2-3 Compact Sets(1)

2.31 Definition

By an open cover of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of x such that $E \subset \cup_\alpha G_\alpha$.

2.32 Definition

A subset K of a metric space X is said to be *compact* if every open cover of K contains a *finite* subcover.

More explicitly,the requirement is that if $\{G_\alpha\}$ is an open cover of K,then there are finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $K \subset G_{\alpha_1} \cup \ldots \cup G_{\alpha_n}$.

2.33 Theorem

Suppose $K \subset Y \subset X$.Then K is *compact relative to X if and only if* K is compact to Y.

Proof

The "if and only if" condition is equivalent to "sufficient and necessary".

(1)Suppose K is compact relative to X,We have

$$K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n} \tag{1}$$

Let

$$V_\alpha = Y \cup G_\alpha$$

Then we have

$$K \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \ldots \cup V_{\alpha_n} \tag{2}$$

Thus we prove that K is compact relative to Y.

(2)Suppose K is compact relative to Y.Similarly Let $\{G_n\}$ be some finite collection of open subsets of X which covers K,and put $V_\alpha = Y \cap G_\alpha$.Then we get (2),which implies (1),and therefore K is compact relative to X.

2.34 Theorem

Compact subsets of metric spaces are closed.

Proof

这一证明表现出了紧性将局部推广至全局的作用.

我们等价于证明，“度量空间的紧子集K之补集K^c为开集”.

我们考虑紧集合中的任意一点a和其补集的任意一点b.现在为了方便我们不妨固定点b,考虑有限个子覆盖(subcover)$\{A_n\}$及点b的有限个邻域$(B_n\}$,使得

$$K \subset \cup_{i=1}^{n} A_n$$
$$V = \cap_{i=1}^{n} B_n$$

其中，这两个邻域的“中心点”距离记为$d(a_n, b)$.并令所有对应的领域A_n与B_n满足半径$r_n < \frac{d(a_n,b)}{2}$.

这样就有$V \cap K = \varnothing$.(局部不相交推广至全局不相交).因此对于补集中的任意一点b,总能以此法构造领域,使得$N_r(b) \cap K = \varnothing, N_r(b) \subset K^c$.

因此等价命题得证，原命题得证.

2.35 Theorem

Closed subsets of compact sets are compact.

Proof

Let K be a compact subset of a metric space X and let $F \subseteq K$ be closed(relative to X).

Let $\{G_n\}$ be an open cover of F,then $\{G_\alpha \cap F^c\}$ is an open cover of K.**(Since F^c is open)**

Corollary

If F is closed and K is compact,then $F \cap K$ is compact.

The intersection of finite compact sets is compact.

2.36 Definition

Let $\{S_n\}$ be a collection of subsets of a metric space X.We say $\{S_n\}$ satisfies ***finite intersection condition*** if the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty.

2.37 Theorem

If $\{K_\alpha\}$ is a collection of **compact** subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty,then $\cap K_\alpha$ is nonempty.

Proof

The statement is equivalent to "If $\cap K_\alpha = \varnothing$,then $\cap_{i=1}^{n} K_i = \varnothing$ ".

We may fix a member K_1 such that $K_1 \cap (\cap_{i=2}^{\infty} K_i) = \varnothing, K_1 \subset (\cap_{i=2}^{\infty} K_i)^c, K_1 \subset (\cup_{i=2}^{\infty} K_i^c)$.

Since K_1 is compact,$K_1 \subset (\cup_{i=2}^{n} K_i^c)$,$\cap_{i=1}^{n} K_i = \varnothing$.

Corollary

If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$.(n=1,2,3,...),then $\cap_1^{\infty} K_n$ is not empty.