METRIC SPACES

2.15 Definition

A set X is said to be a metric space if two points a and b in X associate a real number function d(p,q) such that

> $$\begin{aligned} &Non-negativity: && d(p,q)>0\, if\; p\neq q, d(p,p)=0;\\ &Symmetry: && d(p,q)=d(q,p);\\ &Triangle\ Inequality: && d(p,q)\leq d(p,r)+d(r,q), for\ any\ r\in X.\end{aligned}$$

Any function with these three properties is called a distance function,or a metric.

2.16 Definition

The **unit ball** is a set defined by

$$\{\mathrm{x}\in R^n|d(\mathrm{x},0)\leq 1\}$$

2.17 Definition

Let X be a metric space,all points and sets mentioned below are elements and subsets of X.

> 1. A **neighborhood** of p is a set $N_r(p)$ consisting of all q such that $d(p,q)<r$,for some $r>0$.The number ***r*** is called the radius of $N_r(p)$.(random r)
> 2. A point p is a ***limit point*** of the set E if *every* neighborhood of p contains a point $q\neq p$ such that $q\in E$.(no one but myself).$i.e. \forall r>0, N_r(p)\cap E\neq\varnothing$
> 3. If $p\in E$ and p is not a limit point of E,then p is called an ***isolated point*** of E.
> 4. A point is an ***interior point*** of E if there is a neighborhood N of p such that $N\subset E$. $i.e. \exists r>0, N_r(p)\subseteq E$.
> 5. E is closed if **every limit point** of E is a point of E.
> 6. E is open if every point of E is a **interior point**.$\forall p\in E, \exists r>0, s.t. N_r(p)\subseteq E$.
> 7. The **complement** of E,denoted by E^c,is the set of all points $p\in X$ such that $p\notin E$.
> 8. E is ***perfect*** if E is closed and if every point of E is a limit point of E.
> 9. E is ***bounded*** if there is a real number M and a point $q\in X$ such that $d(p,q)<M$ for all $p\in E$.
> 10. E is ***dense*** in X if every point of X is a limit point of E,or a point of E(or both).

Notice that an interior point is not necessarily a limit point.

2.18 Facts

Let X be a metric space,all points and sets mentioned below are elements and subsets of X.

> 1. If E is open,E^c is closed;
> 2. If E is closed,E^c is open.

Proof:

1.We need to show that if p is a limit point of E^c then $p\in E^c$.

Let p be a limit point,$i.e. \forall r > 0, N_r(p) \cap E^c \neq \varnothing.$

$\therefore \forall p \in E, \forall r > 0, N_r(p)$ dosen't $\subseteq E.$

Therefore p is not an interior point of E. However,every point in E is an interior point. $i.e. p \notin E, p \in E^c.$

2.The statement is equivalent to **"If E is not open,then E^c is not closed"**,which is trivial.

Since E is not open,that is $\exists p \in E, \forall r > 0, s.t. N_r(p) \cap E^C \neq \varnothing.$

Therefore E^c is not closed.

2.19 Facts

Let A_n, B_n be a open set and a closed set which are both subsets of a metric space X. $(n = 1, 2, 3...)$

> 1. $\cup_{i=1}^{n} A_n$ is open. $\cup_{i=1}^{n} B_n$is closed.$\cap_{i=1}^{n} A_n$is open,$\cap_{i=1}^{n} B_n$is closed.
> 2. $\cup_{i=1}^{\infty} A_n$ is open.$\cup_{i=1}^{\infty} B_n$ is closed.
> 3. $\cap_{i=1}^{\infty} A_n$ may NOT be open.$\cup_{i=1}^{\infty} B_n$ may NOT be closed.

2.20 Theorem

If p is a limit point of a set E,then every neighbourhood of p contains **infinitely** many points of E.

Corollary

A finite point set has no limit points.

Hence,a finite set is closed.

2.22 Theorem

Let $\{E_n\}$ be a (finite or infinite)collection of sets E_α.Then $(\cup_\alpha E_\alpha)^c = \cap_\alpha (E_\alpha^c)$

2.23 Definition

If X is a metric space,if $E \subset X$,and if E' denotes the set of all limit points of E,then the *closure* of E is the set $\overline{E} = E \cup E'$.

2.24 Theorem

If X is a metric space and $E \subset X$,then

> (a) $\overline{E}$ is closed,
>
> (b) $\overline{E} = E$ if and only if E is closed,
>
> (c) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

2.25 Definition

If E is a metric space and E is a closed set,E is perfect if and only if $E' = E$.