Two Stage Treatment Regime Draft

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1 Define a constrained optimal dynamic treatment regime

Set-up

We focus on the case of a two-stage treatment regime with two competing outcomes and binary treatment options. The observed data are denoted by $\{(\boldsymbol{X}_{1,i}, A_{1,i}, \boldsymbol{X}_{2,i}, A_{2,i} Y_i, Z_i)\}_{i=1}^n$, which comprises n identically, independently distributed patient trajectories $\{(\boldsymbol{X}_1, A_1, \boldsymbol{X}_2, A_2, Y, Z)\}$. Capital letters denote random variables; lower case letters denote realized values of these random variables. Let $\boldsymbol{X}_1$ be a patient baseline covariate, A_1 be the first-stage treatment variable, $\boldsymbol{X}_2$ be the patient covariate collected between frist decision point and second decision point, A_2 be the second-stage treatment variable. The primary and secondary competing outcomes are denoted by Y and Z, respectively. The outcome Y is coded so that higher values are better, and the outcome Z is coded so that the lower values are better. A dynamic treatment regime, $\boldsymbol{d} = (d_1, d_2)$, is a pair of decision rules. For each $t = 1, 2$, let $\boldsymbol{H}_t$ denote the patient history information up to the decision point t, i.e., $\boldsymbol{H}_1^\intercal = (1, \boldsymbol{X}_1^\intercal)$ and $\boldsymbol{H}_2^\intercal = (1, \boldsymbol{X}_1^\intercal, A_1, \boldsymbol{X}_2^\intercal)$. For each $t = 1, 2$, $d_t : \mathrm{dom}(\boldsymbol{H}_t) \to \mathrm{dom}(A_t)$, is a function that maps from the space of $\boldsymbol{H}_t$ to the space of feasible treatment A_t at the time point t. In summary, the observed data trajectory is denoted

$$\boldsymbol{W} = \{(\boldsymbol{X}_1, A_1, \boldsymbol{X}_2, A_2, Y, Z)\},$$

$$\boldsymbol{H}_1^\intercal = (1, \boldsymbol{X}_1^\intercal)$$

$$\boldsymbol{H}_2^\intercal = (1, \boldsymbol{X}_1^\intercal, A_1, \boldsymbol{X}_2^\intercal) = (\boldsymbol{H}_1^\intercal, A_1, \boldsymbol{X}_2^\intercal)$$

To construct an estimand of interest and make inference from observed data, we follow the potential outcomes or counter-factual framework, which quantifies treatment effects of dynamic treatment regimes, proposed by Neyman, Rubin and Robins. For an arbitrary treatment sequence (a_1, a_2), the set of potential outcomes of the patient is

$$W^*(a_1, a_2) = \left\{ \boldsymbol{X}_2^*(a_1), Y^*(a_1, a_2), Z^*(a_1, a_2) : (a_1, a_2) \in \{-1, 1\}^2 \right\}$$

$$\boldsymbol{H}_2^*(a_1) = \{1, \boldsymbol{X}_1^\intercal, a_1, \boldsymbol{X}_2^{*\intercal}(a_1)\}^\intercal$$

Furthermore, the potential outcomes of the patient following a treatment regime $\boldsymbol{d}$ are denoted as $Y^*(\boldsymbol{d}) = Y^*(d_1, d_2)$ and $Z^*(\boldsymbol{d}) = Z^*(d_1, d_2)$.

Define a constrained optimal regime

Our goal is to find a optimal constrained regime, $\boldsymbol{d}_\kappa^0$, which, at the population level, maximizes the expectation of the primary potential outcome $\mathbb{E}Y^*(\boldsymbol{d})$, subject to an upper bound on the expectation of the secondary potential outcome $\mathbb{E}Z^*(\boldsymbol{d})$. Therefore, the two-stage constrained optimal regime problem is defined with respect to potential outcomes as

$$\underset{\boldsymbol{d}\in\mathcal{D}}{\text{maximize}} \quad \mathbb{E}Y^*(\boldsymbol{d})$$

$$\text{subject to} \quad \mathbb{E}Z^*(\boldsymbol{d}) \leq \kappa,$$

where $\boldsymbol{d} = (d_1, d_2)$ is the two stage decision rule and $\mathcal{D}$ is the class of two-stage regimes under consideration. As we are restricted to the class of the linear decision rule, the class of decision rules, indexed by $\boldsymbol{\tau} = (\tau_1, \tau_2)$, is $\mathcal{D} = \{\boldsymbol{d} = (d_1, d_2)\}$ where $d_1(\boldsymbol{h}_1; \boldsymbol{\tau}_1) = \text{sgn}(\boldsymbol{h}_1^\intercal \boldsymbol{\tau}_1) = \text{sgn}\left\{r_1(\boldsymbol{h}_1; \boldsymbol{\tau}_1)\right\}$, and $d_2(\boldsymbol{h}_2; \boldsymbol{\tau}_2) = \text{sgn}(\boldsymbol{h}_2^\intercal \boldsymbol{\tau}_2) = \text{sgn}\{r_2(\boldsymbol{h}_2; \boldsymbol{\tau}_2)\}$. Let $r_1(\boldsymbol{h}_1; \boldsymbol{\tau}_1) = \boldsymbol{h}_1^\intercal \boldsymbol{\tau}_1$ and $r_2(\boldsymbol{h}_2; \boldsymbol{\tau}_2) = \boldsymbol{h}_2^\intercal \boldsymbol{\tau}_2$. $\boldsymbol{\tau}^{0\intercal} = (\boldsymbol{\tau}_1^{0\intercal}, \boldsymbol{\tau}_2^{0\intercal})$ are the indexing parameters for the constrained optimal dynamic treatment regime defined. Hence, both $\mathbb{E}Y^*(\boldsymbol{d})$ and $\mathbb{E}Z^*(\boldsymbol{d})$ can be considered as functions of $\boldsymbol{\tau}$. As we will be solving them over $\boldsymbol{\tau} = (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$, we use the notations of $\mathbb{E}Y^*(\boldsymbol{\tau})$ and $\mathbb{E}Z^*(\boldsymbol{\tau})$ from now on. As only the directions of

$\boldsymbol{h}_t^{\mathsf{T}}\boldsymbol{\tau}_t$, $t = 1, 2$, matters, we restrict $\boldsymbol{\tau}_t$ to be unit vectors, i.e., $\boldsymbol{\tau}_t^{\mathsf{T}}\boldsymbol{\tau}_t = 1$. Then, the problem above can be written as

$$\begin{aligned}
\underset{\boldsymbol{\tau}}{\text{maximize}} \quad & \mathbb{E}Y^*(\boldsymbol{\tau}) \\
\text{subject to} \quad & \kappa - \mathbb{E}Z^*(\boldsymbol{\tau}) \le 0, \\
& \boldsymbol{\tau}_1^{\mathsf{T}}\boldsymbol{\tau}_1^{\mathsf{T}} - 1 = 0, \text{ and } \boldsymbol{\tau}_2^{\mathsf{T}}\boldsymbol{\tau}_2^{\mathsf{T}} - 1 = 0.
\end{aligned} \tag{1}$$

A constrained optimal dynamic treatment regime is defined to be a solution to the problem above, and is denoted by $\boldsymbol{d}_\kappa^0 = (d_{\kappa,1}^0, d_{\kappa,2}^0)$, where $d_{\kappa,1}^0(\boldsymbol{h}_1) = \text{sgn}(\boldsymbol{h}_1^{\mathsf{T}}\boldsymbol{\tau}_{1,\kappa}^0) = \text{sgn}\left\{r_1(\boldsymbol{h}_1; \boldsymbol{\tau}_{1,\kappa}^0)\right\}$ and $d_{\kappa,2}^0(\boldsymbol{h}_2) = \text{sgn}(\boldsymbol{h}_2^{\mathsf{T}}\boldsymbol{\tau}_{2,\kappa}^0) = \text{sgn}\left\{r_2(\boldsymbol{h}_2; \boldsymbol{\tau}_{2,\kappa}^0)\right\}$.

2 Re-formalize the problem using barrier penalty

Reformalize the problem using penalty barrier

Barrier penalty function is used to re-formalize Problem (1), with additional restrictions on both the euclidean norms of $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ to be 1 for identification, for a decreasing sequence $\{\mu\}$, where $\mu \to 0$,

$$\max_{\boldsymbol{\tau}} \mathbb{E}Y^*(\boldsymbol{\tau}) + \mu \log\left\{\kappa - \mathbb{E}Z^*(\boldsymbol{\tau})\right\} - \frac{1}{2\mu}\left\{(\boldsymbol{\tau}_1^{\mathsf{T}}\boldsymbol{\tau}_1 - 1)^2 + (\boldsymbol{\tau}_2^{\mathsf{T}}\boldsymbol{\tau}_2 - 1)^2\right\} \tag{2}$$

Let $\boldsymbol{\tau}_\kappa^{*\mathsf{T}}(\mu) = \left\{\boldsymbol{\tau}_{\kappa,1}^{*\mathsf{T}}(\mu), \boldsymbol{\tau}_{\kappa,2}^{*\mathsf{T}}(\mu)\right\}$ denote a solution to Problem (2), and $\boldsymbol{\tau}_{\kappa,\mu}^{*\mathsf{T}} = \left\{\boldsymbol{\tau}_{\kappa,\mu,1}^{*\mathsf{T}}, \boldsymbol{\tau}_{\kappa,\mu,2}^{*\mathsf{T}}\right\}$ for short. The corresponding regime is denoted by $\boldsymbol{d}_\kappa^*(\mu) = \left\{d_{1,\kappa}^*(\mu), d_{2,\kappa}^*(\mu)\right\}$, and $\boldsymbol{d}_{\kappa,\mu}^* = \left\{d_{\kappa,\mu,1}^*, d_{\kappa,\mu,2}^*\right\}$ for short. Then, $d_{\kappa,\mu,1}^*(\boldsymbol{h}_1) = \text{sgn}(\boldsymbol{h}_1^{\mathsf{T}}\boldsymbol{\tau}_{\kappa,\mu,1}^*)$, and $d_{\kappa,\mu,2}^*(\boldsymbol{h}_2) = \text{sgn}(\boldsymbol{h}_2^{\mathsf{T}}\boldsymbol{\tau}_{\kappa,\mu,2}^*)$. For short notation, we let $S^*(\boldsymbol{\tau}, \mu) = \mathbb{E}Y^*(\boldsymbol{\tau}) + \mu \log\left\{\kappa - \mathbb{E}Z^*(\boldsymbol{\tau})\right\} - \frac{1}{2\mu}\left\{(\boldsymbol{\tau}_1^{\mathsf{T}}\boldsymbol{\tau}_1 - 1)^2 + (\boldsymbol{\tau}_2^{\mathsf{T}}\boldsymbol{\tau}_2 - 1)^2\right\}$.

Convergence of the Penalty-Barrier Trajectory $\boldsymbol{\tau}_\kappa^*(\mu)$ to $\boldsymbol{\tau}_\kappa^0$

We again examine the conditions under which the penalty-barrier trajectory $\boldsymbol{\tau}_\kappa^*(\mu)$ converges to the original constrained maximizer $\boldsymbol{\tau}_\kappa^0$, of which details are provided Appendix 1 Proof Draft 1. We specify the conditions needed for our problem as follows. For notation simplicity in this section, we let $f(\boldsymbol{\tau}) = \mathbb{E}Y^*(\boldsymbol{\tau})$, $c_1(\boldsymbol{\tau}) = \kappa - \mathbb{E}Z^*(\boldsymbol{\tau})$, $c_2(\boldsymbol{\tau}) = \boldsymbol{\tau}_1^{\mathsf{T}}\boldsymbol{\tau}_1 - 1$, and $c_3(\boldsymbol{\tau}) = \boldsymbol{\tau}_2^{\mathsf{T}}\boldsymbol{\tau}_2 - 1$. Let $g(\boldsymbol{\tau})$ denote the

gradient of $f(\boldsymbol{\tau})$, i.e., $g(\boldsymbol{\tau}) = \nabla f(\boldsymbol{\tau}) = \nabla \mathbb{E} Y^*(\boldsymbol{\tau})$. Also, let $\boldsymbol{c}(\boldsymbol{\tau})$ be the vector of constraint functions $\{c_i(\boldsymbol{\tau})\}$, $i = 1, 2, 3$. The Jacobian matrix $\boldsymbol{c}'(\boldsymbol{\tau})$ of first derivative of $\boldsymbol{c}(\boldsymbol{\tau})$ has row $\{\nabla c_i(\boldsymbol{\tau})\}^{\mathsf{T}}$, and we use $J(\boldsymbol{\tau})$ to denote this Jacobian for concise.

Lemma 1 (Conditions for the trajectory $\{\boldsymbol{\tau}_\kappa^*(\mu)\}$ converging to $\boldsymbol{\tau}_\kappa^0$[6, 1, 3]). *Assume*

1. *the functions f, c_1, c_2, and c_3 are twice differentiable with respect to $\boldsymbol{\tau}$;*

2. *the gradients ∇c_1, ∇c_2, and ∇c_3 are linearly independent, where the gradients are taken with respect to $\boldsymbol{\tau}$;*

3. *strict complementarity holds for $\lambda_1^0 c_1(\boldsymbol{\tau}_\kappa^0) = 0$, where λ_1^0 is the Lagrangian multiplier of the inequality constraint c_1;*

4. *the sufficient conditions under which $\boldsymbol{\tau}_\kappa^0$ be an isolated local constrained minimum of Problem (2) are satisfied by $(\boldsymbol{\tau}_\kappa^0, \boldsymbol{\lambda}^0)$, where $\boldsymbol{\lambda}^0 = (\lambda_1^0, \lambda_2^0, \lambda_3^0)^{\mathsf{T}}$ of which λ_2^0 is the Lagrangian multipliers for the constraint c_2, and λ_3 for c_3. The sufficient conditions for optimality are*

   (a) *$\boldsymbol{\tau}_\kappa^0$ is feasible and the LICQ (Linear Independence Constraint Qualification) holds at $\boldsymbol{\tau}_\kappa^0$, i.e., the Jacobian matrix of active constraints at $\boldsymbol{\tau}_\kappa^0$, $J_{\mathcal{A}}(\boldsymbol{\tau}_\kappa^0)$, has full row rank;*

   (b) *$\boldsymbol{\tau}_\kappa^0$ is a KKT point and strict complementarity holds, i.e, the (necessarily unique) multiplier $\boldsymbol{\lambda}^0$ has the property that $\lambda_i^0 > 0$, for all $i \in \mathcal{A}_{\mathcal{I}}(\boldsymbol{\tau}_\kappa^0)$, the set of indices of active inequality constraints at $\boldsymbol{\tau}_\kappa^0$;*

   (c) *for all nonzero vectors $\boldsymbol{p}$ satisfying $J_{\mathcal{A}}(\boldsymbol{\tau}_\kappa^0)\boldsymbol{p} = 0$, there exists $\omega > 0$ such that $\boldsymbol{p}^{\mathsf{T}} H(\boldsymbol{\tau}_\kappa^0, \boldsymbol{\lambda}^0)\boldsymbol{p} \geq \omega\|\boldsymbol{p}\|^2$., where $H(\boldsymbol{\tau}_\kappa^0, \boldsymbol{\lambda}^0)$ is the hessian of the Lagrangian at $\boldsymbol{\tau}_\kappa^0$ and $\boldsymbol{\lambda}^0$, where $\boldsymbol{\lambda}^0$ is the vector of the Lagrangian multipliers, $\boldsymbol{\lambda}^0 = (\lambda_1^0, \lambda_2^0, \lambda_3^0)^{\mathsf{T}}$.*

   then there is a positive neighborhood about $\mu = 0$ for which a unique-isolated differentiable function $\boldsymbol{\tau}_\kappa^*(\mu)$ exists. It describes a unique isolated trajectory of local maxima of $S^*(\boldsymbol{\tau}, \mu)$, where $\boldsymbol{\tau}_\kappa^*(\mu) \to \boldsymbol{\tau}_\kappa^0$ as $\mu \to 0$.

To find $\boldsymbol{\tau}^*_\kappa(\mu)$, we need to examine its stationarity. That is $\nabla S^*(\boldsymbol{\tau}, \mu) = 0$ is satisfied at $\boldsymbol{\tau}^*_\kappa(\mu)$. Its equivalent system of non-linear equations is

$$F^\mu(\boldsymbol{\tau}, \boldsymbol{\lambda}) = \begin{pmatrix} g(\boldsymbol{\tau}) - J(\boldsymbol{\tau})\boldsymbol{\lambda} \\ c_1(\boldsymbol{\tau})\lambda_1 - \mu \\ c_2(\boldsymbol{\tau}) + \mu\lambda_2 \\ c_3(\boldsymbol{\tau}) + \mu\lambda_3 \end{pmatrix} = 0$$

We also define $\chi_1 \triangleq \mu/c_1(\boldsymbol{\tau})$, $\chi_2 \triangleq -c_2(\boldsymbol{\tau})/\mu$ and $\chi_3 \triangleq -c_3(\boldsymbol{\tau})/\mu$ which can be considered as approximates of the Lagrangian multipliers under μ-perturbed KKT conditions.

3   Estimation of the barrier trajectory

Modeling for and estimation of the distributions of potential outcomes

Under the conditions of Lemma 1, we can solve Problem (2) under a sequence of μ converging to zero, and the solution $\lim_{\mu \to 0} \boldsymbol{\tau}^*_\kappa(\mu) = \boldsymbol{\tau}^0_\kappa$. However, as the distribution of potential outcomes are unknown, the two major quantities under an arbitrary regime $\boldsymbol{d}$ involved, $\mathbb{E}Y^*(\boldsymbol{d})$ and $\mathbb{E}Z^*(\boldsymbol{d})$, need to be estimated from the observed data. Our strategy for estimating these two quantities is to model the marginal distribution of each potential outcome, and then draw Monte-Carlo random samples from the estimated marginal distributions to calculate their expectations numerically.

To connect observed data with potential outcomes, three necessary causal inference assumptions are needed.

1. Consistency: $Y = Y^*(A_1, A_2)$.

2. Sequential ignorability: $A_t \perp\!\!\!\perp W^* | \boldsymbol{H}_t$ for $t = 1, 2$.

3. Positivity: $\exists \epsilon > 0$ for which $\epsilon < \Pr(A_t = a_t | \boldsymbol{H}_t) < 1 - \epsilon$ with probabilty one for all a_t, $t = 1, 2$.

Following the G-computation formula by Robins etc. [4], we have, for any arbitrary regime $\boldsymbol{d} = (d_1, d_2)$, that

$$\Pr\{Y^*(\boldsymbol{d}) \leq y\} = \mathbb{E}_{\boldsymbol{X}_1}\left\{\mathbb{E}_{\boldsymbol{X}_2}\left[\Pr\left\{Y \leq y \mid \boldsymbol{X}_2, A_2 = d_2(\boldsymbol{X}_2), \boldsymbol{X}_1, A_1 = d_1(\boldsymbol{X}_1)\right\} \mid \boldsymbol{X}_1, A_1 = d_1(\boldsymbol{X}_1)\right]\right\}$$

and similarly,

$$\Pr\{Z^*(\boldsymbol{d}) \leq z\} = \mathbb{E}_{\boldsymbol{X}_1}\left\{\mathbb{E}_{\boldsymbol{X}_2}\left[\Pr\left\{Z \leq z \mid \boldsymbol{X}_2, A_2 = d_2(\boldsymbol{X}_2), \boldsymbol{X}_1, A_1 = d_1(\boldsymbol{X}_1)\right\} \mid \boldsymbol{X}_1, A_1 = d_1(\boldsymbol{X}_1)\right]\right\}$$

Hence, we can estimate the probabilty function of the potential outcomes under a regime $\boldsymbol{d}$, $\Pr\{Y^*(\boldsymbol{d}) \leq y\}$ and $\Pr\{Z^*(\boldsymbol{d}) \leq z\}$, using observed data by modeling and estimating the conditional distributions involved, and hence, $\mathbb{E}Y^*(\boldsymbol{d})$ and $\mathbb{E}Z^*(\boldsymbol{d})$.

For now, we follow the modeling tactic in Interactive Q-learning for Quantiles by Linn et al. We assume the following model,

$$Y = \mathbb{E}(Y|\boldsymbol{H}_2, A_2) + \varepsilon_Y,$$

$$\mathbb{E}(Y|\boldsymbol{H}_2, A_2) = m_Y(\boldsymbol{H}_2) + A_2 c_Y(\boldsymbol{H}_2),$$

$$\text{where } \mathbb{E}(\varepsilon_Y) = 0, \text{Var}(\varepsilon_Y) = \sigma^2, \text{and } \varepsilon_Y \perp\!\!\!\perp (\boldsymbol{H}_2, A_2).$$

Define $F_{\varepsilon_Y}(\cdot)$ to be the distribution of ε_Y; $F_{\boldsymbol{H}_2|\boldsymbol{H}_1, A_1}(\cdot \mid \boldsymbol{h}_1, a_1)$ to be the conditional distribution of $\boldsymbol{H}_2$ given $\boldsymbol{H}_1 = \boldsymbol{h}_1$ and $A_1 = a_1$; $F_{\boldsymbol{H}_1}(\cdot)$ to be the distribution of $\boldsymbol{H}_1$. Again, we have $\boldsymbol{H}_1^\mathsf{T} = (1, \boldsymbol{X}_1^\mathsf{T})$, $d_1(\boldsymbol{H}_1)$, $\boldsymbol{H}_2 = \{\boldsymbol{H}_1^\mathsf{T}, d_1(\boldsymbol{H}_1), \boldsymbol{X}_2^\mathsf{T}\}^\mathsf{T}$.

$$\Pr^{\boldsymbol{d}} \{Y \leq y \mid \boldsymbol{H}_2 = \boldsymbol{h}_2, d_2(\boldsymbol{H}_2) = d_2(\boldsymbol{h}_2)\}$$

$$= \Pr^{\boldsymbol{d}} \{m(\boldsymbol{H}_2) + d_2(\boldsymbol{H}_2)c_Y(\boldsymbol{H}_2) + \varepsilon_Y \leq y \mid \boldsymbol{H}_2 = \boldsymbol{h}_2, d_2(\boldsymbol{H}_2) = d_2(\boldsymbol{h}_2)\}$$

$$= \Pr^{\boldsymbol{d}} \{\varepsilon_Y \leq y - m(\boldsymbol{H}_2) - d_2(\boldsymbol{H}_2)c_Y(\boldsymbol{H}_2) \mid \boldsymbol{H}_2 = \boldsymbol{h}_2, d_2(\boldsymbol{H}_2) = d_2(\boldsymbol{h}_2)\}$$

$$= F_{\varepsilon_Y} \{y - m(\boldsymbol{h}_2) - d_2(\boldsymbol{h}_2)c_Y(\boldsymbol{h}_2)\}$$

$$= F_{\varepsilon_Y} [y - m(\boldsymbol{h}_2) - \text{sgn} \{r_2(\boldsymbol{h}_2; \boldsymbol{\tau}_2)\} c_Y(\boldsymbol{h}_2)]$$

Hence, we have

$$\Pr \{Y^*(\boldsymbol{d}) \leq y\}$$

$$= \iint \Pr^{\boldsymbol{d}} \{Y \leq y \mid \boldsymbol{H}_2 = \boldsymbol{h}_2, d_2(\boldsymbol{H}_2) = d_2(\boldsymbol{h}_2)\} \, dF_{\boldsymbol{H}_2 | \boldsymbol{H}_1, A_1} \{\boldsymbol{h}_2 | d_1(\boldsymbol{h}_1), \boldsymbol{h}_1\} \, dF_{\boldsymbol{H}_1}(\boldsymbol{h}_1)$$

$$= \iint F_{\varepsilon_Y} \{y - m(\boldsymbol{h}_2) - d_2(\boldsymbol{h}_2)c_Y(\boldsymbol{h}_2)\} \, dF_{\boldsymbol{H}_2 | \boldsymbol{H}_1, A_1} \{\boldsymbol{h}_2 \mid \boldsymbol{h}_1, d_1(\boldsymbol{h}_1)\} \, dF_{\boldsymbol{H}_1}(\boldsymbol{h}_1)$$

$$= \iint F_{\varepsilon_Y} [y - m(\boldsymbol{h}_2) - \text{sgn} \{r_2(\boldsymbol{h}_2; \boldsymbol{\tau}_2)\} c_Y(\boldsymbol{h}_2)] \, dG_Y \{m_Y, c_Y, r_2 | \boldsymbol{h}_1, d_1(\boldsymbol{h}_1)\} \, dF_{\boldsymbol{H}_1}(\boldsymbol{h}_1)$$

$$= \iint F_{\varepsilon_Y} [y - m(\boldsymbol{h}_2) - \text{sgn}(r_2)c_Y(\boldsymbol{h}_2)] \, dG_Y \{m_Y, c_Y, r_2 | \boldsymbol{h}_1, d_1(\boldsymbol{h}_1)\} \, dF_{\boldsymbol{H}_1}(\boldsymbol{h}_1)$$

where $G_Y \{m_Y, c_Y, r_2 \mid \boldsymbol{h}_1, a_1\}$ is the joint conditional distribution of $m_Y(\boldsymbol{H}_2)$, $c_Y(\boldsymbol{H}_2)$ and $r_2(\boldsymbol{H}_2; \boldsymbol{\tau}_2)$ given $\boldsymbol{H}_1 = \boldsymbol{h}_1$ and $A_1 = a_1$. The second equality is due to $\int z(x, y) \, dF_{X|Y}(x|y) = \mathbb{E}(z|y) = \int z \, dF_{Z|Y}(z|y)$.

w Same applies to Z:

$$Z = \mathbb{E}(Z|\boldsymbol{H}_2, A_2) + \epsilon,$$

$$\text{where } \mathbb{E}(\epsilon) = 0, \text{Var}(\epsilon) = \sigma^2, \text{and } \epsilon \perp\!\!\!\perp (\boldsymbol{H}_2, A_2)$$

$$\mathbb{E}(Z|\boldsymbol{H}_2, A_2) = m_Z(\boldsymbol{H}_2) + A_2 c_Z(\boldsymbol{H}_2)$$

$$\Pr\left\{Z^*(\boldsymbol{d}) \le z\right\}$$

$$= \iint \Pr^{\boldsymbol{d}}\left\{Z \le z \mid \boldsymbol{H}_2 = \boldsymbol{h}_2, d_2(\boldsymbol{H}_2) = d_2(\boldsymbol{h}_2)\right\} dF_{\boldsymbol{H}_2|\boldsymbol{H}_1,A_1}\left\{\boldsymbol{h}_2|d_1(\boldsymbol{h_1}), \boldsymbol{h}_1\right\} dF_{\boldsymbol{H}_1}(\boldsymbol{h}_1)$$

$$= \iint F_{\varepsilon_Z}\left\{z - m(\boldsymbol{h}_2) - d_2(\boldsymbol{h}_2)c_Z(\boldsymbol{h}_2)\right\} dF_{\boldsymbol{H}_2|\boldsymbol{H}_1,A_1}\left\{\boldsymbol{h}_2 \mid \boldsymbol{h}_1, d_1(\boldsymbol{h}_1)\right\} dF_{\boldsymbol{H}_1}(\boldsymbol{h}_1)$$

$$= \iint F_{\varepsilon_Z}\left[z - m(\boldsymbol{h}_2) - \mathrm{sgn}\left\{r_2(\boldsymbol{h}_2; \boldsymbol{\tau}_2)\right\}c_Z(\boldsymbol{h}_2)\right] dG_Z\left\{m_Z, c_Z, r_2|\boldsymbol{h}_1, d_1(\boldsymbol{h}_1)\right\} dF_{\boldsymbol{H}_1}(\boldsymbol{h}_1)$$

$$= \iint F_{\varepsilon_Z}\left[z - m(\boldsymbol{h}_2) - \mathrm{sgn}(r_2)c_Z(\boldsymbol{h}_2)\right] dG_Z\left\{m_Z, c_Z, r_2|\boldsymbol{h}_1, d_1(\boldsymbol{h}_1)\right\} dF_{\boldsymbol{H}_1}(\boldsymbol{h}_1)$$

where $G_Z\left\{m_Z, c_Z, r_2 \mid \boldsymbol{h}_1, a_1\right\}$ is the joint conditional distribution of $m_Z(\boldsymbol{H}_2)$, $c_Z(\boldsymbol{H}_2)$ and $r_2(\boldsymbol{H}_2; \boldsymbol{\tau}_2)$ given $\boldsymbol{H}_1 = \boldsymbol{h}_1$ and $A_1 = a_1$.

Details of estimation and modeling follows "Estimation of dynamic treatment regimes for complex outcomes: Balancing benefits and risks" by Linn et al.

Estimation of the barrier trajectory

Once we have the estimators of $\mathbb{E}Y^*(\boldsymbol{\tau})$ and $\mathbb{E}Z^*(\boldsymbol{\tau})$, denoted by $\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau})$ and $\widehat{\mathbb{E}}Z_n(\boldsymbol{\tau})$, we plug those estimators into Problem (2), and get Problem (3) as

$$\max_{\boldsymbol{\tau}} \widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) + \mu\log\left\{\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau})\right\} - \frac{1}{2\mu}\left\{(\boldsymbol{\tau}_1^{\mathsf{T}}\boldsymbol{\tau}_1 - 1)^2 + (\boldsymbol{\tau}_2^{\mathsf{T}}\boldsymbol{\tau}_2 - 1)^2\right\} \tag{3}$$

Denote a solution to Problem (3) by $\widehat{\boldsymbol{\tau}}_\kappa(\mu) = (\widehat{\boldsymbol{\tau}}_{\kappa,\mu,1}^{\mathsf{T}}, \widehat{\boldsymbol{\tau}}_{\kappa,\mu,2}^{\mathsf{T}})^{\mathsf{T}}$, and $\widehat{S}(\boldsymbol{\tau}, \mu) = \widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) + \mu\log\left\{\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau})\right\} - \frac{1}{2\mu}\left\{(\boldsymbol{\tau}_1^{\mathsf{T}}\boldsymbol{\tau}_1 - 1)^2 + (\boldsymbol{\tau}_2^{\mathsf{T}}\boldsymbol{\tau}_2 - 1)^2\right\}$. The following figure is an replication of the simulation result in the "Estimation of dynamic treatment regimes for complex outcomes" chapter.
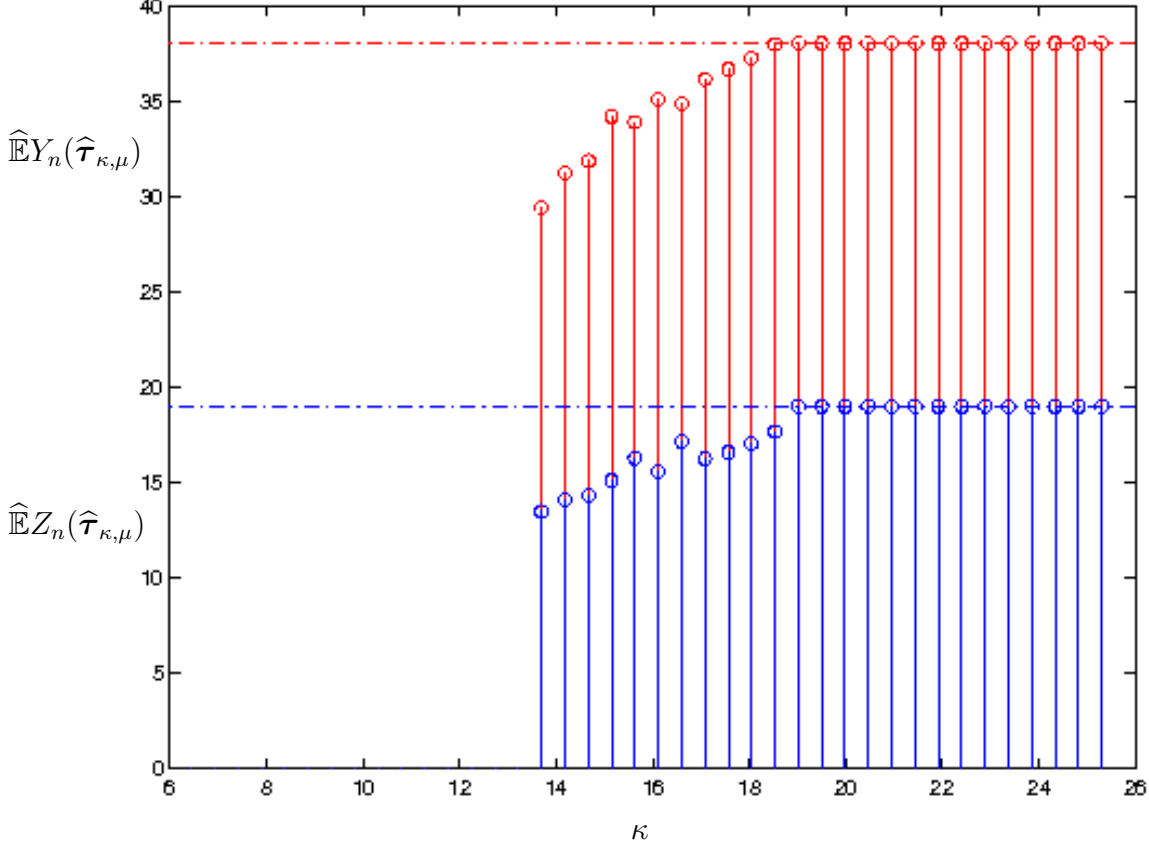
The red bars are the estimated means of Y under the estimated constrained optimal regimes vs. constraint value κ. The blue bars are the estimated means of Z under the estimated constrained optimal regime vs. constrain value κ. The red dash line is the estimated maximal of Y without constraint, and the blue dash line is the estimated minimal of Z without constraint. For the constrained problems, Matlab fmincon solver is used with 'Algorithm' option set to 'interior-point' method, and 'FinDiffRelStep' to 1e-2. For unconstrained problems, Matlab fminunc solver is used, where 'Algorithm' options is set to 'quasi-newton', and 'FinDiffRelStep' to 1e-2. MultiStart function is called for 5 random starts, where 'StartPointsToRun' option is set to 'all'.

4   Consistency of $\widehat{\boldsymbol{\tau}}_\kappa(\mu)$ to $\boldsymbol{\tau}^*_\kappa(\mu)$

Convergence in probabilty of the estimator to $\hat{\tau}_\kappa(\mu)$ to the penalty-barrier trajectory $\boldsymbol{\tau}^*_\kappa(\mu)$

Here, we examine the consistency of $\widehat{\boldsymbol{\tau}}_\kappa(\mu)$ to $\boldsymbol{\tau}^*_\kappa(\mu)$, and the following lemmas are needed. As

$S^*(\boldsymbol{\tau}, \mu) = \mathbb{E}Y^*(\boldsymbol{\tau}) + \mu \log\{\kappa - \mathbb{E}Z^*(\boldsymbol{\tau})\} - \frac{1}{2\mu}\{(\boldsymbol{\tau}_1^\intercal \boldsymbol{\tau}_1 - 1)^2 + (\boldsymbol{\tau}_2^\intercal \boldsymbol{\tau}_2 - 1)^2\}$, and $\widehat{S}(\boldsymbol{\tau}, \mu) = \widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) +$ $\mu \log\left\{\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau})\right\} - \frac{1}{2\mu}\{(\boldsymbol{\tau}_1^\intercal \boldsymbol{\tau}_1 - 1)^2 + (\boldsymbol{\tau}_2^\intercal \boldsymbol{\tau}_2 - 1)^2\}$, then

$$\widehat{S}(\boldsymbol{\tau}, \mu) - S^*(\boldsymbol{\tau}, \mu) = \widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \mathbb{E}Y^*(\boldsymbol{\tau}) + \mu \log \frac{\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau})}{\kappa - \mathbb{E}Z^*(\boldsymbol{\tau})}$$

For notation simplicity, we denote $Pr\{Y^*(\boldsymbol{d}) \le y\} = F_{Y^*(\boldsymbol{\tau})}(y)$ and $Pr\{Z^*(\boldsymbol{d}) \le z\} = F_{Z^*(\boldsymbol{\tau})}(z)$. Their estimators are denoted by $\widehat{Pr}\{Y_n(\boldsymbol{d}) \le y\} = \widehat{F}_{Y_n(\boldsymbol{\tau})}(y)$ and $\widehat{Pr}\{Z_n(\boldsymbol{d}) \le z\} = \widehat{F}_{Z_n(\boldsymbol{\tau})}(z)$. Also, $Y^*(\boldsymbol{\tau})$ denotes the random variable with the cumulative distribution function $F_{Y^*(\boldsymbol{\tau})}(y)$, and $Y_n(\boldsymbol{\tau})$ denotes the random variable with the cumulative distribution function $\widehat{F}_{Y_n(\boldsymbol{\tau})}(y)$. Same applies to $Z^*(\boldsymbol{\tau})$ and $Z_n(\boldsymbol{\tau})$.

Lemma 2. *For any fixed $\boldsymbol{\tau} : \boldsymbol{\tau}_1^\intercal \boldsymbol{\tau}_1 - 1 = 0$, and $\boldsymbol{\tau}_2^\intercal \boldsymbol{\tau}_2 - 1 = 0$. We assume that, for each $\boldsymbol{\tau}$,*

1. $Y_n(\boldsymbol{\tau}) \xrightarrow{d} Y^*(\boldsymbol{\tau})$, i.e., $\lim\limits_{n\to\infty} \widehat{F}_{Y_n(\boldsymbol{\tau})}(y) = F_{Y^*(\boldsymbol{\tau})}(y)$, for all y at which $F_{Y^*(\boldsymbol{\tau})}(y)$ is continuous;

2. If there exists some random variable T, such that $|Y_n(\boldsymbol{\tau})| \le |T|$ for all n and $\mathbb{E}|T| \le \infty$,

Also, we assume the same assumptions for $Z(\boldsymbol{\tau})$. Then, we have

$$\sup_{\boldsymbol{\tau}:\boldsymbol{\tau}_1^\intercal \boldsymbol{\tau}_1 = 1, \boldsymbol{\tau}_2^\intercal \boldsymbol{\tau}_2 = 1} \left|\widehat{S}_\kappa(\boldsymbol{\tau}, \mu) - S_\kappa^*(\boldsymbol{\tau}, \mu)\right| = o_p(1).$$

Proof. For any fixed $\boldsymbol{\tau} : \boldsymbol{\tau}_1^\intercal \boldsymbol{\tau}_1 = 1$ and $\boldsymbol{\tau}_2^\intercal \boldsymbol{\tau}_2 = 1$, we have

$$\begin{aligned}
&\widehat{S}_\kappa(\boldsymbol{\tau}, \mu) - S_\kappa^*(\boldsymbol{\tau}, \mu) \\
=& \widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \mathbb{E}Y^*(\boldsymbol{\tau}) + \mu \log\left\{\frac{\kappa - \widehat{\mathbb{E}}Z(\boldsymbol{\tau})}{\kappa - \mathbb{E}Z^*(\boldsymbol{\tau})}\right\} \\
=& \int y \, d\widehat{F}_{Y(\boldsymbol{\tau})}(y) - \int y \, dF_{Y^*(\boldsymbol{\tau})}(y) + \mu \log\left\{\frac{\kappa - \int z \, d\widehat{F}_{Z_n(\boldsymbol{\tau})}(z)}{\kappa - \int z \, dF_{Z^*(\boldsymbol{\tau})}(z)}\right\}
\end{aligned}$$

For $\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \mathbb{E}Y^*(\boldsymbol{\tau})$, we have $Y_n(\boldsymbol{\tau}) \sim \widehat{F}_{Y_n(\boldsymbol{\tau})}(y)$, and $Y^*(\boldsymbol{\tau}) \sim F_{Y^*(\boldsymbol{\tau})}(y)$. Under the assumptions 1 and 2, by dominated convergence theorem, we have we have $\lim\limits_{n\to\infty} \widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) = \mathbb{E}Y^*(\boldsymbol{\tau})$. Similarily, we have $\lim\limits_{n\to\infty} \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}) = \mathbb{E}Z^*(\boldsymbol{\tau})$. The two terms embedded in log operator are forced to be strictly greater

than zero. Together with continuous mapping theorem, we have

$$\widehat{S}_\kappa(\boldsymbol{\tau}, \mu) - S_\kappa(\boldsymbol{\tau}, \mu) = o_p(1).$$

As this holds for any fixed value of $\boldsymbol{\tau}$ and constants μ sufficiently small, we have

$$\sup_{\boldsymbol{\tau}: \boldsymbol{\tau}_1^\mathsf{T} \boldsymbol{\tau}_1 = 1, \boldsymbol{\tau}_2^\mathsf{T} \boldsymbol{\tau}_2 = 1} \left| \widehat{S}_\kappa(\boldsymbol{\tau}, \mu) - S_\kappa^*(\boldsymbol{\tau}, \mu) \right| = o_p(1)$$

. ∎

Now, we examine the consistency of $\widehat{\boldsymbol{\tau}}_\kappa(\mu)$ in the following corollary.

Corollary 1. *For any sufficiently small μ, suppose all the assumptions in Lemma 2 hold, and we also make the following assumptions,*

1. *$\widehat{S}_\kappa(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) \geq \sup_{\boldsymbol{\tau}} \widehat{S}_\kappa(\boldsymbol{\tau}, \mu) + o_p(1), \forall \boldsymbol{\tau} \neq \widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \boldsymbol{\tau}_1^\mathsf{T} \boldsymbol{\tau}_1 = 1,$ and $\boldsymbol{\tau}_2^\mathsf{T} \boldsymbol{\tau}_2 = 1$;*

2. *$\forall \epsilon > 0, \exists \delta > 0,$ s.t. $S_\kappa^*(\boldsymbol{\tau}_{\kappa,\mu}^*, \mu) > S_\kappa^*(\boldsymbol{\tau}, \mu) + \delta, \forall \|\boldsymbol{\tau} - \boldsymbol{\tau}_{\kappa,\mu}^*\| \geq \epsilon, \boldsymbol{\tau}_1^\mathsf{T} \boldsymbol{\tau}_1 = 1,$ and $\boldsymbol{\tau}_2^\mathsf{T} \boldsymbol{\tau}_2 = 1.$*

Then, we have that $\widehat{\boldsymbol{\tau}}_\kappa(\mu)$ is consistent in probability, i.e., $\widehat{\boldsymbol{\tau}}_\kappa(\mu) \xrightarrow{p} \boldsymbol{\tau}_\kappa^*(\mu).$

Here, assumption 1 states that $\widehat{\boldsymbol{\tau}}_\kappa(\mu)$ is a maximizer of $\widehat{S}_\kappa(\boldsymbol{\tau}, \mu)$. Assumption 2 says that $S_\kappa^*(\boldsymbol{\tau}, \mu)$ has a unique maximizer, $\boldsymbol{\tau}_\kappa^*(\mu)$.

Proof. We expand $S_\kappa^*(\boldsymbol{\tau}_{\kappa,\mu}^*, \mu)$ as,

$$
\begin{aligned}
S_\kappa^*(\boldsymbol{\tau}_{\kappa,\mu}^*, \mu) &= \widehat{S}_\kappa(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) + \{S_\kappa^*(\boldsymbol{\tau}_{\kappa,\mu}^*, \mu) - \widehat{S}_\kappa(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu)\} \\
&\leq \widehat{S}_\kappa(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) + \{S_\kappa^*(\boldsymbol{\tau}_{\kappa,\mu}^*, \mu) - \widehat{S}_\kappa(\boldsymbol{\tau}_{\kappa,\mu}^*, \mu)\} \\
&\leq \widehat{S}_\kappa(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) + |S_\kappa^*(\boldsymbol{\tau}_{\kappa,\mu}^*, \mu) - \widehat{S}_\kappa(\boldsymbol{\tau}_{\kappa,\mu}^*, \mu)| \\
&\leq \widehat{S}_\kappa(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) + \sup_{\boldsymbol{\tau}} |S_\kappa^*(\boldsymbol{\tau}, \mu) - \widehat{S}_\kappa(\boldsymbol{\tau}, \mu)| \\
&= S_\kappa^*(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) + \{\widehat{S}_\kappa(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) - S_\kappa^*(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu)\} + \sup_{\boldsymbol{\tau}} |S_\kappa^*(\boldsymbol{\tau}, \mu) - \widehat{S}_\kappa(\boldsymbol{\tau}, \mu)| \\
&\leq S_\kappa^*(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) + |\widehat{S}_\kappa(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) - S_\kappa^*(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu)| + \sup_{\boldsymbol{\tau}} |S_\kappa^*(\boldsymbol{\tau}, \mu) - \widehat{S}_\kappa(\boldsymbol{\tau}, \mu)| \\
&\leq S_\kappa^*(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) + 2\sup_{\boldsymbol{\tau}} |S_\kappa^*(\boldsymbol{\tau}, \mu) - \widehat{S}_\kappa(\boldsymbol{\tau}, \mu)|,
\end{aligned}
$$

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where for any $\boldsymbol{\tau}$, we have $\boldsymbol{\tau}_1^\mathsf{T}\boldsymbol{\tau}_1 = 1$, and $\boldsymbol{\tau}_2^\mathsf{T}\boldsymbol{\tau}_2 = 1$. Moreover, the first inequality holds by assumption 1 that states $\widehat{\boldsymbol{\tau}}_\kappa(\mu)$ as a maximizer of $\widehat{S}_\kappa(\boldsymbol{\tau}, \mu)$. Then, by Lemma 2 above, we have,

$$S_\kappa^*(\boldsymbol{\tau}_{\kappa,\mu}^*, \mu) \leq S_\kappa^*(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) + o_p(1)$$

Suppose $\widehat{\boldsymbol{\tau}}_{\kappa,\mu} \not\to \boldsymbol{\tau}_{\kappa,\mu}^*$. We take $\limsup$ on both sides, and get

$$S_\kappa^*(\boldsymbol{\tau}_{\kappa,\mu}^*, \mu) \leq \limsup_{n\to\infty} S_\kappa^*(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu).$$

However, by assumption 2, we have

$$S_\kappa^*(\boldsymbol{\tau}_{\kappa,\mu}^*, \mu) > \limsup_{n\to\infty} S_\kappa^*(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu),$$

because $\boldsymbol{\tau}_{\kappa,\mu}^*$ is supposed to by the unique maximizer of $S_\kappa^*(\boldsymbol{\tau}_{\kappa,\mu}^*, \mu)$. These are contradictory. Therefore, it has to be that

$$\widehat{\boldsymbol{\tau}}_\kappa(\mu) \xrightarrow{p} \boldsymbol{\tau}_\kappa^*(\mu), \ \text{as } n \to \infty.$$

■

5 Asymptotic distribution of $\widehat{\boldsymbol{\tau}}_\kappa(\mu)$

Limiting distribution of $\widehat{\boldsymbol{\tau}}$

Before we derive the limiting distribution of the estimator $\widehat{\boldsymbol{\tau}}_{\kappa,\mu}$, we need to examine, for any fixed value of $\boldsymbol{\tau} : \boldsymbol{\tau}_1^\mathsf{T}\boldsymbol{\tau}_1 = 1$ and $\boldsymbol{\tau}_2^\mathsf{T}\boldsymbol{\tau}_2 = 1$, the limiting distribution of $\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau})$.

$$\begin{aligned}
\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) &= \frac{\partial}{\partial\boldsymbol{\tau}} \int y\, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y) \\
&= \frac{\partial}{\partial\boldsymbol{\tau}} \int y\, d\left[\frac{1}{n}\sum_{i=1}^n \widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right] \\
&= \frac{1}{n}\sum_{i=1}^n \frac{\partial}{\partial\boldsymbol{\tau}} \int y\, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})
\end{aligned}$$

Lemma 3. *In addition to the assumptions in corollary 1, we also suppose the following conditions hold.*

1. $\boldsymbol{H}_1^\intercal \boldsymbol{\tau}_1$ *and* $\boldsymbol{H}_2^\intercal \boldsymbol{\tau}_2$ *are both bounded away from 0.*

2. $\forall \boldsymbol{a} \in \mathbb{R}^p, \exists \delta > 0$ *,such that*

   (a) $\mathbb{E} \left| \boldsymbol{a}^\intercal \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right|^{2+\delta} < \infty$

   (b) $\left\{ \boldsymbol{a}^\intercal V \left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right] \boldsymbol{a} \right\}^{1+\frac{\delta}{2}} < \infty.$

Then, we have, for any fixed $\boldsymbol{\tau}$ and $\boldsymbol{\beta}_{Y1}$,

$$\sqrt{n} \left[\nabla \widehat{\mathbb{E}} Y_n(\boldsymbol{\tau}) - \mathbb{E}\left(\nabla \widehat{\mathbb{E}} Y_n(\boldsymbol{\tau}) \right) \right] \xrightarrow{d} \mathcal{N}\left(0, AV \left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right] \right)$$

The proof of this is similar to the proof of Lemma 1 in the one-stage problem.

Proof. For any $\boldsymbol{a} \in \mathbb{R}^p$, we let $W_{ni} = \boldsymbol{a}^\intercal \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$. For each value of n, $w_{n1}, w_{n2}, \cdots, w_{nn}$ are i.i.d, and functions of the sample size n. This is because that $\boldsymbol{X}_i$ are assumed to be i.i.d., and h is a function of sample size n. Then, we have

$$\mu_n := \mathbb{E} W_{ni} = \mathbb{E}\left[\boldsymbol{a}^\intercal \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right],$$

and

$$\sigma_n^2 := V(W_{ni}) = \boldsymbol{a}^\intercal V \left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right] \boldsymbol{a}$$

We let $G_{ni} = W_{ni} - \mu_n$, and $T_n = \sum_{i=1}^n G_{ni}$. Also, we let $s_n^2 = V(T_n) = \sum_{i=1}^n V(G_{ni}) = \sum_{i=1}^n \sigma_n^2 = n\sigma_n^2$, where the second equality is because of independence, and the last equality is due to identicalness. Therefore, T_n/s_n has mean 0, and variance 1. If we can show G_{ni} satisfying the Lyapunov condition, then we have

$$\frac{T_n}{s_n} \xrightarrow{d} \mathcal{N}(0,1), \text{ as } n \to \infty$$

,

Now, we check the Lyapunov condition, that is, [2, 5]

$$\exists \delta > 0, \text{ such that } \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} \mathbb{E} \mid G_{n,i} \mid^{2+\delta} \to 0, \text{ as } n \to 0.$$

We define, for any $\boldsymbol{a}$,

$$C_1 \triangleq \mathbb{E} \left| G_{ni} \right|^{2+\delta} = \mathbb{E} \left| W_{ni} - \mu_n \right|^{2+\delta} = \mathbb{E} \left| \boldsymbol{a}^\intercal \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) - \mu_n \right|^{2+\delta},$$

and

$$C_2 \triangleq s_n^{2+\delta} = n^{1+\frac{\delta}{2}} \sigma_n^{2+\delta} = n^{1+\frac{\delta}{2}} \left\{ \boldsymbol{a}^\intercal V \left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right] \boldsymbol{a} \right\}^{1+\frac{\delta}{2}}.$$

Then, we have

$$
\frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} \mathbb{E} \mid G_{n,i} \mid^{2+\delta}
$$
$$
= \frac{\mathbb{E} \left| \boldsymbol{a}^\intercal \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) - \mu_n \right|^{2+\delta}}{n^{\frac{\delta}{2}} \left\{ \boldsymbol{a}^\intercal V \left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right] \boldsymbol{a} \right\}^{1+\frac{\delta}{2}}}
$$
$$
= \frac{C_1}{n^{\frac{\delta}{2}} C_2}.
$$

As long as $\delta > 0$, for finite C_1 and finite C_2, we have $C_1 / n^{\frac{\delta}{2}} C_2 \to 0$, as $n \to \infty$. This means that the Lyapunov condition is satisfied, if $\mathbb{E} \left| G_{ni} \right|^{2+\delta}$ and $s_n^{2+\delta}$ are finite. Then, by Lyapunov Central Limit Theorem, we have

$$\frac{T_n}{s_n} \xrightarrow{d} \mathcal{N}(0, 1).$$

As this hold for any arbitary non-random vector $\boldsymbol{a} \in \mathbb{R}^p$, we have, by Cramer-Wold Theorem, that

$$\sqrt{n} \left[\frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) - \mathbb{E} \left\{ \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right\} \right] \xrightarrow{d} \mathcal{N} \left(0, V \left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right] \right),$$

as $n \to \infty$. We denote $\boldsymbol{L}_{ni} = \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$, then this is written as

$$\sqrt{n} \left[\frac{1}{n} \sum_{i=1}^{n} \boldsymbol{L}_{ni} - \mathbb{E}\boldsymbol{L}_{n1} \right] \overset{d}{\to} \mathcal{N}\left(0, V\left[\boldsymbol{L}_{n1}\right]\right).$$

Then, we have

$$\frac{1/n \sum_{i=1}^{n} \boldsymbol{L}_{ni} - \mathbb{E}\boldsymbol{L}_{n1}}{[V(\boldsymbol{L}_{n1})/n]^{1/2}} \frac{[V(\boldsymbol{L}_{n1})/n]^{1/2}}{[AV(\boldsymbol{L}_{n1})/n]^{1/2}} \overset{d}{\to} \mathcal{N}(0,1).$$

As $n \to \infty$,

$$\frac{V(\boldsymbol{L}_{n1})^{1/2}}{AV(\boldsymbol{L}_{n1})^{1/2}} \to 1,$$

then we have

$$\frac{1/n \sum_{i=1}^{n} \boldsymbol{L}_{ni} - \mathbb{E}\boldsymbol{L}_{n1}}{[AV(\boldsymbol{L}_{n1})/n]^{1/2}} \overset{d}{\to} \mathcal{N}(0,1),$$

i.e.,

$$\sqrt{n} \left[1/n \sum_{i=1}^{n} \boldsymbol{L}_{ni} - \mathbb{E}\boldsymbol{L}_{n1} \right] \overset{d}{\to} N\left(0, AV(\boldsymbol{L}_{n1})\right).$$

As $\frac{1}{n} \sum_{i=1}^{n} \boldsymbol{L}_{ni} = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) = \nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau})$, we have

$$\sqrt{n} \left[\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \mathbb{E}\left\{ \nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) \right\} \right] \overset{d}{\to} \mathcal{N}\left(0, AV \left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right] \right)$$

∎

Assume $\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$, the following lemma shows that the estimations do not effect the limiting distribution obtained above.

Corollary 2. *Suppose all the assumptions in Lemma 3 hold, and $\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$ is a consistent*

estimator of $F_{Y^*(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$. *Then, we have*

$$\sqrt{n}\left[\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \nabla\mathbb{E}Y_n^*(\boldsymbol{\tau})\right] \xrightarrow{d} \mathcal{N}\left(0, AV\left[\frac{\partial}{\partial\boldsymbol{\tau}}\int y\,dF_{Y^*(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right]\right)$$

Proof. We write

$$\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \nabla\mathbb{E}Y^*(\boldsymbol{\tau})$$

$$=\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \mathbb{E}\left\{\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau})\right\} + \mathbb{E}\left\{\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau})\right\} - \nabla\mathbb{E}Y^*(\boldsymbol{\tau}),$$

where $\mathbb{E}\left\{\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau})\right\} - \nabla\mathbb{E}Y^*(\boldsymbol{\tau}) = \mathbb{E}\left\{\frac{\partial}{\partial\boldsymbol{\tau}}\int y\,d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right\} - \mathbb{E}\frac{\partial}{\partial\boldsymbol{\tau}}\int y\,dF_{Y^*x(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) = o_p(1)$ is due to the consistency of $\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$ and dominated convergence theorem. By lemma 1, we have

$$\sqrt{n}\left[\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \mathbb{E}\left\{\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau})\right\}\right] \xrightarrow{d} \mathcal{N}\left(0, AV\left[\frac{\partial}{\partial\boldsymbol{\tau}}\int y\,d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right]\right)$$

As $\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$ is consistent, we have

$$\frac{AV\left[\frac{\partial}{\partial\boldsymbol{\tau}}\int y\,d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right]}{AV\left[\frac{\partial}{\partial\boldsymbol{\tau}}\int y\,dF_{Y^*(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right]} \xrightarrow{p} 1.$$

Then, we have

$$\sqrt{n}\left[\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \nabla\mathbb{E}Y^*(\boldsymbol{\tau})\right] \xrightarrow{d} \mathcal{N}\left(0, AV\left[\frac{\partial}{\partial\boldsymbol{\tau}}\int y\,dF_{Y_n^*(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right]\right)$$

∎

Limiting distribution of $\widehat{\boldsymbol{\tau}}_{\kappa,\mu}$

Now, we investigate the limiting distribution of $\widehat{\boldsymbol{\tau}}_{\kappa,\mu}$.

Theorem 1. *Suppose all the assumptions above hold. Then we have, as* $n \to \infty$

$$\sqrt{n}(\widehat{\boldsymbol{\tau}}_{\kappa,\mu} - \boldsymbol{\tau}_{\kappa,\mu}^*) \xrightarrow{d} \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{\Sigma}^*\right),$$

where $\boldsymbol{\Sigma}^* = \boldsymbol{D}^{*-1}\boldsymbol{C}^*\boldsymbol{D}^{*-1}$,

$$\boldsymbol{C}^* := AV\left[\frac{\partial}{\partial \boldsymbol{\tau}}\int y\, dF_{Y^*(\boldsymbol{\tau})}(y|\boldsymbol{H}_1 = \boldsymbol{h}_1)\right],$$

and

$$\boldsymbol{D}^* := \nabla^2 \mathbb{E}Y^*(\boldsymbol{\tau}^*_{\kappa,\mu}) - \mu\frac{\nabla^2 \mathbb{E}Z^*(\boldsymbol{\tau}^*_{\kappa,\mu})\left[\kappa - \mathbb{E}Z^*(\boldsymbol{\tau}^*_{\kappa,\mu})\right] + \{\nabla\mathbb{E}Z^*(\boldsymbol{\tau}^*_{\kappa,\mu})\}^2}{\left\{\kappa - \mathbb{E}Z^*(\boldsymbol{\tau}^*_{\kappa,\mu})\right\}^2}$$

$$= \nabla^2 S^*(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu).$$

Proof. Taylor expansion, for each μ

$$\nabla\widehat{S}(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu) = \nabla\widehat{S}(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) - \nabla^2\widehat{S}(\tilde{\boldsymbol{\tau}}_{\kappa,\mu}, \mu)(\widehat{\boldsymbol{\tau}}_{\kappa,\mu} - \boldsymbol{\tau}^*_{\kappa,\mu}),$$

where $\tilde{\boldsymbol{\tau}}_{\kappa,\mu}$ is a vector in between $\widehat{\boldsymbol{\tau}}_{\kappa,\mu}$ and $\boldsymbol{\tau}^*_{\kappa,\mu}$. As $\widehat{\boldsymbol{\tau}}_\kappa(\mu)$ is the maximizer of $\widehat{S}(\boldsymbol{\tau}, \mu)$, it satisfies the first order conditions such that $\nabla\widehat{S}(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) = 0$. Then,

$$\sqrt{n}\nabla\widehat{S}(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu) = -\sqrt{n}\nabla^2\widehat{S}(\tilde{\boldsymbol{\tau}}_{\kappa,\mu}, \mu)(\widehat{\boldsymbol{\tau}}_{\kappa,\mu} - \boldsymbol{\tau}^*_{\kappa,\mu})$$

$$\nabla\widehat{S}(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu) = \nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}^*_{\kappa,\mu}) - \mu\frac{\nabla\widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})}{\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})} + +\frac{2}{\mu}(\boldsymbol{\tau}^{*\intercal}_{\kappa,\mu}\boldsymbol{\tau}^*_{\kappa,\mu} - 1)\boldsymbol{\tau}^*_{\kappa,\mu}, \text{ where } \boldsymbol{\tau}^{*\intercal}_{\kappa,\mu}\boldsymbol{\tau}^*_{\kappa,\mu} - 1 = 0.$$

$$\nabla^2\widehat{S}(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu) = \nabla^2\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}^*_{\kappa,\mu}) - \mu\frac{\nabla^2\widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})\left[\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})\right] + \{\nabla\widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})\}^2}{\left\{\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})\right\}^2}$$

Then, we have

$$\nabla\widehat{S}(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu) = \nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}^*_{\kappa,\mu}) - \mu\frac{\nabla\widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})}{\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})},$$

By Lemma 3, we have the first term on the right hand side as

$$\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) \xrightarrow{d} \mathcal{N}\left\{\nabla\mathbb{E}Y^*(\boldsymbol{\tau}), \frac{1}{n}AV\left[\frac{\partial}{\partial\boldsymbol{\tau}}\int y\, dF_{Y^*(\boldsymbol{\tau})}(y|\boldsymbol{H}_1 = \boldsymbol{h}_1)\right]\right\}$$

For the second term on the right hand side, we also have that

$$\mu\frac{\nabla\widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}_{\kappa,\mu}^*)}{\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}_{\kappa,\mu}^*)} \overset{p}{\to} \mu\frac{\nabla\mathbb{E}Z(\boldsymbol{\tau}_{\kappa,\mu}^*)}{\kappa - \mathbb{E}Z(\boldsymbol{\tau}_{\kappa,\mu}^*)},,$$

where we assume both $\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}_{\kappa,\mu}^*) > 0$ and $\kappa - \mathbb{E}Z(\boldsymbol{\tau}_{\kappa,\mu}^*) > 0$. This convergence in probabilty is due to the consistency of $\widehat{F}_{Z_n(\tau)}(z|\boldsymbol{H}_1 = \boldsymbol{h}_1)$ and dominated convergence theorem. Together, by Sluskty's theorem and the stationarity of $\boldsymbol{\tau}_{\kappa,\mu}^*$ of $S^*(\boldsymbol{\tau},\mu)$, we have

$$\sqrt{n}\nabla\widehat{S}(\boldsymbol{\tau}_{\kappa,\mu}^*,\mu) \overset{d}{\to} \mathcal{N}\left(0,\boldsymbol{C}^*\right),$$

where

$$\boldsymbol{C}^* := AV\left[\frac{\partial}{\partial\boldsymbol{\tau}}\int y\,dF_{Y^*(\tau)}(y|\boldsymbol{H}_1 = \boldsymbol{h}_1)\right]$$

We have that $\nabla^2\widehat{S}(\tilde{\boldsymbol{\tau}}_{\kappa,\mu},\mu) = \nabla^2\widehat{S}(\boldsymbol{\tau}_{\kappa,\mu}^*,\mu) + o_p(1)$, as $\tilde{\boldsymbol{\tau}}_{\kappa,\mu}$ is in between $\widehat{\boldsymbol{\tau}}_{\kappa,\mu}$ and $\boldsymbol{\tau}_{\kappa,\mu}^*$ and $\widehat{\boldsymbol{\tau}}_{\kappa,\mu} - \boldsymbol{\tau}_{\kappa,\mu}^* = o_p(1)$. Therefore, we have

$$
\begin{aligned}
\boldsymbol{D}^* &\triangleq p\lim_{n\to\infty}\nabla^2\widehat{S}_\kappa(\boldsymbol{\tau}_{\kappa,\mu}^*,\mu)\\
&= p\lim_{n\to\infty}\left[\nabla^2\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}_{\kappa,\mu}^*) - \mu\frac{\nabla^2\widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}_{\kappa,\mu}^*)\left[\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}_{\kappa,\mu}^*)\right] + \{\nabla\widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}_{\kappa,\mu}^*)\}^2}{\left\{\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}_{\kappa,\mu}^*)\right\}^2}\right]\\
&= \nabla^2\mathbb{E}Y^*(\boldsymbol{\tau}_{\kappa,\mu}^*) - \mu\frac{\nabla^2\mathbb{E}Z^*(\boldsymbol{\tau}_{\kappa,\mu}^*)\left[\kappa - \mathbb{E}Z^*(\boldsymbol{\tau}_{\kappa,\mu}^*)\right] + \{\nabla\mathbb{E}Z^*(\boldsymbol{\tau}_{\kappa,\mu}^*)\}^2}{\left\{\kappa - \mathbb{E}Z^*(\boldsymbol{\tau}_{\kappa,\mu}^*)\right\}^2}\\
&= \nabla^2 S^*(\boldsymbol{\tau}_{\kappa,\mu}^*,\mu).
\end{aligned}
$$

■

We can estimate $\boldsymbol{\Sigma}^*$ by plug in the corresponding estimators stated, and denote the estimator $\widehat{\boldsymbol{\Sigma}}$, that is, $\widehat{\boldsymbol{\Sigma}} = \widehat{\boldsymbol{D}}^{-1}\widehat{\boldsymbol{C}}\widehat{\boldsymbol{D}}^{-1}$, where

$$\widehat{C} = \widehat{V}\left\{\frac{\partial}{\partial \boldsymbol{\tau}} \int y\, d\widehat{F}_{Y_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu})}(y|\boldsymbol{H}_1 = \boldsymbol{h}_1)\right\}$$

$$= \widehat{V}\left[\nabla\widehat{\mathbb{E}}\left\{Y_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}) \mid \boldsymbol{H}_1 = \boldsymbol{h_1}\right\}\right]$$

and

$$\widehat{\boldsymbol{D}} = \nabla^2\widehat{S}(\widehat{\boldsymbol{\tau}}_{\kappa,\mu},\mu) = \nabla^2\widehat{\mathbb{E}}Y_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}) - \mu\frac{\nabla^2\widehat{\mathbb{E}}Z_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu})\left[\kappa - \widehat{\mathbb{E}}Z_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu})\right] + \{\nabla\widehat{\mathbb{E}}Z_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu})\}^2}{\left\{\kappa - \widehat{\mathbb{E}}Z_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu})\right\}^2},$$

As $\widehat{C}$, $\widehat{\boldsymbol{D}}$ and $\widehat{\boldsymbol{\Sigma}}$ maybe have complicated forms, bootstrap based estimators can be used for practical dataset, and Monte Carlo estimators for simulated replicates.

Confidence set of $\widehat{\boldsymbol{\tau}}$

As Theorem 1 that

$$\boldsymbol{\Sigma}^{*-1/2}(\widehat{\boldsymbol{\tau}}_{\kappa,\mu} - \boldsymbol{\tau}^*_{\kappa,\mu}) \overset{d}{\to} \mathcal{N}\left(\mathbf{0}, \mathbf{I}/n\right), \text{as } n \to \infty,$$

we have

$$(\widehat{\boldsymbol{\tau}}_{\kappa,\mu} - \boldsymbol{\tau}^*_{\kappa,\mu})\boldsymbol{\Sigma}^{*-1}(\widehat{\boldsymbol{\tau}}_{\kappa,\mu} - \boldsymbol{\tau}^*_{\kappa,\mu}) \overset{d}{\to} \chi^2_{p+1}/n^2, \text{as } n \to \infty.$$

Thus, with a consistent estimator of $\widehat{\boldsymbol{\Sigma}}$, the $(1 - \alpha) \times 100\%$ confidence set is

$$\boldsymbol{\mathcal{C}}_{1-\alpha}\left(\boldsymbol{\tau}^*_{\kappa,\mu}\right) = \left\{\boldsymbol{\tau}_{\kappa,\mu} : (\widehat{\boldsymbol{\tau}}_{\kappa,\mu} - \boldsymbol{\tau}_{\kappa,\mu})\widehat{\boldsymbol{\Sigma}}^{-1}(\widehat{\boldsymbol{\tau}}_{\kappa,\mu} - \boldsymbol{\tau}_{\kappa,\mu}) \leq \chi^2_{p+1}(1-\alpha)/n^2\right\}, \text{ as } n \to \infty,$$

where $\chi^2_{p+1}(1 - \alpha)$ is the $1 - \alpha$ quantiles of a χ^2_{p+1} random variable, and $p + 1$ is the dimension of $\boldsymbol{\tau}$.

Projection Confidence Interval of $\mathbb{E}Y^*(\boldsymbol{\tau}^*_{\kappa,\mu})$

For each $\boldsymbol{\tau}_{\kappa,\mu} \in \boldsymbol{\mathcal{C}}_{1-\alpha}\left(\boldsymbol{\tau}^*_{\kappa,\mu}\right)$, we estimate $\mathbb{E}Y^*(\boldsymbol{\tau}_{\kappa,\mu})$ by $\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}_{\kappa,\mu}) = \frac{1}{n}\sum_{i=1}^n \int y\, d\widehat{F}_{Y_n(\boldsymbol{\tau}_{\kappa,\mu})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$. Thus, assuming $\widehat{F}_{Y_n(\boldsymbol{\tau}_{\kappa,\mu})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$ is a consistent estimator of $F_{Y^*(\boldsymbol{\tau}_{\kappa,\mu})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$,

$\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}_{\kappa,\mu})$ has a limiting distribution as

$$\sqrt{n}\left\{\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}_{\kappa,\mu}) - \mathbb{E}Y^*(\boldsymbol{\tau}_{\kappa,\mu})\right\} \sim \mathcal{N}\left[0, AV\left\{\int y\, dF_{Y^*(\boldsymbol{\tau}_{\kappa,\mu})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right\}\right].$$

For each $\boldsymbol{\tau}_{\kappa,\mu} \in \mathcal{C}_{1-\alpha}\left(\boldsymbol{\tau}^*_{\kappa,\mu}\right)$, we can construct a $(1-\eta) \times 100\%$ confidence interval

$$\mathcal{I}_{1-\eta}\left\{\mathbb{E}Y^*(\boldsymbol{\tau}_{\kappa,\mu})\right\}$$
$$= \left\{\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}_{\kappa,\mu}) - Z_{1-\eta/2}\widehat{\sigma}_{Y^*(\boldsymbol{\tau}_{\kappa,\mu})}, \widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}_{\kappa,\mu}) + Z_{1-\eta/2}\widehat{\sigma}_{Y^*(\boldsymbol{\tau}_{\kappa,\mu})}\right\}$$

where $\widehat{\sigma}_{Y^*(\boldsymbol{\tau}_{\kappa,\mu})}$ is the estimator for $AV\left\{\int y\, dF_{Y^*(\boldsymbol{\tau}_{\kappa,\mu})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right\}$, which can also be calculated by bootstrap estimator for practical dataset, or Monte Carlo estimator for simulated replicates. $Z_{1-\eta/2}$ is the upper $1-\eta/2$ critical value for the standard normal distribution. Alternatively, we can also construct this confidence interval using standard methods such as percentile bootstrap. Then, we can construct the projection confidence set for $\mathbb{E}Y^*(\boldsymbol{\tau}_{\kappa,\mu})$ by taking the union of $\mathcal{I}_{1-\eta}\left\{\mathbb{E}Y^*(\boldsymbol{\tau}_{\kappa,\mu})\right\}$ over all $\boldsymbol{\tau}_{\kappa,\mu} \in \mathcal{C}_{1-\alpha}\left(\boldsymbol{\tau}^*_{\kappa,\mu}\right)$, i.e.,

$$\mathcal{U}_{1-\alpha-\eta}\left\{\mathbb{E}Y^*(\boldsymbol{\tau}_{\kappa,\mu})\right\} = \bigcup_{\boldsymbol{\tau}_{\kappa,\mu} \in \mathcal{C}_{1-\alpha}\left(\boldsymbol{\tau}^*_{\kappa,\mu}\right)} \mathcal{I}_{1-\eta}\left\{\mathbb{E}Y^*(\boldsymbol{\tau}_{\kappa,\mu})\right\}.$$

Discussion:

1. Simulation Setup? Generative models other than linear models? Higher-dimension?

2. Monte Carlo/Bootstrap estimators for $\widehat{\Sigma}$ and $\widehat{\sigma}_{Y^*(\tau)}$?

3. Consistency conditions for the modeling of conditional distributions?

4. Non-parametric estimators for the conditional distributions?

5. Infinite-stage? .

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