

Notes

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1 Notations

Consider a weighted, undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, W\}$, where \mathcal{V} is the set N vertices, \mathcal{E} is the set of edges and W is the weighted adjacency matrix. Let D be the diagonal matrix of vertex degrees. Denote \mathcal{L} as the graph Laplacian, where $\mathcal{L} = D - W$. Since \mathcal{L} is real, symmetric and semi-definite, we can diagonalize it as $\mathcal{L} = U\Lambda U^*$, where Λ is the diagonal matrix of its real eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ and the columns u_0, u_1, \dots, u_{n-1} of U are corresponding orthonormal eigenvectors of \mathcal{L} . We use $U_{\mathcal{R}}$ to denote the submatrix formed by taking the columns of U associated with the Laplacian eigenvalues indexed by $\mathcal{R} \subseteq \{0, 1, \dots, n-1\}$. And we use $U_{\mathcal{S}, \mathcal{R}}$ to denote the submatrix formed by taking the rows of $U_{\mathcal{R}}$ associated with the vertices indexed by the set $\mathcal{S} \subseteq \{1, 2, \dots, N\}$. $\delta_i \in \mathbb{R}^n$ denotes the i th column of the identity matrix $I \in \mathbb{R}^{n \times n}$.

2 Definitions

Definition. A signal $\mathbf{f} \in \mathbb{R}^n$ defined on the nodes of the graph \mathcal{G} is \mathcal{R} -concentrated if $\mathbf{f} \in \text{span}(U_{\mathcal{R}})$.

Definition. Let $\mathbf{p} \in \mathbb{R}^n$ represent a sampling distribution on $\{1, 2, \dots, n\}$. The graph weighted coherence of order $|\mathcal{R}|$ for the pair $(\mathcal{G}, \mathbf{p})$ is

$$v_{\mathbf{p}}^k = \max_{1 \leq i \leq n} \{\mathbf{p}_i^{-1/2} \|U_{\mathcal{R}}^T \delta_i\|_2\}$$

3 Theorem

Theorem ([?], Theorem 3.2). Consider any graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, W\}$ with its Laplacian $\mathcal{L} = U\Lambda U^*$. Notice that \mathcal{L} is real, symmetric and positive semi-definite and thus the columns of U are orthonormal and its eigenvalues are non-negative. We propose the following sampling procedure:

1. Sampling Distribution. Let $\mathcal{R} \subseteq \{0, 1, \dots, n-1\}$. Define $\mathbf{p} \in \mathbb{R}^n$ as the sampling distribution on vertices $\{1, 2, \dots, n\}$ such that

$$\mathbf{p}_i = \frac{\|U_{\mathcal{R}}^T \delta_i\|_2^2}{|\mathcal{R}|}, \text{ for } i = 1, 2, \dots, n. \quad (1)$$

The sampling distribution \mathbf{p} minimizes the graph weighted coherence. We associate the matrix $P = \text{diag}(\mathbf{p}) \in \mathbb{R}^{n \times n}$.

2. Subsampling Matrix. Let $\Omega = \{\omega_1, \omega_2, \dots, \omega_m\}$ be the subset of nodes drew independently from the set $\{1, 2, \dots, n\}$ according to the sampling distribution \mathbf{p} . Define M as the random subsampling matrix with the sampling distribution \mathbf{p} such that

$$M_{ij} = \begin{cases} 1 & \text{if } j = \omega_i \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

for all $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$.

From the m samples obtained using the sampling method above, we can reconstruction all \mathcal{R} -concentrated signals accurately by solving the optimization problem

$$\min_{\mathbf{f} \in \text{span}(U_{\mathcal{R}})} \|P_{\Omega}^{-1/2} (M\mathbf{f} - \mathbf{y})\|_2, \quad (3)$$

which estimates signal $\mathbf{f} \in \mathbb{R}^n$ from $\mathbf{y} \in \mathbb{R}^m$.

Let $\epsilon, \delta \in (0, 1)$. Let \mathbf{f}^* be the solution of the problem (3). With probability at least $1 - \epsilon$, the following holds for all $\mathbf{f} \in \text{span}(U_{\mathcal{R}})$ and all $\mathbf{n} \in \mathbb{R}^m$:

$$\|x^* - x\|_2 \leq \frac{2}{\sqrt{m(1-\delta)} \|P_{\Omega}^{-1/2} \mathbf{n}\|_2} \quad (4)$$

provided that

$$m \geq \frac{3}{\delta^2} v_{\mathbf{p}}^k \log\left(\frac{2k}{\epsilon}\right). \quad (5)$$

Theorem.