Section2 Basic Topology

FINITE,COUNTABLE,AND UNCOUNTABLE SETS

2.1 Definition

Consider two sets ,A and B and a mapping from A to B ,which we may call it 'f'.The 'f' should be some manner,or be said to be a ***function***(mapping).

The elements f(x) for all x in A,are called ***values***.

The set of all values is called the ***range*** of f.

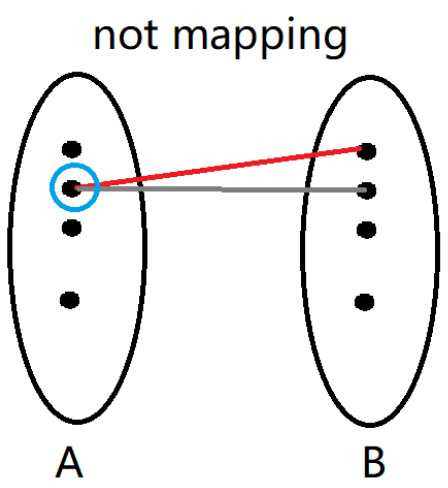
2.2 Definition

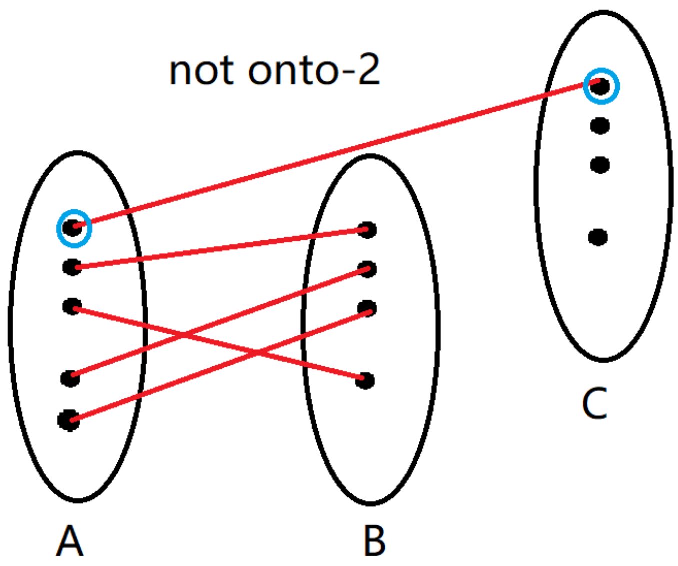
Clearly,$f(A)$ is the range of f and $f(A) \subseteq B$.

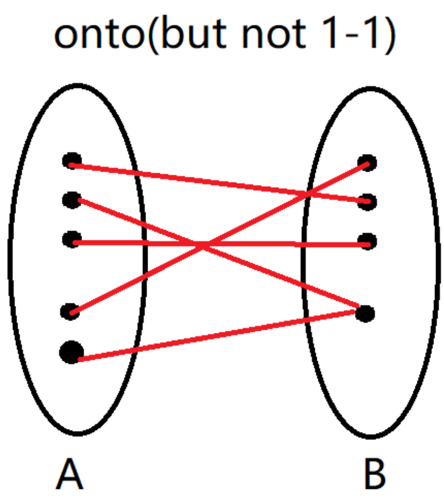
We say that f maps A ***onto*** B when $f(A) = B$.

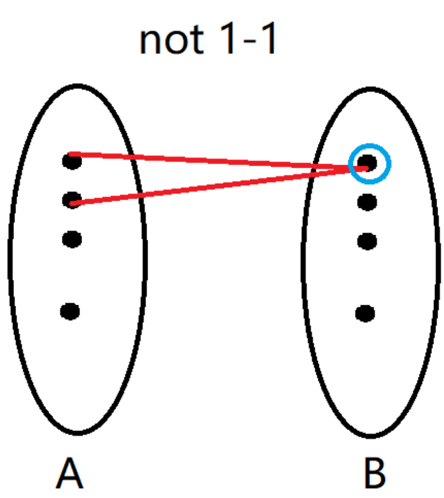
For each $y \in B$,when $f^{-1}(y)$ covers all the elements of A,f is said to be ***1-1(one to one)*** mapping of A into B.

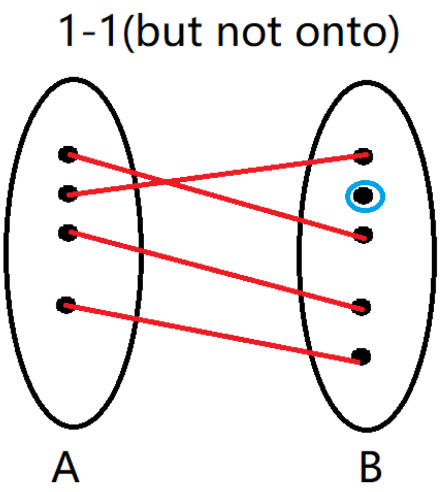
We may illustrate the definition with these graphs below:

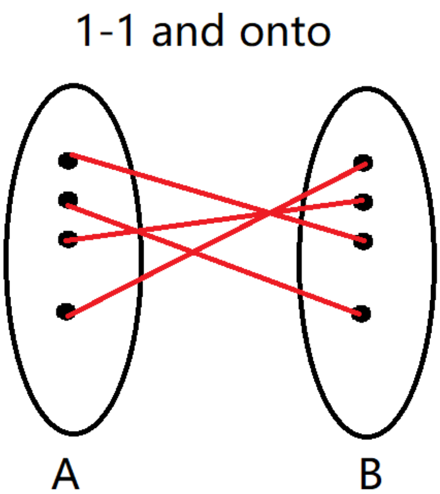



not onto-2
A
B
C


onto(but not 1-1)
A
B


not 1-1
A
B





2.3 Definition

Any relation with these properties is called an equivalence relation:

> $$\begin{aligned}&reflexive: A \sim A.\\&symmestric: If\ A \sim B, then\ B \sim A.\\&transitive: If\ A \sim B\ and\ B \sim C, then\ A \sim C.\end{aligned}$$

If there exists a 1-1 mapping of A onto B,we say that A and B are equivalent,that is, $A \sim B$.

2.4 Remark

> if $J_m \sim J_n, m = n$.
>
> if $m = n, J_m \sim J_n$.

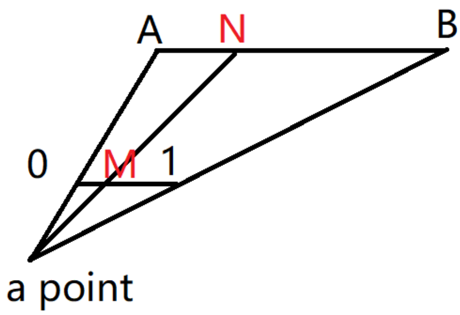
where J_n is the set whose elements are the integers 1,2, . . . ,n;

2.5 Definition

For any set A,we may say:

> a. A is **finite** if $A \sim J_n$ for some n.(empty set $\varnothing$ is considered to be finite)
>
> b. A is **infinite** if A is not finite.
>
> c. A is **countable** if $A \sim J$.
>
> d. A is **uncountable** if A is neither **finite** or **countable**.
>
> e. A is **at most countable**(至多可数) if A is **finite** or **countable**.

2.6 Remark

> a.$N \sim Z$
>
> > For :$0,1,2,3,...\to 0,-1,1,-2,...$
> >
> > We may even find out an explicit formula.
>
> b. $\forall a,b \in R,(a,b)\sim(0,1)$:
>
> > Graphic explanations:
> >
> > 
> >
> >
> > $\forall x$ in (a,b),denoted by N on the axis AB,has some linear pattern corresponding to M in axis 01.
>
> c.$(0,1)\sim R$
>
> > We may even find the explicit function of it:

> > $$f(x):=\begin{cases}\dfrac{x-\dfrac{1}{2}}{x^2}, & 0<x\le\dfrac{1}{2}\\[2ex] \dfrac{x-\dfrac{1}{2}}{1-x^2}, & \dfrac{1}{2}<x<1\end{cases}$$
>
> d. $\forall a,b\in R,(a,b)\sim R$:
>
> > An equivalence relation is transitive,so it is clear.
> >
> > In fact,we could replace the definition of infinite set 2.5.b by the statement below:
> >
> > **A is infinite if A is equivalent to one of its proper subsets.**

2.7 Definition

We may sort of order or label the elements in a countable set A in the way of 'sequence in A':

$A=\{x_n\},n\in J.$

The elements in the set arranged in this way need not to be distinct.(For example,we may assume:$f(x):=x^2,x_1=(-1)^2=1,x_2=(+1)^2=1,x_1=x_2$,which is valid)

2.8 Theorem

Every infinite subset of a countable set A is countable.

No uncountable set can be a subset of a countable set.

Proof:

The key operation is to find out the 'net mapping' of J_n,the subset of J:

k	n_k	x_{n_k}
1	n_1	x_{n_1}
2	n_2	x_{n_2}
...	...	...
n	n_n	n_{n_k}

In this way we found a mapping function from J_n to A such that n_k is the smallest integer greater than $n_{k-1}$and $x_{n_k} \in E$,where $f(k) = x_{n_k}$.

####

2.9 Theorem

Let $\{E_n\}, n = 1, 2, 3, \ldots$,be a sequence of countable sets,and put

$$S = \cup_{n=1}^{\infty} E_n$$

Then S is countable.

2.10 Remark

Let $A = (0, 1)$ and $E_\alpha = (0, \alpha)$,then

> $$(1) \cup_{\alpha \in A} E_\alpha = (0, 1)$$
> $$(2) \cap_{\alpha \in A} E_\alpha = \varnothing.$$

(2) is clear,for:

$$\forall y > x > 0, y \notin E_x.$$

Hence $\forall y > 0, y \notin \cup_{x \in A} E_x$.

To prove (1),we let $M = \cup_{\alpha \in A} E_\alpha . \forall x \in M, x \in (0, 1), M \subseteq (0, 1)$.

Suppose $\forall \beta \in (0, 1)$, $\forall \alpha \in (0, 1), \beta \notin E_\alpha$,then $\beta >= 1$.However $\beta < 1$.

Hence,$\forall$ $\beta \in (0, 1), \exists \alpha \in (0, 1), s.t. \beta \in E_\alpha$,whereas $(0, 1) \subseteq M$.

Since $(0, 1) \subseteq M, M \subseteq (0, 1), M = \cup_{\alpha \in A} E_\alpha = (0, 1)$.

2.11 Facts

A、B are two sets.

> (1)If A has a 1-1 mapping f into B,$\exists$ $g : B \to A, s.t. (gof) = idA.$
>
> (2)If f maps A onto B,$\exists$ $g : B \to A, s.t. (fog) = idB.$

2.12 Theorem

Let $\{E_n\}, n = 1, 2, 3, \ldots$,be a sequence of countable sets,and put $S = \cup_{n=1}^{\infty} E_n$.Then S is countable.

Proof:

Rearrange the order of the terms of the sets T in a sequence $\{x_{nk}\}$:

$x_{11} \rightarrow x_{21} \rightarrow x_{12} \rightarrow x_{31} \rightarrow x_{22} \rightarrow x_{13} \rightarrow \ldots$

It is valid,because for each element on the specific diagonal

,the sum of the indexes ranges from (2) to n+k with step 1.

However,some entries may appear more than once,so what we can assure is that it goes that

$$S \sim T \subseteq N$$

Hence S and T are **at most countable**. Since S is infinite,S is countable.

Corollary

Suppose A is at most countable,and, for every $\alpha \in A, B_\alpha$is at most countable. Put

$$T = \cup_{\alpha \in A} B_\alpha$$

Then T is at most countable.

For T is the subset of the at most countable set $S = \cup_{n=1}^{\infty} E_n$.

2.13 Theorem

Let A be a countable set,and let B_n be the set of all n-tuples $(\alpha_1, \ldots, \alpha_n)$,where $\alpha_k \in A (k = 1, \ldots, n)$,and the elements $\alpha_1, \ldots, \alpha_n$ need not to be distinct. Then B_n is countable.

Proof:

$B_n = \underbrace{A \times \ldots \times A}_{n}$.By induction and Theorem 2.12,we know that $A \times A$ is countable,thus $A \times A \times A$ is countable,...,B_n is countable.

Corollary

The set of all **rational numbers** is countable.

That is because $\mathbb{Q} \subset \mathbb{N} \times \mathbb{N}$,$\mathbb{N} \times \mathbb{N}$ is countable and $\mathbb{Q}$ is at most countable but infinite,and thus countable.

2.14 Theorem

Let A be the set of all sequences whose elements are the digits 0 and 1.This set A is uncountable.

Outline of the proof:

To prove something wrong is usually hard,so we may try to prove it by contradictions.

We may assign E as a sequence of countless sequences:

$$E = \{s_1, s_2, s_3, \dots\}$$

,where $s_k, k = 1, 2, \dots$ is a sequence.

If a set is countable,it can be rearranged to a sequence like that:

$$\begin{aligned} E = \{&\{a_{11}, \dots\}, \\ &\{a_{21}, a_{22}, \dots\}, \\ &\{a_{31}, a_{32}, a_{33}, \dots\}, \dots\} \end{aligned}$$

Let's DIY a sequence to create the contradiction in the way below:

Consider a sequence $\{b_k\}$ like that:

$$b_k = \begin{cases} 1, a_{kk} = 0 \\ 0, a_{kk} = 1 \end{cases}$$

The sequence $b_k \neq E_n, \forall n \in N$.However $b_k \in E$.

Thus A is uncountable.

Corollary

The set of all real numbers is uncountable.

METRIC SPACES

2.15 Definition

A set X is said to be a metric space if two points a and b in X associate a real number function d(p,q) such that

> $$\begin{aligned} &Non-negativity: & d(p,q) > 0 \, if \, p \neq q, d(p,p) = 0; \\ &Symmetry: & d(p,q) = d(q,p); \\ &Triangle\ Inequality: & d(p,q) \leq d(p,r) + d(r,q), \, for \; any \; r \in X. \end{aligned}$$

Any function with these three properties is called a distance function,or a metric.

2.16 Definition

The **unit ball** is a set defined by

$$\{\mathrm{x} \in R^n | d(\mathrm{x}, 0) \leq 1\}$$

2.17 Definition

Let X be a metric space,all points and sets mentioned below are elements and subsets of X.

> 1. A **neighborhood** of p is a set $N_r(p)$ consisting of all q such that $d(p,q) < r$,for some $r > 0$.The number ***r*** is called the radius of $N_r(p)$.(random r)
> 2. A point p is a ***limit point*** of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.(no one but myself).$i.e. \forall r > 0, N_r(p) \cap E \neq \varnothing$
> 3. If $p \in E$ and p is not a limit point of E,then p is called an ***isolated point*** of E.
> 4. A point is an ***interior point*** of E if there is a neighborhood N of p such that $N \subset E$. $i.e. \exists r > 0, N_r(p) \subseteq E$.
> 5. E is closed if **every limit point** of E is a point of E.
> 6. E is open if every point of E is a **interior point**.$\forall p \in E, \exists r > 0, s.t. N_r(p) \subseteq E$.
> 7. The **complement** of E,denoted by E^c,is the set of all points $p \in X$ such that $p \notin E$.
> 8. E is ***perfect*** if E is closed and if every point of E is a limit point of E.
> 9. E is ***bounded*** if there is a real number M and a point $q \in X$ such that $d(p,q) < M$ for all $p \in E$.
> 10. E is ***dense*** in X if every point of X is a limit point of E,or a point of E(or both).

Notice that an interior point is not necessarily a limit point.

2.18 Facts

Let X be a metric space,all points and sets mentioned below are elements and subsets of X.

> 1. If E is open,E^c is closed;
> 2. If E is closed,E^c is open.

Proof:

1.We need to show that if p is a limit point of E^c then $p \in E^c$.

Let p be a limit point,$i.e. \forall r > 0, N_r(p) \cap E^c \neq \varnothing$.

$\therefore \forall p \in E, \forall r > 0, N_r(p)$ dosen't $\subseteq E$.

Therefore p is not an interior point of E. However,every point in E is an interior point.
$i.e. p \notin E, p \in E^c$.

2.The statement is equivalent to **"If E is not open,then E^c is not closed"**,which is trivial.

Since E is not open,that is $\exists p \in E, \forall r > 0, s.t. N_r(p) \cap E^C \neq \varnothing$.

Therefore E^c is not closed.

2.19 Facts

Let A_n, B_n be a open set and a closed set which are both subsets of a metric space X. $(n = 1, 2, 3...)$

> 1. $\cup_{i=1}^n A_n$ is open. $\cup_{i=1}^n B_n$is closed.$\cap_{i=1}^n A_n$is open,$\cap_{i=1}^n B_n$is closed.
> 2. $\cup_{i=1}^\infty A_n$ is open.$\cup_{i=1}^\infty B_n$ is closed.
> 3. $\cap_{i=1}^\infty A_n$ may NOT be open.$\cup_{i=1}^\infty B_n$ may NOT be closed.

2.20 Theorem

If p is a limit point of a set E,then every neighbourhood of p contains **infinitely** many points of E.

Corollary

A finite point set has no limit points.

Hence,a finite set is closed.

2.22 Theorem

Let $\{E_n\}$ be a (finite or infinite)collection of sets E_α.Then $(\cup_\alpha E_\alpha)^c = \cap_\alpha (E_\alpha^c)$

2.23 Definition

If X is a metric space,if $E \subset X$,and if E' denotes the set of all limit points of E,then the *closure* of E is the set $\overline{E} = E \cup E'$.

2.24 Theorem

If X is a metric space and $E \subset X$,then

> (a) $\overline{E}$ is closed,
>
> (b) $\overline{E} = E$ if and only if E is closed,
>
> (c) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

2.25 Definition

If E is a metric space and E is a closed set,E is perfect if and only if $E' = E$.

2.26 Theorem

Let E be a nonempty set of real numbers which is bounded above. Let $y = supE$.Then $y \in \overline{E}$.

Proof

Assume $y \notin E$.For each $h > 0$ there exists a point $x \in E$,such that $y - h < x < y$,otherwise $y - h$ would be the upper bound of E.Thus y is a limit point and $y \in \overline{E}$.

(We omit some new definitions)

Compact Sets

2.31 Definition

By an open cover of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of x such that $E \subset \cup_\alpha G_\alpha$.

2.32 Definition

A subset K of a metric space X is said to be *compact* if every open cover of K contains a *finite* subcover.

More explicitly,the requirement is that if $\{G_\alpha\}$ is an open cover of K,then there are finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $K \subset G_{\alpha_1} \cup \ldots \cup G_{\alpha_n}$.

2.33 Theorem

Suppose $K \subset Y \subset X$.Then K is *compact relative to X if and only if* K is compact to Y.

Proof

The "if and only if" condition is equivalent to "sufficient and necessary".

(1)Suppose K is compact relative to X,We have

$$K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n} \tag{1}$$

Let

$$V_\alpha = Y \cup G_\alpha$$

Then we have

$$K \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \ldots \cup V_{\alpha_n} \tag{2}$$

Thus we prove that K is compact relative to Y.

(2)Suppose K is compact relative to Y.Similarly Let $\{G_n\}$ be some finite collection of open subsets of X which covers K,and put $V_\alpha = Y \cap G_\alpha$.Then we get (2),which implies (1),and therefore K is compact relative to X.

2.34 Theorem

Compact subsets of metric spaces are closed.

Proof

这一证明表现出了紧性将局部推广至全局的作用.

我们等价于证明，“度量空间的紧子集K之补集K^c为开集”.

我们考虑紧集合中的任意一点a和其补集的任意一点b.现在为了方便我们不妨固定点b,考虑有限个子覆盖(subcover)$\{A_n\}$及点b的有限个邻域$(B_n\}$,使得

$$\begin{gathered} K \subset \cup_{i=1}^n A_n \\ V = \cap_{i=1}^n B_n \end{gathered}$$

其中，这两个邻域的“中心点”距离记为$d(a_n,b)$.并令所有对应的领域A_n与B_n满足半径$r_n < \frac{d(a_n,b)}{2}$.

这样就有$V \cap K = \varnothing$.(局部不相交推广至全局不相交).因此对于补集中的任意一点b,总能以此法构造领域,使得$N_r(b) \cap K = \varnothing, N_r(b) \subset K^c$.

因此等价命题得证，原命题得证.

2.35 Theorem

Closed subsets of compact sets are compact.

Proof

Let K be a compact subset of a metric space X and let $F \subseteq K$ be closed(relative to X).

Let {G_n} be an open cover of F,then {$G_\alpha \cap F^c$} is an open cover of K.**(Since F^c is open)**

Corollary

If F is closed and K is compact,then $F \cap K$ is compact.

The intersection of finite compact sets is compact.

2.36 Definition

Let {S_n} be a collection of subsets of a metric space X.We say {S_n} satisfies ***finite intersection condition*** if the intersection of every finite subcollection of {K_α} is nonempty.

2.37 Theorem

If {K_α} is a collection of **compact** subsets of a metric space X such that the intersection of every finite subcollection of {K_α} is nonempty,then $\cap K_\alpha$ is nonempty.

Proof

The statement is equivalent to "If $\cap K_\alpha = \varnothing$,then $\cap_{i=1}^n K_i = \varnothing$ ".

We may fix a member K_1 such that $K_1 \cap (\cap_{i=2}^\infty K_i) = \varnothing, K_1 \subset (\cap_{i=2}^\infty K_i)^c, K_1 \subset (\cup_{i=2}^\infty K_i^c)$.

Since K_1 is compact,$K_1 \subset (\cup_{i=2}^n K_i^c)$,$\cap_{i=1}^n K_i = \varnothing$.

Corollary

If {K_n} is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$.(n=1,2,3,...),then $\cap_1^\infty K_n$ is not empty.

2.38 Theorem

If E is a infinite subset of a compact set K,then E has a limit point in K.

Proof

The key idea is,let I be the k-cell,$\exists x^* \in I$,s.t. if $y \in K_n$, and $K_{n+1} \subset K_n, K_1 \subset I$,then $|y - x^*|$ must be covered by some neighborhood. Namely, $\forall r > 0, \exists n \in N, y \in K_n, s.t. |y - x^*| < r$ .That is to say,there is always some finite subcollection of {G_n} capable of covering I_n .Suppose $E' \cap E = \varnothing$.That is to say, $\forall p \in E, \forall \epsilon \in R, N_\epsilon p \backslash \{p\} = \varnothing$ .Then there is no finite subcollection of {V_q} can cover E.Since $E \subset K$,therefore K is not compact. This contradicts the compactness of K.

我们应当指出，定理2.38可以用于证明**闭区间套**定理。

更严格的证明如下：

We prove it by contradiction.

Suppose no point of K is a limit point of a compact set K,then for each $q \in K$, $\exists r > 0, s.t. N_r(p) \cap E = \varnothing, \{p\}. i.e.$each point in K would have a neighborhood $N_r(p)$ containing at most one point of E.

Therefore,E cannot be covered by infinite open sets,and the same is true for K,since $E \subset K$,which contradicts the compactness of K.

2.39 Theorem

If {I_n} is a sequence of intervals in R^1,such that $I_n \supset I_{n+1}$(n=1,2,3,...),then $\cap_1^\infty I_n$ is not empty.

Proof

令$I_n = [a_n, b_n]$,由于实数集的稠密性，总能找到一个{a_n}的上确界α使得$\alpha \in I_n, (n = 1, 2, 3, \dots)$.故原命题得证.

It is known that R has the least-upper-bound property,and thus has the greatest-lower-bound property. Let $\{a_n\}$ be set of all the lower bounds of each interval I_n,and $\{b_n\}$ be the set of all the upper bounds of each interval I_n.

It is apparent that:

$$\exists x \in R, s.t. x = sub\{a_n\}.$$
$$\forall n \in N^*, a_n < x \le b_1 \le b_2 \le \dots \le b_n$$

Therefore we find that $\forall n \in N^*, x \in I_n, x \in \cap_1^\infty I_n$.

2.40 Theorem

Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_n + 1$(n=1,2,3,...),then $\cap_1^\infty I_n$ is not empty.

Proof

我们对k维的每一维空间都作类似2.39定理的操作即可得证。

2.41 Theorem

Every k-cells is compact.

Proof

Let I be a k-cell,consisting of all points $x = (x_1, x_2., \dots x_n)$ such that $a_j < x_j < b_j(1 \le j \le k)$.Put

$$\delta = \sqrt{\sum_1^k (b_j - a_j)^2}$$

Then $|x - y| \le \delta$,if $x \in I, y \in I$.

To get a contradiction,we suppose that there exists a open cover $\{G_n\}$ of I which contains no finite subcover of I.

Put $c_j = \frac{a_j + b_j}{2}$.The intervals will be separated into 2^k k-cells of Q_i whose union is I.

According the hypothesis,there must be at least one subset cannot be covered by $\{G_n\}$,call it I_1.

Corollary:

Let $k \in N^*$,If $\{I_n\}$ is a sequence of k-cells such that $I_{n+1} \subset I_n$(n=1,2,3,...),then $\cap_1^\infty I_n$ is not empty.

Proof:

For each dimension,apply Theorem 2.39.We then obtain $x^* = (x_1^*, x_2^*, \dots, x_k^*)$.
$x^* \in I_n, n = 1, 2, 3...$,which makes the corollary follow.

2.42 Theorem(3 个等价命题)

需要指出的是，该定理中(a)与(b)的等价性被称作Heine-Borel theorem（海涅-博雷尔定理）：

If a set E in R^k has one of the following three properties,then it has the other two:

> (a) E is closed and bounded.
>
> (b) E is compact.
>
> (c) Every infinite subset of E has a limit point in E.

Proof:

(a) -> (b):

$E \subset I$ for some k-cell I.Since a closed subset of a compact set is compact(Theorem 2.35) and each k-cell is compact(Theorem 2.40,E is compact.

(b) -> (c):

If E is a infinite subset of a compact set K,then E has a limit point in K.(Theorem 2.38)

That implies (b) to (c).

(c) -> (a):

Suppose E is not bounded,then some subset of E(call it S) contains points x_n with $|x_n| > n, n \in N^*$.

Thus S clearly has no limit point,contradicting with (c).

Suppose E is not closed,consider a limit point of E but not in E.Let $S = \{x_n | x_n \in E, |x_n - x_0| < \frac{1}{n}\}$.

Thus S is infinite, and S has no limit point but x_0 in R^k.

Let's prove it:

Fix $y \in R^k, y \neq x_0$,then $|x_n - y| \geq |x_0 - y| + |x_n - x_0|$

$$\begin{aligned} &\geq |x_0 - y| - \frac{1}{n} \\ &\geq \frac{1}{2}|x_0 - y| \end{aligned}$$

Therefore x_0 is the only limit point of S.

Thus S has no limit point in E,contradicting with (c).

Hence it must be closed if (c) holds.

2.43 Weierstrass Theorem(魏尔斯特拉斯定理)

Every bounded infinite subset of R^k has a limit point in R^k.

Proof:

Let the set be S.Since it is bounded,there is some k-cell I_k s.t. $S \subset I_k$.I_k is compact,thus S has a limit point in I_k.Namely the theorem holds.

Perfect Sets and Connected Sets

2.44 Definition

A set E is **perfect** if and only if $E' = E$.

2.45 Theorem

Let P be a nonempty perfect set in R^k.Then P is uncountable.

Proof

Suppose P is countable,consider a sequence of neighborhood $\{V_n\}$ of these points $x_1, x_2, x_3, \ldots$.

Since $P' = P, V_n \cap P \neq \varnothing$.We may construct a sequence of $\{V_n\}$ following three properties below:

> 1. $\overline{V}_{n+1} \subset V_n$
> 2. $x_n \notin \overline{V}_{n+1}$
> 3. $V_{n+1} \cap P \neq \varnothing$

Put $K_n = \overline{V_n} \cap P$.Since $\overline{V}_n$ is closed and bounded,it is compact.

By (2),no points in P lies in $\cap_1^\infty K_n$.$K_n \subset P$,$\cap_1^\infty K_n = \varnothing$.

But each K_n is nonempty,by (3).

It contradicts the theorem "If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$.(n=1,2,3,...),then $\cap_1^\infty K_n$ is not empty."

2.46 Cantor Set

Note:

It is a perfect set in R^1 which contains no segment.

2.47 Definition

Two subsets A and B of a metric space X are said to be separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty, i.e.,if no point of A lies in $\overline{B}$ and no points of B lies in $\overline{A}$.

A set $E \subset X$ is said to be connected if E is not a union of two nonempty separated sets.

If a set is NOT connected,it is a union of two nonempty separated sets.

Separated sets are disjoint,but disjoint sets need not to be separated.