrm{ind}(\gamma) := \frac{\Delta\theta}{2\pi} \in \frac{\mathbb{Z}}{2}$

in particular, if γ is a small loop arround a cone pt of order k,

$\mathrm{ind}(\gamma) = 1 + \frac{k}{2}$



∘ Core of flat surf. w/ h.o.p.

<u>Def 3.4</u> $E \subset (X, q)$: subset is convex $\Leftrightarrow$ $\forall x, y \in E$, $\forall \gamma \subset X$: geod. joining x, y is contained in E.

the core of (X, q) is the convex hull of Λ.
denoted by $\mathrm{core}(X)$.

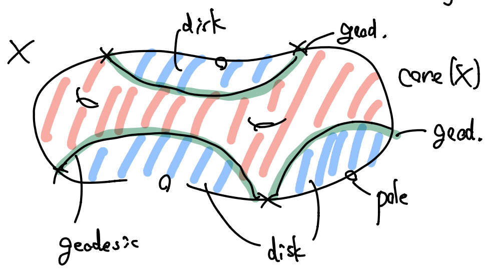
$IC(X) :=$ interior of $\mathrm{core}(X)$ and $\partial C(X) = \mathrm{core}(X) \setminus IC(X) = \partial\,\mathrm{core}(X)$.

<u>Lem 3.5</u> (Haiden, Katzarkov, Kontsevich 2014)

$X \setminus \mathrm{core}(X)$ consists of $p = \#\Delta$ connected components and each of them is a top. disk arround higher pole.

<u>Lem 3.6</u> (Haiden, Katzarkov, Kontsevich 2014)

$\partial C(X)$ is a finite union of saddle connections.

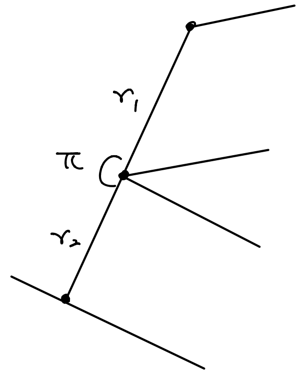


<u>Def</u> When $IC = \phi$ (core $= \partial C$: graph), $\mathrm{core}(X)$ is said to be degenerated.

@ Discriminant & walls-and-chambers str.

'discriminant'

Def 3.8. (X, q) belongs to the discriminant of the stratum



$$\Longleftrightarrow \ {}^\exists \gamma_1, \gamma_2 : \text{saddle conn s.t.} \begin{cases} \gamma_1, \gamma_2 \subset \partial \mathcal{C}(X) \\ \gamma_1, \gamma_2 : \text{consecutive} \\ \qquad \& \text{ non parallel} \\ \gamma_1, \gamma_2 \text{ share an angle } \pi \end{cases}$$

Chambers are defined to be conn. comps. of (Strata) \ (Discriminant)

Lemma 3.9. *The discriminant is a $GL^+(2, \mathbb{R})$-invariant hypersurface of real codimension one in the stratum.*

The topological map on a flat surface with poles of higher order (X, q) defined by the embedded graph $\partial\mathcal{C}(X)$ is invariant along the chambers. The qualitative shape of the core and in particular the number of saddle connections of its boundary depend only on the chamber (see Proposition 4.13 in [12] for details).

3.8. **Dynamics and decomposition into invariant components.** We essentially follow the definitions Strebel gives in [11].

Definition 3.10. Depending on the direction, a trajectory starting from a regular point is of one of the four following types:
- regular closed geodesic (the trajectory is periodic),
- critical trajectory (the trajectory reaches a conical singularity in finite time),
- trajectory finishing at a pole (the trajectory converges to a pole of higher order as $t \to +\infty$),
- recurrent trajectory (infinite trajectory nonconverging to a pole of higher order).

Theorem 3.11 describes how the directional flow decomposes flat surfaces into a finite number of invariant components. This theorem is proved as Theorem 2.3 in [12].

Theorem 3.11. *Let (X, q) be a flat surface with poles of higher order. Cutting along all saddle connections sharing a given direction θ, we obtain finitely many connected components called invariant components. There are four types of invariant components:*
- ***finite volume cylinders*** *where the leaves are periodic with the same period,*
- ***minimal components*** *of finite volume where the foliation is minimal, the directions are recurrent and whose dynamics are given by a nontrivial interval exchange map,*
- ***infinite volume cylinders*** *bounding a simple pole and where the leaves are periodic with the same period,*
- ***free components*** *of infinite volume where generic leaves go from a pole to another or return to the same pole.*
Finite volume components belong to core(X).

§4. Classification of VG.

Rem VG of flat surf. w/ h.o.p. : very different from usual trans. surfs.

- either very big : NOT discrete
  or very small : Infinite index

$\longrightarrow$ No Veech surfaces w/ h.o.p.

Theorem 2.1. *The Veech group of a flat surface with poles of higher order belongs to one of the following three types of subgroups of $GL^+(2,\mathbb{R})/\{\pm Id\}$:*
- *Finite type: conjugated to a finite rotation group ;*
- *Cyclic parabolic type: conjugated to* $\left\{\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z}\right\}$;
- *Continuous type: conjugated to* $\left\{\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{R}_+^*, b \in \mathbb{R}\right\}$.

It should be noted that flat cones that belong to strata $\mathcal{Q}(a, -a-4)$ with $a \in \{-1\} \cup \mathbb{N}^*$ have the whole $GL^+(2,\mathbb{R})/\{\pm Id\}$ as Veech group. We exclude these trivial cases by assuming $n + p \geq 3$.

pf) Let (X, q) : flat surf w/ h.o.p.

by Lem 3.6 (Haiden, Katzarkov, Kontsevich 2014),

there are finite S.C.s in $\partial C(X)$. different from origamis!!

$hol(V) = \{ hol(\gamma) \mid \gamma \in SC(X) \} \subset \mathbb{C}/\{\pm 1\}$: finite set

... invariant under VG-action.

$\longrightarrow$ either (i) $hol(V)$ share the same direction

or (ii) VG is a finite group.

$\hookrightarrow$ finite subgrps in $GL^+(2,\mathbb{R})$ are rotation grps.

$\rightarrow$ We say that such a VG is of finite type.

We consider the case (i). VG should pres the direction shared in $hol(V)$.

$\rightarrow$ VG is conjugated to a subgrp of $\left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \middle| \begin{matrix} a > 0 \\ b \in \mathbb{R} \end{matrix} \right\} = Stab F_h$
$< GL^+(2,\mathbb{R})$.

Next disjunction: $\begin{cases} \text{core is degenerated} & (\mathcal{IC} = \phi) \\ \text{non-degenerated.} & (\quad \neq \quad) \end{cases}$

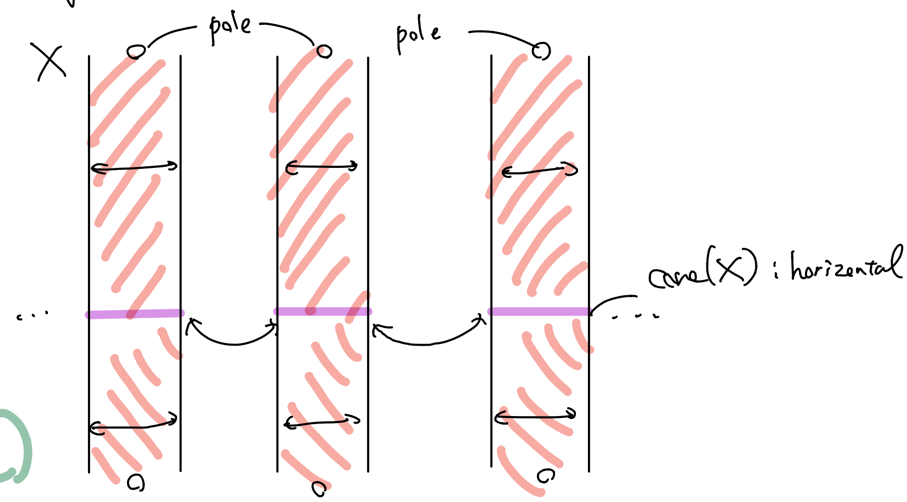
(※ usual : core(X))

(I) If $ZC(X) = \phi$, then every S.C. belongs to $\partial C(X)$ and they belong to the same direction; we may assume horizontal.

by Lem 3.5 (Haiden, Katzarkov, Kontsevich 2014) ?

the total geometry of X is that of infinite vertical strips (cylinders ?)

i.e.



thus, $\mathrm{Stab}_{GL^+(2,\mathbb{R})} \mathcal{F}_h$ leaves (X, q) unchanged.

$\rightsquigarrow VG \underset{conj}{\simeq} \mathrm{Stab}_{GL^+(2,\mathbb{R})} \mathcal{F}_h$

If core(X) is non degenerated $\Rightarrow VG < SL(2,\mathbb{R})/\{\pm I\}$.

: since Area(core(X)) $< \infty$ is preserved by VG-action.

Besides, non-deg. core implies that $\exists\, \gamma_1, \gamma_2 \in SC(X)$: in different directions.

$\rightarrow$ since $hol(V) \subset \mathbb{C}/\{\pm I\}$ contains vectors in at least 2 directions,

$\forall A \in VG$ is completely determined by how A acts on $hol(V)$. (finite set, stabilized!)

$\Rightarrow VG$ is a discrete subgrp of $P\cancel{GL^+(2,\mathbb{R})}$ $PSL(2,\mathbb{R})$!

Furthermore we know that $VG \underset{conj}{\simeq} \mathrm{Stab}\, \mathcal{F}_h = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \middle| \begin{array}{l} a > 0 \\ b \in \mathbb{R} \end{array} \right\}$

So we conclude that $VG \underset{conj}{\simeq} \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \middle| k \in \mathbb{Z} \right\}$.

or it is trivial.

$\boxtimes$

~~Rem.~~ One can find a realization for every VG of the classification provided by Thm 2.1.

For example, any finite cyclic type is realized by following exm:

Example 4.1. Gluing infinite cylinders on the edges of a regular $2k$-gon, we get a surface in $\mathcal{Q}(4k-4, -2^{2k})$ whose Veech group is conjugated to $\mathbb{Z}/k\mathbb{Z}$, see Figure 1.

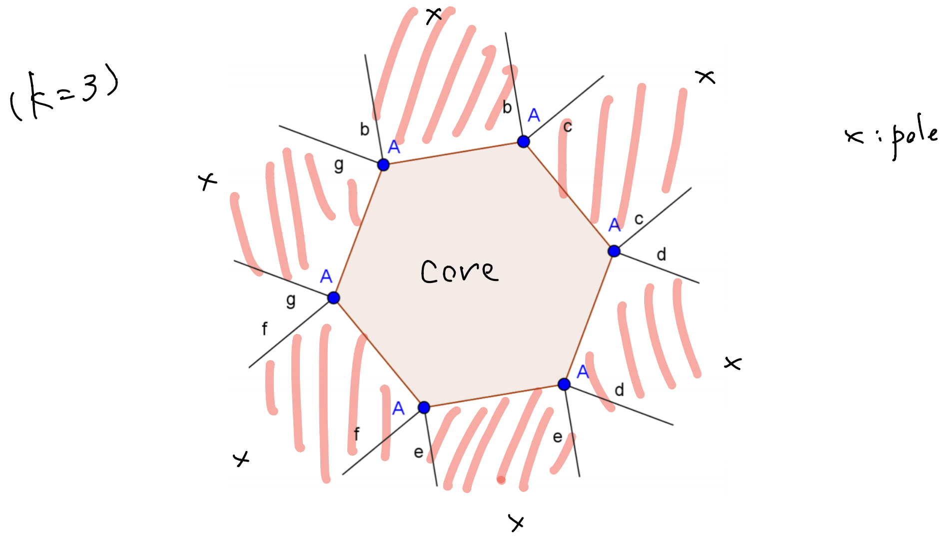


FIGURE 1. A flat surface in $\mathcal{Q}(8, -2^6)$ with a discrete rotational symmetry.

Example 4.3. We can construct many surfaces with a Veech group of cyclic parabolic type starting from a square-tiled core, see Figure 2.

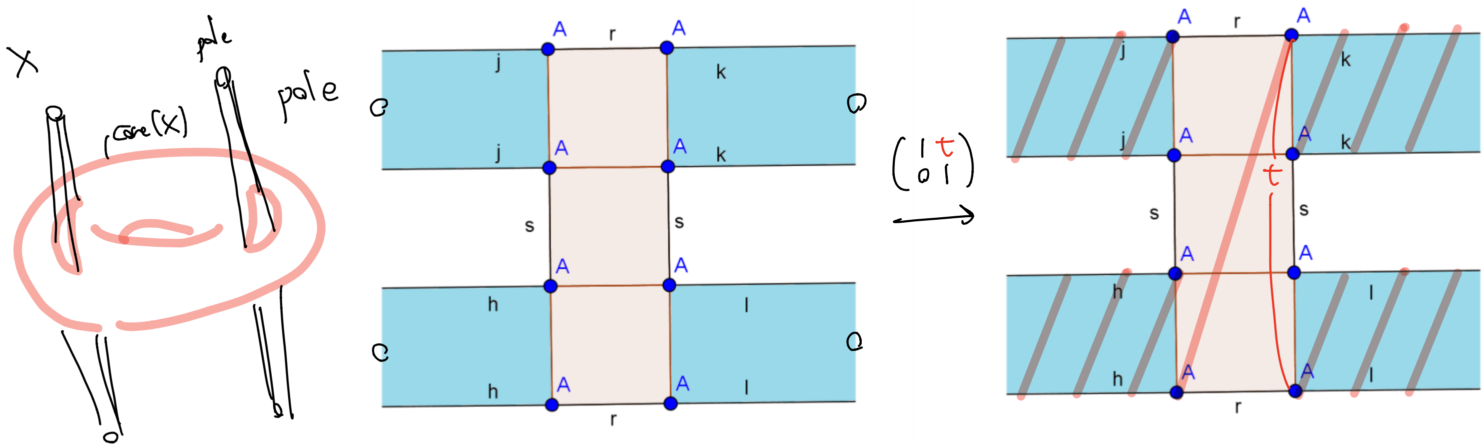


FIGURE 2. A flat surface in $\mathcal{Q}(8, -2^4)$ whose Veech group is of cyclic parabolic type.

Example 4.4. Three flat planes with a vertical slit in each of them and connected cyclically provide a surface in $\mathcal{Q}(4^2, -4^3)$ with a Veech group of continuous type and that does not belong to the discriminant.

§5 Closedness of orbits

In classical cases : $\exists$ Veech surfaces (Teich. curves) $\hookrightarrow \mathcal{M}_{g,n}$

$\leftarrow$ since their $GL^+(2,\mathbb{R})$-orbits are closed

In cases of flat surf w/ h.o.p : $\uparrow$ this occurs only when the VG is of conti. type!

Theorem 2.2. *The following statements are equivalent for a flat surface with poles of higher order (X, q):*
(i) The Veech group of (X, q) is of continuous type. Conj. to $\mathrm{Stab}_{GL^+(2,\mathbb{R})} \mathcal{F}_h$
(ii) All saddle connections of (X, q) share the same direction.
(iii) The $GL^+(2,\mathbb{R})$-orbit of (X, q) is closed in the ambient stratum. w.r.t. period coords (closed set)

pf) (i) $\Rightarrow$ (iii) : Let $G \underset{\text{conj}}{\sim} \mathrm{Stab}_{GL^+} \mathcal{F}_h$: VG of (X, q) of conti. type

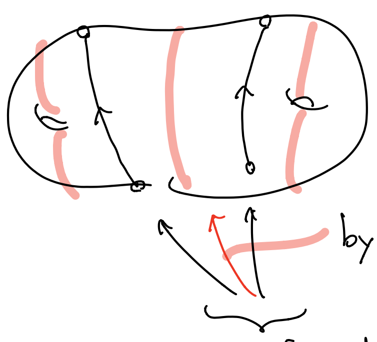
$\mathbb{C}^* < GL^+(2,\mathbb{R})$: grp of matrices of dilation & rotation

since $(G, \mathbb{C}^*)$ generate $GL^+(2,\mathbb{R})$,

rotation & dilation provide an affine parametrization of the $GL^+(2,\mathbb{R})$-orbit.

$\rightarrow$ the $GL^+(2,\mathbb{R})$-orbit is closed set in the stratum...?

$\neg$(ii) $\Rightarrow$ $\neg$(iii)



by taking other directions and applying the contraction flow in this direction,

SC.s of two different directions

we will obtain the limit surface (X_0, q_0) which doesn't belong to the orbit. //

(ii) $\Rightarrow$ (i) see pf of thm 2.1 //