2-3 Compact Sets

2.31 Definition

By an open cover of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of x such that $E \subset \cup_\alpha G_\alpha$.

2.32 Definition

A subset K of a metric space X is said to be *compact* if every open cover of K contains a *finite* subcover.

More explicitly,the requirement is that if $\{G_\alpha\}$ is an open cover of K,then there are finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $K \subset G_{\alpha_1} \cup \ldots \cup G_{\alpha_n}$.

2.33 Theorem

Suppose $K \subset Y \subset X$.Then K is *compact relative to* X *if and only if* K is compact to Y.

Proof

The "if and only if" condition is equivalent to "sufficient and necessary".

(1)Suppose K is compact relative to X,We have

$$K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n} \tag{1}$$

Let

$$V_\alpha = Y \cup G_\alpha$$

Then we have

$$K \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \ldots \cup V_{\alpha_n} \tag{2}$$

Thus we prove that K is compact relative to Y.

(2)Suppose K is compact relative to Y.Similarly Let $\{G_n\}$ be some finite collection of open subsets of X which covers K,and put $V_\alpha = Y \cap G_\alpha$.Then we get (2),which implies (1),and therefore K is compact relative to X.

2.34 Theorem

Compact subsets of metric spaces are closed.

Proof

这一证明表现出了紧性将局部推广至全局的作用.

我们等价于证明，“度量空间的紧子集K之补集K^c为开集”.

我们考虑紧集合中的任意一点a和其补集的任意一点b.现在为了方便我们不妨固定点b,考虑有限个子覆盖(subcover)$\{A_n\}$及点b的有限个邻域$(B_n\}$,使得

$$K \subset \cup_{i=1}^{n} A_n$$
$$V = \cap_{i=1}^{n} B_n$$

其中，这两个邻域的"中心点"距离记为$d(a_n, b)$.并令所有对应的领域A_n与B_n满足半径$r_n < \frac{d(a_n,b)}{2}$.

这样就有$V \cap K = \varnothing$.(局部不相交推广至全局不相交).因此对于补集中的任意一点b,总能以此法构造领域,使得$N_r(b) \cap K = \varnothing, N_r(b) \subset K^c$.

因此等价命题得证，原命题得证.

2.35 Theorem

Closed subsets of compact sets are compact.

Proof

Let K be a compact subset of a metric space X and let $F \subseteq K$ be closed(relative to X).

Let $\{G_n\}$ be an open cover of F,then $\{G_\alpha \cap F^c\}$ is an open cover of K.**(Since F^c is open)**

Corollary

If F is closed and K is compact,then $F \cap K$ is compact.

The intersection of finite compact sets is compact.

2.36 Definition

Let $\{S_n\}$ be a collection of subsets of a metric space X.We say $\{S_n\}$ satisfies ***finite intersection condition*** if the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty.

2.37 Theorem

If $\{K_\alpha\}$ is a collection of **compact** subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty,then $\cap K_\alpha$ is nonempty.

Proof

The statement is equivalent to "If $\cap K_\alpha = \varnothing$,then $\cap_{i=1}^{n} K_i = \varnothing$ ".

We may fix a member K_1 such that $K_1 \cap (\cap_{i=2}^{\infty} K_i) = \varnothing, K_1 \subset (\cap_{i=2}^{\infty} K_i)^c, K_1 \subset (\cup_{i=2}^{\infty} K_i^c)$.

Since K_1 is compact,$K_1 \subset (\cup_{i=2}^{n} K_i^c)$,$\cap_{i=1}^{n} K_i = \varnothing$.

Corollary

If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$.(n=1,2,3,...),then $\cap_1^{\infty} K_n$ is not empty.

2.38 Theorem

If E is a infinite subset of a compact set K,then E has a limit point in K.

Proof

The key idea is,let I be the k-cell,$\exists x^* \in I$,s.t. if $y \in K_n$, and $K_{n+1} \subset K_n, K_1 \subset I$,then $|y - x^*|$ must be covered by some neighborhood. Namely, $\forall r > 0, \exists n \in N, y \in K_n, s.t. |y - x^*| < r$ .That is to say,there is always some finite subcollection of {G_n} capable of covering I_n .Suppose $E' \cap E = \varnothing$.That is to say, $\forall p \in E, \forall \epsilon \in R, N_\epsilon p \backslash \{p\} = \varnothing$ .Then there is no finite subcollection of {V_q} can cover E.Since $E \subset K$,therefore K is not compact. This contradicts the compactness of K.

我们应当指出，定理2.38可以用于证明**闭区间套**定理。

更严格的证明如下：

We prove it by contradiction.

Suppose no point of K is a limit point of a compact set K,then for each $q \in K$, $\exists r > 0, s.t. N_r(p) \cap E = \varnothing, \{p\}. i.e.$each point in K would have a neighborhood $N_r(p)$ containing at most one point of E.

Therefore,E cannot be covered by infinite open sets,and the same is true for K,since $E \subset K$,which contradicts the compactness of K.

2.39 Theorem

If {I_n} is a sequence of intervals in R^1,such that $I_n \supset I_{n+1}$(n=1,2,3,...),then $\cap_1^\infty I_n$ is not empty.

Proof

令$I_n = [a_n, b_n]$,由于实数集的稠密性，总能找到一个{a_n}的上确界α使得$\alpha \in I_n, (n = 1, 2, 3, \ldots)$.故原命题得证.

It is known that R has the least-upper-bound property,and thus has the greatest-lower-bound property. Let $\{a_n\}$ be set of all the lower bounds of each interval I_n ,and $\{b_n\}$ be the set of all the upper bounds of each interval I_n.

It is apparent that:

$$\exists x \in R, s.t. x = sub\{a_n\}.$$
$$\forall n \in N^*, a_n < x \le b_1 \le b_2 \le \ldots \le b_n$$

Therefore we find that $\forall n \in N^*, x \in I_n, x \in \cap_1^\infty I_n$.

2.40 Theorem

Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_n + 1$(n=1,2,3,...),then $\cap_1^\infty I_n$ is not empty.

Proof

我们对k维的每一维空间都作类似2.39定理的操作即可得证。

2.41 Theorem

Every k-cells is compact.

Proof

Let I be a k-cell,consisting of all points $x = (x_1, x_2., \dots x_n)$ such that $a_j < x_j < b_j (1 \le j \le k)$.Put

$$\delta = \sqrt{\sum_1^k (b_j - a_j)^2}$$

Then $|x - y| \le \delta$,if $x \in I, y \in I$.

To get a contradiction,we suppose that there exists a open cover $\{G_n\}$ of I which contains no finite subcover of I.

Put $c_j = \frac{a_j + b_j}{2}$.The intervals will be separated into 2^k k-cells of Q_i whose union is I.

According the hypothesis,there must be at least one subset cannot be covered by $\{G_n\}$,call it I_1.

Corollary:

Let $k \in N^*$,If $\{I_n\}$ is a sequence of k-cells such that $I_{n+1} \subset I_n$ (n=1,2,3,...),then $\cap_1^\infty I_n$ is not empty.

Proof:

For each dimension,apply Theorem 2.39.We then obtain $x^* = (x_1^*, x_2^*, \dots, x_k^*)$.
$x^* \in I_n, n = 1, 2, 3...$,which makes the corollary follow.

2.42 Theorem(3 个等价命题)

需要指出的是，该定理中(a)与(b)的等价性被称作Heine-Borel theorem（海涅-博雷尔定理）：

If a set E in R^k has one of the following three properties,then it has the other two:

> (a) E is closed and bounded.
>
> (b) E is compact.
>
> (c) Every infinite subset of E has a limit point in E.

Proof:

(a) -> (b):

$E \subset I$ for some k-cell I.Since a closed subset of a compact set is compact(Theorem 2.35) and each k-cell is compact(Theorem 2.40,E is compact.

(b) -> (c):

If E is a infinite subset of a compact set K,then E has a limit point in K. (Theorem 2.38)

That implies (b) to (c).

(c) -> (a):

Suppose E is not bounded,then some subset of E(call it S) contains points x_n with $|x_n| > n, n \in N^*$.

Thus S clearly has no limit point,contradicting with (c).

Suppose E is not closed,consider a limit point of E but not in E.Let $S = \{x_n | x_n \in E, |x_n - x_0| < \frac{1}{n}\}$.

Thus S is infinite, and S has no limit point but x_0 in R^k.

Let's prove it:

Fix $y \in R^k, y \neq x_0$,then $|x_n - y| \geq |x_0 - y| + |x_n - x_0|$

$$\geq |x_0 - y| - \frac{1}{n}$$

$$\geq \frac{1}{2}|x_0 - y|$$

Therefore x_0 is the only limit point of S.

Thus S has no limit point in E,contradicting with (c).

Hence it must be closed if (c) holds.

2.42 Weierstrass Theorem(魏尔斯特拉斯定理)

Every bounded infinite subset of R^k has a limit point in R^k.

Proof:

Let the set be S.Since it is bounded,there is some k-cell I_k s.t. $S \subset I_k$.I_k is compact,thus S has a limit point in I_k.Namely the theorem holds.