

# **Study on general origamis and Veech groups of flat surfaces**

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Doctor of Philosophy

by

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## Abstract

In this century, an origami (a square-tiled translation surface) is intensively studied as an object with special properties of its translation structure and its  $PSL(2, \mathbb{R})$ -orbit embedded in the moduli space, particularly in the context of the study of the absolute Galois group and the Teichmüller geodesic flow.

We formulate the concept of origamis generalized in the language of flat surfaces arising naturally in the Teichmüller theory. The family of flat surfaces with two cylindrical directions that induce a fixed origami as a combinatorial structure is parametrized in the Euclidian space. The  $PSL(2, \mathbb{R})$ -orbits of such flat surfaces are observed in terms of origamis.

Furthermore, we present some calculation results on origamis and discuss the Galois conjugacy of the  $PSL(2, \mathbb{R})$ -orbits of origamis.



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# Nomenclature

## Notation

Symbol	Description
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the ring of integers
$\mathbb{Q}$	the field of rational numbers
$\bar{\mathbb{Q}}$	the algebraic closure of $\mathbb{Q}$ , the field of algebraic numbers over $\mathbb{Q}$
$\mathbb{R}$	the field of real numbers
$\mathbb{K}_{>0}$	$\{x \in \mathbb{K} \mid x > 0\}$ ; the subset defined by the subscript condition
$i$	$\sqrt{-1}$ ; the imaginary unit
$\mathbb{C}$	the complex plane, the field of complex numbers
$\hat{\mathbb{C}}$	$\mathbb{C} \cup \{\infty\}$ ; the Riemann sphere
$\operatorname{Re}(z)$	the real part of $z \in \mathbb{C}$
$\operatorname{Im}(z)$	the imaginary part of $z \in \mathbb{C}$
$\bar{z}$	the complex conjugate of $z \in \mathbb{C}$
$\mathbb{H}$	$\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ ; the upper half plane
$\mathbb{L}$	$\{z \in \mathbb{C} \mid \operatorname{Im}(z) < 0\}$ ; the lower half plane
$[\cdot]$	assignment of the class or the quotient under given equivalence
$SL(2, G)$	$\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in G, ad - bc = 1\}$ ; the special linear group $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
$PSL(2, G)$	$\{[A] = \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \mid a, b, c, d \in G, ad - bc = 1\}$ ; the projective special linear group
$\bar{A}, \bar{G}$	the mirror conjugate of a matrix and a group (e.g. $\bar{A} = JAJ$ )
$F_2$	$\langle x, y \rangle$ ; the free group of rank 2

$I_d$	$\{1, 2, \dots, d\}$ ; the set of $d$ indices
$\bar{I}_d$	$\{\pm 1, \pm 2, \dots, \pm d\}$ ; the double of the set of $d$ indices
$\mathfrak{S}_d$	$\text{Sym}(I_d)$ ; the symmetric group
$\tilde{\mathfrak{S}}_d$	$\{\bar{\sigma} \in \text{Sym}(\bar{I}_d) \mid \bar{\sigma}(-i) = -\bar{\sigma}(i), i \in \bar{I}_d\}$ ; the rotation-symmetric permutation group



# Chapter 1

## Introduction

### Background

A Riemann surface is a connected, complex 1-dimensional manifold. A Riemann surface is called analytically finite type  $(g, n)$  if it has genus  $g$  and precisely  $n$  boundary components that are points. Poincaré-Klein-Koebe's uniformization theorem [36–39, 50] in 1907 classifies the complex structures of universal Riemann surfaces into the three cases: the Riemann sphere  $\hat{\mathbb{C}}$  ('elliptic'), the complex plane  $\mathbb{C}$  ('parabolic'), and the upper plane  $\mathbb{H}$  or equivalently the unit disk  $\mathbb{D}$  ('hyperbolic'). These three cases are distinguished by the type of each Riemann surface. In most cases where  $2g - 2 + n > 0$ , Riemann surfaces are hyperbolic. This thesis thinks of hyperbolic Riemann surfaces of analytically finite type. Such a Riemann surface is represented by a Fuchsian model  $\mathbb{H}/\Gamma$  where  $\Gamma$  is a group acting on  $\mathbb{H}$  properly discontinuously by the Möbius transformations.

Biholomorphism classes of Riemann surfaces are called moduli. Analysis on the space  $M_{g,n}$  of moduli of Riemann surfaces of type  $(g, n)$  is worked out in terms of its universal covering space  $T_{g,n}$ , the Teichmüller space introduced by Teichmüller in the 1930s. The Teichmüller space  $T_{g,n}$  parameterizes deformations of a fixed Riemann surface of type  $(g, n)$  under quasiconformal mappings. A quasiconformal mapping is a homeomorphism defined by the Beltrami equation  $f_{\bar{z}} = \mu f_z$ , where  $\mu$  is a bounded measurable  $(1, -1)$ -form with norm less than one called the Beltrami differential. Ahlfors and Bers [2] showed the existence and uniqueness theorem for the Beltrami equation on the Riemann sphere. Simultaneous uniformization leads to the Bers embedding of the Teichmüller space  $T_{g,n}$  into a complex Banach space of dimension  $3g - 3 + n$ . The embedded image of Teichmüller space  $T_{g,n}$  in the complex Euclidian space  $\mathbb{C}^{3g-3+n}$  is a bounded domain homeomorphic to the ambient space. The covering transformation group of the universal covering  $T_{g,n} \rightarrow M_{g,n}$  is given by

the action of quasiconformal self-mappings of a type  $(g, n)$ -surface  $R$  up to homotopically-trivial mappings. Such a group  $\text{Mod}_{g,n}$  is called the Teichmüller-modular group or the mapping class group of type  $(g, n)$ .

Holomorphic quadratic differentials on Riemann surfaces plays a significant role in Teichmüller theory. Consider the Banach space  $\mathcal{Q}^\infty(R)$  of uniformly bounded holomorphic quadratic differentials or (in analytically finite case) equivalently the space  $\mathcal{Q}(R)$  of integrable holomorphic quadratic differentials (admitting simple poles at the boundary) on a Riemann surface  $R$ . Then the space  $\mathcal{Q}^\infty(R)$  embeds the Teichmüller space  $T_{g,n}$  and the space  $\mathcal{Q}(R)$  appears to be the cotangent space  $T_{g,n}^*$  as a dual of  $\mathcal{Q}^\infty(R)$ . The space  $\mathcal{Q}(R)$  is naturally stratified in terms of specified data of singular orders and primitivity. Each stratum is a complex analytic orbifold parametrized on the cohomology group relative to singularities by the period coordinates [4, 30, 57].

A Riemann surface  $R$  together with an integrable holomorphic quadratic differential  $\phi$  is called a flat surface. The coordinates defined by line integral in the locally defined differential  $\sqrt{\phi}$  form an atlas any of whose transition map is half-translation. Such a structure induces the notion of locally-affine geometry with a flat metric with finitely many conical singularities. Teichmüller's theorem states that every quasiconformal mapping is uniquely represented by some flat structures as an extremal affine deformation that attains the bound of norm of its Beltrami differential. For each fixed flat surface  $(R, \phi)$ , we obtain a holomorphic and isometric embedding  $\hat{\iota}_\phi : \mathbb{D} \hookrightarrow T_{g,n}$  of the upper half plane into the Teichmüller space. The action of Teichmüller-modular group on the embedded disk is described by the Möbius transformation of derivative of locally affine self-mapping of  $(R, \phi)$ . The group of such action is called the Veech group  $\Gamma(R, \phi)$ , and the embedded disk projects into the moduli space as an orbifold  $\mathbb{H}/\Gamma(R, \phi)$ . If  $\Gamma(R, \phi)$  has a finite covolume, the orbifold  $\mathbb{H}/\Gamma(R, \phi)$  is an algebraic curve, and we obtain a curve family embedded in the moduli space  $M_{g,n}$  as an algebraic curve called the Teichmüller curve induced from  $(R, \phi)$ .

Veech groups were introduced by Veech in the context of the study of the geodesic flow on a flat surface. In his paper [56] in 1989, Veech showed a dichotomy of billiard (reflecting at boundary) geodesic flows on polygonal regions on the plane. He also presented the first nontrivial example of Veech group. Earle and Gardiner [10] reformulated the theory of flat surfaces in terms of the Teichmüller spaces. Veech groups of flat surfaces are studied in terms of combinatorial objects invariant under affine self-mappings, such as [7, 11, 51, 54].

An (abelian, or formerly known as "oriented") origami [25] is a typical example of a flat surface with a combinatorial structure. It is a finite covering of the unit square torus branched over precisely one point, which comes down to a combinatorial structure. An origami has

a Veech group as a subgroup of  $SL(2, \mathbb{Z})$  of finite index, and induces a Teichmüller curve defined over  $\bar{\mathbb{Q}}$  called an origami curve. Schmithüsen [51] showed that the universal Veech group  $SL(2, \mathbb{Z})$  of an abelian origami  $\mathcal{O}$  acts automorphically on the free group  $F_2$  and its Veech group is the stabilizer of the fundamental group of  $\mathcal{O}$  under this action. She also presented an algorithm to calculate the Veech group of given origami using the Reidemeister-Schreier method [43]. Ellenberg and McReynolds [12] showed a sufficient condition for a group to be the Veech group of an abelian origami. A significant aspect of origami is the compatibility of the Galois actions on embedding origami curves.

The absolute Galois group  $G_{\mathbb{Q}} = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ , the group of field automorphisms of an algebraic closure  $\bar{\mathbb{Q}}$  of the field of rational numbers has been intensively studied in many areas for a long time. A remarkable progress of this study started from the theorem of Belyĭ [5] in 1979. He showed a purely combinatorial or analytical condition ‘three-times branched covering of the projective line’ (Belyĭ covering) for an algebraic curve to be defined over the field  $\bar{\mathbb{Q}}$ . By the fundamental theory of coverings, every Belyĭ covering is identified with a combinatorial object called a dessin d’enfants. The absolute Galois group  $G_{\mathbb{Q}}$  acts faithfully the set of dessins, and some  $G_{\mathbb{Q}}$ -invariants are visualized in terms of dessins. Using the idea of dessins to approach the absolute Galois group  $G_{\mathbb{Q}}$  is a project contained in the Grothendieck’s programme [18] in 1984. Drinfel’d [8] and Ihara [31] showed that the Grothendieck-Teichmüller group  $\widehat{GT}$  realizes the absolute Galois group  $G_{\mathbb{Q}}$  as a subgroup, the “ $\widehat{GT}$ -relation”. The Grothendieck-Teichmüller group  $\widehat{GT}$  is defined pure combinatorially and is related to the study of profinite mapping class groups [53].

Origamis are well-behaved in this context. In 2005, Möller [45] showed that the  $G_{\mathbb{Q}}$ -action on origamis respects the embedding origami curves into the moduli spaces. In his paper, Möller presented an application to the  $\widehat{GT}$ -relation by considering the  $G_{\mathbb{Q}}$ -action on an origami curve induced from a degree 4 origami. The  $G_{\mathbb{Q}}$ -action on the embedding origami curve is compared with the action on a Teichmüller tower [24] that is defined by a collection of mapping class groups linked by certain natural homomorphisms coming from inclusions of underlying surfaces. The faithfulness of the  $G_{\mathbb{Q}}$ -action on origamis are shown by the “M-origami” construction [45, 48] using dessins.

## Results of this thesis

In this thesis, we consider flat surfaces whose Veech groups can be dealt with similarly to origamis. To do this, we introduce a generalization of origamis in the category of flat surfaces. Such a general origami is regarded as a covering with a prescribed branching behavior like an abelian origami, but the situation is a bit different from abelian case in

the sense of combinatorial characterizations and its Veech group. Theorem 5.1.7 shows combinatorial characterizations and equivalence of general origamis. A characterization is given by regluing an abelian origami after inverting prescribed squares is a better tool for the calculation of Veech group.

Earle and Gardiner [10] showed that every flat surface with two finite-cylindrical directions (Jenkins-Strebel directions) admits a decomposition into finitely many aligned parallelograms. We may regard this decomposition as an origami-like decomposition given by aligned parallelograms of specified moduli replacing unit square cells of a general origami. In terms of Theorem 5.1.7, we observe that a general origami gives a finite system of linear equations whose solution space presents the set of moduli for which the replacement of cells successes. Theorem 5.2.3 shows that there is a one-to-one correspondence between flat surfaces with fixed two finite-cylindrical directions and origamis with compatible moduli lists up to equivalences. It leads to Corollary 5.2.4 that the family of flat surfaces inducing a general origami in prescribed directions is parametrized in the quotient of the solution space of the system of linear equations by a finite group.

As a consequence of Theorem 5.2.3, we may compare two origami-like decompositions of a flat surface to determine whether each matrix belongs to the Veech group, as in the way shown in Corollary 5.3.1. Every origami-like decomposition combinatorially projects via a covering with a prescribed branching behavior concerning the singularities, and thus the inclusion of Veech groups holds for origami-like flat surfaces in such a covering relation. Theorem 5.3.5 observes such a situation as the Veech group of the lower surface acting on the set of monodromies, where the Veech group of the upper surface is the stabilizer of the class of original monodromy. This result is a generalization of [51, 52, 54].

We present a set of algorithms summarized in Theorem 6.1.7 for calculating the Veech groups of general origamis. It has a different structure than [51, 54], which computes the Veech group from a single object. Our algorithm makes an exhaustive and simultaneous calculation of the Veech groups of all origamis of a fixed degree. We create a list of equivalence classes of general origamis according to Theorem 5.1.7 and compute the action of the universal Veech group  $PSL(2, \mathbb{Z})$  on these classes defined by the re-decomposition of the origami-like decomposition. Each  $PSL(2, \mathbb{Z})$ -orbit corresponds to the square-tiled points of the Teichmüller curves induced from origamis, and elements of Veech groups are represented by closed cycles.

We have calculated up to  $d = 7$ , classified the result by Galois invariants, and summarized the possibility of Galois conjugation as in Theorem 6.2.1.

# Chapter 2

## Concepts in Complex Analysis

### 2.1 Covering

In this section, we present basic theory of coverings of topological surfaces. In Section 2.2 a branched covering appears as a morphism in the category of Riemann surfaces. The theory of covering is an important tool in discussions about dessins (Section 3.2) and origami (Section 4.4.2 and 5.1). This section is based on [14, 17, 49].

We say that a *topological surface* is a connected, oriented, two-dimensional real manifold.

**Definition 2.1.1.** A mapping  $f : R \rightarrow S$  between two topological surfaces  $R$  and  $S$  is called a *branched covering* if there exists nowhere dense subset  $B \subset S$  and a mapping  $v : R \rightarrow \mathbb{Z}_{\geq 0}$  with  $R \setminus \text{supp}(v - 1) = f^{-1}(B)$  such that:

- (1) every point  $p \in S \setminus B$  has an *evenly covered* neighborhood, an open neighborhood  $U \subset S \setminus B$  such that  $f^{-1}(U)$  is the disjoint union of open subsets in  $R$ , each of which restricts  $f$  to a homeomorphism onto  $U$ .
- (2) around every point  $p \in R$ ,  $f$  is locally represented by the form  $z \mapsto z^{v(p)}$ .

We say that  $\text{Br}(f) = B \subset S$  is the set of *branched points* and  $\text{Crit}(f) = f^{-1}(B) \subset R$  is the set of *critical points*. The integer  $v(p) =: \text{mult}_p(f)$  is called the *multiplicity* of  $f$  at  $p \in R$ . The mapping  $f$  is called an *unbranched covering* (or simply a *covering*) if  $\text{Br}(f) = \emptyset$ . The restriction of a branched covering to the unbranched region  $R^* = R \setminus f^{-1}(B)$  is an unbranched covering.

**Definition 2.1.2.** Two coverings  $f_i : R_i \rightarrow S_i$  ( $i = 1, 2$ ) are called *equivalent* if there exists two homeomorphisms  $\varphi : R_1 \rightarrow R_2$  and  $\psi : S_1 \rightarrow S_2$  such that  $\psi \circ f_1 = f_2 \circ \varphi$ .

**Definition 2.1.3.** Let  $R, S$  be topological surfaces.

- (1) A *path* in  $R$  is a continuous mapping  $\gamma : [0, 1] \rightarrow R$ . The points  $\gamma(0), \gamma(1) \in R$  are the *endpoints* of  $\gamma$ . A path is called a *loop* if the two endpoints coincide.
- (2) A *homotopy* joining two continuous mappings  $f_0, f_1 : R \rightarrow S$  is a continuous mapping  $F : R \times [0, 1] \rightarrow S$  such that  $F(x, 0) = f_0(x)$  and  $F(x, 1) = f_1(x)$  for any  $x \in R$ . We say that  $f_0, f_1$  are *homotopic* if there exists a homotopy joining them. By *fixed-endpoint homotopy* we mean a homotopy  $F$  of paths such that every  $F(t, s), s \in [0, 1]$  has the same endpoints.

A covering  $\pi_R : \tilde{R} \rightarrow R$  with  $\tilde{R}$  simply connected is called a *universal covering* of  $R$ . The name ‘universal’ comes from the *universal property* that for any covering  $f : R' \rightarrow R$  there exists a covering  $\pi' : \tilde{R} \rightarrow R'$  such that  $\pi' = \pi \circ f$ .

**Lemma 2.1.4.** *For every topological surface  $R$ , one can construct a universal covering  $\pi_R : \tilde{R} \rightarrow R$  of  $R$  in the following way.*

- (i) fix a base point  $p$  on  $R$ .
- (ii) let  $\tilde{R}$  be the space of all paths  $\gamma : [0, 1] \rightarrow R$  with  $\gamma(0) = p$  up to fixed-endpoint homotopy.
- (iii) define  $\pi_R : \tilde{R} \rightarrow R$  by  $[\gamma] \mapsto \gamma(1)$ .

**Proposition 2.1.5** (lifting property). *Let  $R$  and  $S$  be topological surfaces. Then we have the following:*

- (1) For any path  $\gamma : [0, 1] \rightarrow R$  and  $\tilde{p} \in \tilde{R}$  with  $\gamma(0) = \pi_R(\tilde{p})$ , there exists a unique path  $\tilde{\gamma} : [0, 1] \rightarrow \tilde{R}$  such that  $\pi_R \circ \tilde{\gamma} = \gamma$  and  $\tilde{\gamma}(0) = \tilde{p}$ .
- (2) For any continuous mapping  $f : R \rightarrow S$  and  $\tilde{p} \in \tilde{R}, \tilde{q} \in \tilde{S}$  with  $f(\pi_R(\tilde{p})) = \pi_S(\tilde{q})$ , there exists a unique continuous mapping  $\tilde{f} : \tilde{R} \rightarrow \tilde{S}$  such that  $f \circ \pi_R = \pi_S \circ \tilde{f}$  and  $\tilde{f}(\tilde{p}) = \tilde{q}$ .

We say that  $\tilde{\gamma}$  ( $\tilde{f}$ , respectively) is a *lift* of  $\gamma$  ( $f$ , respectively).

As a consequence of Proposition 2.1.5, the cardinality of  $f^{-1}(p)$  defines the *degree*  $\deg(f)$  of a covering  $f : R \rightarrow S$  independent of choices of  $p \in R$ . If  $\deg(f) = d \in \mathbb{N}$ , we say that  $f$  is a *finite covering* or a *d-fold covering*.

**Definition 2.1.6.** Let  $R, S$  be topological surfaces and  $f : R \rightarrow S$  be a covering. Fix base points  $p \in R$  and  $q \in S$ .

- (1) The *fundamental group*  $\pi_1(R, p)$  of  $R$  with base point  $p$  is the group of homotopy classes of all loops on  $R$  based on  $p$ , with the group structure defined by post-composition of loops.

- (2) The *monodromy map* of  $f$  is a homeomorphism  $m_f : \pi_1(S, q) \rightarrow \text{Sym}(f^{-1}(q))$  that assigns each loop  $\gamma \in \pi_1(S, q)$  to the permutation defined by the lift of  $\gamma$  starting from each point in  $f^{-1}(q)$ . The image of the monodromy map is called the *monodromy group* of  $f$ . Note that  $m_f$  can be defined as the homomorphism into the symmetric group  $\mathfrak{S}_{\deg(f)}$  up to conjugacy, independent of the choices of base point  $q \in S$ .
- (3) A *covering transformation* (or *deck transformation*) of  $f$  is a homeomorphism  $\varphi : R \rightarrow R$  satisfying that  $f \circ \varphi = f$ . The covering transformations of  $f$  form a group under post-composition, denoted by  $\text{Deck}(f)$  or  $\text{Deck}(R/S)$  (if  $f$  is understood).

**Remark 2.1.7.** Let  $f : R \rightarrow S$  be a covering and fix  $p \in R$ ,  $q = f(p) \in S$ . Then the stabilizer  $\text{Stab}_{m_f}(p)$  is the image of  $\pi_1(R, p)$  embedded in  $\pi_1(S, q)$ , via the natural homomorphism  $f_\#$  induced from  $f$ . Once a subgroup  $H < \pi_1(S, q)$  given, one can reconstruct a covering  $f : R_H \rightarrow S$  so that the embedded image of  $\pi_1(R_H, p)$  in  $\pi_1(S, q)$  is  $H$ , in a similar way to Lemma 2.1.4. In this construction, the monodromy  $m_f$  is identified with the action of  $H < \pi_1(S, q)$  on the coset representatives in  $\pi_1(S, q)/H$ , where  $\text{Stab}_{m_f}(1) = H$ .

Note about the equivalence as follows. Suppose that two equivalent coverings  $f_i : R_i \rightarrow S_i$  ( $i = 1, 2$ ) are joined via homeomorphisms  $\varphi : R_1 \rightarrow R_2$  and  $\psi : S_1 \rightarrow S_2$  as in Definition 2.1.2. Fix a base point  $p_1 \in R_1$  and let  $p_2 = \varphi(p_1) \in R_2$ ,  $q_1 = f_1(p_1) \in S_1$ , and  $q_2 = f_2(p_2) \in S_2$ . Then, we have isomorphisms  $\varphi^\# : \pi_1(R_1, p_1) \rightarrow \pi_1(R_2, p_2)$  and  $\psi^\# : \pi_1(S_1, q_1) \rightarrow \pi_1(S_2, q_2)$  commutative with the embeddings  $f_i^\# : \pi_1(R_i, p_i) \hookrightarrow \pi_1(S_i, q_i)$  ( $i = 1, 2$ ). We say that  $f_i$  ( $i = 1, 2$ ) are in *covering equivalence* over a topological surface  $S$  if  $S_1 = S_2 = S$  and  $\psi = id$ . In this case, the isomorphism  $\varphi$  is a covering transformation and the embedded image of  $\pi_1(R_i, p_i)$ ,  $i = 1, 2$  are conjugated in  $\pi_1(S_1, q_1)$ . The monodromy homomorphisms  $m_{f_i}$ ,  $i = 1, 2$  are identified by the pullback of the isomorphism  $\varphi^\#$ .

Let  $\pi : \tilde{R} \rightarrow R$  be the universal covering constructed as in Lemma 2.1.4. Then every covering transformation of  $\pi$  is represented by the action of  $\pi_1(R, p)$  on  $\tilde{R}$  by the pre-compositions of paths. (See [32, p.31].)

**Proposition 2.1.8.** *The following are equivalent for a covering  $f : R \rightarrow S$ ; in this case we say that  $f$  is a Galois covering.*

- (1)  $\text{Deck}(R/S)$  acts transitively on  $f^{-1}(p) \subset R$  for a fixed base point  $p \in S$ .
- (2) The monodromy map  $m_f$  is an isomorphism onto the monodromy group.
- (3) The fundamental group of  $S$  embeds the fundamental group of  $R$  as a normal subgroup via the natural homomorphism induced from  $f$ .



**Lemma 2.1.9.** *Let  $f : R \rightarrow S$  be a Galois covering. Then  $\text{Deck}(R/S)$  acts properly discontinuously on  $R$ , that is, for any compact subset  $K \subset R$ , there are at most finitely many elements  $\varphi \in \text{Deck}(R/S)$  such that  $\varphi(K) \cap K \neq \emptyset$ . Furthermore, each element in  $\text{Deck}(R/S)$  except for the identity has no fixed points.*

Let  $R$  be a topological surface and  $G$  be a group of self-homeomorphisms of  $R$  acting properly discontinuously on  $R$ . Then the natural projection  $\pi_G : R \rightarrow R/G$  induces a topological structure on the quotient space  $R/G$ . With this structure,  $\pi_G$  becomes a Galois covering with  $\text{Deck}(\pi_G) = G$ .

**Proposition 2.1.10.** *A Galois covering  $f : R \rightarrow S$  is equivalent to  $\pi_{\text{Deck}(f)}$ .*

## 2.2 Riemann surface

This section is based on [32].

A one-dimensional connected complex manifold is called a *Riemann surface*. In other words, a Riemann surface is a connected Hausdorff space  $R$  with a one-dimensional complex structure, a maximal atlas  $\mathcal{A}_R = \{(U_i, \varphi_i)\}_{i \in I}$  satisfying the following properties.

- (1) Every  $U_i$  is an open subset of  $R$  and  $R = \bigcup_{i \in I} U_i$ .
- (2) Every  $\varphi_i$  is a homeomorphism of  $U_i$  onto an open subset of  $\mathbb{C}$ .
- (3) For every  $U_i, U_j$  with  $U_i \cap U_j \neq \emptyset$ , the transition mapping

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j) \quad (2.1)$$

is a biholomorphism on the plane.

**Example 1.** The Riemann sphere  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  with  $\mathcal{A}_{\hat{\mathbb{C}}} = \{(\mathbb{C}, z), (\hat{\mathbb{C}} \setminus \{0\}, \frac{1}{z})\}$ .

**Example 2.** A torus  $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$  ( $\tau \in \mathbb{H}$ ) with

$$\mathcal{A}_{E_\tau} = \left\{ \left( \left\{ [z] \mid |z - z_0| < \min\left\{0, \frac{|\tau|}{2}\right\} \right\}, z \right) \mid z_0 \in \mathbb{C} \right\}. \quad (2.2)$$

**Definition 2.2.1.** Let  $R, S$  be Riemann surfaces.

- (1) A continuous mapping  $f : R \rightarrow S$  is called *holomorphic* if for every  $(U, \varphi) \in \mathcal{A}_R$  and  $(V, \psi) \in \mathcal{A}_S$ , the local representation

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V) \quad (2.3)$$

is a holomorphic mapping on the plane.



- (2) A bijective holomorphic mapping  $f : R \rightarrow S$  is called a *biholomorphism*. We say that  $R, S$  are *biholomorphically equivalent* ( $R$  is *biholomorphic* to  $S$ ) if there exists a biholomorphism  $f : R \rightarrow S$ .
- (3) A Riemann surface  $R$  is called of *analytically finite type*  $(g, n)$  if  $R$  is obtained from a compact Riemann surface of genus  $g$  by removing  $n$  points. The removed points are called the *marked points*.

Local analysis on Riemann surfaces is reduced to analysis on domains in the plane via their complex structures. Local properties of holomorphic functions on the plane [16, III.3], such as the open mapping theorem, the identity theorem, and the maximum principle also hold for holomorphic mappings on Riemann surfaces.

**Lemma 2.2.2.** *Let  $R, S$  be Riemann surfaces. Then a non-constant holomorphic mapping  $f : R \rightarrow S$  is a branched covering.*

*Proof.* Let  $p \in R$  be an arbitrary point and  $(V, \varphi_V) \in \mathcal{A}_S$  be a chart around  $f(p) \in S$ . Since  $f$  is holomorphic, we may take a chart  $(U, \varphi') \in \mathcal{A}_R$  around  $p \in R$  so that  $f$  is locally represented by a convergent Taylor series

$$\varphi_V \circ f \circ \varphi'^{-1}(z) = \sum_{i=n}^{\infty} a_i z^i = z^n g(z), \quad z \in U, \quad (2.4)$$

where  $a_i \in \mathbb{C}$ ,  $i = n, n+1, \dots$  with  $a_n \neq 0$ . By taking  $U$  sufficiently small, the restriction  $g|_{\varphi'(U)}$  admits a holomorphic, non-zero branch of  $n$ -th root. If we set

$$\varphi_U = \varphi' \cdot \sqrt[n]{g \circ \varphi'} \quad \text{on } U, \quad (2.5)$$

the chart  $(U, \varphi_U)$  has a biholomorphic transition with  $(U, \varphi')$  by the inverse function theorem. Thus everywhere the mapping  $f$  is locally represented as  $\varphi_V \circ f \circ \varphi_U^{-1}(z) = z^n$ , and the claim follows.  $\square$

The complex structure of a Riemann surface naturally lifts via a covering. The arguments on coverings in Section 2.1 is translated in the language of holomorphic mappings on Riemann surfaces.

**Theorem 2.2.3** (Uniformization theorem, Poincaré-Klein-Koebe [36–39, 50]). *Every simply connected Riemann surface is biholomorphically equivalent to one of the three Riemann surfaces  $\hat{\mathbb{C}}$ ,  $\mathbb{C}$ , or  $\mathbb{H}$ .*

Thus every Riemann surface  $R$  has a universal covering  $\pi : \tilde{R} \rightarrow R$  where  $\tilde{R}$  is biholomorphically equivalent to either  $\hat{\mathbb{C}}$ ,  $\mathbb{C}$ , or  $\mathbb{H}$ . By Theorem 2.1.10,  $R$  is represented by  $\tilde{R}/\Gamma$  for a subgroup  $\Gamma$  acting biholomorphically on  $\tilde{R}$ . The following classification of universal Riemann surfaces holds.

**Lemma 2.2.4.** *Let  $\pi : \tilde{R} \rightarrow R$  be the universal covering of a Riemann surface  $R$ . We denote by  $\text{Aut}(\tilde{R})$  the group of all automorphisms (self-biholomorphisms) on  $\tilde{R}$ .*

- (a)  $\tilde{R} = \hat{\mathbb{C}}$  if and only if  $R$  is biholomorphically equivalent to  $\hat{\mathbb{C}}$ . Moreover,  $\text{Aut}(\hat{\mathbb{C}}) = \left\{ z \mapsto \frac{az+b}{cz+d} \mid a, b, c, d, \in \mathbb{C}, ad - bc = 1 \right\} \cong PSL(2, \mathbb{C})$  holds.
- (b)  $\tilde{R} = \mathbb{C}$  if and only if  $R$  is biholomorphically equivalent to one of  $\mathbb{C}$ ,  $\mathbb{C} \setminus \{0\}$ , or tori. Moreover,  $\text{Aut}(\mathbb{C}) = \{z \mapsto az + b \mid a, b \in \mathbb{C}, a \neq 0\}$  holds.

In particular, for a Riemann surface of analytically finite type  $(g, n)$ , its universal covering space is  $\mathbb{H}$  if and only if  $2g - 2 + n > 0$ . In this case, It follows that  $\text{Aut}(\mathbb{H}) = \left\{ z \mapsto \frac{az+b}{cz+d} \mid a, b, c, d, \in \mathbb{R}, ad - bc = 1 \right\} \cong PSL(2, \mathbb{R})$ .

Let  $z_i \in \hat{\mathbb{C}}$  ( $i = 1, 2, 3$ ) be pairwise distinct three points. Then, the mapping  $\gamma_{z_1, z_2, z_3} \in \text{Aut}(\hat{\mathbb{C}})$  defined by

$$\gamma_{z_1, z_2, z_3}(z) = \frac{(z_1 - z_2)(z - z_1)}{(z_3 - z_2)(z - z_3)}, \quad z \in \hat{\mathbb{C}} \quad (2.6)$$

maps  $(z_1, z_2, z_3)$  to  $(0, 1, \infty)$ . If  $z_i \in \partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$  ( $i = 1, 2, 3$ ), the mapping  $\gamma_{z_1, z_2, z_3}$  belongs to  $\text{Aut}(\mathbb{H})$ . In this way,  $\text{Aut}(\hat{\mathbb{C}})$  ( $\text{Aut}(\mathbb{H})$ , respectively) acts sharply 3-transitively on  $\hat{\mathbb{C}}$  ( $\partial\mathbb{H} = \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ , respectively).

From now on in this chapter, we consider a Riemann surface of analytically finite type  $(g, n)$  with  $2g - 2 + n > 0$ . Such a Riemann surface is represented by the quotient of  $\tilde{R} = \mathbb{H}$  by a subgroup of  $PSL(2, \mathbb{R})$  acting properly discontinuously on  $\mathbb{H}$ , by the formula:

$$A \cdot z = \frac{az+b}{cz+d}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{R}), \quad z \in \mathbb{H}. \quad (2.7)$$

**Definition 2.2.5.** The automorphism on  $\mathbb{H}$  defined by the formula (2.7) is called a *Möbius transformation*. A subgroup of  $PSL(2, \mathbb{R})$  acting properly discontinuously on  $\mathbb{H}$  is called a *Fuchsian group*. A Fuchsian group representing a Riemann surface  $R$  is called a *Fuchsian model* of  $R$ .

A biholomorphism  $f : R \rightarrow S$  lifts to an automorphism on  $\mathbb{H}$  via the universal coverings of  $R$  and  $S$ . For each biholomorphism class of a Riemann surface  $R$ , its Fuchsian model  $\Gamma_R$  is uniquely determined up to conjugacy in  $PSL(2, \mathbb{R})$ .

**Lemma 2.2.6** (classification of Möbius transformations, [32, Section 2.3.3]). *Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{R})$  and  $\text{tr}(A) = a + d$ . Then, one of the following holds for the Möbius transformation  $\gamma_A$  of  $A$  up to conjugacy in  $PSL(2, \mathbb{R})$ .*

- (a) (parabolic)  $\gamma_A$  is conjugated to  $z \mapsto z + c$  for some  $c \in \mathbb{C} \setminus \{0\}$  if and only if  $\text{tr}(A)^2 = 4$ .
- (b) (elliptic)  $\gamma_A$  is conjugated to  $z \mapsto e^{i\theta} z$  for some  $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$  if and only if  $0 \leq \text{tr}(A)^2 < 4$ .
- (c) (hyperbolic)  $\gamma_A$  is conjugated to  $z \mapsto kz$  for some  $k \in \mathbb{R}_{>0}$  if and only if  $\text{tr}(A)^2 > 4$ .
- (d) (loxodromic)  $\gamma_A$  is conjugated to  $z \mapsto \lambda z$  for some  $\lambda \in \mathbb{C}$ ,  $|\lambda| \neq 1$ ,  $\lambda \notin [0, \infty)$  if and only if  $\text{tr}(A)^2 \in \mathbb{C} \setminus [0, 4]$ .

The upper half plane  $\mathbb{H}$  and the unit disk  $\mathbb{D}$  are biholomorphic under the mapping

$$h : \mathbb{H} \rightarrow \mathbb{D} : z \mapsto \frac{z - i}{z + i}. \quad (2.8)$$

Each of them admits a hyperbolic metric

$$ds_{\mathbb{D}} = \frac{|dz|}{1 - |z|^2}, \quad z \in \mathbb{D}, \quad (2.9)$$

$$ds_{\mathbb{H}} = h^* ds_{\mathbb{D}} = \frac{|dz|}{2\text{Im}z}, \quad z \in \mathbb{H}, \quad (2.10)$$

for which every automorphism is an isometry. The distance function on  $\mathbb{D}$  is given by

$$d_{\mathbb{D}}(z_1, z_2) = \log \left\{ \left( 1 + \left| \frac{z_2 - z_1}{1 - \overline{z_1} z_2} \right| \right) \left( 1 - \left| \frac{z_2 - z_1}{1 - \overline{z_1} z_2} \right| \right)^{-1} \right\}, \quad z_1, z_2 \in \mathbb{D}. \quad (2.11)$$

## 2.3 Teichmüller space

This section is based on [1, 32, 47].

Let  $L^\infty(D)$  denote the complex Banach space of all uniformly bounded measurable functions on a domain  $D \subset \hat{\mathbb{C}}$  with the norm

$$\|\mu\|_\infty = \text{ess.sup}_{z \in D} |\mu(z)| < \infty, \quad \mu \in L^\infty(D). \quad (2.12)$$

**Definition 2.3.1.** Let  $D \subset \hat{\mathbb{C}}$  be a domain. An orientation-preserving homeomorphism  $f$  on  $D$  into  $\hat{\mathbb{C}}$  is called a *quasiconformal mapping* if the following holds.

- (1) The mapping  $f$  is absolutely continuous on lines (ACL):  $f$  is continuous on every horizontal or vertical local segments on  $D$ . Or equivalently,  $f$  admits distributional partial derivatives  $f_z, f_{\bar{z}}$  almost everywhere on  $D$ .

- (2) The Beltrami equation  $f_{\bar{z}} = \mu f_z$  holds almost everywhere on  $D$ , for some  $\mu \in L^\infty(D)$  with  $0 \leq \|\mu\|_\infty < 1$ .

The function  $\mu_f = \mu = \frac{f_{\bar{z}}}{f_z} \in L^\infty(D)$  is called the *Beltrami coefficient* of  $f$ . The constant  $K(f) = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty} \geq 1$  is called the *maximal dilatation* of  $f$ .

**Example 3.** For  $0 \leq k < 1$ , the mapping  $f(z) = \frac{z + k\bar{z}}{1 - k}$  defined on the plane is a quasiconformal mapping such that  $f(i) = i$ ,  $\mu_f = k$ , and  $K(f) = \frac{1 + k}{1 - k}$ .

**Lemma 2.3.2.** Let  $f : D_1 \rightarrow D_2$ ,  $g : D_2 \rightarrow D_3$  be quasiconformal mappings. Then the following holds:

$$\mu_g \circ f \cdot \frac{\overline{f_z}}{f_z} = \frac{\mu_{g \circ f} - \mu_f}{1 - \overline{\mu_f} \mu_{g \circ f}}. \quad (2.13)$$

In particular, for biholomorphisms  $\varphi : D_1 \rightarrow D_2$  and  $\psi : D_2 \rightarrow D_3$ , it follows that

$$\mu_{g \circ \varphi} = \mu_g \circ \varphi \cdot \frac{\overline{\varphi_z}}{\varphi_z}, \quad \mu_{\psi \circ f} = \mu_f. \quad (2.14)$$

As a consequence of Lemma 2.3.2, for two quasiconformal mappings  $f_i : D_1 \rightarrow D_2$  ( $i = 1, 2$ ), we have

$$\begin{aligned} \|\mu_{f_2 \circ f_1^{-1}}\|_\infty &= \text{ess. sup}_{f_1^{-1}(\mathbb{H})=\mathbb{H}} \left| \frac{\mu_{f_2} - \mu_{f_1}}{1 - \overline{\mu_{f_1}} \mu_{f_2}} \right|, \text{ and} \\ \log K(f_2 \circ f_1^{-1}) &= \text{ess. sup}_{\mathbb{H}} \log \left\{ \left( 1 + \left| \frac{\mu_{f_2} - \mu_{f_1}}{1 - \overline{\mu_{f_1}} \mu_{f_2}} \right| \right) \left( 1 - \left| \frac{\mu_{f_2} - \mu_{f_1}}{1 + \overline{\mu_{f_1}} \mu_{f_2}} \right| \right)^{-1} \right\} \\ &= \text{ess. sup}_{\mathbb{H}} d_{\mathbb{D}}(\mu_{f_1}, \mu_{f_2}). \end{aligned} \quad (2.15)$$

**Lemma 2.3.3.** Let  $f : D_1 \rightarrow D_2$ ,  $g : D_2 \rightarrow D_3$  be two quasiconformal mappings. Then,

- (1)  $g \circ f : D_1 \rightarrow D_3$  is a quasiconformal mapping with  $K(g \circ f) \leq K(g)K(f)$ ,
- (2)  $K(f) = 1$  if and only if  $f$  is a biholomorphism.

Note that the concept of quasiconformal mappings naturally extends to Riemann surfaces. The maximal dilatation is an invariant under biholomorphisms.

Let  $R$  be a Riemann surface of analytically finite type  $(g, n)$  with  $2g - 2 + n > 0$ . We consider an arbitrary tuple  $(R, S, f)$  (or  $(S, f)$  if  $R$  is understood) of a Riemann surface  $S$  and a quasiconformal mapping  $f : R \rightarrow S$ , called a *marked Riemann surface* based on  $R$ .

Every  $f$  lifts via the universal coverings to a quasiconformal mapping  $\tilde{f} : \mathbb{H} \rightarrow \mathbb{H}$  such that  $\tilde{f} \circ \gamma \circ \tilde{f}^{-1} \in \Gamma_S$  for any  $\gamma \in \Gamma_R$ , since the following holds on  $\mathbb{H}$ :

$$\begin{aligned} \pi_S \circ \tilde{f} \circ \gamma \circ \tilde{f}^{-1} &= f \circ \pi_R \circ \gamma \circ \tilde{f}^{-1} \\ &= f \circ \pi_R \circ \tilde{f}^{-1} \\ &= f \circ f^{-1} \circ \pi_S \\ &= \pi_S. \end{aligned} \tag{2.16}$$

The conjugation  $\theta_{\tilde{f}}(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$  defines a group isomorphism  $\theta_{\tilde{f}} : \Gamma_R \rightarrow \Gamma_S$ . It depends on the way choosing a lift  $\tilde{f}$ , that is unique up to pre-compositions in  $\Gamma_R$  and post-compositions in  $\Gamma_S$ . We say that two isomorphisms  $\theta_i : \Gamma_1 \rightarrow \Gamma_2$  ( $i = 1, 2$ ) are *equivalent* if they arise from essentially the same quasiconformal mappings. That is, for some  $\delta_1 \in \Gamma_1$ ,  $\delta_2 \in \Gamma_2$ ,

$$\theta_2(\gamma) = \delta_2^* \circ \theta_1 \circ \delta_1^*(\gamma), \quad \gamma \in \Gamma_1. \tag{2.17}$$

**Lemma 2.3.4.** *Two marked Riemann surfaces  $(R, S, f_i)$  ( $i = 1, 2$ ) determine equivalent isomorphisms if and only if  $f_1, f_2$  are homotopic.*

*Proof.* Let  $\tilde{f}_i : \mathbb{H} \rightarrow \mathbb{H}$  be a lift of  $f_i$  ( $i = 1, 2$ ). If  $f_1$  and  $f_2$  are homotopic,  $\tilde{f}_1$  and  $\tilde{f}_2$  are also joined by a homotopy  $\tilde{F}_t$ ,  $t \in [0, 1]$ . For each  $z \in \mathbb{H}$  and  $t \in [0, 1]$ , the orbit of  $z$  under  $\theta_{\tilde{F}_t}(\Gamma_S) = \Gamma_S$  is a discrete set invariant for  $t$ . It cannot be moved continuously and we have  $\theta_{\tilde{F}_t} = id_{\Gamma_S}$ .

Conversely, if  $f_1$  and  $f_2$  determine equivalent isomorphisms, we may choose  $\tilde{f}_i$  ( $i = 1, 2$ ) so that  $\theta_{\tilde{f}_1} = \theta_{\tilde{f}_2} =: \theta$  holds. For each  $z \in \mathbb{H}$  and  $t \in [0, 1]$ , define  $\tilde{F}_t(z)$  as the point on the hyperbolic geodesic between  $\tilde{f}_1(z)$  and  $\tilde{f}_2(z)$  dividing by the ratio  $t : (1 - t)$ . Then we have

$$\tilde{F}_t \circ \gamma(z) = \theta(\gamma) \circ \tilde{F}_t(z), \quad z \in \mathbb{H}, \gamma \in \Gamma_R, \tag{2.18}$$

and thus the mapping  $\tilde{F}_t$  projects to a homotopy joining  $f_1$  and  $f_2$ .  $\square$

**Definition 2.3.5.** Fix a Riemann surface  $R$  of analytically finite type  $(g, n)$  with  $2g - 2 + n > 0$ .

- (1) We say that two marked Riemann surfaces  $(S_1, f_1), (S_2, f_2)$  are *Teichmüller equivalent* if there exists a biholomorphism  $\varphi : S_1 \rightarrow S_2$  homotopic to  $f_2 \circ f_1^{-1} : S_1 \rightarrow S_2$ . We call the set of Teichmüller equivalence classes of all marked Riemann surfaces based on  $R$  the *Teichmüller space* of  $R$  and denote by  $T(R)$ .
- (2) We say that two marked Riemann surfaces  $(S_1, f_1), (S_2, f_2)$  are *biholomorphically equivalent* if  $f_2 \circ f_1^{-1} : S_1 \rightarrow S_2$  is a biholomorphism. We call the set of biholomorphically equivalence classes of all marked Riemann surfaces the *moduli space* of  $R$  and denote by  $M(R)$ .

For any two points  $x_i = [R, S_i, f_i] \in T(R)$  ( $i = 1, 2$ ), we set

$$d_T(x_1, x_2) := \inf\{\log K(g) \mid g : S_1 \rightarrow S_2 \text{ is homotopic to } f_2 \circ f_1^{-1}\}. \quad (2.19)$$

It follows from (2.15) and Lemma 2.3.3 that the mapping  $d_T$  defines a distance on  $T(R)$ , called the *Teichmüller distance*.

For every quasiconformal mapping  $g : R_1 \rightarrow R_2$ , the pre-composition

$$\rho_g([R_1, S, f]) := [R_2, S, f \circ g^{-1}], \quad [R_1, S, f] \in T(R_1) \quad (2.20)$$

defines a isometry  $T(R_1) \rightarrow T(R_2)$  called a *geometric isomorphism*. In this way, quasiconformal self-mappings of  $R$  acts on the Teichmüller space  $T(R)$ . The group of all geometric automorphisms of  $T(R)$  is called the *Teichmüller modular group* of  $R$  and denoted by  $\text{Mod}(R)$ . Every biholomorphically equivalent marked Riemann surfaces are related by a suitable geometric automorphism, and  $M(R) = T(R)/\text{Mod}(R)$  holds.

**Remark 2.3.6.** The action of quasiconformal self-mappings on the Teichmüller space is faithful up to factor of homotopically trivial mappings except for few exceptional types  $(2, 0)$ ,  $(1, 2)$ ,  $(1, 1)$ ,  $(0, 4)$ , and  $(0, 3)$  [47, Proposition 2.3.10]. Quasiconformal mappings approximate orientation preserving homeomorphisms. Teichmüller spaces are equivalently defined by the space of classes of orientation preserving homeomorphisms, and then the Teichmüller modular group  $\text{Mod}(R)$  is identified with the *mapping class group* of  $R$ .

Via the geometric isomorphisms,  $T(R)$  are mutually homeomorphic for all Riemann surfaces  $R$  of the same analytically finite type  $(g, n)$ . The Teichmüller space of such  $R$  is independent of the base point, and we denote such a space by  $T_{g,n}$ . Similarly we may denote by  $\text{Mod}_{g,n}$  and  $M_{g,n}$  up to identifications of spaces under the change of base points.

**Theorem 2.3.7** (measurable Riemann mapping theorem, Ahlfors-Bers, [2]). *For any  $\mu \in L^\infty(\hat{\mathbb{C}})$  with  $\|\mu\|_\infty < 1$ , there exists a unique quasiconformal mapping  $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  with Beltrami coefficient  $\mu$  that leaves  $0, 1, \infty$  fixed.*

For  $D = \hat{\mathbb{C}}, \mathbb{H}$ , or  $\mathbb{L}$ , let  $B_1(D)$  denote the unit ball in  $L^\infty(D)$ . By Lemma 2.3.2, for a marked Riemann surface  $(R, S, f)$ , it follows that

$$\mu_f = \mu_{f \circ \gamma} = \mu_f \circ \gamma \cdot \frac{\overline{\gamma'}}{\gamma'} \text{ a.e. on } \mathbb{H}, \text{ for every } \gamma \in \Gamma_R. \quad (2.21)$$

We say that  $B_1(D, \Gamma_R) := \{\mu \in B_1(D) \mid \mu = \mu \circ \gamma \cdot \frac{\overline{\gamma'}}{\gamma'} \text{ a.e. on } \mathbb{H}, \text{ for every } \gamma \in \Gamma_R\}$  is the set of *Beltrami differentials* on  $D$  with respect to  $\Gamma_R$ . Every  $\mu \in B_1(\mathbb{H}, \Gamma_R)$  extends to  $B_1(\hat{\mathbb{C}}, \Gamma_R)$  in the following two ways:

(1) (symmetric extension)  $\mu = 0$  on  $\hat{\mathbb{R}}$  and  $\mu(t) = \overline{\mu(\bar{t})}$  for each  $t \in \mathbb{L}$ ,

(2) (holomorphic extension)  $\mu = 0$  on  $\mathbb{L}$ .

Let  $f^\mu$  ( $f_\mu$ , respectively) be the unique quasiconformal mapping given by Theorem 2.3.7 corresponding to the extension (1) ((2), respectively) of  $\mu$ . As the compositions with each element in  $\Gamma_R$  gives an another normalized solution of the Beltrami equation,  $f^\mu$  and  $f_\mu$  are compatible with the action of  $\Gamma_R$ .

**Lemma 2.3.8.**  $f^\mu = f^\nu$  holds on  $\hat{\mathbb{R}}$  if and only if  $f_\mu = f_\nu$  holds on  $\mathbb{L}$ .

*Proof.* If  $f^\mu = f^\nu$  holds on  $\hat{\mathbb{R}}$ , the mapping  $f = (f^\mu)^{-1} \circ f^\nu : \mathbb{H} \rightarrow \mathbb{H}$  extends by the identity on  $\hat{\mathbb{R}} \cup \mathbb{L}$  as to be ACL. Then, the mapping  $g = f^\mu \circ f \circ (f^\nu)^{-1} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  is a quasiconformal mapping whose Beltrami coefficient vanishes, and hence a Möbius transformation by Lemma 2.3.3. By Theorem 2.3.7 it must be identity and thus  $f_\mu = f_\nu$  holds on  $\mathbb{L}$ .

Conversely, if  $f_\mu = f_\nu$  holds on  $\mathbb{L}$ , then  $f_\mu = f_\nu$  still holds on  $\hat{\mathbb{R}} \cup \mathbb{L}$ . As before, the mapping  $h = f^\mu \circ (f_\mu)^{-1} \circ f_\nu \circ (f^\nu)^{-1} : \mathbb{H} \rightarrow \mathbb{H}$  extends by the identity on  $\hat{\mathbb{R}} \cup \mathbb{L}$  to be a quasiconformal mapping whose Beltrami coefficient vanishes. Again by Theorem 2.3.7, it must be identity and thus  $f^\mu = f^\nu$  holds on  $\hat{\mathbb{R}}$ .  $\square$

By definition, the mapping  $f_\mu$  is biholomorphic on the lower half-plane  $\mathbb{L}$ . For each biholomorphism  $f$  on  $\mathbb{L}$ , we define the *Schwarzian derivative*  $\mathcal{S}(f)$  of  $f$  by

$$\mathcal{S}(f)(z) := \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \quad (2.22)$$

$$= (\log f'(z))'' - \frac{1}{2} \{(\log f'(z))'\}^2, \quad z \in \mathbb{L}. \quad (2.23)$$

**Lemma 2.3.9.** *The following holds for any holomorphic mappings  $f, g$  on  $\mathbb{L}$ :*

- (1)  $\mathcal{S}(f) = 0$  if and only if  $f$  is a Möbius transformation,
- (2)  $\mathcal{S}(g \circ f)(z) = \mathcal{S}(g)(f(z)) \cdot f'(z)^2 + \mathcal{S}(f)(z), \quad z \in \mathbb{L}.$

*Proof.* By solving  $\mathcal{S}(f) = 0$  with equation (2.23), we will observe that such a  $f$  is a Möbius transformation. The rest follows from a direct calculation.  $\square$

Thus the Schwarzian derivative of  $f_\mu, \mu \in B_1(\mathbb{H}, \Gamma_R)$  is regarded as a bounded *holomorphic quadratic differential* (see Definition 4.1.1) on  $\mathbb{L}/\Gamma_R$  that is the mirror image of  $R$ . The space  $\mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)$  of hyperbolically bounded, holomorphic quadratic differentials on  $\mathbb{L}/\Gamma_R$  is a complex Banach space equipped with the hyperbolic  $L^\infty$  norm (of weight  $-2$ ), whose

dimension is  $3g - 3 + n$  by Riemann-Roch theorem. The mapping  $\mathcal{B} : T(R) \rightarrow \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)$  defined by  $\mathcal{B}([S, g]) := S(f_{\mu_g})$  is injective by Lemma 2.3.8 and Lemma 2.3.9. It induces a complex structure from  $\mathcal{Q}^\infty(\mathbb{L}, \Gamma_R) \cong \mathbb{C}^{3g-3+n}$ , and the mapping  $\mathcal{B}$  is called the *Bers' embedding*.

The composition of the surjective mapping  $\mathcal{P} : B_1(\mathbb{H}, \Gamma_R) \rightarrow T(R) : \mu \mapsto [f^\mu(R), f^\mu]$  and the Bers' embedding is called the *Bers' projection*  $\Phi = \mathcal{B} \circ \mathcal{P} : B_1(\mathbb{H}, \Gamma_R) \rightarrow \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)$ . The above is summarized in the following diagram, where  $QC(\hat{\mathbb{C}}, \Gamma_R)$  denotes the group of quasiconformal mappings of  $\hat{\mathbb{C}}$  compatible with  $\Gamma_R$ .

$$\begin{array}{ccccccc}
 B_1(\mathbb{H}, \Gamma_R) & \xrightarrow{\mathcal{P}} & T(R) & \xrightarrow{\quad \quad \quad} & B_1(\mathbb{H}, \Gamma_R) & \longrightarrow & QC(\hat{\mathbb{C}}, \Gamma_R) \longrightarrow \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R) \\
 \psi & & \psi & & \psi & & \psi \\
 \mu & \longmapsto & [f^\mu(R), f^\mu = g] & \longmapsto & \mu_g & \longmapsto & f_{\mu_g} \longmapsto S(f_{\mu_g})
 \end{array}$$

$\mathcal{B}$

It is proved by Ahlfors and Weill [3] [32, Theorem 6.9] that the *harmonic Beltrami differential* operator defined by

$$\mathcal{Q}^\infty(\mathbb{L}, \Gamma_R) \rightarrow B_1(\mathbb{H}, \Gamma_R) : \alpha \mapsto (\mu_\alpha(z) = -2(\operatorname{Im} z)^2 \alpha(\bar{z}), \quad z \in \mathbb{H}) \quad (2.24)$$

gives a local inverse mapping of the Bers' projection.

The following analysis of the projection  $\Phi$  is due to Bers [6] [47, Chapter 3]. The derivative of the Bers' projection  $\Phi$  in the direction  $\nu \in L^\infty(\mathbb{H}, \Gamma_R)$  at  $\mu \in B_1(\mathbb{H}, \Gamma_R)$  is defined by

$$\Phi_\mu[\nu] = \lim_{t \rightarrow 0} \frac{\Phi(\mu + t\nu) - \Phi(\mu)}{t}. \quad (2.25)$$

The limit (2.25) does exist and is represented as a bounded linear mapping on  $B_1(\mathbb{H}, \Gamma_R)$  by the formula [32, Section 6.2.2]:

$$\Phi_\mu[\nu] = \left( z \mapsto -\frac{6}{\pi} \iint_{\mathbb{H}} \frac{\nu(\zeta) \left( \frac{df_\mu}{d\zeta}(\zeta) \right)^2}{(f_\mu(\zeta) - f_\mu(z))^4} d\xi d\eta \cdot f'_\mu(z)^2, \quad z \in \mathbb{L} \right). \quad (2.26)$$

The mapping (2.26) surjects onto  $\mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)$  and its kernel is a direct summand in  $B_1(\mathbb{H}, \Gamma_R)$ . Finally, the Bers' projection  $\Phi$  is a holomorphic *submersion*; everywhere the mapping  $\Phi : B_1(\mathbb{H}, \Gamma_R) \rightarrow \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)$  locally is a projection

$$U_1 \times U_2 \rightarrow U_1 \hookrightarrow V, \quad U_1, U_2 \subset B_1(\mathbb{H}, \Gamma_R), \quad V \subset \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R). \quad (2.27)$$

Then, the Banach space  $L^\infty(\mathbb{H}, \Gamma_R)$  splits into the kernel  $\ker \Phi_\mu$  and the tangent space  $T_{\mathcal{P}(\mu)}T(R) = \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)$ . The dual space  $T_{\mathcal{P}(\mu)}^*T(R) = \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)^*$  is represented [47,



Theorem 1.4.3] as to be the space  $\mathcal{Q}(\mathbb{L}, \Gamma_R) = \mathcal{Q}(\bar{R})$  of integrable holomorphic quadratic differentials (Definition 4.1.2), by the following *Weil-Peterson inner product* pairing:

$$(\alpha, \phi)_{\mathbb{L}/\Gamma_R} = \iint_{\mathbb{L}/\Gamma_R} \alpha(z) \overline{\phi(z)} \frac{|dz|^2}{(2\operatorname{Im}z)^2}, \quad \alpha \in \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R), \quad \phi \in \mathcal{Q}(\mathbb{L}, \Gamma_R), \quad (2.28)$$

where the integral refers to any fundamental domain for  $\Gamma_R$  in  $\mathbb{L}$ . Remark that the spaces  $\mathcal{Q}(R)$  and  $\mathcal{Q}^\infty(R)$  are equal for a Riemann surface  $R$  of analytically finite type. We have the following.

**Proposition 2.3.10.** *Let  $R$  be a Riemann surface and  $[S, f] \in T(R)$ . Then, the cotangent space  $T_{[S, f]}^* T(R)$  is isomorphic to the space  $\mathcal{Q}(\bar{R})$  of integrable holomorphic quadratic differentials on  $\bar{R}$ .*



# Chapter 3

## Concepts in Algebraic Geometry

### 3.1 Variety

In this section, we make some notes to grasp an outline of the algebraic-geometric aspects of the subject described in chapter 2. This section is based on [21].

Let  $\mathbb{K}$  be a field and  $\mathbb{K}[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over  $\mathbb{K}$ . Assume that  $\mathbb{K}$  is an *algebraically closed* field i.e. every non-constant polynomial in  $\mathbb{K}[x]$  has a root in  $\mathbb{K}$ . For a subset  $T \subset \mathbb{K}[x_1, \dots, x_n]$ , Let  $Z(T)$  denote the set of common zeros of all elements in  $T$ :

$$Z(T) = \{(a_1, \dots, a_n) \in \mathbb{K}^n \mid f(a_1, \dots, a_n) = 0 \text{ for all } f \in T\}. \quad (3.1)$$

A subset  $Y \subset \mathbb{K}^n$  is called an *algebraic set* if  $Y = Z(T)$  for some  $T \subset \mathbb{K}[x_1, \dots, x_n]$ . Every algebraic set  $Z(T)$ ,  $T \subset \mathbb{K}[x_1, \dots, x_n]$  is uniquely represented by the ideal generated by  $T$ .

The collection of algebraic sets in  $\mathbb{K}^n$  satisfies the axiom of closed sets. We consider the *Zariski topology* on  $\mathbb{K}^n$  defined by taking the closed subsets to be the algebraic sets. An irreducible closed subset in  $\mathbb{K}^n$  is called an *affine variety*. An open subset in an affine variety is called a *quasi-affine variety*.

**Definition 3.1.1.** For a subset  $Y \subset \mathbb{K}^n$ , define the *ideal*  $I(Y)$  of  $Y$  by

$$I(Y) = \{f \in \mathbb{K}[x_1, \dots, x_n] \mid f(a_1, \dots, a_n) = 0 \text{ for all } (a_1, \dots, a_n) \in Y\}. \quad (3.2)$$

**Theorem 3.1.2** (Hilbert's Nullstellensatz [21, Theorem 1.3A]). *Let  $\mathbb{K}$  be an algebraically closed field, let  $\mathfrak{a} \subset \mathbb{K}[x_1, \dots, x_n]$  be an ideal, and  $f \in \mathbb{K}[x_1, \dots, x_n]$  be a polynomial which vanishes at all points of  $Z(\mathfrak{a})$ . Then, the polynomial  $f$  belongs to the radical ideal*

$$\sqrt{\mathfrak{a}} = \{g \in \mathbb{K}[x] \mid g^r \in \mathfrak{a} \text{ for some integer } r > 0\}. \quad (3.3)$$

**Corollary 3.1.3** ([21, Corollary 1.4]). *There is a one-to-one inclusion-reversing correspondence between algebraic sets in  $\mathbb{K}^n$  and radical ideals in  $\mathbb{K}[x_1, \dots, x_n]$ , given by  $Y \mapsto I(Y)$  and  $\mathfrak{a} \mapsto Z(\mathfrak{a})$ . Furthermore, an algebraic set is irreducible if and only if its ideal is a prime ideal.*

*Proof.* The correspondences  $I$  and  $Z$  give inclusion-reversing by their definitions. Let  $\mathfrak{a} \subset \mathbb{K}[x_1, \dots, x_n]$  be an arbitrary ideal. Theorem 3.1.2 implies  $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ . For any algebraic set  $Y = Z(\mathfrak{a})$ , applying  $Z$  to  $\mathfrak{a} \subset \sqrt{\mathfrak{a}} = I(Z(\mathfrak{a}))$ , we obtain  $Y \supset Z(I(Y))$ . The converse inclusion clearly holds.

Let  $Y \subset \mathbb{K}^n$  be an irreducible algebraic set and  $fg \in I(Y)$ . Then  $Y$  is written by

$$Y = Y \cap (Z(fg)) = Y \cap (Z(f) \cup Z(g)) = (Y \cap Z(f)) \cup (Y \cap Z(g)), \quad (3.4)$$

which is the union of two proper closed subsets. By irreducibility, we have either  $Y \subset Z(f)$  or  $Y \subset Z(g)$ , and hence either  $f \in I(Y)$  or  $g \in I(Y)$ . Conversely, suppose that  $Z(\mathfrak{p}) = Y_1 \cup Y_2$  for a prime ideal  $\mathfrak{p} \subset \mathbb{K}[x_1, \dots, x_n]$ . Then  $\mathfrak{p} = I(Y_1 \cup Y_2) = I(Y_1) \cup I(Y_2)$ , so either  $\mathfrak{p} = I(Y_1)$  or  $\mathfrak{p} = I(Y_2)$  holds. By applying  $Z$  we conclude the irreducibility.  $\square$

The affine space  $\mathbb{K}^n$  is known to be a *noetherian* topology space, that is, any (strictly) descending chain  $Y_1 \supset Y_2 \supset \dots$  of closed subsets stops in finite index:  $Y_n = Y_{n+1} = \dots$  for some integer  $n$ . In particular, a maximal ideal  $\mathfrak{m} \subset \mathbb{K}[x_1, \dots, x_n]$  corresponds to a minimal irreducible component of  $\mathbb{K}^n$  that must be a point.

**Definition 3.1.4.** Let  $Y \in \mathbb{K}^n$  be an algebraic set. The (*Krull*) *dimension*  $\dim(Y)$  of  $Y$  is the supremum of the height  $h$  of a descending chain  $\mathfrak{p}_1 \supset \mathfrak{p}_2 \supset \dots \supset \mathfrak{p}_h$  in  $\text{Spec}(I(Y))$ . The variety  $Y$  is called a *curve* if it has dimension 1. The *affine coordinate ring*  $A(Y)$  of  $Y$  is the quotient ring  $\mathbb{K}[x_1, \dots, x_n]/I(Y)$ .

**Definition 3.1.5.** Let  $Y$  be an affine variety.

- (1) A function  $f : Y \rightarrow \mathbb{K}$  on  $Y$  is called *regular* at  $p \in Y$  if there exists a open neighborhood  $U \subset Y$  of  $p$  and polynomials  $g, h \in \mathbb{K}[x_1, \dots, x_n]$  such that  $h$  is nowhere zero on  $U$ , and  $f = g/h$  on  $U$ . Let  $\mathcal{O}(Y)$  be the ring of all regular functions on  $Y$ .
- (2) A *germ* of a regular function at  $p \in Y$  is the class of a pair  $(U, f)$  of a open neighborhood  $U \subset Y$  of  $p$  and a regular function  $f : U \rightarrow \mathbb{K}$  under the equivalence relation defined by

$$(U, f) \sim_p (V, g) \Leftrightarrow f = g \text{ on } U \cap V. \quad (3.5)$$

For each  $p \in Y$ , the ring  $\mathcal{O}_p$  of all germs of regular functions at  $p$  is called the *local ring* of  $p$  on  $Y$ .

- (3) A *rational function* on  $Y$  is the class of a pair  $(U, f)$  of a nonempty open set  $U \subset Y$  and a regular function  $f : U \rightarrow \mathbb{K}$  under the equivalence relation defined by

$$(U, f) \sim (V, g) \Leftrightarrow f = g \text{ on } U \cap V. \quad (3.6)$$

The field  $K(Y)$  of all rational functions on  $Y$  is called the *function field* on  $Y$ .

A continuous mapping  $\varphi : Y_1 \rightarrow Y_2$  between two affine varieties is called a *morphism* if  $\varphi \circ f : U \rightarrow \mathbb{K}$  is regular for any regular function  $f : U \rightarrow \mathbb{K}$  on an open set  $U \subset Y_1$ . A *rational map* is the class of a pair  $(U, \varphi)$  of a nonempty open set  $U \subset Y_1$  and a morphism  $\varphi : Y_1 \rightarrow Y_2$  under the equivalence relation defined by

$$(U, \varphi) \sim (V, \psi) \Leftrightarrow \varphi = \psi \text{ on } U \cap V. \quad (3.7)$$

A rational map  $[U, \varphi]$  is called *dominant* if for some (and hence every) representative  $(U, \varphi)$ , the image  $\varphi(U) \subset Y$  is dense.

**Theorem 3.1.6** ([21, Theorem 3.2]). *Let  $Y \subset \mathbb{K}^n$  be an affine variety with affine coordinate ring  $A(Y)$ . Then the following holds:*

- (1) *The ring  $\mathcal{O}(Y)$  is isomorphic to the affine coordinate ring  $A(Y)$ .*
- (2) *For each point  $p \in Y$ , let  $\mathfrak{m}_p \subset A(Y)$  be the ideal of polynomials vanishing at  $p$ . Then,  $p \mapsto \mathfrak{m}_p$  gives a one-to-one correspondence between the points of  $Y$  and maximal ideals of  $A(Y)$ .*
- (3) *For each point  $p \in Y$ ,  $\mathcal{O}_p \cong A(Y)/\mathfrak{m}_p$  and  $\dim(\mathcal{O}_p) = \dim(Y)$  hold.*
- (4) *The function field  $K(Y)$  is isomorphic to the quotient field of  $A(Y)$ , and hence  $K(Y)$  is a finitely generated extension field of  $\mathbb{K}$ , of transcendence degree  $\dim(Y)$ .*

An affine variety  $Y \subset \mathbb{K}^n$  is called *nonsingular* at  $p \in Y$  if

$$\text{rank} \left( \frac{\partial f_j}{\partial x_i}(p) \right)_{\substack{i=1, \dots, m \\ j=1, \dots, n}} = n - \dim(Y), \quad (3.8)$$

where  $\{f_1, \dots, f_m\}$  is a generating system of the ideal  $I(Y)$  of  $Y$ . The variety  $Y$  is called nonsingular if it is nonsingular at any point  $p \in Y$ . If  $\mathbb{K} = \mathbb{C}$ , a nonsingular variety  $Y$  is a complex manifold of dimension  $\dim(Y)$ .

For an affine variety  $Y \subset \mathbb{K}^n$ , we may take the *projective* model defined in the projective  $n$ -space  $\mathbb{K}^{n+1}/\mathbb{K}^\times$  and in terms of homogenous polynomials. There is a similar discussion above for the projective varieties.

**Theorem 3.1.7** ([21, Corollary 6.12]). *The following three categories are equivalent:*

- (1) *the category of nonsingular projective curves with dominant rational maps,*
- (2) *the category of compact Riemann surfaces with holomorphic mappings, and*
- (3) *the category of function fields of dimension 1 over  $\mathbb{C}$  with  $\mathbb{C}$ -algebra homomorphisms.*

For a commutative ring  $R$ , the set of all prime ideals of  $R$  is called the *spectrum* of  $R$  and denoted by  $\text{Spec}(R)$ . By Corollary 3.1.3, the spectrum  $\text{Spec}(R)$  of  $R$  is identified with the set of all irreducible algebraic sets on which every  $f \in R$  vanishes. The Zariski topology on  $\text{Spec}(R)$  is defined by an open basis  $\{U_f = \{\mathfrak{p} \in \text{Spec}(R) \mid f \notin \mathfrak{p}\} \mid f \in R\}$ .

**Definition 3.1.8** (localization). For a commutative ring  $R$  and a multiplicative closed set  $S$ , we define an  $R$ -module  $S^{-1}R$  as follows.

- (1) Let  $S^{-1}R$  be the quotient of  $S \times R$  by the equivalence relation:

$$(s_1, r_1) \sim (s_2, r_2) \Leftrightarrow \exists m \in S \text{ s.t. } m(s_1 r_2 - s_2 r_1) = 0. \quad (3.9)$$

We denote the equivalence class of  $(s, r) \in S^{-1}R$  by  $r/s$ .

- (2) For  $r_1/s_1, r_2/s_2 \in S^{-1}R$  and  $c \in R$ , define an addition and a scalar multiplication by

$$r_1/s_1 + r_2/s_2 := (r_1 s_2 + r_2 s_1)/(s_1 s_2), \quad (3.10)$$

$$c \cdot r_1/s_1 := (c r_1)/s_1. \quad (3.11)$$

For each  $\mathfrak{p} \in \text{Spec}(R)$ , the  $R$ -module  $R_{\mathfrak{p}} := (R \setminus \mathfrak{p})^{-1}R$  has a unique maximal ideal  $\mathfrak{p}R_{\mathfrak{p}}$  and is called the local ring of  $\text{Spec}(R)$  at  $\mathfrak{p}$ . The quotient field  $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$  is called the *residue field* of  $\text{Spec}(R)$  at  $\mathfrak{p}$ .

There exists a unique way to glue the local rings of  $\text{Spec}(R)$  to form the ring  $R_f := \{1, f, f^2, \dots\}^{-1}R = R[x]/(fx - 1)$  on  $U_f, f \in R$  [21, Proposition 2.2]. A spectrum with the structure (*sheaf* of local rings) constructed in this way is called an *affine scheme*.

**Example 4.** The spectrum of a field  $\mathbb{K}$  is the one point scheme  $\text{Spec}(\mathbb{K}) = \{0\}$ . Its local ring is  $\mathcal{O}_{\{0\}} = \mathbb{K}$ .

Let  $\mathbb{K}_1 \subset \mathbb{K}$  be a subfield. Fix an embedding  $\varphi : \mathbb{K}_1 \hookrightarrow \mathbb{K}[x_1, \dots, x_n]$ . Then, for  $f \in \mathbb{K}[x_1, \dots, x_n]$  and  $(p_1, \dots, p_n) \in \mathbb{K}_1^n$ , the mapping

$$A(U_f) = \mathbb{K}[x_1, \dots, x_n]/(f) \rightarrow \mathbb{K} : x_j \mapsto p_j, \quad j = 1, \dots, n \quad (3.12)$$

is well-defined as to be compatible with the embedding  $\mathbb{K}_1 \hookrightarrow \mathbb{K}$  if and only if  $f(p_1, \dots, p_n)$  vanishes. We may reproduce the algebraic set  $Z(f)$  from the the affine coordinate ring  $A(U_f)$  in this way. More generally, if we introduce the notion of *schemes* that are defined by gluing locally defined ‘spaces with local rings’ each of which isomorphic to an affine scheme, every scheme  $S$  with a fixed morphism  $\varphi$  to a fixed affine scheme  $\text{Spec}(\mathbb{K})$  is locally represented by a variety over  $\mathbb{K}$ . Now, the embedding  $\varphi$  controls the coefficients arising in the local representation by Theorem 3.1.6. Each of the categories stated in Theorem 3.1.7 is identified with the category of schemes. We say that a scheme  $S$  admitting a morphism to  $\text{Spec}(\mathbb{K})$  is *defined over*  $\mathbb{K}$ , and a fixed morphism  $\varphi : S \rightarrow \text{Spec}(\mathbb{K})$  is called a *structure morphism*.

## 3.2 Dessin d’enfant

This section is based on [28, 33].

The *absolute Galois group*  $G_{\mathbb{Q}} = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  is the group of field automorphisms on the algebraic closure  $\bar{\mathbb{Q}}$  of  $\mathbb{Q}$ . The absolute Galois group  $G_{\mathbb{Q}}$  acts on the category of nonsingular projective curves defined over  $\bar{\mathbb{Q}}$  by post-composing with fixed structure morphism.

**Theorem 3.2.1** (Belyĭ [5], 1979). *A nonsingular projective curve  $C$  is defined over  $\bar{\mathbb{Q}}$  if and only if it admits a covering  $\beta : C \rightarrow \hat{\mathbb{C}}$  branched at most over three points.*

**Definition 3.2.2.** A compact Riemann surface  $R$  admitting a meromorphic function  $\beta : R \rightarrow \hat{\mathbb{C}}$  branched over at most three points  $0, 1, \infty \in \hat{\mathbb{C}}$  is called a *Belyĭ surface*. Such  $\beta$  is called a *Belyĭ covering* and a pair  $(R, \beta)$  is called a *Belyĭ pair*.

The if-part of Theorem 3.2.1 is sketched in Belyĭ’s paper [5], and known as an "obvious part" to those who are familiar with the results of Weil’s paper [58]. Köck [35] reformulated the proof of this part. The only-if-part of Theorem 3.2.1 is worked out by the following *Belyĭ’s algorithm* [33, Section 1.4.4]:

- (1) Take a projection  $\pi : C \rightarrow \hat{\mathbb{C}}$ . Since  $C$  is nonsingular, the mapping  $\pi$  is holomorphic and has finite set of branched points  $B := \text{Br}(\pi) \subset \bar{\mathbb{Q}} \cup \{\infty\}$ .
- (2) Take a minimal polynomial  $P \in \mathbb{Q}[x]$  of  $B$ . As  $P$  and  $P'$  are invariant under the  $G_{\mathbb{Q}}$ -action, so are  $\text{Crit}(P)$  and  $\text{Br}(P)$ . Let  $B' := \text{Br}(P \circ \pi) \subset P(\text{Br}(\pi)) \cup \text{Br}(P)$ . The number of branched points outside  $\mathbb{Q}$  can be strictly reduced by repeatedly composing a minimal polynomial of the branched set.
- (3) Let  $i = 0$ ,  $P_0 = P$ , and  $\text{Br}(P_i \circ \pi) = \{p_1, \dots, p_n\} \subset \mathbb{Q}$ .

- (4) Take a branched point  $p \in \text{Br}(P_i \circ \pi) \subset \mathbb{Q}$ . We may assume  $p \in [0, 1] \cap \mathbb{Q}$  up to the automorphisms  $z \mapsto 1 - z$ ,  $z \mapsto z^{-1}$  that preserve  $\{0, 1, \infty\}$ . For  $p = \frac{n}{m} \in [0, 1] \cap \mathbb{Q}$ , let  $P_j := Q_{m,n} \circ P_{j-1}$  where

$$Q_{m,n}(z) := \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n, \quad z \in \mathbb{C}. \quad (3.13)$$

The polynomial  $Q_{m,n}$  sends  $0, 1 \mapsto 0$ ,  $p \mapsto 1$ ,  $\infty \mapsto \infty$ , and  $\text{Crit}(Q_{m,n})$  leaves in  $\{0, 1, \infty, p\}$ .

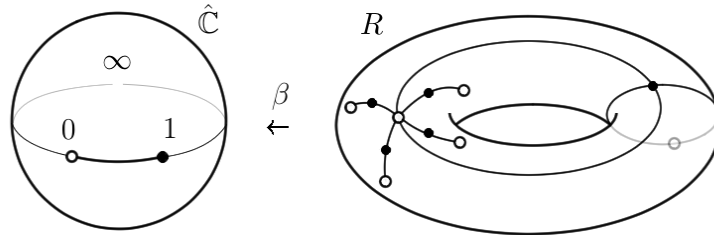
- (5) As the cardinality of  $\text{Br}(P_j \circ \pi) \setminus \{0, 1, \infty\} \subset Q(\text{Br}(\pi)) \cup \text{Br}(P) \subset \text{Br}(P_j \circ \pi)$  strictly decreases for  $j$ , we can repeat the step (4) in finite time  $n$  so that  $\text{Br}(P_n \circ \pi) \subset \{0, 1, \infty\}$ . Then,  $P_n \circ \pi$  is a Belyi covering.

**Definition 3.2.3.** A *dessin d'enfants* (or simply a *dessin*) is a bipartite, connected, filling graph-embedding. i.e. an embedding a pair of finite sets  $\mathcal{G} = (\mathcal{V} = \mathcal{V}_\circ \sqcup \mathcal{V}_\bullet, \mathcal{E})$  into a topological surface  $R$  as follows:

- (1) the image of every vertex  $v \in \mathcal{V}$  is a point on  $R$ ,
- (2) the image of every edge  $e \in \mathcal{E}$  is a path intersecting no edges,
- (3) every edge has one endpoint in  $\mathcal{V}_\circ$  and the other in  $\mathcal{V}_\bullet$ ,
- (4) every two vertices are connected via finitely many edges, and
- (5) every components of  $R \setminus \mathcal{G}$  (*faces*) are homeomorphic to an open disk.

The number of edges is called the *degree* of a dessin d'enfants. We say that two dessins d'enfants  $f_i : \mathcal{G}_i \hookrightarrow R_i$  ( $i = 1, 2$ ) are *equivalent* if there exists a homeomorphism  $g : R_1 \rightarrow R_2$  such that  $f_2 = g \circ f_1$ .

**Example 5.** By assigning  $\mathcal{V}_\circ = \{0\}$ ,  $\mathcal{V}_\bullet = \{1\}$ ,  $\mathcal{E} = \{[0, 1]\}$ , we obtain the trivial dessin on the Riemann sphere  $\hat{\mathbb{C}}$ . Every Belyi covering  $\beta : R \rightarrow \hat{\mathbb{C}}$  induces a dessin d'enfants on  $R$  as the pullback of the trivial dessin (see Fig. 3.A).



**Fig. 3.A** Example of a dessin d'enfants induced from a Belyi covering  $\beta : R \rightarrow \hat{\mathbb{C}}$ .



**Proposition 3.2.4.** *A dessin d'enfant of degree  $d$  is up to equivalence (in brackets) uniquely determined by each of the following.*

- (a) *A Belyĭ pair  $(R, \beta)$  of degree  $d$  [up to covering equivalence over  $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ ].*
- (b) *A pair of two permutations  $x, y \in \mathfrak{S}_d$  generating a transitive permutation group [up to conjugation in  $\mathfrak{S}_d$ ].*
- (c) *A subgroup  $H$  of the free group  $F_2$  of index  $d$  [up to conjugation in  $F_2$ ].*

*Proof.* As we have seen in Remark 2.1.7, the objects in (a-c) are in a one-to-one correspondence up to equivalence. For a Belyĭ pair  $(R, \beta)$ , the pullback of the trivial dessin under  $\beta$  gives a dessin on  $R$ . The covering equivalence of Belyĭ pairs corresponds to the equivalence of dessins given by this construction. Conversely, once a dessin  $(\mathcal{V}_\circ \sqcup \mathcal{V}_\bullet, \mathcal{E}) \hookrightarrow R$  of degree  $d$  given up to equivalence, we may define two permutations  $x \in \text{Sym}(\mathcal{E})$  ( $y \in \text{Sym}(\mathcal{E})$ , respectively) by the anti-clockwise permutation of edges around all the vertices in  $\mathcal{V}_\circ$  ( $\mathcal{V}_\bullet$ , respectively). We obtain a permutation group as in (b) by numbering edges, the way of which is unique up to conjugation in  $\mathfrak{S}_d$ .  $\square$

**Definition 3.2.5.** We say that a branched covering  $f : R \rightarrow S$  has the *valency list*

$$(k_1^{p_1}, \dots, k_{l_1}^{p_1} \mid \dots \mid k_1^{p_n}, \dots, k_{l_n}^{p_n}), \quad k_1^{p_1} \leq k_2^{p_1} \leq \dots \leq k_{l_1}^{p_1} \quad (3.14)$$

if  $\text{Br}(f) = \{p_1, \dots, p_n\}$  and the pullback  $f^{-1}(p)$  consists precisely of  $n_p$  points of orders  $k_1^p, \dots, k_{n_p}^p$  for every  $p \in \text{Br}(f)$ .

For a Belyĭ pair  $(R, \beta)$ , the multiplicity function  $\text{mult}_\bullet(\beta) : R \rightarrow \mathbb{Z}_{\geq 0}$  locally is represented by the multiplicity of a rational function, and it is invariant under the  $G_{\mathbb{Q}}$ -action. Thus the valency list of a Belyĭ covering  $\beta$  is a  $G_{\mathbb{Q}}$ -invariant.

The genus  $g$  of  $R$  is obtained by Euler characteristic calculation as

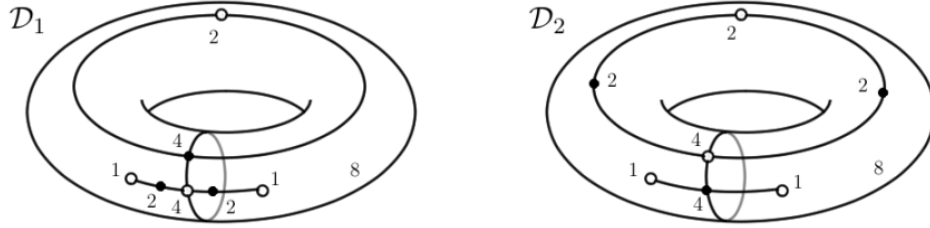
$$g = 1 + \frac{l_0 + l_1 + l_\infty - d}{2}, \quad (3.15)$$

where  $d$  is the degree of  $\beta$  and  $(k_1^0, \dots, k_{l_0}^0 \mid k_1^1, \dots, k_{l_1}^1 \mid k_1^\infty, \dots, k_{l_\infty}^\infty)$  is the valency list of  $\beta$ . For a dessin  $D = (x, y)$ ,  $x, y \in \mathfrak{S}_d$ , its *automorphism group*  $\text{Aut}(D)$  is defined by the centralizer  $\text{Cent}_{\mathfrak{S}_d}\langle x, y \rangle$ . Every  $\sigma \in \text{Aut}(D)$  corresponds to an orientation-preserving self-homeomorphism respecting the graph embedding and an automorphism of the Belyĭ surface compatible with Belyĭ covering. In particular, the isomorphism class of the automorphism group of a dessin is a  $G_{\mathbb{Q}}$ -invariant. The realization of a group in terms of automorphism group of dessin is studied by Jones [34] and Hidalgo [29].

**Example 6.** Fig. 3.B shows the two dessins  $\mathcal{D}_i$  ( $i = 1, 2$ ) defined by the Belyĭ pairs

$$y^2 = x(x-1)\left(x - (-1)^i\sqrt{2}\right), \quad \beta(x, y) = \frac{4}{x^2} \left(1 - \frac{1}{x^2}\right), \quad i = 1, 2. \quad (3.16)$$

They are conjugated by the element  $(\sqrt{2} \mapsto -\sqrt{2}) \in G_{\mathbb{Q}}$ , and have the same valency list  $(1^2, 2, 4 \mid 2^2, 4 \mid 8)$ .



**Fig. 3.B** Example of Galois conjugate dessins.

**Example 7.** Let  $d \in \mathbb{Z}_{>0}$ . The cycle graph with  $d$  edges defines a dessin  $\mathcal{C}_d = (x, y)$  where

$$x = (1\ 2)(3\ 4) \cdots (2d-1\ 2d), \quad y = (2\ 3)(4\ 5) \cdots (2d\ 1) \in \mathfrak{S}_d. \quad (3.17)$$

Its automorphism group is generated by a cyclic permutation  $(1\ 3 \cdots 2d-1)(2\ 4 \cdots 2d)$  and an involutive permutation  $(1\ 2d)(2\ 2d-1)(3\ 2d-2) \cdots (d\ d+1)$ . So the automorphism group is  $\text{Aut}(\mathcal{C}_d) \cong \{\pm 1\} \rtimes C_d$  where  $C_d$  is the cyclic group of order  $d$ .

We mention to the following formulation of the  $G_{\mathbb{Q}}$ -action on dessins [31, Appendix]. Let  $\mathcal{N}$  be the set of finite index normal subgroups of the free group  $F_2$ . The *profinite free group*  $\hat{F}_2$  is defined by

$$\hat{F}_2 = \{(g_N N)_{N \in \mathcal{N}} \mid g_N \in F_2, \ g_N N = g_{N'} N \text{ for any } N, N' \in \mathcal{N} \text{ with } N > N'\}. \quad (3.18)$$

Fix a tangential base point  $\vec{u} = \vec{0}1$  on  $\hat{\mathbb{C}}$ . For each Belyĭ pair  $(R, \beta)$ , fix a point  $v \in \beta^{-1}(0)$  of order  $m$  and choose a local coordinate  $z$  around  $v$  such that  $\beta(z) = z^m$ . Consider the map

$$\varphi_v(f) = \sum_{n=-k}^{\infty} a_n z^{\frac{n}{m}}, \quad f \in \bar{\mathbb{Q}}(R), \quad (3.19)$$

where  $\sum_{n=-k}^{\infty} a_n z^n$  is the Laurent series expansion of  $f$  around 0. Around the point  $v$ ,  $\varphi_v(f)$  has precisely  $m$  branches

$$\sum_{n=k}^{\infty} a_n \zeta_m^{kn} z^{\frac{n}{m}}, \quad k = 1, \dots, m, \quad (3.20)$$

where  $\zeta_m$  is a  $m$ -th root of unity. The mapping  $\varphi = \varphi_v$  defines an embedding of  $\bar{\mathbb{Q}}(R)$  into the space  $P_{\vec{u}}$  of *convergent Puiseux series* of the form (3.20) with coefficients in  $\bar{\mathbb{Q}}$  based on  $\vec{u}$ . The monodromy  $x$  around  $0 \in \hat{\mathbb{C}}$  acts on the image by the rotational permutation of these branches. The absolute Galois group  $G_{\mathbb{Q}}$  acts on  $P_{\vec{u}}$  as in the following diagram: for  $\sigma \in G_{\mathbb{Q}}$ ,

$$\begin{array}{ccccc}
 \bar{\mathbb{Q}}(R) & \xhookrightarrow{\varphi} & P_{\vec{u}} & & P_{\vec{u}} & \xleftarrow{x(\varphi)} & \bar{\mathbb{Q}}(R) \\
 \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi \\
 f \dashv & \longrightarrow & \sum_{n=k}^{\infty} a_n z^{\frac{n}{m}} & \xrightarrow{x} & \sum_{n=k}^{\infty} a_n \zeta_m^n z^{\frac{n}{m}} & \xleftarrow{\quad} & f \dashv \\
 & & \downarrow \sigma & & \downarrow \sigma & & \\
 f^{\sigma} \dashv & \longrightarrow & \sum_{n=k}^{\infty} \sigma(a_n) z^{\frac{n}{m}} & \xrightarrow{\sigma \cdot x} & \sum_{n=k}^{\infty} \sigma(a_n) \sigma(\zeta_m)^n z^{\frac{n}{m}} & \xleftarrow{\quad} & f^{\sigma} \dashv \\
 \downarrow \cap & & \downarrow \cap & & \downarrow \cap & & \downarrow \cap \\
 \bar{\mathbb{Q}}(R^{\sigma}) & \xhookrightarrow{\varphi^{\sigma}} & P_{\vec{u}}^{\sigma} & & P_{\vec{u}}^{\sigma} & \xleftarrow{\sigma \cdot x(\varphi^{\sigma})} & \bar{\mathbb{Q}}(R^{\sigma})
 \end{array}$$

In particular, the  $\sigma$ -image of the monodromy  $x$  is described by the formula

$$\sigma \cdot x = x^{\lambda_{\sigma}}, \quad (3.21)$$

where  $\zeta_m^{\lambda_{\sigma}} = \sigma(\zeta_m)$ . For the monodromy  $y$  around  $1 \in \hat{\mathbb{C}}$ , we can make the same argument except that there is a conjugate path  $t$  connecting 0 to 1. The path  $t$  can be interpreted as acting on embeddings  $\bar{\mathbb{Q}}(R) \hookrightarrow \mathcal{P}_{\vec{u}}$ . By working with the "fundamental groupoid", we can calculate the action of  $\sigma$  on  $t$ , and  $f_{\sigma} \in t^{-1}(\sigma \cdot t)$  belongs to  $F_2$ . The  $\sigma$ -image of the monodromy  $y$  is described by the formula

$$\begin{aligned}
 \sigma \cdot y &= \sigma \cdot (t^{-1} x t) \\
 &= \left( (\sigma \cdot t^{-1}) t \right) \left( t^{-1} (\sigma \cdot x) t \right) \left( t^{-1} (\sigma \cdot t) \right) \\
 &= f_{\sigma}^{-1} y^{\lambda_{\sigma}} f_{\sigma}.
 \end{aligned} \quad (3.22)$$

The *Grothendieck-Teichmüller group*  $\widehat{GT}$  introduced by Drinfel'd [8] is a group acting on  $\hat{F}_2$  in the same way to  $G_{\mathbb{Q}}$  (formulae (3.21) and (3.22)). The group  $\widehat{GT}$  is defined in a purely combinatorial way, and is known as the automorphism group of the "tower" of profinite mapping class groups [24, 53].

**Theorem 3.2.6** ( $\widehat{GT}$ -relation, Drinfel'd [8] and Ihara [31]). *The absolute Galois group  $G_{\mathbb{Q}}$  is embedded in  $\text{Aut}(\hat{F}_2)$  as a subgroup of the Grothendieck-Teichmüller group  $\widehat{GT}$ .*



# Chapter 4

## Flat surfaces and origamis

### 4.1 Flat surface

Let  $R$  be a Riemann surface of finite analytic type  $(g, n)$  with  $2g - 2 + n > 0$ .

**Definition 4.1.1.** A *holomorphic quadratic differential* on  $R$  is a family  $\phi = \{\phi_\alpha\}$  such that:

- (1) for each  $\alpha = (U, \varphi) \in \mathcal{A}$ ,  $\phi_\alpha : \varphi(U) \rightarrow \mathbb{C}$  is a nonconstant holomorphic mapping,
- (2) for each  $\alpha_i = (U_i, \varphi_i) \in \mathcal{A}_R$  ( $i = 1, 2$ ) with  $U_1 \cap U_2 \neq \emptyset$ ,

$$\phi_{\alpha_1}(z) = \phi_{\alpha_2} \circ \varphi_1^{-1}(z) \cdot \left( \frac{d\varphi_2 \circ \varphi_1^{-1}(z)}{dz} \right)^2, \text{ for any } z \in \varphi_1(U_1 \cap U_2). \quad (4.1)$$

A *holomorphic abelian differential* on  $R$  is defined similarly but (2) replaced by

- (2') for each  $\alpha_i = (U_i, \varphi_i) \in \mathcal{A}_R$  ( $i = 1, 2$ ) with  $U_1 \cap U_2 \neq \emptyset$ ,

$$\phi_{\alpha_1}(z) = \phi_{\alpha_2} \circ \varphi_1^{-1}(z) \cdot \left( \frac{d\varphi_2 \circ \varphi_1^{-1}(z)}{dz} \right), \text{ for any } z \in \varphi_1(U_1 \cap U_2). \quad (4.2)$$

A pair  $(R, \phi)$  of Riemann surface  $R$  and a holomorphic quadratic differential  $\phi$  on  $R$  is called a *flat surface*. We say that the set  $\text{Sing}(R, \phi)$  of the marked points of  $R$  and the zeros of  $\phi$  is the set of the *singularities* of  $(R, \phi)$ . Define the *order* of  $\phi$  by

$$\text{ord}_p(\phi) := \begin{cases} \text{mult}_p(\phi) & \text{if } p \in \text{Sing}(R, \phi) \\ 0 & \text{otherwise} \end{cases}, \text{ for each } p \in R. \quad (4.3)$$

Let  $p_0 \in R^* = R \setminus \text{Sing}(R, \phi)$  and  $\alpha = (U, z) \in \mathcal{A}_R$  be a chart around  $p_0$ . Then  $\phi$  defines a natural coordinate of  $\phi$  ( $\phi$ -coordinate) defined by

$$\zeta_\phi(p) = \int_{p_0}^p \sqrt{\phi_\alpha(z)} dz, \quad p \in U, \quad (U, z) \in \mathcal{A}_R, \quad (4.4)$$

for which  $\phi = (d\zeta_\phi)^2$  holds. The  $\phi$ -coordinates give an atlas  $\mathcal{A}_\phi$  on  $R^*$  any of whose transition map is a *half-translation*  $\zeta \mapsto \pm\zeta + c$  ( $c \in \mathbb{C}$ ). Such a structure, an atlas any of whose coordinate transformation is a half-translation is called a *flat structure*. The atlas  $\mathcal{A}_\phi$  extends to each singularity  $p_0 \in \text{Sing}(R, \phi)$  of multiplicity  $m$ , with local representation

$$\zeta_\phi(p) = \int_{p_0}^p \sqrt{z^m} dz = z(p)^{\frac{m}{2}+1}, \quad p \in U \setminus \{p_0\}, \quad (4.5)$$

for a suitable chart  $(U, z) \in \mathcal{A}_R$  around  $p_0$ .

A flat structure on  $R$  defines a holomorphic quadratic differential  $\phi = (dz)^2$  conversely. Note that  $\mathcal{A}_\phi$  is biholomorphically equivalent to  $\mathcal{A}_R$  as a complex structure on  $R$ . In this way, we identify  $\phi$  with the flat surface  $(R, \phi)$ .

**Definition 4.1.2.**

- (1) We say that two flat surfaces  $(R, \phi)$  and  $(S, \psi)$  are *isomorphic* if there exists a homeomorphism  $f : (R, \phi) \rightarrow (S, \psi)$  that is locally a half-translation.
- (2) We say that a flat surface  $(R, \phi)$  is *abelian* if  $\phi$  becomes the square of an abelian differential on  $R$  and otherwise *non-abelian* (or *primitive*).
- (3) Let  $g \geq 0$ ,  $m_1, \dots, m_n$  be integers and  $R$  be a Riemann surface. Define as follows.

$$\mathcal{Q}(R) := \{\phi : \text{holomorphic quadratic differential on } R\},$$

$$\mathcal{Q}_g := \{\phi \in \mathcal{Q}(S) \mid S : \text{Riemann surface of genus } g, \int_R |\phi| < \infty\},$$

$$\mathcal{Q}_g(m_1, \dots, m_n) := \{\phi \in \mathcal{Q}_g \mid \phi \text{ has precisely } n \text{ singularities of orders } m_1, \dots, m_n\}.$$

Let  $\mathcal{Q}^a$  ( $\mathcal{Q}^p$ , respectively) be the symbols that assign a set of abelian (non-abelian, respectively) differentials in place of  $\mathcal{Q}$ . (e.g.  $\mathcal{Q}^a(R) = \{\phi \in \mathcal{Q}(R) : \text{abelian}\}$ .)

**Definition 4.1.3.** Let  $(R, \phi), (S, \psi)$  be flat surfaces in  $\mathcal{Q}_g$ .

- (1) For  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ , we denote by  $[A] = \left[ \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right]$  the quotient class of  $A$  in  $PSL(2, \mathbb{R})$ . We define an affine map on the plane as follows:

$$T_A(x + iy) = (ax + cy) + i(bx + dy), \quad x, y \in \mathbb{R}. \quad (4.6)$$

- (2) A homeomorphism  $f : (R, \phi) \rightarrow (S, \psi)$  is called *locally affine* or an *affine deformation* if everywhere  $f$  is locally represented by  $z \mapsto T_A(z) + c$ , for some  $A \in SL(2, \mathbb{R})$  and  $c \in \mathbb{R}^2$  with respect to the natural coordinates of  $\phi$  and  $\psi$ , around everywhere on  $R^*$ .
- (3) For a locally affine homeomorphism  $f : (R, \phi) \rightarrow (S, \psi)$ , the local derivative  $A$  is constant up to a factor  $\mathbb{R}_{\neq 0}$ , independent of the natural coordinates of  $\phi$  and  $\psi$ . We call  $D(f) := [A] \in PSL(2, \mathbb{R})$  the *derivative* of  $f$ .
- (4) The group  $\text{Aff}^+(R, \phi) := \{f : (R, \phi) \rightarrow (R, \phi) : \text{locally affine}\}$  is called the *affine group* of  $(R, \phi)$ . The group  $\Gamma(R, \phi) = \{D(f) \in PSL(2, \mathbb{R}) \mid f \in \text{Aff}^+(R, \phi)\}$  is called the *Veech group* of  $(R, \phi)$ .

For every affine deformation  $f : (R, \phi) \rightarrow (S, \psi)$ ,

**Theorem 4.1.4** (Teichmüller's existence & uniqueness theorem, [47, Theorem 2.6.4]). *Let  $2g - 2 + n > 0$  and  $[R_1, R_2, f] \in T_{g,n}$ . Then, there exist  $0 \leq k < 1$ ,  $\phi, \psi \in \mathcal{Q}_g$ , and a quasiconformal mapping  $f_T : R_1 \rightarrow R_2$  such that the following holds.*

(1)  $\phi \in \mathcal{Q}(R_1)$  and  $\psi = f_T^* \phi \in \mathcal{Q}(R_2)$ .

(2)  $f_T$  is homotopic to  $f$ .

(3)  $f_T$  is an affine deformation locally represented as  $\zeta_\psi \circ f_T \circ \zeta_\phi^{-1}(z) = \frac{z + k\bar{z}}{1 - k}$ .

Moreover,  $f_T$  is a unique mapping that attains the minimal dilatation  $K(f_T) = \frac{1+k}{1-k}$  in the homotopy class of  $f$ .

By Lemma 2.3.2, the Beltrami differential of the mapping  $f_T$  in Theorem 4.1.4 is given by

$$\mu_{f_T} = \mu_{\zeta_\psi^{-1} \circ (z \mapsto \frac{z+k\bar{z}}{1-k}) \circ \zeta_\phi} = \mu_{(z \mapsto \frac{z+k\bar{z}}{1-k})} \circ \zeta_\phi \cdot \frac{(\overline{\zeta_\phi})_z}{(\zeta_\phi)_z} = k \frac{\bar{\phi}}{|\phi|}. \quad (4.7)$$

Let  $(R, \phi) \in \mathcal{Q}_g$  be a flat surface and  $t \in \mathbb{H}$ . Consider the flat surface  $(R, \phi_t)$  whose natural coordinates are deformed by the formula

$$\zeta_{\phi_t} = \text{Re}(\zeta_\phi) + t \cdot \text{Im}(\zeta_\phi) = \frac{1 - it}{2} \zeta_\phi + \frac{1 + it}{2} \bar{\zeta}_\phi. \quad (4.8)$$

Define  $\hat{t}_\phi : \mathbb{H} \rightarrow T(R)$  by  $\hat{t}_\phi(t) := [R_t = (R_{\text{top}}, \mathcal{A}_{\phi_t}), f_t = id_{R_{\text{top}}}]$ , where  $R_{\text{top}}$  is the underlying topological surface of  $R$ . The mapping  $f_t : R \rightarrow R_t$  is an affine deformation locally represented by the formula (4.8) with Beltrami coefficient  $h(t) \frac{\bar{\phi}_1}{|\phi_1|}$  where  $h(t) = \frac{t - i}{t + i} \in \mathbb{D}$ . It follows from equation (2.15) that  $\hat{t}_\phi$  is an isometric embedding with respect to

the hyperbolic metric and the Teichmüller metric. Since the mapping  $\hat{\iota}_\phi$  is the composition of the holomorphic mapping  $\mathbb{H} \rightarrow B_1(\mathbb{R}, \Gamma_R)$  and the projection  $\mathcal{P} : B_1(\mathbb{R}, \Gamma_R) \rightarrow T(R)$ , it is holomorphic with respect to the complex structure of  $T(R)$ .

The embedded image  $D_\phi := \hat{\iota}_\phi(\mathbb{H}) \subset T(R)$  is called the *Teichmüller disk* induced from  $\phi$ .

**Lemma 4.1.5** ([9, Lemma10.1]). *Let  $g : R_1 \rightarrow R_2$  be a quasiconformal mapping between Riemann surfaces of analytically finite type and  $\phi \in \mathcal{Q}(R_1)$ . If the image of  $D_\phi$  under  $\rho_g : T(R_1) \rightarrow T(R_2)$  contains the base point of  $T(R_2)$ , then there exists  $s \in \mathbb{H}$ ,  $\psi \in \mathcal{Q}(R_2)$  and an affine deformation  $f : R_1 \rightarrow R_2$  such that the following holds.*

- (1)  $\rho_g = \rho_f$ ,
- (2)  $\psi = g^* \phi$ ,
- (3)  $\mu_g = -h(s) \frac{\bar{\phi}}{|\phi|}$ ,
- (4)  $\rho_g(\hat{\iota}_\phi(t)) = \hat{\iota}_\psi \left( h^{-1} \left( \frac{h(t) - h(s)}{1 - h(t)\bar{h}(s)} \right) \right)$  for any  $t \in \mathbb{H}$ .

In particular,  $g(D_\phi) = D_\psi$ .

As a consequence of Lemma 4.1.5, the maximal subgroup of the Teichmüller-modular group acting on  $D_\phi$  is the affine group  $\text{Aff}^+(R, \phi)$ . The action of  $f \in \text{Aff}^+(R, \phi)$  with derivative  $D(f) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is described by

$$\rho_f(\hat{\iota}_\phi(t)) = \hat{\iota}_\phi \left( \frac{-at + b}{ct - d} \right), \quad t \in \mathbb{H}, \quad (4.9)$$

which is the Möbius transformation of  $\overline{D(f)} = J^{-1}D(f)J$ ,  $J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ . Also by Lemma 4.1.5, the embedding  $\hat{\iota}_\phi : \mathbb{D} \hookrightarrow T(R)$  is unique up to the natural  $\mathbb{C} \setminus \{0\}$ -action for each  $\phi$ .

The mapping  $\iota_\phi : \mathbb{H} \xrightarrow{\hat{\iota}_\phi} T(R) \xrightarrow{\text{proj.}} M(R)$  results in an orbifold  $C_\phi \cong \mathbb{H}/\overline{\Gamma(R, \phi)}$  embedded in the moduli space  $M(R)$ . In the case that  $\Gamma(R, \phi)$  has a finite covolume,  $C_\phi \subset M(R)$  is an algebraic curve called the *Teichmüller curve* induced from  $(R, \phi)$ . The above is summarized in the following diagram.

$$\begin{array}{ccccc} \mathbb{H} & \xrightarrow[\cong]{\hat{\iota}_\phi} & D_\phi & \xrightarrow{\text{incl.}} & T(R) \\ \downarrow \text{proj.} & & \downarrow \text{proj.} & & \downarrow \text{proj.} \\ \mathbb{H}/\overline{\Gamma(R, \phi)} & \xrightarrow[\cong]{\iota_\phi} & C_\phi & \xrightarrow{\text{incl.}} & M(R) \end{array}$$



## 4.2 Stratum of holomorphic quadratic differentials

Let  $R$  be a Riemann surface of analytically finite type  $(g, n)$  with  $2g - 2 + n > 0$ . By Definition 4.1.2, we can say the following about the orders of singularities of a flat surface:

$$\begin{cases} m_j > 0 \text{ and even} & \text{if } (R, \phi) \in \mathcal{Q}_g^a(m_1, \dots, m_k), \\ m_j \geq -1, \neq 0 & \text{if } (R, \phi) \in \mathcal{Q}_g^p(m_1, \dots, m_k). \end{cases} \quad (4.10)$$

Note that any singularity of order  $-1$  is regarded as a marked point. For a *geometric triangulation*, a triangulation of  $R$  such that the vertices are singularities and the edges are saddle connections, by Euler characteristic calculation it follows that

$$\sum_{j=1}^n m_j = 4g - 4. \quad (4.11)$$

The set  $\mathcal{Q}_g(m_1, \dots, m_k)$  splits into the disjoint union

$$\begin{aligned} \mathcal{Q}_g &= \mathcal{Q}_g^a \sqcup \mathcal{Q}_g^p \\ &= \left( \bigsqcup \mathcal{Q}_g^a(m_1, \dots, m_k) \right) \sqcup \left( \bigsqcup \mathcal{Q}_g^p(m_1, \dots, m_k) \right), \end{aligned} \quad (4.12)$$

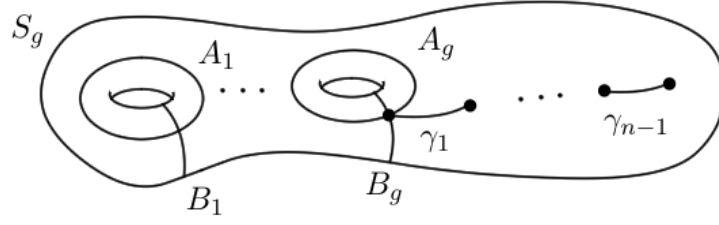
where  $(m_1, \dots, m_k)$  runs possible tuples of integers satisfying (4.10) and (4.11). The equality (4.12) is called a *stratification* into strata  $\mathcal{Q}_g^\bullet(m_1, \dots, m_k)$ . Two different strata contain a common flat surface only if they differ by nonsingular marked points.

It is observed [30, 57] that every stratum  $\mathcal{Q}_g^a(m_1, \dots, m_n)$  is complex analytic and of dimension  $2g - 1 + n$ . Its parametrization is given in terms of relative homology [23, Section 2.1]. The homology group of the genus  $g$  surface  $S_g (= R_{\text{top}})$  relative to  $n$  points  $p_1, \dots, p_n \in S_g$  is defined by

$$H_m(S_g, \{p_1, \dots, p_n\}, \mathbb{Z}) = \text{im}(\partial_{m+1}) / \ker(\partial_m), \quad m = 0, 1, \dots, \quad (4.13)$$

where  $C_m(S_g, n) = C_m(S_g) / C_m(\{p_1, \dots, p_n\})$  is the quotient group of  $m$ -simplices and  $\partial_m : C_m(S_g, n) \rightarrow C_{m-1}(S_g, n)$  is the boundary homomorphism. The trivial element in  $H_m(S_g, \{p_1, \dots, p_n\}, \mathbb{Z})$  is represented by the form  $\partial\gamma + \gamma'$ , for some  $\gamma \in C_{m+1}(S_g)$  and  $\gamma' \in C_m(\{p_1, \dots, p_n\})$ .

The first relative homology group  $H_1(S_g, \{p_1, \dots, p_n\}, \mathbb{Z})$  is a free abelian group of dimension  $2g - 1 + n$ , whose basis can be taken as a standard basis  $\{A_1, B_1, \dots, A_g, B_g\}$  together with a choice of paths  $\gamma_1, \dots, \gamma_{n-1}$  joining  $n$  marked points as shown in Fig. 4.A. Local coor-



**Fig. 4.A** A basis of the first relative homology group  $H_1(S_g, \{p_1, \dots, p_n\}, \mathbb{Z})$ . The black dots indicate the points  $p_1, \dots, p_n$ , and the path  $\gamma_i$  joins  $p_i$  and  $p_{i+1}$ .

ordinates of a stratum of abelian differentials are given by the *period map*  $\Psi$  of  $\mathcal{Q}_g^a(m_1, \dots, m_k)$  to the first relative cohomology group  $H^1(S_g, \{p_1, \dots, p_n\}, \mathbb{C}) \cong \mathbb{C}^{2g-1+n}$  defined by

$$\Psi(R, \omega^2) = \left( [\gamma_j] \mapsto \int_{\gamma_j} \omega \right), \quad j = 1, \dots, 2g-1+n, \quad (R, \omega^2) \in \mathcal{Q}_g^a(m_1, \dots, m_k), \quad (4.14)$$

where  $\gamma_1, \dots, \gamma_{2g-1+n}$  is a fixed basis of  $H_1(S_g, \{p_1, \dots, p_n\}, \mathbb{Z})$ .

**Proposition 4.2.1** ([42, Construction1]). *For a flat surface  $(R, \phi) \in \mathcal{Q}_g^p$ , the analytic continuation of the two branches of  $\sqrt{\phi}$  gives a branched covering  $\pi_\phi : (\hat{R}, \psi) \rightarrow (R, \phi)$  such that:*

- (1)  $\psi = \pi_\phi^* \phi$  is abelian,
- (2) the branched points of  $\pi_\phi$  are precisely the singularities of  $\phi$  of odd orders, and
- (3)  $\pi_\phi$  is the minimal covering in the sense of (1).

We say that  $(\hat{R}, \psi)$  is the *canonical double* and  $\pi_\phi$  is the *canonical double covering* of  $(R, \phi)$ . The canonical double admits an involution  $\tau : \hat{R} \rightarrow \hat{R}$  interchanging every two preimages of  $\pi_\phi$ .

For a singularity  $p$  of  $(R, \phi)$ , it follows that

$$\pi_\phi^{-1}(p) \text{ consists of } \begin{cases} \text{two points of order } \text{ord}_p(\phi) & \text{if } \text{ord}_p(\phi) \text{ is even,} \\ \text{one point of order } 2\text{ord}_p(\phi) + 2 & \text{if } \text{ord}_p(\phi) \text{ is odd.} \end{cases} \quad (4.15)$$

If there are  $l$  singularities of odd order on  $(R, \phi)$ , it follows from equation (4.11) that the genus  $\hat{g}$  of  $\hat{R}$  satisfies that

$$\begin{aligned} 4\hat{g} - 4 &= 2 \sum_{\text{even order}} \text{ord}_p(\phi) + \sum_{\text{odd order}} (2\text{ord}_p(\phi) + 2) \\ &= 2(4g - 4) + 2l. \end{aligned} \quad (4.16)$$

Thus  $(\hat{R}, \psi)$  has genus  $\hat{g} = 2g - 2 + \frac{l}{2}$  and  $\hat{n} = 2n - l$  singularities.

The mapping  $\tau$  induces an involutive linear map  $\tau^*$  on  $H^1(\hat{R}, \{\hat{p}_1, \dots, \hat{p}_{\hat{n}}\})$ . The image of a neighborhood  $U$  of  $(\hat{R}, \psi)$  in  $\mathcal{Q}_{\hat{g}}^a$  decomposes into two eigenspaces  $E_{\pm 1}$  with eigenvectors  $\pm 1$  for  $\tau^*$ . As any abelian differential has eigenvalue  $-1$ ,  $U$  is bijectively mapped into  $E_{-1} \cong \mathbb{C}^{2g-2+n}$ , and so locally is the non-abelian stratum  $\mathcal{Q}_g^p$ .

### 4.3 $\phi$ -metric

Let  $(R, \phi) \in \mathcal{Q}_g$  be a flat surface. The Euclidian metric lifts via  $\phi$ -coordinates to a flat metric on  $R$ , called the  $\phi$ -metric. A geodesic of  $\phi$ -metric is called a  $\phi$ -geodesic. Via the  $\phi$ -coordinates, a  $\phi$ -geodesic is locally a line segment on the plane whose *direction* is uniquely determined in  $\mathbb{R}/\pi\mathbb{Z}$ .

**Definition 4.3.1.** Let  $(R, \phi) \in \mathcal{Q}_g$  be a flat surface.

- (1) A  $\phi$ -geodesic joining two singularities is called a *saddle connection* on  $(R, \phi)$ .
- (2) The  $\phi$ -cylinder generated by a  $\phi$ -geodesic  $\gamma$  is the union of all  $\phi$ -geodesics parallel (with same direction) and free homotopic to  $\gamma$ . We define the direction of a  $\phi$ -geodesic by the one of its generator.
- (3)  $\theta \in \mathbb{R}/\pi\mathbb{Z}$  is called *Jenkins-Strebel direction* of  $(R, \phi)$  if almost every point in  $R$  lies on some closed  $\phi$ -geodesic in the direction  $\theta$ . Let  $J(R, \phi)$  denote the set of Jenkins-Strebel directions of  $(R, \phi)$ .

A  $\phi$ -cylinder  $C$  on  $(R, \phi)$  admits an isomorphism  $C \rightarrow \{0 < |\text{Im}(z)| < h\} / \langle z \mapsto z + w \rangle$  for some *height*  $h > 0$  and *width*  $w > 0$ . (See Section 5.2.) Note that any Jenkins-Strebel direction of flat surface in  $\mathcal{Q}_g$  is *finite*, namely there are at most finitely many  $\phi$ -cylinders of that direction in  $R$ .

We say that a system  $\gamma = (\gamma_1, \dots, \gamma_p)$  of Jordan curves on  $R$  is *admissible* if none of the curves is homotopically trivial and any two distinct  $\gamma_i, \gamma_j$  are neither crossing nor (freely) homotopic. For the existence of a holomorphic quadratic differential with one Jenkins-Strebel direction, the following result is known.

**Theorem 4.3.2** (Strebel, [55, Theorem 21.1]). *Let  $\gamma = (\gamma_1, \dots, \gamma_p)$  be an admissible curve system on  $R$ , which satisfies bounded moduli condition for  $\gamma$ . Then for any  $b = (b_1, \dots, b_p) \in \mathbb{R}_{>0}^p$ , there exists a holomorphic quadratic differential  $\phi$  on  $R$  such that  $0 \in J(R, \phi)$  and  $(R, \phi)$  is decomposed into cylinders  $(V_1, \dots, V_p)$ , where each  $V_j$  has homotopy type  $\gamma_j$  and height  $b_j$ .*

Let  $f \in \text{Aff}^+(R, \phi)$  be an affine mapping with derivative  $D(f) = [A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PSL}(2, \mathbb{R})$ . Then,  $f$  maps any line segment in the direction  $\theta \in \mathbb{R}/\pi\mathbb{Z}$  to a line segment in the direction  $A\theta := \arg(T_A(e^{i\theta}))$ . We may observe that  $f$  maps a  $\phi$ -cylinder of modulus  $M$  to a  $\phi$ -cylinder of modulus  $M/\sqrt{a^2 + c^2}$ . Since the list of moduli of  $\phi$ -cylinders of one direction are uniquely determined up to order, the following holds.

**Lemma 4.3.3.** *Let  $J(R, \phi) \neq \emptyset$  and  $(M_1^\theta, \dots, M_{n_\theta}^\theta) \in \mathbb{R}_{>0}^{n_\theta}$  be the list of moduli of the  $\phi$ -cylinders in the direction  $\theta \in J(R, \phi)$  sorted in ascending order. If  $[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PSL}(2, \mathbb{R})$  belongs to  $\Gamma(R, \phi)$ , for any  $\theta \in J(R, \phi)$ , it follows that*

- (1)  $A\theta \in J(R, \phi)$ ,
- (2)  $n_\theta = n_{A\theta} = n \in \mathbb{Z}_{>0}$ , and
- (3)  $M_j^{A\theta} = M_j^\theta / \sqrt{a^2 + c^2}$  for  $j = 1, \dots, n$ .

## 4.4 Origami

For an abelian flat surface  $(R, \phi^2) \in \mathcal{Q}_g^a$ , the holomorphic abelian differential  $\phi$  on  $R$  defines natural coordinates without taking square-roots. These coordinates form an atlas any of whose transition map is a translation  $\zeta \mapsto \zeta + c$  ( $c \in \mathbb{C}$ ), called a *translation structure*. In this case, the derivative of an affine deformation is defined by distinguishing a factor of  $\{\pm 1\}$ , and the Veech group is defined as a subgroup of  $SL(2, \mathbb{Z})$ . An isomorphism of translation surfaces is defined similarly.

In this section, we will present an introduction to origamis and some results on them related to the absolute Galois group  $G_{\mathbb{Q}}$ . An (abelian) origami is a special example of abelian flat surface that induces a Teichmüller curve defined over  $\bar{\mathbb{Q}}$ . In 2005, Möller [45] proved that the  $G_{\bar{\mathbb{Q}}}$ -action on origamis respects the embedding of the Teichmüller curve in the moduli space, leading to an application to the  $\widehat{GT}$ -relation.

**Definition 4.4.1.** An (abelian) *origami* of degree  $d$  is an abelian flat surface obtained from  $d$  copies of the Euclidian unit squares by gluing them in such a way that

- each left edge is glued to a right edge,
- each upper edge is glued to a lower edge, and
- the resulting closed surface is connected.

The above gluing rule is called an *origami-rule*. An *origami* refers to an abelian origami in this section.

Every origami has a natural cellular decomposition. If an origami  $\mathcal{O}$  has degree  $d$  and  $n$  singularities, it follows from Euler characteristic calculation that the genus  $g$  of  $\mathcal{O}$  is

$$g = 1 + \frac{d - n}{2}. \quad (4.17)$$

**Lemma 4.4.2.** *An origami of degree  $d$  is up to equivalence (in brackets) uniquely determined by each of the following.*

- (a) *A connected, oriented graph  $(\mathcal{V}, \mathcal{E})$  with  $|\mathcal{V}| = d$  such that each vertex has precisely one incoming edge and one outgoing edge labelled one with  $x$  and one with  $y$ , respectively [up to equivalence of the natural filling graph embedding].*
- (b) *A  $d$ -fold branched covering  $p : R \rightarrow E$  of a torus  $E$  branched at most over one point  $\infty \in E$  [up to covering equivalence over  $E \setminus \{\infty\}$ ].*
- (c) *A pair of two permutations  $x, y \in S_d$  generating a transitive permutation group [up to conjugation in  $S_d$ ].*
- (d) *A subgroup  $H$  of the free group  $F_2$  of index  $d$  [up to conjugation in  $F_2$ ].*

*Proof.* An origami naturally corresponds to a graph (a), by assigning the unit square cells to vertices and the adjacency of cells to edges referring to the directions on the plane. By defining permutations  $x, y \in S_d$  by the permutations of vertices along edges labelled with  $x, y$ . Transitivity follows from the connectedness of the resulting surface. As we have seen in Remark 2.1.7, the objects in (b-d) are in a one-to-one correspondence up to suitable equivalence where  $\pi_1(E^*, \cdot) \cong F_2$ . A covering (b) induces an abelian differential  $\phi = p^*dz$  on  $R$ , which makes  $(R, \phi)$  an origami.  $\square$

Let  $\mathcal{O} = (p : (R, \phi) \rightarrow (E, dz))$  be an origami and  $\psi = \pi_{R^*}^* \phi = \pi_{E^*}^* dz$  be the abelian differential on  $\mathbb{H}$  induced from the universal covering. We fix a continuation of  $\psi$ -coordinates on  $\mathbb{H}$  leading to a mapping  $\zeta : (\mathbb{H}, \psi) \rightarrow \mathbb{C} \setminus \mathbb{Z} + i\mathbb{Z}$ . Every lift of  $f \in \text{Aff}^+(R, \phi)$  on  $\mathbb{H}$  projects through  $\zeta$  to an affine mapping  $z \mapsto T_A(z) + c$  for some  $A \in SL(2, \mathbb{R})$ ,  $c \in \mathbb{C}$ . As it is continued to an affine mapping  $f^{\text{dev}}$  on  $\mathbb{C} \setminus \mathbb{Z} + i\mathbb{Z}$ , it follows that  $A \in SL(2, \mathbb{Z})$  and  $c \in \mathbb{Z} + i\mathbb{Z}$ . We say that  $f \in \text{Aff}^+(R, \phi)$  is *developed* to  $f^{\text{dev}} \in \text{Aff}^+(\mathbb{C} \setminus \mathbb{Z} + i\mathbb{Z})$ . Note that every affine mapping on  $(R, \phi)$  is developed in this way and  $\Gamma(\mathbb{H}, \psi) = \Gamma(E, dz) = SL(2, \mathbb{Z})$  holds.

**Lemma 4.4.3** (Schmithüsen [51, Lemma 2.8]). *Let  $\mathcal{O} = (R, \phi) = (p : R \rightarrow E)$  be an origami, and*

$$\text{Aut}^+(F_2) = \{\sigma : \text{orientation-preserving automorphism of } F_2 \cong \pi_1(E^*, \cdot)\}, \quad (4.18)$$

$$\text{Inn}(F_2) = \{z^* = (w \mapsto z^{-1}wz) \in \text{Aut}^+(F_2) \mid z \in F_2\} \cong F_2, \quad (4.19)$$

$$\text{Out}^+(F_2) = \text{Aut}^+(F_2)/\text{Inn}(F_2). \quad (4.20)$$

Then, we have the following exact (in horizontal direction) and commutative diagram.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \text{Deck}(\mathbb{H}/E^*) & \hookrightarrow & \text{Aff}^+(\mathbb{H}, \psi) & \xrightarrow{D} & \text{SL}_2(\mathbb{Z}) \longrightarrow 1 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 1 & \longrightarrow & \text{Inn}(F_2) & \hookrightarrow & \text{Aut}^+(F_2) & \longrightarrow & \text{Out}^+(F_2) \longrightarrow 1
 \end{array}$$

Furthermore, the subgroup of  $\text{Aut}^+(F_2)$  that corresponds to  $\text{Aff}^+(\mathcal{O}) < \text{Aff}^+(\mathbb{H}, \psi)$  in this diagram is the stabilizer  $\text{Stab}_{\text{Aut}^+(F_2)}[H]$  of the class of the fundamental group  $H < F_2$  of the origami  $\mathcal{O}$ .

*Proof.* Take an arbitrary  $f \in \text{Aff}^+(\mathbb{H}, \psi)$  with  $D(f) = I$ . The developed mapping  $f^{\text{dev}}$  should be a translation in  $\mathbb{Z} + i\mathbb{Z}$ , and thus a covering transformation of  $\pi_{E^*} : \mathbb{H} \rightarrow E^*$ . Converly, as every covering transformation preserves the induced translation  $\psi = \pi_{E^*}^* dz$  on  $\mathbb{H}$ , it is a translation on  $(\mathbb{H}, \psi)$ . So the group  $\text{Deck}(\mathbb{H}/E^*) \cong \pi_1(E^*, \cdot) \cong F_2$  is embedded in  $f \in \text{Aff}^+(\mathbb{H}, \psi)$  as the kernel of the derivative  $D$ . Next, for each  $f \in \text{Aff}^+(\mathbb{H}, \psi)$  we define

$$f^\star(g) = f^{-1} \circ g \circ f, \quad g \in \text{Deck}(\mathbb{H}/E^*). \quad (4.21)$$

The mapping  $f^\star(g)$  is an affine mapping with derivative  $I$ , and so  $\star : f \mapsto f^\star$  defines a homomorphism of  $\text{Aff}^+(\mathbb{H}, \psi)$  into  $\text{Aut}^+(F_2)$ . We observe that  $f^\star$  is isomorphism by showing the commutativity. The left half diagram commutes by definition. For the right half diagram, the group  $\text{Aut}^+(F_2)$  surjects onto  $SL(2, \mathbb{Z})$  by the homomorphism defined by

$$\beta : \text{Aut}^+(F_2) \rightarrow SL(2, \mathbb{Z}) : \gamma \mapsto \begin{pmatrix} \#_x \gamma(x) & \#_x \gamma(y) \\ \#_y \gamma(x) & \#_y \gamma(y) \end{pmatrix}, \quad (4.22)$$

where  $\#_\bullet w$ ,  $w \in F_2$  counts the (signed) number of  $x$  or  $y$  appearing in  $w$ . Indeed, the two generators  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  of  $SL(2, \mathbb{Z})$  have pullback  $\gamma_T, \gamma_S \in \text{Aut}^+(F_2)$  defined by

$$\gamma_T(x, y) = (x, xy), \quad \gamma_S(x, y) = (y, x^{-1}). \quad (4.23)$$

One can observe that for an arbitrary  $g \in \text{Deck}(\mathbb{H}/E^*)$ , the images of the two vector  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  under the developed affine mapping  $(f^\star(g))^{\text{dev}}$  is precisely given by the matrix  $\beta(f^\star)$ . Thus  $\beta$  commutes with  $\star$ , the kernel of  $\beta$  is  $\text{Inn}(F_2)$ , and hence the claim follows.

Finally, we show about the discription of  $\text{Aff}^+(\mathcal{O}) < \text{Aff}^+(\mathbb{H}, \psi)$  in  $\text{Aut}^+(F_2)$ . An affine mapping  $f \in \text{Aff}^+(\mathbb{H}, \psi)$  projects via the universal covering  $\pi_{R^*} : \mathbb{H} \rightarrow R^*$  if and only if there exists an automorphism  $\sigma$  of  $\text{Deck}(\mathbb{H}/R^*) = \pi_1(R^*, \cdot)$  such that

$$f \circ \gamma = \sigma(\gamma) \circ f, \quad (4.24)$$

for any  $\gamma \in \text{Deck}(\mathbb{H}/R^*)$ . On the other hand, the automorphism  $\sigma = f^* \in \text{Aut}^+(F_2)$  satisfies (4.24) for any  $\gamma \in \text{Deck}(\mathbb{H}/E^*)$ . It follows from Lemma 2.1.9 that the automorphism  $\sigma$  have to be of the form  $f^*$ . Thus the claim follows.  $\square$

By the fact that the Veech group of an origami is a stabilizer of a finite-index subgroup of  $F_2$ , the following finiteness follows. (See Corollary 5.3.1 for the proof.)

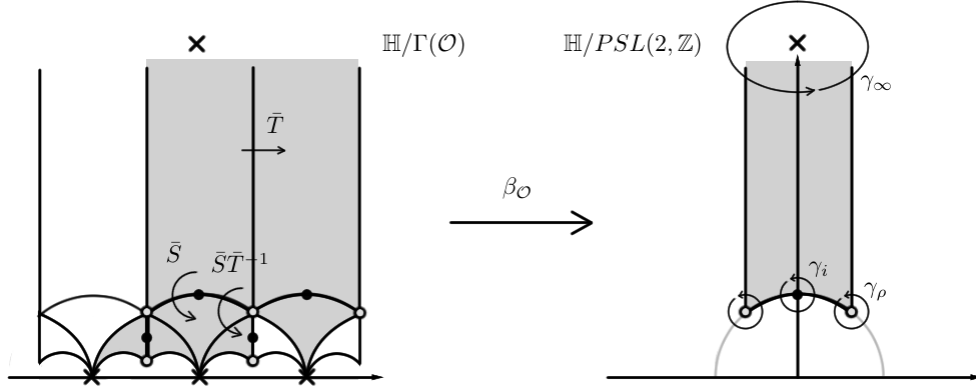
**Lemma 4.4.4.** *The Veech group of an origami is a subgroup of  $SL(2, \mathbb{Z})$  of finite index.*

In her paper [51], Schmithüsen presented the following algorithm for finding the Veech group of an origami based on Lemma 4.4.3.

**Algorithm 4.4.5** (Schmithüsen [51, Algorithm 1-4]). Let  $\mathcal{O} = (R, \phi) = (x, y)$  be an origami of degree  $d$  and fix a generating system  $C \subset F_2$  of the fundamental group  $\pi_1(R^*) = H < F_2$ . We obtain a generating system **Gen** and a representative set **Rep** of the Veech group  $\Gamma(\mathcal{O}) < SL(2, \mathbb{Z})$  in the following steps:

- (1) Let  $m = n = N = 0$  and  $R_0 = I$ .
- (2) For a word  $W = W(T, S)$  in  $T$  and  $S$ , let  $\gamma_W = W(\gamma_T, \gamma_S) \in \text{Aut}^+(F_2)$  be the composition of  $\gamma_T, \gamma_S$  according to  $W$ . For each  $n' \leq N$ , check whether  $\gamma_{R_n T R_{n'}^{-1}}(c) \in F_2$  defines a closed path in  $\mathcal{O}$  starting at some  $i \in I_d$  (i.e. it defines a permutation  $z$  with  $z(i) = i$ ) for all  $c \in C$ . If there exists such  $n'$ , let  $G_{m+1} = R_n T R_{n'}^{-1}$  and increment  $m$  by one. Otherwise let  $R_{N+1} = R_n T$  and increment  $N$  by one.
- (3) Do the same as (2) for  $S$  instead of  $T$ .
- (4) If  $n < N$  then go back to (2) for the next  $n$ . Otherwise finish the loop and let **Gen** =  $\{G_1, \dots, G_m\}$  and **Rep** =  $\{R_1, \dots, R_n\}$ .

**Remark 4.4.6.** The surface  $\mathbb{H}/PSL(2, \mathbb{Z})$  is an orbifold of genus 0 and with three singularities of order 2 at  $[i]$ , 3 at  $[\rho] = [e^{\pi i/3}]$ , and  $\infty$  at the infinity. Lemma 4.4.4 implies that an arbitrary origami  $\mathcal{O} = (p : R \rightarrow E)$  induces a Teichmüller curve  $C(\mathcal{O})$  as a Belyĭ surface, and thus defined over  $\bar{\mathbb{Q}}$ . If  $[\Gamma(\mathcal{O})]$  denotes the projected image of  $\Gamma(\mathcal{O})$  in  $PSL(2, \mathbb{Z})$ , the Belyĭ covering is given by the projection  $\mathbb{H}/[\Gamma(\mathcal{O})] \rightarrow \mathbb{H}/PSL(2, \mathbb{Z})$ . Its monodromy is given by the natural action of  $[\Gamma(\mathcal{O})]$  on the coset representatives in  $SL(2, \mathbb{Z})$  (see Remark 2.1.7 and Fig. 4.B). We have the same formula as (4.17) for the genus  $g$  of Teichmüller curve  $C(\mathcal{O})$ , where  $d$  is the index  $[PSL(2, \mathbb{Z}) : [\Gamma(\mathcal{O})]]$  and  $n$  is the number of the singularities. On the other hands, an origami  $\mathcal{O}$  itself is a Belyĭ surface as a covering of the Belyĭ pair  $(C_0 = \{y = 4x^3 - x\}, \beta_0(x, y) = 4x^2)$ .



**Fig. 4.B** The Belyi covering  $\beta_{\mathcal{O}}$  of the Teichmüller curve induced from an origami  $\mathcal{O}$ . The monodromy around  $[i]$  ( $[\rho]$ ,  $[\infty]$ , respectively) is given by the action of matrix  $[S]$  ( $[ST^{-1}]$ ,  $[T]$ , respectively) on the coset representatives of  $PSL(2, \mathbb{Z})/\Gamma(\mathcal{O})$ .

**Proposition 4.4.7** (Möller, [45]). *Let  $\sigma \in G_{\mathbb{Q}}$  and  $\mathcal{O}$  be an origami of genus  $g$ . Then,  $\mathcal{O}^{\sigma}$  is again an origami and the Belyi surfaces  $C(\mathcal{O})$  and  $C(\mathcal{O}^{\sigma})$  are conjugated by  $\sigma$  as embedded curves in  $M_g$ .*

We mention to the  $G_{\mathbb{Q}}$ -conjugacy of embedded curves in  $M_{g,n}$  as follows. The moduli space  $M_{g,n}$  has a structure of *stack* for which  $M_{g,n}$  satisfies the universal property among families of schemes of type  $(g, n)$ . It parametrizes all schemes of type  $(g, n)$  with structure morphism to each assigned scheme as a contravariant functor  $(\text{Schemes}/\mathbb{Z}) \rightarrow (\text{Sets})$ . Restricted to the subcategory of schemes defined over  $\bar{\mathbb{Q}}$ , say  $M_{g,n}^{\mathbb{Q}} \otimes \bar{\mathbb{Q}}$  where  $M_{g,n}^{\mathbb{Q}} = M_{g,n}(\text{Spec}(\mathbb{Q}))$ , its *algebraic* fundamental group  $\pi_1^{\text{alg}}(M_{g,n}^{\mathbb{Q}} \otimes \bar{\mathbb{Q}}, \cdot)$  is known to be the profinite mapping class group  $\widehat{\text{Mod}}_{g,n}$ . It gives an exact sequence

$$1 \rightarrow \pi_1^{\text{alg}}(M_{g,n}^{\mathbb{Q}} \otimes \bar{\mathbb{Q}}, \cdot) \rightarrow \pi_1^{\text{alg}}(M_{g,n}^{\mathbb{Q}}, \cdot) \rightarrow G_{\mathbb{Q}} \rightarrow 1 \quad (4.25)$$

and enables us to relate the absolute Galois group  $G_{\mathbb{Q}}$  to the moduli spaces in terms of profinite mapping class groups. See [44, Section 4] and [15] for instance.

In his paper [45], Möller observed the  $G_{\mathbb{Q}}$ -action on the Teichmüller curve induced from a degree 4 origami called the two-steps origami. The (orbifold) fundamental group of the Teichmüller curve is embedded in the profinite mapping class group  $\widehat{\text{Mod}}_{2,0}$ . The actions of  $\widehat{GT}$  and  $G_{\mathbb{Q}}$  on  $\widehat{\text{Mod}}_{2,0}$  were compared, and the compatibility with the Teichmüller curve induces a relation of the elements in  $G_{\mathbb{Q}}$  embedded in  $\widehat{GT}$ .



# Chapter 5

## Main results

This chapter is based on [40, 41].

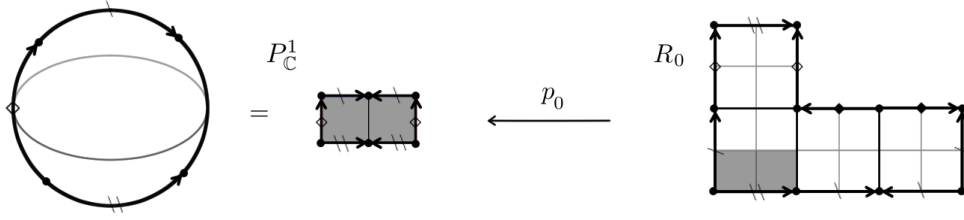
### 5.1 General origamis

**Definition 5.1.1.** A *general origami* of degree  $d$  is a flat surface obtained from  $d$  copies of the Euclidian unit squares by gluing along edges. An *origami* refers to a general origami in this chapter.

In the non-abelian case, similar arguments to Section 4.4.2 are valid for the genus (formula (4.17)), the development of affine maps (Corollary 4.4.4), and the Teichmüller curves (Remark 4.4.6). For Proposition 4.4.7, non-abelian origamis are mentioned but reduced from the argument by the existence of the canonical double cover which are abelian origamis. Note that non-abelian origamis are well expected to satisfy the same statement as Proposition 4.4.7.

**Example 8.** The *pillowcase sphere*  $P = \mathbb{C}/\langle z + 2, z + 2i, -z \rangle$  is a degree 2, non-abelian origami in the stratum  $\mathcal{Q}_0^p(-1^4)$ . It is isomorphic to the elliptic involution quotient of the unit square torus and represented by the algebraic curve  $C_0 : y = 4x^3 - x$ . By Lemma 4.4.2, every abelian origami of degree  $d$  is a  $2d$ -fold covering  $R \rightarrow E \rightarrow P$  of the pillowcase sphere.

Fig. 5.A shows an example of a non-abelian origami. We will observe in Theorem 5.1.7 that every origami of degree  $d$  is a  $2d$ -fold covering of the sphere over four points with the valency list  $(2^d \mid 2^d \mid 2^d \mid *)$  over  $\{[0], [1], [i], [1 + i]\} \subset P$ . Every critical point over the three branched points, say  $[1], [i], [1 + i] \in P$ , has multiplicity two and so is nonsingular. The rest one branched point  $[0] \in P$  pulls back the singularities of origami.



**Fig. 5.A** An origami of degree 4: edges with the same character are glued so that the arrows match. It admits a 8-fold covering of the pillowcase sphere  $P$  with valency list  $(2^4 \mid 2^4 \mid 1^2, 3^2)$ .

**Notation 5.1.2.** Let  $I_d = \{1, \dots, d\}$  be the set of  $d$  indices and  $\bar{I}_d = \{\pm 1, \dots, \pm d\}$  be its double. Let  $\mathcal{E}_d := \{\varepsilon \in \{\pm 1\}^{\bar{I}_d} \mid \varepsilon(-i) = \varepsilon(i), i \in \bar{I}_d\}$  be the set of symmetric signs on  $\bar{I}_d$ . Let  $\tilde{\mathfrak{S}}_d := \{\bar{\sigma} \in \text{Sym}(\bar{I}_d) \mid \bar{\sigma}(-i) = -\bar{\sigma}(i), i \in \bar{I}_d\}$  be the group of permutations with rotational symmetry, which naturally embeds the symmetric group  $\mathfrak{S}_d$ . For each  $x \in \mathfrak{S}_d$  and  $\varepsilon \in \mathcal{E}_d$ , define a mapping  $x^\varepsilon : \bar{I}_d \rightarrow \bar{I}_d$  by

$$x^\varepsilon(i) = \begin{cases} x(i) & \text{if } \varepsilon(i) = +1 \\ x^{-1}(i) & \text{if } \varepsilon(i) = -1 \end{cases} \quad \text{for each } i \in \bar{I}_d. \quad (5.1)$$

**Definition 5.1.3.** Let  $\Omega_d := \mathfrak{S}_d \times \mathfrak{S}_d$  be the set of (possibly disconnected) abelian origamis of degree  $d$ ,  $\Omega_{2d}^0 := \{\mathcal{O} \in \Omega_{2d} \mid \text{there exists an origami whose canonical double is } \mathcal{O}\}$ , and  $\tilde{\Omega}_d := \Omega_d \times \mathcal{E}_d$ . For each  $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$ , define  $(\mathbf{x}_\mathcal{O}, \mathbf{y}_\mathcal{O}) \in \Omega_{2d}$  by

$$\begin{cases} \mathbf{x}_\mathcal{O}(i) &= x^{\text{sign}(i)}(i) \\ \mathbf{y}_\mathcal{O}(i) &= \varepsilon(i) \cdot y^\varepsilon(i) \cdot \varepsilon(y^\varepsilon(i)) \end{cases} \quad \text{for each } i \in \bar{I}_d. \quad (5.2)$$

Every monodromy  $\mathbf{z} \in \langle \mathbf{x}_\mathcal{O}, \mathbf{y}_\mathcal{O} \rangle < \tilde{\mathfrak{S}}_d$  satisfies the following rotational symmetry with respect to the canonical double:

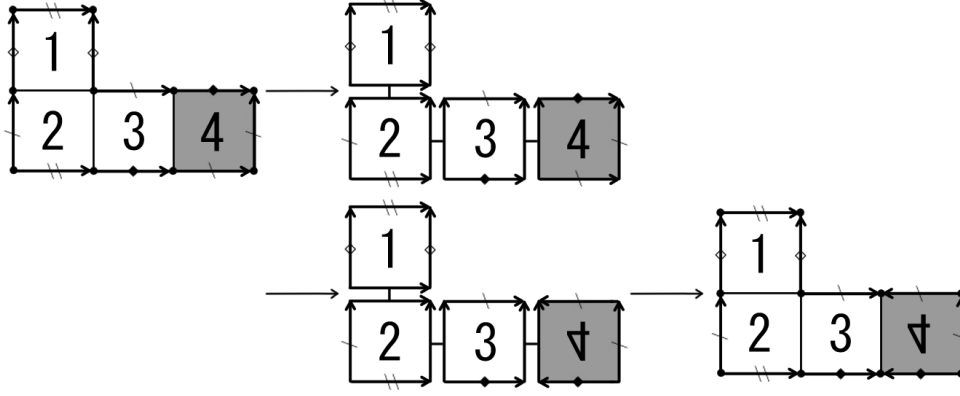
$$\mathbf{z}(-i) = -\mathbf{z}^{-1}(i) \quad \text{for each } i \in \bar{I}_d. \quad (5.3)$$

**Lemma 5.1.4.** Any  $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$  corresponds to an origami of degree  $d$  whose canonical double covering is the abelian origami  $(\mathbf{x}_\mathcal{O}, \mathbf{y}_\mathcal{O}) \in \Omega_{2d}$ . In particular,  $\mathcal{O} \mapsto (\mathbf{x}_\mathcal{O}, \mathbf{y}_\mathcal{O})$  gives a 1-1 correspondence  $\tilde{\Omega}_d \rightarrow \Omega_{2d}^0$  up to equivalence.

*Proof.* Consider the following construction for given  $\mathcal{O} \in \tilde{\Omega}_d$ :

- (A1) Cut the resulting surface of  $\mathcal{O}$  at all edges (with the edge-pairings remembered).
- (A2) Apply the vertical reflection to all cells with  $\varepsilon = -1$ .
- (A3) Glue all paired edges in such a way that with the natural coordinates, the quadratic differential  $(dz)^2$  is globally defined on the resulting surface.

This produces a new origami which can be non-abelian as shown in Fig. 5.B. By taking double and gluing to the half-rotated copy in place of each cell with  $\varepsilon = -1$  at (3), we obtain the canonical double represented by  $(\mathbf{x}_\mathcal{O}, \mathbf{y}_\mathcal{O})$ .



**Fig. 5.B** The construction of the origami in Fig. 5.A. It is given by  $(x, y, \varepsilon)$  where  $x = (1)(2\ 3\ 4)$ ,  $y = (1\ 2)(3\ 4)$ ,  $\varepsilon = (+, +, +, -)$ . We obtain the origami by regluing the abelian origami  $(x, y)$  after inverting squares of negative sign.

Conversely, consider the following construction for given  $(\mathbf{x}, \mathbf{y}) \in \Omega_{2d}^0$ :

(B1) Fix orientations of all horizontal and vertical cylinders in the resulting surface. For each  $i \in \bar{I}_d$ , let  $h_i$  ( $v_i$ , respectively) be an oriented, horizontal (vertical, respectively) closed geodesic crossing the cell with label  $i$ .

(B2) Define

$$\varepsilon(i) = \begin{cases} 1 & \text{if } v_i \text{ intersects to } h_i \text{ in positive crossing} \\ -1 & \text{if } v_i \text{ intersects to } h_i \text{ in negative crossing} \end{cases}, \quad i \in \bar{I}_d.$$

(B3) Do the same operation as (A2-A3). (i.e. cut all edges, apply reflections to all cells such that  $\varepsilon = -1$ , and reglue them.)

It will be shown in Lemma 5.1.5 that the above procedure recovers  $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$  from  $(\mathbf{x}_\mathcal{O}, \mathbf{y}_\mathcal{O})$ . Note that  $\varepsilon$  at (B2) depends on the choice of directions at (B1), but the resulting surface is uniquely determined up to half-translation.  $\square$

**Lemma 5.1.5.** *Let  $\mathbf{y} \in \tilde{\mathfrak{S}}_d$  and  $\mathbf{y} = c_1 c'_1 \cdots c_n c'_n$  be a cycle decomposition of  $\mathbf{y}$ , where  $(a_1\ a_2\ \cdots\ a_m)' := (-a_1\ -a_2\ \cdots\ -a_m)^{-1}$ . For  $j = 1, \dots, n$ , define  $\varepsilon_j \in \mathcal{E}_d$  so that  $\varepsilon_j(i) \cdot i$  belongs to the cycle  $c_j$  for each  $i \in \bar{I}_d$ . Then, the pair  $(y, \varepsilon)$  that correspond to  $\mathbf{y}_\mathcal{O}$  under the formula (5.2) is given by*

$$\begin{cases} y = \bar{\mathbf{y}} := |c_1| \cdots |c_n| \in \tilde{\mathfrak{S}}_d \\ \varepsilon = \varepsilon_{\mathbf{y}} := \varepsilon_1 \cdots \varepsilon_n \in \mathcal{E}_d, \end{cases} \quad (5.4)$$

where  $|(a_1 a_2 \cdots a_m)| := (|a_1| |a_2| \cdots |a_m|)$ . In particular, the inverse image of  $(\mathbf{x}, \mathbf{y}) \in \Omega_{2d}^0$  under (A1-A3) in the proof of Lemma 5.1.4 is  $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \varepsilon_{\mathbf{y}}) \in \tilde{\Omega}_d$ .

*Proof.* Suppose  $n = 1$ . We denote  $\mathbf{y} = cc' = (a_1 a_2 \cdots a_d)(-a_1 -a_2 \cdots -a_d)^{-1}$ ,  $y = (|a_1| |a_2| \cdots |a_d|)(-|a_1| -|a_2| \cdots -|a_d|)$ , and  $a_{d+1} = a_1$ . By definition, we have  $\varepsilon(a_i) = 1$  and  $\mathbf{y}(a_i) = \varepsilon(|a_{i+1}|)|a_{i+1}|$ , for all  $i \in I_d$ . We will show that  $\mathbf{y}(y, \varepsilon) := \varepsilon \cdot y^\varepsilon \cdot \varepsilon(y^\varepsilon)$  equals to  $\mathbf{y}$ . For each  $i \in I_d$ ,

$$\begin{aligned} \mathbf{y}(y, \varepsilon)(a_i) &= \varepsilon(a_i) \cdot y^{\varepsilon(a_i)}(a_i) \cdot \varepsilon(y^{\varepsilon(a_i)}(a_i)) \\ &= y(\text{sign}(a_i)|a_i|) \cdot \varepsilon(y(\text{sign}(a_i)|a_i|)) \\ &= \text{sign}(a_i)|a_{i+1}| \cdot \text{sign}(a_i)\varepsilon(|a_{i+1}|) \\ &= \varepsilon(|a_{i+1}|)|a_{i+1}| \\ &= \mathbf{y}(a_i). \end{aligned} \tag{5.5}$$

Applying this result to each cycle in  $\mathbf{y} = c_1 c'_1 \cdots c_n c'_n$ , we obtain the claim for general  $n$ .  $\square$

**Proposition 5.1.6.** *Let  $\mathcal{O}_j = (x_j, y_j, \varepsilon_j) \in \tilde{\Omega}_d$  ( $j = 1, 2$ ) be two origamis. Then  $\mathcal{O}_1, \mathcal{O}_2$  are isomorphic as flat surfaces if and only if there exists  $\bar{\sigma} = \delta\sigma \in \tilde{\mathfrak{S}}_d$  ( $\delta = \text{sign}(\bar{\sigma}) \in \{\pm 1\}^{I_d}$ ,  $\sigma \in \mathfrak{S}_d$ ) such that the following holds on  $I_d$ :*

- (1)  $\delta = \delta \circ x_1$ ,
- (2)  $x_2 = \sigma^\#(x_1^\delta)$ ,
- (3)  $\xi(y_2, \delta \circ \sigma^{-1} \cdot \varepsilon_1 \circ \sigma^{-1} \cdot \varepsilon_2) = 1$  where  $\xi(\tau, \lambda) := \lambda \cdot \lambda(\tau) \in \mathcal{E}_d$ ,
- (4)  $y_2 = \sigma^\#(y_1^{\delta \cdot \varepsilon_1 \cdot \varepsilon_2 \circ \sigma})$ .

*Proof.* Assume the existence of an isomorphism between  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . Then its lift via their canonical double coverings induces a cell-to-cell correspondence  $\bar{\sigma} \in \tilde{S}_d$ , such that  $\mathbf{x}_2(i) = \bar{\sigma}^\# \mathbf{x}_1(i)$  and  $\mathbf{y}_2(i) = \bar{\sigma}^\# \mathbf{y}_1(i)$  for  $i \in I_d$ . By the symmetry of  $\mathbf{x}_1, \mathbf{x}_2$ , it follows that  $\mathbf{x}_2(\bar{\sigma}(-i)) = \mathbf{x}_2(-\bar{\sigma}(i))$  for each  $i \in I_d$  and thus  $\bar{\sigma} \in \tilde{S}_d$ . For  $i \in I_d$  and  $\varepsilon \in \{\pm 1\}$ , we have

the following:

$$\begin{aligned}
 \bar{\sigma}(\mathbf{x}_1)(\varepsilon i) &= (\delta\sigma)(x_1^{\text{sign}(\varepsilon i)}(\varepsilon i)) \\
 &= \varepsilon\delta(x_1^\varepsilon(i))\sigma(x^\varepsilon(i)), & \dots (a) \\
 \mathbf{x}_2(\bar{\sigma}(\varepsilon i)) &= x_2^{\text{sign}(\sigma(\varepsilon i))}(\varepsilon\delta(i)\sigma(i)) \\
 &= \varepsilon\delta(i)x_2^{\varepsilon\delta(i)}(\sigma(i)), & \dots (b) \\
 \bar{\sigma}(\mathbf{y}_1)(\varepsilon i) &= (\delta\sigma)(\xi(y_1, \varepsilon_1)(\varepsilon i) \cdot y_1^{\varepsilon_1}(\varepsilon i)) \\
 &= \varepsilon\xi(y_1^{\varepsilon\varepsilon_1}, \varepsilon_1)(i) \cdot \delta(y_1^{\varepsilon\varepsilon_1(i)}(i)) \cdot \sigma(y_1^{\varepsilon\varepsilon_1(i)}(i)), & \dots (c) \\
 \mathbf{y}_2(\bar{\sigma}(\varepsilon i)) &= \xi(y_2^{\varepsilon_2}, \varepsilon_2)(\varepsilon\delta(i)\sigma(i)) \cdot y_2^{\varepsilon_2}(\varepsilon\delta(i)\sigma(i)) \\
 &= \varepsilon\delta(i)\xi(y_2^{\varepsilon\delta\circ\sigma^{-1}\varepsilon_2}, \varepsilon_2)(\sigma(i)) \cdot y_2^{\varepsilon\delta(i)\varepsilon_2(\sigma(i))}(\sigma(i)). & \dots (d)
 \end{aligned}$$

By comparing both sides of  $\mathbf{x}_2(\bar{\sigma}(\varepsilon\sigma^{-1}(i))) = \bar{\sigma}(\mathbf{x}_1(\varepsilon\sigma^{-1}(i)))$ , we obtain (1) and (2). Similarly for  $\mathbf{y}_1, \mathbf{y}_2$ , setting  $\varepsilon = \varepsilon_2(i) \cdot \delta \circ \sigma^{-1}(i)$ , we obtain (4) and the following:

$$\begin{aligned}
 \delta \circ \sigma^{-1}(i) \cdot \xi(y_2^{\varepsilon\delta\circ\sigma^{-1}\varepsilon_2}, \varepsilon_2)(i) &= \xi(y_1^{\varepsilon\varepsilon_1}, \varepsilon_1)(\sigma^{-1}(i)) \cdot \delta(y_1^{\varepsilon\varepsilon_1}(\sigma^{-1}(i))) \\
 &= \xi(\sigma^\# y_1^{\varepsilon\varepsilon_1}, \varepsilon_1 \circ \sigma^{-1})(i) \cdot \delta \circ \sigma^{-1}(\sigma^\# y_1^{\varepsilon\varepsilon_1(i)}(i)).
 \end{aligned}$$

With (4)  $\sigma^\# y_1^{\varepsilon\varepsilon_1(i)}(i) = y_2(i)$ , we conclude (3).

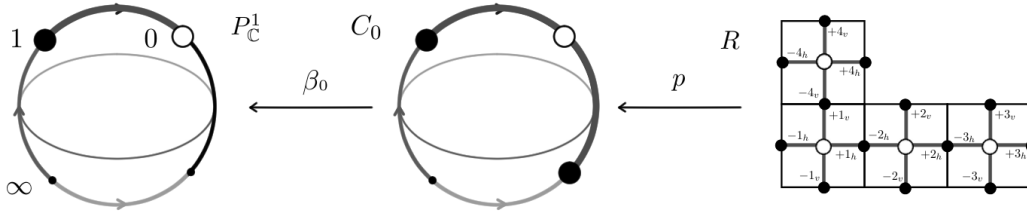
Suppose (1)-(4) conversely. Then for each  $i \in I_d$ , we have  $(a) = (b)$  and  $(c) = (d)$  for one of  $\varepsilon \in \{\pm 1\}$ . We may fill the equations for the other  $\varepsilon \in \{\pm 1\}$  as follows. First, the signs of (a), (b) coincide by (1). The equality of the other parts of (a), (b) follows from (2) taking inverse mappings of both sides. We can say the same for the other parts of (c), (d). Finally, the equality of the signs of (c), (d) follows from (3) for each  $y_2^{-1}(i) = \sigma^\#(y_1^{-\varepsilon_1 \cdot \varepsilon_2 \circ \sigma^{-1}}(i))$ ,  $i \in I_d$ . The above observation completes the proof.  $\square$

**Theorem 5.1.7.** *An origami of degree  $d$  is up to equivalence (mentioned in Remark 5.1.8) uniquely determined by each of the following.*

- (a) A  $2d$ -fold covering  $p : R \rightarrow P_C^1$  with the valency list  $(2^d \mid 2^d \mid 2^d \mid *)$ .
- (b) A pair of abelian origami of degree  $d$  and a  $d$ -tuples of signs.
- (c) A connected tripartite graph  $(\mathcal{V} = \mathcal{V}_c \sqcup \mathcal{V}_h \sqcup \mathcal{V}_v, \mathcal{E})$  with  $|\mathcal{V}_c| = |\mathcal{V}_h| = |\mathcal{V}_v| = d$  such that each edge connects vertices in  $\mathcal{V}_c$  and either  $\mathcal{V}_h$  or  $\mathcal{V}_v$ , and each vertex in  $\mathcal{V}_c, \mathcal{V}_h, \mathcal{V}_v$  has valency 4, 2, 2 respectively.
- (d) A pair of permutations  $\mu, \nu \in \text{Sym}(\bar{I}_d)$  which are fixed-point-free, of order 2, and together with sign inversion generate a transitive permutation group.

*Proof.* (origami  $\Leftrightarrow$  (a)  $\Leftrightarrow$  (b)) A covering (a) uniquely lifts the flat structure of the pillowcase sphere. The equivalence between origamis and (b) follows from Proposition 5.1.6. The construction (A1-A3) in the proof of Lemma 5.1.4 shows that by inverting some vertical monodromies of an abelian origami  $R' \rightarrow E \rightarrow P$ , one obtains a covering (a).

(origami  $\Leftrightarrow$  (c)  $\Leftrightarrow$  (d)) A graph (c) defines an origami by assigning a unit square cell to each vertex in  $\mathcal{V}_c$ , a horizontal edge to each vertex in  $\mathcal{V}_h$ , a vertical edge to each vertex in  $\mathcal{V}_h$ , and the adjacency between a cell and an edge to each edge in  $\mathcal{E}$ . Conversely, by composing a covering (a) to the Belyi pair  $(C_0 = \{y = 4x^3 - x\}, \beta_0(x, y) = 4x^2)$ , one obtains a dessin d'enfants on  $R$  as a graph (c). The rest of proof follows from Proposition 3.2.4.  $\square$



**Fig. 5.C** (The origami in Fig. 5.A, 5.B.) Suppose the bipartite graph  $\beta^{-1}([0, 1])$  embedded in  $R$ . The monodromy group of  $\beta$  is generated by the two permutation  $\iota, \sigma$  of edges around the white, black vertices respectively. Each edge is labelled by the index of the square it belongs to and its direction. For example, the horizontal edge adjacent to the right (left, respectively) side of  $i$ -th square is labelled by  $+i_h$  ( $-i_h$ , respectively).

Fig. 5.C shows an example of the  $4d$ -fold Belyi covering  $\beta = \beta_0 \circ p : R \rightarrow P_{\mathbb{C}}^1$ . The monodromy group of  $\beta$  is generated by two permutations  $\iota, \sigma = \sigma_{\mu, \nu} \in \text{Sym}(\bar{I}_d^h \sqcup \bar{I}_d^v)$  defined by

$$\begin{cases} \iota(\pm i_h) = \pm i_v & \iota(\pm i_v) = \mp i_h \\ \sigma(\pm i_h) = \mu(\pm i)_h & \sigma(\pm i_v) = \nu(\pm i)_v \end{cases} \quad \text{for each } i \in I_d, \quad (5.6)$$

where  $\bar{I}_d^\bullet = \{\pm 1_\bullet, \dots, \pm d_\bullet\}$  denotes a copy of  $\bar{I}_d$ . The permutation  $\iota\sigma$  arranges the edges clockwise around each of the centers of cells in  $R \setminus \beta^{-1}([0, 1])$ , which are the singularities of  $(R, \phi)$ . The permutation  $\iota\sigma$  has even order, as does that of the pillowcase sphere.

**Remark 5.1.8.** Note about Theorem 5.1.7 as follows. The equivalence of origamis is defined by an isomorphism of flat surfaces. It corresponds to the covering equivalence of (a) over  $P_{\mathbb{C}}^1 \setminus \{0, \pm 1, \infty\}$ . Lemma 5.1.6 presents a formula to determine the equivalence of origamis in terms of their canonical double coverings, which will be used in Section 6.1.

The equivalence of graph embeddings respecting the notion of ‘horizontal, vertical’ (i.e. the coloring of vertices) gives the equivalence of (c). In terms of dessins (d), it is described by the conjugacy in  $\mathfrak{S}_d$ .

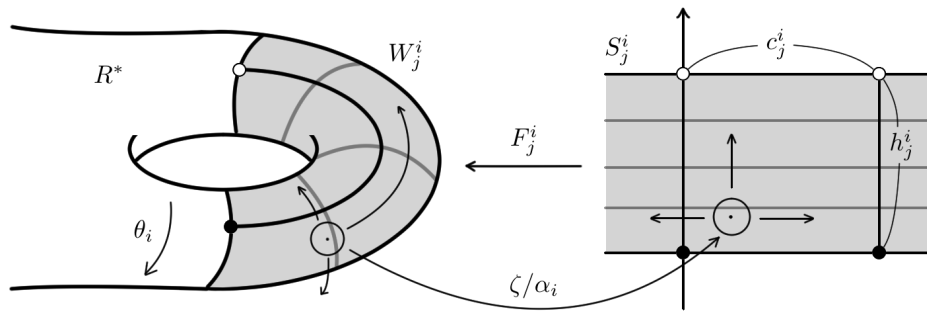
We may observe that the Veech group of an origami is a finite-index subgroup of  $PSL(2, \mathbb{Z})$  by the same arguments as in Section 4.4.

**Theorem 5.1.9** (=Theorem 6.1.7). *There exists two permutations  $\sigma_T, \sigma_S \in \mathfrak{S}_d$  such that the Veech group of an origami of degree  $d$  is the stabilizer of its equivalence class under the action of  $PSL(2, \mathbb{Z})$  on  $\tilde{\Omega}_d$  defined by  $[A](x, y, \varepsilon) := \theta^{-1}(\sigma_A^* \gamma_A(\theta(\mathcal{O})))$ ,  $A = T, S$ .*

## 5.2 Origami with moduli list

In this section, we consider a flat surface  $(R, \phi) \in \mathcal{Q}_g$  with two distinct finite Jenkins-Strebel directions  $\theta_1, \theta_2 \in J(R, \phi)$ . Then  $R$  is obtained by finite collections of parallelograms in the way presented in [10, Theorem2], in which we conclude  $R$  is finite analytic type even for more general settings. We review that construction.

Let  $i = 1, 2$  and  $\alpha_i = e^{i\theta_i} \in \mathbb{R}/\pi\mathbb{Z}$ . We have a decomposition of  $R$  into the  $\phi$ -cylinders  $W_1^i, \dots, W_{n_i}^i$  in direction  $\theta_i$ . For each  $i, j$ , an analytic continuation of local inverse of  $\phi$ -coordinates gives a holomorphic covering  $F_j^i : S_j^i \rightarrow W_j^i$  on a strip region  $S_j^i = \{0 < \text{Im} z < h_j^i\} \subset \mathbb{C}$  and  $\text{Deck}(F_j^i) = \langle z \mapsto z + c_j^i \rangle$  for some  $h_j^i, c_j^i > 0$  (see Fig. 5.D). We denote by  $z_j, w_j$  the local  $\phi$ -coordinates in  $W_j^1, W_j^2$ . By construction,  $F_j^{1*}(\alpha_i \phi) = dz_j^2$  and  $F_j^{2*}(\alpha_i \phi) = dw_j^2$  hold.

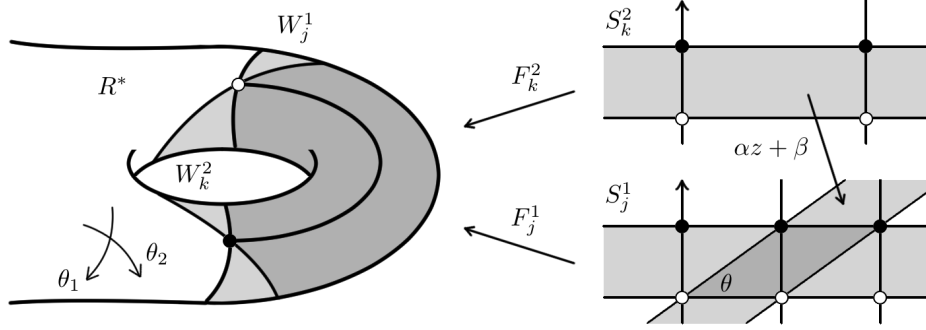


**Fig. 5.D** The  $\phi$ -cylinder  $W_j^i$  and the covering  $F_j^i$ .

For any  $p \in S_j^1$ , there is a neighborhood  $U$  in which  $F_j^1 = F_k^2 \circ f$  for some  $k$  and some holomorphic function  $f : U \rightarrow S_k^2$ . By the formula

$$f^* dw_k^2 = f^*(F_k^{2*}(\alpha_2 \phi)) = F_j^{1*}(\alpha_2 \phi) = (\alpha_2 / \alpha_1) dz_j^2, \quad (5.7)$$

$f$  is continued on  $S_j^1$  by the form  $f(z_j^1) = \alpha z_j^1 + \beta$  where  $\alpha = \pm\sqrt{\alpha_2/\alpha_1}$  and  $\beta \in \mathbb{C}$  (see Fig. 5.E). The intersection  $V_{j,k} = S_j^1 \cap f^{-1}(S_k^2)$  is a parallelogram isometrically mapped to  $W_j^1 \cap W_k^2$ . The collection  $(V_{j,k})_{j=1}^{n_1}$  fills the strip region  $S_j^1$  by translations in  $\text{Deck}(F_j^1) = \langle z \mapsto z + c_j^1 \rangle$  ( $j = 1, \dots, n_1$ ). The same can be said for  $(f^{-1}(V_{j,k}))_{k=1}^{n_2}$  filling the strip region  $S_k^2$ .



**Fig. 5.E** A parallelogram as an intersection of two  $\phi$ -cylinders.

Thus the surface  $R$  is decomposed into the collection of regions  $(W_j^1 \cap W_k^2)_{j,k}$ , each of which is empty or isomorphic to a parallelogram on the plane. Suppose  $(j, k)$  in the latter case. Such a parallelogram  $V_{j,k}$  is uniquely determined up to half-translations. We call them the  $(\theta_1, \theta_2)$ -parallelograms of  $(R, \phi)$ . Via  $F_j^1$  and  $F_k^2$ , the isomorphism between  $W_j^1 \cap W_k^2$  and  $V_{j,k}$  is continued over the boundary. Thus  $(R, \phi)$  is isomorphic to the surface obtained by gluing  $(\theta_1, \theta_2)$ -parallelograms along boundary edges in the way that respects the adjacencies determined by the continuations of the local isomorphisms.

A  $(\theta_1, \theta_2)$ -parallelogram  $V_{j,k}$  has boundary edges in the directions  $\theta_1, \theta_2$  and a modulus  $M(V_{j,k}) = (h_j^1/h_k^2) \sin |\theta_1 - \theta_2|$ . On the plane, an affine map with derivative  $A \in SL(2, \mathbb{Z})$  maps a  $(\theta_1, \theta_2)$ -parallelogram to an  $(A\theta_1, A\theta_2)$ -parallelogram whose modulus is a scalar multiple of  $\rho_{A, \theta_1, \theta_2} := |T_A(e^{i\theta_2})|/|T_A(e^{i\theta_1})|$ . The same holds for  $(R, \phi)$  and we have the following in connection with Lemma 4.3.3.

**Lemma 5.2.1.** *Let  $(R, \phi) \in \mathcal{Q}_g$ ,  $\theta_1, \theta_2$  be two distinct directions in  $J(R, \phi)$ , and  $\{V_i\}_{i=1}^d$  be the  $(\theta_1, \theta_2)$ -parallelograms of  $(R, \phi)$ . If  $[A] \in PSL(2, \mathbb{R})$  belongs to  $\Gamma(R, \phi)$  then  $M(f(V_j)) = \rho_{A, \theta_1, \theta_2}(M(V_1))$  holds for  $j = 1, \dots, d$ .*

Stretching and rotating  $\phi$ -cylinders leads to a homeomorphism from  $(R, \phi)$  to an origami which respects the markings determined by boundaries of parallelograms. In this way,  $(R, \phi)$  and  $\theta_1, \theta_2 \in J(R, \phi)$  determines a unique origami with additional data of moduli list  $\mathbf{M} = (M_i)_{i=1}^d$  of the  $(\theta_1, \theta_2)$ -parallelograms and directions  $\theta_1, \theta_2$ . Conversely, an origami



$\mathcal{O}$  and a moduli list  $\mathbf{M} = (M_i)_{i=1}^d$  compatible with  $\mathcal{O}$  is supposed to give a flat surface with a decomposition as above for each pair  $(\theta_1, \theta_2)$  of distinct directions assigned.

Recall that an origami can be seen as a dessin given by a pair of arbitrary  $\mu, \nu \in \tilde{\mathfrak{S}}_d$ , by Theorem 5.1.7. We will define the compatibility of  $\mathbf{M} = (M_i)_{i=1}^d \in \mathbb{R}_{>0}^d$  with an origami  $\mathcal{O} = (\mu, \nu)$ , which purposes that we can glue  $d$  rectangles  $V_1, \dots, V_d$  with  $M(V_i) = M_i$  along edges to form a flat surface  $(R, \phi)$  in the same way as  $\mathcal{O}$ .

Let  $|\kappa| = i$  for each  $\kappa = \pm i \in \{\pm 1_h, \dots, \pm d_h\} \sqcup \{\pm 1_v, \dots, \pm d_v\}$ . Then  $|\mu(\kappa)|$  ( $|\nu(\kappa)|$ , respectively) represents the rectangle adjacent to the right (upper, respectively) side of  $|\kappa|$ -th rectangle. Then the lengths of their horizontal (vertical, respectively) edges should be related by a factor of  $K_{\kappa, \mu} = M_{|\kappa|}/M_{|\mu(\kappa)|}$  ( $K_{\kappa, \nu} = M_{|\nu(\kappa)|}/M_{|\kappa|}$ , respectively). When we go along a path  $\gamma$  on  $R^*$  joining two rectangles, indices of rectangles we pass through and directions of entry are interpreted as a path in the bipartite graph  $\beta^{-1}([0, 1])$ . It is described in terms of monodromy of the form  $(\iota^{k_1} \sigma) \cdots (\iota^{k_m} \sigma) \in \tilde{\mathfrak{S}}_d$ , which is a word of  $\iota^k \sigma_k$  ( $k = 0, 1, 2, 3$ ). We may set starting edge as  $+i_h$ , then we have  $\sigma_k = \mu$  for  $k_j = 0, 2$  and  $\sigma_k = \nu$  for  $k_j = 1, 3$ . We define as follows.

$$K_{\mathcal{O}}(\gamma, \mathbf{M}) := \prod_{j=1}^m K_{(\iota^{k_1} \sigma) \cdots (\iota^{k_j} \sigma)(+i_h), \sigma_{k_j}} \quad (5.8)$$

**Definition 5.2.2.** Let  $\mathcal{O} = (\mu, \nu)$ ,  $\mathcal{O}_i = (\mu_i, \nu_i)$  be origamis of degree  $d$ .

- (1) We call  $\mathbf{M} = (M_i)_{i=1}^d \in \mathbb{R}_{>0}^d$  a *moduli list compatible with  $\mathcal{O}$*  if  $K_{\mathcal{O}}(\gamma, \mathbf{M}) = 1$  for any  $\gamma \in \pi_1(\mathcal{O}^*)$ . ( $\mathcal{O}^*$  is the flat surface  $\mathcal{O}$  punctured at all the singularities.)
- (2) Let  $\mathbf{M}_i = (M_i^i)_{i=1}^d \in \mathbb{R}_{>0}^d$  be a moduli list compatible with  $\mathcal{O}_i$  for  $i = 1, 2$ . We say that  $(\mathcal{O}_1, \mathbf{M}_1)$  and  $(\mathcal{O}_2, \mathbf{M}_2)$  are equivalent if there exists  $\tau \in \tilde{\mathfrak{S}}_d$  such that the following holds for  $i = 1, \dots, d$ .

$$\mu_1 = \tau^* \mu_2, \quad \nu_1 = \tau^* \nu_2, \quad M_i^1 = M_{|\tau(i)|}^2 \quad (5.9)$$

Observe that an isomorphism between two flat surfaces with two finite Jenkins-Strebel directions naturally induces an equivalence between two origamis with compatible moduli lists. The mapping  $K_{\mathcal{O}}(\cdot, \mathbf{M})$  defines a group homomorphism  $\pi_1(\mathcal{O}^*) \rightarrow \mathbb{R}_{>0}$ . The compatibility of lengths of the rectangles placed along a path  $\gamma$  on  $R^*$  fails only when  $\gamma$  contains a loop. So we may determine the compatibility from finite generator of  $\pi_1(\mathcal{O}^*)$ .

From above, we can conclude the following.

**Theorem 5.2.3.** *Let  $\theta_1, \theta_2 \in \mathbb{R}/\pi\mathbb{Z}$  be two distinct directions. A flat surface  $(R, \phi)$  such that  $\theta_1, \theta_2 \in J(R, \phi)$  is up to equivalence uniquely determined by an origami with a compatible moduli list.*

We say that a flat surface  $(R, \phi)$  is *origami-like* if  $J(R, \phi)$  has cardinality at least 2. Let  $\mathcal{Q}^{2JS}$  be the symbol that assign the set of origami-like flat surfaces in place of  $\mathcal{Q}$  in Definition 4.1.2. For each  $(R, \phi) \in \mathcal{Q}_g^{2JS}$  and  $\theta_1, \theta_2 \in J(R, \phi)$ , let  $P(R, \phi, \theta_1, \theta_2)$  be the origami with compatible moduli list given by Theorem 5.2.3. For two distinct directions  $\theta_1, \theta_2 \in \mathbb{R}/\pi\mathbb{Z}$  and an origami  $\mathcal{O} = (\mu, \nu) \in \tilde{\Omega}_d$ , consider the set

$$\mathcal{Q}_{\theta_1, \theta_2}^{2JS}(\mathcal{O}) = \{(R, \phi) \in \mathcal{Q}_g^{2JS} \mid \theta_1, \theta_2 \in JS(R, \phi), P(R, \phi, \theta_1, \theta_2) = (\mathcal{O}, \cdot)\}. \quad (5.10)$$

The mapping  $K_{\mathcal{O}}(\gamma, \cdot)$  can be regarded as a linear map via the conjugation by the logarithm. By taking a basis of the fundamental group  $\pi_1(R^*, \cdot)$ , we obtain an integer matrix  $A_{\mathcal{O}}$  with  $d$  rows representing a finite system of linear equations to ensure compatibility. Thus we may define

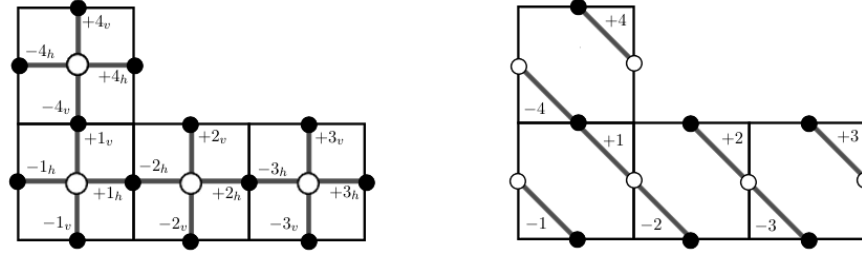
$$o : \mathbb{R}^d \supset \ker A_{\mathcal{O}} \rightarrow \mathcal{Q}_{\theta_1, \theta_2}^{2JS}(\mathcal{O}) : (x_1, \dots, x_d) \mapsto (\mathcal{O}, (e^{x_1}, \dots, e^{x_d})). \quad (5.11)$$

By Theorem 5.2.3, the mapping  $o$  is bijective up to the factor  $\text{Stab}_{\mathcal{O}} := \text{Cent}_{\tilde{\Omega}_d}(\mu, \nu)$ . The group  $\text{Stab}_{\mathcal{O}}$  equals the automorphism group of the natural dessin  $(\iota, \sigma_{\mu, \nu})$  of origami  $\mathcal{O}$ . In particular, the list of isomorphism classes of  $\text{Stab}_{A_{\mathcal{O}}}$ ,  $A \in \text{PSL}(2, \mathbb{Z})/\Gamma(\mathcal{O})$  is a  $G_{\mathbb{Q}}$ -invariant of the Teichmüller curve  $C(\mathcal{O})$ . Note that the group  $\text{Stab}_{\mathcal{O}}$  is invariant under the deformations by two matrices  $J, S$  since their images are the origamis given by  $(\nu \circ \iota, \mu)$ ,  $(\mu \circ \iota, \nu)$  respectively.

As the group  $\text{Stab}_{\mathcal{O}}$  trivially conjugates the mapping  $K_{\mathcal{O}}(\gamma, \cdot)$ , it acts on  $\mathbb{R}^d$  by permutations of coordinates compatible with  $A_{\mathcal{O}}$ . We summarize as follows.

**Corollary 5.2.4.** *Let  $\theta_1, \theta_2 \in \mathbb{R}/\pi\mathbb{Z}$  be two distinct directions and  $\mathcal{O} = (\mu, \nu)$  be an origami of degree  $d$ . Then, the family  $\mathcal{Q}_{\theta_1, \theta_2}^{2JS}(\mathcal{O})$  is globally parametrized in  $\ker A_{\mathcal{O}}/\text{Stab}_{\mathcal{O}}$ .*

The group  $\text{Cent}_{\text{Sym}(\bar{I}_d)}(\mu, \nu)$  is the automorphism group of the (possibly disconnected) dessin  $(\mu, \nu)$  of degree  $2d$ . The graph of  $(\mu, \nu)$  is the disjoint union of cycle graphs (Example 7) each components of which corresponds to the  $\phi$ -cylinders in the direction  $[\frac{3}{2}\pi] \in \mathbb{R}/\pi\mathbb{Z}$ , as shown in Fig. 5.F. The automorphism group  $\text{Aut}(\mu, \nu)$  is generated by finitely many groups of the form  $\{\pm 1\} \rtimes C_l$ ,  $l \in \mathbb{N}$  and permutations of cycles of the same lengths. The group  $\text{Stab}_{\mathcal{O}}$  is given by the intersection  $\text{Aut}(\mu, \nu) \cap \tilde{\Omega}_d$ .



**Fig. 5.F** The dessin  $(\mu, \nu)$  is the disjoint union of cycle graphs given by the  $\phi$ -cylinders in the direction  $[\frac{3}{2}\pi] \in \mathbb{R}/\pi\mathbb{Z}$ . We identify the two edges  $\kappa_h, \kappa_v$  for each  $\kappa \in \bar{I}_d$ .

### 5.3 Veech groups in terms of origamis

**Corollary 5.3.1** (to Theorem 5.2.3). *Let  $(R, \phi) \in \mathcal{Q}^{2JS}$  be an origami-like flat surface with  $\theta_1, \theta_2 \in J(R, \phi)$ .  $[A] \in \text{PSL}(2, \mathbb{R})$  belongs to  $\Gamma(R, \phi)$  if and only if the following holds.*

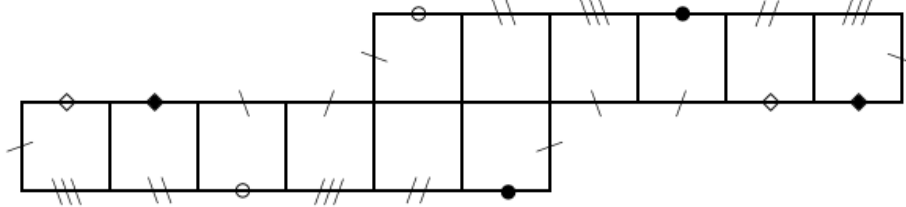
- (1)  $A\theta_1, A\theta_2$  belongs to  $J(R, \phi)$ .
- (2) Let  $(\mathcal{O}, \mathbf{M}), (\mathcal{O}_A, \mathbf{M}_A)$  be the origamis with compatible moduli lists given by the decomposition of  $(R, \phi)$  in  $(\theta_1, \theta_2)$  and  $(A\theta_1, A\theta_2)$  respectively. Then  $(\mathcal{O}, \mathbf{M})$  is equivalent to  $(\mathcal{O}_A, \rho_{A, \theta_1, \theta_2} \cdot \mathbf{M}_A)$ .

*Proof.* An affine map  $f$  on  $(R, \phi)$  with derivative  $[A]$  maps the  $(\theta_1, \theta_2)$ -parallelograms to the  $(A\theta_1, A\theta_2)$ -parallelograms with their adjacency preserved. As their moduli change by the constant multiple in the way described in Lemma 5.2.1, the equivalence follows. Conversely, (a) and (b) imply that  $(R, \phi)$  is represented by two flat surfaces one of which is obtained from the other by the natural affine deformation with derivative  $[A]$ .  $\square$

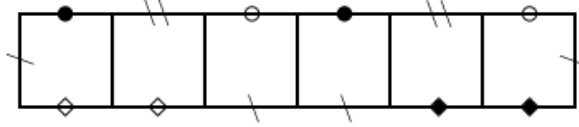
For abelian origamis, the *quaternion origami* and the *Ornithorynque origami* in Fig. 5.G are known as nontrivial origamis of small degree (8 and 12, respectively) with the maximal Veech group  $SL(2, \mathbb{Z})$ . The quaternion origami has been studied for its intrinsic properties in the moduli space [26, 27, 46]. The Ornithorynque origami was focused on [13, 44] in the context of the Teichmüller geodesic flow, the genodesic flow in the Teichmüller space defined by the contractive affine deformation of the matrix  $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ ,  $t \in \mathbb{R}$ .

The following is obtained by the calculation that we will state in Section 6.1.

**Proposition 5.3.2.** *The origami  $\mathcal{D} = ((1, 2, 3, 4, 5, 6), (1, 2, 5, 6, 3, 4), (-, +, -, +, -, +))$  in Fig. 5.H is the unique nontrivial origami with the maximal Veech group  $\text{PSL}(2, \mathbb{Z})$  of the smallest degree 6, which is non-abelian. The canonical double of  $\mathcal{D}$  is the Ornithorynque origami.*



**Fig. 5.G** The Ornithorynque origami.



**Fig. 5.H** The origami  $\mathcal{D} : (x, y, \varepsilon) = ((1, 2, 3, 4, 5, 6), (1, 2, 5, 6, 3, 4), (-, +, -, +, -, +))$ .

**Definition 5.3.3.** Let  $(R, \phi)$  and  $S, \psi$  be origami-like flat surfaces. We say that a finite branched covering  $f : (S, \psi) \rightarrow (R, \phi)$  is an *unbranched covering of origami-like flat surfaces* if  $\psi = f_*\phi$  and  $\text{Crit}(f) \subset f^{-1}(\text{Crit}(\phi)) \subset \text{Crit}(\psi)$  holds.

The condition  $\text{Crit}(f) \subset f^{-1}(\text{Crit}(\phi))$  implies that  $f$  branches at most over the singularities of  $(R, \phi)$ . The condition  $f^{-1}(\text{Crit}(\phi)) \subset \text{Crit}(\psi)$  implies that no singularity on  $(R, \phi)$  is canceled when pulled back. The latter can be replaced by removing all such points, the doubled points of singularities of order  $-1$  on  $(R, \phi)$ . For flat surfaces in covering relation, the commensurability of the Veech groups is known [19]. More strongly, the following holds in our situation.

**Lemma 5.3.4.** Let  $f : (S, \psi) \rightarrow (R, \phi)$  be an unbranched covering of origami-like flat surfaces. Then  $\Gamma(S, \psi)$  is a finite index subgroup of  $\Gamma(R, \phi)$ .

*Proof.* Let  $\theta_1, \theta_2 \in J(R, \phi)$ ,  $p \in R \setminus \text{Crit}(\phi)$ , and  $\gamma$  be a closed  $\phi$ -geodesic in the direction  $\theta_1$  through  $p$ . Then any lift of  $\gamma$  is a  $\phi$ -geodesic joining points in  $f^{-1}(p)$  in the direction  $\theta_1$ . Finite collection of such lifts form a closed  $\psi$ -geodesic and any closed  $\psi$ -geodesic is of this form. Since  $\psi = f_*\phi$  where no singularity on  $(R, \phi)$  is canceled, any pullbacks of a  $\phi$ -cylinder are not laminated together to make a wider cylinder. Thus  $J(R, \phi) = J(S, \psi)$  and each  $(\theta_1, \theta_2)$ -parallelogram are invariant on the plane.

Let  $\mathcal{O}_R$  ( $\mathcal{O}_S$ , respectively) the origami determined by the decomposition  $P(R, \phi, \theta_1, \theta_2)$  ( $P(S, \psi, \theta_1, \theta_2)$ , respectively). We can see that  $\mathcal{O}_S$  is obtained from finite copies of  $\mathcal{O}_R$  by regluing along their edges according to the monodromy of  $f$ . Furthermore  $f$  induces a projection from  $\mathcal{O}_S$  to  $\mathcal{O}_R$  which respects adjacency of squares up to the copies. So if  $(S, \psi)$

satisfies the condition (b) in Corollary 5.3.1, then the same holds for  $(R, \phi)$ . Conversely, for  $[A] \in \Gamma(R, \phi)$ , the origami determined by  $(S, \psi)$  with  $(A\theta_1, A\theta_2)$  is similarly constructed as  $\mathcal{O}_S$  up to difference of monodromy. As it has finitely many possibilities, it coincides with  $\mathcal{O}_S$  up to finite representatives. The same can be said for the decomposition of  $(R, \phi)$  into parallelograms.  $\square$

**Theorem 5.3.5.** *Let  $f : (S, \psi) \rightarrow (R, \phi)$  be an  $N$ -fold, unbranched covering of origami-like flat surfaces with  $(\theta_1, \theta_2) \in J(R, \phi)$ . Fix a base point  $p \in R^*$  and a generating system  $\mathcal{F}$  of  $\pi_1(R^*, \cdot)$ . Define the action of the Veech group  $\Gamma(R, \mu) < PSL(2, \mathbb{R})$  on  $\mathcal{M} = (\mathfrak{S}_N)^{\mathcal{F}}$  so that  $[A] \in \Gamma(R, \mu)$  transforms the monodromy of  $f$  by taking the new decomposition in  $A^{-1}(\theta_1, \theta_2)$ . Then,  $\Gamma(\hat{R}, \psi)$  is the stabilizer of  $\tau_f = m_f(\mathcal{F}) \in \mathcal{M}$  under the equivalence defined by*

- (1) relabeling of sheets of  $f$  (i.e. conjugacy in  $\mathfrak{S}_N$ ), and
- (2) simultaneous conjugation in  $\text{Stab}_{\mathcal{O}_R}$ .

*Proof.* As we have seen in the proof of Lemma 5.3.4, it follows that any lift of  $\gamma \in \pi_1(R^*, \cdot)$  connects two copies of  $P(R, \phi, A^{-1}\theta_1, A^{-1}\theta_2) = P(R, \phi, \theta_1, \theta_2)$  in  $P(S, \psi, A^{-1}\theta_1, A^{-1}\theta_2)$  for each  $[A] \in \Gamma(R, \phi)$ . One obtains the decomposition  $P(S, \psi, A^{-1}\theta_1, A^{-1}\theta_2)$  by patching copies of  $P(R, \phi, \theta_1, \theta_2)$  according to the new monodromy data  $[A]\tau_f$ . It follows from Corollary 5.3.1 that the stabilizer represents the Veech group.  $\square$

**Example 9.** Let  $f : \mathcal{O} \rightarrow \mathcal{D}$  be an  $N$ -fold, unbranched covering of the origami  $\mathcal{D}$  in Proposition 5.3.2. Then,  $\tau_f$  runs over any element of  $\mathcal{M} = (\mathfrak{S}_N)^7$ ,  $\text{Stab}_{\mathcal{D}} = \langle (1\ 3\ 5)(2\ 4\ 6) \rangle \cong C_3$ , and the action of  $\Gamma(\mathcal{D}) = PSL(2, \mathbb{Z}) = \langle [T], [S] \rangle$  on  $\mathcal{M}$  is defined by

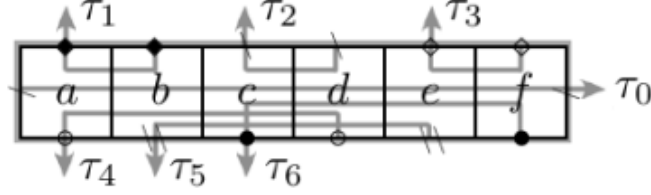
$$[T](\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = (\tau_0, \tau_1, \tau_2, \tau_3, \tau_5, \tau_6, \tau_4^{-1}\tau_0^{-1}), \quad (5.12)$$

$$[S](\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = (\tau_2^{-1}\tau_6\tau_3^{-1}\tau_5^{-1}\tau_1^{-1}\tau_4, \tau_2, \tau_3, \tau_1, \tau_3\tau_6^{-1}\tau_2, \tau_5\tau_3\tau_6^{-1}, \tau_1\tau_5\tau_3\tau_0), \quad (5.13)$$

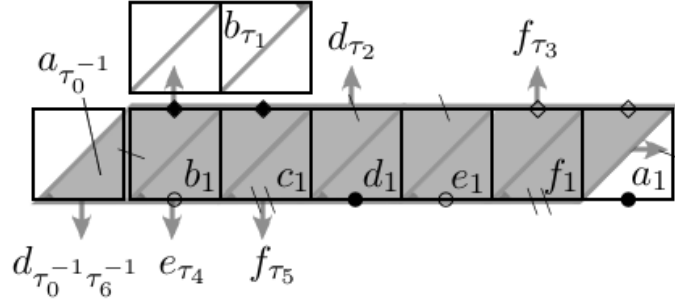
for each  $(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) \in \mathcal{M}$ . The Veech group  $\Gamma(\mathcal{O})$  is the stabilizer of the equivalence class of  $\tau_f$  under the action of  $PSL(2, \mathbb{Z})$ .

The formulae (5.12) and (5.13) are obtained as follows. First, fix a generating system  $\mathcal{F} = \{\tau_i\}_{i=0}^6$  of  $\pi_1(R^*, \cdot)$  and label the cells of the origami  $\mathcal{D}$  as shown in Fig. 5.I. Then, if we fix directions  $(0, \frac{\pi}{2})$  of decomposition, the covering  $f : \mathcal{O} \rightarrow \mathcal{D}$  is uniquely determined by a monodromy  $\tau_f \in (\mathfrak{S}_N)^7$  up to equivalence mentioned in Theorem 5.3.5. For a matrix  $[A] \in \Gamma(\mathcal{D}) = PSL(2, \mathbb{Z})$ ,  $A = T^{-1}S$ , the decomposition  $P(\mathcal{O}, A(0, \frac{\pi}{2}))$  is tiled by the decomposition  $[A]\mathcal{D} = P(\mathcal{D}, A(0, \frac{\pi}{2})) \cong \mathcal{D}$  as shown in Fig. 5.J and 5.K. It is

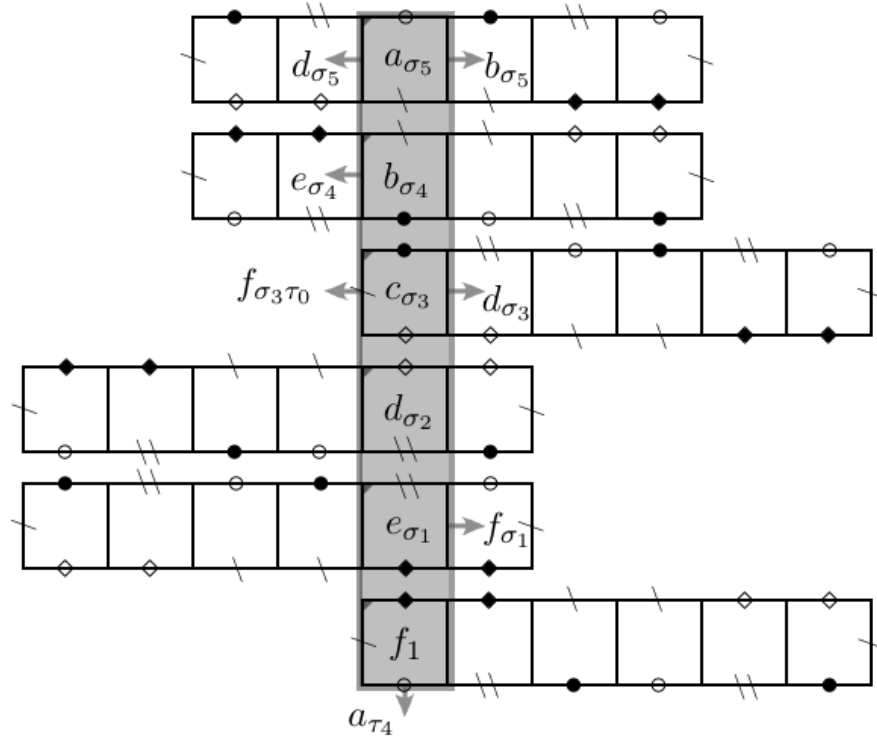
also uniquely determined by some monodromy  $[A^{-1}]\tau_f \in (\mathfrak{S}_N)^7$  up to equivalence. By labelling parallelogram cells in the directions  $A(0, \frac{\pi}{2})$  according to sheets of the original decomposition  $P(\mathcal{O}, (0, \frac{\pi}{2}))$ , we obtain the transformation  $\tau_f \mapsto [A^{-1}]\tau_f$  by the formulae (5.12) and (5.13).



**Fig. 5.I** Fixed generating system  $\mathcal{F} = \{\tau_i\}_{i=0}^6$  of  $\pi_1(R^*, \cdot)$  and labeling of cells of  $\mathcal{D}$ .



**Fig. 5.J** The decomposition  $P(\mathcal{O}, T^{-1}(0, \frac{\pi}{2}))$  is tiled by  $[T^{-1}]\mathcal{D} \cong \mathcal{D}$  (shaded). Each parallelogram cells are labelled according to the sheets of  $\mathcal{D}$ . We obtain the formula  $[T^{-1}](\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = (\tau_0, \tau_1, \tau_2, \tau_3, \tau_0^{-1}\tau_6^{-1}, \tau_4, \tau_5)$ , which leads to the formula (5.12).



**Fig. 5.K** The decomposition  $P(\mathcal{O}, S(0, \frac{\pi}{2}))$  is tiled by  $[S]\mathcal{D} \cong \mathcal{D}$  (shaded). Each parallelogram cells  $s$  are labelled according to the sheets of  $\mathcal{D}$ . The symbols  $\sigma_\bullet$  represent permutations as follows:  $\sigma_1 = \tau_1$ ,  $\sigma_2 = \sigma_1\tau_5$ ,  $\sigma_3 = \sigma_2\tau_3$ ,  $\sigma_4 = \sigma_3\tau_6^{-1}$ , and  $\sigma_5 = \sigma_4\tau_2$ . We obtain the formula (5.13).





# Chapter 6

## Calculation on origamis

This chapter is based on [40]. Throughout this chapter, an *origami* refers to a general origami.

### 6.1 Classification of origamis into components of Teichmüller curves

This section observes Theorem 5.1.9 and states a concrete procedure for implementation. A *partition* of  $d$  is a finite sequence of weakly decreasing positive integers that sum to  $d$ . The *partition number*  $p(d)$ , which counts the number of partitions of  $d$ , defines the following rapidly increasing sequence.

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, ...

(cf. <http://oeis.org/A000041>.) The following asymptotic formula [20] is known:

$$p(d) \sim \frac{1}{4d\sqrt{3}} \cdot e^{\sqrt{2d/3}}.$$

The algorithm in [22] constructs all partitions of given integer. We will accept  $P(d) = \{(j_1, j_2, \dots, j_d) : \text{partition of } d\}$  as a known data.

To describe the isomorphism class of each origami  $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$ , we enumerate all the conjugators  $\bar{\sigma} = \delta\sigma \in \bar{S}_d$  ( $\delta = \text{sign}(\bar{\sigma}) \in \{\pm 1\}^{\bar{I}_d}$ ,  $\sigma \in S_d$ ) satisfying the conditions in Lemma 5.1.6. By (2), up to isomorphisms, we only have to think of  $x$  with the normalized cycle decompositions

$$x = (1 \dots j_1)(j_1 + 1 \dots j_1 + j_2) \dots (j_1 + 1 \dots \sum_{k=1}^n j_k = d) \quad (6.1)$$

according to the partition  $(j_1, j_2, \dots, j_n) \in P(d)$  determined by its cycle lengths.

We will consider the *restricted class* of an origami, the set of origamis with the same ‘ $x$ ’ and isomorphic to it. By Lemma 5.1.6, the restricted class is the conjugacy class in  $\text{Stab}(x) := \{\bar{\sigma} = \delta\sigma \in \bar{S}_d \mid \delta = \delta \circ x \text{ and } x = \sigma^\#(x^\delta) \text{ on } I_d\}$ . Remark that for general  $y \in \bar{S}_d$  and  $\varepsilon \in \mathcal{E}_d$ , the mapping  $y^\varepsilon$  does not belong to  $\bar{S}_d$ . It will be confirmed in (2) of Algorithm 6.1.1.

First, we present an algorithm for describing the  $\text{Stab}(x)$ -conjugacy class of  $(y, \varepsilon)$  for each  $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$  satisfying the conditions in Lemma 5.1.6.

**Algorithm 6.1.1.** For each  $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$ , we construct its restricted class  $[\mathcal{O}] = \{(x, y', \varepsilon') \in \tilde{\Omega}_d \mid (x, y', \varepsilon') \sim (x, y, \varepsilon)\}$  in the following steps:

- (1) Take  $\bar{\sigma} = \delta\sigma \in \text{Stab}(x)$ : with  $\delta = \delta \circ x$  and  $x = \sigma^\#(x^\delta)$  on  $I_d$ .
- (2) For each  $\varepsilon' \in \mathcal{E}_d$ , let  $y_{\bar{\sigma}, \varepsilon'} := \sigma^\#(y^{\varepsilon \cdot \varepsilon' \circ \sigma \cdot \delta})$ . Verify  $\varepsilon' \in \mathcal{E}_d$  such that  $y_{\bar{\sigma}, \varepsilon'} \in S_d$  and  $\xi(y_{\bar{\sigma}, \varepsilon'}, \delta \circ \sigma^{-1} \cdot \varepsilon \circ \sigma^{-1} \cdot \varepsilon') = 1$  on  $I_d$ .
- (3) Let  $C_{\bar{\sigma}} := \{(x, y_{\bar{\sigma}, \varepsilon'}, \varepsilon') \mid \varepsilon' \text{ passes the test in (2)}\}$ .
- (4) Go back to (1) for some other leftover  $\sigma \in \text{Stab}(x)$ . When we have been through all elements in  $\text{Stab}(x)$ , finish the algorithm and we conclude that  $[\mathcal{O}] = \bigcup_{\bar{\sigma} \in \text{Stab}(x)} C_{\bar{\sigma}}$ .

**Algorithm 6.1.2.** Let  $P(d) = \{(j_1, j_2, \dots, j_d) : \text{partition of } d\}$ . We obtain the set  $C\tilde{\Omega}_d$  of the restricted classes of all origamis of degree  $d$  in the following steps.

- (1)  $C\tilde{\Omega}_d := \emptyset$
- (2) Take  $j = (j_1, j_2, \dots, j_d) \in P(d)$ . Define as follows:

$$\begin{aligned} d'_j &:= \max\{k \mid j_k > 0\}, \\ x_j &:= (1 \ 2 \ \dots \ j_1)(j_1 + 1 \ j_1 + 2 \ \dots \ j_1 + j_2) \cdots (\sum_{k=1}^{d'-1} j_k + 1 \ \dots \ d) \in S_d, \\ R_j &:= S_d \times \mathcal{E}_d. \end{aligned}$$

- (3) Take  $(y, \varepsilon) \in R_j$ . Apply Algorithm 6.1.1 to  $(x_j, y, \varepsilon) \in \tilde{\Omega}_d$  to get  $[(x_j, y, \varepsilon)]$ .
- (4) Add  $[(x_j, y, \varepsilon)]$  to  $C\tilde{\Omega}_d$ . After that, remove  $(y(\mathcal{O}), \varepsilon(\mathcal{O}))$  from  $R_j$  for every  $\mathcal{O} = (x_j, y(\mathcal{O}), \varepsilon(\mathcal{O})) \in [(x_j, y, \varepsilon)]$ .
- (5) Go back to (3) until  $R_j = \emptyset$ . If so, go to the next step.
- (6) Go back to (2) for other leftover  $j \in P(d)$ . When we have been through all elements in  $P(d)$ , finish the algorithm.

Next, we calculate the permutations  $\varphi_T, \varphi_S \in \text{Sym}(C\tilde{\Omega}_d)$  which correspond to  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})$  acting on  $\tilde{\Omega}_d$  as decomposing origamis into pairs of directions  $T(0, \frac{\pi}{2}) = (0, \frac{\pi}{4})$  and  $S(0, \frac{\pi}{2}) = (-\frac{\pi}{2}, 0)$ , respectively. Recall that the two automorphisms  $\gamma_T, \gamma_S \in \text{Aut}^+(F_2)$  in Lemma 4.4.3 are defined by:

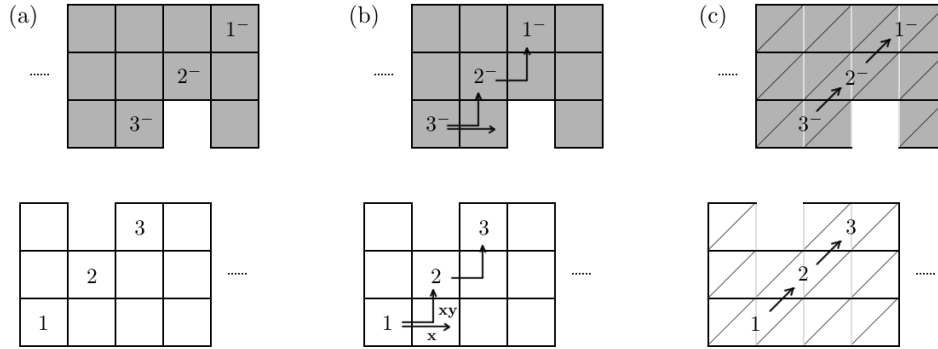
$$\gamma_T(x, y) = (x, xy), \quad \gamma_S(x, y) = (y, x^{-1}). \quad (6.2)$$

Let  $C\tilde{\Omega}_d$  be the output in Algorithm 6.1.2.

1. To obtain the permutation  $\varphi_T$ , we consider as follows:

$$(x, y, \varepsilon) \xrightarrow{\text{Def.5.1.3}} (\mathbf{x}, \mathbf{y}) \xrightarrow{\gamma_T \text{ conj.}} (\mathbf{x}_T, \mathbf{y}_T) \xrightarrow{\text{Lem.5.1.5}} (x_T, y_T, \varepsilon_T).$$

To apply Lemma 5.1.5 we calculate  $\varepsilon_{\mathbf{y}_T}$  and a cycle decomposition of  $\mathbf{y}_T$ . Remark that the decomposition into  $T(0, \frac{\pi}{2}) = (0, \frac{\pi}{4})$  is given by  $\gamma_T$  and the conjugation in  $(-i \mapsto ix^{-1}(i) \mid i \in I_d)$  as shown in Fig. 6.A.



**Fig. 6.A** Decomposition of origami (a) into  $T(0, \frac{\pi}{2}) = (0, \frac{\pi}{4})$ : The desired decomposition (c) is obtained from (b) applying  $\gamma_T$  and the conjugation in  $(-i \mapsto -x^{-1}(i) \mid i \in I_d)$ .

For  $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$ ,  $a \in I_d$ , and  $\varepsilon' \in \{\pm 1\}$ , we have:

$$\begin{aligned} \gamma_T(\mathbf{y}_{\mathcal{O}})(\varepsilon' a) &= \mathbf{y}_{\mathcal{O}} \circ \mathbf{x}_{\mathcal{O}}(\varepsilon' a) \\ &= \mathbf{y}_{\mathcal{O}}(\varepsilon' x^{\varepsilon'}(a)) \\ &= \varepsilon(\varepsilon' x^{\varepsilon'}(a)) \cdot y^{\varepsilon(\varepsilon' x^{\varepsilon'}(a))}(\varepsilon' x^{\varepsilon'}(a)) \cdot \varepsilon(y^{\varepsilon(\varepsilon' x^{\varepsilon'}(a))}(\varepsilon' x^{\varepsilon'}(a))) \\ &= \varepsilon' \varepsilon(x^{\varepsilon'}(a)) \cdot \varepsilon' y^{\varepsilon' \varepsilon(x^{\varepsilon'}(a))}(x^{\varepsilon'}(a)) \cdot \varepsilon' \varepsilon(y^{\varepsilon' \varepsilon(x^{\varepsilon'}(a))}(x^{\varepsilon'}(a))) \\ &= \varepsilon' \varepsilon(x^{\varepsilon'}(a)) \cdot \varepsilon(y^{\varepsilon' \varepsilon(x^{\varepsilon'}(a))}(x^{\varepsilon'}(a))) \cdot y^{\varepsilon' \varepsilon(x^{\varepsilon'}(a))}(x^{\varepsilon'}(a)). \end{aligned} \quad (6.3)$$

**Algorithm 6.1.3.** Let  $C = [(x, y, \varepsilon)] \in C\tilde{\Omega}_d$  be a restricted class. By (6.3), we obtain  $\varphi_T(C)$  in the following steps:

- (1)  $I'_d := I_d, j := 0$ .
- (2)  $a_{0,j} := \min(I'_d), \varepsilon'_{0,j} := 1, i := 0$ .
- (3)  $b_{i,j} := x^{\varepsilon'_{i,j}}(a_{i,j}), a'_{i+1,j} := y^{\varepsilon'_{i,j}\varepsilon(b_{i,j})}(b_{i,j}), \varepsilon'_{i+1,j} := \varepsilon'_{i,j}\varepsilon(b_{i,j})\varepsilon(a_{i+1,j})$ .
- (4) Remove  $a_{i,j}$  from  $I'_d$ . Define:

$$a_{i+1,j} := \begin{cases} a'_{i+1,j} & \text{if } \varepsilon'_{i+1,j} = 1, \\ x^{-1}(a'_{i+1,j}) & \text{otherwise.} \end{cases}$$

- (5) If  $a_{i+1,j} = a_{0,j}$ , let  $c_j := (a_{0,j} \ a_{1,j} \ \dots \ a_{i,j})$ . Otherwise, go back to (3) for the next  $i$ .
- (6) If  $I'_d \neq \emptyset$ , go back to (2) for the next  $j$ . Otherwise, finish the loop and let  $x_T = x$ ,  $y_T := c_1 c_2 \cdots c_j$ , and  $\varepsilon_T := (a_{i,j} \mapsto \varepsilon'_{i,j})$ .
- (7) Search for the isomorphism class  $C_T \in C\tilde{\Omega}_d$  represented by  $(x_T, y_T, \varepsilon_T)$  and we conclude that  $\varphi_T(C) = C_T$ .

2. To obtain the permutation  $\varphi_S$ , we consider as follows:

$$(x, y, \varepsilon) \xrightarrow{\text{Def.5.1.3}} (\mathbf{x}, \mathbf{y}) \xrightarrow{\gamma_S \text{ conj.}} (\mathbf{x}_S, \mathbf{y}_S) \xrightarrow{\text{Lem.5.1.5 conj.}} (x_S, y_S, \varepsilon_S).$$

We use two conjugators in  $\bar{S}_d$ : the former collects signs of cells in each vertical cylinder to apply Lemma 5.1.5, and the latter makes  $x_S$  to be the normalized form (6.1). The former conjugator is given by  $\sigma_\delta := (\pm i \mapsto \pm \delta(i)i \mid i \in I_d) \in \bar{S}_d$  where  $\delta \in \mathcal{E}_d$  satisfies that for every cycle  $c$  in  $\mathbf{x}$ ,  $\{\delta(|i|)i \mid i \in c\}$  forms a cycle either  $c$  or  $c'$ .

For  $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$ ,  $a \in I_d$ , and  $\delta' \in \{\pm 1\}$ , we have:

$$\begin{aligned} \gamma_S(\mathbf{x})(\delta' a) &= \mathbf{y}(\delta' a) \\ &= \varepsilon(\delta' a) \cdot y^{\varepsilon(\delta' a)}(\delta' a) \cdot \varepsilon(y^{\varepsilon(\delta' a)}(\delta' a)) \\ &= \delta' \varepsilon(a) \cdot \delta' y^{\delta' \varepsilon(a)}(a) \cdot \delta' \varepsilon(y^{\delta' \varepsilon(a)}(a)) \\ &= \delta' \varepsilon(a) \varepsilon(y^{\delta' \varepsilon(a)}(a)) \cdot y^{\delta' \varepsilon(a)}(a). \end{aligned} \tag{6.4}$$

**Algorithm 6.1.4.** By (6.4), we obtain  $\delta$  in the following steps:

- (1)  $I'_d := I_d, j := 0$ .
- (2)  $a_{0,j} := \min(I'_d), \delta_{0,j} := 1, i := 0$ .
- (3)  $a_{i+1,j} := y^{\delta_{i,j}\varepsilon(a_{i,j})}(a_{i,j}), \delta_{i+1,j} := \delta_{i,j}\varepsilon(a_{i,j})\varepsilon(a_{i+1,j})$

- (4) Remove  $a_{i,j}$  from  $I'_d$ .
- (5) If  $a_{i+1,j} = a_{0,j}$ , let  $c_j := (a_{0,j} \ a_{1,j} \ \dots \ a_{i,j})$ . Otherwise go back to (3) for the next  $i$ .
- (6) If  $I'_d \neq \emptyset$  then go back to (2) for the next  $j$ . Otherwise finish the loop and let  $x'_S := c_1 c_2 \cdots c_j$  and  $\delta := (a_{i,j} \mapsto \delta_{i,j})$ .

To apply Lemma 5.1.5, we will calculate  $\varepsilon_{\sigma_\delta^\# \mathbf{y}_S}$  and a cycle decomposition of  $\sigma_\delta^\# \mathbf{y}_S$ . After that, we apply the conjugator which makes  $x_S$  to the normalized form (6.1). So in advance, we will prepare the list  $\{\sigma^\# x_p \mid \sigma \in S_d\}$  equipped with information of conjugator for each  $p \in P(d)$ . Note that 'x's of any isomorphic two origamis share the same partition by Lemma 5.1.6. Hence the restricted classes calculated from Algorithm 6.1.1 with this list exhausts all the patterns of origamis.

For  $(x, y, \varepsilon) \in \tilde{\Omega}_d$ ,  $a \in I_d$  and  $\varepsilon' \in \{\pm 1\}$ , we have:

$$\begin{aligned}
 \delta^\#(\mathbf{y}_S)(\varepsilon' a) &= \delta(\mathbf{x}^{-1}(\delta(|\varepsilon' a|)\varepsilon' a)) \\
 &= \delta(|\mathbf{x}^{-1}(\varepsilon' \delta(a)a)|) \cdot \mathbf{x}^{-1}(\varepsilon' \delta(a)a) \\
 &= \varepsilon' \delta(a) \cdot \delta(x^{-\varepsilon' \delta(a)}(a)) \cdot x^{-\varepsilon' \delta(a)}(a).
 \end{aligned} \tag{6.5}$$

**Algorithm 6.1.5.** Let  $C = [(x, y, \varepsilon)] \in C\tilde{\Omega}_d$  be a restricted class. By (6.5), we obtain  $\varphi_S(C)$  in the following steps:

- (1)  $I'_d := I_d$ ,  $j := 0$ .
- (2)  $a_{0,j} := \min(I'_d)$ ,  $\varepsilon'_{0,j} := 1$ ,  $i := 0$
- (3) Remove  $a_{i,j}$  from  $I'_d$ . Let  $a_{i+1,j} := x^{-\varepsilon'_{i,j} \delta(a_{i,j})}(a_{i,j})$ ,  $\varepsilon'_{i+1,j} := \varepsilon'_{i,j} \delta(a_{i,j}) \delta(a_{i+1,j})$ .
- (4) If  $a_{i+1,j} = a_{0,j}$ , let  $c_j := (a_{0,j} \ a_{1,j} \ \dots \ a_{i,j})$ . Otherwise go back to (3) for the next  $i$ .
- (5) If  $I'_d \neq \emptyset$ , go back to (2) for the next  $j$ . Otherwise finish the loop and let  $x'_S := \delta^\# x_S$ ,  $y'_S := c_1 c_2 \cdots c_j$  and  $\varepsilon'_S := (a_{i,j} \mapsto \varepsilon'_{i,j})$ .
- (6) Search for the conjugator  $\sigma \in S_d$  such that  $\sigma^\# x'_S$  is of normalized form. Let  $(x_S, y_S, \varepsilon_S) := (\sigma^\# x'_S, \sigma^\# y'_S, \varepsilon'_S \circ \sigma^{-1})$ .
- (7) Search for the isomorphism class  $C_S \in C\tilde{\Omega}_d$  represented by  $(x_S, y_S, \varepsilon_S)$  and we conclude that  $\varphi_S(C) = C_S$ .

Finally, we present an algorithm for a simultaneous calculation of the Veech groups of origamis in  $\tilde{\Omega}_d$ .

**Algorithm 6.1.6.** Let  $\varphi_T, \varphi_S \in \text{Sym}(C\tilde{\Omega}_d)$ . We obtain the  $\langle \varphi_T^{-1}, \varphi_S^{-1} \rangle$ -orbit decomposition of  $C\tilde{\Omega}_d$  in the following steps.

- (1)  $I'_N := I_N$ .
- (2) For  $t \in \mathbb{N}$ ,  $O_t := \emptyset$ .
- (3) Take  $i \in I'_N$  and add  $i$  to  $O_t$ .
- (4) Take  $j \in O_t$  and let  $O(j) := \{\varphi_T^{-k}(j), \varphi_S^{-k}(j) \mid k \in \mathbb{N}\}$ .
- (5) Add all elements in  $O(j)$  to  $O_t$  and remove them from  $I'_N$ .
- (6) Go back to (4) for other leftover  $j \in O_t$ . When we have been through all elements in  $O_t$ , go to the next step.
- (7) Go back to (2) for the next  $t$  until  $I'_N = \emptyset$ . If so, finish the algorithm.

**Theorem 6.1.7** ([41]). *For each  $d \in \mathbb{N}$ , Algorithm 6.1.1-6.1.6 outputs the orbit decomposition of the action  $PSL(2, \mathbb{Z})$  on  $C\tilde{\Omega}_d$ . Moreover, for each origami  $\mathcal{O} \in \tilde{\Omega}_d$ , the Veech group is the stabilizer  $\text{Stab}_{PSL(2, \mathbb{Z})}[\mathcal{O}]$ .*

*Proof.* Let  $\mathcal{O} \in \tilde{\Omega}_d$ . As seen in Remark 4.4.6, the inclusion  $\Gamma(\mathcal{O}) < PSL(2, \mathbb{Z})$  induces a Belyi covering  $C(\mathcal{O}) = \mathbb{H}/\Gamma(\mathcal{O}) \rightarrow P^1_{\mathbb{C}}$ .  $PSL(2, \mathbb{Z})$  acts on  $\tilde{\Omega}_d$  by linear deformation of the natural coordinates of origamis, which respects the  $PSL(2, \mathbb{Z})$ -action on the Teichmüller disk  $D_{\mathcal{O}}$ . Since the two decompositions  $P(A\mathcal{O}, \Theta_0)$  and  $P(\mathcal{O}, A^{-1}\Theta_0)$  are equal for  $\Theta_0 = (0, \frac{\pi}{2})$  and every  $[A] \in PSL(2, \mathbb{Z})$ , the homomorphism

$$PSL(2, \mathbb{Z}) \rightarrow \text{Sym}(C\tilde{\Omega}_d) : ([T], [S]) \mapsto (\varphi_T^{-1}, \varphi_S^{-1}) \quad (6.6)$$

represents the projected action on  $C\tilde{\Omega}_d \hookrightarrow C_{\mathcal{O}}$ . Algorithm 6.1.1 specifies the equivalence class of each origami by Lemma 5.1.6. The last part follows from Corollary 5.3.1.  $\square$

Note that we may combine the Reidemeister-Schreier method [43, 51] with Algorithm 6.1.6 to obtain the list of generators and the list of representatives of the Veech group of each origami.

## 6.2 Teichmüller curves and Galois conjugacy

In the following, we show some calculation results obtained by the algorithms stated in the previous section. For each degree  $d$ , we number classes of origamis according to Algorithm 6.1.2 (i.e. lexicographic order with respect to permutations and signs). We first note that all classes representing disconnected origamis are removed from the results.

We describe Teichmüller curves in the same way to [51]. Teichmüller curves of origamis are coverings of  $\mathbb{H}/PSL(2, \mathbb{Z})$ , and we denote copies of the standard fundamental domain of  $PSL(2, \mathbb{Z})$  by isosceles triangles where the keen vertices correspond to the cusp. Every two edges with the same symbol are glued so that the cusps match. Every edge with no symbol is glued individually, making a conical point of angle  $\pi/2$ .

Fig.6.B and Fig.6.C show all classes of origamis of degree 4 and their positions in Teichmüller curves. There are 26 classes of abelian origamis summing up to 5 components and 34 classes of non-abelian origamis summing up to 6 components.

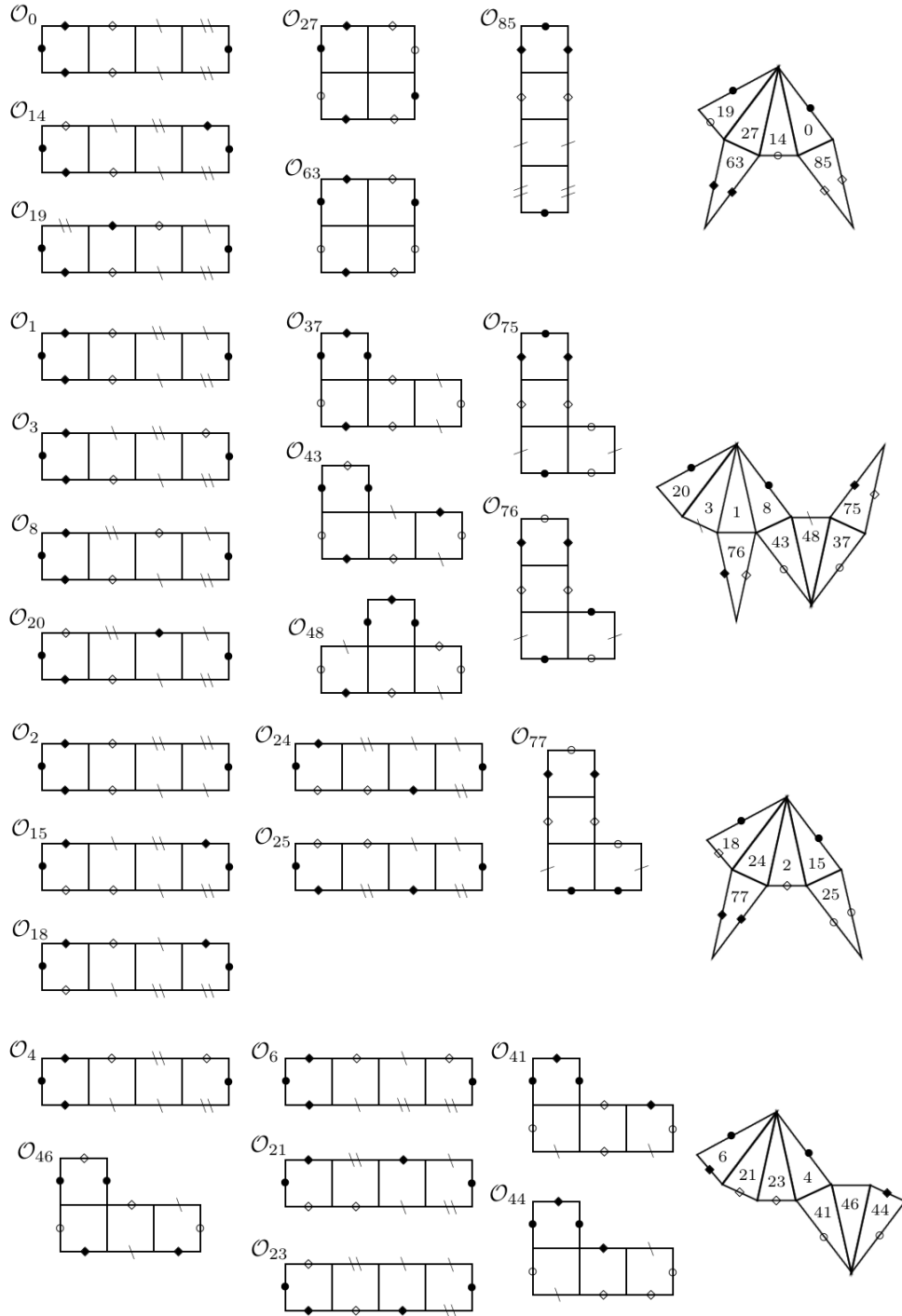
Table.6.D shows the number of classes of origamis, the number of components of Teichmüller curves, the range of genus of Teichmüller curves, and the number of classes of possible  $G_{\mathbb{Q}}$ -conjugacy for degree  $1 \leq d \leq 7$ . Here the possibility of  $G_{\mathbb{Q}}$ -conjugacy is checked by the information of degree, genus, valency list, and stratum of origamis.

	abelian				non-abelian			
$d$	$\#\{\mathcal{O}\}$	$\#\{C(\mathcal{O})\}$	$g(C(\mathcal{O}))$	possible $G_{\mathbb{Q}}$ -conjugacy	$\#\{\mathcal{O}\}$	$\#\{C(\mathcal{O})\}$	$g(C(\mathcal{O}))$	possible $G_{\mathbb{Q}}$ -conjugacy
1	1	1	0	none	0	0	0	none
2	2	1	0	"	1	1	0	"
3	7	2	0	"	4	1	0	"
4	26	5	0	"	34	6	0	"
5	91	8	0	"	227	13	0	"
6	490	28	0	1 class	2316	88	0	13 classes
7	2773	41	0 ~ 1	5 classes	26586	88	0 ~ 11	3 classes

Table 6.D Summary of the result for degree  $1 \leq d \leq 7$

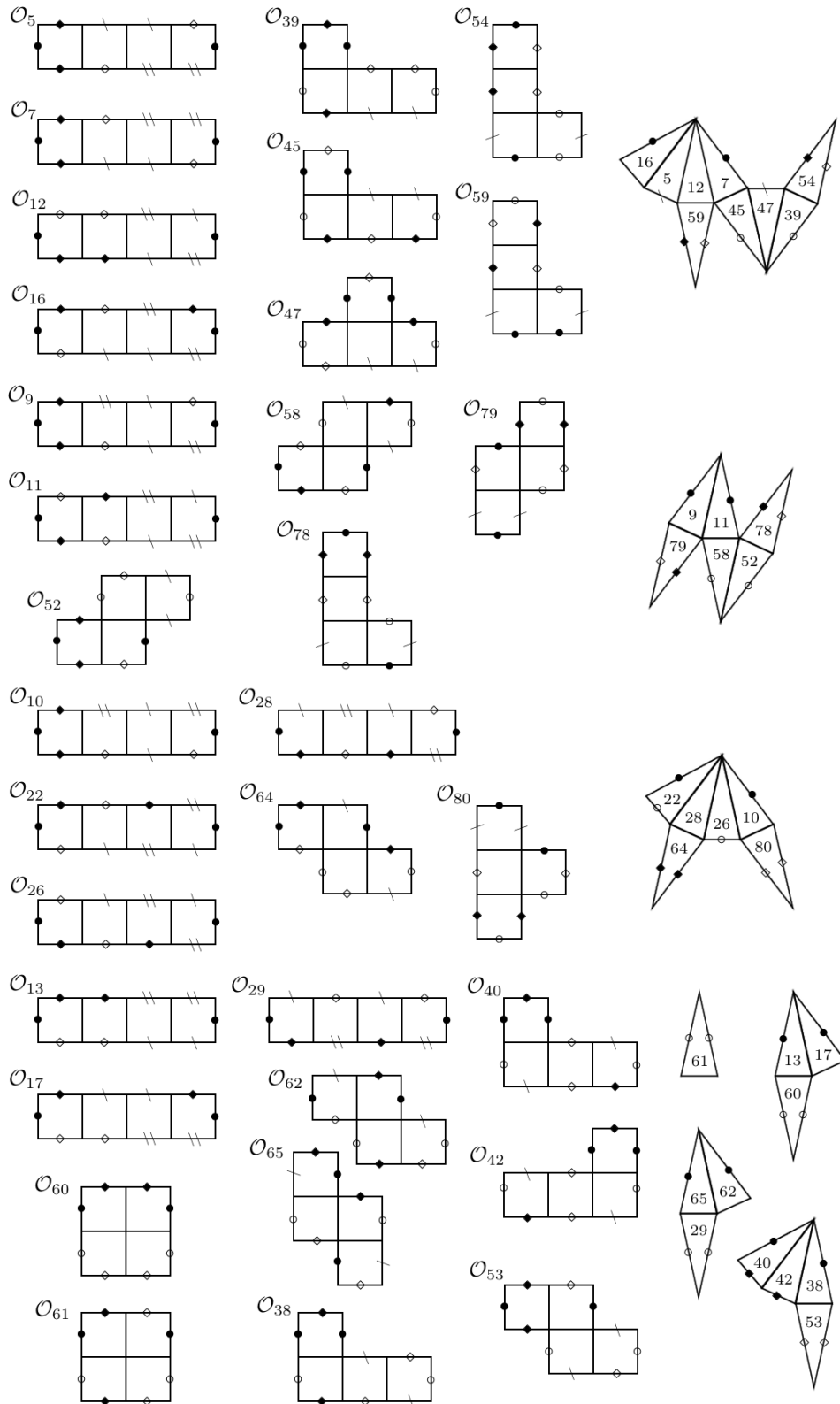
**Theorem 6.2.1.** *All Teichmüller curves induced from origamis of degree  $d \leq 7$  except for the 13 cases in Table.6.E and the 9 cases in Table.6.G are distinguished by Galois invariants. Fig.6.F and Fig.6.H shows origamis that induce Teichmüller curves in each of the exceptional cases.*

**Remark 6.2.2.** The mirror relation implies a Galois conjugacy which induces complex conjugacy. The situation ‘one pair of mirror-symmetric curves, mirroring each other’ is caused by such a Galois conjugacy modifying only the embeddings of Teichmüller curves into the moduli space.

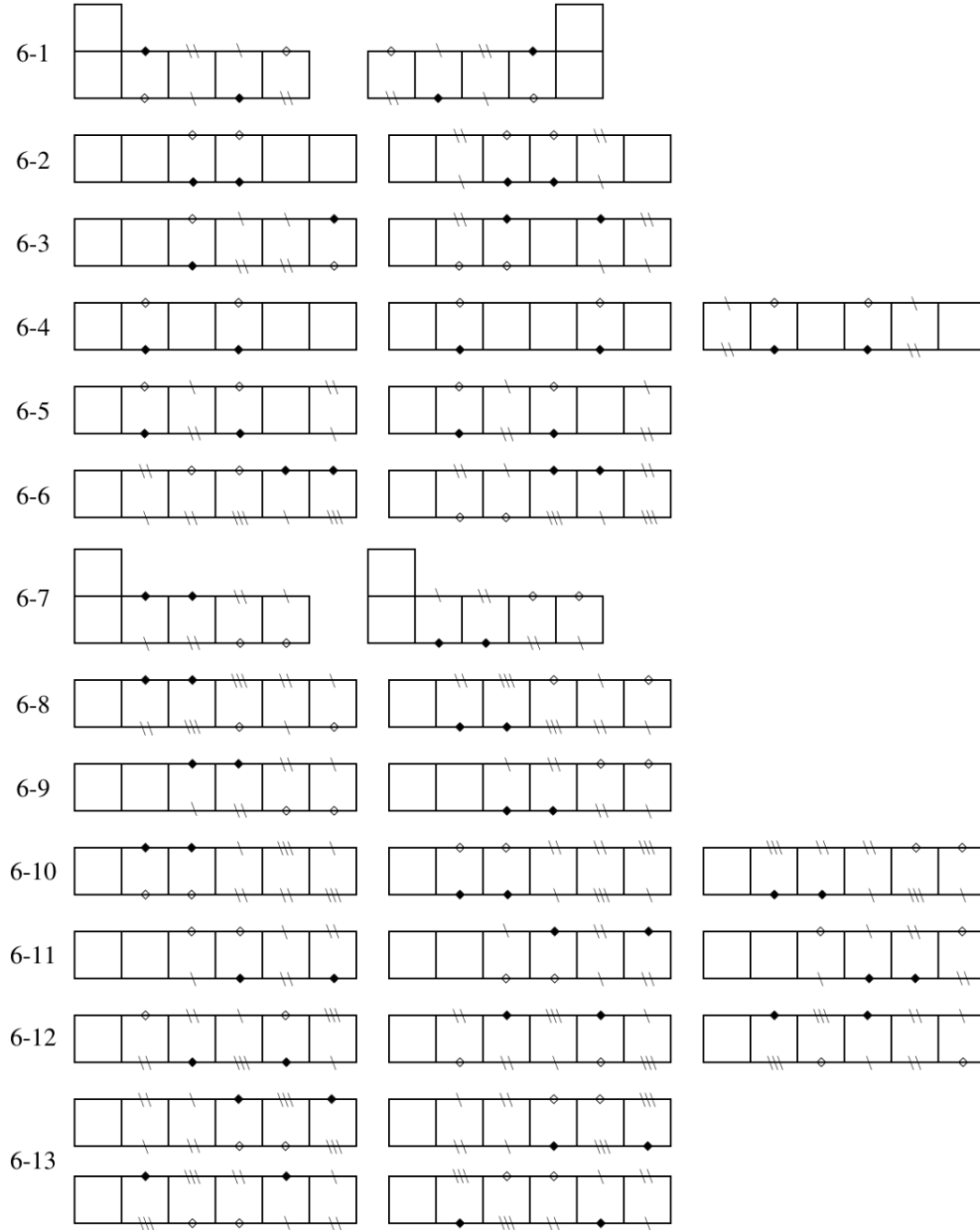


**Fig. 6.B** (Part 1/2) All classes of origamis of degree 4 and their positions in Teichmüller curves.

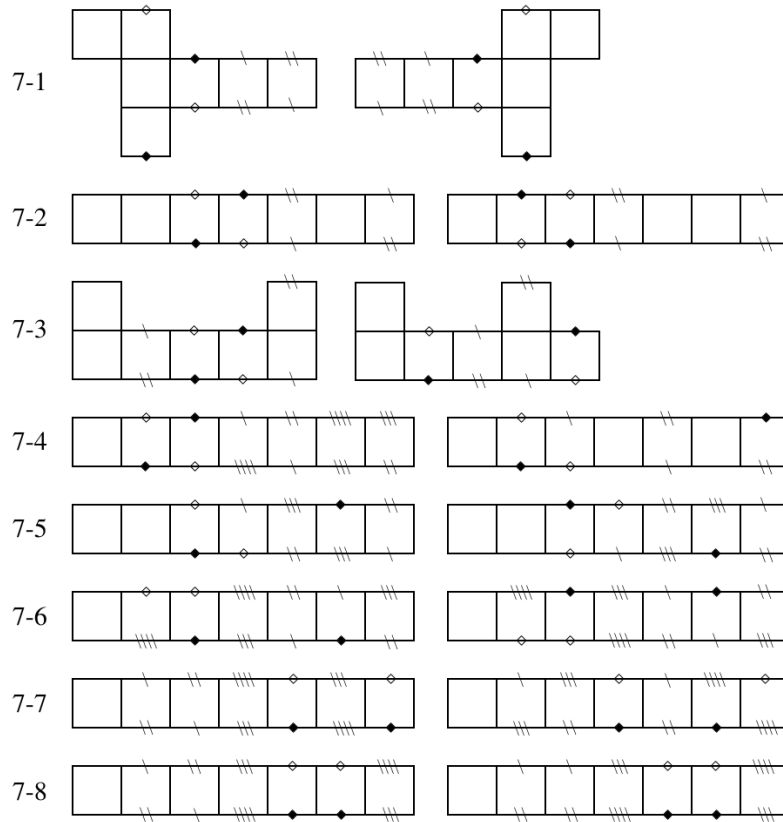




**Fig. 6.C** (Part 2/2) All classes of origamis of degree 4 and their positions in Teichmüller curves.



**Fig. 6.F** Origamis that induce Teichmüller curves in Table.6.E: unmarked edges are glued with the opposite.



**Fig. 6.H** Origamis that induce Teichmüller curves in Table.6.G; unmarked edges are glued with the opposite.

No.	stratum	index	valency list of $C(\mathcal{O})$	relationship between $C(\mathcal{O})$
6-1	$\mathcal{Q}_3^a(0, 8)$	15	$(3^5   2^7, 1   5, 4, 3^2)$	one pair of mirror-symmetric curves, mirroring each other
6-2	$\mathcal{Q}_1^p(-1^2, 0^3, 2)$	12	$(3^4   2^6   6, 3, 2, 1)$	two identical, mirror-closed curves
6-3	$\mathcal{Q}_2^p(-1^2, 0, 6)$	12	$(3^4   2^6   6, 3, 2, 1)$	two identical, mirror-closed curves
6-4	$\mathcal{Q}_2^p(0^2, 2^2)$	12	$(3^4   2^6   6, 3, 2, 1)$	three identical, mirror-closed curves
6-5	$\mathcal{Q}_3^p(2, 6)$	12	$(3^4   2^6   6, 3, 2, 1)$	two identical, mirror-closed curves
6-6	$\mathcal{Q}_2^p(-1^2, 3^2)$	15	$(3^5   2^7, 1   6, 5, 3, 1)$	two distinct, mirror-closed curves
6-7	$\mathcal{Q}_2^p(-1^2, 3^2)$	15	$(3^5   2^7, 1   5, 4, 3^2)$	one pair of mirror-symmetric curves, mirroring each other
6-8	$\mathcal{Q}_3^p(-1, 9)$	22	$(3^7, 1   2^{11}   6, 5, 4^2, 3)$	one pair of mirror-symmetric curves, mirroring each other
6-9	$\mathcal{Q}_2^p(-1^2, 0, 6)$	24	$(3^8   2^{12}   6, 5, 4^2, 3, 2)$	one mirror-conjugate pair
6-10	$\mathcal{Q}_2^p(-1^3, 7)$	27	$(3^9   2^{13}, 1   6^2, 5, 4, 3^2)$	one mirror-conjugate pair & one mirror-closed curve
6-11	$\mathcal{Q}_2^p(-1, 0, 1, 4)$	36	$(3^{12}   2^{18}   6^2, 5^2, 4^2, 3^2)$	one mirror-conjugate pair & one mirror-closed curve
6-12	$\mathcal{Q}_3^p(1, 7)$	54	$(3^{18}   2^{27}   6^4, 5^3, 4^3, 3)$	one mirror-conjugate pair & one mirror-closed curve
6-13	$\mathcal{Q}_3^p(-1, 9)$	66	$(3^{22}   2^{33}   6^6, 5^3, 4^3, 3)$	two mirror-conjugate pairs

 Table 6.E Classes of possible  $G_{\mathbb{Q}}$ -conjugacy for degree 6

No.	stratum	index	valency list of $C(\mathcal{O})$	relationship between $C(\mathcal{O})$
7-1	$\mathcal{Q}_4^a(12)$	7	$(3^2, 1   2^3, 1   4, 3)$	one pair of mirror-symmetric curves, mirroring each other
7-2	$\mathcal{Q}_3^a(0, 2, 6)$	16	$(3^5, 1   2^8   7, 4, 3, 2)$	two distinct, mirror-closed curves
7-3	$\mathcal{Q}_4^a(12)$	21	$(3^7   2^{11}   6, 5, 4, 3^2)$	two distinct, mirror-closed curves
7-4	$\mathcal{Q}_4^a(12)$	42	$(3^{14}   2^{21}   7^2, 5^2, 4^3, 3^2)$	two distinct, mirror-closed curves
7-5	$\mathcal{Q}_3^a(0, 2, 6)$	48	$(3^{16}   2^{24}   7^2, 6, 5^2, 4^3, 3^2)$	one mirror-conjugate pair
7-6	$\mathcal{Q}_2^p(-1, 1^3, 2)$	16	$(3^5, 1   2^8   7, 6, 2, 1)$	one mirror-conjugate pair
7-7	$\mathcal{Q}_4^p(12)$	28	$(3^9, 1   2^{14}   7^2, 6, 3^2, 2)$	two distinct, mirror-closed curves
7-8	$\mathcal{Q}_3^p(-1^2, 10)$	36	$(3^{12}   2^{18}   7^3, 6, 3^2, 2, 1)$	two distinct, mirror-closed curves

 Table 6.G Classes of possible  $G_{\mathbb{Q}}$ -conjugacy for degree 7

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# Appendix: Program codes

The following is the implementation of Algorithms in Section 6.1 that was used to obtain the calculation results in Section 6.2. The program codes are powered by Python (<https://www.python.org/>). Files including program codes and calculation results have been made publicly available at GitHub (<https://github.com/ShunKumagai/origami>). The program codes were originally written by the author, and some of it was arranged with the help of the staff in Cyber Science Center(<https://www.cc.tohoku.ac.jp/>), Tohoku University.

Program1 (Algorithm 6.1.1 and 6.1.2): classification of all patterns of origamis in equivalence classes.

Before running the program, input the degree  $d$  (line 18) and remove the line breaks at ‘#partition data’ (line 29-43).

Program1.py

---

```
1  #coding: UTF-8
2  import time
3  t1=time.time()
4  t0=time.time()
5
6  import itertools as it
7  import numpy as np
8  import math
9  import pickle
10 import copy
11 from functools import reduce
12 import multiprocessing
13 from multiprocessing import Pool
14 from concurrent.futures import ProcessPoolExecutor
15 from concurrent.futures import ThreadPoolExecutor
16
17 #input
18 d=3
19
20 #partition data: remove line breaks for d=>5
21 N0=[[0]],[[1]]
22 #d=2
23 N0.append([ [2]],[[1,1]] )
```

```

24 #d=3
25 N0.append([ [[3]], [[1,2]], [[1,1,1]] ])
26 #d=4
27 N0.append([ [[4]], [[1,3],[2,2]], [[1,1,2]], [[1,1,1,1]] ])
28 #d=5
29 N0.append([ [[5]], [[1,4],[2,3]], [[1,1,3],[1,2,2]], [[1,1,1,2]],
30             [[1,1,1,1,1]] ])
31 #d=6
32 N0.append([ [[6]], [[1,5],[2,4],[3,3]], [[1,1,4],[1,2,3],[2,2,2]],
33             [[1,1,1,3],[1,1,2,2]], [[1,1,1,1,2]], [[1,1,1,1,1,1]] ])
34 #d=7
35 N0.append([ [[7]], [[1,6],[2,5],[3,4]], [[1,1,5],[1,2,4],[1,3,3],
36             [2,2,3]], [[1,1,1,4],[1,1,2,3],[1,2,2,2]], [[1,1,1,1,3],
37             [1,1,1,2,2]], [[1,1,1,1,1,2]], [[1,1,1,1,1,1,1]] ])
38 #d=8
39 N0.append([ [[8]], [[1,7],[2,6],[3,5],[4,4]], [[1,1,6],[1,2,5],
40             [1,3,4],[2,2,4],[2,3,3]], [[1,1,1,5],[1,1,2,4],[1,1,3,3],
41             [1,2,2,3]], [[1,1,1,1,4],[1,1,1,2,3],[1,1,2,2,2]],
42             [[1,1,1,1,1,3],[1,1,1,1,2,2]], [[1,1,1,1,1,1,2]],
43             [[1,1,1,1,1,1,1,1]] ])
44
45 np.set_printoptions(threshold=np.inf)#Do not always omit print of
    numpy matrices
46
47
48 def indices(A):
49     return range(len(A))
50 def M(x):
51     return np.identity(len(x), dtype=int)[: , x]
52 def inv(A):
53     return np.swapaxes(A, -1, -2)
54 def iM(x):
55     return inv(M(x))
56 def get_time():
57     t=time.time()
58     dt=(t-t0)
59     print("total time:",dt)
60 def finish():
61     t=time.time()
62     dt=(t-t0)
63     print("total time:",dt)
64     exit()
65
66 def cycle(v):
67     rest=[i for i in indices(v)]
68     cv=[]
69     while len(rest)!=0:
70         cvi=[rest[0]]
71         rest.remove(rest[0])
72         j=0
73         while True:

```

```

74         cvi.append(v[cvi[j]])
75     try:
76         rest.remove(v[cvi[j]])
77     except ValueError:
78         pass
79     j=j+1
80     if cvi[j]==cvi[0]:
81         cvi.pop(j)
82         break
83     cv.append(cvi)
84     return cv
85
86 def icycle(c,L="no data"):
87     if L=="no data":L=sum([len(c[i]) for i in indices(c)])
88     v=[i for i in range(L)]
89     for i in indices(c):
90         for j in range(len(c[i])-1):
91             v[c[i][j]]=c[i][j+1]
92             v[c[i][len(c[i])-1]]=c[i][0]
93     return v
94
95 def Sym(d):
96     a = np.identity(d, dtype='i')
97     S = np.array([a[np.array(idx)] for idx in it.permutations(
range(d))])
98     return S
99
100 def represent(X,d):
101     p=len(X)
102     N=np.asarray([len(x) for x in X])
103     p0=sum(N)
104     Y=[]
105     for i in range(p):
106         for j in range(N[i]):
107             Xij=[ np.asarray([0]) if X[i][j][k]==1 else np.
asarray([s+1 for s in range(X[i][j][k]-1)]+[0]) for k in range(
i+1)]
108             Dij=[0]+[len(Xij[k]) for k in range(i)]
109             Eij=np.asarray([sum(Dij[0:k+1]) for k in range(i+1)])
110             Yij=np.concatenate([Xij[k]+Eij[k] for k in range(i+1)
])
111             Y.append(Yij)
112     return np.asarray(Y)
113
114 def Sign(d):
115     return np.asarray([(t//(2**s))%2 for s in range(d)] for t in
range(2**d)])
116 def vinv(v):
117     return M(v).dot(e)
118 def conjugate(sigma,x,invsigma=None):
119     if invsigma is None:

```

```

120         return np.array([sigma[ x[vinv(sigma)[j]] ] for j in
range(d)])
121     else:
122         return np.array([sigma[ x[invsigma[j]] ] for j in range(d
)])
123
124 def ytosign(j,sign):#positive if sign=0 or 'False', negative if
sign=1 or 'True'
125     return Vd[j] if sign == 1 else iVd[j]
126 def xtosign(i,sign):#positive if sign=0 or 'False', negative if
sign=1 or 'True'
127     return Xrep[i] if sign == 1 else iXrep[i]
128 def vtosign(v,sign):
129     return v if sign == 1 else vinv(v)
130 def vtosigns(v,signs):
131     return np.array([vtosign(v,not signs[m])[m] for m in Id])
132 def xi(y,e):
133     return np.array([e[m]!=e[vtosign(y,not e[m])[m]] for m in Id
])
134
135 #Algorithm 6.1.1
136 def isom_sub(i, j, k):
137     ret=[]
138     x=Xrep[i]
139     y=Vd[j]
140     invy=vinv(y)
141     yi=iVd[j]
142     eps=Signd[k]
143     for nd in range(lSignd):
144         delta=Signd[nd]
145         vtosign_x=[vtosign(x,not delta[m])[m] for m in Id]
146         if np.any(delta!=np.array([delta[x[m]] for m in Id])):
147             continue
148         for ns in range(lVd):
149             sigma=Vd[ns]
150             isigma=iVd[ns]
151             invsigma = vinv(sigma)
152             if np.any(conjugate(sigma,vtosign_x,invsigma)!=x):
153                 continue
154             Yeesd=np.array([conjugate(sigma,[invy[m] if (eps[m]+
Signd[n][sigma[m]]+delta[m])%2 else y[m] for m in Id], invsigma
) for n in indices(Signd)])
155
156             NYeesd=[NVd[np.all(Vd==Yeesd[n],axis=1)] for n in
NSignd]
157             exNSignd=NSignd[np.array([len(NYeesd[n])!=0 for n in
NSignd])]
158             eta=np.array([np.all(1-xi(Yeesd[n],[delta[isigma[m]
]]+eps[isigma[m]]+Signd[n][m])%2 for m in Id))] for n in
exNSignd])
159             trueNSignd=exNSignd[eta]

```

```

160         if len(trueNSignd) > 0:
161             ret.extend([[i,NYeesd[n][0],n] for n in
trueNSignd])
162         return ret
163
164 #Algorithm 6.1.2
165 def classify(i):
166     rest=np.concatenate([[[j,k]for k in NSignd]for j in NVd])
167     NYE0i=[]
168     while len(rest)>0:
169         ye=[rest[0][0],rest[0][1]]
170         isom_pre2 = isom_sub(i, ye[0], ye[1])
171         Isom=np.unique(isom_pre2,axis=0)
172         NYE0i.append(Isom)
173         rest=rest[[np.all(np.any(ye1!=Isom[:,1:],axis=1)) for ye1
in rest]]
174     return NYE0i
175
176 Signd=Sign(d)
177 NSignd=np.arange(len(Signd))
178
179 S=Sym(d)
180 iS = inv(S)
181
182 e=np.arange(d)#[0,1,2,...,d-1]
183
184 Id=e
185 Vd=S.dot(e)
186 iVd=iS.dot(e)
187 NVd=np.arange(len(Vd))
188
189 Xrep=represent(N0[d],d)
190 iXrep=np.asarray([vinv(x) for x in Xrep])
191 X1=np.concatenate([NVd[np.all(Vd==x,axis=1)] for x in Xrep])
192 NX1=np.arange(len(X1))
193
194
195 lVd=len(Vd)
196 lSignd=len(Signd)
197
198 #multiprocessing
199 n_process=1
200 n_thread=1
201
202 if __name__ == "__main__":
203     print(multiprocessing.cpu_count())
204     p = Pool(n_process)
205     NYE0 = p.map(classify, NX1)
206     p.close()
207     t=time.time()
208     dt=(t-t0)

```

```
209     NYE1=[np.array([[i,nye0[0][1],nye0[0][2]] for nye0 in NYE0[i
    ]]) for i in NX1]
210     YE1=[np.array([[Xrep[i],Vd[nye0[0][1]],Signd[nye0[0][2]]] for
    nye0 in NYE0[i]]) for i in NX1]
211     lYE1=[len(NYE1[i]) for i in NX1]
212     CNYE0=np.concatenate(NYE0)
213     NCO=len(CNYE0)
214     CNYE1=np.concatenate(NYE1)
215     CNNYE1=np.array([i for i in indices(CNYE0)])
216     CYE0=[[np.array([Xrep[a[0]],Vd[a[1]],Signd[a[2]]]) for a in c
    ] for c in CNYE0]
217     t = time.localtime()
218     fname = str(t.tm_mon)+str(t.tm_mday)+str(t.tm_hour)+str(t.
    tm_min)
219     with open('data_d={0}.txt'.format(d), 'w') as f:
220         print('#d={0}'.format(d), file=f)
221         print('import numpy as np'.format(d), file=f)
222         print('NYE0=',NYE0, file=f)
223         t=time.time()
224         dt=(t-t0)
225         print("#total time:",dt,file=f)
226         finish()
```

---

Example of output of Program1.py (input:d=3) is as follows.

data\_d=3.txt

---

```
1 #d=3
2 import numpy as np
3 NYE0= [[np.array([[0, 0, 0],[0, 0, 1],[0, 0, 2],[0, 0, 3],[0, 0,
    4],[0, 0, 5],[0, 0, 6],[0, 0, 7]]),
4     np.array([[0, 1, 0],[0, 1, 1],[0, 1, 6],[0, 1, 7],[0, 2,
    0],[0, 2, 3],[0, 2, 4],[0, 2, 7],[0, 5, 0],[0, 5, 2],[0, 5,
    5],[0, 5, 7]]),
5     np.array([[0, 1, 2],[0, 1, 3],[0, 1, 4],[0, 1, 5],[0, 2,
    1],[0, 2, 2],[0, 2, 5],[0, 2, 6],[0, 5, 1],[0, 5, 3],[0, 5,
    4],[0, 5, 6]]),
6     np.array([[0, 3, 0],[0, 4, 7]]),
7     np.array([[0, 3, 1],[0, 3, 2],[0, 3, 4],[0, 4, 3],[0, 4,
    5],[0, 4, 6]]),
8     np.array([[0, 3, 3],[0, 3, 5],[0, 3, 6],[0, 4, 1],[0, 4,
    2],[0, 4, 4]]),
9     np.array([[0, 3, 7],[0, 4, 0]]),
10    [np.array([[1, 0, 0],[1, 0, 1],[1, 0, 2],[1, 0, 3],[1, 0,
    4],[1, 0, 5],[1, 0, 6],[1, 0, 7]]),
11    np.array([[1, 1, 0],[1, 1, 1],[1, 1, 6],[1, 1, 7]]),
12    np.array([[1, 1, 2],[1, 1, 3],[1, 1, 4],[1, 1, 5]]),
13    np.array([[1, 2, 0],[1, 2, 1],[1, 2, 2],[1, 2, 3],[1, 2,
    4],[1, 2, 5],[1, 2, 6],[1, 2, 7],[1, 5, 0],[1, 5, 1],[1, 5,
    2],[1, 5, 3],[1, 5, 4],[1, 5, 5],[1, 5, 6],[1, 5, 7]])],
```

---

```

14     np.array([[1, 3, 0],[1, 3, 1],[1, 3, 6],[1, 3, 7],[1, 4,
0],[1, 4, 1],[1, 4, 6],[1, 4, 7]]),
15     np.array([[1, 3, 2],[1, 3, 3],[1, 3, 4],[1, 3, 5],[1, 4,
2],[1, 4, 3],[1, 4, 4],[1, 4, 5]]), [np.array([[2, 0,
0],[2, 0, 1],[2, 0, 2],[2, 0, 3],[2, 0, 4],[2, 0, 5],[2, 0,
6],[2, 0, 7]]),
16     np.array([[2, 1, 0],[2, 1, 1],[2, 1, 2],[2, 1, 3],[2, 1,
4],[2, 1, 5],[2, 1, 6],[2, 1, 7],[2, 2, 0],[2, 2, 1],[2, 2,
2],[2, 2, 3],[2, 2, 4],[2, 2, 5],[2, 2, 6],[2, 2, 7],[2, 5,
0],[2, 5, 1],[2, 5, 2],[2, 5, 3],[2, 5, 4],[2, 5, 5],[2, 5,
6],[2, 5, 7]]),
17     np.array([[2, 3, 0],[2, 3, 1],[2, 3, 2],[2, 3, 3],[2, 3,
4],[2, 3, 5],[2, 3, 6],[2, 3, 7],[2, 4, 0],[2, 4, 1],[2, 4,
2],[2, 4, 3],[2, 4, 4],[2, 4, 5],[2, 4, 6],[2, 4, 7]])]

```

---

Program2 (Algorithm 6.1.3, 6.1.4, 6.1.5, and 6.1.6): Calculation of the action of  $PSL(2, \mathbb{Z})$  on  $\tilde{\Omega}_d$  and its orbit decomposition. Finally, it output the data of Galois invariants of origami.

Before running the Program2, do the following: Input the degree  $d$  (line 13). Remove the line breaks at '#partition data' (line 26-41). Replace every string "array" in file data\_d=\*.txt with "asarray", change the filename extension to ".py", and place it in the same directory as the program.

Program2.py

---

```

1  import time
2  t1=time.time()
3  t0=time.time()
4
5  import itertools as it
6  import numpy as np
7  import math
8  import pickle
9  import copy
10 from functools import reduce
11
12 #input
13 d=3
14 import importlib
15 from importlib import import_module
16 data = import_module( 'data_d={}'.format(d))
17
18 #partition data: remove line breaks for d=>5
19 N0=[[[]],[[1]]]
20 #d=2
21 N0.append([ [[2]],[[1,1]] ])
22 #d=3
23 N0.append([ [[3]],[[1,2]],[[1,1,1]] ])

```

```

24 #d=4
25 N0.append([[[4]], [[1, 3], [2, 2]], [[1, 1, 2]], [[1, 1, 1, 1]]])
26 #d=5
27 N0.append([[[5]], [[1, 4], [2, 3]], [[1, 1, 3], [1, 2, 2]], [[1, 1, 1, 2]],
28             [[1, 1, 1, 1, 1]]])
29 #d=6
30 N0.append([[[6]], [[1, 5], [2, 4], [3, 3]], [[1, 1, 4], [1, 2, 3], [2, 2, 2]],
31             [[1, 1, 1, 3], [1, 1, 2, 2]], [[1, 1, 1, 1, 2]], [[1, 1, 1, 1, 1, 1]]])
32 #d=7
33 N0.append([[[7]], [[1, 6], [2, 5], [3, 4]], [[1, 1, 5], [1, 2, 4], [1, 3, 3],
34             [2, 2, 3]], [[1, 1, 1, 4], [1, 1, 2, 3], [1, 2, 2, 2]], [[1, 1, 1, 1, 3],
35             [1, 1, 1, 2, 2]], [[1, 1, 1, 1, 1, 2]], [[1, 1, 1, 1, 1, 1, 1]]])
36 #d=8
37 N0.append([[[8]], [[1, 7], [2, 6], [3, 5], [4, 4]], [[1, 1, 6], [1, 2, 5],
38             [1, 3, 4], [2, 2, 4], [2, 3, 3]], [[1, 1, 1, 5], [1, 1, 2, 4], [1, 1, 3, 3],
39             [1, 2, 2, 3]], [[1, 1, 1, 1, 4], [1, 1, 1, 2, 3], [1, 1, 2, 2, 2]],
40             [[1, 1, 1, 1, 1, 3], [1, 1, 1, 1, 2, 2]], [[1, 1, 1, 1, 1, 1, 2]],
41             [[1, 1, 1, 1, 1, 1, 1, 1]]])
42
43 np.set_printoptions(threshold=np.inf)#Do not always omit print of
    numpy matrices
44
45 def lcm_base(x, y):
46     return (x * y) // math.gcd(x, y)
47 def lcm(*numbers):
48     return reduce(lcm_base, numbers, 1)
49 def lcm_list(numbers):
50     return reduce(lcm_base, numbers, 1)
51 def indices(A):
52     return range(len(A))
53 def Cdelta(a,b):
54     D=1 if a==b else 0
55     return D
56 def M(x):
57     l=len(x)
58     return np.asarray([[Cdelta(j,x[i])for i in range(l)] for j in
59                         range(l)])
60 def inv(A):
61     l=len(A)
62     return np.asarray([[A[i][j] for i in range(l)] for j in range(l)
63                         ]])
64 def iM(x):
65     return inv(M(x))
66 def ttime():
67     t=time.time()
68     dt=(t-t0)
69     print("total time:",dt)
70 def finish():
71     t=time.time()
72     dt=(t-t0)
73     print("total time:",dt)

```



```

72     exit()
73
74 def cycle(v):
75     rest=[i for i in indices(v)]
76     cv=[]
77     while len(rest)!=0:
78         cvi=[rest[0]]
79         rest.remove(rest[0])
80         j=0
81         while True:
82             cvi.append(v[cvi[j]])
83             try:
84                 rest.remove(v[cvi[j]])
85             except ValueError:
86                 pass
87             j=j+1
88             if cvi[j]==cvi[0]:
89                 cvi.pop(j)
90                 break
91         cv.append(cvi)
92     return cv
93 def icycle(c,L="no data"):
94     if L=="no data":L=sum([len(c[i]) for i in indices(c)])
95     v=[i for i in range(L)]
96     for i in indices(c):
97         for j in range(len(c[i])-1):
98             v[c[i][j]]=c[i][j+1]
99             v[c[i][len(c[i])-1]]=c[i][0]
100     return v
101
102 def Sym(d):
103     S=np.asarray([[[Cdelta(i,s[j])for i in range(d)] for j in range
104     (d)] for s in list(it.permutations(list(range(d))))])
105     return S
106 def represent(X,d):
107     p=len(X)
108     N=np.asarray([len(x) for x in X])
109     p0=sum(N)
110     Y=[]
111     for i in range(p):
112         for j in range(N[i]):
113             #Xij[k]=[0] if X[i][j][k]==1 else np.concatenate([s+1
114             for s in range(X[i][j][k]-1)], [0])
115             Xij=[ np.asarray([0]) if X[i][j][k]==1 else np.asarray
116             ([s+1 for s in range(X[i][j][k]-1)]+[0]) for k in range(i+1)]
117             Dij=[0]+[len(Xij[k]) for k in range(i)]
118             Eij=np.asarray([sum(Dij[0:k+1]) for k in range(i+1)])
119             Yij=np.concatenate([Xij[k]+Eij[k] for k in range(i+1)
120             ])
121         Y.append(Yij)
122     return np.asarray(Y)

```

```

119
120 def Sign(d):
121     return np.asarray([[t//(2**s))%2 for s in range(d)] for t in
        range(2**d)])
122
123 Signd=Sign(d)
124 NSignd=np.arange(len(Signd))
125 e=np.arange(d)#[0,1,2,...,d-1]
126 S=Sym(d)
127 iS=np.asarray([inv(A) for A in S])
128
129
130 def vinv(v):
131     return M(v).dot(e)
132
133 Id=e
134 Vd=S.dot(e)
135 iVd=iS.dot(e)
136 NVd=np.arange(len(Vd))
137
138 Xrep=represent(N0[d],d)
139 iXrep=np.asarray([vinv(x) for x in Xrep])
140 X1=np.concatenate([NVd[np.all(Vd==x,axis=1)] for x in Xrep])
141 NX1=np.arange(len(X1))
142
143 def conjugate(v,x):
144     return ([v[ x[vinv(v)[j]] ] for j in range(d)])
145 Xclass=[[conjugate(v,x) for v in Vd] for x in Xrep]
146
147 def ytosign(j,sign):#sign=0,は正、falsesign=1,は負として処理 true
148     return Vd[j] if sign == 1 else iVd[j]
149 def xtosign(i,sign):#sign=0,は正、falsesign=1,は負として処理 true
150     return Xrep[i] if sign == 1 else iXrep[i]
151 def vtosign(v,sign):
152     return v if sign == 1 else vinv(v)
153
154 def bx(x,y,eps,j):
155     pm=j//d
156     i=j%d
157     bx=[pm, vtosign(x,1-pm)[i]]
158     return d*bx[0]+bx[1]
159 def ibx(x,y,eps,j):
160     pm=j//d
161     i=j%d
162     ibx=[pm, vtosign(x,pm)[i]]
163     return d*ibx[0]+ibx[1]
164 def by(x,y,eps,j):
165     pm=j//d
166     i=j%d
167     by=[(pm+eps[i]+eps[vtosign(y,eps[i]==pm)[i]])%2,vtosign(y,eps[
        i]==pm)[i]]

```

```

168     return d*by[0]+by[1]
169 def iby(x,y,eps,j):
170     pm=j//d
171     i=j%d
172     iby=[(pm+eps[i]+eps[vtsign(y,eps[i]!=pm)[i]])%2,vtsign(y,eps
173           [i]!=pm)[i]]
174     return d*iby[0]+iby[1]
175 NYE=np.concatenate([[[j,k] for k in NSignd] for j in NVd])
176 YE=np.array([[ Vd[nye[0]],Signd[nye[1]] ] for nye in NYE])
177
178
179 NYE0=data.NYE0
180 NYE1=[np.array([[i,nye0[0][1],nye0[0][2]] for nye0 in NYE0[i]])
181        for i in NX1]
182 YE1=[np.array([Xrep[i],Vd[nye0[0][1]],Signd[nye0[0][2]]] for nye0
183           in NYE0[i])) for i in NX1]
184
185 def restore(nxye):
186     return [cycle(Xrep[nxye[0]]),cycle(Vd[nxye[1]]),Signd[nxye
187           [2]]]
188
189 lYE1=[len(NYE1[i]) for i in NX1]
190 CNYE0=np.concatenate(NYE0)
191 NCO=len(CNYE0)
192 CNYE1=np.concatenate(NYE1)
193 CNNYE1=np.array([i for i in indices(CNYE0)])
194 CYE0=[[np.array([Xrep[a[0]],Vd[a[1]],Signd[a[2]]])] for a in c] for
195       c in CNYE0]
196 CYE1=[c[0] for c in CYE0]
197
198
199 def Orbit(x,y):
200     if len(x)!=len(y):print("error:Orbitlength")
201     n=len(x)
202     cx=cycle(x)
203     Ncx=np.array(indices(cx))
204     cy=cycle(y)
205     Ncy=np.array(indices(cy))
206     rest=np.array(range(n))
207     decomp=[]
208     while len(rest)!=0:
209         i=rest[0]
210         orbi=[i]
211         resti=[i]
212         donei=[]
213         while len(resti)!=0:
214             j=resti[0]
215             cxj=cx[Ncx[(np.any(np.array(cx[k])==j) for k in
216 indices(cx)))] [0]]
217             cyj=cy[Ncy[(np.any(np.array(cy[k])==j) for k in

```

```

indices(cy))][0]]
213         orbi=np.unique(np.concatenate([orbi,cxj,cyj]))
214         resti=copy.deepcopy(orbi)
215         donei.append(j)
216         for l in donei:
217             resti=resti[resti!=l]
218         for l in orbi:
219             rest=rest[rest!=l]
220         decomp.append(orbi)
221     return decomp
222
223 #Algorithm 6.1.3
224 print("start")
225 def PermT():
226     permT=[[0 for j in NYE1[i]]for i in NX1]
227     for i in NX1:
228         x=Xrep[i]
229         ix=iXrep[i]
230         for j in indices(NYE1[i]):
231             nye1=NYE1[i][j]#[i,j,kの形], i=nye[0]
232             y=Vd[nye1[1]]
233             eps=Signd[nye1[2]]
234             cyT=[]
235             epsT=[0 for i in Id]
236             rest=copy.deepcopy(Id)
237             while len(rest)!=0:
238                 a1=rest[0]
239                 a2=rest[0]
240                 a0=rest[0]
241                 cyT1=[a0]
242                 eps_1=0
243                 while True:
244                     rest=rest[rest!=a2]#remove a1 from rest
245                     a2_=vtosign(y,not eps_1!=eps[vtosign(x,not
eps_1)[a1]])[vtosign(x,not eps_1)[a1]]
246                     eps_2=((eps_1!=eps[vtosign(x,not eps_1)[a1]])
!=eps[a2_])
247                     if eps_2==0:a2=a2_
248                     else:a2=ix[a2_]
249                     if a2==a0:break
250                     cyT1=cyT1+[a2]
251                     epsT[a2]=eps_2
252                     eps_1=eps_2
253                     a1=a2_
254                 cyT=cyT+[cyT1]
255             epsT=np.array(epsT).astype(np.int)
256             yT=icycle(cyT,d)
257             permT[i][j]=np.concatenate([[i],NYE[np.all(np.all(YE
==[yT,epsT],axis=2),axis=1)])[0]])
258     return permT
259

```

```

260 #Algorithm 6.1.5
261 def PermS():
262     debugflag=False
263     permS=[0 for j in NYE1[i]]for i in NX1]
264     for i in NX1:
265         x=Xrep[i]
266         ix=iXrep[i]
267         for j in indices(NYE1[i]):
268             nye1=NYE1[i][j]#=[i,j,k], i=nye[0]
269             y=Vd[nye1[1]]
270             eps=Signd[nye1[2]]
271
272             if debugflag:print(x,y,eps)
273
274     #Algorithm 6.1.4
275     cxS=[]
276     deltaS=[0 for i in Id]
277     rest=copy.deepcopy(Id)
278     while len(rest)!=0:#make cycle representation
279         a1=rest[0]
280         a2=rest[0]
281         a0=rest[0]
282         cxS1=[a0]
283         delta1=0
284         while True:#cycle starting from a0
285             rest=rest[rest!=a2]#remove a2 from rest
286             a2=vtosign(y,not eps[a1]!=delta1)[a1]
287             delta2=(delta1!=eps[a1])!=eps[a2]
288             if a2==a0:break
289             cxS1=cxS1+[a2]
290             deltaS[a2]=delta2
291             delta1=delta2
292             a1=a2
293         cxS=cxS+[cxS1]
294     xS_=icycle(cxS,d)
295
296     cyS=[]
297     epsS_=[0 for i in Id]
298     rest=copy.deepcopy(Id)
299     while len(rest)!=0:#make cycle representation
300         b1=rest[0]
301         b2=rest[0]
302         b0=rest[0]
303         cyS1=[b0]
304         epsS_1=0
305         while True:#cycle starting from b0
306             rest=rest[rest!=b2]#remove b2 from rest
307             b2=vtosign(x,not not epsS_1!=deltaS[b1])[b1]
308             epsS_2=(epsS_1!=deltaS[b1])!=deltaS[b2]
309             if debugflag:print(b2,epsS_2)
310             if b2==b0:

```

```

311             break
312             cyS1=cyS1+[b2]
313             epsS_[b2]=epsS_2
314             epsS_1=epsS_2
315             b1=b2
316             cyS=cyS+[cyS1]
317             epsS_=np.array(epsS_).astype(np.int)
318             yS_=icycle(cyS,d)
319             Xnum=NX1[np.any(np.all(Xclass==np.array(xS_),axis=2),
axis=1)])[0]#xS belongs to Xrep[Xnum]
320             conj=iVd[np.all(Xclass==np.array(xS_),axis=2)[Xnum
]] [0]#conjugator
321             if np.any(xS!=conjugate(conj,xS_)):
322                 print("conjugate error")
323                 finish()
324             yS=conjugate(conj,yS_)
325             epsS=[epsS_[vinv(conj)[i]] for i in Id]
326             permS[i][j]=np.concatenate([[Xnum],NYE[np.all(np.all(
YE==[yS,epsS],axis=2),axis=1)])[0]])
327             return permS
328
329
330 permT=np.concatenate(PermT())
331 print("PermT:done")
332 tttime()
333 permS=np.concatenate(PermS())
334 print("PermS:done")
335 tttime()
336 permT1=[(CNYE1[( [np.any(np.all(CNYE0[j]==permT[i],axis=1)) for j in
indices(CNYE0) )])[0] for i in CNYE1]
337 permS1=[(CNYE1[( [np.any(np.all(CNYE0[j]==permS[i],axis=1)) for j in
indices(CNYE0) )])[0] for i in CNYE1]
338 permTS1=[permT1[permS1[i]] for i in CNYE1]
339
340 print("")
341
342 CpermT=cycle(permT1)
343 CpermS=cycle(permS1)
344 CpermTS=cycle(permTS1)
345
346 #Algorithm 6.1.6
347 Orb=Orbit(permT1,permS1)
348
349 #Output result
350 def output():
351     with open('result_d={0}.txt'.format(d), 'w') as f:
352         print(" ")
353         print("output")
354         print("  d =",d)
355         print("#d =",d, file=f)
356         print(" ")

```

```

357         Num=1
358         for orb in Orb:
359             rep=orb[0]
360             if np.any(np.any(CNIE0[rep]==[-1,-1,0],axis=1)) or np
.any(np.any(CNIE0[rep]==[-1,-1,len(Signd)-1],axis=1)):Abelian=
True
361             else:Abelian=False
362             Nxye=CNIE1[rep]
363             x=Xrep[Nxye[0]]
364             y=Vd[Nxye[1]]
365             eps=Signd[Nxye[2]]
366             decomp=Orbit(x,y)
367             if len(decomp)!=1:
368                 continue
369             if Abelian:
370                 z=[vinv(y)[vinv(x)[y[x[i]]]] for i in Id]
371                 vl=np.sort([4*len(c) for c in cycle(z)])#valency
list
372                 Nv=len(vl)#number of vertices
373             else:
374                 bz=[iby(x,y,eps,ibx(x,y,eps,by(x,y,eps,bx(x,y,eps,
j)))) for j in range(2*d)]
375                 cbz=cycle(bz)
376                 vld=[]#valency list of double-paired vertices
377                 vlr=[]#valency list of ramified vertices
378                 for c in cbz:
379                     bxby0=by(x,y,eps,bx(x,y,eps,c[0]))
380                     cbz0=[c1 for c1 in cbz if np.any(np.array(c1)
==c[0]))[0]]
381                     if np.any(cbz0==bxby0*d+(1-bxby0//d)*d):#
ramified vertex
382                         vlr.append(len(c)*2)
383                     else:#double-paired vertex
384                         vld.append(len(c)*4)
385                     vldr=[np.sort(vld)[2*i] for i in range(len(vld)
//2)]
386                 vl=np.sort(vlr+vldr)#valency list
387                 Nv=len(vl)#number of vertices
388                 CT=[c for c in CpermT if np.any(orb==c[0])]
389                 CS=[c for c in CpermS if np.any(orb==c[0])]
390                 CTS=[c for c in CpermTS if np.any(orb==c[0])]
391                 genus=1-(len(orb)-(3*len(orb)-len([c for c in CS if
len(c)==1])))/2+len(CT)+len(CTS))/2
392                 WL=lcm(*[len(c) for c in CT])
393                 print(" Component No.",Num, file=f)
394                 print(" representatives: ",orb, file=f)
395                 print(" index of VG =",len(orb), file=f)
396                 print(" base: (x,y,eps) = (" ,cycle(x),cycle(y),eps
,")", file=f)
397                 print(" surface type= (" ,int((d-Nv)/2+1)," ,Nv," )",
file=f)

```

```
398         print("  valency list= ",vl , file=f)
399         print("  Abelian:",Abelian, file=f)
400         print("  stratum:", "A_" if Abelian else "Q_",int((d-Nv
)/2+1),[int(v/2-2) for v in vl], file=f)
401         print("  T =",CT, file=f)
402         print("  widths list of T =",[len(c) for c in CT],len(
CT), file=f)
403         print("  S =",CS, file=f)
404         print("  widths list of S =",[len(c) for c in CS],len(
CS), file=f)
405         print("  TS =",CTS, file=f)
406         print("  widths list of TS =",[len(c) for c in CTS],
len(CTS), file=f)
407         print("  genus =",math.floor(genus), file=f)
408         print("  Wolfarht level =",WL, file=f)
409         #list of origamis in component
410         for i in orb:
411             print("      representative No.",i, file=f)
412             Nxyei=CNye1[i]
413             xi=Xrep[Nxyei[0]]
414             yi=Vd[Nxyei[1]]
415             epsi=Signd[Nxyei[2]]
416             print("      (x,y,eps) = (" ,cycle(xi),cycle(yi),
epsi,")", file=f)
417
418         print(" ", file=f)
419         print(" ", file=f)
420         print(" ", file=f)
421         Num=Num+1
422     print("end")
423
424 print("Output")
425 output()
426 finish()
```

---

Example of output of Program2.py (input:d=3) is as follows.

result\_d=3.txt

---

```
1 #d = 3
2 Component No. 1
3 representatives: [ 0  3  6 15]
4 index of VG = 4
5 base: (x,y,eps) = ( [[0, 1, 2]] [[0], [1], [2]] [0 0 0] )
6 surface type= ( 1 , 3 )
7 Abelian: True
8 T = [[0, 3, 6], [15]]
9 widths list = [3, 1] 2
10 S = [[0, 15], [3, 6]]
11 widths list = [2, 2] 2
```



```

12 TS = [[0, 15, 3], [6]]
13 widths list = [3, 1] 2
14 genus = 0
15 Wolfarht level = 3
16     representative No. 0
17     (x,y,eps) = ( [[0, 1, 2]] [[0], [1], [2]] [0 0 0] )
18     representative No. 3
19     (x,y,eps) = ( [[0, 1, 2]] [[0, 1, 2]] [0 0 0] )
20     representative No. 6
21     (x,y,eps) = ( [[0, 1, 2]] [[0, 1, 2]] [1 1 1] )
22     representative No. 15
23     (x,y,eps) = ( [[0], [1], [2]] [[0, 1, 2]] [0 0 0] )
24
25
26
27 Component No. 2
28 representatives: [ 1 10 11]
29 index of VG = 3
30 base: (x,y,eps) = ( [[0, 1, 2]] [[0], [1, 2]] [0 0 0] )
31 surface type= ( 2 , 1 )
32 Abelian: True
33 T = [[1], [10, 11]]
34 widths list = [1, 2] 2
35 S = [[1, 11], [10]]
36 widths list = [2, 1] 2
37 TS = [[1, 10, 11]]
38 widths list = [3] 1
39 genus = 0
40 Wolfarht level = 2
41     representative No. 1
42     (x,y,eps) = ( [[0, 1, 2]] [[0], [1, 2]] [0 0 0] )
43     representative No. 10
44     (x,y,eps) = ( [[0], [1, 2]] [[0, 1], [2]] [0 0 0] )
45     representative No. 11
46     (x,y,eps) = ( [[0], [1, 2]] [[0, 1, 2]] [0 0 0] )
47
48
49
50 Component No. 3
51 representatives: [ 2 4 5 12]
52 index of VG = 4
53 base: (x,y,eps) = ( [[0, 1, 2]] [[0], [1, 2]] [0 1 0] )
54 surface type= ( 2 , 1 )
55 Abelian: False
56 T = [[2, 4, 5], [12]]
57 widths list = [3, 1] 2
58 S = [[2, 12], [4, 5]]
59 widths list = [2, 2] 2
60 TS = [[2, 12, 4], [5]]
61 widths list = [3, 1] 2
62 genus = 0

```

```

63  Wolfarht level = 3
64  representative No. 2
65  (x,y,eps) = ( [[0, 1, 2]] [[0], [1, 2]] [0 1 0] )
66  representative No. 4
67  (x,y,eps) = ( [[0, 1, 2]] [[0, 1, 2]] [1 0 0] )
68  representative No. 5
69  (x,y,eps) = ( [[0, 1, 2]] [[0, 1, 2]] [1 1 0] )
70  representative No. 12
71  (x,y,eps) = ( [[0], [1, 2]] [[0, 1, 2]] [0 1 0] )

```

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