Recitation 2: Solutions for the exercise problems

Shunkei Kakimoto

Solution to Exercise 1

By the definition of CDF, $\Phi(z)$ can be written as $\Phi(z) = Pr(Z \leq z)$.

Answers:

Part (a)

$$Pr(Z \leq b) = \Phi(b)$$

Part (b)

$$Pr(Z \leq b) = \int_{-\infty}^{b} \phi(z)\, dz$$

Part (c)

$$Pr(a \leq Z \leq b) = Pr(Z \leq b) - Pr(Z \leq a) = \Phi(b) - \Phi(a)$$

Part (c)

$$Pr(a \leq Z \leq b) = \int_{a}^{b} \phi(z)\, dz$$

Solution to Exercise 2

Idea We want to derive the PDF of Y which is defined by $Y = X^2$. Since $0 \leq X \leq 1$, $0 \leq Y \leq 1$. Recall the definition of PDF. Let $G(y)$ be the CDF of Y and $g(y)$ be the PDF of Y. Then, $g(y) = \frac{d}{dy}G(y)$ by the definition of PDF. That is, once you get $G(y)$, you can derive $g(y)$. So, let's start with the CDF of Y.

Answers:

By the definition of CDF,

$$G(y) = Pr(Y \leq y).$$

Substituting $Y = X^2$,

$$\begin{aligned} G(y) &= Pr(X^2 \le y) \\ &= Pr(0 \le X \le \sqrt{y}) \end{aligned}$$

(The last equality is because of the fact that $X^2 \le y \iff -\sqrt{y} \le X \le \sqrt{y}$ and the condition $X \ge 0$.)

Since we know that X is uniformly distributed ($X \sim U[0,1]$), the PDF of X is 1 for $0 \le X \le 1$. Also, note that the range $[0, \sqrt{y}]$ is contained in the range of the PDF of X. So,

$$G(y) = Pr(0 \le X \le \sqrt{y}) = \int_0^{\sqrt{y}} 1dx = \sqrt{y}. \tag{1}$$

Thus, PDF of Y is

$$g(y) = \frac{d}{dy}G(y) = \frac{1}{2\sqrt{y}} \quad (0 \le y \le 1)$$

Solution to Exercise 3

Answers:

Part (a)

$$\begin{aligned} Cov(X,Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E\Big[XY - XE[Y] - E[X]Y + E[X]E[Y]\Big] \\ &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] && \text{(linearity of expectation)} \\ &= E[XY] - E[X]E[Y] \\ &= E[XY] && (E[X] = 0 \text{ or } E[Y] = 0) \end{aligned}$$

Also, note that $Cov(X,Y) = E[XY] - E[X]E[Y]$.

Part (b)

By definition,

$$corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var[X]Var[Y]}}$$

If $X \perp\!\!\!\perp Y$, then, $Cov(X,Y) = E[XY] - E[X]E[Y] = E[X]E[Y] - E[X]E[Y] = 0$. (Note that if $X \perp\!\!\!\perp Y$, $E[XY] = E[X]E[Y]$).

Therefore, if $X \perp\!\!\!\perp Y$,

$$corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var[X]Var[Y]}} = 0$$

Part (c)

Here, I show $Var[X+Y] = Var[X] + Var[Y] + 2Cov(X,Y)$ without assuming $E[X] = E[Y] = 0$.

By definition,

$$
\begin{aligned}
Var[X+Y] &= E\Big[\Big(X+Y-E[X+Y]\Big)^2\Big] \\
&= E\Big[\Big(X+Y-E[X]-E[Y]\Big)^2\Big] && \text{(linearity of expectation)} \\
&= E\Big[\Big(X-E[X]) + (Y-E[Y])\Big)^2\Big] \\
&= E\Big[(X-E[X])^2 + (Y-E[Y])^2 + 2(X-E[X])(Y-E[Y])\Big] \\
&= E[(X-E[X])^2] + E[(Y-E[Y])^2] + 2E[(X-E[X])(Y-E[Y])] && \text{(linearity of expectation)} \\
&= Var[X] + Var[Y] + 2Cov(X,Y)
\end{aligned}
$$

Part (d)

If X and Y are uncorrelated, $corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var[X]Var[Y]}} = 0 \iff cov(X,Y) = 0$.

Therefore, using the result of Part (e),

$$
Var[X+Y] = Var[X] + Var[Y] + 2Cov(X,Y) = Var[X] + Var[Y]
$$

Solution to Exercise 4

Part (a)

$$
\begin{aligned}
E[X] &= E\Big[\sum_{i=1}^{k} Z_i^2\Big] \\
&= \sum_{i=1}^{k} E[Z_i^2] && (Z_i \text{ are independent}) \\
&= \sum_{i=1}^{k} (Var[Z_i] + (E[Z_i])^2) && (Var[Z_i] = E[Z_i^2] - (E[Z_i])^2)
\end{aligned}
$$

We know that $E[Z_i] = 0$ and $Var[Z_i] = 1$ because $Z_i \sim N(0,1)$. Therefore,

$$
E[X] = \sum_{i=1}^{k} (1 + 0^2) = \sum_{i=1}^{k} 1 = k
$$

Part (b)

$$
\begin{aligned}
Var[K] &= Var[Z_1^2 + Z_2^2] \\
&= Var[Z_1^2] + Var[Z_2^2] \quad (Z_1 \text{ and } Z_2 \text{ are independent})
\end{aligned}
$$

Because $Var[Z_i^2] = E[Z_i^4] - (E[Z_i^2])^2$. Using the facts that $E[Z_i^2] = 1$ and $E[Z_i^4] = 3$, $Var[Z_i^2] = 3 - 1 = 2$. Therefore,

$$
Var[K] = Var[Z_1^2] + Var[Z_2^2] = 2 + 2 = 4
$$