Notes

February 20, 2018

1 Notations

Consider a weighted, undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, W\}$, where $\mathcal{V}$ is the set N vertices, $\mathcal{E}$ is the set of edges and W is the weighted adjacency matrix. Let D be the diagonal matrix of vertex degrees. Denote $\mathcal{L}$ as the graph Laplacian, where $\mathcal{L} = D - W$. Since $\mathcal{L}$ is real, symmetric and semi-definite, we can diagonalize it as $\mathcal{L} = U\Lambda U^*$, where Λ is the diagonal matrix of its real eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_{n-1}$ and the columns $u_0, u_1 \cdots u_{n-1}$ of U are corresponding orthonormal eigenvectors of $\mathcal{L}$. We use $U_{\mathcal{R}}$ to denote the submatrix formed by taking the columns of U associated with the Laplacian eigenvalues indexed by $\mathcal{R} \subseteq \{0, 1, \cdots, n-1\}$. And we use $U_{\mathcal{S},\mathcal{R}}$ to denote the submatrix formed by taking the rows of $U_{\mathcal{R}}$ associated with the vertices indexed b the set $\mathcal{S} \subseteq \{1, 2, \cdots, N\}$. $\boldsymbol{\delta_i} \in \mathbb{R}^n$ denotes the ith column of the identity matrix $I \in \mathbb{R}^{n \times n}$.

2 Definitions

Definition. A signal $\boldsymbol{f} \in \mathbb{R}^n$ defined on the nodes of the graph $\mathcal{G}$ is $\mathcal{R}$-concentrated if $\boldsymbol{f} \in \text{span}(U_{\mathcal{R}})$.

Definition. Let $\boldsymbol{p} \in \mathbb{R}^n$ represent a sampling distribution on $\{1, 2, \cdots n\}$. The graph weighted coherence of order $|\mathcal{R}|$ for the pair $(\mathcal{G}, \boldsymbol{p})$ is

$$v_{\boldsymbol{p}}^k = \max_{1 \le i \le n} \{\boldsymbol{p_i}^{-1/2} ||U_{\mathcal{R}}^T \boldsymbol{\delta_i}||_2\}$$

3 Theorem

Theorem ([?], Theorem 3.2)**.** Consider any graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, W\}$ with its Laplacian $\mathcal{L} = U\Lambda U^*$. Notice that $\mathcal{L}$ is real, symmetric and positive semi-definite and thus the columns of U are orthonormal and its eigenvalues are non-negative. We propose the following sampling procedure:

1. Sampling Distribution. Let $\mathcal{R} \subseteq \{0, 1, \cdots, n-1\}$. Define $\boldsymbol{p} \in \mathbb{R}^n$ as the sampling distribution on vertices $\{1, 2, \cdots n\}$ such that

$$\boldsymbol{p_i} = \frac{||U_{\mathcal{R}}^T \boldsymbol{\delta_i}||_2^2}{|\mathcal{R}|}, \text{ for } i = 1, 2, \cdots, n. \tag{1}$$

   The sampling distribution $\boldsymbol{p}$ minimizes the graph weighted coherence. We associate the matrix $P = \text{diag}(\boldsymbol{p}) \in \mathbb{R}^{n \times n}$.

2. Subsampling Matrix. Let $\Omega = \{\omega_1, \omega_2, \cdots, \omega_m\}$ be the subset of nodes drew independently from the set $\{1, 2, \cdots n\}$ according to the sampling distribution $\boldsymbol{p}$. Define M as the random subsampling matrix with the sampling distribution $\boldsymbol{p}$ such that

$$M_{ij} = \begin{cases} 1 & \text{if } j = \omega_i \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

   for all $i \in \{1, 2, \cdots, m\}$ and $j \in \{1, 2, \cdots, n\}$.

From the m samples obtained using the sampling method above, we can reconstruction all $\mathcal{R}$-concentrated signals accurately by solving the optimization problem

$$\min_{\boldsymbol{f} \in \text{span}(U_{\mathcal{R}})} ||P_{\Omega}^{-1/2}(M\boldsymbol{f} - \boldsymbol{y})||_2, \tag{3}$$

which estimates signal $\boldsymbol{f} \in \mathbb{R}^n$ from $\boldsymbol{y} \in \mathbb{R}^m$.

Let $\epsilon, \delta \in (0, 1)$. Let $\boldsymbol{f^*}$ be the solution of the problem (3). With probability at lease $1 - \epsilon$, the following holds for all $\boldsymbol{f} \in \text{span}(U_{\mathcal{R}})$ and all $\boldsymbol{n} \in \mathbb{R}^m$:

$$||x^* - x||_2 \le \frac{2}{\sqrt{m(1-\delta)}||P_{\Omega}^{-1/2}\boldsymbol{n}||_2} \tag{4}$$

provided that

$$m \ge \frac{3}{\delta^2} v_{\boldsymbol{p}}^k \log(\frac{2k}{\epsilon}). \tag{5}$$

Theorem (Orthogonality Proof)**.** Let $\{\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_M\}$ be M partitions of the graph Laplacian eigenvalue indices $\{0, 1, \cdots, N\}$ and $\{g_1, g_2, \cdots, g_M\}$ be filters defined on each of the M bands. Then, the subspace spanned by $g_i(\mathcal{L})$ is orthogonal to the subspace spanned by $g_j(\mathcal{L})$ for $i \neq j$

Proof. Notice that the (i, j) entry of matrix $g(\mathcal{L})$ can be represented as

$$g(\mathcal{L})(i, j) = [Ug(\Lambda)U^T](i, j) = \sum_{k=1}^{N} g(\lambda_k)U_k(i)U_k(j)$$

Now consider two filters g_1 and g_2. To show that two subspaces spanned are orthogonal, we show that the dot product of any two vectors is 0.

$$\begin{aligned} g_1(\mathcal{L})(, k) \cdot g_2(\mathcal{L})(, l) = \; & \sum_{j=1}^{N}[\sum_{k=1}^{N} g_1(\lambda_k)U_k(i)U_k(j))(\sum_{l=1}^{N} g_2(\lambda_l)U_l(i)U_l(j))] \\ &= \sum_{k=1}^{N}\sum_{l=1}^{N} g_1(\lambda_k)g_2(\lambda_l)U_k(i)U_l(i)(\sum_{j=1}^{N} U_k(j)U_l(j)) \\ &= \sum_{k=1}^{N} g_1(\lambda_k)g_2(\lambda_k)U_k(i)U_k(i)(\sum_{j=1}^{N} U_k(j)U_k(j)) \end{aligned}$$

Notice that $g_1(\lambda_k)g_2(\lambda_k)$ is always equal to 0, since g_1 and g_2 are defined on disjoint sets of Laplacian eigenvalues.

□