Recitation 4: Solutions for Exercise Problem

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Let me know when you have any questions.

Summer 2013 environment prelim

Two consumers are the only members of an island economy. They have identical preferences over two goods, a private numeraire good x and a pure public good q. Preferences are given by

$$U_i(x_i, q) = \ln x_i + 2\ln q.$$

Each consumer is endowed with $\omega_i = 10$ of the private good, of which x_i is consumed directly and the remainder $z_i = \omega_i - x_i$ is contributed to the provision of the public good. q is produced according to the simple production function $q = z_1 + z_2$.

a. Determine the outcome (x_1, x_2, z_1, z_2), that a benevolent social planner would choose so as to maximize the unweighted sum of preferences, $W = U_1(x_1, q) + U_2(x_2, q)$.

b. Find the voluntary-contribution equilibrium for this economy. Show that the VCE level of the public good, $\hat{q}$ is less than q^*.

c. Find the Lindahl price at which the consumers, taking this price as given when selecting their contribution z_i, will choose the socially optimal level of contribution to the public good.

Solution

Part a

The aggregate endowment is $\omega = \omega_1 + \omega_2 = 20$. The social planner's problem is to allocate this endowmnet to x_1 ,x_2, and q so that the social welfare function $W = U_1(x_1, q) + U_2(x_2, q) = logx_1 + logx_2 + 4log$ is maximized:

$$\begin{aligned} \max_{x_1, x_2, q} \quad & W = logx_1 + logx_2 + 4log\, q \\ \text{s.t.} \quad & q = 20 - x_1 - x_2 \end{aligned}$$

Note that $x_1 > 0$, $x_2 > 0$, and $q > 0$ by the form of U_j. So we should have an interior solution.

$$L = logx_1 + logx_2 + 4log\, q + \lambda(20 - x_1 - x_2 - q)$$

F.O.Cs are

$$\frac{\partial L}{\partial x_1} = \frac{1}{x_1} - \lambda = 0 \quad \cdots (1)$$
$$\frac{\partial L}{\partial x_2} = \frac{1}{x_2} - \lambda = 0 \quad \cdots (2)$$
$$\frac{\partial L}{\partial q} = \frac{4}{q} - \lambda = 0 \quad \cdots (3)$$
$$\frac{\partial L}{\partial \lambda} = 20 - x_1 - x_2 - q = 0 \quad \cdots (4)$$

By the conditions (1) and (2),

$$x_2 = x_1$$

By the the conditions (1) and (3),

$$q = 4x_1$$

Substituting these for condition (4),

$$20 - x_1 - x_1 - 4x_1 = 0$$

Thus, $x_1^{PO} = \frac{10}{3}$. Then, $x_2^{PO} = \frac{10}{3}$ and $q^{PO} = \frac{40}{3}$.

In summary, the socially optimum outcome is $(x_1^{PO}, x_2^{PO}, z_1^{PO}, z_2^{PO}) = (\frac{10}{3}, \frac{10}{3}, \frac{20}{3}, \frac{20}{3})$. The optimal level of public goods is $q^{PO} = \frac{40}{3}$.

Part b

Consumer i's ($i \in 1, 2$) problem is

$$\begin{aligned} \max_{x_i, z_i} \quad & U_i = log x_i + 2 log\, q \\ \text{s.t.} \quad & x_i = 10 - z_i \\ & q = z_i + z_{-i} \end{aligned}$$

By incorporating the contraints into the objective function, this maximiztion problem becomes

$$\max_{z_i} \quad U_i = log(10 - z_i) + 2log\,(z_i + z_{-i})$$

The first order condition for this unconstrained maximization problem is

$$-\frac{1}{10 - z_i} + \frac{2}{z_i + z_{-i}} = 0$$

or

$$z_i = \frac{20 - z_{-i}}{3}$$

(Note that this is person i's response function to the other person's choice on contribution to the public goods.) This condition must hold for $i = 1, 2$. That is, the following must hold

$$z_1 = \frac{20 - z_2}{3} z_2 = \frac{20 - z_1}{3}$$

By solving this equaltions simultaneously, we get $z_1^{VCE} = z_2^{VCE} = 5$. So, $x_1^{VCE} = x_2^{VCE} = 10 - 5 = 5$.

In summary, the VCE is $(x_1^{VCE}, x_2^{VCE}, z_1^{VCE}, z_2^{VCE}) = (5, 5, 5, 5)$, and $q^{VCE} = z_1^{VCE} + z_2^{VCE} = 10$, which is less than q^{PO} as expected.

Part c

Let p_1 and p_2 be the Lindahl prices of the public good for consumer 1 and consumer 2. **Each consumer solves**,

$$\max_{x_i} \quad U_i = log x_i + 2log\, q$$
$$\text{s.t.} \quad x_i + p_i q = 10$$

Again, the solution should be interior.

$$L = log x_i + 2log\, q + \lambda(10 - x_i - p_i q)$$

F.O.Cs are

$$\frac{\partial L}{\partial x_i} = \frac{1}{x_1} - \lambda = 0$$
$$\frac{\partial L}{\partial q} = \frac{2}{q} - \lambda p_i = 0$$
$$\frac{\partial L}{\partial \lambda} = 10 - x_i - p_i q$$

From the first two conditions,

$$\frac{1}{x_1} = \frac{1}{p_i} \cdot \frac{2}{q}$$
$$\iff \quad q = \frac{2}{p_i} x_i \quad \cdots (*1)$$

Substituting this for the last F.O.C.,

$$10 - x_i + p_i \cdot \frac{2}{p_i} x_i = 0$$
$$\iff \quad x_i = \frac{10}{3}$$

So, substituting this for condition $(*1)$, we have,

$$q = \frac{2}{p_i} \cdot \frac{10}{3}$$
$$\iff \quad p_i = \frac{3}{20} q \quad \cdots (*2) \quad (\text{for } i = 1, 2)$$

The producer's problem is

$$\max_{q,z} \quad p \cdot q - z$$
$$\text{s.t.} \quad q = z$$

, where $p = p_1 + p_2$ and $z = z_1 + z_2$.

$$L = p \cdot q - z + \lambda(z - q)$$

F.O.C.

$$\frac{\partial L}{\partial q} = p - \lambda = 0$$
$$\frac{\partial L}{\partial z} = -1 + \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = z - q = 0$$

From the first two conditions, we have $p = 1$, or $p_1 + p_2 = 1$. Together with condition $(*2)$,

$$\begin{aligned} & \frac{3}{20}q + \frac{3}{20}q = 1 \\ \iff \quad & q = \frac{10}{3} \end{aligned}$$

Substituting this for condition $(*2)$, we have the Lindahl price

$$p_i = \frac{1}{2} \quad \text{for } i = 1, 2$$