Voronoi decompositions of flat surfaces and origamis

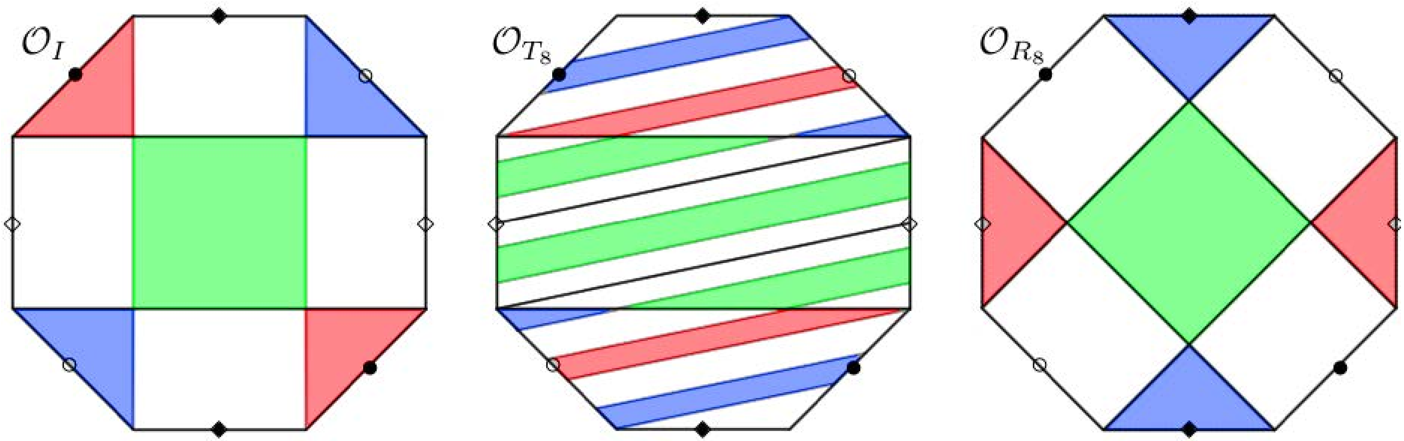
熊谷 駿 (東北大学 (情報) →秋∼ 九州大学 IMI)

リーマン面に関連する位相幾何学 (2023/8/23, 東京大学)

1 Introduction/Background

Main theme: calculation of Veech groups and its calculation process

The **Veech group** of a flat surface is the group of its *affine self-similarity*.



Strategy:

- ▷ tiling-based (parallelogram decomposition, cut-and-paste construction; forklore result / K. [9] '23)
- ▷ covering-based (monodromy, $\mathrm{Aut}(\pi_1(\check{R}^*))$-action; Schmithüsen '04, Freidinger '08 Shinomiya '12)
- ▷ singularity-based (Delaunay/Voronoi; Bowman '10, Mukarel '17, Edwards-Sanderson-Schmidt [4] '22)

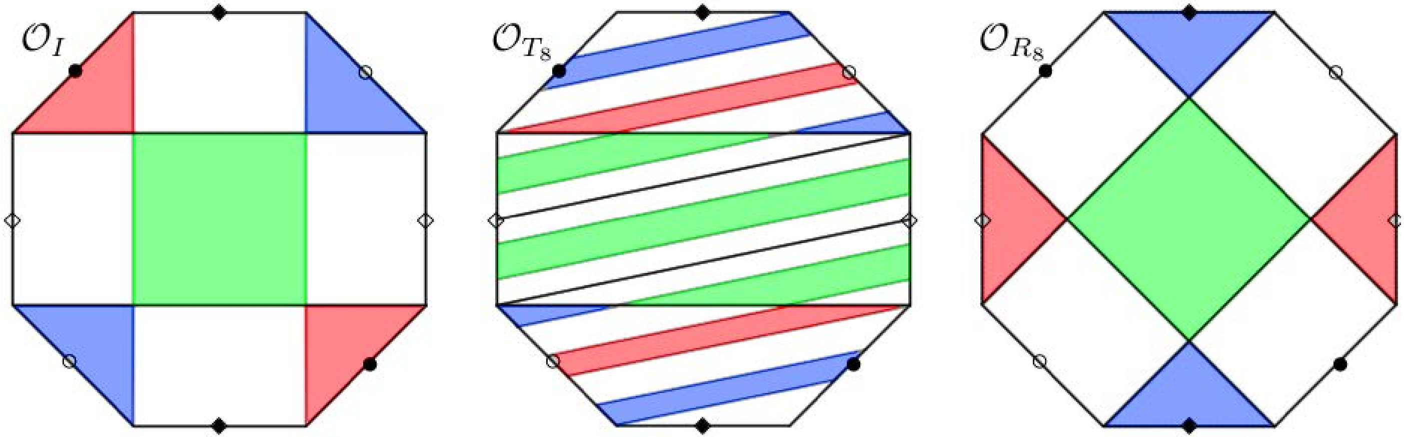
Main result: for **origamis (square-tiled surfaces)**

- ▷ Relationship among *covering, singularity, and modification of tiling*
- ▷ Categorical description of singularity-based method

1 Introduction/Background

Main theme: calculation of Veech groups and its calculation process

The **Veech group** of a flat surface is the group of its *affine self-similarity*.



Strategy:

▷ tiling-based (parallelogram decomposition, cut-and-paste construction; forklore result / K. [9] '23)

▷ covering-based (monodromy, $\mathrm{Aut}(\pi_1(\check{R}^*))$-action; Schmithüsen '04, Freidinger '08 Shinomiya '12)

▷ singularity-based (Delaunay/Voronoi; Bowman '10, Mukarel '17, Edwards-Sanderson-Schmidt [4] '22)

Main result: for **origamis (square-tiled surfaces)**

▷ Relationship among *covering, singularity, and modification of tiling*

▷ Categorical description of singularity-based method

1.1 Teichmüller space

Let $R_\bullet$ be a Riemann surface of type (g, n) with $2g-2+n>0$. (This kind of surface is mainly considered in this talk.)

Definition 1 An orientation-preserving homeomorphism $f : R_0 \to R$ is called a **marked Riemann surface** over R_0. Two marked Riemann surfaces f_1, f_2 are called **Teichmüller equivalent** $f_1 \sim f_2$ if there exists a biholomorphism $h : R_1 \to R_2$ homotopic to the map $f_2 \circ f_1^{-1} : R_1 \to R_2$.

$$\begin{tikzcd}
& R_1 \arrow[dd, "f_2\circ f_1^{-1}\sim"] \arrow[dd, bend left=80, "\exists h:\text{biholo.}"] \\
R_0 \arrow[ur, "f_1"] \arrow[dr, "f_2"'] \arrow[r, phantom, "\circlearrowleft"] & {} \\
& R_2
\end{tikzcd}$$

For each base point R_0 of type (g, n), the **Teichmüller space** $T(R_0) = \{f : R_0 \to R : \text{ marked Riemann surface}\}/_\sim$ is known to be a simply-connected complex manifold of dimension $(3g-3+n)$ (Ahlfors-Bers). The **mapping class group** $MCG(R_0) = \{\gamma : R_0 \to R_0 : \text{orientation preserving homeomorphism}\}/_{\text{homotopy}}$ acts discontinously on $T(R_0)$ by the pullback

$$[\gamma]^* : [f] \mapsto [f \circ \gamma^{-1}],\ [\gamma] \in MCG(R_0),\ [f] \in T(R_0).$$

The **moduli space** $M(R_0) = \{\text{Riemann surface homeomorphic to } R_0\}/_{\text{biholo.}}$ is the quotient orbifold $T(R_0)/MCG(R_0)$ of dimension $(3g-3+n)$.

1.2 Flat surface

Definition 2 A **flat surface** is a G-manifold (i.e. surface with G-atlas) where $G = \mathrm{HTrans}(\mathbb{C}) := \{z \mapsto \pm z + c \mid c \in \mathbb{C}\}$ is the group of **half-translations** on the plane. A flat surface is called **abelian** if it admits a $\{z \mapsto z + c \mid c \in \mathbb{C}\}$-subatlas.

For a Riemann surface R and a holomorphic quadratic differential $\phi = \phi(z)dz^2$ on R, we have the **ϕ-coordinates** arround a point $p_0 \in R^* := R \setminus \mathrm{Zero}(\phi)$ locally defined by

$$p \mapsto \pm z_\phi(p) = \pm \int_{p_0}^{p} \sqrt{\phi(z)}dz.$$

The ϕ-coordinates form a $\mathrm{HTrans}(\mathbb{C})$-atlas on R^*. We call it a flat surface (R, ϕ).

Consider the flat metric on (R^*, ϕ) given by pullback of the Euclidian metric via the ϕ-coordinates. From now on, we assume the finite area $\|\phi\| := \int_R |\phi| < \infty$, and then the $\mathrm{HTrans}(\mathbb{C})$-atlas uniquely extends to the completion $\bar{R}$ in such a way that a transition map arround $p \in \mathrm{Sing}(R, \phi) := \mathrm{Zero}(\phi) \cup \partial R$ is of the form $z \mapsto \frac{2}{k+2} z^{\frac{k+2}{2}}$, $k = \mathrm{ord}_p(\phi) \geq -1$. In particular, p is a cone point of angle $(k+2)\pi$.

Definition 3 An **affine deformation** is an orientation-preserving homeomorphism $F : (R, \phi) \to (S, \psi)$ that is locally affine, i.e. for some $a, b, c, d, e, f \in \mathbb{R}$ a local representation is of the form

$$z_\psi \circ F \circ z_\phi^{-1}(x + iy) = (ax + cy + e) + i(bx + dy + f), \ \forall x + iy \in \mathrm{Im} z_\phi.$$

The **derivative** $D_F = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in PGL_2(\mathbb{R})$ is a $\mathrm{HTrans}(\mathbb{C})$-invariant such that $\|\psi\| = |D_F|\|\phi\|$. So we have $D_F \in PSL_2(\mathbb{R})$ for a self-deformation $F : (R, \phi) \to (R, \phi)$.

1.2 Flat surface

Definition 2 A **flat surface** is a G-manifold (i.e. surface with G-atlas) where $G = \mathrm{HTrans}(\mathbb{C}) := \{z \mapsto \pm z + c \mid c \in \mathbb{C}\}$ is the group of **half-translations** on the plane. A flat surface is called **abelian** if it admits a $\{z \mapsto z + c \mid c \in \mathbb{C}\}$-subatlas.

For a Riemann surface R and a holomorphic quadratic differential $\phi = \phi(z)dz^2$ on R, we have the **ϕ-coordinates** arround a point $p_0 \in R^* := R \setminus \mathrm{Zero}(\phi)$ locally defined by

$$p \mapsto \pm z_\phi(p) = \pm \int_{p_0}^{p} \sqrt{\phi(z)}dz.$$

The ϕ-coordinates form a HTrans($\mathbb{C}$)-atlas on R^*. We call it a flat surface (R, ϕ).

Consider the flat metric on (R^*, ϕ) given by pullback of the Euclidian metric via the ϕ-coordinates. From now on, we assume the finite area $\|\phi\| := \int_R |\phi| < \infty$, and then the HTrans($\mathbb{C}$)-atlas uniquely extends to the completion $\bar{R}$ in such a way that a transition map arround $p \in \mathrm{Sing}(R, \phi) := \mathrm{Zero}(\phi) \cup \partial R$ is of the form $z \mapsto \frac{2}{k+2} z^{\frac{k+2}{2}}$, $k = \mathrm{ord}_p(\phi) \geq -1$. In particular, p is a cone point of angle $(k + 2)\pi$.

Definition 3 An **affine deformation** is an orientation-preserving homeomorphism $F : (R, \phi) \to (S, \psi)$ that is locally affine, i.e. for some $a, b, c, d, e, f \in \mathbb{R}$ a local representation is of the form

$$z_\psi \circ F \circ z_\phi^{-1}(x + iy) = (ax + cy + e) + i(bx + dy + f), \ \forall x + iy \in \mathrm{Im} z_\phi.$$

The **derivative** $D_F = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in PGL_2(\mathbb{R})$ is a HTrans($\mathbb{C}$)-invariant such that $\|\psi\| = |D_F|\|\phi\|$. So we have $D_F \in PSL_2(\mathbb{R})$ for a self-deformation $F : (R, \phi) \to (R, \phi)$.

We say that a trivial ($D_F = I$) affine deformation is a **half-translation (HTrans)**.

Proposition 4 (Teichmüller [1]) For any $[f : R \to S] \in T(R)$, there exist $0 \le k < 1$, $\phi \in QD(R)$, $\psi \in QD(S)$, and a representative $F = F_{[f]} \in [f]$ of Teichmüller equivalence class such that

1. $F : (R, \phi) \to (S, \psi)$ is an affine deformation of the form $z_\psi \circ F \circ z_\phi^{-1}(z) = \dfrac{z + k\bar{z}}{1 - k}$,
2. F attains the minimum of $K(h) := \dfrac{1 + \|h_{\bar{z}}/h_z\|_\infty}{1 - \|h_{\bar{z}}/h_z\|_\infty}$, $h \in [f]$: weakly x, y-differentiable (ACL), where $K(F) = \dfrac{1+k}{1-k}$.

Furthermore, an extremal deformation F in the sense of 2. is unique up to half-translations.

The complete **Teichmüller distance** on $T(R)$ is defined by $d_T([f_1], [f_2]) := \frac{1}{2} \log \inf\{K(h) \mid h \in [f_2 \circ f_1^{-1}] : \mathrm{ACL}\}$ for $[f_1], [f_2] \in T(R)$. For each fixed flat surface (R, ϕ), the following **Teichmüller embedding** ι_ϕ is an isometric embedding by Proposition 4.

$$\iota_\phi : \mathbb{H} \to T(R) : t \mapsto [f_t] \ s.t. \ f_t^* \phi = \mathrm{Re}(\phi) + t\mathrm{Im}(\phi)$$

Proposition 5 (Earle-Gardiner [3]) A mapping class $[\gamma] \in MCG(R)$ satisfies that $[\gamma]^*(\iota_\phi(\mathbb{H})) \cap \iota_\phi(\mathbb{H}) \neq \emptyset$ if and only if $F_{[\gamma]}$ is an affine self-deformation of (R, ϕ). Furthermore, such a $[\gamma]$ acts on $\mathbb{H}$ by

$$F_{[\gamma]}^* \iota_\phi(t) = \iota_\phi\left(\frac{-at + b}{ct - d}\right), \ \forall t \in \mathbb{H}, \ D_{F_{[\gamma]}} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in PSL(2, \mathbb{R}).$$

The group $\Gamma(R, \phi) := \{D_F \mid F : (R, \phi) \xrightarrow{\text{affine}} (R, \phi)\} < PSL(2, \mathbb{R})$ is called the **Veech group** of (R, ϕ). We have an embedded disk-uniformized model $\mathbb{L}/\Gamma(R, \phi) \cong \mathrm{proj}(\iota_\phi(\mathbb{H})) \hookrightarrow M(R)$ of the affine deformations of (R, ϕ).

We say that a trivial ($D_F = I$) affine deformation is a **half-translation (HTrans)**.

Proposition 4 (Teichmüller [1]) For any $[f : R \to S] \in T(R)$, there exist $0 \le k < 1$, $\phi \in QD(R)$, $\psi \in QD(S)$, and a representative $F = F_{[f]} \in [f]$ of Teichmüller equivalence class such that

1. $F : (R, \phi) \to (S, \psi)$ is an affine deformation of the form $z_\psi \circ F \circ z_\phi^{-1}(z) = \dfrac{z + k\bar{z}}{1 - k}$,
2. F attains the minimum of $K(h) := \dfrac{1 + \|h_{\bar{z}}/h_z\|_\infty}{1 - \|h_{\bar{z}}/h_z\|_\infty}$, $h \in [f]$: weakly x, y-differentiable (ACL), where $K(F) = \dfrac{1+k}{1-k}$.

Furthermore, an extremal deformation F in the sense of 2. is unique up to half-translations.

The complete **Teichmüller distance** on $T(R)$ is defined by $d_T([f_1], [f_2]) := \frac{1}{2} \log \inf\{K(h) \mid h \in [f_2 \circ f_1^{-1}] : \text{ACL}\}$ for $[f_1], [f_2] \in T(R)$. For each fixed flat surface (R, ϕ), the following **Teichmüller embedding** ι_ϕ is an isometric embedding by Proposition 4.

$$\iota_\phi : \mathbb{H} \to T(R) : t \mapsto [f_t] \ s.t. \ f_t^* \phi = \mathrm{Re}(\phi) + t\mathrm{Im}(\phi)$$

Proposition 5 (Earle-Gardiner [3]) A mapping class $[\gamma] \in MCG(R)$ satisfies that $[\gamma]^*(\iota_\phi(\mathbb{H})) \cap \iota_\phi(\mathbb{H}) \neq \emptyset$ if and only if $F_{[\gamma]}$ is an affine self-deformation of (R, ϕ). Furthermore, such a $[\gamma]$ acts on $\mathbb{H}$ by

$$F_{[\gamma]}^* \iota_\phi(t) = \iota_\phi\left(\frac{-at + b}{ct - d}\right), \ \forall t \in \mathbb{H}, \ D_{F_{[\gamma]}} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in PSL(2, \mathbb{R}).$$

The group $\Gamma(R, \phi) := \{D_F \mid F : (R, \phi) \xrightarrow{\text{affine}} (R, \phi)\} < PSL(2, \mathbb{R})$ is called the **Veech group** of (R, ϕ). We have an embedded disk-uniformized model $\mathbb{L}/\Gamma(R, \phi) \cong \mathrm{proj}(\iota_\phi(\mathbb{H})) \hookrightarrow M(R)$ of the affine deformations of (R, ϕ).

1.3 Category

A composable concept is regarded as an **arrow** connecting two **objects**. The system of compositions of arrows with certain axioms (identity law, associative law) is called a **category**. A **functor** is a correspondence between two categories C_1, C_2

$$F = (F_{\mathcal{O}} : \mathrm{Obj}_{C_1} \to \mathrm{Obj}_{C_2},\ F_{\mathcal{A}} : \mathrm{Arr}_{C_1} \to \mathrm{Arr}_{C_2})$$

$$\text{compatible with} \begin{cases} \text{domains and codomains} & : F_{\mathcal{A}}(o_1 \xrightarrow{f} o_2) = (F_{\mathcal{O}}(o_1) \xrightarrow{F_{\mathcal{A}}(f)} F_{\mathcal{O}}(o_2)), \\ \text{compositions} & : F_{\mathcal{A}}(o_1 \xrightarrow{f_1} o_2 \xrightarrow{f_2} o_3) = (F_{\mathcal{O}}(o_1) \xrightarrow{F_{\mathcal{A}}(f_1)} F_{\mathcal{O}}(o_2) \xrightarrow{F_{\mathcal{A}}(f_2)} F_{\mathcal{O}}(o_3)), \text{ and} \\ \text{the identity} & : F_{\mathcal{A}}(o \xrightarrow{1_o} o) = (F_{\mathcal{O}}(o) \xrightarrow{1_{F_{\mathcal{O}}(o)}} F_{\mathcal{O}}(o)). \end{cases}$$

Example A (a group) A group is a category such that $\mathrm{Obj} = \{\bullet\}$ and all arrows are invertible.

Example B (category of groups) $\mathrm{Obj} = \{G : \text{group}\}$, $\mathrm{Arr} = \{(G_1 \xrightarrow{f} G_2) : \text{group homomorphism}\}$.

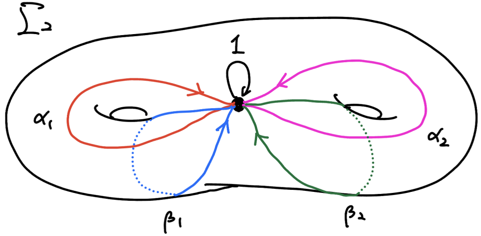
1.3 Category

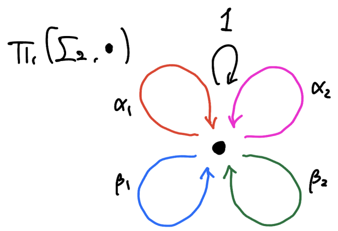
A composable concept is regarded as an **arrow** connecting two **objects**. The system of compositions of arrows with certain axioms (identity law, associative law) is called a **category**. A **functor** is a correspondence between two categories C_1, C_2

$$F = (F_{\mathcal{O}} : \mathrm{Obj}_{C_1} \to \mathrm{Obj}_{C_2},\ F_{\mathcal{A}} : \mathrm{Arr}_{C_1} \to \mathrm{Arr}_{C_2})$$

$$\text{compatible with} \begin{cases} \text{domains and codomains} & : F_{\mathcal{A}}(o_1 \xrightarrow{f} o_2) = (F_{\mathcal{O}}(o_1) \xrightarrow{F_{\mathcal{A}}(f)} F_{\mathcal{O}}(o_2)), \\ \text{compositions} & : F_{\mathcal{A}}(o_1 \xrightarrow{f_1} o_2 \xrightarrow{f_2} o_3) = (F_{\mathcal{O}}(o_1) \xrightarrow{F_{\mathcal{A}}(f_1)} F_{\mathcal{O}}(o_2) \xrightarrow{F_{\mathcal{A}}(f_2)} F_{\mathcal{O}}(o_3)),\ \text{and} \\ \text{the identity} & : F_{\mathcal{A}}(o \xrightarrow{1_o} o) = (F_{\mathcal{O}}(o) \xrightarrow{1_{F_{\mathcal{O}}(o)}} F_{\mathcal{O}}(o)). \end{cases}$$

Example A (a group) A group is a category such that $\mathrm{Obj} = \{\bullet\}$ and all arrows are invertible.





Example B (category of groups) $\mathrm{Obj} = \{G : \text{group}\}$, $\mathrm{Arr} = \{(G_1 \xrightarrow{f} G_2) : \text{group homomorphism}\}$.

Example C (Teichmüller vs Flat) We say that a **2-arrow** connects two arrows (respecting their dom. and cod.).

Teich

$$\begin{tikzcd} R \arrow[loop left, "{[\gamma]}"] \arrow[r, bend left] \arrow[r, "{[f]}" description] \arrow[r, bend right, "{[f\circ\gamma^{-1}]}"'] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow{[\gamma]^*}"] & S \arrow[r] & \cdots \end{tikzcd}
\quad
\begin{cases} \text{obj: Riemann surfaces}/_{\text{biholo.}} \\ \text{arr: ori. pres. homeo.}/_{\text{homotopy}} \\ \text{arr}^2\text{: pullback of MCG} \end{cases}$$

▷ The set of objects associated to R is $M(R)$. We have $\mathrm{Arr}(R,-) = T(R)$ and $\mathrm{Arr}^2([f],-) = \mathrm{Mod}(\mathrm{dom}([f]))$.

Flat

$$\begin{tikzcd} (R,\phi) \arrow[loop above, "F_{[\gamma]}"] \arrow[r, bend left] \arrow[r, "F_{[f]}"'] \arrow[r, bend right] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] \arrow[d, dotted, "\times"] & (S,\psi) \arrow[r] \arrow[d, dotted, "\times"] & \cdots \\ (R,\phi') \arrow[r, "F_{[f\circ\gamma^{-1}]}"'] \arrow[r, phantom, shift left=3, "\Downarrow{[\gamma]^*}"] & (S,\psi') \end{tikzcd}
\quad
\begin{cases} \text{obj: Flat surfaces}/_{\text{HTrans.}} \\ \text{arr: affine deformations} \\ \text{arr}^2\text{: Teichmüller's theorem} \circ \text{pullback of MCG} \end{cases}$$

▷ The **forgetful functor** **Flat**→**Teich** : $(R,\phi) \mapsto R$ embeds all arrows but rotations. For each fixed (R,ϕ), it induces an embedding ι_ϕ of $\mathbb{H} = \{_{\text{HTrans.}}\backslash\text{affine deformations}/_{\text{rotations}}\}$ into $T(R)$.

$\iota_\phi(\mathbb{H})$

$$\begin{tikzcd} (R,\phi) \arrow[loop above, "F_{[\gamma]}"] \arrow[r, bend left] \arrow[r, "f_t" description] \arrow[r, bend right, "f_{D_\gamma^{-1}t}"'] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow{[\gamma]^*}"] & (R_t,\phi_t) \arrow[r] & \cdots \end{tikzcd}
\quad
\begin{cases} \text{obj: Flat surfaces} \\ \text{arr: } _{\text{HTrans.}}\backslash\text{affine deformation}/_{\text{rotation}} \\ \text{arr}^2\text{: pullback of } \mathrm{Arr}((R,\phi),(R,\phi)) \\ \qquad = \text{Möbius transformation of } \Gamma(R,\phi)^* \end{cases}$$

As in the examples above, a category with all arrows invertible is called a **groupoid**.

Example C (Teichmüller vs Flat) We say that a **2-arrow** connects two arrows (respecting their dom. and cod.).

Teich

$$\begin{tikzcd} R \arrow[loop left, "{[\gamma]}"] \arrow[r, "{[f]}"] \arrow[r, bend left, ""{name=U}] \arrow[r, bend right, "{[f\circ\gamma^{-1}]}"'] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow{[\gamma]^*}"] & S \arrow[r] & \cdots \end{tikzcd} \quad \begin{cases} \text{obj: Riemann surfaces}/_{\text{biholo.}} \\ \text{arr: ori. pres. homeo.}/_{\text{homotopy}} \\ \text{arr}^2\text{: pullback of MCG} \end{cases}$$

▷ The set of objects associated to R is $M(R)$. We have $\mathrm{Arr}(R,-) = T(R)$ and $\mathrm{Arr}^2([f],-) = \mathrm{Mod}(\mathrm{dom}([f]))$.

Flat

$$\begin{tikzcd} (R,\phi) \arrow[loop above, "F_{[\gamma]}"] \arrow[r, "F_{[f]}"'] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] \arrow[d, dotted, "\times"] & (S,\psi) \arrow[r] \arrow[d, dotted, "\times"] & \cdots \\ (R,\phi') \arrow[r, "F_{[f\circ\gamma^{-1}]}"'] \arrow[r, phantom, shift left=3, "\Downarrow{[\gamma]^*}"] & (S,\psi') \end{tikzcd} \quad \begin{cases} \text{obj: Flat surfaces}/_{\text{HTrans.}} \\ \text{arr: affine deformations} \\ \text{arr}^2\text{: Teichmüller's theorem} \circ \text{pullback of MCG} \end{cases}$$

▷ The **forgetful functor** **Flat**→**Teich** : $(R,\phi) \mapsto R$ embeds all arrows but rotations. For each fixed (R,ϕ), it induces an embedding ι_ϕ of $\mathbb{H} = \{_{\text{HTrans.}}\backslash\text{affine deformations}/_{\text{rotations}}\}$ into $T(R)$.

$\underline{\iota_\phi(\mathbb{H})}$

$$\begin{tikzcd} (R,\phi) \arrow[loop above, "F_{[\gamma]}"] \arrow[r, "f_t"] \arrow[r, bend left] \arrow[r, bend right, "f_{D_\gamma^{-1}t}"'] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow{[\gamma]^*}"] & (R_t,\phi_t) \arrow[r] & \cdots \end{tikzcd} \quad \begin{cases} \text{obj: Flat surfaces} \\ \text{arr: } _{\text{HTrans.}}\backslash\text{affine deformation}/_{\text{rotation}} \\ \text{arr}^2\text{: pullback of } \mathrm{Arr}((R,\phi),(R,\phi)) \\ \qquad = \text{Möbius transformation of } \Gamma(R,\phi)^* \end{cases}$$

As in the examples above, a category with all arrows invertible is called a **groupoid**.

The GAGA theorem [8] implies **Riemann** $\begin{Bmatrix} \text{obj: cpt. Riemann surfaces} \\ \text{arr: holomorphic coverings} \end{Bmatrix} \sim$ **Curve** $\begin{Bmatrix} \text{obj: nonsing. proj. curves} \\ \text{arr: rational maps} \end{Bmatrix}$.

Arrows of **Teich** (=higher arrows of **Riemann**) are understood as follows.

Proposition A (Benirschke-Serván [15], 2023) In case $2g-2+n \geq 3$, *any* d_T-isometric embedding $T(R) \hookrightarrow T(\hat{R})$ arises from a **covering construction** up to pre-/post- compositions of MCGs. **It** is given by a map

$$h^* : T(R) \to T(\hat{R}) : [R \xrightarrow{f} S] \mapsto [\hat{R} \xrightarrow{\hat{f}} \hat{S}] \ s.t. \ (\hat{S} \xrightarrow{\hat{f}^{-1}} \hat{R} \xrightarrow{h} R \xrightarrow{f} S) : \text{holomorphic}$$

for some holomorphic covering $h : \hat{R} \to R$ of totally marked surfaces ($\partial R \subset \mathrm{Crit}(h)$, $\partial \hat{R} = h^{-1}(\partial R)$), in combination with forgetting marked points $\left(T(S \setminus B) \xrightarrow{h^*} T(\hat{S} \setminus h^{-1}(B))\right) \mapsto \left(T(S) \xrightarrow{h_F^*} T(\hat{S})\right)$.

Proposition B (Haiden-Katzarkov-Kontsevich [17], 2017) The moduli space of marked flat surfaces $M(R, f, \phi)$ is bianalytically embedded into the set of **stability conditions of the Fukaya category** of (R, f, ϕ).

Surface	Triangulated category
mapping class group	group of autoequivalences
simple curve c	spherical object C
intersection number $i(c_1, c_2)$	$\mathrm{dimArr}^*(C_1, C_2)$
flat metric μ	stability condition σ
μ-geodesic	σ-semistable object
$\mathrm{length}_\mu(c)$	$\mathrm{mass}_\sigma(C)$

Table 1 (Heng [16]) stability condition theory : categorical version of Teichmüller theory

The GAGA theorem [8] implies **Riemann** $\left\{\begin{array}{l}\text{obj: cpt. Riemann surfaces}\\ \text{arr: holomorphic coverings}\end{array}\right\} \sim$ **Curve** $\left\{\begin{array}{l}\text{obj: nonsing. proj. curves}\\ \text{arr: rational maps}\end{array}\right\}$.

Arrows of **Teich** (=higher arrows of **Riemann**) are understood as follows.

Proposition A (Benirschke-Serván [15], 2023) In case $2g-2+n \geq 3$, *any* d_T-isometric embedding $T(R) \hookrightarrow T(\hat{R})$ arises from a **covering construction** up to pre-/post- compositions of MCGs. **It** is given by a map

$$h^* : T(R) \to T(\hat{R}) : [R \xrightarrow{f} S] \mapsto [\hat{R} \xrightarrow{\hat{f}} \hat{S}] \ s.t.\ (\hat{S} \xrightarrow{\hat{f}^{-1}} \hat{R} \xrightarrow{h} R \xrightarrow{f} S) : \text{holomorphic}$$

for some holomorphic covering $h : \hat{R} \to R$ of totally marked surfaces ($\partial R \subset \mathrm{Crit}(h)$, $\partial\hat{R} = h^{-1}(\partial R)$), in combination with forgetting marked points $\Big(T(S \setminus B) \xrightarrow{h^*} T(\hat{S} \setminus h^{-1}(B))\Big) \mapsto \Big(T(S) \xrightarrow{h^*_F} T(\hat{S})\Big)$.

Prop

bian

[15] F. Benirschke, C. A. Serván: Isometric embeddings of Teichmüller spaces are covering constructions. arXiv.2305.04153, 2023.

Surface	Triangulated category
mapping class group	group of autoequivalences
simple curve c	spherical object C
intersection number $i(c_1, c_2)$	$\dim \mathrm{Arr}^*(C_1, C_2)$
flat metric μ	stability condition σ
μ-geodesic	σ-semistable object
$\mathrm{length}_\mu(c)$	$\mathrm{mass}_\sigma(C)$

Table 1 (Heng [16]) stability condition theory : categorical version of Teichmüller theory

The GAGA theorem [8] implies **Riemann** $\left\{\begin{array}{l}\text{obj: cpt. Riemann surfaces} \\ \text{arr: holomorphic coverings}\end{array}\right\} \sim$ **Curve** $\left\{\begin{array}{l}\text{obj: nonsing. proj. curves} \\ \text{arr: rational maps}\end{array}\right\}$.

Arrows of **Teich** (=higher arrows of **Riemann**) are understood as follows.

Proposition A (Benirschke-Serván [15], 2023) In case $2g-2+n \geq 3$, *any* d_T-isometric embedding $T(R) \hookrightarrow T(\hat{R})$ arises from a **covering construction** up to pre-/post- compositions of MCGs. **It** is given by a map

$$h^* : T(R) \to T(\hat{R}) : [R \xrightarrow{f} S] \mapsto [\hat{R} \xrightarrow{\hat{f}} \hat{S}] \text{ s.t. } (\hat{S} \xrightarrow{\hat{f}^{-1}} \hat{R} \xrightarrow{h} R \xrightarrow{f} S) : \text{holomorphic}$$

[16] E. X. C. Heng: Categorification and Dynamics in Generalised Braid Groups, *PhD thesis, Australian National University,* 2023.

[17] F. Haiden, L. Katzarkov, M. Kontsevich: Flat surfaces and stability structures. Publ. Math. IHES **126**, 247–318 (2017).

Proposition B (Haiden-Katzarkov-Kontsevich [17], 2017) The moduli space of marked flat surfaces $M(R, f, \phi)$ is bianalytically embedded into the set of **stability conditions of the Fukaya category** of (R, f, ϕ).

Surface	Triangulated category
mapping class group	group of autoequivalences
simple curve c	spherical object C
intersection number $i(c_1, c_2)$	$\dim \mathrm{Arr}^*(C_1, C_2)$
flat metric μ	stability condition σ
μ-geodesic	σ-semistable object
$\mathrm{length}_\mu(c)$	$\mathrm{mass}_\sigma(C)$

Table 1 (Heng [16]) stability condition theory : categorical version of Teichmüller theory

2 Voronoi decomposition and Edwards-Sanderson-Schmidt method

Definition 6 We denote by Q_g the set of flat surfaces of genus g. We denote by $\mathcal{A}_g$ (resp. $\mathcal{Q}_g := Q_g \setminus \mathcal{A}_g$) the subset of abelian (resp. non-abelian) flat surfaces in Q_g. For $\mathcal{H} = Q, \mathcal{A}, \mathcal{Q}$, the **stratum** $\mathcal{H}_g(k_1, ..., k_n) \subset \mathcal{H}$ is defined by assigning singular orders $k_1, ..., k_n$ satisfying the Riemann-Hurwitz formula $\sum k_i = 4g - 4$.

(Hubbard-Masur, Veech[5]) The stratum $\mathcal{A}_g(k_1, ..., k_n)$ is a complex $(2g-1+n)$-dimensional orbifold locally parametrized by the **period map**

$$\Pi(R, \omega) := ([c] \mapsto \int_c \omega) \in H^1(R, \mathrm{Sing}(R, \omega); \mathbb{C}) \cong \mathbb{C}^{2g-1+n}.$$

For each $(R, \phi) \in \mathcal{Q}_g(k_1, ..., k_n)$, an analytic continuation of $\sqrt{\phi}$ defines a **canonical double** $\pi : \hat{R} \to R$ and an abelian flat surface $(\hat{R}, \hat{\phi} = \pi^*\phi) \in \mathcal{A}_{\hat{g}}(\hat{k}_1, ..., \hat{k}_{\hat{n}})$. The Riemann-Hurwitz formula implies $2\hat{g} - 1 + \hat{n} = 4g - 3 + 2n$, and every local image of Π splits into two eigenspaces $\Pi(\mathcal{A})_{\mathrm{local}} \oplus \Pi(\mathcal{Q})_{\mathrm{local}} \subset \mathbb{C}^{2g-1+n} \oplus \mathbb{C}^{2g-2+n}$ w.r.t. the linear involution $\sigma : \sqrt{\phi} \mapsto -\sqrt{\phi}$. In particular, the non-abelian stratum $\mathcal{Q}_g(k_1, ..., k_n)$ is a complex $(2g-2+n)$-dimensional orbifold.

Proposition C (Haiden-Katzarkov-Kontsevich [17], 2017) Let $\mathcal{F}(R, \phi)$ be the Fukaya category of a flat surface (R, ϕ) of finite type. Then there exist a series of subcategories $\left(\mathcal{C}^k \subset H^0(\mathcal{F}(R, \phi))\right)_{k \in \mathbb{R}}$ and the "period map" $Z : K_0(\mathcal{F}(R, \phi)) \to H_1(R, \partial R, \mathbb{Z} \otimes_{\mathbb{Z}/2} \sigma)$ satisfying the axiom of a stability condition.

2 Voronoi decomposition and Edwards-Sanderson-Schmidt method

Definition 6 We denote by Q_g the set of flat surfaces of genus g. We denote by $\mathcal{A}_g$ (resp. $\mathcal{Q}_g := Q_g \setminus \mathcal{A}_g$) the subset of abelian (resp. non-abelian) flat surfaces in Q_g. For $\mathcal{H} = Q, \mathcal{A}, \mathcal{Q}$, the **stratum** $\mathcal{H}_g(k_1, ..., k_n) \subset \mathcal{H}$ is defined by assigning singular orders $k_1, ..., k_n$ satisfying the Riemann-Hurwitz formula $\sum k_i = 4g - 4$.

(Hubbard-Masur, Veech[5]) The stratum $\mathcal{A}_g(k_1, ..., k_n)$ is a complex $(2g-1+n)$-dimensional orbifold locally parametrized by the **period map**

$$\Pi(R, \omega) := ([c] \mapsto \int_c \omega) \in H^1(R, \mathrm{Sing}(R, \omega); \mathbb{C}) \cong \mathbb{C}^{2g-1+n}.$$

For each $(R, \phi) \in \mathcal{Q}_g(k_1, ..., k_n)$, an analytic continuation of $\sqrt{\phi}$ defines a **canonical double** $\pi : \hat{R} \to R$ and an abelian flat surface $(\hat{R}, \hat{\phi} = \pi^* \phi) \in \mathcal{A}_{\hat{g}}(\hat{k}_1, ..., \hat{k}_{\hat{n}})$. The Riemann-Hurwitz formula implies $2\hat{g} - 1 + \hat{n} = 4g - 3 + 2n$, and every local image of Π splits into two eigenspaces $\Pi(\mathcal{A})_{\mathrm{local}} \oplus \Pi(\mathcal{Q})_{\mathrm{local}} \subset \mathbb{C}^{2g-1+n} \oplus \mathbb{C}^{2g-2+n}$ w.r.t. the linear involution $\sigma : \sqrt{\phi} \mapsto -\sqrt{\phi}$. In particular, the non-abelian stratum $\mathcal{Q}_g(k_1, ..., k_n)$ is a complex $(2g-2+n)$-dimensional orbifold.

Proposition C (Haiden-Katzarkov-Kontsevich [17], 2017) Let $\mathcal{F}(R, \phi)$ be the Fukaya category of a flat surface (R, ϕ) of finite type. Then there exist a series of subcategories $(\mathcal{C}^k \subset H^0(\mathcal{F}(R, \phi)))_{k \in \mathbb{R}}$ and the "period map" $Z : K_0(\mathcal{F}(R, \phi)) \to H_1(R, \partial R, \mathbb{Z} \otimes_{\mathbb{Z}/2} \sigma)$ satisfying the axiom of a stability condition.

2 Voronoi decomposition and Edwards-Sanderson-Schmidt method

Definition 6 We denote by Q_g the set of flat surfaces of genus g. We denote by $\mathcal{A}_g$ (resp. $\mathcal{Q}_g := Q_g \setminus \mathcal{A}_g$) the subset of abelian (resp. non-abelian) flat surfaces in Q_g. For $\mathcal{H} = Q, \mathcal{A}, \mathcal{Q}$, the **stratum** $\mathcal{H}_g(k_1, ..., k_n) \subset \mathcal{H}$ is defined by assigning singular orders $k_1, ..., k_n$ satisfying the Riemann-Hurwitz formula $\sum k_i = 4g - 4$.

(Hubbard-Masur, Veech[5]) The stratum $\mathcal{A}_g(k_1, ..., k_n)$ is a complex $(2g-1+n)$-dimensional orbifold locally parametrized by the **period map**

$$\Pi(R,\omega) := ([c] \mapsto \int_c \omega) \in H^1(R, \mathrm{Sing}(R,\omega); \mathbb{C}) \cong \mathbb{C}^{2g-1+n}.$$

For each $(R,\phi) \in \mathcal{Q}_g(k_1, ..., k_n)$, an analytic continuation of $\sqrt{\phi}$ defines a **canonical double** $\pi : \hat{R} \to R$ and an abelian flat surface $(\hat{R}, \hat{\phi} = \pi^*\phi) \in \mathcal{A}_{\hat{g}}(\hat{k}_1, ..., \hat{k}_{\hat{n}})$. The Riemann-Hurwitz formula implies $2\hat{g} - 1 + \hat{n} = 4g - 3 + 2n$, and every local image of Π splits into two eigenspaces $\Pi(\mathcal{A})_{\mathrm{local}} \oplus \Pi(\mathcal{Q})_{\mathrm{local}} \subset \mathbb{C}^{2g-1+n} \oplus \mathbb{C}^{2g-2+n}$ w.r.t. the linear involution $\sigma : \sqrt{\phi} \mapsto -\sqrt{\phi}$. In particular, the non-abelian stratum $\mathcal{Q}_g(k_1, ..., k_n)$ is a complex $(2g-2+n)$-dimensional orbifold.

Proposition C (Haiden-Katzarkov-Kontsevich [17], 2017) Let $\mathcal{F}(R,\phi)$ be the Fukaya category of a flat surface (R,ϕ) of finite type. Then there exist a series of subcategories $\left(\mathcal{C}^k \subset H^0(\mathcal{F}(R,\phi))\right)_{k\in\mathbb{R}}$ and the "period map" $Z : K_0(\mathcal{F}(R,\phi)) \to H_1(R, \partial R, \mathbb{Z} \otimes_{\mathbb{Z}/2} \sigma)$ satisfying the axiom of a stability condition.

[17] F. Haiden, L. Katzarkov, M. Kontsevich: Flat surfaces and stability structures. Publ. Math. IHES **126**, 247–318 (2017).

Definition 7 A geodesic connecting two singularities is called a **saddle connection**. A geometric triangulation (i.e. all edges are geodesics) is called **Delaunay** if no circumcircle contains a vertex in its interior. Delaunay triangles sharing common circumcircle are called **degenerate**. We say that a **flip** replaces two triangles pqr, rsp by qrs, spq.

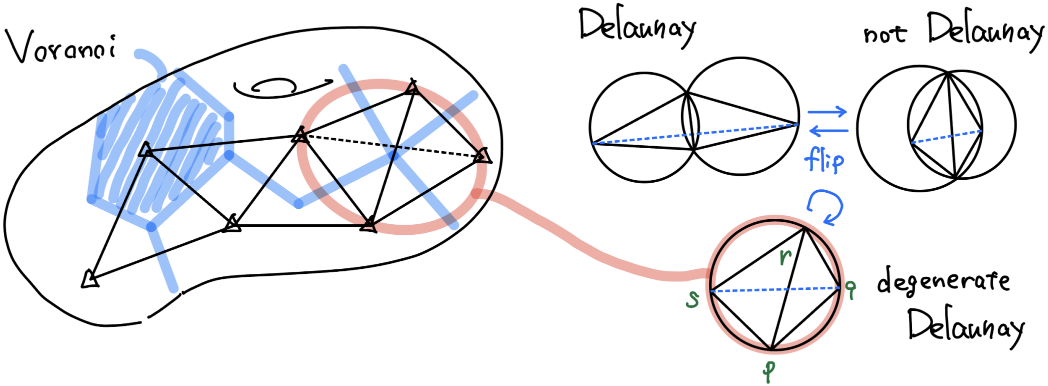


Figure 1 Delaunay triangulation and flip. Voronoi decomposition is obtained by prependicular bisectors of edges.

The **Voronoi decomposition** of a flat surface (R, ϕ) with $\mathrm{Sing}(R, \phi) = \{p_1, ..., p_n\}$ is the cell decomposition whose 2-cells C_{p_i}, $i = 1, ..., n$ are given by

$$C_{p_i} := \left\{ x \in R \;\middle|\; \exists!\text{ shortest saddle connection from } x \text{ to } \mathrm{Sing}(R, \phi) \text{ with terminus } p_i \right\}.$$

It is dual to the Delaunay triangulation of (R, ϕ).

Definition 7 A geodesic connecting two singularities is called a **saddle connection**. A geometric triangulation (i.e. all edges are geodesics) is called **Delaunay** if no circumcircle contains a vertex in its interior. Delaunay triangles sharing common circumcircle are called **degenerate**. We say that a **flip** replaces two triangles pqr, rsp by qrs, spq.

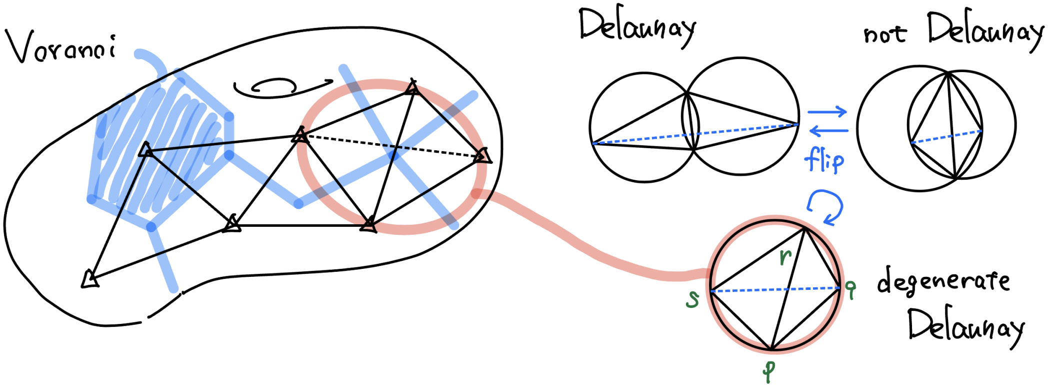


Figure 1 Delaunay triangulation and flip. Voronoi decomposition is obtained by prependicular bisectors of edges.

The **Voronoi decomposition** of a flat surface (R, ϕ) with $\mathrm{Sing}(R, \phi) = \{p_1, ..., p_n\}$ is the cell decomposition whose 2-cells C_{p_i}, $i = 1, ..., n$ are given by

$$C_{p_i} := \Big\{ x \in R \;\Big|\; \exists! \text{ shortest saddle connection from } x \text{ to } \mathrm{Sing}(R, \phi) \text{ with terminus } p_i \Big\}.$$

It is dual to the Delaunay triangulation of (R, ϕ).

Definition 8 The **canonical surface** of the stratum $Q_g(k_1, ..., k_n)$ is the infinite flat surface

$$\Delta = \Delta(k_1, ..., k_n) = \bigsqcup_{i=1}^{n} \Delta_i := \bigsqcup_{i=1}^{n} (\mathbb{C}, z^{k_i} dz^2).$$

We have $\mathrm{Htrans}(\Delta) \cong \prod_{i=1}^{n} C_{2(k_i+1)} \times \prod_{t=1}^{\infty} \mathrm{Sym}\{i \mid k_i = t\}$, where each $C_{2(k_i+1)}$ acts on Δ_i by π-rotation and each $\mathrm{Sym}\{i \mid k_i = t\}$ permutates Δ_i's of the same degree.

Lemma 9 There exists a unque embedding $\iota = \iota_{(R,\phi)}$ of Voronoi 2-cells C_{p_i} into Δ_i modulo $\mathrm{Htrans}(\Delta)$. The embedding ι extends to any star-like region in R^*, and it follows that $(R, \phi) \cong \left(\bigsqcup_{i=1}^{n} \overline{\iota(C_{p_i})}\right)/_\sim$ for a suitable edge identification $\sim$.

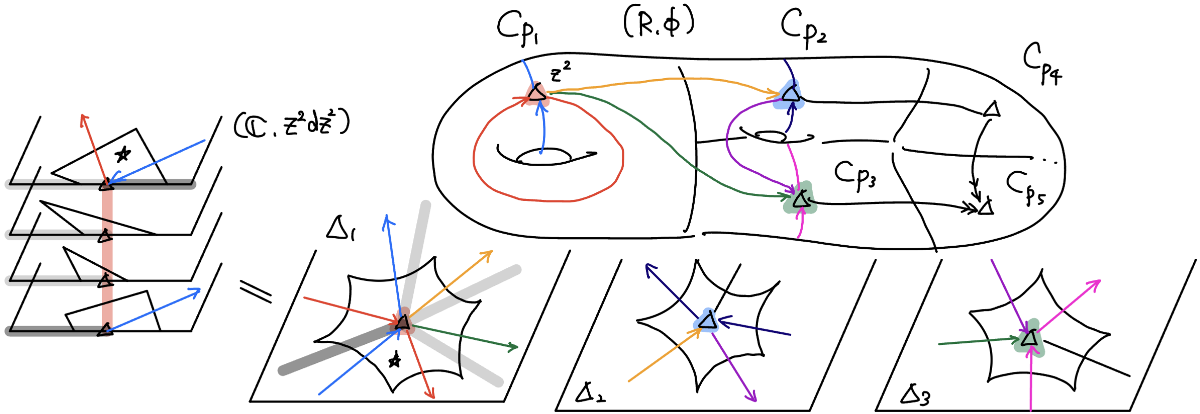


Figure 2 Voronoi decomposition and embedded image of C_{p_i} in Δ_i.

Definition 10 Fix an embedding $\iota = \iota_{(R,\phi)}$ in Lemma 9. For each oriented saddle connection $s(p_i \to p_j)$, let $\iota(s) \in \Delta_i$ be the extension of $\iota\,|_{C_{p_i}}$ along s. The inverse is denoted by $s^{-1}(p_j \to p_i)$ where $\iota(s^{-1}) \in \Delta_j$. We define

$$\widehat{\mathcal{M}}(R,\phi) := \Big\{(\iota(s), \iota(s^{-1})) \in \Delta \times \Delta \;\Big|\; s : \text{ori. saddle conn. on } (R,\phi)\Big\},$$
$$\widehat{\mathcal{S}}(R,\phi) := \Big\{(\iota(s), \iota(s^{-1})) \in \widehat{\mathcal{M}}(R,\phi) \;\Big|\; s : \text{prep. bisector of a Voronoi 1-cell}\Big\}.$$

The finite set $\widehat{\mathcal{S}}(R,\phi)$ is called the set of **Voronoi staples** of (R,ϕ).

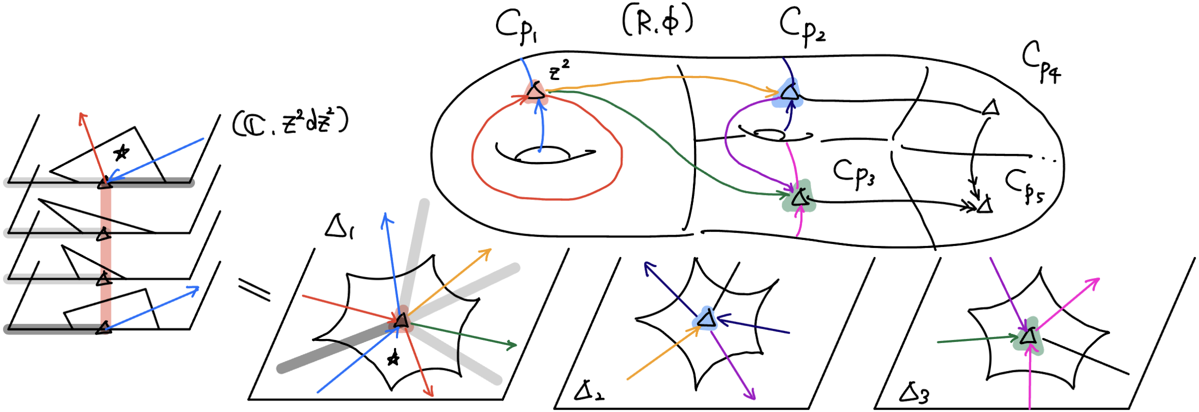
Each $(\iota(s), \iota(s^{-1})) \in \widehat{\mathcal{M}}(R,\phi)$, $s(p_1 \to p_2)$ consists of the following data modulo HTrans(Δ)-action.

- the
- the
- the

Lemma 1

Propositi nd only if

(1)



Furthermore, for $\|\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]\|_{\mathrm{Frob}} := \sqrt{a^2+b^2+c^2+d^2}$ and $\mathrm{Sys}(R,\phi) := \min\{\iota(s) | s : \text{saddle conn. of } (R,\phi)\}$ we have

$$\operatorname{diam}(A \cdot \widehat{\mathcal{S}}(R,\phi)) < \|A\|_{\mathrm{Frob}} \cdot \mathrm{Sys}(R,\phi). \tag{2}$$

Definition 10 Fix an embedding $\iota = \iota_{(R,\phi)}$ in Lemma 9. For each oriented saddle connection $s(p_i \to p_j)$, let $\iota(s) \in \Delta_i$ be the extension of $\iota|_{C_{p_i}}$ along s. The inverse is denoted by $s^{-1}(p_j \to p_i)$ where $\iota(s^{-1}) \in \Delta_j$. We define

$$\widehat{\mathcal{M}}(R,\phi) := \left\{ (\iota(s), \iota(s^{-1})) \in \Delta \times \Delta \;\middle|\; s : \text{ori. saddle conn. on } (R,\phi) \right\},$$
$$\widehat{\mathcal{S}}(R,\phi) := \left\{ (\iota(s), \iota(s^{-1})) \in \widehat{\mathcal{M}}(R,\phi) \;\middle|\; s : \text{prep. bisector of a Voronoi 1-cell} \right\}.$$

The finite set $\widehat{\mathcal{S}}(R,\phi)$ is called the set of **Voronoi staples** of (R,ϕ).

Each $(\iota(s), \iota(s^{-1})) \in \widehat{\mathcal{M}}(R,\phi)$, $s(p_1 \to p_2)$ consists of the following data modulo HTrans(Δ)-action.

- the domain p_1 and the codomain p_2
- the passing sheets of Δ_1 and Δ_2; in $\mathbb{Z}/(k+2)\mathbb{Z}$
- the holonomy vector $hol(s) = \int_s \sqrt{\phi} \in \mathbb{C}$

Lemma 11 $(R_1,\phi_1),(R_2,\phi_2)$ are equal in **Flat** if and only if $\widehat{\mathcal{S}}(R_1,\phi_1), \widehat{\mathcal{S}}(R_2,\phi_2)$ are HTrans(Δ)-equivalent.

Proposition 12 (Edwards-Sanderson-Schmidt [4], 2022) A matrix $A \in PSL(2,\mathbb{R})$ belongs to $\Gamma(R,\phi)$ if and only if

$$\exists \gamma \in \text{Htrans}(\Delta) \text{ s.t. } \gamma(A \cdot \widehat{\mathcal{S}}(R,\phi)) \subset \widehat{\mathcal{M}}(R,\phi). \tag{1}$$

Furthermore, for $\| \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \|_{\text{Frob}} := \sqrt{a^2+b^2+c^2+d^2}$ and $\text{Sys}(R,\phi) := \min\{l(s) | s : \text{saddle conn. of } (R,\phi)\}$ we have

$$\text{diam}(A \cdot \widehat{\mathcal{S}}(R,\phi)) < \|A\|_{\text{Frob}} \cdot \text{Sys}(R,\phi). \tag{2}$$

For each $a > 0$, the set $\Gamma^a(R,\phi) := \{A \in PSL(2,\mathbb{R}) \mid \exists \gamma \in \mathrm{Htrans}(\Delta) \text{ s.t. } \gamma(A \cdot \widehat{\mathcal{S}}(R,\phi)) \subset \widehat{\mathcal{M}}(R,\phi) \text{ and } \|A\|_{\mathrm{Frob}} < a\}$ is a computable, finite subset in $\Gamma(R,\phi)$. Denote its convex body by $\Omega(\Gamma^a) = \bigcap_{A \in \Gamma^a} \{\tau \in \mathbb{H} \mid d_{\mathbb{H}}(i,\tau) \leq d_{\mathbb{H}}(\gamma_A(i),\tau)\}$.

Proposition 13 (Edwards-Sanderson-Schmidt [4], 2022) If $a \geq \sqrt{2}$ satisfies

$$\mu_{\mathbb{H}}(\Omega(\Gamma^a)) < 2\mu_{\mathbb{H}}\left(\Omega(\Gamma^a) \cap B\left(i, \log\sqrt{\frac{a^2 - \sqrt{a^4-4}}{2}}\right)\right) \tag{3}$$

then, $\Gamma^a(R,\phi)$ generates $\Gamma(R,\phi)$. In particular, $\Gamma(R,\phi)$ is a lattice in this case.

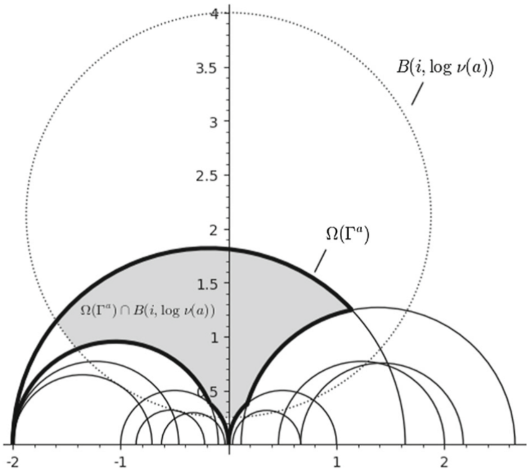


Figure 3 Lattice test (3): cited from Edwards-Sanderson-Schmidt [4, Fig. 7]

3 Main result

3.1 Origami and Delaunay triangulation

Fix $d \in \mathbb{N}$ and let $\Lambda := \{\pm 1, \ldots, \pm d\}, \mathfrak{S} := \mathrm{Sym}(\Lambda)$. A flat surface obtained by gluing d unit square cells at edges by half-translations is called an **origami** of degree d. The Veech group of an origami is a lattice in $PSL(2, \mathbb{Z})$. Möller [13] showed that the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action respects the Teichmüller embedding of the arithmetic curve $\mathbb{L}/\Gamma(\mathcal{O}) \hookrightarrow M_{g,n}$.

Example D For $x, y \in S_d$, an abelian origami (x, y) is defined by the gluing rule (right edge of $i \leftrightarrow$ left edge of $x(i)$) and (upper edge of $i \leftrightarrow$ lower edge of $y(i)$), where the d squares are labelled by $i = 1, ..., d$. Its Veech group is a stabilizer under the following $SL(2, \mathbb{Z}) \cong \mathrm{Out}^+(F_2)$-action (Schmithüsen[14]).

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (x, y) = (x, xy), \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (x, y) = (y, x^{-1}),$$

Figure 4 Abelian origami (x, y) where $x = (1)(2\ 3\ 4)$, $y = (1\ 2)(3\ 4)$.

3 Main result

3.1 Origami and Delaunay triangulation

Fix $d \in \mathbb{N}$ and let $\Lambda := \{\pm 1, \ldots, \pm d\}, \mathfrak{S} := \mathrm{Sym}(\Lambda)$. A flat surface obtained by gluing d unit square cells at edges by half-translations is called an **origami** of degree d. The Veech group of an origami is a lattice in $PSL(2, \mathbb{Z})$. Möller [13] showed that the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action respects the Teichmüller embedding of the arithmetic curve $\mathbb{L}/\Gamma(\mathcal{O}) \hookrightarrow M_{g,n}$.

Example D For $x, y \in S_d$, an abelian origami (x, y) is defined by the gluing rule (right edge of $i \leftrightarrow$ left edge of $x(i)$) and (upper edge of $i \leftrightarrow$ lower edge of $y(i)$), where the d squares are labelled by $i = 1, ..., d$. Its Veech group is a stabilizer under the following $SL(2, \mathbb{Z}) \cong \mathrm{Out}^+(F_2)$-action (Schmithüsen[14]).

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (x, y) = (x, xy), \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (x, y) = (y, x^{-1}),$$

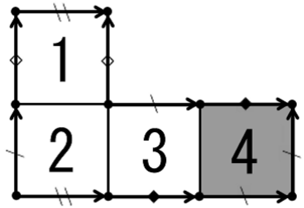


Figure 4 Abelian origami (x, y) where $x = (1)(2\ 3\ 4)$, $y = (1\ 2)(3\ 4)$.

Example E (K. 2023 [9]) All possible patterns of origamis are obtained by considering the **cut-and-paste construction** with respect to the origami (x, y) and all the negative cells, where $x, y \in S_d < \mathfrak{S}_{\mathrm{odd}}$, $\varepsilon \in \{\pm 1\}^{\mathrm{d}}_{\mathrm{odd}}$. In the construction, The canonical double is represented by the abelian origami $(x^{\mathrm{sign}}, \varepsilon y^{\varepsilon} \varepsilon(y^{\varepsilon})) \in \mathfrak{S} \times \mathfrak{S}$.

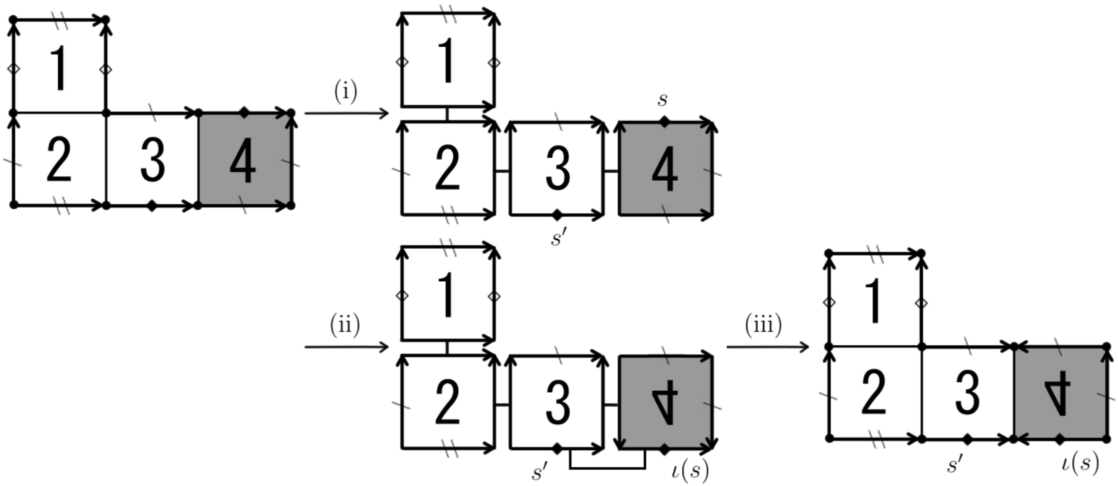


Figure 5 Cut-and-paste construction for the case $x = (1)(2\ 3\ 4)$, $y = (1\ 2)(3\ 4)$, $\varepsilon = (+, +, +, -)$.

- In the step (i), the upper side s of the 4th cell is paired with the lower side s' of the 3rd cell.
- In the step (ii), the vertical reflection ι is applied to the 4th cell.
- In the step (iii), the sides $\iota(s)$ and s' are glued by a half-translation.

Proposition 14 (K. [10]) Let $\mathfrak{S}^i := \{\sigma \in \mathfrak{S} : \text{fixed-point-free, order } 2\}$. There is one-to-one correspondence between HTrans-classes of origamis and $\mathfrak{S}$-conjugacy classes of tuples of $\mu, \nu, \tau \in \mathfrak{S}^i$ with the relationships in the following table. The canonical double of $\mathcal{O}$ is the abelian origami $(x_{\mathcal{O}}, y_{\mathcal{O}})$. We have $\mu = z_{\mathcal{O}} y_{\mathcal{O}}$, $\nu = x_{\mathcal{O}}^{-1} z_{\mathcal{O}}$, $\tau = x_{\mathcal{O}}^{-1} z_{\mathcal{O}} y_{\mathcal{O}}$, and $z_{\mathcal{O}}^2 = x_{\mathcal{O}} y_{\mathcal{O}} x_{\mathcal{O}}^{-1} y_{\mathcal{O}}^{-1}$.

$\lambda \in \Lambda$	half-square(◺ ◹)	μ, ν, τ	reflection along edge
$\lambda \cdot \langle \mu \rangle$	horiz. λ-edge	$x_{\mathcal{O}} := \mu\tau$	horiz. translation
$\lambda \cdot \langle \nu \rangle$	vert. λ-edge	$y_{\mathcal{O}} := \nu\tau$	vert. translation
$\lambda \cdot \langle \tau \rangle$	λ-cell	$z_{\mathcal{O}} := \mu\tau\nu$	π-rotation arr. cone
$\lambda \cdot \langle x_{\mathcal{O}} \rangle$	horiz. λ-cylinder	$\#(\lambda \cdot \langle x_{\mathcal{O}} \rangle)$	cylinder width
$\lambda \cdot \langle y_{\mathcal{O}} \rangle$	vert. λ-cylinder	$\#(\lambda \cdot \langle y_{\mathcal{O}} \rangle)$	cylinder width
$\lambda \cdot \langle z_{\mathcal{O}} \rangle$	λ-cone	$\#(\lambda \cdot \langle z_{\mathcal{O}} \rangle)$	deg(cone)+2

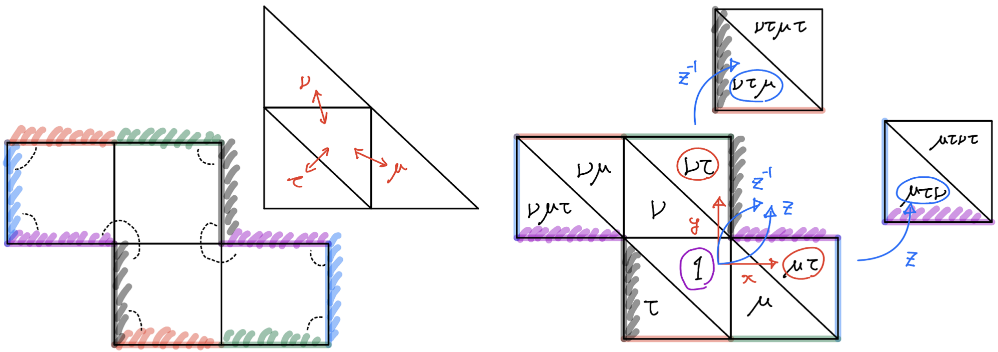


Figure 6 (x, y, z) for the origami (x, y, z) : $\mu\tau\nu(1) = \nu\mu\tau(1)$ in this case.

Let $\Theta := \mathfrak{S}^i \times \mathfrak{S}^i \times \mathfrak{S}^i / \mathfrak{S}^{\mathrm{conj}}$ and $\tau = \tau_0 := (\lambda \mapsto -\lambda) \in \mathfrak{S}^i$. We have $(\mu, \nu, \tau_0) = (x\tau_0, y\tau_0, \tau_0) = (-x, -y, \tau_0)$.

Theorem 15 Identify $\mathcal{O} = (\mu, \nu, \tau) \in \Theta$ with the unique (◺◹)-Delaunay triangulation of the origami $\mathcal{O}$. Then, for each $(\mu, \nu, \tau_0) \in \Theta$ and $\kappa = 1, ..., d$, the flip w.r.t. the degenerate two triangles $\pm\kappa$ is represented by $(-x_{\mathcal{O}}^{\mathrm{sign}}, -\varepsilon_\kappa y_{\mathcal{O}}^{\varepsilon_\kappa} \varepsilon_\kappa (y_{\mathcal{O}}^{\varepsilon_\kappa}), \tau_0) \in \Theta$, where $\varepsilon_\kappa = (\pm\kappa \mapsto \mp\kappa) \in \{\pm 1\}^{\Lambda}_{\mathrm{odd}}$. Furthermore, to be square-tiled is invariant property of triangulations under flips.

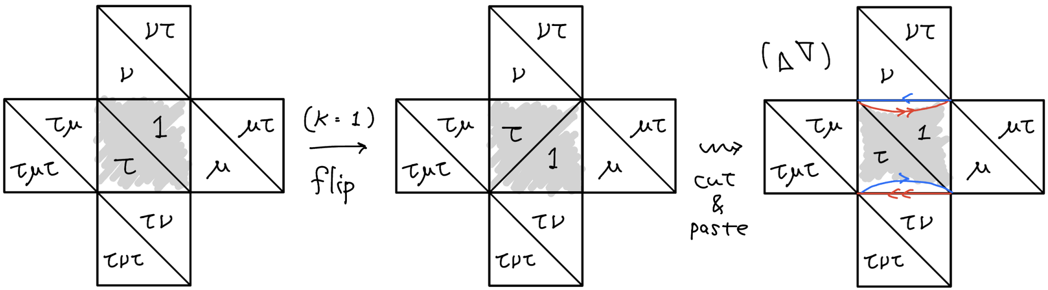


Figure 7 The (◺◹)-Delaunay triangulation of origami (μ, ν, τ). The flip at the square $(1(\kappa), \tau(\kappa))$ is represented by the (◺◹)-Delaunay triangulation of the origami given by the cut-and-paste construction in Example E.

Proposition 16 The universal Veech group $PSL(2, \mathbb{Z}) = \langle T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \rangle$ acts on $\mathcal{O} = (\mu, \nu, \tau) \in \Theta$ as follows.

$$\begin{cases} T(\mu, \nu, \tau) = (\mu^*\tau, \nu, \mu) \\ S(\mu, \nu, \tau) = (\tau^*\nu, \mu, \tau) \\ TS(\mu, \nu, \tau) = (\nu, \tau, \mu) \end{cases} \quad \begin{cases} T(x, y, z) = (x, yx^{-1}, z) \\ S(x, y, z) = (y, x, y^{-1}zy) \\ TS(x, y, z) = (yx^{-1}, x^{-1}, z^{-1}) \end{cases}$$

outline of proof) One gets the formulae for (◺ ◹)-Delaunay triangulation of $T\mathcal{O}, S\mathcal{O}$ from a local picture below.

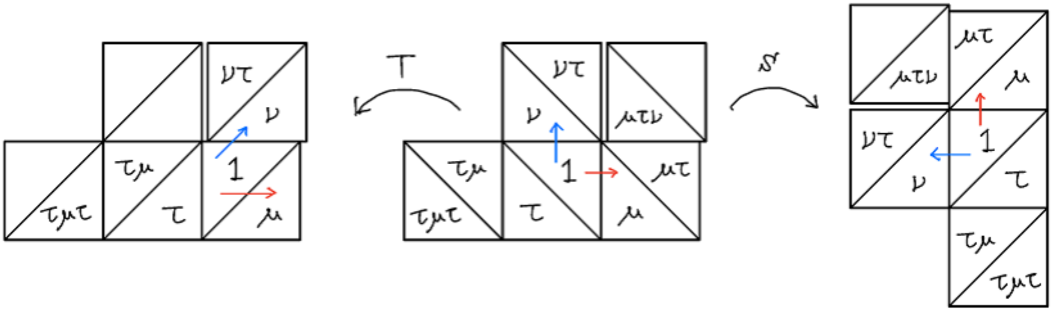


Figure 8　T (resp. S)-affine deformation of (μ, ν, τ) is the mirror of (τ, ν, μ) (resp. (ν, μ, τ)).

The simulatenous flip of (μ, ν, τ) at all cells gives the (◸ ◿)-Delaunay represented by the mirror image of the origami $(\tau^*\mu, \nu, \tau)$. The formulae for x, y, z follows from $\mu = zy$, $\nu = x^{-1}z$, $\tau = x^{-1}zy$.

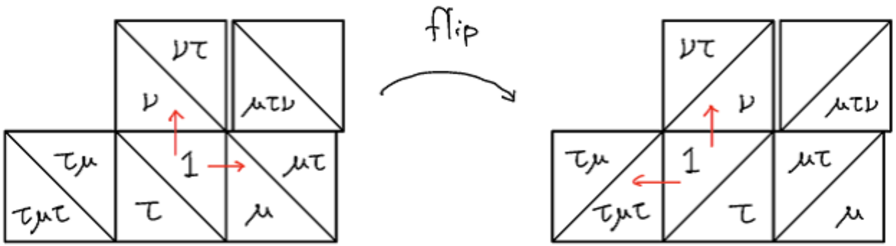


Figure 9　The simulatenous flip of (μ, ν, τ) is the mirror of $(\tau^*\mu, \nu, \tau)$.

3.2 Edwards-Sanderson-Schmidt method in terms of category

We will construct $\widehat{\mathcal{M}}, \widehat{\mathcal{S}}$ for origamis. Assume that all the corner points of squares are marked. Though singularities of order 0 should be removed with Veech groups in mind, our assumption does not matter because these marked points form a $PSL(2,\mathbb{Z})$-invariant set. We can align the Voronoi staples with the sheet cuts in this way.

Definition 17 For an origami $\mathcal{O} = (\mu, \nu, \tau) \in \Theta$, denote $x = x_{\mathcal{O}}$, $y = y_{\mathcal{O}}$, $z = z_{\mathcal{O}}$, $Z := \langle z \rangle$. We define a **groupoid** $\mathcal{G}_{\mathcal{O}}$ equipped with 2-arrows (ribbon-graph structure) and 3-arrows (relabeling) as follows.

$$\begin{cases} \mathrm{Obj}_{\mathcal{G}_{\mathcal{O}}} = \Lambda/Z \\ \mathrm{Arr}_{\mathcal{G}_{\mathcal{O}}} = \langle \lambda Z \xrightarrow{\lambda^\sigma} \lambda\sigma Z \mid \lambda \in \Lambda, \sigma = \mu, \nu, \tau \rangle \\ \mathrm{Arr}^2_{\mathcal{G}_{\mathcal{O}}} = \{\lambda^\sigma \to (\lambda z)^\sigma\} \\ \mathrm{Arr}^3_{\mathcal{G}_{\mathcal{O}}} = \mathfrak{S}\text{-conjugate} \end{cases}$$

Arrows arround vertices λZ, $\lambda \in \Lambda$ are placed as follows. We have $\#\mathrm{Arr}(\lambda Z, -) = 3\#\lambda Z = 3(\mathrm{ord}_{\lambda Z}(\phi) + 2)$.

$$\begin{tikzcd} \lambda\nu Z \arrow[d] & \lambda Z \arrow[l, "\lambda^\nu"'] \arrow[r, "(\lambda z)^\nu"] \arrow[dl, "\lambda^\tau"'] \arrow[d, "\lambda^\mu"'] & \lambda\mu\tau Z \\ \lambda\tau Z & \lambda\mu Z \arrow[l] & \end{tikzcd} \qquad \begin{tikzcd} & \lambda z\mu Z & \lambda z\tau Z \\ \lambda z^2\nu Z & \lambda z Z \arrow[l, "(\lambda z^2)^\nu"] \arrow[u, "(\lambda z)^\mu"] \arrow[ur, "(\lambda z)^\tau"'] \arrow[r, "(\lambda z)^\nu"'] & \lambda\mu\tau Z \end{tikzcd}$$

3.2 Edwards-Sanderson-Schmidt method in terms of category

We will construct $\widehat{\mathcal{M}}, \widehat{\mathcal{S}}$ for origamis. Assume that all the corner points of squares are marked. Though singularities of order 0 should be removed with Veech groups in mind, our assumption does not matter because these marked points form a $PSL(2, \mathbb{Z})$-invariant set. We can align the Voronoi staples with the sheet cuts in this way.

Definition 17 For an origami $\mathcal{O} = (\mu, \nu, \tau) \in \Theta$, denote $x = x_{\mathcal{O}}$, $y = y_{\mathcal{O}}$, $z = z_{\mathcal{O}}$, $Z := \langle z \rangle$. We define a **groupoid** $\mathcal{G}_{\mathcal{O}}$ equipped with 2-arrows (ribbon-graph structure) and 3-arrows (relabeling) as follows.

$$\begin{cases} \mathrm{Obj}_{\mathcal{G}_{\mathcal{O}}} = \Lambda / Z \\ \mathrm{Arr}_{\mathcal{G}_{\mathcal{O}}} = \langle \lambda Z \xrightarrow{\lambda^\sigma} \lambda\sigma Z \mid \lambda \in \Lambda, \sigma = \mu, \nu, \tau \rangle \\ \mathrm{Arr}^2_{\mathcal{G}_{\mathcal{O}}} = \{ \lambda^\sigma \to (\lambda z)^\sigma \} \\ \mathrm{Arr}^3_{\mathcal{G}_{\mathcal{O}}} = \mathfrak{S}\text{-conjugate} \end{cases}$$

Arrows arround vertices λZ, $\lambda \in \Lambda$ are placed as follows. We have $\#\mathrm{Arr}(\lambda Z, -) = 3\#\lambda Z = 3(\mathrm{ord}_{\lambda Z}(\phi) + 2)$.

$$\begin{tikzcd}
\lambda\nu Z \arrow[d] & \lambda Z \arrow[l, "\lambda^\nu"'] \arrow[r, "(\lambda z)^\nu"] \arrow[dl, "\lambda^\tau"'] \arrow[d, "\lambda^\mu"'] & \lambda\mu\tau Z \\
\lambda\tau Z & \lambda\mu Z \arrow[l] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \lambda z\mu Z & \lambda z\tau Z \\
\lambda z^2\nu Z & \lambda z Z \arrow[l, "(\lambda z^2)^\nu"] \arrow[u, "(\lambda z)^\mu"] \arrow[ur, "(\lambda z)^\tau"'] \arrow[r, "(\lambda z)^\nu"'] & \lambda\mu\tau Z
\end{tikzcd}$$

By $\nu = x^{-1}z$, $\mu = y^{-1}z^{-1}$, and proposition 14, arrows are regarded as local-translations.

$$
\begin{tikzcd}
\lambda x^{-1}zZ \arrow[r, equal] & \lambda\nu Z & \lambda Z \arrow[l, "\lambda^\nu"'] \arrow[r, "(\lambda z)^\nu"] \arrow[d, "\lambda^\mu"] \arrow[rr, bend left, "\lambda^x"] \arrow[ll, bend left, "\lambda^{(x^{-1})}"] \arrow[dd, bend right, "\lambda^{(y^{-1})}"'] \arrow[dr, phantom, "\triangledown"] & \lambda\mu\tau Z \arrow[r, equal] & \lambda xZ \\
 & & \lambda\mu Z \arrow[ur, "(\lambda\mu)^\tau"'] \arrow[d, equal] & & \\
 & & \lambda y^{-1}z^{-1}Z & &
\end{tikzcd}
$$

We may define a homomorphism $h : \mathrm{Arr}(\lambda Z, -) \to (\mathbb{C}, +)$ by $h(\lambda^x) = 1$, $h(\lambda^y) = i$ and commutativity $h(⊿) = h(◸) = 0$. We obtain $\widehat{\mathcal{M}}^*(\mathcal{O}) := (\mathcal{G} \times h)_{\mathcal{O}} \in \underline{\mathbf{Groupoid}}\times(\mathbb{C}, +)$ as follows; vector asigned to each arrows

$$
\begin{tikzcd}
 & & \lambda y^2xZ & \\
 {} & \lambda yZ \arrow[l] & & \lambda yx^2Z \\
 \lambda x^{-1}Z \arrow[u] & \lambda Z \arrow[l, "-1", "\lambda^{(x^{-1})}"'] \arrow[r, "1", "\lambda^x"'] \arrow[u, "i", "\lambda^y"'] \arrow[uur, "1+2i", "\lambda^y \bullet \lambda y^y \bullet \lambda yy^x"'] \arrow[urr, bend left, "2+i", "\lambda^y \bullet \lambda y^x \bullet \lambda yx^x"'] & \lambda xZ &
\end{tikzcd}
$$

By replacing the arrow set with $\mathrm{Arr}_{G_{\mathcal{O}}} = \{\lambda Z \xrightarrow{\lambda^\sigma} \lambda\sigma Z \mid \lambda \in \Lambda, \sigma = \mu, \nu, \tau\}$, we get $\widehat{\mathcal{S}}^*(\mathcal{O}) := (G \times h)_{\mathcal{O}} \in \underline{\mathbf{Graph}}\times(\mathbb{C}, +)$.
As explained as adjunction $\underline{\mathbf{Graph}} \rightleftarrows \underline{\mathbf{Category}}$ [12], $\widehat{\mathcal{S}}^*$ is a "finite generating system" of $\widehat{\mathcal{M}}^*$.

By $\nu = x^{-1}z$, $\mu = y^{-1}z^{-1}$, and proposition 14, arrows are regarded as local-translations.

$$\begin{tikzcd}
\lambda x^{-1}zZ \arrow[r, equal] & \lambda\nu Z & \lambda Z \arrow[l, "\lambda^{\nu}"'] \arrow[r, "(\lambda z)^{\nu}"] \arrow[rr, bend left, "\lambda^{x}"] \arrow[ll, bend left, "\lambda^{(x^{-1})}"] \arrow[d, "\lambda^{\mu}"'] \arrow[dd, bend right, "\lambda^{(y^{-1})}"'] \arrow[dr, phantom, "\triangledown"] & \lambda\mu\tau Z \arrow[r, equal] & \lambda x Z \\
 & & \lambda\mu Z \arrow[ur, "(\lambda\mu)^{\tau}"'] \arrow[d, equal] & & \\
 & & \lambda y^{-1}z^{-1}Z & &
\end{tikzcd}$$

We may define a homomorphism $h : \mathrm{Arr}(\lambda Z, -) \to (\mathbb{C}, +)$ by $h(\lambda^x) = 1$, $h(\lambda^y) = i$ and commutativity $h(\triangle) = h(\triangledown) = 0$. We obtain $\widehat{\mathcal{M}}^*(\mathcal{O}) := (\mathcal{G} \times h)_{\mathcal{O}} \in \underline{\textbf{Groupoid}} \times (\mathbb{C}, +)$ as follows; vector asigned to each arrows

$$\begin{tikzcd}
 & & \lambda y^2 x Z & \\
 \cdot & \lambda y Z \arrow[l] & & \lambda y x^2 Z \\
\lambda x^{-1} Z \arrow[u] & \lambda Z \arrow[l, "-1", "\lambda^{(x^{-1})}"'] \arrow[r, "1", "\lambda^{x}"'] \arrow[u, "i", "\lambda^{y}"'] \arrow[uur, "1+2i", "\lambda^{y} \bullet \lambda y^{y} \bullet \lambda y y^{x}"'] \arrow[urr, bend left, "2+i", "\lambda^{y} \bullet \lambda y^{x} \bullet \lambda y x^{x}"'] & \lambda x Z &
\end{tikzcd}$$

By replacing the arrow set with $\mathrm{Arr}_{G_{\mathcal{O}}} = \{\lambda Z \xrightarrow{\lambda^{\sigma}} \lambda\sigma Z \mid \lambda \in \Lambda, \sigma = \mu, \nu, \tau\}$, we get $\widehat{\mathcal{S}}^*(\mathcal{O}) := (G \times h)_{\mathcal{O}} \in \underline{\textbf{Graph}} \times (\mathbb{C}, +)$. As explained as adjunction $\underline{\textbf{Graph}} \underset{\perp}{\rightleftarrows} \underline{\textbf{Category}}$ [12], $\widehat{\mathcal{S}}^*$ is a "finite generating system" of $\widehat{\mathcal{M}}^*$.

Theorem 18 $\widehat{\mathcal{M}}^* = (\mathcal{G} \times h)$ is a functor $\underline{\textbf{Flat}} \supset \Theta$ (*origamis*) $\to \underline{\textbf{Groupoid}} \times (\mathbb{C}, +)$ such that

1. $\widehat{\mathcal{M}}^*(\mathcal{O}_1 \xrightarrow{f} \mathcal{O}_2) = 1 \times (D_f)_{\text{linear}}$,
2. $\mathcal{O}_1 = \mathcal{O}_2 \Leftrightarrow \widehat{\mathcal{M}}^*(\mathcal{O}_1) = \widehat{\mathcal{M}}^*(\mathcal{O}_2) \Leftrightarrow \widehat{\mathcal{S}}^*(\mathcal{O}_1) \subset \widehat{\mathcal{M}}^*(\mathcal{O}_2)$, and
3. $A \in \Gamma(\mathcal{O}) \Leftrightarrow A\widehat{\mathcal{S}}^*(\mathcal{O}) \subset \widehat{\mathcal{M}}^*(\mathcal{O})$.

Note that the inclusions in 2. 3. implies that the arrows are embedded modulo $\cong \mathrm{Htrans}(\Delta)$; 3-arrows preserving 2-arrows.

Edwards-Sanderson-Schmidt method	categorical version for origamis
flat surface (R, ϕ)	origami $\mathcal{O} = (\mu, \nu, \tau)$
singularity p	orbit λZ
Voronoi staple $(\iota(s), \iota(s^{-1}))$	generating arrow $\lambda Z \xrightarrow{\lambda^\sigma} \lambda\sigma Z$
saddle connection $(\iota(s), \iota(s^{-1}))$	arrow (piecewise line)
sheet transition on Δ	arrow^2 $\lambda^\sigma \to (\lambda z)^\sigma$
automorphism (Htrans(Δ))	arrow^3 ($\mathfrak{S}$-conjugate)
combinatorial structure of $\widehat{\mathcal{M}}$, $\widehat{\mathcal{S}}$	groupoid $\mathcal{G}$, group G
holonomy $(hol(s), hol(s^{-1}))$	vector $h(\lambda^\sigma)$
finite bound for $A\widehat{\mathcal{S}}$	finite arrows of bounded length

Table 2 categorical version of Edwards-Sanderson-Schmidt method

Thank you for your attention!!

Bibliography

[1] L. V. Ahlfors, Lectures on Quasiconformal Mappings, *Amer. Math. Soc., Providence, RI,* 1966.

[2] J. P. Bowman: Teichmüller geodesics, Delaunay triangulations, and Veech groups. *Ramanujan Math. Soc. Lect. Notes Ser.,* **10** (2010), 113–129.

[3] C. J. Earle, F. P. Gardiner, Geometric isomorphisms between infinite dimensional Teichmüller spaces, *Trans. Amer. Math. Soc.,* **348** (1996), 1163–1190.

[4] B. Edwards, S. Sanderson, and T. A. Schmidt: Canonical translationsurfaces for computing Veech groups. *Geom. Dedicata*, **216**(5): 60, 2022.

[5] G. Forni and C. Matheus: Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards, *Journal of Modern Dynamics,* **8** (2014), no. 3-4, 271–436.

[6] F. P. Gardiner, N. Lakic: Quasiconformal Teichmüller Theory, *Amer. Math. Soc., Providence, RI,* 2000.

[7] A. Hatcher: Algebraic topology, *Cambridge University Press, Cambridge,* 2002.

[8] F., Herrlich, G. Schmithüsen: Dessins d'enfants and origami curves, *IRMA Lect. Math. Theor. Phys.,* **13**, 767–809 (2009)

[9] S. Kumagai: Calculation of Veech groups and Galois invariants of general origamis. *Accepted for publication of Kodai Math Journal,* arXiv.2006.00905, 2023.

[10] S. Kumagai: General origamis and Veech groups of flat surfaces. arXiv.2111.09654, 2021.

[11] H. Masur and J. Smillie: Hausdorff dimension of sets of nonergodic measured foliations. *Ann. Math.* **134**, 455–543, 1991.

[12] S. MacLane: Categories for the working mathematician. *Graduate texts in mathematics* **5**, *Springer-Verlag,* 1971.

[13] M. Möller: Teichmüller curves, Galois actions and $\widehat{GT}$-relations. *Math. Nachr.* **278**, no.9, 1061–1077 (2005)

[14] G. Schmithüsen: An algorithm for finding the Veech group of an origami. *Experiment. Math.* **13**, no. 4, 459–472, 2004

[15] F. Benirschke, C. A. Serván: Isometric embeddings of Teichmüller spaces are covering constructions. arXiv.2305.04153, 2023.

[16] E. X. C. Heng: Categorification and Dynamics in Generalised Braid Groups, *PhD thesis, Australian National University,* 2023.

[17] F. Haiden, L. Katzarkov, M. Kontsevich: Flat surfaces and stability structures. Publ. Math. IHES **126**, 247–318 (2017).