# VEECH GROUPS OF LOCH NESS MONSTERS

プルテツキ　プシェテツキ
~~プルザイティキ~~　(Polish)

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ABSTRACT. — We classify Veech groups of tame non-compact flat surfaces. In particular we prove that all countable subgroups of $GL_+(2, R)$ avoiding the set of mappings of norm less than 1 appear as Veech groups of tame non-compact flat surfaces which are Loch Ness monsters. Conversely, a Veech group of any tame flat surface is either countable, or one of three specific types.

1. Intro

- precpt flat surf $S$ (non-precpt) : Veech group $\Gamma < SL_2\mathbb{R}$ ($GL_2^+\mathbb{R}$), $\Gamma = \Gamma(S)$

  Jacobian derivatives of $\mathrm{Aff}^+(S)$

  discrete grp acting on

$$\begin{array}{ccccc} \mathbb{H} & \hookrightarrow & T(S) & \twoheadrightarrow & M(S) \\ & {}^{\text{isom}}\searrow & \cup & & \cup \\ & & \Delta & \longrightarrow & C \end{array}$$

  which gives $C \cong \mathbb{H}/R^{-1}\Gamma R$ (R : reflection)

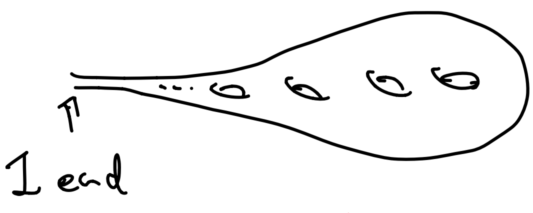
  only for pre cpt cases.

- goal describe all possible Veech grps for tame non-cpt flat surfaces

  coming from the study of billiard dynamics of irrational-angle table.

  cf. [1989, Veech]

- Loch Ness monster

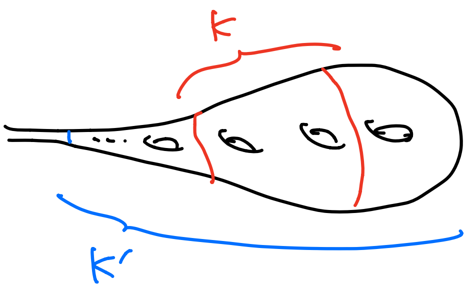



$\Leftrightarrow$ the top. surf. $S$ of infinite genus

s.t. $\forall K \subset S$ cpt $\exists K \subset K' \subset S$ : cpt, $S \setminus K'$ : connected

"1-ended ness"

cplment of sufficiently large cpt set is connected.



$P := \left\{ \begin{pmatrix} 1 & t \\ 0 & s \end{pmatrix} \middle| \ t \in \mathbb{R}, s \in \mathbb{R}_{>0} \right\} < GL_2^+\mathbb{R}$ : all affine deform. fixing $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$P' := \langle P, \{\pm I\} \rangle < GL_2^+\mathbb{R}$

$\mathcal{U} := \{ M \in GL_2^+\mathbb{R} :$ Shrinking i.e. $\|Mv\|_{\mathbb{R}^2} < \|v\|_{\mathbb{R}^2} \ \forall v \in \mathbb{R}^2 \}$

(as linear maps)

Main Results.

THEOREM 1.1. — *Let $G \subset \mathbf{GL}_+(2,\mathbb{R})$ be the Veech group of a tame flat surface. Then one of the following holds.*

(i) *$G$ is countable and disjoint from $\mathcal{U}$.*
(ii) *$G$ is conjugate to $P$.*
(iii) *$G$ is conjugate to $P'$.*
(iv) *$G = \mathbf{GL}_+(2,\mathbb{R})$.*

} uncountable

Conversely, we prove the following.

THEOREM 1.2. — *Any subgroup $G$ of $\mathbf{GL}_+(2,\mathbb{R})$ satisfying assertion (i), (ii) or (iii) of Theorem 1.1 can be realized as a Veech group of a tame flat surface $X$ which is a Loch Ness monster.*

Rem $SL_2^{\pm}\mathbb{R}$ is disjoint from $\mathcal{U}$.

the problem describing all possible Veech grps for precpt flat surfs.

is still open.

2. preliminaries

$(S,\omega)$ : pair of $\left\{ \begin{array}{l} S: \text{ conn. Riem. surf.} \\ \omega: \text{ holo. 1-form on } S \end{array} \right\}$ : flat surf. (also called translation surf.)

$Z(\omega) = \mathrm{Zero}(\omega) \subset S$.

- $(S,\omega) \rightsquigarrow$ G-structure on $S$, where $G = \mathrm{Trans}\,\mathbb{R}^2$. —(*)

$\mathrm{Aff}^+(S) = \mathrm{Aff}^+(S,\omega) := \{ f : (S,\omega) \to (S,\omega) :$ ori. pres. G-map $\}$ (i.e. loc-affine)

pres. marked pts ?

(anyway crit. pts. are setwise-presed.)

$\Gamma(S) = \Gamma(S,\omega) := \{ D(f) \mid f \in \mathrm{Aff}^+(S,\omega) \}$ : Veech group

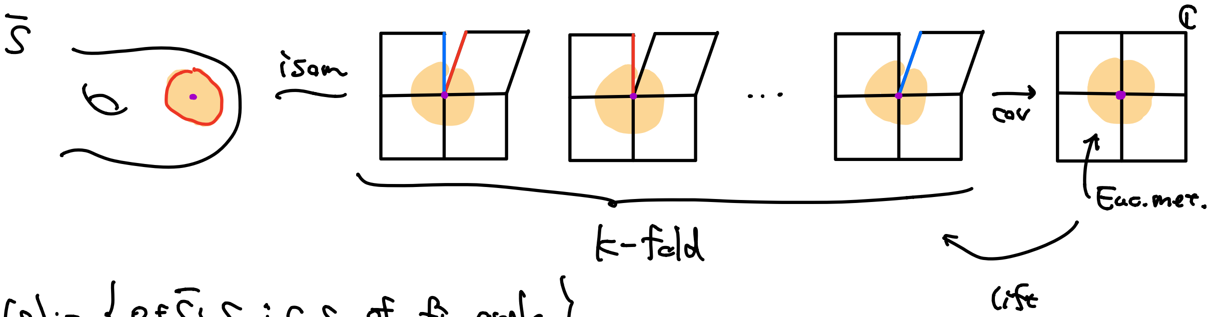
where $D$ : Jacobian derivative

- for $(S,\omega)$, $\bar{S}$ : met. completion of $S$ w.r.t. Euclidian met. coming from (*).

(different from closure $\bar{S}^{top}$)

$(S, \omega)$ is pre cpt $\overset{\text{def}}{\Longleftrightarrow}$ $\overline{S}$ : cpt.

- $p \in \overline{S} \setminus S$ is called a singularity.
  is called $\begin{cases} \text{cone sing. of angle } 2k\pi \ (k \in \mathbb{N}) \Leftrightarrow {}^\exists \text{nbd} \overset{\text{loc. isom.}}{\sim} k\text{-fold cov. of nbd of } 0 \in \mathbb{C} \ (\text{equiped w/ Euc. met.}) \\ \quad " \quad \text{infinite angle} \Leftrightarrow {}^\exists \text{nbd} \overset{\text{loc isom}}{\sim} \text{univ. cov. of nbd of } 0 \in \mathbb{C} \end{cases}$




- $\Sigma_{fin} = \Sigma_{fin}(S) := \{ p \in \overline{S} \setminus S : \text{c.s. of fin. angle} \}$
  $\Sigma_{inf} = \Sigma_{inf}(S) := \{ p \in \overline{S} \setminus S : \text{c.s. of inf. angle} \}$
  $\Sigma = \Sigma(S) := \Sigma_{fin} \cup \Sigma_{inf}$ $\quad \Sigma_0 := \{ p \in \Sigma_{fin} \mid p \text{ is } 0\text{-angle singularity.} \}$
  $\hat{S} := S \cup \Sigma_{fin} = \overline{S} \setminus \Sigma_{inf}$.

- path $\gamma \subset \overline{S}$ : Saddle conn. of $(S, \omega)$ $\Leftrightarrow$ $\gamma$ : geod. joining pts in $\Sigma(S)$.
- $(S, \omega)$ is a tame flat surface $\Leftrightarrow$ $\overline{S} \setminus S = \Sigma$

Rem.

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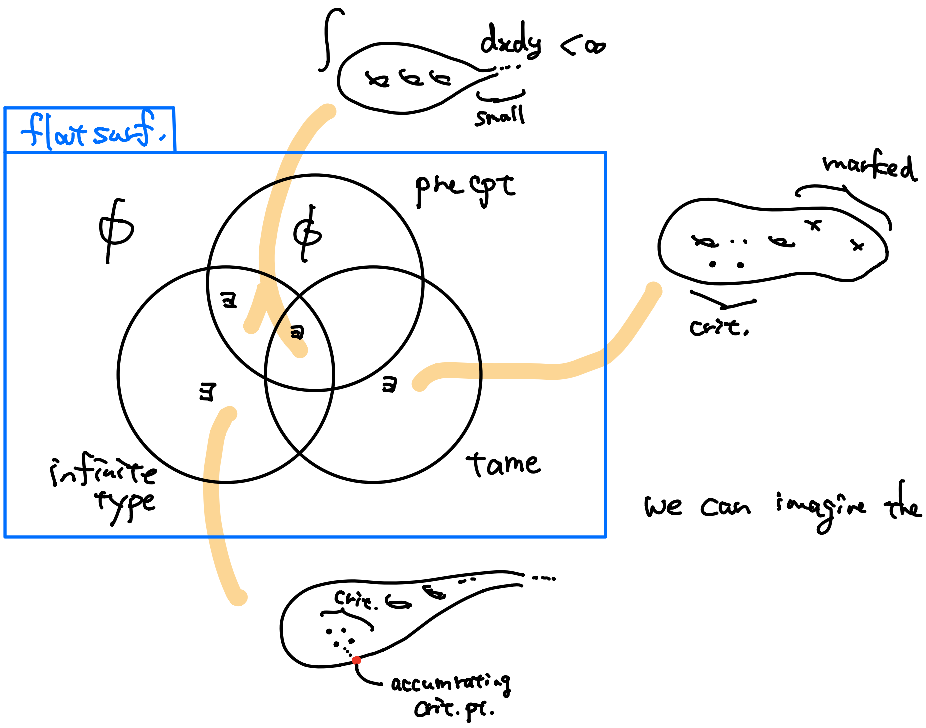



we can imagine the rest cases easily.

- being $(\mathrm{Trans}\, \mathbb{R}^2)$-mfd implies that $D(f) = \text{const}$ ${}^\forall f \in \mathrm{Aff}^+(S)$. $\rightsquigarrow$ $\Gamma(S) \subset GL_2^+\mathbb{R}$
  $S$ : pre cpt $\Rightarrow$ $f \in \mathrm{Aff}^+(S)$ pres. $\mathrm{Area}(S)$ $\rightsquigarrow$ $\Gamma(S) < SL_2^+(\mathbb{R})$

## 3. uncountable Veech grps.

Let $(S,\omega)$ : tame flat surface.

**Prop 3.1** $\Gamma(S,\omega)$ is unc'ble $\Rightarrow$ $\Gamma$ is $\begin{cases} \text{conjugated to } P, \\ \phantom{\text{conjugated to }} P', \text{ or} \\ \text{equal to } GL_2^+\mathbb{R}. \end{cases}$

**Lem 3.2** $\{\text{sdl. conn. on } S\} = \phi$ and $\Gamma(S)$ : unc'ble

$\Rightarrow$ $\Gamma(S) = P'$ or

$\Gamma(S) = GL_2^+\mathbb{R}$ and $S = \mathbb{C}$

pf) first, assume $\Sigma(S) = \phi$.

$\Rightarrow$ the univ. cov $\widetilde{S} = \mathbb{C}$ and $S = \begin{cases} (i)\ \mathbb{C} \\ (ii)\ \mathbb{C}/\langle r \rangle : \text{cylinder} \\ (iii)\ \mathbb{C}/\Lambda : \text{cpt torus} \end{cases}$

(Euc. met. on $\mathbb{C}$ is lifted to $\widetilde{S}$ as to be complete.)

$\Gamma(S)$ : unc'ble $\Rightarrow$ $S$ : non precpt, (iii) is impossible.

if (ii), $S$ is conjugated to $S_0 = \mathbb{C}/\langle z \mapsto z+1 \rangle$

Now we have $\Gamma(S_0) = P'$. ☒

It remains to show:

**Lem 3.3** $S$: Tame flat surf. carrying saddle conns. & $\Gamma(S)$ : unc'ble

$\Rightarrow$ $\Gamma(S)$ is conjugated to $P$ or $P'$.

$\boxed{\text{step 1.}}$ $^\forall$sdl. conns. are parallel.

たぶん $\hat{S}$ 内だから… (or passibly open?.)

☺ $\{[r]_{\text{homot.}} \mid r \subset \hat{S} : \text{arc joining } \Sigma_{fin}\}$ : countable ?

$\Rightarrow$ $SC := \{\text{sdl. conns.}\}$ and $V := hol(SC)$ : countable

If $^\exists s_1, s_2 \in SC$ : non parallel. $v_j := hol(s_j)$ $(j = 1, 2)$

$^\forall g \in \Gamma(S)$. let $\eta(g) := (g(v_1), g(v_2)) \in V \times V$.

since ____, $\eta : \Gamma(S) \hookrightarrow \mathbb{R}^2$ is an embedding.

however the image is both unc'ble and contained in $V \times V$ : countable

($\Gamma(S)$ is…) Contradiction↯.

· We can assume $^\forall r \in SC$ : horizontal (up to conjugation)

Let $Spine(S) := \bigcup \{p \in r \mid r \subset \hat{S} : \text{singular horizontal geod.}\} \subset \hat{S}$

(including sdl. conn.)

We claim: Spine$(S)$ is conn. & complete w.r.t. intrinsic metric. $\left(d(x,y) = \inf\limits_{x \overset{\gamma}{\sim} y} \operatorname{len} \gamma\right)$

follows from ... follows from ← $\hat{S}$ : complete

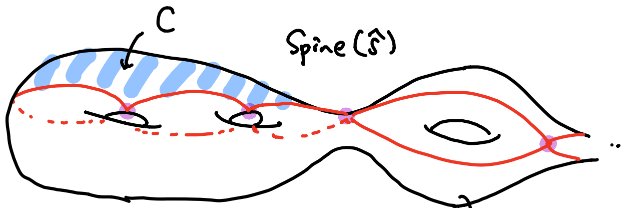
fact $\forall p_1, p_2 \in \Sigma$ $\exists \gamma \in SC$ joining $p_1, p_2$

horizontal!

$\boxed{\text{step 2.}}$ $P \subset \Gamma(S)$

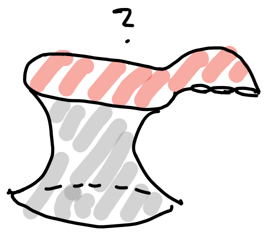
for each component of $\hat{S} \setminus \text{Spine}(S)$,

take the closure and denote by $C$.




$C$ : complete R.S., nonvanishing Abelian diff., w/ horizontal bdry.

$\partial C$ is connected. (otherwise there'd be a non-horiz. sdl. conn.)




$\rightsquigarrow$ $C$ is either $\begin{cases} \text{half-plane} \\ \text{half-cylinder ?} \end{cases}$ w/ horizontal bdry.

$\Rightarrow$ $\forall g \in P$, $\exists \varphi_g^C \in \mathrm{Aff}^+(C)$ w/ $D\varphi_g^C = g$.

Since each $\varphi_g$ stabilizes Spin$(S)$,

$\exists \varphi_g \in \mathrm{Aff}^+(S)$ w/ $D\varphi_g = g$ obtained by gluing $(\varphi_g^C)_C$ //

$\boxed{\text{step 3.}}$ $\Gamma(S) \subset P'$.

claim $\forall g \in \Gamma(S)$ $g(\vec{e}) = \pm \vec{e}$. where $\vec{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

since $\forall \bar{g} \in \mathrm{Aff}^+(S)$ should pres. "horizontal",

$\exists \lambda \in \mathbb{R}$ $\bar{g}(\vec{e}) = \lambda \vec{e}$.

assume $|\lambda| \neq 1$. we consider the action of $\bar{g}$ or $\bar{g}^{-1}$ on Spine$(S) = \Sigma \cup$ (horiz. sing. geods.)

: it is contraction. $= \mathbb{R}$. (Now)

By Banach fixed pt thm, the iterates (くり返し) of singularity under $\bar{g}^{(-1)}$

accumulates to fixed pt of $\bar{g}$.

since $\bar{g}$ pres $\Sigma$, it implies that

$\Sigma$ has accumulation pt. ↯ (Tameness)

$\rightsquigarrow$ possibly $\vec{e} \mapsto \pm \vec{e}$.

- Loch Ness monster w/ Veech grp $P, P'$

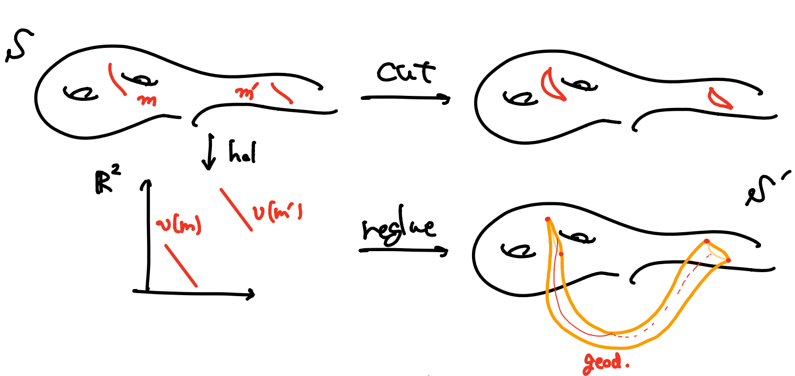
Def 3.4 $S$: Tame flat surf.

a mark on $S$ is an oriented, finite-length geodesic on $S$ which doesn't meet any sings.

rem: if $S$ is simply-conn, $\{\text{mark on } S\} \hookrightarrow S \times S$

the vector of a mark is its holonomy vector (in $\mathbb{R}^2$)

for $m, m'$ : disjoint marks on $S$, w/ same vector,




We say

$S'$ is obtained from $S$ by regluing along $m, m'$.

Let $S_0 := S \setminus (m \sqcup m')$

then there is a natur. embedding $S_0 \hookrightarrow S'$.

If $A \subset S_0$, we say $i(A)$ is inherited by $S'$ from $A$.

Rem 3.5 when $(S,(m,m')) \xrightarrow{\text{reg}} S'$,

$\forall m \neq 2$, $\Sigma_{2m\pi}(S') = \Sigma_{2m\pi}(S)$

$\Sigma_{4\pi}(S') = \Sigma_{4\pi}(S) + 2$. ($\partial m, \partial m'$ consist of nonsing. pts.)

Def 3.6 $\mathcal{M} = (m_n)_{n\in\mathbb{N}}$ $\mathcal{M}' = (m'_n)_{n\in\mathbb{N}}$ : families of marks on $S$ (tame fl. surf.) which are pairwise disjoint and $v(m_n) = v(m'_n)\ \forall n$. do not accumulate on $S$.

Let $S_0 = S$, $(S_{n-1}, (m_n, m'_n)) \xrightarrow{\text{reg.}} S_n$, and $S'$ be the limit of $S_n$ w/ Abel. diff.

⇝ $\exists S'$ since $\mathcal{M}, \mathcal{M}'$: not accumulate but in general $S'$ : not tame.

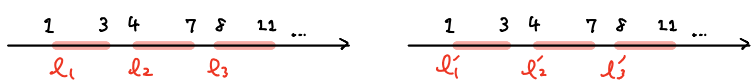
We define regluings $(S, (\mathcal{M}, \mathcal{M}'))$ in the same way as $(S, (m, m'))$.

Convention An oriented flat plane is Euclid. plane w/ a choice of a fixed direction, the vertical direction.

Lemma 3.7 There exists a Loch Ness monster $S$ s.t. $\Gamma(S) = P$.
3.8 $P'$

pf) Let $A, A'$ : ori. fl. plane ($\simeq \mathbb{R}^2$)

$A$ | $\ell_n$ : $(4n+1, 4n+3)$-segment ($n \in \mathbb{N}$). $\mathbb{Z}$ | $A'$




⇝ Let $\hat{A} \xleftarrow{\text{reg.}} (A \sqcup A', (c, c'))$

$P \curvearrowright A, A'$ w/ $\forall \ell_n, \ell'_n$ : fixed. $\overset{\text{carry over}}{⇝}$ $\hat{A}$.

thus we see $P \subset \Gamma(\hat{A})$. ⇝ $\Gamma(\hat{A})$ : uncble.

by Lemma 3.3. $\Gamma(\hat{A}) = P$, $P'$, or $GL_2^+\mathbb{R}$.

however any translations of $-\mathrm{Id}$ cannot preserve $\Sigma$. $\to \Gamma(\hat{A}) = P$ "

it can be easily seen that $-\mathrm{Id} : \hat{A} \circlearrowleft$ and $\Gamma(\hat{A}) \neq GL_2^+\mathbb{R} \to \Gamma(\hat{A}) = P'$ "

one can show that $\hat{A}$ has just 1 end. See Lemma 4.3 for details. $\boxtimes$

# 4. Countable Veech groups

The main part of this section is devoted to the proof of Theorem 1.2 in the case where the group $G \subset \mathbf{GL}_+(2,\mathbb{R})$ is countable. In other words, we prove the following.

Proposition 4.1. — *For any countable subgroup $G$ of $\mathbf{GL}_+(2,\mathbb{R})$ disjoint from $\mathcal{U} = \{g \in \mathbf{GL}_+(2,\mathbb{R}) : \|g\| < 1\}$ there exists a tame flat surface $S = S(G)$, which is a Loch Ness monster, with Veech group $G$.*

for the pf we need this,
but in fact this can be assumed
for countable Veech groups!

maybe $S$ in the construction?

$D : \mathrm{Aff}_+ S \longrightarrow G$ : isomorphic. (see pf of Lemma 4.13.)

In fact the group $\mathrm{Aff}_+(S)$ will map isomorphically onto $G$ under the differential map. This means that the group $G$ will act on $S$ via affine homeomorphisms with appropriate differentials. Here we adopt the convention that an action of a group $G$ on a set $X$ is a mapping $(g,x) \to g \cdot x$ satisfying $(gh)\cdot x = g\cdot(h\cdot x)$ and $\mathrm{Id}\cdot x = x$. ※ $G = \Gamma(S)$

We begin with an outline of the proof of Proposition 4.1. We make use of the fact that any group $G$ acts on its Cayley graph $\Gamma$. We turn $\Gamma$ equivariantly into a flat surface. With each vertex $g$ of $\Gamma$ we associate a flat surface $V_g$ which can be cut into a flat plane $A_g$ and a decorated surface $\widetilde{L}'_g$, whose role is explained later.

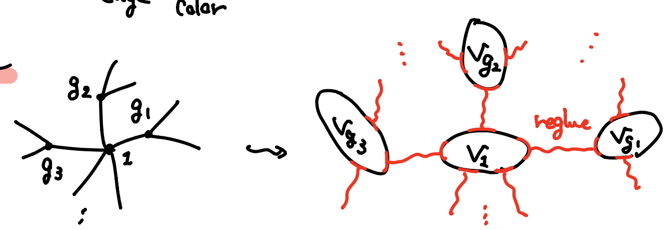
Cayley Graph of a grp. $G$

is : $\Gamma = \Gamma(G, \{a_i\}_{i \in I}) = (\underbrace{V, E}_{\text{graph}}, \underset{\uparrow}{C})$ coloring

defined by
$$\begin{cases} \cdot\ V = G \\ \cdot\ C = \{c_i\}_{i\in I} \\ \cdot\ E = \{(\underbrace{g, ga_i}_{\text{edge}} : \underset{\uparrow\ \text{color}}{c_i}) \mid g \in G, i \in I\} \end{cases}$$

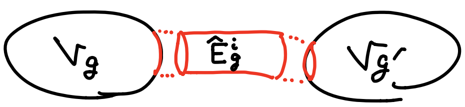
$G \curvearrowright \Gamma$ by $\forall g, h \in G = V(\Gamma)$,
$$\begin{cases} h \cdot g_{\in V} := hg_{\in V} \\ h \cdot (g, ga_i) := (hg, hga_i) \end{cases}$$

main idea




※ $V_g = g \cdot V_1$ : affine deformation

- To obtain tameness, we use "buffer surface" in regluing so that new singularities do not accumurate.

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- We keep the one-endedness in the following way:

  1. Let each $V_g . \hat{E}_g^i$ : one-end.
  2. we glue them so that their ends merge into one end.

- We use "decorated surface" $\tilde{L}_g$ to prevent $\Gamma(\cdot)$ from being richer than $G$.
  To activate this, $\tilde{L}_g$ is decorated w/ special sings.

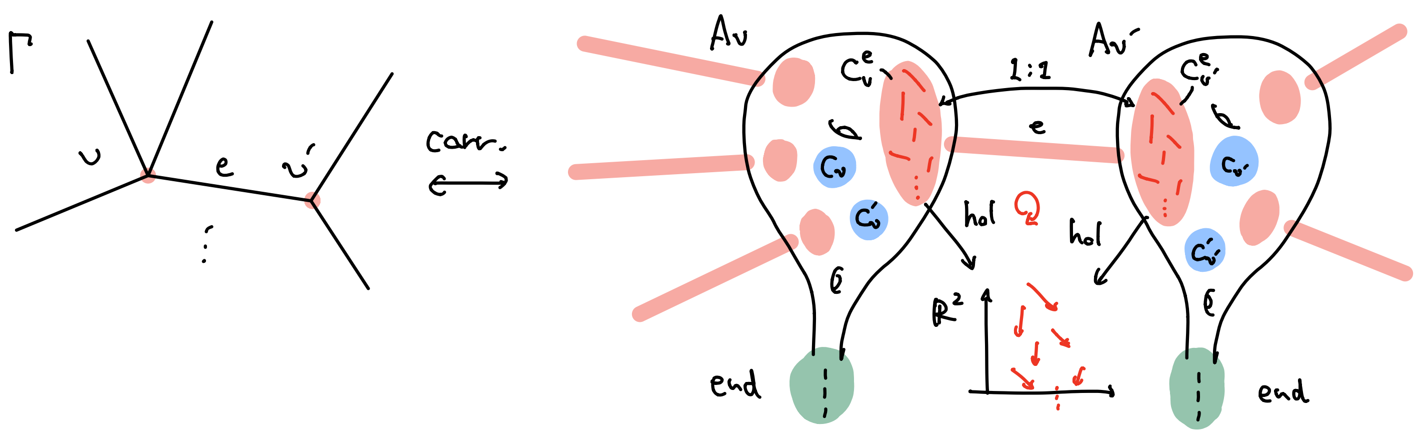
First, we begin by explaining how to obtain a nice action $GL_2^+\mathbb{R} \curvearrowright \bigsqcup_{g\in G} V_g$.

Def 4.2 $\quad S_{Id}$ : tame flat surf.
for each $g \in GL_2^+\mathbb{R}$, Let $S_g$ : affine copy, whose atlas is deformed w/ $g$ (post composing, affine action.)
Now we have canonical affine homeo. $\bar{g} : S_{Id} \to S_g$ which is top. id.
Moreover $GL_2^+\mathbb{R} \curvearrowright \bigsqcup_{g' \in GL_2^+\mathbb{R}} S_{g'}$ s.t. each $g \in GL_2^+\mathbb{R}$ acts as $S_{g'} \xrightarrow{\text{can.}} S_{g'g}$ w/ diff. $g$.

@ 1-endedness

- Let $\Gamma$ : conn. graph. $A := \bigcup_{v\in\Gamma^{(0)}} A_v$ : union along $\Gamma$.
  where $A_v$ : 1-ended tame fl. surf. w/ $\Sigma_{inf}(A_v) = \phi$.
- assume that each $A_v$ is equipped w/ $\left\{ \begin{array}{l} C_v^e : \text{inf. family of marks} \quad (e : \text{edge issuing from } v.) \\ \left.\begin{array}{l} C_v \\ C_v' \end{array}\right\} \text{possibly finite families of marks, w/ same cardinary.} \end{array} \right\}$ disjoint, no accumuration
  $\rightsquigarrow \Gamma^{(1)}$ : countable (we assume?)

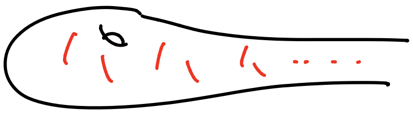
- Moreover, assume that for $e = (v\ v') \in \Gamma^{(1)}$, $C_v^e = \{m_n\}$, $C_{v'}^e = \{m_n'\}$, we have $hol(m_n) = hol(m_n')$



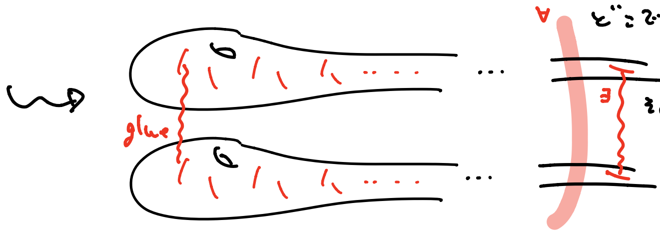

LEMMA 4.3. — *Let $S$ be the surface obtained from $A$ by regluing along $\mathcal{C}_v^e$ and $\mathcal{C}_{v'}^e$, for all edges $e = (v, v')$ in $\Gamma^{(1)}$, and along $\mathcal{C}_v$ and $\mathcal{C}_v'$, for all vertices $v$ in $\Gamma^{(0)}$. Then $S$ is 1-ended. If $\Gamma$ has an edge or if it has only one vertex $v$ but with infinite $\mathcal{C}_v$ (or if $A_v$ has infinite genus), then $S$ has infinite genus.*

**Note** "inf. family of marks w/ no accumulation" を 1-ended surf. で とってるから, その marks は

イメージ



← end の方に流れて分布している.




どこで切っても

その先に mark があって

はり合ってるから. complement は conn.

→ 1-endedness は保存される

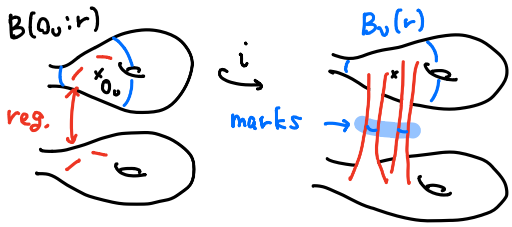
pf) $^\forall v \in \Gamma^{(0)}$, choose $O_v \in A_v$ : base pt.

$B_v(r)$ : closure of $\qquad (B(O_v:r) = \{x \in A_v \mid d(x, O_v) < r\})$

[ $i(B(O_v:r))$ in $S$ w/ $^\forall$marks removed. ]

We take a seq. of all vertices $(v_j)_{j=1}^{\infty}$

For $\ell \geq 1$. $K_\ell := \bigcup_{j=1}^{\ell} B_{v_j}(\ell)$




**Claim** $^\forall \ell \geq 1$. $S \setminus K_\ell$ : connected

Since $\left\{ \begin{array}{l} A_v : \text{complete non-positively curved, 1-ended,} \\ \text{balls and marks we consider : convex, and} \\ \quad \text{marks are disjoint} \end{array} \right.$

$\Rightarrow$ $A'_{v_j} := A_{v_j} \setminus \left( B_{v_j}(\ell) \cup_e C^e_{v_j} \cup C_{O_j} \cup C'_{O_j} \right)$ is connected for $^\forall j \geq 1$.

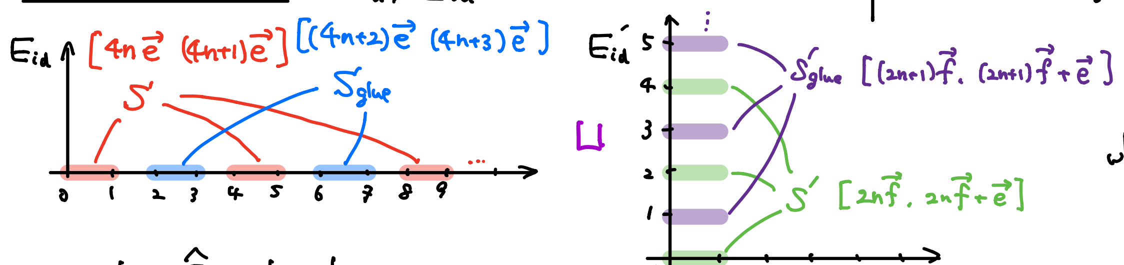
for sufficiently large $\ell \geq 1$.

必要では?

since $\bigcup A'_j$ is dense in $S$, $S \setminus K_\ell$ is conn. for $^\forall \ell \geq 1$.

$\rightsquigarrow$ $S$ : 1-ended.

**Construction 4.4** $E_{id}, E'_{id}$ : 2 oriented flat planes $\sim \mathbb{R}^2$ (equipped w/ origins)




where $\vec{e} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{f} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\rightarrow$ Let $\hat{E}_{id}$ be the

tame flat surf. obtained from $(E_{id} \sqcup E'_{id}, (S_{glue}, S'_{glue}))$.

w/ marks $S, S'$.

We call $\hat{E}_{id}$ the buffer surface.

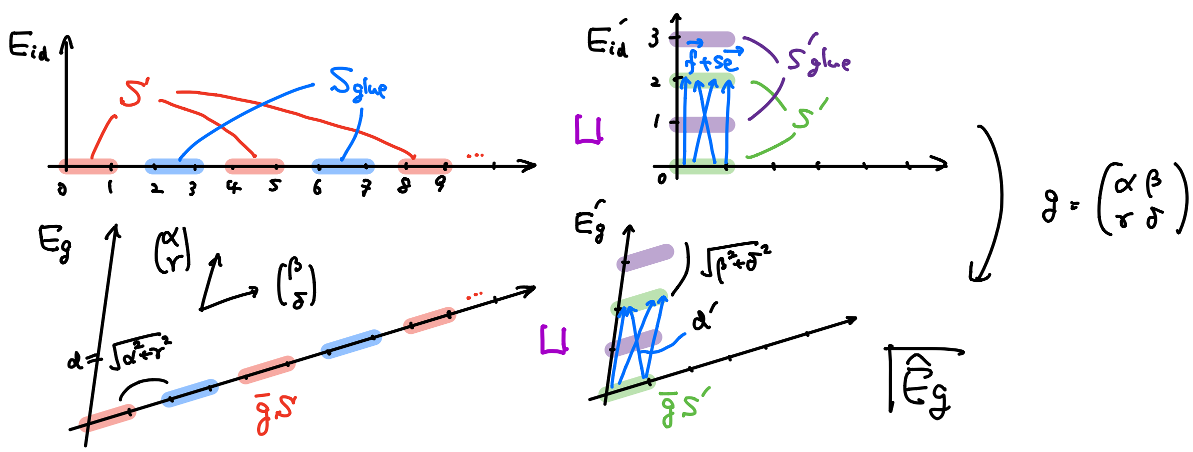
we will make correspondence $\hat{E}_{id} \longleftrightarrow e \in \Gamma^{(1)}$.

**Lemma 4.5** $^\forall g \in GL_2^+\mathbb{R} \setminus \mathcal{U}$ $\quad (\mathcal{U} = \{\|g\| < 1\})$

the distance $\hat{d}$ : in $\hat{E}_g$, between $gS$ and $gS'$ is at least $1/\sqrt{2}$.

*Proof.* — Denote by $\hat{d}$ the distance in $\widehat{E}_g$ between $\overline{g}\mathcal{S}$ and $\overline{g}\mathcal{S}'$. Let $d$ be the distance in $E_g$ between $\overline{g}\mathcal{S}$ and $\overline{g}\mathcal{S}_{\text{glue}}$ and let $d'$ be the distance in $E'_g$ between $\overline{g}\mathcal{S}'_{\text{glue}}$ and $\overline{g}\mathcal{S}'$. Then we have $\hat{d} \geqslant d + d'$. Moreover, $d = |g(\vec{\mathbf{e}})|$ and

$$d' = \min_{|s|\leqslant 1} |g(\vec{\mathbf{f}} + s\vec{\mathbf{e}})|.$$




Let $s \in [-1, 1]$ be such that the minimum is attained, that is $d' = |g(\vec{\mathbf{f}}+s\vec{\mathbf{e}})|$. If $d + d' < \frac{1}{\sqrt{2}}$, then

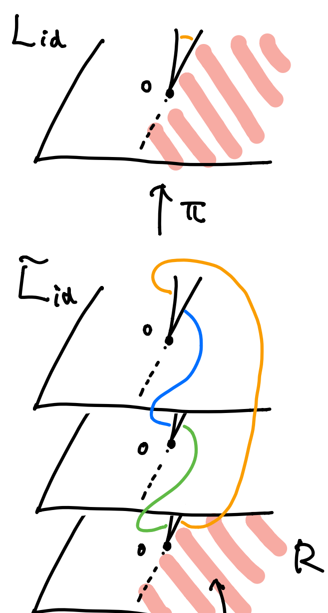
$$|g(\vec{\mathbf{f}})| \leqslant |g(\vec{\mathbf{f}} + s\vec{\mathbf{e}})| + |s||g(\vec{\mathbf{e}})| < \frac{1}{\sqrt{2}}.$$

Hence for any $v = x\vec{\mathbf{e}} + y\vec{\mathbf{f}} \in \mathbb{R}^2$ we have

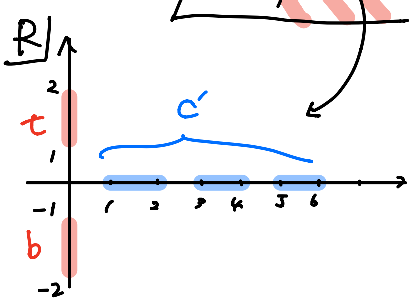
$$|g(v)| \leqslant |x||g(\vec{\mathbf{e}})| + |y||g(\vec{\mathbf{f}})| < \frac{1}{\sqrt{2}}(|x| + |y|) \leqslant \sqrt{x^2 + y^2} = |v|.$$

Thus $\|g\| < 1$. Contradiction. □

Q decoration
↳ to obtain the rigidity of $P(S)$.




Construction 4.6 Let $L_{id}$ : ori. flat plane w/ an origin $O$.
$\tilde{L}_{id}$ : 3-fold cyclic branched cov. br. over $O$.
denote the projection by $\pi : \tilde{L}_{id} \to L_{id}$.
the closure of one component of $\pi^{-1}(\mathbb{R}^2_{x>0})$ by $R$.
Let $\tilde{L}'_{id}$ the same flat surf. obtained from $(\tilde{L}_{id}, (t, b))$
→ We call $\tilde{L}'_{id}$ the decorated surface.
(equipped w/ marks $C'$, w/ vector $\vec{e}$.)
Note that $O \in \tilde{L}'_{id}$ is a $6\pi$-angle sing.

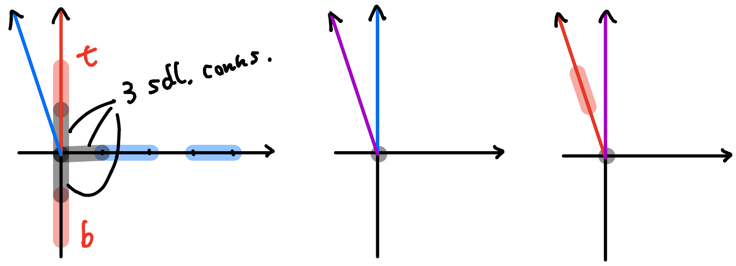


**Rem 4.8** Let $S$ be a tame flat surf. w/ no-accumulating family $C$ of marks w/ vector. $\vec{e}$.

Let $S'$ be obtained from $(S \sqcup \tilde{L}'_{id}, (C, C'))$

⇝ there are only 3 saddle connections issuing from $0$.

$\tilde{L}'_{id}$



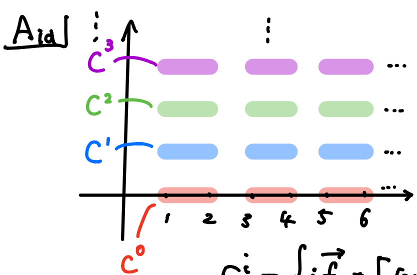

interior pts of these sdle conns. are contained in $\mathbb{R} \setminus (t \cup b \cup C')$.

and their holonomy vectors are $-\vec{f}$, $\vec{e}$, and $\vec{f}$.

hence angles between them are $\frac{\pi}{2}$, $\frac{\pi}{2}$, and $5\pi$.

**Construction 4.9** Let $G = \langle a_i \mid i \in \mathbb{N} \rangle < GL_2^+\mathbb{R} \setminus \mathcal{U}$. (if $G = \{id\}$, let $a_1 = id$.)
(possibly finite, and we can count even all elements in $G$.)

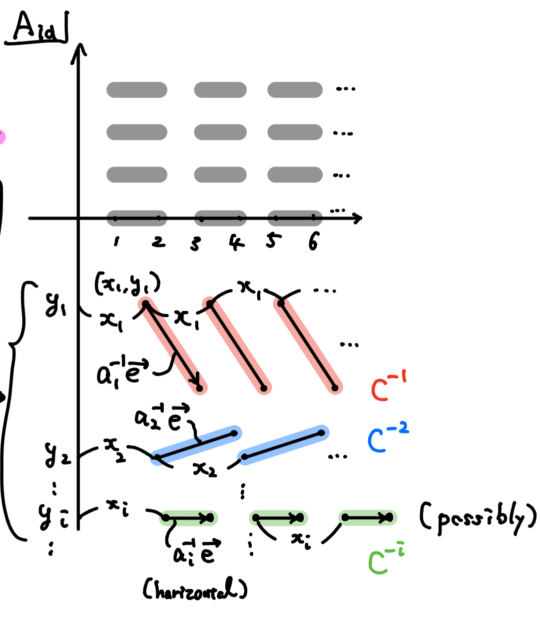
1. Let $A_{id}$ : an oriented flat plane w/ origin $0$.
$A := \bigsqcup_{g \in G} A_g$. (We use AC.)




For $i \geq 1$ (take inductively), $x_i, -y_i > 0$ : sufficiently large s.t. if we take $C^{-i}$ as, they are (domestically / pairwise) disjoint.

$$C^i = \{ i\vec{f} + [(2n-1)\vec{e}, 2n\vec{e}] \mid n \geq 1 \}$$

$$C^{-i} = \{ iy_i \vec{f} + nx_i\vec{e} + [0, a_i^{-1}\vec{e}] \mid n \geq 1 \}$$




2. Let $\tilde{L}_{id}$ : the decorated surface

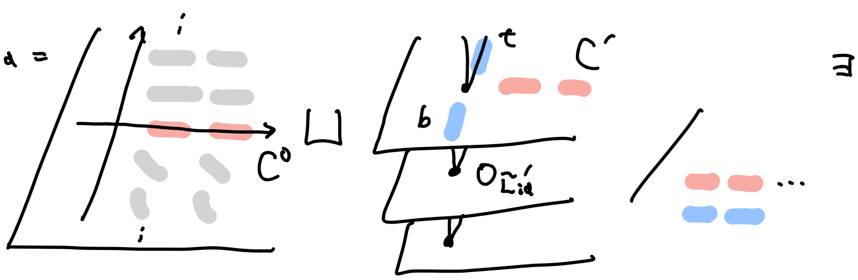
$\tilde{L}' := \bigsqcup_{g \in G} \tilde{L}_g$.

$V_g$ : the tame flat surf. obtained from $(A_g \sqcup \tilde{L}'_g, (\bar{g}C^0, \bar{g}C'))$.

← it is allowed since all of these marks have common hol. vect. $g\vec{e}$.

$V := \bigsqcup_{g \in G} V_g$.

$V_{id} =$




$\exists \curvearrowleft G$ as $\bar{g}(V_{g'}) = V_{gg'}$ $\forall g, g' \in G$.

We keep the notation $C^i$, $i \in \mathbb{Z}^*$ for the families of marks inherited from ones of $A_g$.

3. for each $i \in \mathbb{N}$ take $\widehat{E}^i_{id}$ as a copy of the buffer surface $\widehat{E}_{id}$.

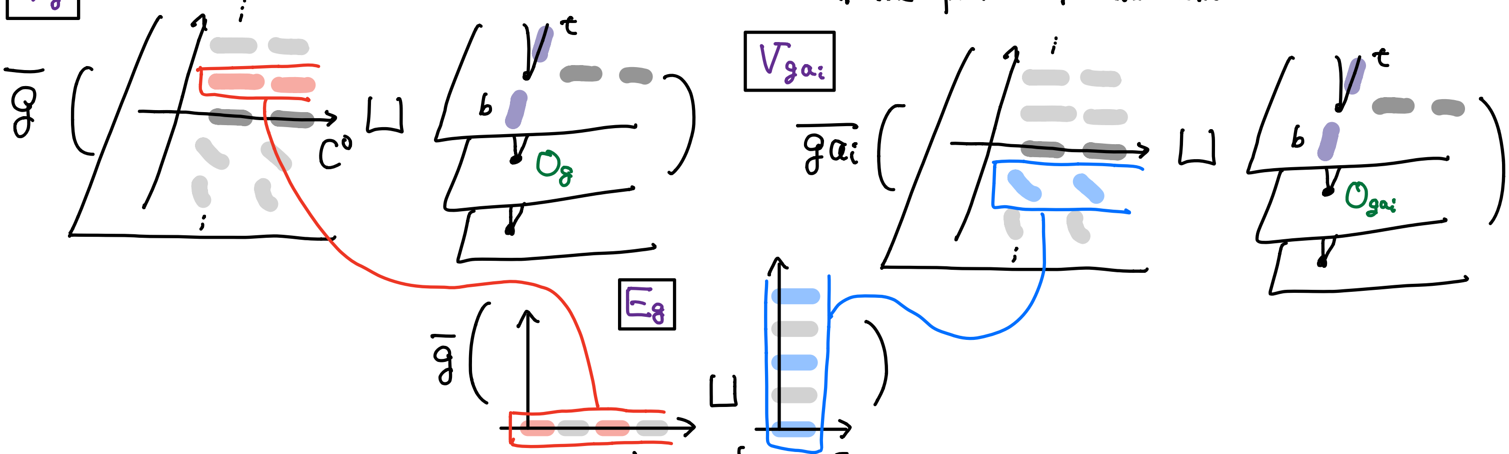
we denote the copies of $S, S'$ in $\widehat{E}^i_{id}$ by $S^i, S'^i$.

Let $E := \bigsqcup_{g \in G}^{i \in \mathbb{N}} \widehat{E}^i_g$

$S = S(G)$ be the flat surf. obtained from $\left(V \cup E, \left(\begin{array}{cc} \{\bar{g}C^i\}_{g\in G} & \{\bar{g}S^i\}_{g\in G} \\ \{\bar{g}S'^i\}_{g\in G}, & \{\overline{ga_i}C^{-i}\}_{g\in G} \end{array}\right)\right)$

← Note that $\forall g \in G$ the marks in ↗ have common vector $g\vec{e}$.

Finally, the action of $G$ carries over to $S$, and we keep the notations for this action.




Rem 4.10 $\Gamma_{6\pi}(S) = \{ O_{\sqcup_g} = \bar{g}(O) \mid g \in G \}$. $(O = O_{\sqcup_{id}})$

they are pairwise different. ( $O$ is disjoint from all marks in $C'$.)

Lemma 4.11. — *$S$ is a Loch Ness Monster.*

*Proof.* — This follows from Lemma 4.3 applied to the graph $\Gamma'$ obtained from the Cayley graph $\Gamma$ of $G = \langle a_i \rangle_{i \geqslant 1}$. We get $\Gamma'$ from $\Gamma$ by subdividing each edge of $\Gamma$ into three parts and by adding for each original vertex $v$ of $\Gamma$ an additional vertex $v'$ and an edge joining $v'$ to $v$. ☐

Lemma 4.12. — *$S$ is a tame flat surface.*

*Proof.* — Let $\bar{V}_g$, respectively $\bar{E}^i_g$, denote the closures in $S$ of the subsets inherited from $V_g \smallsetminus \bar{g}(\bigcup_{i \neq 0} \mathcal{C}^i)$, respectively $\widehat{E}^i_g \smallsetminus \bar{g}(\mathcal{S}^i \cup \mathcal{S}'^i)$.

It is enough to prove that $S$ is complete. Let $(x_k)$ be a Cauchy sequence on $S$. By Lemma 4.5 we may assume that there is some $g \in G$ such that all $x_k$ lie in the union of $\bar{V}_g$ and the adjacent affine buffer surfaces $\bar{E}^i_g$ and $\bar{E}^i_{ga_i^{-1}}$. Since the components of $\bar{V}_g \cap \left(\bigcup_i (\bar{E}^i_g \cup \bar{E}^i_{ga_i^{-1}})\right)$ form a discrete subset in $\bar{V}_g$, we may assume that all $x_k$ lie in $\bar{V}_g$ and in a single adjacent buffer surface. Since both $\bar{V}_g$ and the buffer surface are complete, $(x_k)$ converges, as required. ☐

Lemma 4.13. — *Any orientation preserving affine homeomorphism of $S$ is equal to $\bar{g}$ for some $g \in G$.* ← KEY LEMMA

pf) take $\forall \psi \in \mathrm{Aff}^+(S)$.
since $\psi$ preserve $\Sigma_{6\pi}(S) = \{\bar{g}(0) \mid g \in G\}$, there exists $^\exists g \in G$ s.t. $\psi(0) = \bar{g}(0)$.

claim $\psi = \bar{g}$.
∵) Let $\varphi := \bar{g}^{-1} \circ \psi$. Now we only know $\varphi(0) = 0$.
as in Rem 4.8, there are just 3 sdl. conns. issuing from 0
and they cross by angles $\frac{\pi}{2}$, $\frac{\pi}{2}$, and $5\pi$.

so $\varphi$ should preserve these three sdl. conns..
$\Rightarrow \varphi = id$ near 0,
$\Rightarrow \varphi = id$ globally. $\boxtimes$

※ $4\pi$-angle singularities occur in regluing and hence we need to use $6\pi$-angle ones for decolation.
→ for half-translation cases, we may use simple poles for decolation!

Note that by construction $\Gamma(S) > G$.
So Lem 4.13 implies that $\Gamma(S) = G$.

With Lem 4.11 and 4.12, we complete the pf of Prop 4.1 //.

Conversely, we have the following.

LEMMA 4.15. — *If the Veech group $G$ of a flat surface $S$ is countable, then $G$ is disjoint from $\mathcal{U}$.*

*Proof.* — First consider the case, where $S$ has a singularity $x$. Recall that $\widehat{S}$ denotes the metric completion of $S$ and that the action of the group of orientation preserving affine homeomorphisms of $S$ extends to an action on $\widehat{S}$. Suppose that there is an orientation preserving affine homeomorphism $\varphi$ of $S$ with $D\varphi \in \mathcal{U}$. Then $\varphi$ extends to a contraction on $\widehat{S}$. By the Banach fixed point theorem, the sequence $\varphi^k(x)$ converges in $\widehat{S}$. If $x$ is not the fixed point of $\varphi$, then this contradicts tameness.

Assume now that $x$ is the fixed point of $\varphi$ and the only singularity of $S$. Then $S$ is simply connected. Otherwise by iterating under $\varphi$ a homotopically nontrivial loop going through $x$ we obtain arbitrarily short homotopically nontrivial loops through $x$, which contradicts tameness. Hence $S$ is a cyclic branched covering of $\mathbb{R}^2$ and thus $G = \mathbf{GL}_+(2, \mathbb{R})$ which is not countable, contradiction.

If $S$ does not have singularities, its universal cover is the flat plane. Since $G$ is countable, $S$ must be a flat torus and we have $G \subset \mathbf{SL}(2, \mathbb{R})$ which is disjoint from $\mathcal{U}$. □

This proves Theorem 1.1 in the case where $G$ is countable.