Spirallikeness, strong starlikeness and quasiconformal extension of harmonic univalent functions

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Outline

Outline for section 1

Analytic univalent functions[1,2]

Let $\mathbb{C}$ be the complex plane. A single-valued function f is said to be univalent in a domain $\Omega \subset \mathbb{C}$ if f is injective in Ω. f is called locally univalent at a point z if it is univalent in some neighborhood of $z \in \Omega$. For analytic function f, the local univalence at z is equivalent to that $f'(z) \neq 0$. An analytic univalent function is called a conformal mapping.

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $\mathcal{A}$ denote all analytic functions of $\mathbb{D}$. Consider the following classes of analytic functions: $\mathcal{A}_1 := \{f \in \mathcal{A} : f(0) = f'(0) - 1 = 0\}$; let $\mathcal{S}$ denote the univalent functions in $\mathcal{A}_1$.

For $f \in \mathcal{A}_1$, it has the expansion

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$

[1] C. Pommerenke. "Univalent functions". In: *Vandenhoeck and Ruprecht* (1975).

[2] P. Duren. *Univalent Functions*. Vol. 259. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, New York, 1983.

Bieberbach conjecture

For any $f \in \mathcal{S}$, with the series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D},$$

the coefficients of f satisfy

$$|a_n| \le n, \quad n \ge 2.$$

This conjecture was presented by Bieberbach in 1916, and finally proved by de Branges[3] in 1985. The equality holds for

$$k_\alpha(z) = \frac{z}{(1-\alpha z)^2} = \sum_{n=1}^{\infty} n\alpha^{n-1} z^n,$$

where $|\alpha| = 1$.

[3] L. de Branges. "A proof of the Bieberbach conjecture". In: *Acta Math.* 154.1 (1985), pp. 137–152.

Koebe function

Koebe function:

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n.$$

With a modified form

$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4},$$

it maps the unit disk onto a slit domain $\mathbb{C}\backslash(-\infty, -1/4]$.

Koebe one-quarter theorem states that the range of each function $f \in \mathcal{S}$ contains a disk $\{w : |w| < 1/4\}$ and the constant is sharp. This theorem was conjectured by Koebe in 1907, and proved by Bieberbach in 1916.

Starlike and convex functions

A domain Ω is said to be starlike (with respect to the origin) if the line segment joining the origin and any point in Ω lies entirely in Ω. If Ω is starlike with respect to each point in the domain, then Ω is called a convex domain.

A function $f \in \mathcal{A}_1$ is starlike/convex if f maps the unit disk $\mathbb{D}$ univalently onto a starlike/convex domain. We denote the class of starlike and convex functions by $\mathcal{S}^*$ and $\mathcal{K}$, respectively. The inclusions $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}$ hold.

Starlike and convex functions

Suppose that $f \in \mathcal{A}_1$. Then

(1) $f \in \mathcal{S}^*$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

(2) $f \in \mathcal{K}$ if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

(3) $f \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$.

The condition (3) is called Alexander theorem, proved by Alexander[4] in 1915.

[4] J. W. Alexander. "Functions which map the interior of the unit circle upon simple regions". In: *Ann. of Math.* 17.1 (1915), pp. 12–22.

Strongly starlike functions

Let α be a real number with $0 < \alpha \le 1$.

A function f in $\mathcal{A}_1$ is called strongly starlike of order α if f satisfies

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \le \frac{\pi\alpha}{2}, \quad z \in \mathbb{D}.$$

This notion was first introduced by Stankiewicz[5] and Brannan and Kirwan[6], independently. We denote the class of strongly starlike functions of order α by $\mathcal{SS}(\alpha)$. Clearly, $\mathcal{SS}(1) = \mathcal{S}^*$.

A domain Ω containing the origin is called a strongly starlike domain of order α if $f \in \mathcal{SS}(\alpha)$ maps the unit disk onto Ω.

[5] J. Stankiewicz. "Quelques problèmes extrèmaux dans les classes des fonctions α-angulairement ètoilèes". In: *Ann. Univ. Mariae Curie-Skłodowska, Sect. A* 20 (1966), pp. 59–75.

[6] D. A. Brannan and W. E. Kirwan. "On some classes of bounded univalent functions". In: *J. London Math. Soc.* 2.1 (1969), pp. 431–443.

Spirallike functions

Let λ be a real number with $|\lambda| < \pi/2$.

A plane curve of the form

$$w = w_0 \exp(te^{i\lambda}),\ t \in \mathbb{R},$$

for some $w_0 \in \mathbb{C}\backslash\{0\}$ is called a λ-spiral (about the origin). A λ-spiral segment is formed by

$$[0, w_0]_\lambda := \{w_0 \exp(te^{i\lambda}) : t \leq 0\} \cup \{0\}.$$

Then a domain Ω is called a λ-spirallike domain (with respect to the origin), if for any point $w \in \Omega$, the λ-spiral segment $[0, w]_\lambda$ is contained in Ω. The spiral segment reduces to a line segment if $\lambda = 0$. A 0-spirallike domain is thus a starlike domain.

A function $f \in \mathcal{A}_1$ is called a λ-spirallike function if f maps $\mathbb{D}$ univalently onto a λ-spirallike domain.

Spirallike functions

The notion of spirallike functions was introduced by Špaček[7]. We denote the class of λ-spirallike functions by $\mathcal{SP}(\lambda)$. The class $\mathcal{SP}(0)$ is the class of starlike functions.

Let $f \in \mathcal{A}_1$. Then $f \in \mathcal{SP}(\lambda)$ if and only if

$$\mathrm{Re}\left\{e^{-i\lambda}\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

Together with the inequality, it shows a relation between spirallikeness and strong starlikeness. Let $w = zf'/f$ and let H_λ be a slanted half-plane $\{w \in \mathbb{C} : \mathrm{Re}\{e^{-i\lambda}w\} > 0\}$. Then we obtain

$$\{w : |\arg w| \le \pi\alpha/2\} = H_\lambda \cap H_{-\lambda}$$

if $\lambda = \pi(1-\alpha)/2$ holds for $0 < \alpha \le 1$. Then

$$\mathcal{SS}(\alpha) = \mathcal{SP}\left(\frac{\pi(1-\alpha)}{2}\right) \cap \mathcal{SP}\left(-\frac{\pi(1-\alpha)}{2}\right).$$

[7] L. Špaček. "Příspěvek k teorii funkcí prostých". In: *Časopis pro pěstování matematiky a fysiky* 62.2 (1933), pp. 12–19.

An example of λ-spirallike functions is given by Duren[8] of the form

$$f(z) = z(1-z)^{-2e^{i\lambda}\cos\lambda},$$

which is a generalized form of the Koebe function. It maps the unit disk onto the complement of an arc of an λ-sprial.

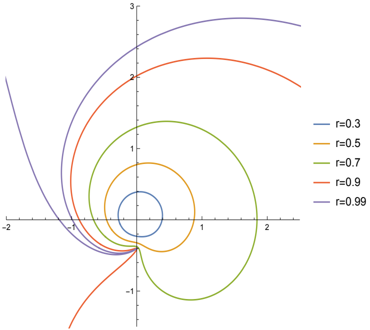
When $\lambda = -\pi/3$, we plot the figure of the function

$$f(z) = z(1-z)^{-\frac{1}{2}+\frac{\sqrt{3}}{2}i}$$

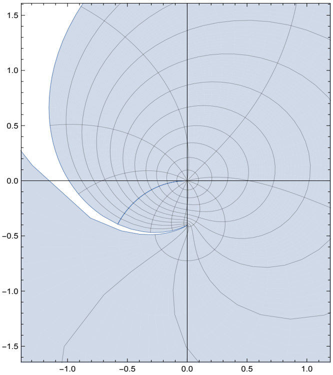
for $|z| \leq 0.99$.

[8] P. Duren. *Univalent Functions*. Vol. 259. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, New York, 1983.

Spirallike functions



Images of $f(|z| = r)$



$(-\frac{\pi}{3})$-spirallike domain

Quasiconformal mappings

Let $w = f(z)$ be a C^1 homeomorphism of Ω, where $w = u + iv$, $z = x + iy$. Then $dw = du + idv = f_z dz + f_{\bar{z}} d\bar{z}$, where

$$f_z = \frac{1}{2}(f_x - i f_y), \quad \text{and} \quad f_{\bar{z}} = \frac{1}{2}(f_x + i f_y).$$

The Jacobian of f is given by

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2.$$

If $J_f(z) > 0$ for $z \in \Omega$, then f is said to be orientation-preserving.

Quasiconformal mappings were first introduced by Grötzsch[9] in 1928, and named by Ahlfors[10] in 1935.

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[9] H. Grötzsch. "Über einige Extremalprobleme der konformen Abbildung". In: *Ber. Verh. sächs. Akad. Wiss. Leipzig, Math.-phys. Kl* 80 (1928), pp. 367–376.

[10] L. V. Ahlfors. "Zur theorie der überlagerungsflächen". In: *Acta Math.* 65.1 (1935), pp. 157–194.

Quasiconformal mappings

An orientation-preserving homeomorphism $f : \Omega \to \Omega'$ is called a K-quasiconformal mapping if $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and satisfies

$$\frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \leq K,$$

for almost every $z \in \Omega$ and some constant $K \geq 1$. A function in C^1 is conformal if and only if it is 1-quasiconformal.

Write $\mu_f(z) = f_{\bar{z}}(z)/f_z(z)$. The above inequality is equivalent to

$$|\mu_f(z)| \leq \frac{K-1}{K+1} = k$$

for some constant $k < 1$.

Harmonic functions

A real-valued function $u = u(x, y)$ is harmonic if u satisfies Laplace's equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A complex-valued continuous function $f = u + iv$ is harmonic in a domain Ω if u and v are both real harmonic in Ω. A harmonic function f has the representation $f = h + \bar{g}$, where h and g are analytic functions in Ω. The Jacobian of f is given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2.$$

A theorem due to Lewy[11] asserts that f is locally univalent in Ω if and only if $J_f(z) \neq 0$ for any $z \in \Omega$.

[11] H. Lewy. "On the non-vanishing of the Jacobian in certain one-to-one mappings". In: *Bull. Amer. Math. Soc.* 42.10 (1936), pp. 689–692.

Harmonic univalent functions

Let $\mathcal{H}$ denote the class of all complex-valued harmonic functions defined in $\mathbb{D}$ and $\mathcal{H}_0$ the subclass of $\mathcal{H}$, normalized by $f(0) = f_z(0) - 1 = 0$. Each function $f = h + \bar{g} \in \mathcal{H}_0$ has the power series expansions for h and g by

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}.$$

Let $\mathcal{S}_{\mathrm{H}}$ be the class of orientation-preserving harmonic univalent functions in $\mathcal{H}_0$ and $\mathcal{S}_{\mathrm{H}}^0$ the subclass of $\mathcal{S}_{\mathrm{H}}$ with $g'(0) = b_1 = 0$.

Consider some subclasses of functions in $\mathcal{S}_{\mathrm{H}}$ and $\mathcal{S}_{\mathrm{H}}^0$, which map $\mathbb{D}$ onto a convex domain and starlike domain, denoted by $\mathcal{K}_{\mathrm{H}}$, $\mathcal{S}_{\mathrm{H}}^*$ and ${\mathcal{K}_{\mathrm{H}}}^0$, ${\mathcal{S}_{\mathrm{H}}^*}^0$, respectively. The inclusions $\mathcal{K}_{\mathrm{H}} \subset \mathcal{S}_{\mathrm{H}}^* \subset \mathcal{S}_{\mathrm{H}}$ and ${\mathcal{K}_{\mathrm{H}}}^0 \subset {\mathcal{S}_{\mathrm{H}}^*}^0 \subset \mathcal{S}_{\mathrm{H}}^0$ hold.

Harmonic Koebe function

Consider the harmonic Koebe function $K(z) = h(z) + \overline{g(z)}$, with

$$h(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}, \quad g(z) = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}.$$

The function K maps the unit disk onto a slit domain $\mathbb{C}\backslash(-\infty, -1/6]$.

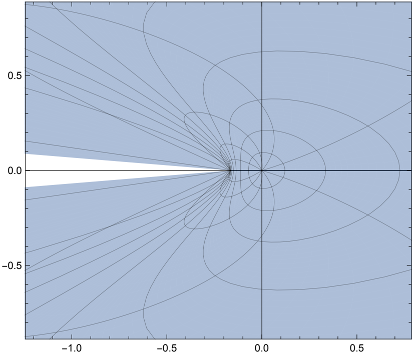


Figure: Image domain of $K(z)$

Harmonic analogue of the Bieberbach conjecture

Let $f \in \mathcal{S}_{\mathrm{H}}^0$. Then

$$\big||a_n| - |b_n|\big| \le n$$

and

$$|a_n| \le \frac{1}{6}(n+1)(2n+1), \quad |b_n| \le \frac{1}{6}(n-1)(2n-1),$$

where $n \ge 2$.

This conjecture was proposed by Clunie and Sheil-Small[12]. Sheil-Small[13] proved it for starlike functions in ${\mathcal{S}_{\mathrm{H}}^*}^0$. The equality holds if f is the harmonic Koebe function $K(z)$.

We still cannot prove it for all harmonic functions in $\mathcal{S}_{\mathrm{H}}^0$.

[12] J. Clunie and T. Sheil-Small. "Harmonic univalent functions". In: *Ann. Acad. Sci. Fenn. Ser. A I Math.* 9.1 (1984), pp. 3–25.

[13] T. Sheil-Small. "Constants for planar harmonic mappings". In: *J. London Math. Soc.* 2.2 (1990), pp. 237–248.

Outline for section 2

Al-Amiri and Mocanu's theorem

For $f \in C^1(\mathbb{D})$, we introduce a differential operator D by

$$Df(z) = zf_z(z) - \bar{z}f_{\bar{z}}(z).$$

Al-Amiri and Mocanu[14] gave a sufficient condition for a function of the class C^1 to be univalent and λ-spirallike in the unit disk $\mathbb{D}$. Let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ denote the subdisk of $\mathbb{D}$ for each $0 < r < 1$. We complete the following theorem.

Theorem

Let λ be a real number with $|\lambda| < \pi/2$. Suppose that a function $f \in C^1(\mathbb{D})$ satisfies that $f(z) = 0$ if and only if $z = 0$, and that $J_f = |f_z|^2 - |f_{\bar{z}}|^2 > 0$ on $\mathbb{D}$. Then f is univalent in $\mathbb{D}$ and $f(\mathbb{D}_r)$ is λ-spirallike for each $0 < r < 1$ if and only if

$$\mathrm{Re}\left\{e^{-i\lambda}\frac{Df(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D}\backslash\{0\}.$$

[14] H. Al-Amiri and P. T. Mocanu. "Spirallike nonanalytic functions". In: *Proc. Amer. Math. Soc.* 82.1 (1981), pp. 61–65.

Hereditary property

If a function f maps the unit disk $\mathbb{D}$ onto a convex domain, then f maps each subdisk $\mathbb{D}_r$ onto a convex domain. Such a property is called a hereditary property. That is, if an analytic function $f \in \mathcal{K}$, then $f_r(z) = f(rz)/r$ belongs to $\mathcal{K}$ for each $0 < r < 1$.

Duren[15] mentioned that convexity is not a hereditary property for harmonic univalent functions. Moreover, starlikeness does not remain a hereditary property under harmonic univalent functions.

[15] P. Duren. *Harmonic Mappings in the Plane*. Vol. 156. Cambridge Tracts in Mathematics. Cambridge university press, 2004.

Definitions

Definition

Let $-\pi/2 < \lambda < \pi/2$. A harmonic function f in $\mathcal{H}_0$ is called hereditarily λ-spirallike if f is orientation-preserving and univalent on $\mathbb{D}$ and $f(\mathbb{D}_r)$ is λ-spirallike for each $0 < r < 1$. The class of such functions will be denoted by $\mathcal{SP}_{\mathrm{H}}(\lambda)$.

Definition

Let $0 < \alpha \leq 1$. A harmonic function $f \in \mathcal{H}_0$ is called hereditarily strongly starlike of order α if it is orientation-preserving and univalent on $\mathbb{D}$ and $f(\mathbb{D}_r)$ is a strongly starlike domain of order α for each $0 < r < 1$. We denote by $\mathcal{SS}_{\mathrm{H}}(\alpha)$ the class of such functions.

Analytic characterizations

Corollary

Let $-\pi/2 < \lambda < \pi/2$ and $0 < \alpha \le 1$. Suppose that a function $f \in \mathcal{H}_0$ satisfies that $f(z) \neq 0$ for $0 < |z| < 1$ and that $J_f = |f_z|^2 - |f_{\bar{z}}|^2 > 0$ on $\mathbb{D}$. Then we have

(i) $f \in \mathcal{SP}_{\mathrm{H}}(\lambda)$ *if and only if*

$$\mathrm{Re}\left\{e^{-i\lambda}\frac{Df(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D}\backslash\{0\};$$

(ii) $f \in \mathcal{SS}_{\mathrm{H}}(\alpha)$ *if and only if*

$$\left|\arg\frac{Df(z)}{f(z)}\right| < \frac{\pi\alpha}{2}, \quad z \in \mathbb{D}\backslash\{0\}.$$

(iii) $\mathcal{SS}_{\mathrm{H}}(\alpha) = \mathcal{SP}_{\mathrm{H}}(\lambda) \cap \mathcal{SP}_{\mathrm{H}}(-\lambda)$, *where* $\lambda = \pi(1-\alpha)/2$.

Back

Uniform boundedness of strong starlikeness

Let $0 < \alpha < 1$. A result given by Brannan and Kirwan[16] shows that each function $f \in \mathcal{SS}(\alpha)$, satisfies

$$|f(z)| < |z| \exp\left\{2\alpha \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+1-\alpha)}\right\} = |z|M(\alpha) < M(\alpha)$$

for $0 < |z| < 1$. We show the uniform boundedness of harmonic strongly starlike functions.

Theorem

If $f \in \mathcal{SS}_{\mathrm{H}}(\alpha)$ with $0 < \alpha < 1$, then

$$|f(z)| \leq \frac{\pi}{2} \exp\left\{\pi \tan\left(\frac{\pi\alpha}{2}\right)\right\} = N(\alpha), \quad z \in \mathbb{D}.$$

[16] D. A. Brannan and W. E. Kirwan. "On some classes of bounded univalent functions". In: *J. London Math. Soc.* 2.1 (1969), pp. 431–443.

Uniform boundedness of strong starlikeness

We plot the following figure and obtain that

$$N(\alpha) \leq 2\pi M(\alpha)$$

for $0 < \alpha < 1$ after some computations.

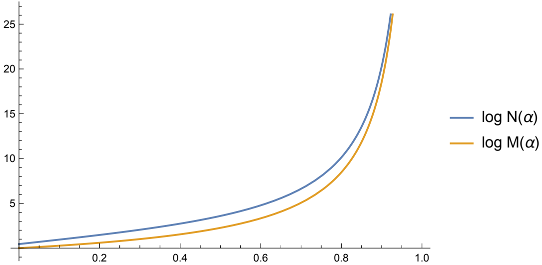


Figure: Graph of $\log M(\alpha)$ and $\log N(\alpha)$

Sufficient conditions for strong starlikeness

Let $f = h + \bar{g} \in \mathcal{H}_0$. Silverman[17] and Jahangiri[18] proved, if f satisfies $\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \le 1$, then $f \in \mathcal{S}_{\mathrm{H}}^*$. For $0 < \alpha \le 1$, we introduce two quantities for integers $n \ge 1$:

$$A_n(\alpha) = n - 1 + |n - e^{-i\pi\alpha}| \quad \text{and} \quad B_n(\alpha) = n + 1 + |n + e^{i\pi\alpha}|.$$

Theorem

Suppose that $f = h + \bar{g} \in \mathcal{H}_0$ satisfies

$$\sum_{n=2}^{\infty} A_n(\alpha)|a_n| + \sum_{n=1}^{\infty} B_n(\alpha)|b_n| \le 2 \sin \frac{\pi\alpha}{2}.$$

Then $f \in \mathcal{SS}_{\mathrm{H}}(\alpha)$.

Back

[17] H. Silverman. "Harmonic univalent functions with negative coefficients". In: *J. Math. Anal. Appl.* 220.1 (1998), pp. 283–289.

[18] J. M. Jahangiri. "Harmonic functions starlike in the unit disk". In: *J. Math. Anal. Appl.* 235.2 (1999), pp. 470–477.

Harmonic convolution

Let $f = h + \bar{g}$ and $F = H + \bar{G}$ be two harmonic functions on $\mathbb{D}$ of the forms

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \quad \text{and} \quad F(z) = \sum_{n=0}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n}.$$

Then the harmonic convolution of f and F is defined as

$$\begin{aligned}(f * F)(z) &= f(z) * F(z) \\ &= h(z) * H(z) + \overline{g(z) * G(z)} \\ &= \sum_{n=0}^{\infty} a_n A_n z^n + \overline{\sum_{n=1}^{\infty} b_n B_n z^n}.\end{aligned}$$

An equivalent condition of harmonic spirallikeness

Let $\mathbb{T}$ denote the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$.

Theorem

Let $-\pi/2 < \lambda < \pi/2$. Suppose that an orientation-preserving $f = h + \overline{g} \in \mathcal{H}_0$ satisfies $f(z) \neq 0$ for $0 < |z| < 1$. Then $f \in \mathcal{SP}_{\mathrm{H}}(\lambda)$ if and only if

$$(f * \varphi_{\lambda,\zeta})(z) \neq 0 \quad \textit{for } z \in \mathbb{D}\backslash\{0\},\ \zeta \in \mathbb{T}\backslash\{-1\},$$

and

$$\varphi_{\lambda,\zeta}(z) = \frac{(1+e^{2i\lambda})z + (\zeta - e^{2i\lambda})z^2}{(1-z)^2} + \frac{(-1+e^{2i\lambda}-2\zeta)\bar{z} + (\zeta - e^{2i\lambda})\bar{z}^2}{(1-\bar{z})^2}.$$

An equivalent condition of harmonic spirallikeness

Taking $\pm\pi(1-\alpha)/2$ as λ, we obtain the following result.

Corollary

Let f be an orientation-preserving harmonic function in $\mathcal{H}_0$ satisfying the condition $f(z) \neq 0$ for $0 < |z| < 1$. For $0 < \alpha \leq 1$, $f \in \mathcal{SS}_{\mathrm{H}}(\alpha)$ if and only if

$$\left(f * \varphi_{\frac{\pi(1-\alpha)}{2},\zeta}\right)(z) \neq 0 \quad and \quad \left(f * \varphi_{-\frac{\pi(1-\alpha)}{2},\zeta}\right)(z) \neq 0$$

for all $z \in \mathbb{D}\backslash\{0\}$ and $\zeta \in \mathbb{T}\backslash\{-1\}$.

A special example

Consider the harmonic function $f_{b,n}$ of the special form

$$f_{b,n}(z) = z + b\overline{z}^n$$

for $b \in \mathbb{C}$ and $n = 1, 2, 3, \ldots$.

Proposition

Let $0 < \alpha \le 1$ and set $\lambda = \pi(1-\alpha)/2$. Then the following are equivalent:

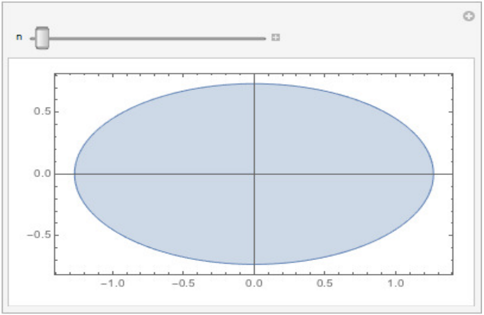
(i) $f_{b,n} \in \mathcal{SS}_{\mathrm{H}}(\alpha)$;

(ii) $f_{b,n} \in \mathcal{SP}_{\mathrm{H}}(\lambda)$;

(iii) $|b| \le C_n(\alpha)$, *where* $C_n(\alpha) = \dfrac{2\sin(\pi\alpha/2)}{n+1+|n+e^{i\pi\alpha}|}$.

A special example

Consider the function $f_{b,n}(z) = z + b\overline{z}^n$ with $b = C_n(1/3)$, where $n \geq 1$.



Outline for section 3

Quasiconformal extension of strongly starlike functions

A well-known result given by Fait, Krzyż and Zygmunt[19] shows that any strongly starlike function of order α $(0 < \alpha < 1)$ has a k-quasiconformal extension to the whole plane. The extension function can be expressed in the form

$$F(z) = \begin{cases} f(z), & |z| \le 1, \\ \left| f\left(\frac{z}{|z|}\right)\right|^2 \Big/ \overline{f\left(\frac{1}{\bar{z}}\right)}, & |z| \ge 1. \end{cases}$$

Moreover, the dilatation μ_F satisfies

$$|\mu_F(z)| \le k = \sin\frac{\pi\alpha}{2} \ a.e.$$

[19] M. Fait, J.G. Krzyż, and J. Zygmunt. "Explicit quasiconformal extensions for some classes of univalent functions". In: *Comment. Math. Helv.* 51.1 (1976), pp. 279–285.

Quasiconformal extension of strongly starlike functions

Theorem (Continued)

Let $0 < \alpha < 1$*. Suppose that a function* $f \in \mathcal{H}_0$ *satisfies the condition. Then* f *belongs to* $\mathcal{SS}_{\mathrm{H}}(\alpha)$ *and has a quasiconformal extension* F *to the whole complex plane, of the form*

$$F(z) = \begin{cases} f(z), & |z| \le 1, \\ z + \sum_{n=2}^{\infty} a_n \bar{z}^{-n} + \sum_{n=1}^{\infty} \overline{b_n} z^{-n}, & |z| \ge 1, \end{cases}$$

with the dilatation

$$|\mu_F(z)| \le \frac{\sin(\pi\alpha/2)}{1 + \cos(\pi\alpha/2)}$$

for $z \in \mathbb{D}$*, and* $|\mu_F(z)| \le \sin(\pi\alpha/2)$ *for* $z \in \mathbb{C} \backslash \overline{\mathbb{D}}$*. Finally,* f *is a* $\sin(\pi\alpha/2)$*-quasiconformal mapping of the whole plane.*

Quasiconformal extension of strongly starlike functions

Example

For a given α with $0 < \alpha < 1$, we consider the strongly starlike function $f_{b,n}(z) = z + b\overline{z}^n$ $(n \geq 2)$, where

$$|b| \leq C_n(\alpha) = \frac{2\sin(\pi\alpha/2)}{(n+1) + |n + e^{i\pi\alpha}|}.$$

Then $f_{b,n}$ can be extended to $\mathbb{C}$, and the mapping of the form

$$F_{b,n}(z) = \begin{cases} z + b\overline{z}^n, & |z| \leq 1, \\ z + bz^{-n}, & |z| \geq 1, \end{cases}$$

is a quasiconformal extension of $f_{b,n}$, with the dilatation

$$|\mu_{F_{b,n}}(z)| \leq \frac{2n\sin(\pi\alpha/2)}{n + 1 + |n + e^{i\pi\alpha}|}$$

for $z \in \mathbb{C}$.

Harmonic univalent functions of the exterior unit disk

Let $\Delta := \{z \in \mathbb{C} : |z| > 1\}$ denote the exterior unit disk. Hengartner and Schober[20] considered a class of orientation-preserving harmonic univalent functions defined on Δ, which map ∞ to ∞. Such functions have the representation

$$f(z) = \alpha z + \beta \overline{z} + \sum_{n=0}^{\infty} a_n z^{-n} + \overline{\sum_{n=1}^{\infty} b_n z^{-n}} + A \log |z|, \quad z \in \Delta,$$

where $0 \leq |\beta| < |\alpha|$ and $A \in \mathbb{C}$, denoted by Σ_{H}.

Let $\Sigma_{\mathrm{H}}(k)$ $(0 < k < 1)$ be the class of functions $f \in \Sigma_{\mathrm{H}}$, which satisfy

$$|\beta| + |A| + \sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq k|\alpha|.$$

[20] W. Hengartner and G. Schober. "Univalent harmonic functions". In: *Trans. of the Amer. Math. Soc.* 299.1 (1987), pp. 1–31.

Quasiconformal extension of the exterior unit disk

Theorem

Let f be in the class $\Sigma_{\mathrm{H}}(k)$ for some $k \in (0,1)$. Then f has a homeomorphic extension to the unit circle. Moreover, the mapping

$$F(z) = \begin{cases} f(z), & |z| \geq 1, \\ \alpha z + \beta \overline{z} + \sum_{n=0}^{\infty} a_n \overline{z}^n + \sum_{n=1}^{\infty} \overline{b_n} z^n, & |z| \leq 1, \end{cases}$$

is a quasiconformal extension of f with the dilatation $|\mu_F(z)| \leq k$ for $z \in \mathbb{C}$.

An example

Example

Consider a function $f \in \Sigma_{\mathrm{H}}$ of the form

$$f(z) = z - \frac{i}{6}\overline{z} + \frac{i}{4}\log|z| - \frac{i}{8}z^{-4}.$$

Then f has a homeomorphic extension to the unit circle, and the mapping

$$F(z) = \begin{cases} z - \frac{i}{6}\overline{z} + \frac{i}{4}\log|z| - \frac{i}{8}z^{-4}, & |z| \geq 1, \\ z - \frac{i}{6}\overline{z} - \frac{i}{8}\overline{z}^4, & |z| \leq 1, \end{cases}$$

is a k-quasiconformal extension of f with $k = 7/9$.

An example

We plot the graph of $F(z)$.

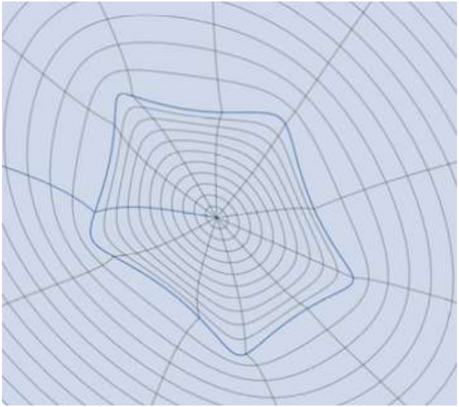


Figure: Image domain of $F(z)$

A convolution theorem

A result of Ruscheweyh and Sheil-Small[21] shows, if $f,\ g \in \mathcal{K}$, then $f * g \in \mathcal{K}$.

Let Σ_k $(0 < k < 1)$ be a class of orientation-preserving homeomorphisms h of the plane $\mathbb{C}$ onto itself, with $h(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}$ analytic univalent in Δ and k-quasiconformal in $\mathbb{C}$. Krzyż[22] proved, for $f_1 \in \Sigma_{k_1}$ and $f_2 \in \Sigma_{k_2}$, the convolution function $f_1 * f_2 \in \Sigma_{k_1 k_2}$.

Theorem

Let $k_1, k_2 \in (0,1)$. If $f_1 \in \Sigma_{\mathrm{H}}(k_1)$ and $f_2 \in \Sigma_{\mathrm{H}}(k_2)$, then $f_1 * f_2 \in \Sigma_{\mathrm{H}}(\sqrt{k_1 k_2})$.

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[21] S. Ruscheweyh and T Sheil-Small. "Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture". In: *Comment. Math. Helv.* 48.1 (1973), pp. 119–135.

[22] J. G. Krzyż. "Convolution and quasiconformal extension". In: *Comment. Math. Helv.* 51.1 (1976), pp. 99–104.

Outline for section 4

Future work

Analytic characterization of harmonic spirallikeness

Question

1. *How to construct a hereditarily spirallike function, which is not strongly starlike?*
2. *How to verify the hereditary spirallikeness in a rigorous way?*

$$\mathcal{THANKS\ SO\ MUCH}$$
$$\mathcal{FOR}$$
$$\mathcal{YOUR\ ATTENTION}!$$
$$(\bullet > \omega < \bullet)$$