1. **Two stochastic processes, z_{1t} and z_{2t}, have theoretical autocovariance functions at lag k of γ_{1k} and γ_{2k}, respectively, where $\gamma_{1k} = 0$ for $k > 2$ and $\gamma_{2k} = 0$ for $k > 3$. Derive the theoretical autocorrelation function (ACF) for the process $z_{3t} = z_{1t} + bz_{2t}$ in terms of the theoretical autocovariance functions for z_{1t} and z_{2t} where b is a constant. Assume that z_{1t} and z_{2t} are independent of one another.**

$$\gamma_{1k} = \text{cov}[z_{1t}, z_{1t+k}] = E[(z_{1t} - \mu_1)(z_{1t+k} - \mu_1)] \qquad [5.1]$$

$$\gamma_{2k} = \text{cov}[z_{2t}, z_{2t+k}] = E[(z_{2t} - \mu_2)(z_{2t+k} - \mu_2)] \qquad [5.2]$$

$$\begin{aligned}
\gamma_{3k} &= \text{cov}[z_{3t}, z_{3t+k}] = E[(z_{3t} - \mu_3)(z_{3t+k} - \mu_3)] \\
&= E\{[z_{1t} + bz_{2t} - (\mu_1 + b\mu_2)][z_{1t+k} + bz_{2t+k} - (\mu_1 + b\mu_2)]\} \\
&= E\{[(z_{1t} - \mu_1) + b(z_{2t} - \mu_2)][(z_{1t+k} - \mu_1) + b(z_{2t+k} - \mu_2)]\} \\
&= E[(z_{1t} - \mu_1)(z_{1t+k} - \mu_1) + b(z_{1t} - \mu_1)(z_{2t+k} - \mu_2) + b(z_{2t} - \mu_2)(z_{1t+k} - \mu_1) + b^2(z_{2t} - \mu_2)(z_{2t+k} - \mu_2)] \\
&= E[(z_{1t} - \mu_1)(z_{1t+k} - \mu_1)] + bE[(z_{1t} - \mu_1)(z_{2t+k} - \mu_2)] + bE[(z_{2t} - \mu_2)(z_{1t+k} - \mu_1)] + \\
&\quad b^2E[(z_{2t} - \mu_2)(z_{2t+k} - \mu_2)] \\
&= \text{cov}[z_{1t}, z_{1t+k}] + b\text{cov}[z_{1t}, z_{2t+k}] + b\text{cov}[z_{2t}, z_{1t+k}] + b^2\text{cov}[z_{2t}, z_{2t+k}] \\
&= \gamma_{1k} + b\text{cov}[z_{1t}, z_{2t+k}] + b\text{cov}[z_{2t}, z_{1t+k}] + b^2\gamma_{2k} \qquad [5.3]
\end{aligned}$$

Since z_{1t} and z_{2t} are independent of one another,

$$\text{cov}[z_{1t}, z_{2t+k}] = 0 \qquad [5.4]$$

$$\text{cov}[z_{2t}, z_{1t+k}] = 0 \qquad [5.5]$$

Hence,

$$\gamma_{3k} = \gamma_{1k} + b^2\gamma_{2k} \qquad [5.6]$$

Since $\gamma_{1k} = 0$ for $k > 2$ and $\gamma_{2k} = 0$ for $k > 3$

The theoretical autocovariance function of the process z_{3t} is

$$\gamma_{3k} = \begin{cases} \gamma_{1k} + b^2\gamma_{2k}, & 0 < k \le 2 \\ b^2\gamma_{2k}, & 2 < k \le 3 \\ 0, & k > 3 \end{cases} \qquad [5.7]$$

$$\rho_{3k} = \gamma_{3k} / \gamma_{30} = (\gamma_{1k} + b^2\gamma_{2k}) / (\gamma_{10} + b^2\gamma_{20}) \qquad [5.8]$$

Finally, the theoretical autocorrelation function of the process z_{3t} is

$$\rho_{3k} = \begin{cases} \dfrac{\gamma_{1k} + b^2\gamma_{2k}}{\gamma_{10} + b^2\gamma_{20}}, & 0 < k \leq 2 \\ \dfrac{b^2\gamma_{2k}}{\gamma_{10} + b^2\gamma_{20}}, & 2 < k \leq 3 \\ 0, & k > 3 \end{cases} \qquad [5.9]$$

2. **A constrained AR(3) model without the second AR parameter, ϕ_2, is written as**

$$(1 - \phi_1 B - \phi_3 B^3)(z_t - \mu) = a_t$$

From basic principles, derive the Yule-Walker equations for this specific AR model.

$$(1 - \phi_1 B - \phi_3 B^3)(z_t - \mu) = a_t \quad [7.1]$$

$$z_t - \mu - \phi_1 B(z_t - \mu) - \phi_3 B^3(z_t - \mu) = a_t \quad [7.2]$$

$$z_t - \mu = \phi_1 B(z_t - \mu) + \phi_3 B^3(z_t - \mu) + a_t \quad [7.3]$$

$$z_t - \mu = \phi_1(z_{t-1} - B\mu) + \phi_3(z_{t-3} - B^3\mu) + a_t \quad [7.4]$$

$B\mu = B^3\mu = \mu$ since the mean level is a constant at all times. Hence,

$$z_t - \mu = \phi_1(z_{t-1} - \mu) + \phi_3(z_{t-3} - \mu) + a_t \quad [7.5]$$

Multiply both sides of [7.5] by $(z_{t-k} - \mu)$ to obtain

$$(z_{t-k} - \mu)(z_t - \mu) = \phi_1(z_{t-k} - \mu)(z_{t-1} - \mu) + \phi_3(z_{t-k} - \mu)(z_{t-3} - \mu) + (z_{t-k} - \mu)a_t \quad [7.6]$$

By taking expected values of [7.6], the difference equation for the autocovariance function of the AR(3) process is

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_3\gamma_{k-3} + E[(z_{t-k} - \mu)a_t],\ k > 0 \quad [7.7]$$

The term $E[(z_{t-k} - \mu)a_t]$ is zero for $k > 0$ because z_{t-k} is only a function of the disturbances up to time t-k and a_t is uncorrelated with these shocks. Hence,

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_3\gamma_{k-3},\ \ k > 0 \quad [7.8]$$

To determine an expression for the theoretical ACF for the AR(3) process, divide [7.8] by γ_0 to obtain

$$\rho_k = \phi_1\rho_{k-1} + \phi_3\rho_{k-3},\ k > 0 \quad [7.9]$$

$$\rho_k - \phi_1\rho_{k-1} - \phi_3\rho_{k-3} = 0, \qquad k > 0 \quad [7.10]$$

$$\rho_k - \phi_1 B\rho_k - \phi_3 B^3\rho_k = 0, \qquad k > 0 \quad [7.11]$$

$$(1 - \phi_1 B - \phi_3 B^3)\rho_k = \phi(B)\rho_k = 0, \qquad k > 0 \quad [7.12]$$

By substituting k = 1, 2, 3 into [7.12], parameters can be expressed in terms of the theoretical ACF. The resulting set of linear equations are called the Yule-Walker equations and are given by

$$\rho_1 = \phi_1 + \phi_3\rho_2 \qquad [7.13]$$

$$\rho_2 = \phi_1\rho_1 + \phi_3\rho_1 \qquad [7.14]$$

$$\rho_3 = \phi_1\rho_2 + \phi_3 \qquad [7.15]$$

By writing the Yule-Walker equations in matrix form, the relationship for the AR parameters is

$$\underline{\phi} = \underline{P}_3^{-1}\underline{\rho}_3 \qquad [7.16]$$

where

$$\underline{\phi} = \begin{bmatrix} \phi_1 \\ 0 \\ \phi_3 \end{bmatrix}, \quad \underline{\rho}_3 = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix}, \quad \underline{P}_3 = \begin{bmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{bmatrix} \qquad [7.17]$$

3. From first principles, derive the theoretical ACF for a MA(2) process. Using the Yule-Walker equations, determine the theoretical PACF for this process.

The MA(2) process is written as

$z_t - \mu = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$ [8.1]

The autocovariance function of the MA(2) process is

$\gamma_k = E[(z_t - \mu)(z_{t-k} - \mu)]$ [8.2]

Replace $z_t - \mu$ and $z_{t-k} - \mu$ by the MA(2) process

$\gamma_k = E[(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})(a_{t-k} - \theta_1 a_{t-k-1} - \theta_2 a_{t-k-2})]$

$= E(a_t a_{t-k} - \theta_1 a_t a_{t-k-1} - \theta_2 a_t a_{t-k-2} - \theta_1 a_{t-1} a_{t-k} + \theta_1\theta_1 a_{t-1} a_{t-k-1} + \theta_1\theta_2 a_{t-1} a_{t-k-2} - \theta_2 a_{t-2} a_{t-k} + \theta_1\theta_2 a_{t-2}\, a_{t-k-1} + \theta_2\theta_2 a_{t-2} a_{t-k-2})$

$= E(a_t a_{t-k}) - E(\theta_1 a_t a_{t-k-1}) - E(\theta_2 a_t a_{t-k-2}) - E(\theta_1 a_{t-1} a_{t-k}) + E(\theta_1\theta_1 a_{t-1} a_{t-k-1}) + E(\theta_1\theta_2 a_{t-1} a_{t-k-2}) - E(\theta_2 a_{t-2} a_{t-k}) + E(\theta_1\theta_2 a_{t-2} a_{t-k-1}) + E(\theta_2\theta_2 a_{t-2} a_{t-k-2})$ [8.3]

Since

$$\theta_i\theta_j E\left(a_{t-i}a_{t-j}\right) = \begin{cases} \theta_i\theta_j\sigma_a^2, & i = j \\ 0, & i \neq j \end{cases}$$ [8.4]

$\gamma_0 = E(a_t a_t) - E(\theta_1 a_t a_{t-1}) - E(\theta_2 a_t a_{t-2}) - E(\theta_1 a_{t-1} a_t) + E(\theta_1\theta_1 a_{t-1} a_{t-1}) + E(\theta_1\theta_2 a_{t-1} a_{t-2}) - E(\theta_2 a_{t-2} a_t) + E(\theta_1\theta_2 a_{t-2} a_{t-1}) + E(\theta_2\theta_2 a_{t-2} a_{t-2})$

$= \sigma_a^2 + \theta_1^2\sigma_a^2 + \theta_2^2\sigma_a^2$

$= (1 + \theta_1^2 + \theta_2^2)\sigma_a^2$ [8.5]

$\gamma_1 = E(a_t a_{t-1}) - E(\theta_1 a_t a_{t-2}) - E(\theta_2 a_t a_{t-3}) - E(\theta_1 a_{t-1} a_{t-1}) + E(\theta_1\theta_1 a_{t-1} a_{t-2}) + E(\theta_1\theta_2 a_{t-1} a_{t-3}) - E(\theta_2 a_{t-2} a_{t-1}) + E(\theta_1\theta_2 a_{t-2} a_{t-2}) + E(\theta_2\theta_2 a_{t-2} a_{t-3})$

$= -\theta_1\sigma_a^2 + \theta_1\theta_2\sigma_a^2$

$= (-\theta_1 + \theta_1\theta_2)\sigma_a^2$ [8.6]

$\gamma_2 = E(a_t a_{t-2}) - E(\theta_1 a_t a_{t-3}) - E(\theta_2 a_t a_{t-4}) - E(\theta_1 a_{t-1} a_{t-2}) + E(\theta_1\theta_1 a_{t-1} a_{t-3}) + E(\theta_1\theta_2 a_{t-1} a_{t-4}) - E(\theta_2 a_{t-2} a_{t-2}) + E(\theta_1\theta_2 a_{t-2} a_{t-3}) + E(\theta_2\theta_2 a_{t-2} a_{t-4})$

$= -\theta_2\sigma_a^2$ [8.7]

$$\gamma_k = E(a_t a_{t-k}) - E(\theta_1 a_t a_{t-k-1}) - E(\theta_2 a_t a_{t-k-2}) - E(\theta_1 a_{t-1} a_{t-k}) + E(\theta_1\theta_1 a_{t-1} a_{t-k-1}) + E(\theta_1\theta_2 a_{t-1} a_{t-k-2}) - E(\theta_2 a_{t-2} a_{t-k}) + E(\theta_1\theta_2 a_{t-2} a_{t-k-1}) + E(\theta_2\theta_2 a_{t-2} a_{t-k-2})$$

$$= 0 \text{ for } k > 2 \qquad [8.8]$$

By dividing the autocovariance function by the variance γ_0, the theoretical ACF for a MA(2) process is found to be

$$\rho_0 = \gamma_0 / \gamma_0 = 1 \qquad [8.9]$$

$$\rho_1 = \gamma_1 / \gamma_0 = [(-\theta_1 + \theta_1\theta_2)\sigma_a^2] / [(1 + \theta_1^2 + \theta_2^2)\sigma_a^2] = (-\theta_1 + \theta_1\theta_2) / (1 + \theta_1^2 + \theta_2^2) \qquad [8.10]$$

$$\rho_2 = \gamma_2 / \gamma_0 = (-\theta_2\sigma_a^2) / [(1 + \theta_1^2 + \theta_2^2)\sigma_a^2] = -\theta_2 / (1 + \theta_1^2 + \theta_2^2) \qquad [8.11]$$

$$\rho_k = \gamma_k / \gamma_0 = 0 / \gamma_0 = 0 \quad \text{for} \quad k > 2 \qquad [8.12]$$

The Yule-Walker equations can be written as

$$\begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} \\ \rho_2 & \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} \\ \rho_3 & \rho_2 & \rho_1 & 1 & \cdots & \rho_{k-4} \\ . & . & . & . & \cdots & . \\ . & . & . & . & \cdots & . \\ . & . & . & . & \cdots & . \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \rho_{k-4} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ \phi_{k3} \\ \phi_{k4} \\ . \\ . \\ . \\ \phi_{kk} \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ . \\ . \\ . \\ \rho_k \end{bmatrix} \qquad [8.13]$$

The coefficient ϕ_{kk} is a function of the lag k and is called the theoretical partial autocorrelation function (PACF).

$$\begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} \\ \rho_2 & \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} \\ \rho_3 & \rho_2 & \rho_1 & 1 & \cdots & \rho_{k-4} \\ . & . & . & . & \cdots & . \\ . & . & . & . & \cdots & . \\ . & . & . & . & \cdots & . \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \rho_{k-4} & \cdots & 1 \end{bmatrix} = P_k \qquad [8.14]$$

According to the Cramer's rule,

$\phi_{kk} = |P_k^*| / |P_k|$ [8.15]

where P_k^* is equal to the matrix P_k, in which the kth column is replaced with ρ_k. Besides, |.| indicates the determinant.

Since the theoretical ACF for a MA(2) process has been derived, the values of ρ_k, k = 1, 2, … are known. Then, the values of ϕ_{kk}, k = 1, 2, … can be calculated.

For order 1

$\phi_{11} = \rho_1 / 1 = \rho_1$ [8.16]

For order 2

$$\varphi_{22} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} \quad [8.17]$$

For order 3

$$\varphi_{33} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}} = \frac{\rho_3 + \rho_1\rho_2^2 + \rho_1^3 - \rho_1\rho_2 - \rho_1\rho_2 - \rho_1^2\rho_3}{1 + \rho_1^2\rho_2 + \rho_1^2\rho_2 - \rho_2^2 - \rho_1^2 - \rho_1^2} = \frac{\rho_1^3 - \rho_1\rho_2(2 - \rho_2)}{1 - \rho_2^2 - 2\rho_1^2(1 - \rho_2)} \quad [8.18]$$

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The PACF of a MA(2) process tails off as an exponential decay or a damped sine wave depending on the signs and magnitudes of θ_1 and θ_2 or equivalently the roots of $(1 - \theta_1 B - \theta_2 B^2) = 0$. The PACF will be damped sine wave if the roots of $(1 - \theta_1 B - \theta_2 B^2) = 0$ are complex.

4. **For the ARMA(1,1) process in [3.4.2], derive the two main equations that are required to solve for γ_k, the theoretical autocovariance function of this process. Use these equations to solve for γ_k, k = 0, 1, 2, … .**

The ARMA(1,1) process is written as

$$(z_t - \mu) - \phi_1(z_{t-1} - \mu) = a_t - \theta_1 a_{t-1} \qquad [9.1]$$

Multiply both sides of [9.1] by $(z_{t-k} - \mu)$ to obtain

$$(z_{t-k} - \mu)(z_t - \mu) - \phi_1(z_{t-k} - \mu)(z_{t-1} - \mu) = (z_{t-k} - \mu)a_t - \theta_1(z_{t-k} - \mu)a_{t-1} \qquad [9.2]$$

Take expectations of [9.2] to obtain

$$\gamma_k - \phi_1\gamma_{k-1} = \gamma_{za}(k) - \theta_1\gamma_{za}(k-1) \qquad [9.3]$$

where $\gamma_k = E[(z_{t-k} - \mu)(z_t - \mu)]$ is the theoretical autocovariance function and $\gamma_{za}(k) = E[(z_{t-k} - \mu)a_t]$ is the cross covariance function between z_{t-k} and a_t.

Since z_{t-k} is dependent only upon the shocks which have occurred up to time t-k, it follows that

$$\gamma_{za}(k) = 0, \quad k > 0$$
$$\gamma_{za}(k) \neq 0, \quad k \leq 0 \qquad [9.4]$$

Hence,

$$\gamma_0 = \phi_1\gamma_{-1} + \gamma_{za}(0) - \theta_1\gamma_{za}(-1) = \phi_1\gamma_1 + \gamma_{za}(0) - \theta_1\gamma_{za}(-1) \qquad [9.5]$$
$$\gamma_1 = \phi_1\gamma_0 + \gamma_{za}(1) - \theta_1\gamma_{za}(0) = \phi_1\gamma_0 - \theta_1\gamma_{za}(0) \qquad [9.6]$$
$$\gamma_k = \phi_1\gamma_{k-1} + \gamma_{za}(k) - \theta_1\gamma_{za}(k-1) = \phi_1\gamma_{k-1}, \quad k \geq 2 \qquad [9.7]$$

Because of the $\gamma_{za}(k)$ terms in [9.3], it is necessary to derive other relationships before it is possible to solve for the autocovariance.

Multiply both sides of [9.1] by a_{t-k} to obtain

$$(z_t - \mu)a_{t-k} - \phi_1(z_{t-1} - \mu)a_{t-k} = a_t a_{t-k} - \theta_1 a_{t-1}a_{t-k} \qquad [9.8]$$

Take expectations of [9.8] to obtain

$$\gamma_{za}(-k) - \phi_1\gamma_{za}(-k+1) = E(a_t a_{t-k}) - E(\theta_1 a_{t-1}a_{t-k}) = -[\theta_k]\sigma_a^2 \qquad [9.9]$$

where

$$[\theta_k] = \begin{cases} \theta_k, & k = 1 \\ -1, & k = 0 \\ 0, & \text{otherwise} \end{cases} \quad [9.10]$$

and $E(a_t a_{t-k})$ is defined by

$$E(a_t a_{t-k}) = \begin{cases} \sigma_a^2, & k = 0 \\ 0, & k \neq 0 \end{cases} \quad [9.11]$$

Hence,

$\gamma_{za}(0) - \phi_1\gamma_{za}(1) = \sigma_a^2$ [9.12]

$\gamma_{za}(0) = \sigma_a^2$ [9.13]

$\gamma_{za}(-1) - \phi_1\gamma_{za}(0) = -\theta_1\sigma_a^2$ [9.14]

$\gamma_{za}(-1) = \phi_1\sigma_a^2 - \theta_1\sigma_a^2 = (\phi_1 - \theta_1)\sigma_a^2$ [9.15]

Substitute $\gamma_{za}(0)$ and $\gamma_{za}(-1)$ into [9.5] and [9.6] to obtain

$\gamma_0 = \phi_1\gamma_1 + \gamma_{za}(0) - \theta_1\gamma_{za}(-1) = \phi_1\gamma_1 + \sigma_a^2 - \theta_1(\phi_1 - \theta_1)\sigma_a^2$ [9.16]

$\gamma_1 = \phi_1\gamma_0 - \theta_1\gamma_{za}(0) = \phi_1\gamma_0 - \theta_1\sigma_a^2$ [9.17]

Rearrange [9.16] and [9.17] to obtain

$\gamma_0 = (1 - 2\phi_1\theta_1 + \theta_1^2)\sigma_a^2 / (1 - \phi_1^2)$ [9.18]

$\gamma_1 = (1 - \phi_1\theta_1)(\phi_1 - \theta_1)\sigma_a^2 / (1 - \phi_1^2)$ [9.19]

Combine [9.18] and [9.19] with [9.7] to obtain the theoretical autocovariance function of the ARMA(1,1) process

$\gamma_0 = (1 - 2\phi_1\theta_1 + \theta_1^2)\sigma_a^2 / (1 - \phi_1^2)$ [9.20]

$\gamma_1 = (1 - \phi_1\theta_1)(\phi_1 - \theta_1)\sigma_a^2 / (1 - \phi_1^2)$ [9.21]

$\gamma_k = \phi_1\gamma_{k-1} + \gamma_{za}(k) - \theta_1\gamma_{za}(k-1) = \phi_1\gamma_{k-1}, \quad k \geq 2$ [9.22]

5. **Express the model given by**

$$\mathbf{(1 - 0.6\mathit{B})(\mathit{z_t} - 15) = (1 - 0.8\mathit{B})\mathit{a_t}}$$

in

a) random shock form, and

b) inverted form.

a)

For the given ARMA(1,1) model,

$\phi(B)\psi_k = -\theta_k$ [10.1]

$(1 - 0.6B)\psi_k = -\theta_k$ [10.2]

where $\theta_k = 0$ for $k > 1$ and $\psi_0 = 1$.

When k = 1

$(1 - 0.6B)\psi_1 = -0.8$ or $\psi_1 - 0.6\psi_0 = -0.8$ [10.3]

Therefore, $\psi_1 = 0.6 - 0.8 = -0.2$ [10.4]

For k = 2

$(1 - 0.6B)\psi_2 = 0$ or $\psi_2 - 0.6\psi_1 = 0$ [10.5]

Hence, $\psi_2 = 0.6\psi_1 = 0.6(-0.2) = -0.12$ [10.6]

When k = 3

$(1 - 0.6B)\psi_3 = 0$ or $\psi_3 - 0.6\psi_2 = 0$ [10.7]

Therefore, $\psi_3 = 0.6\psi_2 = 0.6(-0.12) = -0.072$ [10.8]

The general expression for ψ_k is

$\psi_k = 0.6\psi_{k-1} = (0.6)^{k-1}\psi_1, \quad k > 0$ [10.9]

Due to the form of this equation, ψ_k will decrease in absolute value for increasing lag. When the ARMA(1,1) model is expressed using the ψ coefficients, the random shock form of the model is

$z_t - 15 = (1 - 0.2B - 0.12B^2 - 0.072B^3 + \ldots)a_t$ [10.10] b)

For the given ARMA(1,1) model,

$\theta(B)\pi_k = \phi_k$ [10.11]

$(1 - 0.8B)\pi_k = \phi_k$ [10.12]

where $\phi_k = 0$ for $k > 1$ and $\pi_0 = -1$.

When $k = 1$

$(1 - 0.8B)\pi_1 = 0.6$ or $\pi_1 - 0.8\pi_0 = 0.6$ [10.13]

Therefore, $\pi_1 = -0.8 + 0.6 = -0.2$ [10.14]

For $k = 2$

$(1 - 0.8B)\pi_2 = 0$ or $\pi_2 - 0.8\pi_1 = 0$ [10.15]

Hence, $\pi_2 = 0.8\pi_1 = 0.8(-0.2) = -0.16$ [10.16]

When $k = 3$

$(1 - 0.8B)\pi_3 = 0$ or $\pi_3 - 0.8\pi_2 = 0$ [10.17]

Therefore, $\pi_3 = 0.8\pi_2 = 0.8(-0.16) = -0.128$ [10.18]

The general expression for π_k is

$\pi_k = 0.8\pi_{k-1} = (0.8)^{k-1}\pi_1, \quad k > 0$ [10.19]

It can be seen from this equation that π_k will decrease in absolute value for increasing lag. When the ARMA(1,1) model is written using the π parameters, the inverted form of the model is

$(1 + 0.2B + 0.16B^2 + 0.128B^3 - \ldots)(z_t - 15) = a_t$ [10.20]

1. **An ARIMA(1,2,1) model is written as**

   $$(1 - B)^2(1 - 0.8B)z_t = (1 - 0.5B)a_t$$

   Write this model in the random shock and inverted forms. Determine at least seven random shock and inverted parameters.

Treating (1 - 0.8B), (1 - 0.5B), and $(1 - B)^2$ as algebraic operators, the random shock form of the ARIMA(1,2,1) process is

$$\begin{aligned} z_t^{(\lambda)} &= [(1 - 0.8B)(1 - B)^2]^{-1}(1 - 0.5B)a_t \\ &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots \\ &= (1 + \psi_1 B + \psi_2 B^2 + \ldots)a_t \\ &= \psi(B)a_t \end{aligned} \qquad [2.1]$$

where $\psi(B)$ is the random shock or infinite MA operator and ψ_i is the ith parameter or weight of $\psi(B)$.

To develop a relationship for ascertaining the ψ parameters, first multiply [2.1] by (1 - 0.8B)$(1 - B)^2$ to obtain

$$(1 - 0.8B)(1 - B)^2 z_t^{(\lambda)} = (1 - 0.8B)(1 - B)^2\psi(B)a_t \qquad [2.2]$$

Since

$$\omega_t = (1 - B)^d z_t^{(\lambda)} \qquad [2.3]$$

$$\phi(B)\omega_t = \theta(B)a_t \qquad [2.4]$$

Substitute [2.3] into [2.4] to obtain

$$\phi(B)(1 - B)^d z_t^{(\lambda)} = \theta(B)a_t \qquad [2.5]$$

According to [2.5]

$$(1 - 0.8B)(1 - B)^2 z_t^{(\lambda)} = (1 - 0.5B)a_t \qquad [2.6]$$

Substitute [2.6] into [2.2] to obtain

$$(1 - 0.5B)a_t = (1 - 0.8B)(1 - B)^2\psi(B)a_t \qquad [2.7]$$

$$1 - 0.5B = (1 - 0.8B)(1 - B)^2\psi(B) \qquad [2.8]$$

The ψ weights can be readily determined by expressing [2.8] as

$$(1 - 0.8B)(1 - B)^2\psi_k = -\theta_k \quad [2.9]$$

where B operates on k, $\psi_0 = 1$, $\psi_k = 0$ for $k < 0$ and $\theta_k = 0$ if $k > 1$. The ψ weights can be recursively calculated by solving [2.9] for k = 1, 2, …, q', where q' is the number of ψ parameters that are required.

When k = 1

$$(1 - 0.8B)(1 - B)^2\psi_1 = -\theta_1 \quad [2.10]$$

$$(1 - 2.8B + 2.6B^2 - 0.8B^3)\psi_1 = -0.5 \quad [2.11]$$

$$\psi_1 - 2.8\psi_0 + 2.6\psi_{-1} - 0.8\psi_{-2} = -0.5 \quad [2.12]$$

$$\psi_1 = 2.8\psi_0 - 2.6\psi_{-1} + 0.8\psi_{-2} - 0.5$$

$$= 2.8 - 0.5 = 2.3 \quad [2.13]$$

When k = 2

$$(1 - 0.8B)(1 - B)^2\psi_2 = -\theta_2 \quad [2.14]$$

$$(1 - 2.8B + 2.6B^2 - 0.8B^3)\psi_2 = 0 \quad [2.15]$$

$$\psi_2 - 2.8\psi_1 + 2.6\psi_0 - 0.8\psi_{-1} = 0 \quad [2.16]$$

$$\psi_2 = 2.8\psi_1 - 2.6\psi_0 + 0.8\psi_{-1}$$

$$= 2.8*2.3 - 2.6 = 3.84 \quad [2.17]$$

When k = 3

$$(1 - 0.8B)(1 - B)^2\psi_3 = -\theta_3 \quad [2.18]$$

$$(1 - 2.8B + 2.6B^2 - 0.8B^3)\psi_3 = 0 \quad [2.19]$$

$$\psi_3 - 2.8\psi_2 + 2.6\psi_1 - 0.8\psi_0 = 0 \quad [2.20]$$

$$\psi_3 = 2.8\psi_2 - 2.6\psi_1 + 0.8\psi_0$$

$$= 2.8*3.84 - 2.6*2.3 + 0.8 = 5.572 \quad [2.21]$$

When k = 4

$$(1 - 0.8B)(1 - B)^2\psi_4 = -\theta_4 \quad [2.22]$$

$$(1 - 2.8B + 2.6B^2 - 0.8B^3)\psi_4 = 0 \quad [2.23]$$

$$\psi_4 - 2.8\psi_3 + 2.6\psi_2 - 0.8\psi_1 = 0 \quad [2.24]$$

$$\psi_4 = 2.8\psi_3 - 2.6\psi_2 + 0.8\psi_1$$

$$= 2.8*5.572 - 2.6*3.84 + 0.8*2.3 = 7.4576 \quad [2.25]$$

When $k = 5$

$$(1 - 0.8B)(1 - B)^2\psi_5 = -\theta_5 \quad [2.26]$$

$$(1 - 2.8B + 2.6B^2 - 0.8B^3)\psi_5 = 0 \quad [2.27]$$

$$\psi_5 - 2.8\psi_4 + 2.6\psi_3 - 0.8\psi_2 = 0 \quad [2.28]$$

$$\psi_5 = 2.8\psi_4 - 2.6\psi_3 + 0.8\psi_2$$

$$= 2.8*7.4576 - 2.6*5.572 + 0.8*3.84 = 9.46608 \quad [2.29]$$

When $k = 6$

$$(1 - 0.8B)(1 - B)^2\psi_6 = -\theta_6 \quad [2.30]$$

$$(1 - 2.8B + 2.6B^2 - 0.8B^3)\psi_6 = 0 \quad [2.31]$$

$$\psi_6 - 2.8\psi_5 + 2.6\psi_4 - 0.8\psi_3 = 0 \quad [2.32]$$

$$\psi_6 = 2.8\psi_5 - 2.6\psi_4 + 0.8\psi_3$$

$$= 2.8*9.46608 - 2.6*7.4576 + 0.8*5.572 = 11.572864 \quad [2.33]$$

When $k = 7$

$$(1 - 0.8B)(1 - B)^2\psi_7 = -\theta_7 \quad [2.34]$$

$$(1 - 2.8B + 2.6B^2 - 0.8B^3)\psi_7 = 0 \quad [2.35]$$

$$\psi_7 - 2.8\psi_6 + 2.6\psi_5 - 0.8\psi_4 = 0 \quad [2.36]$$

$$\psi_7 = 2.8\psi_6 - 2.6\psi_5 + 0.8\psi_4$$

$$= 2.8*11.572864 - 2.6*9.46608 + 0.8*7.4576 = 13.7582912 \quad [2.37]$$

When the ARIMA(1,2,1) model is expressed using the ψ coefficients, the random shock form of the model is

$$z_t^{(\lambda)} = (1 + 2.3B + 3.84B^2 + 5.572B^3 + 7.4576B^4 + 9.46608B^5 + 11.572864B^6 + 13.7582912B^7 + \ldots)a_t \quad [2.38]$$

In order to write the inverted form of the process, the ARIMA(1,2,1) process is reformulated as

$$\begin{aligned} a_t &= (1 - 0.5B)^{-1}(1 - 0.8B)(1 - B)^2 z_t^{(\lambda)} \\ &= z_t^{(\lambda)} - \pi_1 z_{t-1}^{(\lambda)} - \pi_2 z_{t-2}^{(\lambda)} - \ldots \\ &= (1 - \pi_1 B - \pi_2 B^2 - \ldots)z_t^{(\lambda)} \\ &= \pi(B) z_t^{(\lambda)} \end{aligned} \quad [2.39]$$

where $\pi(B)$ is the inverted or infinite AR operator and π_i is the ith parameter or weight of $\pi(B)$.

To determine a relationship for calculating the π parameters, multiply [2.39] by (1 - 0.5B) to get

$$(1 - 0.5B)a_t = (1 - 0.5B)\pi(B)z_t^{(\lambda)} \quad [2.40]$$

Substitute [2.6] into [2.40] to obtain

$$(1 - 0.8B)(1 - B)^2 z_t^{(\lambda)} = (1 - 0.5B)\pi(B)z_t^{(\lambda)} \quad [2.41]$$

$$(1 - 0.8B)(1 - B)^2 = (1 - 0.5B)\pi(B) \quad [2.42]$$

The π coefficients can be easily ascertained by expressing the above equation as

$$(1 - 0.5B)\pi_k = (1 - B)^2\phi_k \quad [2.43]$$

where $\pi_0 = -1$ and $\phi_0 = -1$ when using [2.43] to calculate π_k for $k > 0$, $\pi_k = 0$ for $k < 0$, and $\phi_k = 0$ if $k > 1$ or $k < 0$. By solving [2.43] for k = 1, 2, …, p', where p' is the number of π parameters that are needed, the π weights can be recursively calculated.

When k = 1

$$(1 - 0.5B)\pi_1 = (1 - B)^2\phi_1 \quad [2.44]$$

$$\pi_1 - 0.5\pi_0 = \phi_1 - 2\phi_0 + \phi_{-1} \quad [2.45]$$

$$\pi_1 - 0.5*(-1) = 0.8 - 2*(-1) \quad [2.46]$$

$$\pi_1 = 2.3 \quad [2.47]$$

When k = 2

$$(1 - 0.5B)\pi_2 = (1 - B)^2\phi_2 \quad [2.48]$$

$$\pi_2 - 0.5\pi_1 = \phi_2 - 2\phi_1 + \phi_0 \quad [2.49]$$

$$\pi_2 - 0.5*2.3 = -2*0.8 + (-1) \quad [2.50]$$

$$\pi_2 = -1.45 \quad [2.51]$$

When k = 3

$$(1 - 0.5B)\pi_3 = (1 - B)^2\phi_3 \quad [2.52]$$

$$\pi_3 - 0.5\pi_2 = \phi_3 - 2\phi_2 + \phi_1 \quad [2.53]$$

$$\pi_3 - 0.5*(-1.45) = 0.8 \quad [2.54]$$

$$\pi_3 = 0.075 \quad [2.55]$$

When $k = 4$

$$(1 - 0.5B)\pi_4 = (1 - B)^2\phi_4 \quad [2.56]$$

$$\pi_4 - 0.5\pi_3 = \phi_4 - 2\phi_3 + \phi_2 \quad [2.57]$$

$$\pi_4 - 0.5*0.075 = 0 \quad [2.58]$$

$$\pi_4 = 0.0375 \quad [2.59]$$

When $k = 5$

$$(1 - 0.5B)\pi_5 = (1 - B)^2\phi_5 \quad [2.60]$$

$$\pi_5 - 0.5\pi_4 = \phi_5 - 2\phi_4 + \phi_3 \quad [2.61]$$

$$\pi_5 - 0.5*0.0375 = 0 \quad [2.62]$$

$$\pi_5 = 0.01875 \quad [2.63]$$

When $k = 6$

$$(1 - 0.5B)\pi_6 = (1 - B)^2\phi_6 \quad [2.64]$$

$$\pi_6 - 0.5\pi_5 = \phi_6 - 2\phi_5 + \phi_4 \quad [2.65]$$

$$\pi_6 - 0.5*0.01875 = 0 \quad [2.66]$$

$$\pi_6 = 0.009375 \quad [2.67]$$

When $k = 7$

$$(1 - 0.5B)\pi_7 = (1 - B)^2\phi_7 \quad [2.68]$$

$$\pi_7 - 0.5\pi_6 = \phi_7 - 2\phi_6 + \phi_5 \quad [2.69]$$

$$\pi_7 - 0.5*0.009375 = 0 \quad [2.70]$$

$$\pi_7 = 0.0046875 \quad [2.71]$$

When the ARIMA(1,2,1) model is written using the π parameters, the inverted form of the model is

$$(1 - 2.3B + 1.45B^2 - 0.075B^3 - 0.0375B^4 - 0.01875B^5 - 0.009375B^6 - 0.0046875B^7 - \ldots)z_t^{(\lambda)} = a_t \quad [2.72]$$

2. For an ARMA(2,1) model, calculate MMSE forecasts up to a lead time of 10.

An ARMA(2,1) model is written in its original difference equation form for time t+l as

$$(1 - \phi_1 B - \phi_2 B^2) z_{t+l} = (1 - \theta_1 B) a_{t+l} \quad [7.1]$$

or

$$z_{t+l} - \phi_1 z_{t+l-1} - \phi_2 z_{t+l-2} = a_{t+l} - \theta_1 a_{t+l-1} \quad [7.2]$$

or

$$z_{t+l} = \phi_1 z_{t+l-1} + \phi_2 z_{t+l-2} + a_{t+l} - \theta_1 a_{t+l-1} \quad [7.3]$$

By taking conditional expectations of each term in [7.3], the MMSE forecasts of an ARMA(2,1) model are

$$[z_{t+l}] = \phi_1 [z_{t+l-1}] + \phi_2 [z_{t+l-2}] + [a_{t+l}] - \theta_1 [a_{t+l-1}] \quad [7.4]$$

Using the rules listed in [7.5] to [7.8], one can calculate the MMSE forecasts for various lead times l from origin t.

$$[z_{t-j}] = E_t[z_{t-j}] = z_{t-j}, \qquad j = 0, 1, 2, \ldots \quad [7.5]$$

Because an observation at or before time t is known, the conditional expectation of this known value or constant is simply the observation itself.

$$[z_{t+j}] = E_t[z_{t+j}] = \hat{z}_t(j), \qquad j = 1, 2, \ldots \quad [7.6]$$

The conditional expectation of a time series value after time t is the MMSE forecasts that one wishes to calculate for lead time j from origin t.

$$[a_{t-j}] = E_t[a_{t-j}] = a_{t-j}, \qquad j = 0, 1, 2, \ldots \quad [7.7]$$

Since an innovation at or before time t is known, the conditional expectation of this known value is the innovation itself.

$$[a_{t+j}] = E_t[a_{t+j}] = 0, \qquad j = 1, 2, \ldots \quad [7.8]$$

In the definition of the ARMA model, the a_t's are assumed to be independently distributed and have a mean of zero and variance of σ_a^2. Consequently, the expected value of the unknown a_t's after time t is zero because they have not yet taken place.

Lead Time l = 1:

Substitute l = 1 into [7.4] to get

$$[z_{t+1}] = \phi_1[z_t] + \phi_2[z_{t-1}] + [a_{t+1}] - \theta_1[a_t] \tag{7.9}$$

After applying the forecasting rules, the one step ahead forecast is

$$\hat{z}_t(1) = \phi_1 z_t + \phi_2 z_{t-1} + 0 - \theta_1 a_t = \phi_1 z_t + \phi_2 z_{t-1} - \theta_1 a_t \tag{7.10}$$

In [7.10], all of the parameters and variable values on the right hand side are known, so one can determine $\hat{z}_t(1)$.

Lead Time l = 2:

After substituting l = 2 into [7.4], one obtains

$$[z_{t+2}] = \phi_1[z_{t+1}] + \phi_2[z_t] + [a_{t+2}] - \theta_1[a_{t+1}] \tag{7.11}$$

Next one uses the rules from [7.5] to [7.8] to get

$$\hat{z}_t(2) = \phi_1\hat{z}_t(1) + \phi_2 z_t + 0 - \theta_1(0) = \phi_1\hat{z}_t(1) + \phi_2 z_t \tag{7.12}$$

where the one step ahead forecast is known from the previous step for lead time l = 1.

Lead Time l = 3:

After substituting l = 3 into [7.4], one obtains

$$[z_{t+3}] = \phi_1[z_{t+2}] + \phi_2[z_{t+1}] + [a_{t+3}] - \theta_1[a_{t+2}] \tag{7.13}$$

Next one uses the rules from [7.5] to [7.8] to get

$$\hat{z}_t(3) = \phi_1\hat{z}_t(2) + \phi_2\hat{z}_t(1) + 0 - \theta_1(0) = \phi_1\hat{z}_t(2) + \phi_2\hat{z}_t(1) \tag{7.14}$$

where the one step and two steps ahead forecasts are known from the previous steps for lead time l = 1 and 2.

Lead Time l = 4:

After substituting l = 4 into [7.4], one obtains

$$[z_{t+4}] = \phi_1[z_{t+3}] + \phi_2[z_{t+2}] + [a_{t+4}] - \theta_1[a_{t+3}] \tag{7.15}$$

Next one uses the rules from [7.5] to [7.8] to get

$$\hat{z}_t(4) = \phi_1\hat{z}_t(3) + \phi_2\hat{z}_t(2) + 0 - \theta_1(0) = \phi_1\hat{z}_t(3) + \phi_2\hat{z}_t(2) \qquad [7.16]$$

where the two and three steps ahead forecasts are known from the previous steps for lead time l = 2 and 3.

Lead Time l = 5:

After substituting l = 5 into [7.4], one obtains

$$[z_{t+5}] = \phi_1[z_{t+4}] + \phi_2[z_{t+3}] + [a_{t+5}] - \theta_1[a_{t+4}] \qquad [7.17]$$

Next one uses the rules from [7.5] to [7.8] to get

$$\hat{z}_t(5) = \phi_1\hat{z}_t(4) + \phi_2\hat{z}_t(3) + 0 - \theta_1(0) = \phi_1\hat{z}_t(4) + \phi_2\hat{z}_t(3) \qquad [7.18]$$

where the three and four steps ahead forecasts are known from the previous steps for lead time l = 3 and 4.

Lead Time l = 6:

After substituting l = 6 into [7.4], one obtains

$$[z_{t+6}] = \phi_1[z_{t+5}] + \phi_2[z_{t+4}] + [a_{t+6}] - \theta_1[a_{t+5}] \qquad [7.19]$$

Next one uses the rules from [7.5] to [7.8] to get

$$\hat{z}_t(6) = \phi_1\hat{z}_t(5) + \phi_2\hat{z}_t(4) + 0 - \theta_1(0) = \phi_1\hat{z}_t(5) + \phi_2\hat{z}_t(4) \qquad [7.20]$$

where the four and five steps ahead forecasts are known from the previous steps for lead time l = 4 and 5.

Lead Time l = 7:

After substituting l = 7 into [7.4], one obtains

$$[z_{t+7}] = \phi_1[z_{t+6}] + \phi_2[z_{t+5}] + [a_{t+7}] - \theta_1[a_{t+6}] \qquad [7.21]$$

Next one uses the rules from [7.5] to [7.8] to get

$$\hat{z}_t(7) = \phi_1\hat{z}_t(6) + \phi_2\hat{z}_t(5) + 0 - \theta_1(0) = \phi_1\hat{z}_t(6) + \phi_2\hat{z}_t(5) \qquad [7.22]$$

where the five and six steps ahead forecasts are known from the previous steps for lead time l = 5 and 6.

Lead Time l = 8:

After substituting l = 8 into [7.4], one obtains

$$[z_{t+8}] = \phi_1[z_{t+7}] + \phi_2[z_{t+6}] + [a_{t+8}] - \theta_1[a_{t+7}] \quad [7.23]$$

Next one uses the rules from [7.5] to [7.8] to get

$$\hat{z}_t(8) = \phi_1\hat{z}_t(7) + \phi_2\hat{z}_t(6) + 0 - \theta_1(0) = \phi_1\hat{z}_t(7) + \phi_2\hat{z}_t(6) \quad [7.24]$$

where the six and seven steps ahead forecasts are known from the previous steps for lead time l = 6 and 7.

Lead Time l = 9:

After substituting l = 9 into [7.4], one obtains

$$[z_{t+9}] = \phi_1[z_{t+8}] + \phi_2[z_{t+7}] + [a_{t+9}] - \theta_1[a_{t+8}] \quad [7.25]$$

Next one uses the rules from [7.5] to [7.8] to get

$$\hat{z}_t(9) = \phi_1\hat{z}_t(8) + \phi_2\hat{z}_t(7) + 0 - \theta_1(0) = \phi_1\hat{z}_t(8) + \phi_2\hat{z}_t(7) \quad [7.26]$$

where the seven and eight steps ahead forecasts are known from the previous steps for lead time l = 7 and 8.

Lead Time l = 10:

After substituting l = 10 into [7.4], one obtains

$$[z_{t+10}] = \phi_1[z_{t+9}] + \phi_2[z_{t+8}] + [a_{t+10}] - \theta_1[a_{t+9}] \quad [7.27]$$

Next one uses the rules from [7.5] to [7.8] to get

$$\hat{z}_t(10) = \phi_1\hat{z}_t(9) + \phi_2\hat{z}_t(8) + 0 - \theta_1(0) = \phi_1\hat{z}_t(9) + \phi_2\hat{z}_t(8) \quad [7.28]$$

where the eight and nine steps ahead forecasts are known from the previous steps for lead time l = 8 and 9.

3. Determine MMSE forecasts for an ARIMA(1,2,1) model up to a lead time of 10.

Following the general form of the ARIMA model, an ARIMA(1,2,1) model is written at time t+l as

$$(1 - B)^2(1 - \phi_1 B)z_{t+l} = (1 - \theta_1 B)a_{t+l} \quad [8.1]$$

or

$$[1 - (\phi_1 + 2)B + (2\phi_1 + 1)B^2 - \phi_1 B^3]z_{t+l} = (1 - \theta_1 B)a_{t+l} \quad [8.2]$$

or

$$z_{t+l} - (\phi_1 + 2)z_{t+l-1} + (2\phi_1 + 1)z_{t+l-2} - \phi_1 z_{t+l-3} = a_{t+l} - \theta_1 a_{t+l-1} \quad [8.3]$$

After taking conditional expectations of each term in [8.3], the forecasting equation is

$$[z_{t+l}] - (\phi_1 + 2)[z_{t+l-1}] + (2\phi_1 + 1)[z_{t+l-2}] - \phi_1[z_{t+l-3}] = [a_{t+l}] - \theta_1[a_{t+l-1}] \quad [8.4]$$

or

$$[z_{t+l}] = (\phi_1 + 2)[z_{t+l-1}] - (2\phi_1 + 1)[z_{t+l-2}] + \phi_1[z_{t+l-3}] + [a_{t+l}] - \theta_1[a_{t+l-1}] \quad [8.5]$$

By employing the rules given in [8.6] to [8.9], one can determine the MMSE forecasts for lead times l = 1, 2, …, from origin t.

$$[z_{t-j}] = E_t[z_{t-j}] = z_{t-j}, \qquad j = 0, 1, 2, \ldots \quad [8.6]$$

Because an observation at or before time t is known, the conditional expectation of this known value or constant is simply the observation itself.

$$[z_{t+j}] = E_t[z_{t+j}] = \hat{z}_t(j), \qquad j = 1, 2, \ldots \quad [8.7]$$

The conditional expectation of a time series value after time t is the MMSE forecasts that one wishes to calculate for lead time j from origin t.

$$[a_{t-j}] = E_t[a_{t-j}] = a_{t-j}, \qquad j = 0, 1, 2, \ldots \quad [8.8]$$

Since an innovation at or before time t is known, the conditional expectation of this known value is the innovation itself.

$$[a_{t+j}] = E_t[a_{t+j}] = 0, \qquad j = 1, 2, \ldots \tag{8.9}$$

In the definition of the ARMA model, the a_t's are assumed to be independently distributed and have a mean of zero and variance of σ_a^2. Consequently, the expected value of the unknown a_t's after time t is zero because they have not yet taken place.

Lead Time l = 1:

Substitute l = 1 into [8.5] to obtain

$$[z_{t+1}] = (\phi_1 + 2)[z_t] - (2\phi_1 + 1)[z_{t-1}] + \phi_1[z_{t-2}] + [a_{t+1}] - \theta_1[a_t] \tag{8.10}$$

After invoking the forecasting rules, the one step ahead forecast is

$$\begin{aligned}\hat{z}_t(1) &= (\phi_1 + 2)z_t - (2\phi_1 + 1)z_{t-1} + \phi_1 z_{t-2} + 0 - \theta_1 a_t \\ &= (\phi_1 + 2)z_t - (2\phi_1 + 1)z_{t-1} + \phi_1 z_{t-2} - \theta_1 a_t \end{aligned} \tag{8.11}$$

Because all entries on the right hand side of [8.11] are known, one can calculate $\hat{z}_t(1)$.

Lead Time l = 2:

After assigning l = 2 in [8.5], one gets

$$[z_{t+2}] = (\phi_1 + 2)[z_{t+1}] - (2\phi_1 + 1)[z_t] + \phi_1[z_{t-1}] + [a_{t+2}] - \theta_1[a_{t+1}] \tag{8.12}$$

In the next step, one uses the rules for calculating conditional expectations in order to obtain

$$\begin{aligned}\hat{z}_t(2) &= (\phi_1 + 2)\hat{z}_t(1) - (2\phi_1 + 1)z_t + \phi_1 z_{t-1} + 0 - \theta_1(0) \\ &= (\phi_1 + 2)\hat{z}_t(1) - (2\phi_1 + 1)z_t + \phi_1 z_{t-1} \end{aligned} \tag{8.13}$$

where the one step ahead forecast is determined in the previous iteration for which l = 1.

Lead Time l = 3:

Substitute l = 3 into [8.5] to obtain

$$[z_{t+3}] = (\phi_1 + 2)[z_{t+2}] - (2\phi_1 + 1)[z_{t+1}] + \phi_1[z_t] + [a_{t+3}] - \theta_1[a_{t+2}] \tag{8.14}$$

After applying the rules for calculating conditional expectations, [8.14] becomes

$$\begin{aligned}\hat{z}_t(3) &= (\phi_1 + 2)\hat{z}_t(2) - (2\phi_1 + 1)\hat{z}_t(1) + \phi_1 z_t + 0 - \theta_1(0) \\ &= (\phi_1 + 2)\hat{z}_t(2) - (2\phi_1 + 1)\hat{z}_t(1) + \phi_1 z_t \end{aligned} \tag{8.15}$$

where the one step and two steps ahead forecasts from origin t are determined in the previous two iterations.

Lead Time l = 4:

Substitute l = 4 into [8.5] to obtain

$$[z_{t+4}] = (\phi_1 + 2)[z_{t+3}] - (2\phi_1 + 1)[z_{t+2}] + \phi_1[z_{t+1}] + [a_{t+4}] - \theta_1[a_{t+3}] \qquad [8.16]$$

After applying the rules for calculating conditional expectations, [8.16] becomes

$$\hat{z}_t(4) = (\phi_1 + 2)\hat{z}_t(3) - (2\phi_1 + 1)\hat{z}_t(2) + \phi_1\hat{z}_t(1) + 0 - \theta_1(0)$$

$$= (\phi_1 + 2)\hat{z}_t(3) - (2\phi_1 + 1)\hat{z}_t(2) + \phi_1\hat{z}_t(1) \qquad [8.17]$$

where the one step, two steps, and three steps ahead forecasts from origin t are determined in the previous three iterations.

Lead Time l = 5:

Substitute l = 5 into [8.5] to obtain

$$[z_{t+5}] = (\phi_1 + 2)[z_{t+4}] - (2\phi_1 + 1)[z_{t+3}] + \phi_1[z_{t+2}] + [a_{t+5}] - \theta_1[a_{t+4}] \qquad [8.18]$$

After applying the rules for calculating conditional expectations, [8.18] becomes

$$\hat{z}_t(5) = (\phi_1 + 2)\hat{z}_t(4) - (2\phi_1 + 1)\hat{z}_t(3) + \phi_1\hat{z}_t(2) + 0 - \theta_1(0)$$

$$= (\phi_1 + 2)\hat{z}_t(4) - (2\phi_1 + 1)\hat{z}_t(3) + \phi_1\hat{z}_t(2) \qquad [8.19]$$

where the two, three, and four steps ahead forecasts from origin t are determined in the previous three iterations.

Lead Time l = 6:

Substitute l = 6 into [8.5] to obtain

$$[z_{t+6}] = (\phi_1 + 2)[z_{t+5}] - (2\phi_1 + 1)[z_{t+4}] + \phi_1[z_{t+3}] + [a_{t+6}] - \theta_1[a_{t+5}] \qquad [8.20]$$

After applying the rules for calculating conditional expectations, [8.20] becomes

$$\hat{z}_t(6) = (\phi_1 + 2)\hat{z}_t(5) - (2\phi_1 + 1)\hat{z}_t(4) + \phi_1\hat{z}_t(3) + 0 - \theta_1(0)$$

$$= (\phi_1 + 2)\hat{z}_t(5) - (2\phi_1 + 1)\hat{z}_t(4) + \phi_1\hat{z}_t(3) \qquad [8.21]$$

where the three, four, and five steps ahead forecasts from origin t are determined in the previous three iterations.

Lead Time l = 7:

Substitute l = 7 into [8.5] to obtain

$$[z_{t+7}] = (\phi_1 + 2)[z_{t+6}] - (2\phi_1 + 1)[z_{t+5}] + \phi_1[z_{t+4}] + [a_{t+7}] - \theta_1[a_{t+6}] \qquad [8.22]$$

After applying the rules for calculating conditional expectations, [8.22] becomes

$$\hat{z}_t(7) = (\phi_1 + 2)\hat{z}_t(6) - (2\phi_1 + 1)\hat{z}_t(5) + \phi_1\hat{z}_t(4) + 0 - \theta_1(0)$$

$$= (\phi_1 + 2)\hat{z}_t(6) - (2\phi_1 + 1)\hat{z}_t(5) + \phi_1\hat{z}_t(4) \quad [8.23]$$

where the four, five, and six steps ahead forecasts from origin t are determined in the previous three iterations.

Lead Time l = 8:

Substitute l = 8 into [8.5] to obtain

$$[z_{t+8}] = (\phi_1 + 2)[z_{t+7}] - (2\phi_1 + 1)[z_{t+6}] + \phi_1[z_{t+5}] + [a_{t+8}] - \theta_1[a_{t+7}] \quad [8.24]$$

After applying the rules for calculating conditional expectations, [8.24] becomes

$$\hat{z}_t(8) = (\phi_1 + 2)\hat{z}_t(7) - (2\phi_1 + 1)\hat{z}_t(6) + \phi_1\hat{z}_t(5) + 0 - \theta_1(0)$$

$$= (\phi_1 + 2)\hat{z}_t(7) - (2\phi_1 + 1)\hat{z}_t(6) + \phi_1\hat{z}_t(5) \quad [8.25]$$

where the five, six, and seven steps ahead forecasts from origin t are determined in the previous three iterations.

Lead Time l = 9:

Substitute l = 9 into [8.5] to obtain

$$[z_{t+9}] = (\phi_1 + 2)[z_{t+8}] - (2\phi_1 + 1)[z_{t+7}] + \phi_1[z_{t+6}] + [a_{t+9}] - \theta_1[a_{t+8}] \quad [8.26]$$

After applying the rules for calculating conditional expectations, [8.26] becomes

$$\hat{z}_t(9) = (\phi_1 + 2)\hat{z}_t(8) - (2\phi_1 + 1)\hat{z}_t(7) + \phi_1\hat{z}_t(6) + 0 - \theta_1(0)$$

$$= (\phi_1 + 2)\hat{z}_t(8) - (2\phi_1 + 1)\hat{z}_t(7) + \phi_1\hat{z}_t(6) \quad [8.27]$$

where the six, seven, and eight steps ahead forecasts from origin t are determined in the previous three iterations.

Lead Time l = 10:

Substitute l = 10 into [8.5] to obtain

$$[z_{t+10}] = (\phi_1 + 2)[z_{t+9}] - (2\phi_1 + 1)[z_{t+8}] + \phi_1[z_{t+7}] + [a_{t+10}] - \theta_1[a_{t+9}] \quad [8.28]$$

After applying the rules for calculating conditional expectations, [8.28] becomes

$$\hat{z}_t(10) = (\phi_1 + 2)\hat{z}_t(9) - (2\phi_1 + 1)\hat{z}_t(8) + \phi_1\hat{z}_t(7) + 0 - \theta_1(0)$$

$$= (\phi_1 + 2)\hat{z}_t(9) - (2\phi_1 + 1)\hat{z}_t(8) + \phi_1\hat{z}_t(7) \qquad [8.29]$$

where the seven, eight, and nine steps ahead forecasts from origin t are determined in the previous three iterations.

4. **Suppose that one wishes to simulate 10 values from an ARMA(2,2) model using WASIM1. Write down how each of these 10 values are calculated using the WASIM1 algorithm.**

Let w_t be a stationary w_t series for time t = 1, 2, …, n, to which an ARMA(2,2) model is fitted as

$$(1 - \phi_1 B - \phi_2 B^2)w_t = (1 - \theta_1 B - \theta_2 B^2)a_t \qquad [9.1]$$

where ϕ_1 and ϕ_2 are the nonseasonal AR parameters, θ_1 and θ_2 are the nonseasonal MA parameters, the a_t's are identically independently distributed innovations with mean 0 and variance σ_a^2 [IID(0,σ_a^2] and often the disturbances are assumed to be normally independently distributed [NID(0,σ_a^2)]. Models with a non-zero mean (or any other type of deterministic component) are simulated by first generating the corresponding zero-mean process and then adding on the mean component.

Suppose that the w_t's are expanded in terms of a pure MA process. This is termed the random shock form of an ARMA(2,2) process and is written as

$$w_t = (1 - \theta_1 B - \theta_2 B^2)a_t / (1 - \phi_1 B - \phi_2 B^2)$$
$$= \psi(B)a_t = (1 + \psi_1 B + \psi_2 B^2 + \ldots)a_t \qquad [9.2]$$

To develop a relationship for determining the ψ parameters or weights, first multiply [9.2] by $(1 - \phi_1 B - \phi_2 B^2)$ to obtain

$$(1 - \phi_1 B - \phi_2 B^2)w_t = (1 - \phi_1 B - \phi_2 B^2)(1 + \psi_1 B + \psi_2 B^2 + \ldots)a_t \qquad [9.3]$$

From [9.1], $(1 - \theta_1 B - \theta_2 B^2)a_t$ can be substituted for $(1 - \phi_1 B - \phi_2 B^2)w_t$ in [9.3] to get

$$(1 - \theta_1 B - \theta_2 B^2)a_t = (1 - \phi_1 B - \phi_2 B^2)(1 + \psi_1 B + \psi_2 B^2 + \ldots)a_t \qquad [9.4]$$
$$(1 - \phi_1 B - \phi_2 B^2)(1 + \psi_1 B + \psi_2 B^2 + \ldots) = 1 - \theta_1 B - \theta_2 B^2 \qquad [9.5]$$

The ψ weights can be conveniently determined by expressing [9.5] as

$$(1 - \phi_1 B - \phi_2 B^2)\psi_k = -\theta_k \qquad [9.6]$$

where B operates on k, $\psi_0 = 1$, $\psi_k = 0$ for k < 0, and $\theta_k = 0$ if k > 2.

When $k = 1$

$$(1 - \phi_1 B - \phi_2 B^2)\psi_1 = -\theta_1$$

$$\psi_1 - \phi_1\psi_0 - \phi_2\psi_{-1} = -\theta_1$$

$$\psi_1 - \phi_1 - 0 = -\theta_1$$

$$\psi_1 = \phi_1 - \theta_1 \qquad [9.7]$$

Since ϕ_1 and θ_1 are known, ψ_1 can be calculated.

When $k = 2$

$$(1 - \phi_1 B - \phi_2 B^2)\psi_2 = -\theta_2$$

$$\psi_2 - \phi_1\psi_1 - \phi_2\psi_0 = -\theta_2$$

$$\psi_2 - \phi_1\psi_1 - \phi_2 = -\theta_2$$

$$\psi_2 = \phi_1\psi_1 + \phi_2 - \theta_2 \qquad [9.8]$$

Since ϕ_1, ϕ_2, and θ_2 are known and ψ_1 is obtained, ψ_2 can be calculated.

When $k =3$

$$(1 - \phi_1 B - \phi_2 B^2)\psi_3 = -\theta_3$$

$$\psi_3 - \phi_1\psi_2 - \phi_2\psi_1 = -\theta_3$$

$$\psi_3 - \phi_1\psi_2 - \phi_2\psi_1 = 0$$

$$\psi_3 = \phi_1\psi_2 + \phi_2\psi_1 \qquad [9.9]$$

Since ϕ_1 and ϕ_2 are known and ψ_1 and ψ_2 are obtained, ψ_3 can be calculated.

The general expression for ψ_k is

$$\psi_k = \phi_1\psi_{k-1} + \phi_2\psi_{k-2}, \qquad k \geq 3 \qquad [9.10]$$

Since the AR operator is present, $\psi(B)$ forms an infinite series and therefore must be approximated by the finite series

$$\psi(B) \approx 1 + \psi_1 B + \psi_2 B^2 + \ldots + \psi_{q'} B^{q'} \qquad [9.11]$$

It is necessary to choose q' such that $\psi_{q'+1}$, $\psi_{q'+2}$, …, are all negligible. Since the model is stationary, this can be accomplished by selecting q' sufficiently large such that the error given below is kept as small as desired.

$$\gamma_0 - \sum_{i=0}^{q'} \psi_i^2 < error \qquad [9.12]$$

where γ_0 is the theoretical variance of the given ARMA(2,2) process with $\sigma_a^2 = 1$; error is the chosen error level (ex. error = 10^{-5}).

To obtain a synthetic series of 10 observations, first generate (10+q') white noise terms, namely $a_{-q'+1}$, $a_{-q'+2}$, …, a_0, a_1, a_2, …, a_{10}. Next, calculate

$$w_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots + \psi_{q'} a_{t-q'} \qquad [9.13]$$

where t = 1 and 2.

The remaining w_t are easily determined from [9.1] as

$$w_t = \phi_1 w_{t-1} + \phi_2 w_{t-2} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} \qquad [9.14]$$

where t = 3, 4, …, 10.

The use of [9.14] avoids the truncation error present in [9.11]. Nevertheless if an AR operator is present (i.e., p > 0), there will be some systematic error in the simulated data due to the approximation involved in [9.13]. However, this bias can be kept to a tolerable level by selecting the "error" term in [9.12] to have a specified minimum value. Of course, if the model is a pure MA(q) model, then set q' = q and [9.13] is exact and can be utilized to generate all of the synthetic data.

5. (a) Suppose that one wishes to use WASIM2 to simulate 10 values from an ARMA(2,1) model. Show all the calculations for generating this data.

(b) Prove that the random starting values in part (a) are from an ARMA(2,1) process.

(a) The calculations for an ARMA(2,1) model using WASIM2 are as follows:

2) For an ARMA(2,1) model p = 2 and, hence, one only has to calculate γ_0 and γ_1 at step 1.

An ARMA(2,1) model is written as

$$w_t - \phi_1 w_{t-1} - \phi_2 w_{t-2} = a_t - \theta_1 a_{t-1} \quad [10.1]$$

Multiply both sides of [10.1] by w_{t-k} and take expectations to obtain

$$\gamma_k - \phi_1\gamma_{k-1} - \phi_2\gamma_{k-2} = \gamma_{wa}(k) - \theta_1\gamma_{wa}(k-1) \quad [10.2]$$

where $\gamma_k = E(w_{t-k}w_t)$ is the theoretical autocovariance function and $\gamma_{wa}(k) = E(w_{t-k}a_t)$ is the cross covariance function between w_{t-k} and a_t.

Since w_{t-k} is dependent only upon the shocks which have occurred up to time t-k, it follows that

$$\gamma_{wa}(k) = 0, \quad k > 0$$
$$\gamma_{wa}(k) \neq 0, \quad k \leq 0 \quad [10.3]$$

Hence,

$$\gamma_0 = \phi_1\gamma_{-1} + \phi_2\gamma_{-2} + \gamma_{wa}(0) - \theta_1\gamma_{wa}(-1) = \phi_1\gamma_1 + \phi_2\gamma_2 + \gamma_{wa}(0) - \theta_1\gamma_{wa}(-1) \quad [10.4]$$

$$\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_{-1} + \gamma_{wa}(1) - \theta_1\gamma_{wa}(0) = \phi_1\gamma_0 + \phi_2\gamma_1 - \theta_1\gamma_{wa}(0) \quad [10.5]$$

$$\gamma_2 = \phi_1\gamma_1 + \phi_2\gamma_0 + \gamma_{wa}(2) - \theta_1\gamma_{wa}(1) = \phi_1\gamma_1 + \phi_2\gamma_0 \quad [10.6]$$

Because of the $\gamma_{wa}(k)$ terms in [10.2], it is necessary to derive other relationships before it is possible to solve for the autocovariances. This can be effected by multiplying [10.1] by a_{t-k} and taking expectations to get

$$\gamma_{wa}(-k) - \phi_1\gamma_{wa}(-k+1) - \phi_2\gamma_{wa}(-k+2) = -[\theta_k]\sigma_a^2 \quad [10.7]$$

where

$$[\theta_k] = \begin{cases} \theta_k, & k = 1 \\ -1, & k = 0 \\ 0, & otherwise \end{cases} \qquad [10.8]$$

and $E(a_t a_{t-k})$ is defined by

$$E(a_t a_{t-k}) = \begin{cases} \sigma_a^2, & k = 0 \\ 0, & k \neq 0 \end{cases} \qquad [10.9]$$

When k = 0

$\gamma_{wa}(0) - \phi_1\gamma_{wa}(1) - \phi_2\gamma_{wa}(2) = \sigma_a^2$

$\gamma_{wa}(0) = \sigma_a^2$ [10.10]

When k = 1

$\gamma_{wa}(-1) - \phi_1\gamma_{wa}(0) - \phi_2\gamma_{wa}(1) = -\theta_1\sigma_a^2$

$\gamma_{wa}(-1) = \phi_1\sigma_a^2 - \theta_1\sigma_a^2 = (\phi_1 - \theta_1)\sigma_a^2$ [10.11]

Substitute $\gamma_{wa}(0)$ and $\gamma_{wa}(-1)$ into [10.4] and [10.5] to obtain

$\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma_a^2 - \theta_1(\phi_1 - \theta_1)\sigma_a^2$ [10.12]

$\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1 - \theta_1\sigma_a^2$ [10.13]

Rearrange [10.12], [10.13], and [10.6] to obtain

$$\gamma_0 = \frac{1 - \phi_2 - 2\phi_1\theta_1 + \theta_1^2 - \phi_2\theta_1^2}{1 - \phi_2 - \phi_1^2 - \phi_2^2 - \phi_1^2\phi_2 + \phi_2^3}\sigma_a^2 \qquad [10.14]$$

$$\gamma_1 = \frac{\phi_1 - \theta_1 - \phi_1\phi_2 + \phi_2\theta_1 + \phi_1\theta_1^2 - \phi_1^2\theta_1 + \phi_2^2\theta_1 - \phi_1\phi_2\theta_1^2 + \phi_1^2\phi_2\theta_1 - \phi_2^3\theta_1}{(1 - \phi_2)(1 - \phi_2 - \phi_1^2 - \phi_2^2 - \phi_1^2\phi_2 + \phi_2^3)}\sigma_a^2$$

[10.15]

Letting $\sigma_a^2 = 1$

$$\gamma_0' = \frac{1 - \phi_2 - 2\phi_1\theta_1 + \theta_1^2 - \phi_2\theta_1^2}{1 - \phi_2 - \phi_1^2 - \phi_2^2 - \phi_1^2\phi_2 + \phi_2^3} \quad [10.16]$$

$$\gamma_1' = \frac{\phi_1 - \theta_1 - \phi_1\phi_2 + \phi_2\theta_1 + \phi_1\theta_1^2 - \phi_1^2\theta_1 + \phi_2^2\theta_1 - \phi_1\phi_2\theta_1^2 + \phi_1^2\phi_2\theta_1 - \phi_2^3\theta_1}{(1 - \phi_2)(1 - \phi_2 - \phi_1^2 - \phi_2^2 - \phi_1^2\phi_2 + \phi_2^3)}$$

[10.17]

3) Using the identity in [10.18] one can calculate the random shock coefficients. Only $\psi_0 = 1$is required for an ARMA(2,1) model.

$$(1 - \phi_1 B - \phi_2 B^2)\psi_k = -\theta_k \quad [10.18]$$

4) Form the covariance matrix $\Delta\sigma_a^2$ of (w_2, w_1, a_2).

$$\Delta = \begin{bmatrix} \gamma_0' & \gamma_{-1}' & \psi_0 \\ \gamma_1' & \gamma_0' & \psi_{-1} \\ \psi_0 & \psi_{-1} & \delta_{1,1} \end{bmatrix} = \begin{bmatrix} \gamma_0' & \gamma_1' & 1 \\ \gamma_1' & \gamma_0' & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad [10.19]$$

5) Determine the lower triangular matrix M by Cholesky decomposition such that

$$\Delta = M\,M^T \quad [10.20]$$

For the case of an ARMA(2,1) model

$$\Delta = \begin{bmatrix} \gamma_0' & \gamma_1' & 1 \\ \gamma_1' & \gamma_0' & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{11} & 0 & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} & m_{31} \\ 0 & m_{22} & m_{32} \\ 0 & 0 & m_{33} \end{bmatrix} \quad [10.21]$$

or

$$\begin{bmatrix} \gamma_0' & \gamma_1' & 1 \\ \gamma_1' & \gamma_0' & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{11}^2 & m_{11}m_{21} & m_{11}m_{31} \\ m_{21}m_{11} & m_{21}^2 + m_{22}^2 & m_{21}m_{31} + m_{22}m_{32} \\ m_{31}m_{11} & m_{31}m_{21} + m_{32}m_{22} & m_{31}^2 + m_{32}^2 + m_{33}^2 \end{bmatrix}$$

[10.22]

By equating (i,j) entries in the matrices on the lest and right hand sides of [10.22], one can calculate m_{11}, m_{21}, m_{22}, m_{31}, m_{32}, and m_{33}.

In particular, for the (1,1) element:

$$m_{11}^2 = \gamma_0' \qquad [10.23]$$

Therefore,

$$m_{11} = \sqrt{\gamma_0'} \qquad [10.24]$$

For the (1,2) entry:

$$m_{11}m_{21} = \gamma_1' \qquad [10.25]$$

Therefore,

$$m_{21} = \frac{\gamma_1'}{m_{11}} = \frac{\gamma_1'}{\sqrt{\gamma_0'}} \qquad [10.26]$$

For the (2,2) element:

$$m_{21}^2 + m_{22}^2 = \gamma_0' \qquad [10.27]$$

Therefore,

$$m_{22}^2 = \gamma_0' - m_{21}^2 = \gamma_0' - \frac{\gamma_1'^2}{\gamma_0'} = \frac{\gamma_0'^2 - \gamma_1'^2}{\gamma_0'} \qquad [10.28]$$

Therefore,

$$m_{22} = \sqrt{\frac{\gamma_0'^2 - \gamma_1'^2}{\gamma_0'}} \qquad [10.29]$$

For the (1,3) entry:

$$m_{11}m_{31} = 1 \qquad [10.30]$$

Therefore,

$$m_{31} = \frac{1}{m_{11}} = \frac{1}{\sqrt{\gamma_0'}} \qquad [10.31]$$

Fore the (2,3) element:

$$m_{21}m_{31} + m_{22}m_{32} = 0 \qquad [10.32]$$

Therefore,

$$m_{32} = -\frac{m_{21}m_{31}}{m_{22}} = -\frac{\frac{\gamma_1'}{\sqrt{\gamma_0'}}\frac{1}{\sqrt{\gamma_0'}}}{\sqrt{\frac{\gamma_0'^2-\gamma_1'^2}{\gamma_0'}}} = -\frac{\gamma_1'\sqrt{\gamma_0'}}{\gamma_0'\sqrt{\gamma_0'^2-\gamma_1'^2}} \qquad [10.33]$$

For the (3,3) element:

$$m_{31}^2 + m_{32}^2 + m_{33}^2 = 1 \qquad [10.34]$$

Therefore,

$$m_{33}^2 = 1 - m_{31}^2 - m_{32}^2 = 1 - \frac{1}{\gamma_0'} - \frac{\gamma_1'^2\gamma_0'}{\gamma_0'^2(\gamma_0'^2-\gamma_1'^2)} = \frac{\gamma_0'^2-\gamma_1'^2-\gamma_0'}{\gamma_0'^2-\gamma_1'^2} \qquad [10.35]$$

Therefore,

$$m_{33} = \sqrt{\frac{\gamma_0'^2-\gamma_1'^2-\gamma_0'}{\gamma_0'^2-\gamma_1'^2}} \qquad [10.36]$$

Hence,

$$M = \begin{bmatrix} \sqrt{\gamma_0'} & 0 & 0 \\ \frac{\gamma_1'}{\sqrt{\gamma_0'}} & \sqrt{\frac{\gamma_0'^2-\gamma_1'^2}{\gamma_0'}} & 0 \\ \frac{1}{\sqrt{\gamma_0'}} & -\frac{\gamma_1'\sqrt{\gamma_0'}}{\gamma_0'\sqrt{\gamma_0'^2-\gamma_1'^2}} & \sqrt{\frac{\gamma_0'^2-\gamma_1'^2-\gamma_0'}{\gamma_0'^2-\gamma_1'^2}} \end{bmatrix} \qquad [10.37]$$

6) Generate (e_1, e_2, e_3) and (a_3, a_4, …, a_{10}) where e_t and a_t sequences are NID(0,σ_a^2).

7) The starting value w_1 and w_2 are calculated using [10.38] as

$$w_{2+1-t} = \sum_{j=1}^{t} m_{t,j} e_j, \qquad t = 1 \text{ and } 2 \qquad [10.38]$$

where $m_{t,j}$ is the t,j entry in the matrix M.

When t = 1

$$w_2 = m_{11} e_1 = \sqrt{\gamma_0'} e_1 \qquad [10.39]$$

When t = 2

$$w_1 = m_{21} e_1 + m_{22} e_2 = \frac{\gamma_1'}{\sqrt{\gamma_0'}} e_1 + \sqrt{\frac{\gamma_0'^2 - \gamma_1'^2}{\gamma_0'}} e_2 \qquad [10.40]$$

8) Determine the initial value for a_2, from

$$a_{2+1-t} = \sum_{j=1}^{2+t} m_{t+2,j} e_j, \qquad t = 1 \qquad [10.41]$$

When t =1

$$a_2 = m_{31} e_1 + m_{32} e_2 + m_{33} e_3 = \frac{1}{\sqrt{\gamma_0'}} e_1 - \frac{\gamma_1' \sqrt{\gamma_0'}}{\gamma_0' \sqrt{\gamma_0'^2 - \gamma_1'^2}} e_2 + \sqrt{\frac{\gamma_0'^2 - \gamma_1'^2 - \gamma_0'}{\gamma_0'^2 - \gamma_1'^2}} e_3$$

[10.42]

9) Use the given definition of the ARMA(2,1) model in [10.1] to get w_3, w_4, …, w_{10}. More specifically,

$$w_3 = \phi_1 w_2 + \phi_2 w_1 + a_3 - \theta_1 a_2 \qquad [10.43]$$

where the starting values for w_1, w_2, and a_2 are calculated in steps 6 and 7, respectively. The values for a_3 and also a_4, a_5, …, a_{10} are determined in step 5.

Next

$$w_4 = \phi_1 w_3 + \phi_2 w_2 + a_4 - \theta_1 a_3 \quad [10.44]$$

where w_3 is calculated in the previous iteration, w_2 is calculated in step 6, and a_4 and a_3 are generated at step 5.

By following the same procedure

$$w_5 = \phi_1 w_4 + \phi_2 w_3 + a_5 - \theta_1 a_4 \quad [10.45]$$

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$$w_{10} = \phi_1 w_9 + \phi_2 w_8 + a_{10} - \theta_1 a_9 \quad [10.46]$$

If w_t has a mean level, this can be added to the above generated values. Notice that the starting values w_1, w_2, and a_2 are randomly generated from the underlying ARMA(2,1) process. Therefore, the simulated sequence is not biased because of fixed starting values.

For a particular ARMA model, it is only necessary to calculate the matrix M once, no matter how many simulated series are synthesized. Therefore, WASIM2 is economical with respect to computer time required, especially when many time series of the same length are generated.

(b) Proof

In step 3, one forms the covariance matrix $\Delta\sigma_a^2$ of the starting values w_2, w_1, a_2, which are contained in a vector W. Next, one determines the lower triangular matrix M for Δ in [10.20]. Following steps 6 and 7, the starting values contained in the vector W can be generated using

$$W = M\,e \quad [10.47]$$

where the e_t's contained in the vector e are NID($0,\sigma_a^2$). In order to simulate exactly the starting values contained in W, the covariance matrix of W must be $\Delta\sigma_a^2$. This can be easily proven as follows:

$$\begin{aligned} Var[M\,e] &= E[M\,e(M\,e)^T] \\ &= E[M\,e\,e^T\,M^T] = MVar(e)M^T \\ &= \sigma_a^2 M\,M^T = \Delta\sigma_a^2 \quad [10.48] \end{aligned}$$

where σ_a^2 is a diagonal matrix for which each diagonal entry is σ_a^2.