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DERIVED EQUIVALENCES BETWEEN DEFECTIVE RECTANGLES

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Abstract. In this article, we provide the quiver realization of two classes of algebras which is derived equivalent to the upper triangulated matrix algebras. We investigate the defective rectangle algebras, and four kinds of defective rectangle algebras turn out to be derived equivalent. Moreover, we prove the derived equivalences between the defective rectangle algebras with the Nakayama algebras.

1. Introduction

Derived categories are introduced and developed by Grothendieck and his student Verdier [20] after 1960. In the representation theory, it is a cruial question to decide whether two derived categories are equivalent. We refer to [6, 7, 15, 16] for derived equivalences. By [15], Rickard states that two rings R, S are derived equivalent if and only if there exists a tilting complex $T \in \mathrm{D}^b(R)$ such that $\mathrm{End}_{\mathrm{D}^b(R)}(T) \simeq S$. But it is sometimes difficult to determine whether a tilting complex exists. Even if it exists, the construction is troublesome.

The Nakayama algebras are introduced by Nakayama [14] as a generalization of Artinian principal ideal rings in 1940. It is an algebra such that each indecomposable projective module as well as each indecomposable injective module is uniserial, that is, it has a unique composition series. The study of the Nakayama algebras is closely related to many mathematical subjects, such as Gorenstein projective modules [17], incidence algebras [10], singularity theory [13, 18, 19], weighted projective lines and stable categories of vector bundles [9, 8, 13]

In our paper we focus on a special class of the Nakayama algebras $N(n, r)$ for $n > r \geq 2$ as following. It is the bound quiver algebra of the equioriented quiver

$$1 \xrightarrow{x} 2 \xrightarrow{x} 3 \xrightarrow{x} \cdots \xrightarrow{x} n-1 \xrightarrow{x} n$$

of type $\vec{A}_n$ subject to all relations $x^r = 0$. It is well known that the Nakayama algebras $N(n, r)$ are representation-finite algebras whose representation theory is completely understood. However the bounded derived category $\mathrm{D}^b\big(N(n, r)\big)$ is not clear. In fact, most of these algebras are derived wild. By [4], Happle and Seidel determine that which pairs (n, r) make the Nakayama algebra $N(n, r)$ is piecewise hereditary. Further, Lenzing, Meltzer and Ruan classify all the Nakayama algebras of Fuchsian type according to different types of their bounded derived categories in [13].

Recently, the study of the derived equivalences between Nakayama algebras generates a lot of interest. For example, [4] shows Happel-Seidel symmetry and [13] extends this symmetry. Lenzing, Meltzer and Ruan have also raised a conjecture that two Nakayama algebras $N(n, a)$ and $N(n, b)$ are derived equivalent if and only if they share the same Coxeter polynomial in [13].

To investigate this conjecture, it is important to calculate the Coxeter polynomials of the Nakayama algebras. For this purpose, we need some derived equivalences between Nakayama algebra $N(n, r)$ and some bound quiver algebra whose Coxeter polynomial is easy to calculate. Ladkani [12] revealed the interesting derived equivalences between the Nakayama algebra $N(2u, u+1)$, the stable Auslander algebra of $\vec{A}_{2u+1}$ and the tensor product algebra $\mathbf{k}\vec{A}_2 \otimes \mathbf{k}\vec{A}_u$ whose Coxeter polynomial is calculated by Hille and Müller [5]. Base on it, Dong, Lin and Ruan [3] provide the derived equivalences between

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a large family of Nakayama algebras and the incidence algebras arising from one-branch extensions of "rectangles" and obtain the Coxeter polynomials for this class of Nakayama algebras.

In this article, we provide the quiver realization of two classes of the upper triangulated matrix algebras which play a crucial role. We focus on the bound quiver $Q(m,n)$ for $m,n \geq 1$ with all commutativity relations which is the so-called "rectangle" as following.

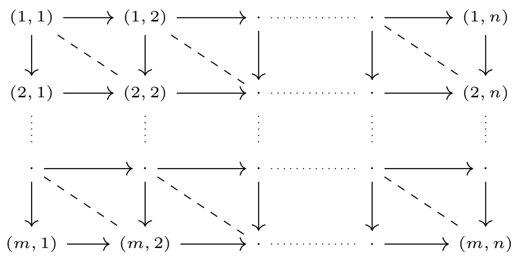

FIGURE 1. the quiver which is a "rectangle"

Denote by $Q(m,n)^{-a}$, ${}^{-a}Q(m,n)$, ${}_{-a}Q(m,n)$, $Q(m,n)_{-a}$ the defective rectangles obtained from the quiver $Q(u,v)$ by deleting vertices $\{(1,n-i+1)\}_{1\leq i\leq a}$, $\{(1,i)\}_{1\leq i\leq a}$, $\{(m,i)\}_{1\leq i\leq a}$, $\{(m,n-i+1)\}_{1\leq i\leq a}$, respectively. In order to investigate the derived equivalence between them, we introduce the gluing "rectangles" (see Construction 4.1). We also study the derived equivalence between the defective rectangle algebras and the Nakayama algebras which provides an effective method to investigate the Nakayama algebras in the forthcoming paper.

Denote the algebra associated to the bound quiver Q by $A(Q)$. Then the main theorem of this article is as following.

Theorem 1.1. (See Corollary 4.5 and Theorem 5.3) *We have the following derived equivalences:*

$$\begin{aligned}\mathrm{D}^b(A[{}^{-a}Q(m,n)]) \simeq \mathrm{D}^b(A[Q(m,n)^{-a}]) \simeq \mathrm{D}^b(A[{}_{-a}Q(m,n)]) \simeq \mathrm{D}^b(A[Q(m,n)_{-a}])\\ \simeq \mathrm{D}^b(N(mn-a,m+1)).\end{aligned}$$

The paper is organized as follows. In Section 2, we recall the definition of the quiver and the algebra, and give description of two S-R-bimodule in more detail. In Section 3, we provide the quiver realization of two classes of algebras in [11] which is derived equivalent to the upper triangulated matrix algebras, and induce a new derived equivalence (see Corollary 3.4). In Section 4, we introduce the gluing "rectangles" and show a derived equivalence between different gluing "rectangles". As a corollary, we prove a part of Theorem 1.1. In Section 5, we investigate the derived equivalence between the defective rectangle algebras and the Nakayama algebra.

2. PRELIMINARY

2.1. **Algebras and quivers.** Throughout this paper, $\mathbf{k}$ is an algebraically closed field of characteristic zero and the algebras are considered as basic and connected finite dimensional $\mathbf{k}$-algebras.

Let $\{e_i | i = 1, 2, \cdots, n\}$ be a complete set of primitive orthogonal idempotents of an algebra A. Then A can be naturally regarded as a standard matrix algebra $A = (e_iAe_j)_{1\leq i,j\leq n}$ and any element $a \in A$ is naturally identified as the linear combination $a = \sum_{i,j=1}^{n} e_iae_j$.

Let $Q = (Q_0, Q_1, I)$ be a quiver with the vertex set Q_0, the arrow set Q_1 and the admissible ideal I. We call $\mathbf{k}Q/I$ the algebra associated to the bound quiver Q which will be denoted by $A(Q)$. By Gabriel's theorem, there exists unique finite connected quiver $Q_A = (Q_{A,0}, Q_{A,1})$ and some admissible ideal I such that $A \cong kQ_A/I$.

A relation on Q is a non-zero $\mathbf{k}$-linear combination of path of length at least 2 with the same start vertex and the same end vertex. A relation is called a *commutativity relation* (resp. *zero relation*) if it is of the form $p_1 - p_2$ (resp. p_1) where p_1, p_2 are paths. If the admissible ideal is generated by a relation set $\rho = \{\rho_i\}_i$, we say Q is the bound quiver with relations.

In our paper we focus on the bound acyclic quiver $Q = (Q_0, Q_1, I)$ with all commutativity relations. It implies that any two path with the same start vertex and the same end vertex are equal, that is, there exists at most one path from i to j for each vertices $i, j \in Q_0$ which is denoted by $p_{i,j}$ if it exists. Then we have $p_{i,k} = p_{i,j}p_{j,k}$ for non-zero paths $p_{i,j}, p_{j,k}, p_{i,k}$. (Here we allow $p_{i,j} = 0$ to represent that there exists a path from i to j which is a zero relation.)

Since Q is acyclic, there is a natural partial ordering $\leq$ on Q_0: $i \leq j$ if and only if there is a path from i to j in (Q_0, Q_1) without considering relations. Then the algebra associated to the bound quiver Q can be seen as an upper triangular matrix algebra, that is, $e_iAe_j = 0$ for $i > j$. We say that the partial ordering is compatible with relations on a subset X of Q_0 if $i \leq j$ implies the path $p_{i,j}$ is non-zero for each $i, j \in X$.

For example, let Q be the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ with the relation $\alpha\beta$, then the natural partial ordering is not compatible with relations on Q_0 because $1 \leq 3$ and $p_{1,3} = \alpha\beta = 0$; let Q' be the quiver

$$\begin{tikzcd} \bullet \arrow[r, "\alpha_1"] \arrow[d, "\alpha_3"'] & \bullet \arrow[d, "\alpha_4"] \\ \bullet \arrow[r, "\alpha_2"] & \bullet \end{tikzcd}$$

with the relation $\alpha_1\alpha_4 - \alpha_3\alpha_2$, then the natural partial ordering is compatible with relations on Q'_0.

2.2. **Shadow.** Now we define the shadow of a vertex in some subset.

Definition 2.1. For a vertex $i \in Q_0$ and a subset $T \subseteq Q_0$, we say j is the *shadow* of i in T if there exists the unique vertex j in T such that each non-zero path $p_{i,t}$ with t in T can factor through non-zero path $p_{i,j}$.

Denote by $\Omega(i, T) = \{t \in T | \text{there exists a non-zero path from } i \text{ to } t \text{ in } Q\}$. The next lemma gives a description of the shadow.

Proposition 2.2. *For a subset $T \subseteq Q_0$, let i be the vertex such that $\Omega(i, T)$ is non-empty. Then j is the shadow of i in T if and only if j is the minimum element in $\Omega(i, T)$.*

Proof. Assume that j is the shadow of i in T, then $j \in \Omega(i, T)$. For each $t \in \Omega(i, T)$, there exists a non-zero path $p_{i,t}$ in Q. By the definition of the shadow, we have $p_{i,t} = p_{i,j}a$ for some non-zero element $a \in A$. Then $e_j a e_t \neq 0$. Hence there exists a non-zero path from j to t, that is, $j \leq t$. Hence j is the minimum element in $\Omega(i, T)$.

Conversely, assume that there exists the minimum element j in $\Omega(i, T)$. For each non-zero path $p_{i,t}$, we have $t \in \Omega(i, T)$. It implies $j \leq t$ by the minimum property of j. Then there exists a path p from j to t in (Q_0, Q_1). Since Q is a quiver with all commutativity relations, we get $p_{i,t} = p_{i,j}p$. Here p is non-zero for $p_{i,t}, p_{i,j}$ are both non-zero. Hence $p_{i,t}$ can factor through $p_{i,j}$, that is, j is the shadow of i in T. □

Assume that T is the subset of Q_0 which satisfies that $\Omega(i, T)$ has the minimum element for each vertex i with non-empty set $\Omega(i, T)$. We define a map f_T on Q_0 with respect to T as following:

(1) $f_T(i)$ is the shadow of i in T when $\Omega(i, T) \neq \emptyset$; (2) $f_T(i) = i$ when $\Omega(i, T) = \emptyset$.

For subset X of Q_0, denote by $f_T(X)$ the image set of $f_T|_X$, and by X^T the subset of X consisting of the stable vertices under f_T. That is,

$$X^T = \{i \in X | f_T(i) = i\} = \{i \in X | \Omega(i, T) = \emptyset\}.$$

Lemma 2.3. *If the partial ordering is compatible with relations on $X \cup f_T(X)$, the map f_T restricting to $X \backslash X^T$ preserves the partial ordering.*

Proof. Let $i, j \in X \backslash X^T$ with $i \leq j$. There exists a non-zero path $p_{i,j}$. For $i, j \notin X^T$, we have that $\Omega(i, T), \Omega(j, T)$ are non-empty. There exist non-zero paths $p_{i,f_T(i)}$ and $p_{j,f_T(j)}$. It implies $i \leq j \leq f_T(j)$. Then $p_{i,f_T(j)} = p_{i,j}p_{j,f_T(j)}$ is non-zero path by the compatibility condition. We get $f_T(j) \in \Omega(i, T)$. By the definition of the shadow, $p_{i,f_T(j)}$ can factor through $p_{i,f_T(i)}$. Hence there exists a non-zero path $p_{f_T(i),f_T(j)}$ which implies $f_T(i) \leq f_T(j)$. We finish the proof. □

2.3. **Triangular matrix algebras.** Let R, S be rings and ${}_RM_S$ an R-S-bimodule. We denote by $T(R,S;M)$ the upper triangular matrix ring

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}.$$

From now on, we consider the algebra A assumed in Subsection 2.1 as the triangular matrix algebra $T(R,S;M)$ where R,S are algebras. Then we may assume that $1_R = \sum_{i\in Q_{R,0}} e_i$ and $1_S = \sum_{j\in Q_{S,0}} e_j$ with $Q_{A,0} = Q_{R,0} \cup Q_{S,0}$ where $\{e_i | i \in Q_{R,0}\}$ (resp. $\{e_j | j \in Q_{S,0}\}$) is a complete set of primitive orthogonal idempotents of R (resp. S). Then M is generated by all non-zero path from the vertex in $Q_{R,0}$ to the vertex in $Q_{S,0}$ as a $\mathbf{k}$-space since $M = 1_R \cdot A \cdot 1_S$.

2.3.1. *Description of* $\mathrm{Hom}_S(M,S)$. In this subsection, we always assume that M is projective as a right S-module and $Q_{S,0}$ satisfies that each non-empty set $\Omega(i, Q_{S,0})$ for $i \in Q_{A,0}$ has the minimum element for convenience. Here we give some description of $\mathrm{Hom}_S(M,S)$ for future needs.

Lemma 2.4. *There exists left S-module isomorphism*

$$\mathrm{Hom}_S(M,S) \cong \bigoplus_{i\in Q_{R,0}} Se_{f_S(i)}$$

where f_S is the map in Subsection 2.2 with respect to the subset $Q_{S,0}$.

Proof. For each vertex i in $Q_{R,0}$ with $\Omega(i, Q_{S,0})$ empty, there exists no non-zero path from i to the vertex in $Q_{S,0}$. For each vertex i in $Q_{R,0}$ with $\Omega(i, Q_{S,0})$ non-empty, each non-zero path from i to the vertex in $Q_{S,0}$ can factor through a non-zero path $p_{i,f_S(i)}$ where $f_S(i) \in Q_{S,0}$ by the definition of the shadow. It implies M is generated by all non-zero path from i to $f_S(i)$ for each i as right S-module. Because M is projective as a right S-module, we have $M \cong \bigoplus_{i\in U} e_{f_S(i)}S$ as a right S-module isomorphism where U consists of the vertice in $Q_{R,0}$ with $\Omega(i, Q_{S,0})$ non-empty. Observe that $e_{f_S(i)}S = 0$ when $\Omega(i, Q_{S,0}) = \emptyset$. Therefore $M \cong \bigoplus_{i\in Q_{R,0}} e_{f_S(i)}S$. By [1, Lemma I.4.2], we have left S-module isomorphism

$$\mathrm{Hom}_S(M,S) \cong \bigoplus_{i\in Q_{R,0}} \mathrm{Hom}_S(e_{f_S(i)}S, S) \cong \bigoplus_{i\in Q_{R,0}} Se_{f_S(i)}.$$

□

The following lemma exhibits the $\mathbf{k}$-basis of S-R-module $\mathrm{Hom}_S(M,S)$. Note that M is generated by all non-zero path from the vertex t in $Q_{R,0}$ to the vertex $f_S(t)$ as right S-module. For each $i \in Q_{R,0}$ and $j \in Q_{S,0}$, we may define the right S-module homomorphism $g_{j,i}$ in $\mathrm{Hom}_S(M,S)$ by $g_{j,i}(p_{t,f_S(t)}) = \delta_{it} p_{j,f_S(i)}$ for each $t \in Q_{R,0}$ where δ_{it} is the Kronecker function as

$$\delta_{it} = \begin{cases} 1 & i = t \\ 0 & i \neq t \end{cases}.$$

Lemma 2.5. *We have that the set $\{g_{j,i} | i \in Q_{R,0}, f_S(i) \in Q_{S,0}, j \leq f_S(i)\}$ form a $\mathbf{k}$-basis of S-R-module* $\mathrm{Hom}_S(M,S)$. *Moreover, $e_{j'} g_{j,i} e_{i'} = \delta_{ii'} \delta_{jj'} g_{j,i}$ for each $i' \in Q_{R,0}$ and $j' \in Q_{S,0}$.*

Proof. By the S-R-module structure on $\mathrm{Hom}_S(M,S)$, we have $e_{j'} g_{j,i} e_{i'}(m) = e_{j'} \cdot g_{j,i}(e_{i'} \cdot m)$ for each $m \in M$. Then $e_{j'} g_{j,i} e_{i'}(p_{t,f_S(t)}) = p_{j,f_S(i)}$ when $i = i' = t$, $j = j' \leq f_S(i)$, and $e_{j'} g_{j,i} e_{i'}(p_{t,f_S(t)}) = 0$ otherwise. Hence the map $\mathrm{Hom}_S(M,S) \to \bigoplus_{i\in Q_{R,0}} Se_{f_S(i)}$ defined by $g_{j,i} \mapsto p_{j,f_S(i)}$ for $i \in Q_{R,0}, f_S(i) \in Q_{S,0}, j \leq f_S(i)$ is a bijection as $\mathbf{k}$-spaces. Hence we can get the result from Lemma 2.4. □

Next we will exhibit the generators of $\mathrm{Hom}_S(M,S)$ as an S-R-module.

Proposition 2.6. *Assume that the partial ordering is compatible with relations on $Q_{R,0} \cup f_S(Q_{R,0})$ and there exists the minimum vertex $v(t)$ in the pre-image set $f_S^{-1}(t)$ for each $t \in Q_{S,0} \cap f_S(Q_{R,0})$. Then $\{g_{t,v(t)} \mid t \in Q_{S,0} \cap f_S(Q_{R,0})\}$ generates* $\mathrm{Hom}_S(M,S)$ *as S-R-bimodule.*

Proof. It is clear that $S \cdot g_{t,v(t)} \cdot R \subseteq \mathrm{Hom}_S(M,S)$ for each $t \in Q_{S,0} \cap f_S(Q_{R,0})$. For $i \in Q_{R,0}$ with $f_S(i) \in Q_{S,0}$ and $j \le f_S(i)$, we have $g_{j,i} = p_{j,f_S(i)} \cdot g_{f_S(i),i}$ by the left S-module structure on $\mathrm{Hom}_S(M,S)$. Since $i \in f_S^{-1}\big(f_S(i)\big)$, we have $v\big(f_S(i)\big) \le i$ for the minimum property of $v\big(f_S(i)\big)$. Based on the assumption of the compatibility, the path $p_{v(f_S(i)),i}$ is non-zero. Then $g_{f_S(i),i} = g_{f_S(i),v(f_S(i))} \cdot p_{v(f_S(i)),i}$ by the right R-module structure on $\mathrm{Hom}_S(M,S)$. Hence we have

$$g_{j,i} = p_{j,f_S(i)} \cdot g_{f_S(i),v(f_S(i))} \cdot p_{v(f_S(i)),i}.$$

By Lemma 2.5, $\mathrm{Hom}_S(M,S) \subseteq \sum_{t\in Q_{S,0}\cap f_S(Q_{R,0})} S \cdot g_{t,v(t)} \cdot R$. Thus we finish the proof. □

2.3.2. *Description of* $\mathrm{D}(M)$. Let $\mathrm{D} = \mathrm{Hom}_{\mathbf{k}}(-,\mathbf{k})$ be the standard duality. For $i, i' \in Q_{R,0}, j, j' \in Q_{S,0}$, let $p^*_{i,j}$ be the element of $\mathrm{D}(M)$ defined by $p^*_{i,j}(p_{i',j'}) = \delta_{ii'}\delta_{jj'}$. Obviously, we have that $p_{j',j} \cdot p^*_{i,j} \cdot p_{i,i'} = p^*_{i',j'}$ and $\{p^*_{i,j} | i \in Q_{R,0}, j \in Q_{S,0}, p_{i,j} \neq 0\}$ is the dual basis of $\mathrm{D}(M)$. We say that a non-zero path $p_{i,j}$ for $i \in Q_{R,0}, j \in Q_{S,0}$ is the maximal path in M if $p_{i',i}p_{i,j}p_{j,j'}$ for $i' \in Q_{R,0}, j' \in Q_{S,0}$ is non-zero implies $i' = i$ and $j' = j$. Denote by Γ_M the set of all maximal paths in M. We have the following description of S-R-bimodule $\mathrm{D}(M)$.

Lemma 2.7. *We have that* $\{p^* \mid p \in \Gamma_M\}$ *generates* $\mathrm{D}(M)$ *as* S-R*-bimodule.*

Proof. Clearly, $S \cdot p^* \cdot R \subseteq \mathrm{D}(M)$ holds for each $p \in \Gamma_M$. Conversely, for each non-zero path $p_{i,j}$ with $i \in Q_{R,0}, j \in Q_{S,0}$, there exist some $i' \in Q_{R,0}, j' \in Q_{S,0}$ such that $p_{i',i}p_{i,j}p_{j,j'} \in \Gamma_M$ for $A = T(R,S;M)$ is the upper triangular matrix algebra. Because $p^*_{i,j} = p_{j,j'} \cdot (p_{i',i}p_{i,j}p_{j,j'})^* \cdot p_{i',i}$, we have $\mathrm{D}(M) \subseteq \sum_{p\in\Gamma_M} S \cdot p^* \cdot R$. We finish the proof. □

3. Quiver realizations of two upper triangular matrix algebras

3.1. **Derived equivalences between upper triangular matrix algebras.** In this subsection, we will recall some derived equivalences between triangular matrix algebras and show an unexpected derived equivalence as a corollary.

First, Ladkani deduced a derived equivalence which will play a key role in our paper between triangular matrix algebras by constructing a tilting complex.

Proposition 3.1. [11, Theorem 4.5] *Let* R, S *be rings and* T_S *a tilting* S*-module. Let* ${}_RM_S$ *be an* R-S*-bimodule such that* $M_S \in \mathrm{per}\, S$*, that is,* M_S *is a right* S*-module and quasi-isomorphic to a bounded complex of finitely generated projective* S*-modules in* $\mathrm{Der}^b(S)$*, and* $\mathrm{Ext}^n_S(M_S, T_S) = 0$ *for all* $n > 0$*. Then the triangular matrix rings*

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{End}_S(T_S) & \mathrm{Hom}_S(M, T_S) \\ 0 & R \end{pmatrix}$$

are derived equivalent.

According to this, Ladkani also showed the following result by taking $T_S = S$ and $T_S = \mathrm{D}(S)$.

Proposition 3.2. [11, Theorem 4.9 and Corollary 4.11] *Let* R, S *be algebras and* ${}_RM_S$ *an* R-S*-bimodule.*

(1) *If* $M_S \in \mathrm{per}\, S$ *and* $\mathrm{Ext}^n_S(M_S, S) = 0$ *for all* $n > 0$*. Then there is a derived equivalence between the triangular matrix algebras* $T(R,S;M)$ *and* $T\big(S,R;\mathrm{Hom}_S(M,S)\big)$*;*
(2) *If* S *is an Artin algebra with* $\mathrm{gl.dim}\, S < \infty$ *and* M *is finitely generated as an* S*-module. Then there is a derived equivalence between* $T(R,S;M)$ *and* $T\big(S,R;\mathrm{D}(M)\big)$*.*

As a byproduct, we provide a corollary of Proposition 3.1 when R, S are both the tensor products of two algebra where one of them is the path algebra of type $\vec{A}$. Our convention is that $\otimes$ (without subscript) will always mean tensor product over $\mathbf{k}$. In order to calculate the endomorphism algebra of tilting object, we need the Cartan-Eilenberg isomorphism as following.

Lemma 3.3. [2, Theorem XI.3.1] *Let* A, B *be algebras over* $\mathbf{k}$*. For* A*-modules* M, M' *and* B*-modules* N, N'*, there is an isomorphism*

$$\mathrm{Hom}_A(M,M') \otimes \mathrm{Hom}_B(N,N') \simeq \mathrm{Hom}_{A\otimes B}(M \otimes N, M' \otimes N').$$

Corollary 3.4. *Let R, S be algebras over $\mathbf{k}$. Assume that M is projective S-module and N is projective $\mathbf{k}\vec{A}_n$-module.*

(1) *If $M_S \in \operatorname{per} S$ and $\operatorname{Ext}^n_S(M_S, S) = 0$ for all $n > 0$. Then there exists a derived equivalence between the triangular matrix algebras $T(R \otimes \mathbf{k}\vec{A}_m, S \otimes \mathbf{k}\vec{A}_n; M \otimes N)$ and $T\big(S \otimes \mathbf{k}\vec{A}_n, R \otimes \mathbf{k}\vec{A}_m; \operatorname{Hom}_S(M, S) \otimes \mathrm{D}(N)\big)$;*

(2) *If S is an Artin algebra with $\operatorname{gl.dim} S < \infty$ and M is finitely generated as an S-module. Then there exists a derived equivalence between the triangular matrix algebras $T(R \otimes \mathbf{k}\vec{A}_m, S \otimes \mathbf{k}\vec{A}_n; M \otimes N)$ and $T\big(S \otimes \mathbf{k}\vec{A}_n, R \otimes \mathbf{k}\vec{A}_m; \mathrm{D}(M) \otimes \operatorname{Hom}_{\mathbf{k}\vec{A}_n}(N, \mathbf{k}\vec{A}_n)\big)$.*

Proof. (1) Since $\mathrm{D}(\mathbf{k}\vec{A}_n)$ is a tilting $\mathbf{k}\vec{A}_n$-module whose endomorphism algebra is isomorphic to $\mathbf{k}\vec{A}_n$ and S is a tilting S-module, we have $S \otimes \mathrm{D}(\mathbf{k}\vec{A}_n)$ is a tilting $S \otimes \mathbf{k}\vec{A}_n$-module by [12, Theorem A] whose endomorphism algebra is isomorphic to $S \otimes \mathbf{k}\vec{A}_n$. According to the Hom-tensor adjointness and Lemma 3.3, we have

$$\begin{aligned}
\operatorname{Hom}_{S\otimes\mathbf{k}\vec{A}_n}(M \otimes N, S \otimes \mathrm{D}(\mathbf{k}\vec{A}_n)) &= \operatorname{Hom}_S(M, S) \otimes \operatorname{Hom}_{\mathbf{k}\vec{A}_n}(N, \mathrm{D}(\mathbf{k}\vec{A}_n)) \\
&= \operatorname{Hom}_S(M, S) \otimes \operatorname{Hom}_{\mathbf{k}\vec{A}_n}\big(N, \operatorname{Hom}_{\mathbf{k}}(\mathbf{k}\vec{A}_n, \mathbf{k})\big) \\
&= \operatorname{Hom}_S(M, S) \otimes \operatorname{Hom}_{\mathbf{k}}(N \otimes_{\mathbf{k}\vec{A}_n} \mathbf{k}\vec{A}_n, \mathbf{k}) \\
&= \operatorname{Hom}_S(M, S) \otimes \mathrm{D}(N).
\end{aligned}$$

Then the upper triangular matrix algebra $T(R \otimes \mathbf{k}\vec{A}_m, S \otimes \mathbf{k}\vec{A}_n; M \otimes N)$ is derived equivalent to $T\big(S \otimes \mathbf{k}\vec{A}_n, R \otimes \mathbf{k}\vec{A}_m; \operatorname{Hom}_S(M, S) \otimes \mathrm{D}(N)\big)$ by Proposition 3.1.

(2) Under the assumption, $\mathrm{D}(S)$ is a tilting S-module whose endomorphism algebra is isomorphic to S, (see the proof in [11, Theorem 4.9]). Then $\mathrm{D}(S) \otimes \mathbf{k}\vec{A}_n$ is a tilting $S \otimes \mathbf{k}\vec{A}_n$-module by [12, Theorem A] whose endomorphism algebra is isomorphic to $S \otimes \mathbf{k}\vec{A}_n$. By the Hom-tensor adjointness and Lemma 3.3, we have

$$\begin{aligned}
\operatorname{Hom}_{S\otimes\mathbf{k}\vec{A}_n}(M \otimes N, \mathrm{D}(S) \otimes \mathbf{k}\vec{A}_n) &= \operatorname{Hom}_S(M, \mathrm{D}(S)) \otimes \operatorname{Hom}_{\mathbf{k}\vec{A}_n}(N, \mathbf{k}\vec{A}_n) \\
&= \operatorname{Hom}_S(M, \operatorname{Hom}_{\mathbf{k}}(S, \mathbf{k})) \otimes \operatorname{Hom}_{\mathbf{k}\vec{A}_n}(N, \mathbf{k}\vec{A}_n) \\
&= \operatorname{Hom}_{\mathbf{k}}(M \otimes_S S, \mathbf{k}) \otimes \operatorname{Hom}_{\mathbf{k}\vec{A}_n}(N, \mathbf{k}\vec{A}_n) \\
&= \mathrm{D}(M) \otimes \operatorname{Hom}_{\mathbf{k}\vec{A}_n}(N, \mathbf{k}\vec{A}_n)
\end{aligned}$$

Then the result holds by Proposition 3.1 immediately. □

In the rest of this section, let $Q = (Q_0, Q_1, \rho)$ be an acyclic quiver with all commutativity relations such that the associated algebra $A = \mathbf{k}Q/(\rho)$ is an upper triangular matrix algebra $T(R, S; M)$ as in Subsection 2.3. We will denote by $\widetilde{A} = T\big(S, R; \operatorname{Hom}_S(M, S)\big)$ and $\widehat{A} = T\big(S, R; \mathrm{D}(M)\big)$ for simplicity.

3.2. **Quiver realization of $T\big(S, R; \mathbf{Hom}_S(M, S)\big)$.** In this subsection, we assume that M is projective as a right S-module and each non-empty set $\Omega(i, Q_{S,0})$ has the minimum element. Moreover we assume that the partial ordering is compatible with relations on $Q_{R,0} \cup f_S(Q_{R,0})$ and there exists the minimum vertex $v(t)$ in the pre-image set $f_S^{-1}(t)$ for each $t \in Q_{S,0} \cap f_S(Q_{R,0})$ as in Proposition 2.6. Now we will provide a quiver realization of $\widetilde{A}$. Before this, we need the following Lemma to describe the relations in the quiver of $\widetilde{A}$.

Lemma 3.5. *Let $t \in Q_{S,0} \cap f_S(Q_{R,0})$. For each $i \in Q_{R,0}$ with non-zero path $p_{v(t),i}$ and each $j \in Q_{S,0}$ with non-zero path $p_{j,t}$, we have the following facts*

(1) *$p_{j,t} \cdot g_{t,v(t)}$ is non-zero;*

(2) *$g_{t,v(t)} \cdot p_{v(t),i} = 0$ if and only if $\Omega(i, Q_{S,0}) = \emptyset$.*

Proof. (1) By the left S-module structure on $\operatorname{Hom}_S(M, S)$, we have $p_{j,t} \cdot g_{t,v(t)} = g_{j,v(t)}$ which maps $p_{v(t),t}$ to $p_{j,t}$. It is obviously non-zero since $p_{j,t}$ is non-zero.

(2) Since $g_{t,v(t)} \cdot p_{v(t),i}(m) = g_{t,v(t)}(p_{v(t),i} \cdot m)$ for $m \in M$, we know that $g_{t,v(t)} \cdot p_{v(t),i}$ maps $p_{i,t}$ (if exists) to $p_{t,t}$. Then the path in M which is mapped to non-zero element under $g_{t,v(t)} \cdot p_{v(t),i}$ must be of form $p_{i,j'}$ for some $j' \in Q_{S,0}$ satisfying that $p_{i,j'} = p_{i,t}p_{t,j'}$ is non-zero. Hence $g_{t,v(t)} \cdot p_{v(t),i} \neq 0$ if and only if there exists a non-zero path $p_{i,t}$, that is, $\Omega(i, Q_{S,0})$ is non-empty. So $g_{t,v(t)} \cdot p_{v(t),i} = 0$ if and only if $\Omega(i, Q_{S,0}) = \emptyset$.

□

Construction 3.6. *We construct* $\widetilde{Q} = (\widetilde{Q}_0, \widetilde{Q}_1, \widetilde{\rho})$ *as following.*

- $\widetilde{Q}_0 = Q_0 = Q_{R,0} \cup Q_{S,0}$;
- $\widetilde{Q}_1 = Q_{R,1} \cup Q_{S,1} \cup \{\alpha_t : t \to v(t) | t \in Q_{S,0} \cap f_S(Q_{R,0})\}$;
- $\widetilde{\rho}$ *consists of the relations in* ρ, *the zero relation* $\alpha_t \cdot p_{v(t),i}$ *for each* t *and each* $i \in Q_{R,0}$ *with* $\Omega(i, Q_{S,0}) = \emptyset$, *and all the commutativity relations from the vertex in* $Q_{S,0}$ *to the vertex in* $Q_{R,0}$.

Proposition 3.7. *We have that* $\widetilde{A}$ *is an algebra associated to the bound quiver* $\widetilde{Q}$. *Moreover, there is an equivalence* $\mathrm{Der}^b\big(\mathbf{k}Q/(\rho)\big) \simeq \mathrm{Der}^b\big(\mathbf{k}\widetilde{Q}/(\widetilde{\rho})\big)$.

Proof. Denote by $Q_{\widetilde{A}}$ the quiver of $\widetilde{A}$. Obviously, $1_{\widetilde{A}} = 1_R + 1_S = \sum_{i \in Q_{R,0}} e_i + \sum_{j \in Q_{S,0}} e_j$. We get that $Q_{\widetilde{A},0} = Q_{R,0} \cup Q_{S,0} = \widetilde{Q}_0$.

Note that $\mathrm{rad}(\widetilde{A}) = \mathrm{rad}(R) \oplus \mathrm{rad}(S) \oplus \mathrm{Hom}_S(M, S)$ as $\mathbf{k}$-spaces. By the multiplication in the upper triangular matrix algebra and Proposition 2.6,

$$\mathrm{rad}(\widetilde{A})/\mathrm{rad}^2(\widetilde{A}) \cong \big(\mathrm{rad}(R)/\mathrm{rad}^2(R)\big) \oplus \big(\mathrm{rad}(S)/\mathrm{rad}^2(S)\big) \oplus \mathrm{span}_{\mathbf{k}}\{\overline{g}_{t,v(t)} | t \in Q_{S,0} \cap f_S(Q_{R,0})\}.$$

By Lemma 2.5, $\overline{g}_{t,v(t)}$ induces a new arrow α_t from the vertex t to the vertex $v(t)$. Hence $Q_{\widetilde{A},1} = \widetilde{Q}_1$.

Next we consider the relation set $\rho_{\widetilde{A}}$ of $Q_{\widetilde{A}}$. It is clear that $\rho \subseteq \rho_{\widetilde{A}}$. Besides these relations, the relations in $\rho_{\widetilde{A}}$ are induced by the added arrows α_t for $t \in Q_{S,0} \cap f_S(Q_{R,0})$. On the one hand, all the zero relations are $\alpha_t \cdot p_{v(t),i}$ for each t and each $i \in Q_{R,0}$ with $\Omega(i, Q_{S,0}) = \emptyset$ by Lemma 3.5. On the other hand, for $i \in Q_{R,0}$ and $j \in Q_{S,0}$, if there exist $t, t' \in Q_{S,0} \cap f_S(Q_{R,0})$ with non-zero paths $p_{v(t),i}, p_{v(t'),i}, p_{j,t}, p_{j,t'}$, one can check that $p_{j,t}g_{t,v(t)}p_{v(t),i} = g_{j,i} = p_{j,t'}g_{t',v(t')}p_{v(t'),i}$ similar as the proof in Proposition 2.6. Then $\rho_{\widetilde{A}}$ contains all the commutativity relations from the vertex in $Q_{S,0}$ to the vertex in $Q_{R,0}$. Hence $\rho_{\widetilde{A}} = \widetilde{\rho}$. To sum up, $\widetilde{A}$ is an algebra associated to the bound quiver $\widetilde{Q}$.

Because M is a projective S-module under the assumption, we have $M_S \in \mathrm{per}\, S$ and $\mathrm{Ext}^n_S(M_S, S) = 0$ for all $n > 0$. By Proposition 3.2(1), there exist equivalences

$$\mathrm{Der}^b\big(\mathbf{k}Q/(\rho)\big) = \mathrm{Der}^b(A) \simeq \mathrm{Der}^b(\widetilde{A}) = \mathrm{Der}^b\big(\mathbf{k}\widetilde{Q}/(\widetilde{\rho})\big).$$

□

3.3. Quiver realization of $T\big(S, R; \mathbf{D}(M)\big)$.

In this subsection, we provide a quiver realization of $\widehat{A}$.

Construction 3.8. *We construct* $\widehat{Q} = (\widehat{Q}_0, \widehat{Q}_1, \widehat{\rho})$ *as following.*

- $\widehat{Q}_0 = Q_0 = Q_{R,0} \cup Q_{S,0}$;
- $\widehat{Q}_1 = Q_{R,1} \cup Q_{S,1} \cup \{\beta_{j,i} : j \to i | p_{i,j} \in \Gamma_M\}$;
- $\widehat{\rho}$ *consists of the relations in* ρ, *the zero relation* $p_{j',j} \cdot \beta_{j,i} \cdot p_{i,i'}$ *whenever there exists no non-zero path from* $i' \in Q_{R,0}$ *to* $j' \in Q_{S,0}$, *and all the commutativity relations from the vertex in* $Q_{S,0}$ *to the vertex in* $Q_{R,0}$.

Proposition 3.9. *We have* $\widehat{A}$ *is an algebra associated to the bound quiver* $\widehat{Q}$. *Moreover, we have* $\mathrm{Der}^b\big(\mathbf{k}Q/(\rho)\big) \simeq \mathrm{Der}^b\big(\mathbf{k}\widehat{Q}/(\widehat{\rho})\big)$.

Proof. Denote by $Q_{\widehat{A}}$ the quiver of $\widehat{A}$. Obviously, $1_{\widehat{A}} = 1_R + 1_S = \sum_{i \in Q_{R,0}} e_i + \sum_{j \in Q_{S,0}} e_j$. We get that $Q_{\widehat{A},0} = Q_{R,0} \cup Q_{S,0} = \widehat{Q}_0$.

Note that $\mathrm{rad}(\widehat{A}) = \mathrm{rad}(R) \oplus \mathrm{rad}(S) \oplus \mathrm{D}(M)$ as $\mathbf{k}$-spaces. By the multiplication in the upper triangular matrix algebra and Lemma 2.7,

$$\mathrm{rad}(\widehat{A})/\mathrm{rad}^2(\widehat{A}) \cong \big(\mathrm{rad}(R)/\mathrm{rad}^2(R)\big) \oplus \big(\mathrm{rad}(S)/\mathrm{rad}^2(S)\big) \oplus \mathrm{span}_{\mathbf{k}}\{\overline{p_{i,j}^*} | p_{i,j} \in \Gamma_M\}.$$

Here $\overline{p_{i,j}^*}$ induces a new arrow $\beta_{j,i}$ from the vertex j to the vertex i for $e_j p_{i,j}^* e_i = p_{i,j}^*$. Hence $Q_{\widehat{A},1} = \widehat{Q}_1$.

For the relation set $\rho_{\widehat{A}}$ of $Q_{\widehat{A}}$, $\rho \subseteq \rho_{\widehat{A}}$. Besides these, the relations in $\rho_{\widehat{A}}$ are induced by the added arrows $\beta_{j,i}$ for $p_{i,j} \in \Gamma_M$. On the one hand, if there exists no non-zero path from $i' \in Q_{R,0}$ to $j' \in Q_{S,0}$, we have $p_{j',j} \cdot p_{i,j}^* \cdot p_{i,i'} = 0$. It implies that all the zero relations are $p_{j',j} \cdot \beta_{j,i} \cdot p_{i,i'}$ whenever there exists no non-zero path from $i' \in Q_{R,0}$ to $j' \in Q_{S,0}$. On the other hand, for $i'' \in Q_{R,0}$ and $j'' \in Q_{S,0}$ with $p_{i'',j''}$ non-zero, if there exist $p_{i,j}, p_{i',j'} \in \Gamma_M$ with non-zero paths $p_{i,i''}, p_{i',i''}, p_{j'',j}, p_{j'',j'}$, we know that $p_{j'',j} p_{i,j}^* p_{i,i''} = p_{i'',j''}^* = p_{j'',j'} p_{i',j'}^* p_{i',i''}$. Then $\rho_{\widehat{A}}$ contains all the commutativity relations from the vertex in $Q_{S,0}$ to the vertex in $Q_{R,0}$. Hence $\rho_{\widehat{A}} = \widehat{\rho}$. To sum up, $\widehat{A}$ is an algebra associated to the bound quiver $\widehat{Q}$.

Finally, since there is the natural partial ordering $\leq$ on $Q_{S,0}$ by the assumption, we know the global dimension of S is finite. Note that A is finite dimensional. We have M is finitely generated as an S-module. By Proposition 3.2(2), we have

$$\mathrm{Der}^b\big(\mathbf{k}Q/(\rho)\big) = \mathrm{Der}^b(A) \simeq \mathrm{Der}^b(\widehat{A}) = \mathrm{Der}^b\big(\mathbf{k}\widehat{Q}/(\widehat{\rho})\big).$$

□

4. Derived equivalences between defective rectangle algebras

In this section, we will apply the Constructions 3.6,3.8 and use Proposition 3.7,3.9 to get some derived equivalences between "rectangles" under the different gluing methods. Further we will prove the half part of the main theorem in our paper.

4.1. **Gluing "rectangles".** We denote by $Q(m,n)$ the bound quiver in Figure 1 with commutative relations for $m, n \geq 1$. The algebra associated to the bound quiver $Q(m,n)$ is isomorphic to the tensor algebra $\mathbf{k}\vec{A}_m \otimes \mathbf{k}\vec{A}_n$. We will call $Q(m,n)$ (resp. the corresponding algebra $A\big(Q(m,n)\big)$) the "rectangle" (resp. the "rectangle" algebra) for simplicity. Obviously, the algebra associated to the bound quiver $Q(m,n)$ has finite global dimension. Now we provide a gluing construction of "rectangles"

Construction 4.1. *For positive integers u_{ij} with $u_{ij} = u_{ji} \leq \min\{m_i, m_j\}$ where $1 \leq i \neq j \leq r$, let $\mathbf{l}_{ij} = \{l_{ij}^{(k)} | 1 \leq l_{ij}^{(1)} \leq l_{ij}^{(2)} \cdots \leq l_{ij}^{(u_{ij})} \leq m_i\}$ be an integer sequence. Denote by $\mathbf{l}_i = \{\mathbf{l}_{ij} | j \neq i\}$. We define by $\#_{i=1}^r Q(m_i, n_i; \mathbf{l}_i) = Q(m_1, n_1; \mathbf{l}_1) \# Q(m_2, n_2; \mathbf{l}_2) \# \cdots \# Q(m_r, n_r; \mathbf{l}_r)$ the quiver by gluing quivers $Q(m_i, n_i)$ $(1 \leq i \leq r)$ as following:*

- *the vertex set is $\bigcup_{i=1}^r Q(m_i, n_i)$; and in order to distinguish the vertices in different rectangles, we will use $(a, b; i)$ to represent the vertex (a, b) in the i-th rectangle $Q(m_i, n_i)$;*
- *we reserve the arrows in all rectangles, and add the arrow from $(l_{ij}^{(k)}, n_i; i)$ to $(l_{ji}^{(k)}, 1; j)$ for each $1 \leq i < j \leq r$;*
- *we reserve the relations in all rectangles, and add the relations which makes all the paths from $(l_{ij}^{(k)}, n_i; i)$ to $(l_{ji}^{(k')}, 1; j)$ commute for each $1 \leq i < j \leq r$ and $k \leq k'$.*

For convenience, we will denote by $\mathbf{l}_{(m)}$ the integer sequence $\{1, 2, \cdots, m\}$.

Example 4.2. (1) If $r = 2$, the following figure shows the added arrows and the added relations in the bound quiver $Q(m_1, n_1; \mathbf{l}_{12}) \# Q(m_2, n_2; \mathbf{l}_{21})$.

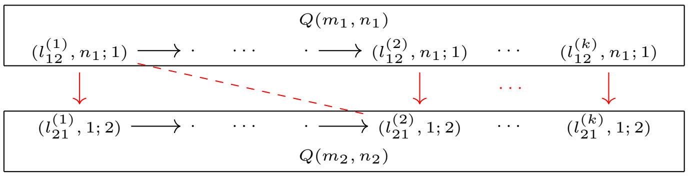

FIGURE 2. the quiver $Q(m_1, n_1; \mathbf{l}_{12})\#Q(m_2, n_2; \mathbf{l}_{21})$

(2) If $r = 3$ and $m_1 = m_3 \geq m_2$, we take $\mathbf{l}_{13} = \mathbf{l}_{31} = \mathbf{l}_{(m_1)}$, $\mathbf{l}_{21} = \mathbf{l}_{23} = \mathbf{l}_{(m_2)}$, $\mathbf{l}_{12} = \mathbf{l}_{32} = \{l_1, l_2, l_3, \cdots, l_{m_2}\}$. The bound quiver $Q(m_1, n_1; \mathbf{l}_{12}, \mathbf{l}_{13})\#Q(m_2, n_2; \mathbf{l}_{21}, \mathbf{l}_{23})\# Q(m_3, n_3; \mathbf{l}_{31}, \mathbf{l}_{32})$ is the quiver as following with all commutativity relations.

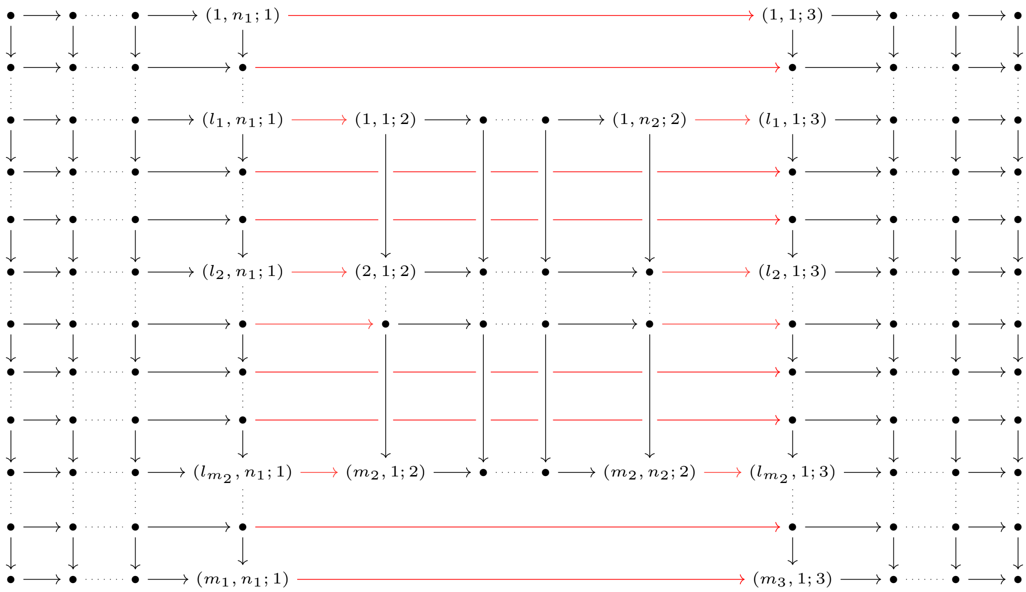

FIGURE 3. the quiver $Q(m_1, n_1; \mathbf{l}_{12}, \mathbf{l}_{13})\#Q(m_2, n_2; \mathbf{l}_{21}, \mathbf{l}_{23})\#Q(m_3, n_3; \mathbf{l}_{31}, \mathbf{l}_{32})$

4.2. **A derived equivalence on gluing "rectangles".** In this subsection, we focus on the gluing "rectangles" in Example 4.2(2), and we always assume that $m_1 \geq m_2$. Denote by A_0 the algebra associated to $Q(m_2, n_2; \mathbf{l}_{(m_2)})\#Q(m_1, n_1; \mathbf{l})$ where $\mathbf{l} = \{l_1, l_2, l_3, \cdots, l_{m_2}\}$. Also we denote by A_t the algebra associated to $Q(m_1, t; \mathbf{l}, \mathbf{l}_{(m_1)})\#Q(m_2, n_2; \mathbf{l}_{(m_2)}, \mathbf{l}_{(m_2)})\#Q(m_1, n_1 - t; \mathbf{l}_{(m_1)}, \mathbf{l})$ for $1 \leq t \leq n_1$. The following derived equivalences are important.

Proposition 4.3. *There exist the derived equivalences* $\mathrm{Der}^b(A_s) \simeq \mathrm{Der}^b(A_t)$ *for each* $0 \leq s, t \leq n_1$.

Proof. Denote by Q the quiver $Q(m_1, t; \mathbf{l}, \mathbf{l}_{(m_1)})\#Q(m_2, n_2; \mathbf{l}_{(m_2)}, \mathbf{l}_{(m_2)})\#Q(m_1, n_1 - t; \mathbf{l}_{(m_1)}, \mathbf{l})$ for $0 \leq t < n_1$. First, we divide Q_0 into $Q_{R,0}$ and $Q_{S,0}$ such that the assumptions in Subsection 3.2 hold.

We set $Q_{S,0} = \{(u, n_1 - t; 3) | 1 \leq u \leq m_1\}$ and $Q_{R,0} = Q_0 \backslash Q_{S,0}$. Denote by R, S the algebras associated to the full subquiver Q_R, Q_S. Hence A_t is isomorphic to the algebra $T(R, S; M)$, where M is the projective S-module $\oplus_{1 \leq u \leq m_1} e_{(u, n_1 - t; 3)} S$. In this partition, each set $\Omega(x, Q_{S,0})$ for $x \in Q_{R,0}$ has the minimum element, and the natural partial ordering is compatible with relations on $Q_{R,0} \cup f_S(Q_{R,0})$. That is, all the assumptions in Subsection 3.2 hold.

For $t \neq 0$, we have $v(u, n_1 - t; 3) = (u, 1; 1)$. For $t = 0$, we have

$$v(u, n_1; 3) = \begin{cases} (i, 1; 2) & u = l_i, \\ (u, 1; 3) & \text{otherwise} \end{cases}$$

By Construction 3.6, we can construct a quiver $\widetilde{Q}$ for both t which is just the quiver

$$Q(m_1, t+1; \mathbf{l}, \mathbf{l}_{(m_1)}) \# Q(m_2, n_2; \mathbf{l}_{(m_2)}, \mathbf{l}_{(m_2)}) \# Q(m_1, n_1 - t - 1; \mathbf{l}_{(m_1)}, \mathbf{l}).$$

By Proposition 3.7, we know that A_t is derived equivalent to A_{t+1}. The result holds by induction. $\square$

Remark 4.4. We omit the case $n_1 = 1$ because the only difference in the proof is $M = \oplus_{1 \le u \le m_2} e_{(l_u,1;3)} S$.

As a consequence, we have the following result.

Corollary 4.5. *For positive integer $a < n$, there exist the equivalences*

$$\mathrm{Der}^b(A[{}^{-a}Q(m,n)]) \simeq \mathrm{Der}^b(A[Q(m,n)^{-a}]) \simeq \mathrm{Der}^b(A[{}_{-a}Q(m,n)]) \simeq \mathrm{Der}^b(A[Q(m,n)_{-a}]).$$

Proof. Denote by $\mathbf{l} = \{1, 2, \cdots, n-a\}$. By Proposition 4.3, we have

$$\begin{aligned} \mathrm{Der}^b(A[Q(m,n)_{-a}]) &= \mathrm{Der}^b(A[Q(n, m-1; \mathbf{l}) \# Q(n-a, 1; \mathbf{l})]) \\ &\simeq \mathrm{Der}^b(A[Q(n-a, 1; \mathbf{l}) \# Q(n, m-1; \mathbf{l})]) \\ &= \mathrm{Der}^b(A[Q(m,n)^{-a}]). \end{aligned}$$

Similarly, we have $\mathrm{Der}^b(A[{}^{-a}Q(m,n)]) \simeq \mathrm{Der}^b(A[{}_{-a}Q(m,n)])$.

Moreover, denote by $\mathbf{l}' = \{1, 2, \cdots, m-1\}$. By Proposition 4.3, we have

$$\begin{aligned} \mathrm{Der}^b(A[Q(m,n)_{-a}]) &= \mathrm{Der}^b(A[Q(m, n-a; \mathbf{l}') \# Q(m-1, a; \mathbf{l}')]) \\ &\simeq \mathrm{Der}^b(A[Q(m-1, a; \mathbf{l}') \# Q(m, n-a; \mathbf{l}')]) \\ &= \mathrm{Der}^b(A[{}_{-a}Q(m,n)]). \end{aligned}$$

Hence we finish the proof. $\square$

5. The derived equivalences for studying the Nakayama algebras

In this section, we will show the derived equivalence between the defective rectangle algebras and the Nakayama algebras.

Recall that the Nakayama algebra $N(n,r)$ is the bound quiver algebra of the equioriented quiver

$$1 \xrightarrow{x} 2 \xrightarrow{x} 3 \xrightarrow{x} \cdots \xrightarrow{x} n-1 \xrightarrow{x} n$$

of type $\vec{A}_n$ subject to all relations $x^r = 0$. To distinguish from the quiver $Q(n,1)$, we regard $Q^{(r)}(n,1)$ as the quiver $\vec{A}_n$ with admissible ideal (x^r). Similar to Construction 4.1, we may define the quiver $Q(u,v;\mathbf{l}_{12}) \# Q^{(r)}(n,1;\mathbf{l}_{21})$ by the quiver $Q(u,v;\mathbf{l}_{12}) \# Q(n,1;\mathbf{l}_{21})$ adding the corresponding zero relations x^r in $Q(n,1;\mathbf{l})$.

Denote by $\mathbf{l} = \{1, 2, \cdots, r-1\}$. We have the following results.

Lemma 5.1. *There exist the following equivalences*

1. $\mathrm{Der}^b\big(A[Q^{(r)}(n,1;\mathbf{l}) \# Q(r-1, v; \mathbf{l})]\big) \simeq \mathrm{Der}^b\big(A[Q(r-1, v-1; \mathbf{l}) \# Q^{(r)}(n+r-1, 1; \mathbf{l})]\big)$;
2. $\mathrm{Der}^b\big(A[Q(r-1, v; \mathbf{l}) \# Q^{(r)}(n,1;\mathbf{l})]\big) \simeq \mathrm{Der}^b\big(A[Q^{(r)}(n,1;\mathbf{l}) \# Q(r-1, v; \mathbf{l})]\big)$.

Proof. (1) Denote by Q the quiver $Q^{(r)}(n,1;\mathbf{l}) \# Q(r-1,v;\mathbf{l})$. We devide Q_0 into $Q_{R,0}$ and $Q_{S,0}$, where $Q_{R,0} = Q^{(r)}(n,1;\mathbf{l})_0$ and $Q_{S,0} = Q_0 \backslash Q_{R,0}$. Then $A(Q)$ is isomorphic to $T(R,S;M)$, where Γ_M has a unique path from $(1,1;1)$ to $(r-1, v-1; 2)$. By Construction 3.8, we obtain a quiver $\widehat{Q}$ which is just $Q(r-1, v-1; \mathbf{l}) \# Q^{(r)}(n+r-1, 1; \mathbf{l})$. The result follows from Proposition 3.9 immediately.

(2) Denote by Q' the quiver $Q(r-1, v; \mathbf{l}) \# Q^{(r)}(n,1;\mathbf{l})$. We devide Q'_0 into $Q_{R',0}$ and $Q_{S',0}$, where $Q_{S',0} = Q^{(r)}(n,1;\mathbf{l})_0$ and $Q_{R',0} = Q_0 \backslash Q_{S',0}$. Then $A(Q')$ is isomorphic to the algebra $T(R', S'; M')$, where M' is the projective S'-module $\oplus_{1 \le i \le r-1} e_{(i,1;2)} S'$. It is clear that the assumptions in Subsection 3.2 hold under this partition. Note that $v(i,1;2) = (i,1;1)$ for each $1 \le i \le r-1$. We obtain a quiver $\widetilde{Q}'$ by Construction 3.6 which is just $Q^{(r)}(n,1;\mathbf{l}) \# Q(r-1,v;\mathbf{l})$. The equivalence follows from Proposition 3.7 immediately. $\square$

Remark 5.2. This lemma allows that $n \le r$. In fact, the proof for $n \le r$ is similar and so it is omitted.

The another half part of the main theorem is as following.

Theorem 5.3. *Let a be an positive integer with $a < m$. Then there exists an equivalence*

$$\mathrm{Der}^b\big(A[Q(m,n)^{-a}]\big) \simeq \mathrm{Der}^b(N(mn-a, m+1)).$$

Proof. By Lemma 5.1, we have

$$\begin{aligned}\mathrm{Der}^b\big(A[Q(m,n)^{-a}]\big) &= \mathrm{Der}^b\big(A[Q^{(m+1)}(m-a,1;\mathbf{1}_{(m-a)})\#Q(m,n-1;\mathbf{1}_{(m-a)})]\big)\\ &\simeq \mathrm{Der}^b\big(A[Q(m,n-2;\mathbf{1}_{(m)})\#Q^{(m+1)}(2m-a,1;\mathbf{1}_{(m)})]\big)\\ &\simeq \mathrm{Der}^b\big(A[Q^{(m+1)}(2m-a,1;\mathbf{1}_{(m)})\#Q(m,n-2;\mathbf{1}_{(m)})]\big).\end{aligned}$$

We complete this proof by induction. □

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