Voronoi decompositions of flat surfaces and origamis

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1 Introduction/Background

Main theme: calculation of Veech groups and its calculation process

The **Veech group** of a flat surface is the group of its *affine self-similarity*.

Strategy:

- ▷ covering-based (monodromy, $\mathrm{Aut}(\pi_1(\check{R}^*))$-action; Schmithüsen '04, Freidinger '08 Shinomiya '12)
- ▷ singularity-based (Delaunay/Voronoi; Bowman '10, Mukarel '17, Edwards-Sanderson-Schmidt [5] '22)
- ▷ tiling-based (parallelogram decomposition, cut-and-paste construction; forklore result / K. [12] '23)

Main result: for **origamis (square-tiled surfaces)**

- ▷ Relationship among *covering, singularity, and modification of tiling*
- ▷ Categorical description of singularity-based method

1.1 Category

A composable concept is regarded as an **arrow** connecting two **objects**. The system of compositions of arrows with certain axioms is called a **category**. In short, given a category, the following are assumed:

1. the **domain** R and the **codomain** S for every arrow $(R \xrightarrow{f} S)$
2. the **composition** $(R \xrightarrow{f}\xrightarrow{g} T)$ of every (composable) two arrows $(R \xrightarrow{f} S)$, $(S \xrightarrow{g} T)$
3. the **associative law**; $(\xrightarrow{f}(\xrightarrow{g}\xrightarrow{h})) = ((\xrightarrow{f}\xrightarrow{g})\xrightarrow{h})$
4. the **identity law**; every object R admits an **identity arrow** $(R \xrightarrow{1_R} R)$; $(R \xrightarrow{1_R}\xrightarrow{f} S) = (R \xrightarrow{f} S) = (R \xrightarrow{f}\xrightarrow{1_S} S)$

A **functor** is a correspondence between two categories C_1, C_2

$$F = (F_{\mathcal{O}} : \mathrm{Obj}_{C_1} \to \mathrm{Obj}_{C_2},\ F_{\mathcal{A}} : \mathrm{Arr}_{C_1} \to \mathrm{Arr}_{C_2})$$

$$\text{compatible with} \begin{cases} \text{domains and codomains} & : F_{\mathcal{A}}(R \xrightarrow{f} S) = (F_{\mathcal{O}}(R) \xrightarrow{F_{\mathcal{A}}(f)} F_{\mathcal{O}}(S)), \\ \text{composition} & : F_{\mathcal{A}}(R \xrightarrow{f}\xrightarrow{g} S) = (F_{\mathcal{O}}(R) \xrightarrow{F_{\mathcal{A}}(f)}\xrightarrow{F_{\mathcal{A}}(g)} F_{\mathcal{O}}(S)), \text{ and} \\ \text{identity arrows} & : F_{\mathcal{A}}(R \xrightarrow{1_R} R) = (F_{\mathcal{O}}(R) \xrightarrow{1_{F_{\mathcal{O}}(R)}} F_{\mathcal{O}}(R)). \end{cases}$$

We say that an **n-arrow** connects two $(n-1)$-arrows respecting structures of all the lower $(1, 2, ..., n-1\text{-})$ arrows.

Example A (a group) A group is a category such that Obj $= \{\bullet\}$ and all arrows are invertible.

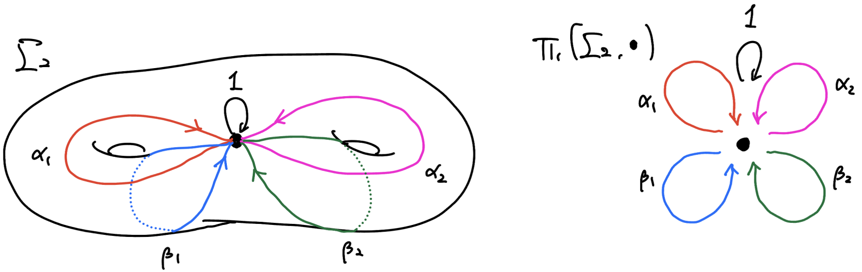


Figure 1 the fundamental group $\pi_1(\Sigma_2, \bullet)$ consist of arrows $\alpha_1, \beta_1, \alpha_2, \beta_2 : \bullet \to \bullet$ and their inverse arrows and compisitions.

The **ribbon graph structure** ordering arrows around vertices embedded on the surface is regarded as a system of 2-arrows.

Example B (category of groups) Obj $= \{G : \text{group}\}$, Arr $= \{(G_1 \xrightarrow{f} G_2) : \text{group homomorphism}\}$.

1.2 Quick review of Teichmüller theory & related results

Example C (Teichmüller vs Flat)

<u>**Teich**</u>

$$\begin{tikzcd} R \arrow[loop left, "{[\gamma]}"] \arrow[r, "{[f]}"] \arrow[r, bend left=60] \arrow[r, bend right=60, "{[f\circ\gamma^{-1}]}"'] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow{[\gamma]^*}"] \arrow[rr, bend left=70, "{[g]}"] & S \arrow[r, "{[g\circ f^{-1}]}"] & T \; \cdots \end{tikzcd} \qquad \left\{\begin{array}{l} \text{obj: Riemann surfaces}/_{\text{biholo.}} \\ \text{arr: qc. homeo.}/_{\text{homotopy}} \\ \text{arr}^2\text{: pullback of MCG} \end{array}\right.$$

The set $\mathrm{Arr}(R, -)$ of arrows associated to R is called the **Teichmüller space** $T(R)$. It is known that

- $T(R)$ is a simply-connected complex manifold of dimension $(3g - 3 + n)$ (Ahlfors-Bers).
- The **Teichmüller distance** that measures the minimal dilatation of the natural arrow $[g \circ f^{-1}]$ between arrows $[f], [g]$ coincide with the Kobayashi distance wrt. the complex structure of $T(R)$.
- The set of arrows associated to R modulo 2-arrows is the moduli space $M(R)$.

Proposition 4 (Teichmüller's theorem [1]) For any arrow $(R \xrightarrow{[f]} S)$ in <u>**Teich**</u>, there uniquely exists an arrow $((R, \phi) \xrightarrow{F_{[f]}} (S, \psi))$ in <u>**Flat**</u> (defined next) such that

1. $F : (R, \phi) \to (S, \psi)$ is an affine deformation of the form $z_\psi \circ F \circ z_\phi^{-1}(z) = \dfrac{z + k\bar{z}}{1 - k}$,
2. F attains the minimum of $K(h) := \dfrac{1 + \|h_{\bar{z}}/h_z\|_\infty}{1 - \|h_{\bar{z}}/h_z\|_\infty}$, $h \in [f]$: weakly x, y-differentiable (ACL), where $K(F) = \dfrac{1 + k}{1 - k}$.

Flat

$$\begin{tikzcd}
(R,\phi) \arrow[loop above, "F_{[\gamma]}"] \arrow[r, bend left] \arrow[r, "F_{[f]}"'] \arrow[r, bend right] \arrow[r, phantom, shift left=2ex, "\Downarrow"] \arrow[r, phantom, shift right=2ex, "\Downarrow"] \arrow[d, dotted, "\times"] & (S,\psi) \arrow[r] \arrow[d, dotted, "\times"] & \cdots \\
(R,\phi') \arrow[r, "F_{[f\circ\gamma^{-1}]}"'] \arrow[r, phantom, shift left=3ex, "\Downarrow{[\gamma]^*}"] & (S,\psi') &
\end{tikzcd}$$

$$\begin{cases} \text{obj: flat surfaces} = (\text{Riemann surface, h.q.d.})/_{\text{HTrans.}} \\ \text{arr: affine deformations} \\ \text{arr}^2\text{: Teichmüller's theorem} \circ \text{pullback of MCG} \end{cases}$$

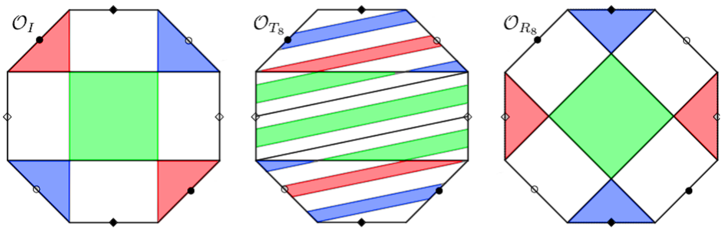
By Teichmüller's theorem, the **forgetful functor** **Flat**$\to$**Teich**$\colon (R,\phi) \mapsto R$ embeds all arrows but rotations. For each fixed (R,ϕ), it induces an embedding ι_ϕ of $\mathbb{H} = \{_{\text{HTrans.}}\backslash\text{affine deformations}/_{\text{rotations}}\}$ into $T(R)$.

$\iota_\phi(\mathbb{H})$

$$\begin{tikzcd}
(R,\phi) \arrow[loop above, "F_{[\gamma]}"] \arrow[r, bend left] \arrow[r, "f_t"] \arrow[r, bend right, "f_{D_\gamma^{-1}t}"'] \arrow[r, phantom, shift left=2ex, "\Downarrow"] \arrow[r, phantom, shift right=2ex, "\Downarrow{[\gamma]^*}"] & (R_t,\phi_t) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{cases} \text{obj: Flat surfaces} \\ \text{arr: } _{\text{HTrans.}}\backslash\text{affine deformation}/_{\text{rotation}} \\ \text{arr}^2\text{: pullback of } \mathrm{Arr}((R,\phi),(R,\phi)) \\ \qquad = \text{Möbius transformation of } \Gamma(R,\phi)^* \end{cases}$$

The group $\Gamma(R,\phi) := D(\mathrm{Arr}(R,\phi;R,\phi)) < PSL(2,\mathbb{R})$ that acts on $\mathbb{H} = \mathrm{Arr}(R,\phi;-) \hookrightarrow T(R)$ as 2-arrows is called the **Veech group** of (R,ϕ). It is the group of matrices of affine self-symmetry.



The GAGA theorem [11] implies $\underline{\textbf{Riemann}}\left\{\begin{array}{l}\text{obj: cpt. Riemann surfaces}\\ \text{arr: holomorphic coverings}\end{array}\right\} \sim \underline{\textbf{Curve}}\left\{\begin{array}{l}\text{obj: nonsing. proj. curves}\\ \text{arr: rational maps}\end{array}\right\}$.

Arrows of **Teich** (=higher arrows of **Riemann**) are understood as follows.

Proposition 6 (Benirschke-Serván [3], 2023) In case $2g-2+n \geq 3$, *any* d_T-isometric embedding $T(R) \hookrightarrow T(\hat{R})$ arises from a **covering construction** up to pre-/post- compositions of MCGs. **It** is given by a map

$$h^* : T(R) \to T(\hat{R}) : [R \xrightarrow{f} S] \mapsto [\hat{R} \xrightarrow{\hat{f}} \hat{S}] \ s.t.\ (\hat{S} \xrightarrow{\hat{f}^{-1}} \hat{R} \xrightarrow{h} R \xrightarrow{f} S) : \text{holomorphic}$$

for some holomorphic covering $h : \hat{R} \to R$ of totally marked surfaces ($\partial R \subset \mathrm{Crit}(h)$, $\partial \hat{R} = h^{-1}(\partial R)$), in combination with forgetting marked points $\Big(T(S \setminus B) \xrightarrow{h^*} T(\hat{S} \setminus h^{-1}(B))\Big) \mapsto \Big(T(S) \xrightarrow{h^*_F} T(\hat{S})\Big)$.

Proposition 7 (Haiden-Katzarkov-Kontsevich [9], 2017) For a marked surface of finite type, the moduli space of marked flat structures is bianalytically embedded into the set of **stability conditions of the Fukaya category** of S.

Surface	Triangulated category
mapping class group	group of autoequivalences
simple curve c	spherical object C
intersection number $i(c_1, c_2)$	$\mathrm{dimArr}^*(C_1, C_2)$
flat metric μ	stability condition σ
μ-geodesic	σ-semistable object
$\mathrm{length}_\mu(c)$	$\mathrm{mass}_\sigma(C)$

Table 1 (Heng [10]) stability condition theory : categorical version of Teichmüller theory

2 Voronoi decomposition and Edwards-Sanderson-Schmidt method

Definition 8 We denote by Q_g the set of flat surfaces of genus g. We denote by $\mathcal{A}_g$ (resp. $\mathcal{Q}_g := Q_g \setminus \mathcal{A}_g$) the subset of abelian (resp. non-abelian) flat surfaces in Q_g. For $\mathcal{H} = Q, \mathcal{A}, \mathcal{Q}$, the **stratum** $\mathcal{H}_g(k_1, ..., k_n) \subset \mathcal{H}$ is defined by assigning singular orders $k_1, ..., k_n$ satisfying the Riemann-Hurwitz formula $\sum k_i = 4g - 4$.

(Hubbard-Masur, Veech[6]) The stratum $\mathcal{A}_g(k_1, ..., k_n)$ is a complex $(2g-1+n)$-dimensional orbifold locally parametrized by the **period map**

$$\Pi(R, \omega) := ([c] \mapsto \int_c \omega) \in H^1(R, \mathrm{Sing}(R, \omega); \mathbb{C}) \cong \mathbb{C}^{2g-1+n}.$$

For each $(R, \phi) \in \mathcal{Q}_g(k_1, ..., k_n)$, an analytic continuation of $\sqrt{\phi}$ defines a **canonical double** $\pi : \hat{R} \to R$ and an abelian flat surface $(\hat{R}, \hat{\phi} = \pi^*\phi) \in \mathcal{A}_{\hat{g}}(\hat{k}_1, ..., \hat{k}_{\hat{n}})$. The Riemann-Hurwitz formula implies $2\hat{g} - 1 + \hat{n} = 4g - 3 + 2n$, and every local image of Π splits into two eigenspaces $\Pi(\mathcal{A})_{\mathrm{local}} \oplus \Pi(\mathcal{Q})_{\mathrm{local}} \subset \mathbb{C}^{2g-1+n} \oplus \mathbb{C}^{2g-2+n}$ w.r.t. the linear involution $\sigma : \sqrt{\phi} \mapsto -\sqrt{\phi}$. In particular, the non-abelian stratum $\mathcal{Q}_g(k_1, ..., k_n)$ is a complex $(2g-2+n)$-dimensional orbifold.

flat surface	stability condition
direction in the canonical trivialization	grading
local 2π-rotation z	shift functor [1]
geodesics of direction θ; $\xrightarrow{z} \theta + 2\pi$	semistable objects of phase $\frac{\theta}{2\pi}$; $\xrightarrow{[1]} \frac{\theta}{2\pi} + 1$
period map Π	central charge Z
return to canonical double	base change $\mathbb{Z} \mapsto \mathbb{Z} \otimes_{\mathbb{Z}_2} \sigma$

Definition 9 A geodesic connecting two singularities is called a **saddle connection**. A geometric triangulation (i.e. all edges are geodesics) is called **Delaunay** if no circumcircle contains a vertex in its interior. Delaunay triangles sharing common circumcircle are called **degenerate**. We say that a **flip** replaces two triangles pqr, rsp by qrs, spq.

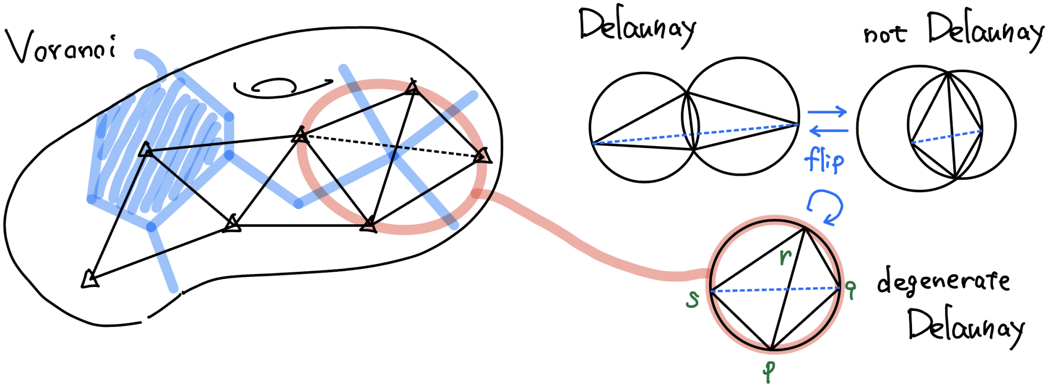


Figure 2 Delaunay triangulation and flip. Voronoi decomposition is obtained by prependicular bisectors of edges.

The **Voronoi decomposition** of a flat surface (R, ϕ) with $\mathrm{Sing}(R, \phi) = \{p_1, ..., p_n\}$ is the cell decomposition whose 2-cells C_{p_i}, $i = 1, ..., n$ are given by

$$C_{p_i} := \left\{ x \in R \;\middle|\; \exists! \text{ shortest saddle connection from } x \text{ to } \mathrm{Sing}(R, \phi) \text{ with terminus } p_i \right\}.$$

It is combinatorially dual to the Delaunay triangulation of (R, ϕ).

Definition 10 The **canonical surface** of the stratum $Q_g(k_1, ..., k_n)$ is the infinite flat surface

$$\Delta = \Delta(k_1, ..., k_n) = \bigsqcup_{i=1}^{n} \Delta_i := \bigsqcup_{i=1}^{n} (\mathbb{C}, z^{k_i} dz^2).$$

We have $\mathrm{Htrans}(\Delta) \cong \prod_{i=1}^{n} C_{2(k_i+1)} \times \prod_{t=1}^{\infty} \mathrm{Sym}\{i \mid k_i = t\}$, where each $C_{2(k_i+1)}$ acts on Δ_i by π-rotation and each $\mathrm{Sym}\{i \mid k_i = t\}$ permutates Δ_i's of the same degree.

Lemma 11 There exists a unque embedding $\iota = \iota_{(R,\phi)}$ of Voronoi 2-cells C_{p_i} into Δ_i modulo $\mathrm{Htrans}(\Delta)$. The embedding ι extends to any star-like region in R^*, and it follows that $(R, \phi) \cong \left(\bigsqcup_{i=1}^{n} \overline{\iota(C_{p_i})}\right)/_\sim$ for a suitable edge identification $\sim$.

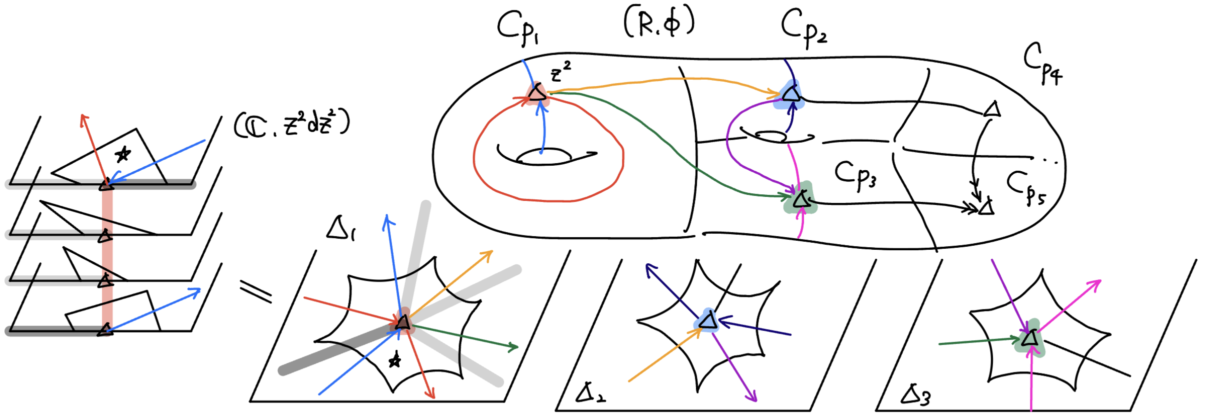


Figure 3 Voronoi decomposition and embedded image of C_{p_i} in Δ_i.

Definition 12 Fix an embedding $\iota = \iota_{(R,\phi)}$ in Lemma 11. For each oriented saddle connection $s(p_i \to p_j)$, let $\iota(s) \in \Delta_i$ be the extension of $\iota\mid_{C_{p_i}}$ along s. The inverse is denoted by $s^{-1}(p_j \to p_i)$ where $\iota(s^{-1}) \in \Delta_j$. We define

$$\widehat{\mathcal{M}}(R,\phi) := \Big\{(\iota(s),\iota(s^{-1})) \in \Delta \times \Delta \;\Big|\; s : \text{ori. saddle conn. on } (R,\phi)\Big\},$$
$$\widehat{\mathcal{S}}(R,\phi) := \Big\{(\iota(s),\iota(s^{-1})) \in \widehat{\mathcal{M}}(R,\phi) \;\Big|\; s : \text{prep. bisector of a Voronoi 1-cell}\Big\}.$$

The finite set $\widehat{\mathcal{S}}(R,\phi)$ is called the set of **Voronoi staples** of (R,ϕ).

Each $(\iota(s),\iota(s^{-1})) \in \widehat{\mathcal{M}}(R,\phi)$, $(p_1 \xrightarrow{s} p_2)$ consists of the following data modulo HTrans(Δ)-action.

- the domain p_1 and the codomain p_2
- the passing sheets of Δ_1 and Δ_2; in $\mathbb{Z}/(k+2)\mathbb{Z}$
- the holonomy vector $hol(s) = \int_s \sqrt{\phi} \in \mathbb{C}$

Lemma 13 $(R_1,\phi_1),(R_2,\phi_2)$ are equal in **Flat** if and only if $\widehat{\mathcal{S}}(R_1,\phi_1),\widehat{\mathcal{S}}(R_2,\phi_2)$ are HTrans(Δ)-equivalent.

Proposition 14 (Edwards-Sanderson-Schmidt [5], 2022) A matrix $A \in PSL(2,\mathbb{R})$ belongs to $\Gamma(R,\phi)$ if and only if

$$\exists\gamma \in \text{Htrans}(\Delta) \text{ s.t. } \gamma(A \cdot \widehat{\mathcal{S}}(R,\phi)) \subset \widehat{\mathcal{M}}(R,\phi). \tag{1}$$

Furthermore, for $\| \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \|_{\text{Frob}} := \sqrt{a^2+b^2+c^2+d^2}$ and $\text{Sys}(R,\phi) := \min\{l(s) | s : \text{saddle conn. of } (R,\phi)\}$ we have

$$\text{diam}(A \cdot \widehat{\mathcal{S}}(R,\phi)) < \|A\|_{\text{Frob}} \cdot \text{Sys}(R,\phi). \tag{2}$$

Example D (Cathedral polygon [McMullen-Mukarel-Wright, 2017]) The cathedral polygon $\hat{C}(a,b)$, $a,b>0$ is a flat surface in $\mathcal{A}_4(4^3,0^3)$. It is the canonical double of the flat surface $C(a,b)$ in $\mathcal{Q}_1(-1^3,1^3)$ shown below.

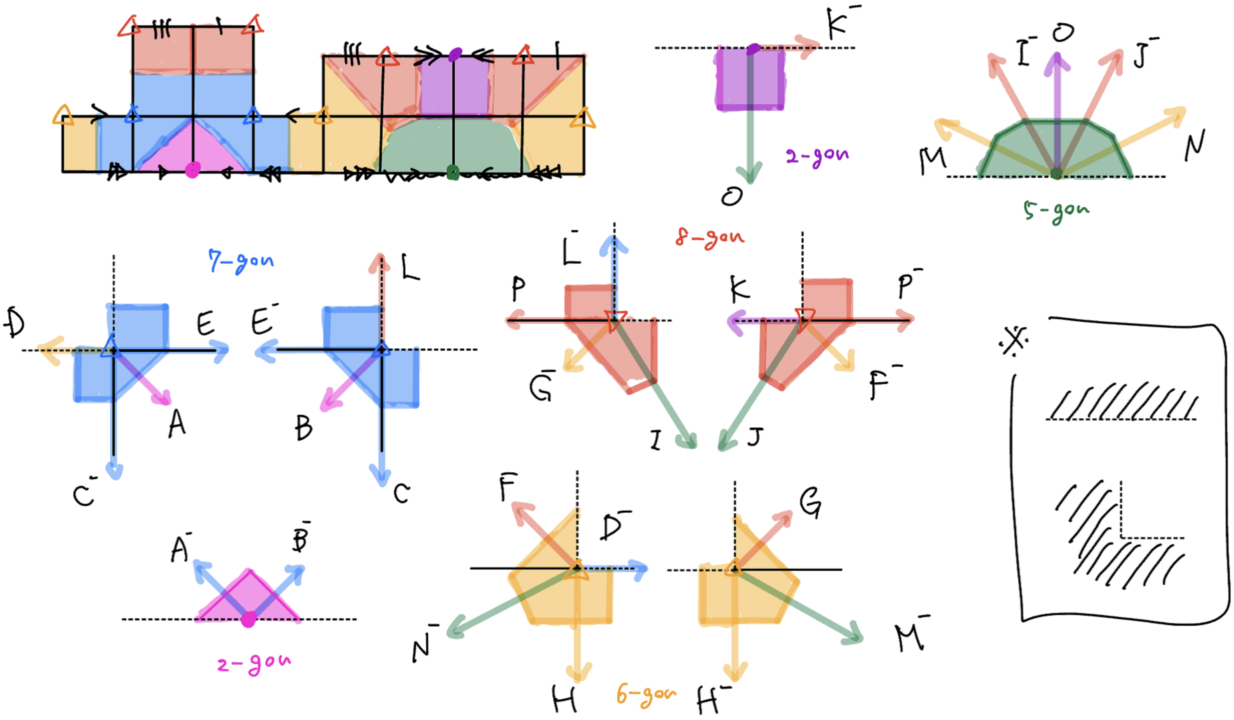


Figure 4 Voronoi decomposition of $C(a,b)$. There are just n staples emanating from an n-gon Voronoi 2-cell.

For each $a > 0$, the set $\Gamma^a(R,\phi) := \{A \in PSL(2,\mathbb{R}) \mid \exists\gamma \in \mathrm{Htrans}(\Delta) \text{ s.t. } \gamma(A \cdot \widehat{\mathcal{S}}(R,\phi)) \subset \widehat{\mathcal{M}}(R,\phi) \text{ and } \|A\|_{\mathrm{Frob}} < a\}$ is a computable, finite subset in $\Gamma(R,\phi)$. Denote its convex hull by $\Omega(\Gamma^a) = \bigcap_{A\in\Gamma^a}\{\tau \in \mathbb{H} \mid d_{\mathbb{H}}(i,\tau) \leq d_{\mathbb{H}}(\gamma_A(i),\tau)\}$.

Proposition 6 (Edwards-Sanderson-Schmidt [5], 2022) If $a \geq \sqrt{2}$ satisfies

$$\mu_{\mathbb{H}}(\Omega(\Gamma^a)) < 2\mu_{\mathbb{H}}\left(\Omega(\Gamma^a) \cap B\left(i, \log\sqrt{\frac{a^2-\sqrt{a^4-4}}{2}}\right)\right) \tag{3}$$

then, $\Gamma^a(R,\phi)$ generates $\Gamma(R,\phi)$. In particular, $\Gamma(R,\phi)$ is a lattice in this case.

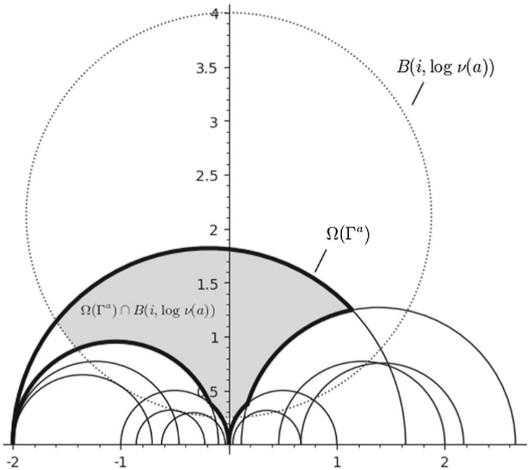


Figure 5 Lattice test (3): cited from Edwards-Sanderson-Schmidt [5, Fig. 7]

3 Main result

3.1 Origami and Delaunay triangulation

Fix $d \in \mathbb{N}$ and let $\Lambda := \{\pm 1, \ldots, \pm d\}, \mathfrak{S} := \mathrm{Sym}(\Lambda)$. A flat surface obtained by gluing d unit square cells at edges by half-translations is called an **origami** of degree d. The Veech group of an origami is a lattice in $PSL(2, \mathbb{Z})$. Möller [16] showed that the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action respects the Teichmüller embedding of the arithmetic curve $\mathbb{L}/\Gamma(\mathcal{O}) \hookrightarrow M_{g,n}$.

Example D For $x, y \in S_d$, an abelian origami (x, y) is defined by the gluing rule (right edge of $i \leftrightarrow$ left edge of $x(i)$) and (upper edge of $i \leftrightarrow$ lower edge of $y(i)$), where the d squares are labelled by $i = 1, ..., d$. Its Veech group is a stabilizer under the following $SL(2, \mathbb{Z}) \cong \mathrm{Out}^+(F_2)$-action (Schmithüsen[17]).

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}(x, y) = (x, xy), \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(x, y) = (y, x^{-1}),$$

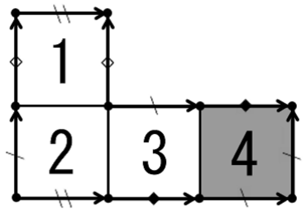


Figure 6 Abelian origami (x, y) where $x = (1)(2\ 3\ 4)$, $y = (1\ 2)(3\ 4)$.

Example E (K. 2023 [12]) All possible patterns of origamis are obtained by considering the **cut-and-paste construction** with respect to the origami (x, y) and all the negative cells, where $x, y \in S_d < \mathfrak{S}_{\mathrm{odd}}$, $\varepsilon \in \{\pm 1\}^{\Lambda}_{\mathrm{odd}}$. In the construction, The canonical double is represented by the abelian origami $(x^{\mathrm{sign}}, \varepsilon y^{\varepsilon} \varepsilon(y^{\varepsilon})) \in \mathfrak{S} \times \mathfrak{S}$.

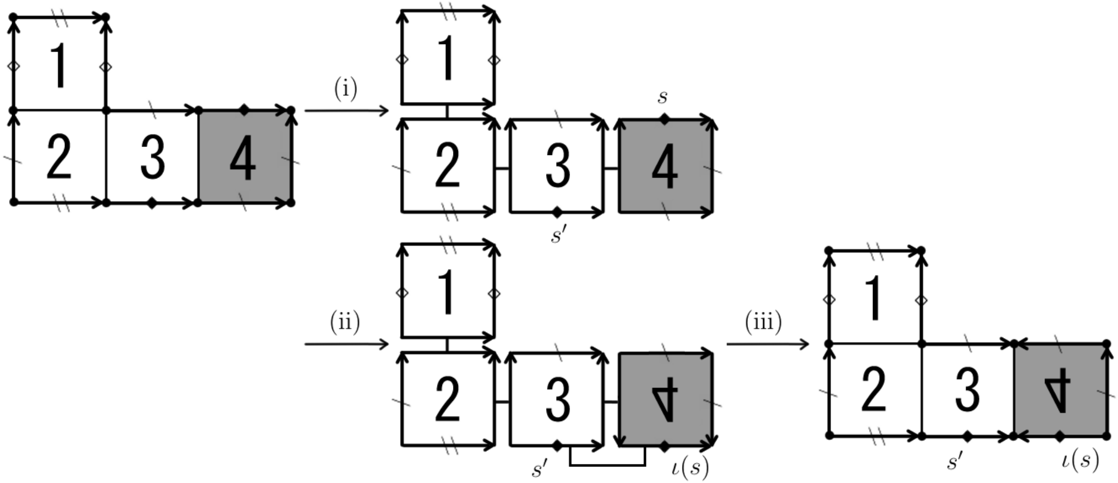


Figure 7 Cut-and-paste construction for the case $x = (1)(2\ 3\ 4)$, $y = (1\ 2)(3\ 4)$, $\varepsilon = (+, +, +, -)$.

- In the step (i), the upper side s of the 4th cell is paired with the lower side s' of the 3rd cell.
- In the step (ii), the vertical reflection ι is applied to the 4th cell.
- In the step (iii), the sides $\iota(s)$ and s' are glued by a half-translation.

Proposition 7 (K. [13]) Let $\mathfrak{S}^i := \{\sigma \in \mathfrak{S} : \text{fixed-point-free, order } 2\}$. There is one-to-one correspondence between HTrans-classes of origamis and $\mathfrak{S}$-conjugacy classes of tuples of $\mu, \nu, \tau \in \mathfrak{S}^i$ with the relationships in the following table. The canonical double of $\mathcal{O}$ is the abelian origami $(x_{\mathcal{O}}, y_{\mathcal{O}})$. We have $\mu = z_{\mathcal{O}} y_{\mathcal{O}}$, $\nu = x_{\mathcal{O}}^{-1} z_{\mathcal{O}}$, $\tau = x_{\mathcal{O}}^{-1} z_{\mathcal{O}} y_{\mathcal{O}}$, and $z_{\mathcal{O}}^2 = x_{\mathcal{O}} y_{\mathcal{O}} x_{\mathcal{O}}^{-1} y_{\mathcal{O}}^{-1}$.

$\lambda \in \Lambda$	half-square(◺ ◹)	μ, ν, τ	reflection along edges
$\lambda \cdot \langle \mu \rangle$	horiz. λ-edge	$x_{\mathcal{O}} := \mu\tau$	horiz. translation
$\lambda \cdot \langle \nu \rangle$	vert. λ-edge	$y_{\mathcal{O}} := \nu\tau$	vert. translation
$\lambda \cdot \langle \tau \rangle$	λ-cell	$z_{\mathcal{O}} := \mu\tau\nu$	π-rotation arr. cone
$\lambda \cdot \langle x_{\mathcal{O}} \rangle$	horiz. λ-cylinder	$\#(\lambda \cdot \langle x_{\mathcal{O}} \rangle)$	cylinder width
$\lambda \cdot \langle y_{\mathcal{O}} \rangle$	vert. λ-cylinder	$\#(\lambda \cdot \langle y_{\mathcal{O}} \rangle)$	cylinder width
$\lambda \cdot \langle z_{\mathcal{O}} \rangle$	λ-cone	$\#(\lambda \cdot \langle z_{\mathcal{O}} \rangle)$	deg(cone)+2

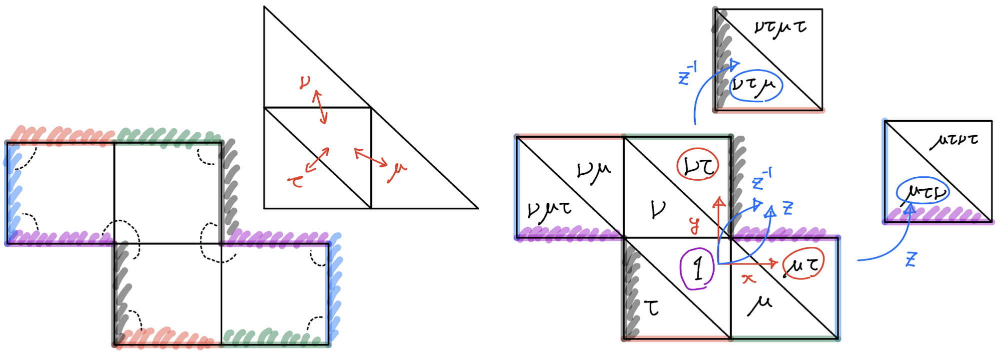


Figure 8 (x, y, z) for the origami (x, y, z) : $\mu\tau\nu(1) = \nu\mu\tau(1)$ in this case.

Let $\Theta := \mathfrak{S}^i \times \mathfrak{S}^i \times \mathfrak{S}^i / \mathfrak{S}^{\mathrm{conj}}$ and $\tau = -1 := (\lambda \mapsto -\lambda) \in \mathfrak{S}^i$. Then,we have $(\mu, \nu, \tau) = (x\tau, y\tau, \tau) = (-x, -y, -1)$.

Theorem 8 Identify $\mathcal{O} = (\mu, \nu, \tau) \in \Theta$ with the unique (◺ ◹)-Delaunay triangulation of the origami $\mathcal{O}$. Then, for each $(\mu, \nu, -1) \in \Theta$ and $\kappa = 1, \ldots, d$, the flip w.r.t. the degenerate two triangles $\pm\kappa$ is represented by $(-x_{\mathcal{O}}^{\mathrm{sign}}, -\varepsilon_\kappa y_{\mathcal{O}}^{\varepsilon_\kappa} \varepsilon_\kappa (y_{\mathcal{O}}^{\varepsilon_\kappa}), -1) \in \Theta$, where $\varepsilon_\kappa = (\pm\kappa \mapsto \mp\kappa) \in \{\pm 1\}_{\mathrm{odd}}^{\Lambda}$. Furthermore, to be square-tiled is an invariant property of triangulations under flips.

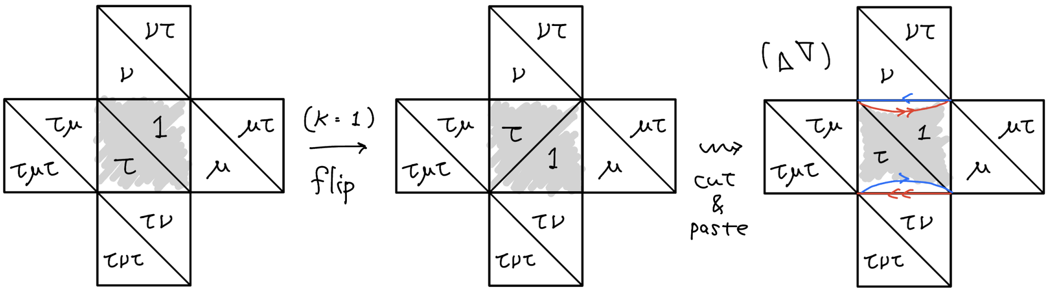


Figure 9 The (◺ ◹)-Delaunay triangulation of origami (μ, ν, τ). The flip at the square $(1(\kappa), \tau(\kappa))$ is represented by the (◺ ◹)-Delaunay triangulation of the origami given by the cut-and-paste construction in Example E.

Proposition 9 The universal Veech group $PSL(2, \mathbb{Z}) = \langle T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \rangle$ acts on $\mathcal{O} = (\mu, \nu, \tau) \in \Theta$ as follows.

$$\begin{cases} T(\mu, \nu, \tau) = (\mu^* \tau, \nu, \mu) \\ S(\mu, \nu, \tau) = (\tau^* \nu, \mu, \tau) \\ TS(\mu, \nu, \tau) = (\nu, \tau, \mu) \end{cases} \quad \begin{cases} T(x, y, z) = (x, yx^{-1}, z) \\ S(x, y, z) = (y, x, y^{-1} zy) \\ TS(x, y, z) = (yx^{-1}, x^{-1}, z^{-1}) \end{cases}$$

outline of proof) One gets the formulae for (◺ ◹)-Delaunay triangulation of $T\mathcal{O}, S\mathcal{O}$ from a local picture below.

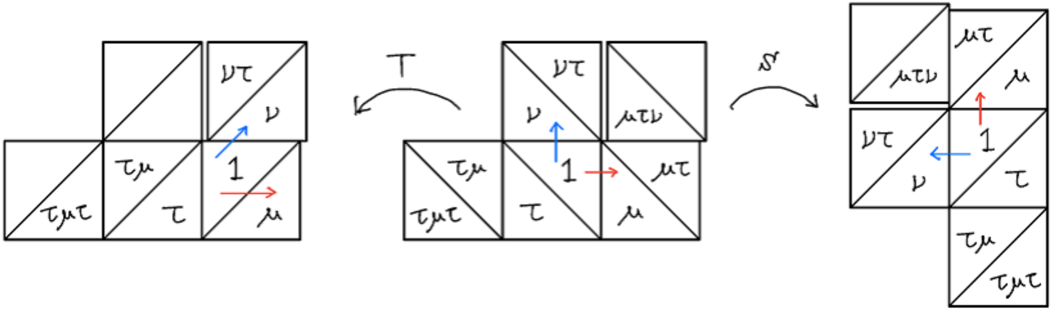


Figure 10 T (resp. S)-affine deformation of (μ, ν, τ) is the mirror of (τ, ν, μ) (resp. (ν, μ, τ)).

The simulatenous flip of (μ, ν, τ) at all cells gives the (◸ ◿)-Delaunay represented by the mirror image of the origami $(\tau^*\mu, \nu, \tau)$. The formulae for x, y, z follows from $\mu = zy$, $\nu = x^{-1}z$, $\tau = x^{-1}zy$.

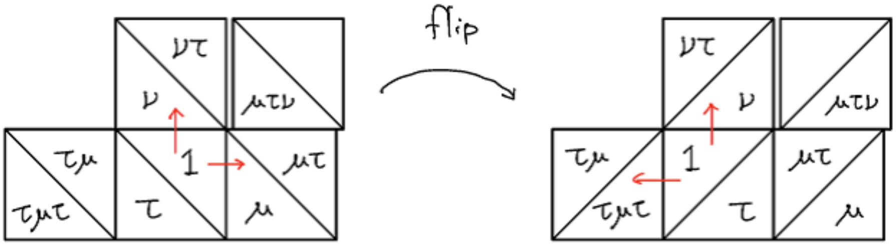


Figure 11 The simulatenous flip of (μ, ν, τ) is the mirror of $(\tau^*\mu, \nu, \tau)$.

3.2 Edwards-Sanderson-Schmidt method in terms of category

We will construct $\widehat{\mathcal{M}}, \widehat{\mathcal{S}}$ for origamis. Assume that all the corner points of squares are marked. Though singularities of order 0 should be removed with Veech groups in mind, our assumption does not matter because these marked points form a $PSL(2,\mathbb{Z})$-invariant set. We can align the Voronoi staples with the sheet cuts in this way.

Definition 10 For an origami $\mathcal{O} = (\mu, \nu, \tau) \in \Theta$, denote $x = x_{\mathcal{O}}$, $y = y_{\mathcal{O}}$, $z = z_{\mathcal{O}}$, $Z := \langle z \rangle$. We define a **groupoid** $\mathcal{G}_{\mathcal{O}}$ equipped with 2-arrows (ribbon-graph structure) and 3-arrows (relabeling) as follows.

$$\begin{cases} \mathrm{Obj}_{\mathcal{G}_{\mathcal{O}}} = \Lambda/Z \\ \mathrm{Arr}_{\mathcal{G}_{\mathcal{O}}} = \langle \lambda Z \xrightarrow{\lambda^\sigma} \lambda\sigma Z \mid \lambda \in \Lambda, \sigma = \mu, \nu, \tau \rangle \\ \mathrm{Arr}^2_{\mathcal{G}_{\mathcal{O}}} = \{\lambda^\sigma \to (\lambda z)^\sigma\} \\ \mathrm{Arr}^3_{\mathcal{G}_{\mathcal{O}}} = \mathfrak{S}\text{-conjugate} \end{cases}$$

Arrows arround vertices λZ, $\lambda \in \Lambda$ are placed as follows. We have $\#\mathrm{Arr}(\lambda Z, -) = 3\#\lambda Z = 3(\mathrm{ord}_{\lambda Z}(\phi) + 2)$.

$$\begin{tikzcd} \lambda\nu Z \arrow[d] & \lambda Z \arrow[l, "\lambda^\nu"'] \arrow[r, "(\lambda z)^\nu"] \arrow[dl, "\lambda^\tau"'] \arrow[d, "\lambda^\mu"'] & \lambda\mu\tau Z \\ \lambda\tau Z & \lambda\mu Z \arrow[l] & \end{tikzcd} \qquad \begin{tikzcd} & \lambda z\mu Z & \lambda z\tau Z \\ \lambda z^2\nu Z & \lambda z Z \arrow[l, "(\lambda z^2)^\nu"] \arrow[u, "(\lambda z)^\mu"] \arrow[ur, "(\lambda z)^\tau"] \arrow[r, "(\lambda z)^\nu"'] & \lambda\mu\tau Z \end{tikzcd}$$

By $\nu = x^{-1}z$, $\mu = y^{-1}z^{-1}$, and proposition 16, arrows are regarded as local-translations.

$$
\begin{tikzcd}
\lambda x^{-1}zZ \arrow[r, equal] & \lambda\nu Z & \lambda Z \arrow[l, "\lambda^{\nu}"'] \arrow[r, "(\lambda z)^{\nu}"] \arrow[rr, bend left, "\lambda^{x}"] \arrow[ll, bend left, "\lambda^{(x^{-1})}"] \arrow[d, "\lambda^{\mu}"'] \arrow[dd, bend right, "\lambda^{(y^{-1})}"'] \arrow[dr, phantom, "◸"] & \lambda\mu\tau Z \arrow[r, equal] & \lambda x Z \\
 & & \lambda\mu Z \arrow[ur, "(\lambda\mu)^{\tau}"'] \arrow[d, equal] & & \\
 & & \lambda y^{-1}z^{-1}Z & &
\end{tikzcd}
$$

We may define a homomorphism $h : \mathrm{Arr}(\lambda Z, -) \to (\mathbb{C}, +)$ by $h(\lambda^x) = 1$, $h(\lambda^y) = i$ and commutativity $h(⊿) = h(◸) = 0$. We obtain $\widehat{\mathcal{M}}^*(\mathcal{O}) := (\mathcal{G} \times h)_{\mathcal{O}} \in$ **Groupoid**$\times(\mathbb{C}, +)$ as follows; vector asigned to each arrows

$$
\begin{tikzcd}
 & & \lambda y^2 x Z & \\
{} & \lambda y Z \arrow[l] & & \lambda y x^2 Z \\
\lambda x^{-1} Z \arrow[u] & \lambda Z \arrow[l, "-1"', "\lambda^{(x^{-1})}"] \arrow[r, "1", "\lambda^{x}"'] \arrow[u, "i", "\lambda^{y}"'] \arrow[uur, "1+2i", "\lambda^{y} \bullet \lambda y^{y} \bullet \lambda y y^{x}"'] \arrow[urr, bend left, "2+i", "\lambda^{y} \bullet \lambda y^{x} \bullet \lambda y x^{x}"'] & \lambda x Z &
\end{tikzcd}
$$

By replacing the arrow set with $\mathrm{Arr}_{G_{\mathcal{O}}} = \{\lambda Z \xrightarrow{\lambda^{\sigma}} \lambda\sigma Z \mid \lambda \in \Lambda, \sigma = \mu, \nu, \tau\}$, we get $\widehat{\mathcal{S}}^*(\mathcal{O}) := (G \times h)_{\mathcal{O}} \in$ **Graph**$\times(\mathbb{C}, +)$. As explained as adjunction **Graph** $\underset{\perp}{\rightleftarrows}$ **Category** [15], $\widehat{\mathcal{S}}^*$ is a "finite generating system" of $\widehat{\mathcal{M}}^*$.

Theorem 11 $\widehat{\mathcal{M}}^* = (\mathcal{G} \times h)$ is a functor $\underline{\textbf{Flat}} \supset \Theta$ $(origamis) \to \underline{\textbf{Groupoid}} \times (\mathbb{C}, +)$ such that

1. $\widehat{\mathcal{M}}^*(\mathcal{O}_1 \xrightarrow{f} \mathcal{O}_2) = 1 \times (D_f)_{\text{linear}}$,
2. $\mathcal{O}_1 = \mathcal{O}_2 \Leftrightarrow \widehat{\mathcal{M}}^*(\mathcal{O}_1) = \widehat{\mathcal{M}}^*(\mathcal{O}_2) \Leftrightarrow \widehat{\mathcal{S}}^*(\mathcal{O}_1) \subset \widehat{\mathcal{M}}^*(\mathcal{O}_2)$, and
3. $A \in \Gamma(\mathcal{O}) \Leftrightarrow A\widehat{\mathcal{S}}^*(\mathcal{O}) \subset \widehat{\mathcal{M}}^*(\mathcal{O})$.

Note that the inclusions in 2. 3. implies that the arrows are embedded modulo $\cong$ Htrans(Δ); 3-arrows preserving 2-arrows.

Edwards-Sanderson-Schmidt method	categorical version for origamis
flat surface (R, ϕ)	origami $\mathcal{O} = (\mu, \nu, \tau)$
singularity p	orbit λZ
Voronoi staple $(\iota(s), \iota(s^{-1}))$	generating arrow $\lambda Z \xrightarrow{\lambda^\sigma} \lambda\sigma Z$
saddle connection $(\iota(s), \iota(s^{-1}))$	arrow (piecewise line)
sheet transition on Δ	arrow^2 $\lambda^\sigma \to (\lambda z)^\sigma$
automorphism (Htrans(Δ))	arrow^3 ($\mathfrak{S}$-conjugate)
combinatorial structure of $\widehat{\mathcal{M}}$, $\widehat{\mathcal{S}}$	groupoid $\mathcal{G}$, group G
holonomy $(hol(s), hol(s^{-1}))$	vector $h(\lambda^\sigma)$
finite bound for $A\widehat{\mathcal{S}}$	finite arrows of bounded length

Table 2 categorical version of Edwards-Sanderson-Schmidt method

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