Cboe Volatility Index® Mathematics Methodology

Contents

Introduction

This document covers the mathematics of calculations for the VIX® Index and other Cboe volatility indices that use this methodology. It is intended to be read in conjunction with a family of White Papers that provide specific attributes for each Cboe volatility index, including the constituent options of the index, publication times, and other characteristics.

While there are several methods to create volatility indices, the methodology used to calculate the Cboe VIX Index and other Cboe volatility indices is based on theoretical work in pricing variance swaps to isolate exposure to volatility of an asset, independent of market conditions.[1] Cboe thanks Sandy Rattray, Devesh Shah, and Tim Klassen for their significant contributions to the development of the Cboe Volatility Index.

A key feature of Cboe volatility indices is that constituent options are weighted inversely proportional to the square of their strike (K^2). The weighting scheme used in the calculation of Cboe volatility indices matches the weighting scheme used to replicate variance swap payoffs with option portfolios. This, along with other elements of the methodology that seek to replicate volatility exposure using a portfolio of options, allows for the creation of volatility index derivatives with constant vega over a wide span of market movements.

1. Constituent Option Series Selection

Depending on the family of volatility indices, either the (a) Bracket Method with Constant Maturity Term or the (b) Nearest Term Method for Exclusion Criteria is used to select the “near-term” and “next-term” option series inputs for a Cboe volatility index given the specified target timeframe of expected volatility:

(a) Bracket Method

While each Cboe volatility index with “near-term” and “next-term” option series inputs seeks to measure a targeted time period of expected volatility, volatility indices that use the **Bracket Method** specify a “Constant Maturity Term,” (e.g., 30 days, 3 months, 6 months, etc.) as an element in the option series selection process. The length of the Constant Maturity Term for a particular Cboe volatility index is set forth in the relevant family of White Papers.

In addition to this Constant Maturity Term, the inputs for this method also include the set of option expirations that are candidates for near- and next-term expirations:

- The “near-term” options are defined to be the options within the provided set with days to expiration less than or equal to the Constant Maturity Term. If no options under this condition are found, then “near-term” options are defined to be options within the provided set expiring closest to the current date.

- The “next-term” options are defined to be the options within the provided set expiring closest to and after the “near-term” options expiration date.

[1] See Neuberger, 1996; Carr & Madan, 1998; Demeterfi, Derman, et al., 1999.

(b) Nearest Term Method

The inputs for this method are the set of option expirations that are candidates for near- and next-term expirations as well as the exclusion criteria, which is a rule that determines which expiration dates should be excluded from this initial set.

The first step is to *exclude* from the provided set all option series where the exclusion criteria applies. For example, if options with a minimum of seven days to expiration are required as near-term option constituents, options that expire in fewer than seven days are excluded from the universe of candidate constituent options.

- The "near-term" options are defined to be the options within the remaining set expiring closest to the current date.

- The "next-term" options are defined to be the options within the remaining set expiring closest to and after the "near-term" options expiration date.

2. Interest Rate Calculation

(a) Bounded Cubic Spline APY Rate

The risk-free interest rate, r_t, is calculated based on U.S. Treasury yield curve rates. The calculation process captures constant maturity Treasury (CMT) yields (i.e., bond equivalent yields) available on the U.S. Treasury website. Next a cubic spline is applied to interpolate/extrapolate a yield for each date between maturities, the bond equivalent yields (BEY) are converted to annualized percentage yields (APY), and then these yields are converted to continuously compounded interest rates for use in the Cboe volatility index calculation engine.

Bounded Cubic Spline Interpolation

The CMT yields (CMT_i) for the most recent business day are retrieved from the U.S. Treasury website. From this set, all null data points are excluded. A natural cubic spline method is applied to derive the bond equivalent yield (BEY) for any given time t. The corresponding number of days (t_i) used in the natural cubic spline interpolation for each fixed maturity found on the website are as follows:

Fixed maturity	1 Mo	2 Mo	3 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
Number of days	30	60	91	182	365	730	1095	1825	2555	3650	7300	10950

The upper bound and lower bound for the BEY calculation is defined below:

- For interpolated periods $t_i < t < t_{i+1}$, where t_i and t_{i+1} are any two consecutive CMT maturities,
  - Lower bound is given by $\min(CMT_i,\ CMT_{i+1})$
  - Upper bound is given by $\max\ (CMT_i,\ CMT_{i+1})$

- For extrapolated periods $t < t_1$, where t_1 is the shortest available CMT maturity,
  - Lower bound b^{lower} is given by the equation $r = m_0^{\text{lower}} \times t + b^{\text{lower}}$. Moreover,

$$m_0^{\text{lower}} = \frac{CMT_x - CMT_1}{t_x - t_1}; \quad b^{\text{lower}} = CMT_1 - M_0^{\text{lower}} \times t_1$$

where

- (t_1, CMT_1) is the shortest available CMT maturity data point;

- (t_x, CMT_x) is the next shortest maturity data point such that $CMT_x \geq CMT_1$. If there is no such point (in the case of a complete inversion of the term structure – almost impossible), then let $m_0^{\text{lower}} = 0$.

- Upper bound b^{upper} is given by the equation $r = m_0^{\text{upper}} \times t + b^{\text{upper}}$.
  Moreover,

$$m_0^{\text{upper}} = \frac{CMT_z - CMT_1}{t_z - t_1}; \quad b^{\text{upper}} = CMT_1 - M_0^{\text{upper}} \times t_1$$

  where

  - (t_1, CMT_1) is the shortest available CMT maturity data point;
  - (t_z, CMT_z) is the next shortest maturity data point such that $CMT_z \leq CMT_1$. If there is no such point (in the case of no inversion in the term structure – a frequent occurrence), then let $m_0^{\text{upper}} = 0$.

Converting the BEY Rate to a Continuously Compounded APY Rate

Once BEY_t is calculated using the respective lower bound and upper bound, the risk-free interest rate r_t is calculated as follows:

$$APY_t = \left(1 + \frac{BEY_t}{2}\right)^2 - 1$$

$$r_t = \ln(1 + APY_t)$$

3. Volatility Index Calculation

(a) Single Term

The inputs for the single term volatility index calculation are the expiration date, interest rate, and the corresponding bid, ask, and option price for all selected options series. The generalized formula used in the volatility calculation is:

$$\sigma^2 = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2 \qquad (1)$$

where

σ	**Volatility Index** $= \sigma \times 100$	K_0	First strike equal to or otherwise immediately below F
T	Time to expiration (in years)	K_i	Strike price of the i^{th} out-of-the-money (OTM) option; a call if $K_i > K_0$ and a put if $K_i < K_0$; both put and call if $K_i = K_0$
F	Option-implied forward price	ΔK_i	Interval between strike spreads:

			• Highest OTM Strike K_i: $K_i - K_{i-1}$ • Lowest OTM Strike K_i: $K_{i+1} - K_i$ • Otherwise: $(K_{i+1} - K_{i-1})\ /\ 2$
R	Risk-free interest rate to expiration	$\mathrm{Q}(K_i)$	Option price of the OTM option with strike K_i; $\mathrm{Q}(K_0)$ is the average of the K_0 put option price and K_0 call option price

(i) Time to Expiration

The Cboe volatility index calculation measures time to expiration of a constituent option series, T, in calendar years. It is calculated by dividing the number of minutes until expiration of the selected options (rounded down to the nearest minute) by the number of minutes in a year. The time to expiration, T, is given by the following:

$$T = \left(M_{\text{Time to Expiry}}\right)\ /\ M_{365}$$

where

$M_{\text{Time to Expiry}}$	Number of minutes from time of calculation until expiration
M_{365}	Number of minutes in a 365-day year (365 x 1,440 = 525,600)

(ii) Forward Price and K_0

Determine the option-implied forward price level, F, by identifying the options strike price at which the absolute difference between the call price and the put price is smallest. If there are multiple put-call pairs with the same minimum absolute difference value, select the lowest strike price of these pairs. This strike is defined as the **at-the-money (ATM) strike**. In this subsection, the call and put prices reflect the midpoint of each candidate constituent option series' bid / ask quotes. Series with null quotes or bid price higher than ask price are not candidates to be the ATM strike.

Using the ATM strike call and put, the forward price, F, for the given term's constituent options is:

$$F = \text{Strike Price} + e^{RT} \times (\text{Call Price} - \text{Put Price})$$

Next, determine K_0, the strike price equal to or otherwise immediately below the forward price, F, for the near- and next-term candidate constituent options. If quotes of the K_0 put option or the K_0 call option are null or the bid price is higher than the ask price, then the Cboe volatility index cannot be calculated.

(iii) Strike Selection

First, remove all option strikes with null quotes from both the put and the call option series.

Then, select out-of-the-money put options with strike prices less than K_0**.** Start with the put option strike immediately lower than K_0 and move to successively lower strike prices. Exclude any put option that has a bid price equal to zero (i.e., is not bid). Once two put options with consecutive strike prices are found to have zero bid prices, exclude the observed put option(s) and consider no put options with lower strikes for inclusion. If all the out-of-the-money put options have been excluded, then the Cboe volatility index cannot be calculated.

Next, select out-of-the-money call options with strike prices greater than K_0. Start with the call option strike immediately higher than K_0 and move to successively higher strike prices. Exclude any call option that has a bid price of zero (i.e., is not bid). As with the put options, once two call options with consecutive strike prices are found to have zero bid prices, exclude the observed call option(s) and consider no call options with higher strikes for inclusion. If all the out-of-the-money call options have been excluded, then the Cboe volatility index cannot be calculated.

Finally, select **both** the put and call options with strike price K_0. Notice that two options are selected at K_0, while a single option, either a put or a call, is used for every other strike price.

The set of option series selected in this subsection comprise the constituent options for the volatility index calculation.

(iv) Calculating Volatility

The volatility index calculation combines the information reflected in the prices of all the selected constituent options. The contribution of a single option is proportional to ΔK and the price of that option, and inversely proportional to the square of the option's strike price.

Calculate the contribution of each strike by following these steps:

- Determine ΔK for each strike included in the calculation. Generally, ΔK_i is half the difference between the strike prices on either side of K_i. At the upper and lower edges of any given set of options, ΔK_i is simply the difference between K_i and the adjacent strike price.
- Compute the contribution by strike for each included option. For all puts $K_i < K_0$, the contribution is $e^{RT} \times \mathrm{Q}(K_i) \times (\Delta K_i / K_i^2)$. For all calls $K_i > K_0$, the contribution is $e^{RT} \times \mathrm{Q}(K_i) \times (\Delta K_i / K_i^2)$.
- For the strike K_0, the contribution is given by $e^{RT} \times \mathrm{Q}(K_0) \times (\Delta K_0 / K_0^2)$.

The contributions for each option strike are the summands in the sigma term of formula (1). Applying (1) to the given term's options with a time to expiration of T yields the following variance term:

$$\sigma^2 = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T} \left[\frac{F}{K_0} - 1 \right]^2$$

The Cboe volatility index value for the single term is therefore given by:

$$\textbf{Volatility Index} = \sigma \times 100$$

(b) Constant Maturity Term

Given the constant maturity term, expiration dates, and the variance (σ^2) for both terms, the formula used in the interpolated Cboe volatility index calculation is

$$\textbf{Volatility Index} = 100 \times \sqrt{\left\{T_1\sigma_1^2\left[\frac{M_{T_2} - M_{\text{CM}}}{M_{T_2} - M_{T_1}}\right] + T_2\sigma_2^2\left[\frac{M_{\text{CM}} - M_{T_1}}{M_{T_2} - M_{T_1}}\right]\right\} \times \frac{M_{365}}{M_{\text{CM}}}}$$

where

M_{T_1}	The number of minutes until expiration of the near-term options
M_{T_2}	The number of minutes until expiration of the next-term options
M_{CM}	The number of minutes in the given constant maturity term
M_{365}	The number of minutes in a 365-day year (365 x 1,440 = 525,600)
T_i	M_{T_i} / M_{365}
σ_i^2	Variance of the i^{th} term

4. Calculation, Dissemination and Republication of Volatility Index Spot Values

(a) Index Level Filtering Algorithm

For any given trading session, the Filtering Algorithm, which is applied at the volatility index level, requires both a maximum republication interval, the "threshold period," and a threshold level (x volatility points).

The Index Filtering Algorithm operates as follows:

1. The first volatility index spot value calculated and disseminated during each trading session is deemed to be the "baseline" volatility index spot value.

2. Any volatility index spot value calculated after and within the threshold period (e.g., 5 minutes) of the baseline that is higher than the baseline value or lower than the baseline value by less than x volatility points becomes the new baseline volatility index spot value. That new baseline volatility index spot value will be disseminated.

3. If volatility index spot values calculated after and within the threshold period of a baseline are lower than the baseline volatility index spot value by x volatility points or more, then the baseline volatility index spot value will be republished as the volatility index spot value. Calculated volatility index spot values above the baseline are not filtered.

4. If the published volatility index spot values remain the same for the threshold period because the calculated values are x or more volatility points lower than the baseline, the first volatility index spot value calculated after the threshold period becomes the new baseline volatility index spot value. That new baseline volatility index spot value will be disseminated.

5. The Index Filtering Algorithm does not apply to the first volatility index spot value calculated and disseminated during the provided trading session. All other volatility index spot values calculated during the same trading session are subject to the filtering process. If the Filtering Algorithm is triggered, the calculated volatility index spot value will not be disseminated. Instead, the last published valid volatility index spot value will be republished.

(b) Volatility Index Spot Value Cannot be Calculated

1. As stated in the **Forward Price and K_0** section of this document, if quotes of the K_0 put option or the K_0 call option are null or the bid price is higher than the ask price, then the volatility index spot value cannot be calculated.
2. As stated in the **Strike Selection** section of this document, if all out-of-the-money call options have been excluded or all out-of-the-money put options have been excluded, then the volatility index spot value cannot be calculated.

In both cases where the volatility index spot value cannot be calculated, the last valid volatility index spot value is republished until a new valid volatility index spot value can be calculated and disseminated.

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