Section2 Basic Topology

FINITE,COUNTABLE,AND UNCOUNTABLE SETS

2.1 Definition

Consider two sets ,A and B and a mapping from A to B ,which we may call it 'f'.The 'f' should be some manner,or be said to be a ***function***(mapping).

The elements f(x) for all x in A,are called ***values***.

The set of all values is called the ***range*** of f.

2.2 Definition

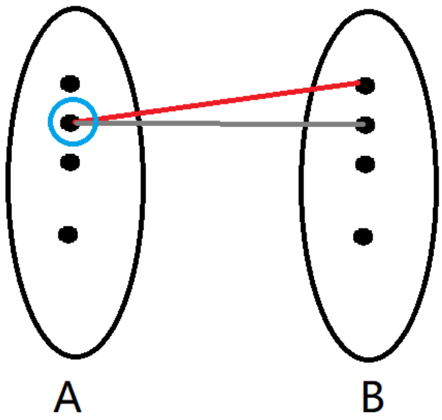
Clearly,$f(A)$ is the range of f and $f(A) \subseteq B$.

We say that f maps A ***onto*** B when $f(A) = B$.

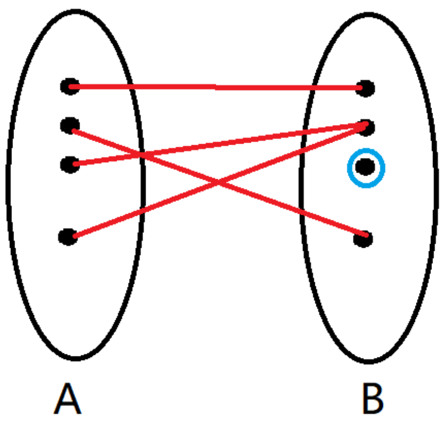
For each $y \in B$,when $f^{-1}(y)$ covers all the elements of A,f is said to be ***1-1(one to one)*** mapping of A into B.

We may illustrate the definition with these graphs below:

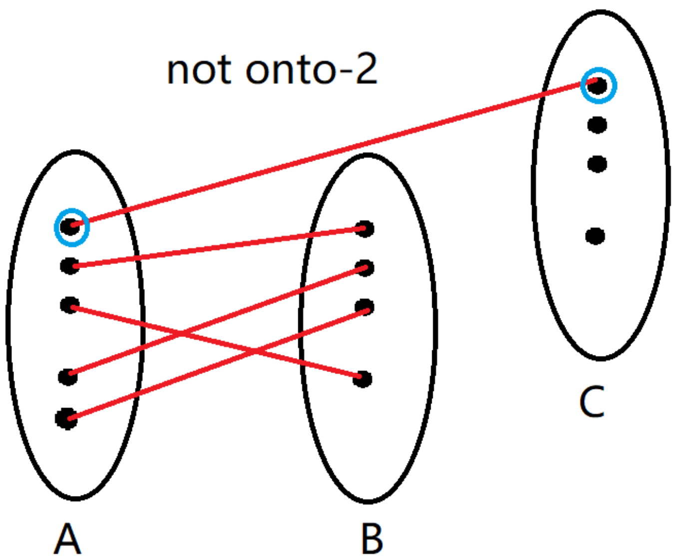
not mapping



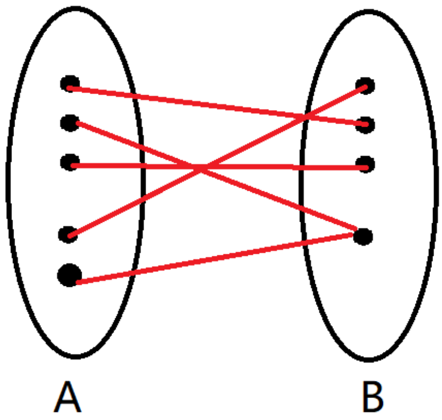
not onto-1



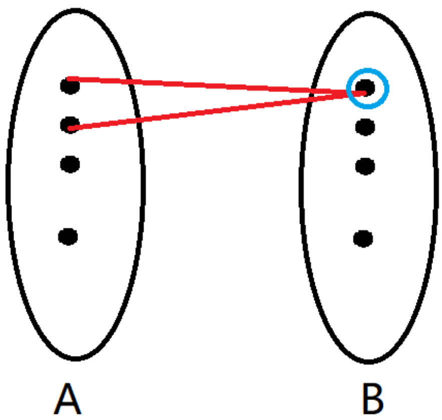
not onto-2



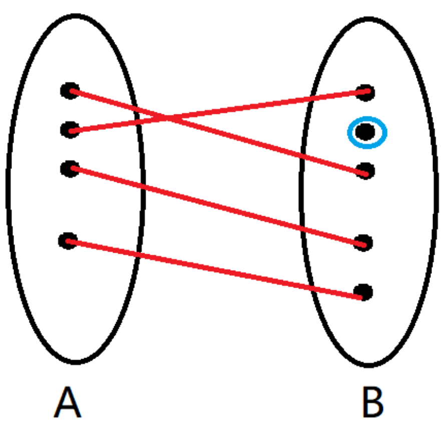
onto(but not 1-1)

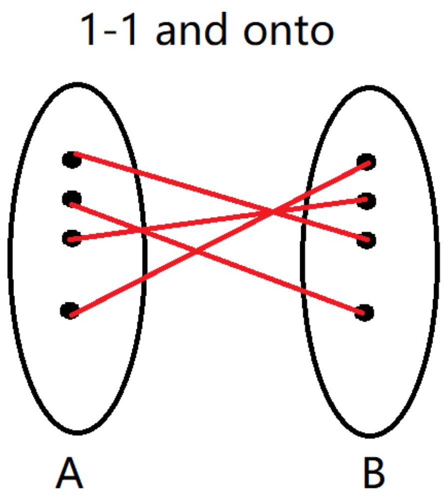


not 1-1



1-1(but not onto)





2.3 Definition

Any relation with these properties is called an equivalence relation:

> $$reflexive: A \sim A.$$
> $$symmestric: If\ A \sim B, then\ B \sim A.$$
> $$transitive: If\ A \sim B\ and\ B \sim C, then\ A \sim C.$$

If there exists a 1-1 mapping of A onto B,we say that A and B are equivalent,that is, $A \sim B$.

2.4 Remark

> if $J_m \sim J_n, m = n.$
>
> if $m = n, J_m \sim J_n.$

where J_n is the set whose elements are the integers 1,2, . . . ,n;

2.5 Definition

For any set A,we may say:

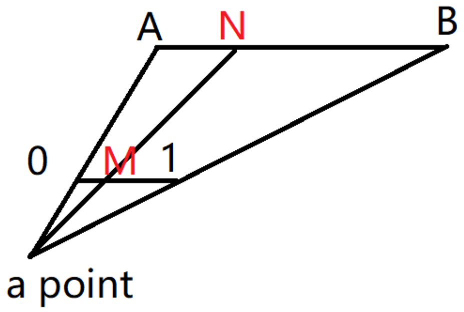
> a. A is **finite** if $A \sim J_n$ for some n.(empty set $\varnothing$ is considered to be finite)
>
> b. A is **infinite** if A is not finite.
>
> c. A is **countable** if $A \sim J.$
>
> d. A is **uncountable** if A is neither **finite** or **countable**.
>
> e. A is **at most countable**(至多可数) if A is **finite** or **countable**.

2.6 Remark

a.$N \sim Z$

> For :$0, 1, 2, 3, \ldots \to 0, -1, 1, -2, \ldots$
>
> We may even find out an explicit formula.

b. $\forall a, b \in R, (a, b) \sim (0, 1)$:

> Graphic explanations:
>
> 
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>
> $\forall x$ in (a,b),denoted by N on the axis AB,has some linear pattern corresponding to M in axis 01.

c.$(0, 1) \sim R$

> We may even find the explicit function of it:
>
> $$f(x) := \begin{cases} \dfrac{x - \frac{1}{2}}{x^2}, & 0 < x \le \dfrac{1}{2} \\[2ex] \dfrac{x - \frac{1}{2}}{1 - x^2}, & \dfrac{1}{2} < x < 1 \end{cases}$$

> d. $\forall a,b \in R, (a,b) \sim R$:
>
> > An equivalence relation is transitive,so it is clear.
> >
> > In fact,we could replace the definition of infinite set 2.5.b by the statement below:
> >
> > **A is infinite if A is equivalent to one of its proper subsets.**

2.7 Definition

We may sort of order or label the elements in a countable set A in the way of 'sequence in A':

$A = \{x_n\}, n \in J.$

The elements in the set arranged in this way need not to be distinct.(For example,we may assume:$f(x) := x^2, x_1 = (-1)^2 = 1, x_2 = (+1)^2 = 1, x_1 = x_2$,which is valid)

2.8 Theorem

Every infinite subset of a countable set A is countable.

No uncountable set can be a subset of a countable set.

Proof:

The key operation is to find out the 'net mapping' of J_n,the subset of J:

k	n_k	x_{n_k}
1	n_1	x_{n_1}
2	n_2	x_{n_2}
...	...	...
n	n_n	n_{n_k}

In this way we found a mapping function from J_n to A such that n_k is the smallest integer greater than $n_{k-1}$and $x_{n_k} \in E$,where $f(k) = x_{n_k}$.

####

2.9 Theorem

Let $\{E_n\}, n = 1, 2, 3, \ldots$,be a sequence of countable sets,and put

$$S = \cup_{n=1}^{\infty} E_n$$

Then S is countable.

2.10 Remark

Let $A = (0,1)$ and $E_\alpha = (0, \alpha)$,then

> $$(1) \cup_{\alpha \in A} E_\alpha = (0,1)$$
> $$(2) \cap_{\alpha \in A} E_\alpha = \varnothing.$$

(2) is clear,for:

$$\forall y > x > 0, y \notin E_x.$$

Hence $\forall y > 0, y \notin \cup_{x\in A} E_x$.

To prove (1),we let $M = \cup_{\alpha\in A} E_\alpha . \forall x \in M, x \in (0,1), M \subseteq (0,1)$.

Suppose $\forall \beta \in (0,1)$, $\forall \alpha \in (0,1), \beta \notin E_\alpha$,then $\beta >= 1$.However $\beta < 1$.

Hence,$\forall\ \beta \in (0,1), \exists \alpha \in (0,1), s.t. \beta \in E_\alpha$,whereas $(0,1) \subseteq M$.

Since $(0,1) \subseteq M, M \subseteq (0,1), M = \cup_{\alpha\in A} E_\alpha = (0,1)$.

2.11 Facts

A、B are two sets.

> (1)If A has a 1-1 mapping f into B,$\exists\ g : B \to A, s.t. (gof) = idA.$
>
> (2)If f maps A onto B,$\exists\ g : B \to A, s.t. (fog) = idB.$

2.12 Theorem

Let $\{E_n\}, n = 1, 2, 3, \ldots$,be a sequence of countable sets,and put $S = \cup_{n=1}^{\infty} E_n$.Then S is countable.

Proof:

Rearrange the order of the terms of the sets T in a sequence $\{x_{nk}\}$:

$x_{11} \to x_{21} \to x_{12} \to x_{31} \to x_{22} \to x_{13} \to \ldots$

It is valid,because for each element on the specific diagonal

,the sum of the indexes ranges from (2) to n+k with step 1.

However,some entries may appear more than once,so what we can assure is that it goes that

$$S \sim T \subseteq N$$

Hence S and T are **at most countable**. Since S is infinite,S is countable.

Corollary

Suppose A is at most countable,and, for every $\alpha \in A, B_\alpha$is at most countable. Put

$$T = \cup_{\alpha\in A} B_\alpha$$

Then T is at most countable.

For T is the subset of the at most countable set $S = \cup_{n=1}^{\infty} E_n$.

2.13 Theorem

Let A be a countable set,and let B_n be the set of all n-tuples $(\alpha_1, \ldots, \alpha_n)$,where $\alpha_k \in A (k = 1, \ldots, n)$,and the elements $\alpha_1, \ldots, \alpha_n$ need not to be distinct. Then B_n is countable.

Proof:

$B_n = \underbrace{A\times \ldots \times A}_{n}$.By induction and Theorem 2.12,we know that $A \times A$ is countable,thus $A \times A \times A$ is countable,...,B_n is countable.

Corollary

The set of all **rational numbers** is countable.

That is because $\mathbb{Q} \subset \mathbb{N} \times \mathbb{N}$,$\mathbb{N} \times \mathbb{N}$ is countable and $\mathbb{Q}$ is at most countable but infinite,and thus countable.

2.14 Theorem

Let A be the set of all sequences whose elements are the digits 0 and 1.This set A is uncountable.

Outline of the proof:

To prove something wrong is usually hard,so we may try to prove it by contradictions.

We may assign E as a sequence of countless sequences:

$$E = \{s_1, s_2, s_3, \ldots\}$$

,where $s_k, k = 1, 2, \ldots$ is a sequence.

If a set is countable,it can be rearranged to a sequence like that:

$$\begin{aligned} E = \{&\{a_{11}, \ldots\}, \\ &\{a_{21}, a_{22}, \ldots\}, \\ &\{a_{31}, a_{32}, a_{33}, \ldots\}, \ldots\} \end{aligned}$$

Let's DIY a sequence to create the contradiction in the way below:

Consider a sequence $\{b_k\}$ like that:

$$b_k = \begin{cases} 1, a_{kk} = 0 \\ 0, a_{kk} = 1 \end{cases}$$

The sequence $b_k \neq E_n, \forall n \in N$.However $b_k \in E$.

Thus A is uncountable.

Corollary

The set of all real numbers is uncountable.