Before we derive the limiting distribution of the estimator $\hat{\boldsymbol{\tau}}_{\kappa,\mu}$, we need to examine, for any fixed value of $\boldsymbol{\tau} : \boldsymbol{\tau}_1^\intercal \boldsymbol{\tau}_1 = 1$ and $\boldsymbol{\tau}_2^\intercal \boldsymbol{\tau}_2 = 1$, the limiting distribution of $\nabla \widehat{\mathbb{E}} Y_n(\boldsymbol{\tau})$.

$$
\begin{aligned}
\nabla \widehat{\mathbb{E}} Y_n(\boldsymbol{\tau}) =& \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y) \\
=& \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d \left[\frac{1}{n} \sum_{i=1}^{n} \widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right] \\
=& \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})
\end{aligned}
$$

Lemma 1. *In addition to the assumptions in corollary 1, we also suppose the following conditions hold.*

1. $\boldsymbol{X}^\intercal \boldsymbol{\tau}$ *is bounded away from 0.*
2. $\forall \boldsymbol{a} \in \mathbb{R}^p, \exists \delta > 0$ *,such that*
   (a) $\mathbb{E} \left| \boldsymbol{a}^\intercal \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right|^{2+\delta} < \infty$
   (b) $\left\{ \boldsymbol{a}^\intercal V \left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right] \boldsymbol{a} \right\}^{1+\frac{\delta}{2}} < \infty.$

Then, we have, for any fixed $\boldsymbol{\tau}$ *and* $\boldsymbol{\beta}_{Y1}$*,*

$$
\sqrt{n} \left[\nabla \widehat{\mathbb{E}} Y_n(\boldsymbol{\tau}) - \mathbb{E} \left(\nabla \widehat{\mathbb{E}} Y_n(\boldsymbol{\tau}) \right) \right] \xrightarrow{d} N \left(0, AV \left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right] \right)
$$

The proof of this is similar to the proof of Lemma 1 in the one-stage problem.

Proof. For any $\boldsymbol{a} \in \mathbb{R}^p$, we let $W_{ni} = \boldsymbol{a}^\intercal \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$. For each value of n, $w_{n1}, w_{n2}, \cdots, w_{nn}$ are i.i.d, and functions of the sample size n. This is because that $\boldsymbol{X}_i$ are assumed to be i.i.d., and h is a function of sample size n. Then, we have

$$
\mu_n := \mathbb{E} W_{ni} = \mathbb{E} \left[\boldsymbol{a}^\intercal \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y | \boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right],
$$

and

$$\sigma_n^2 := V(W_{ni}) = \boldsymbol{a}^\intercal V \left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right] \boldsymbol{a}$$

We let $G_{ni} = W_{ni} - \mu_n$, and $T_n = \sum_{i=1}^n G_{ni}$. Also, we let $s_n^2 = V(T_n) = \sum_{i=1}^n V(G_{ni}) = \sum_{i=1}^n \sigma_n^2 = n\sigma_n^2$, where the second equality is because of independence, and the last equality is due to identicalness. Therefore, T_n/s_n has mean 0, and variance 1. If we can show G_{ni} satisfying the Lyapunov condition, then we have

$\frac{T_n}{s_n} \xrightarrow{d} N(0,1)$,

as $n \to \infty$.

Now, we check the Lyapunov condition, that is, [**Lindsay1995**, **Hunter2014**]

$$\exists \delta > 0, \text{ such that } \frac{1}{s_n^{2+\delta}} \sum_{i=1}^n \mathbb{E} \mid G_{n,i} \mid^{2+\delta} \to 0, \text{ as } n \to 0.$$

We define, for any $\boldsymbol{a}$,

$$C_1 \triangleq \mathbb{E} \left| G_{ni} \right|^{2+\delta} = \mathbb{E} \left| W_{ni} - \mu_n \right|^{2+\delta} = \mathbb{E} \left| \boldsymbol{a}^\intercal \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) - \mu_n \right|^{2+\delta},$$

and

$$C_2 \triangleq s_n^{2+\delta} = n^{1+\frac{\delta}{2}} \sigma_n^{2+\delta} = n^{1+\frac{\delta}{2}} \left\{ \boldsymbol{a}^\intercal V \left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right] \boldsymbol{a} \right\}^{1+\frac{\delta}{2}}.$$

Then, we have

$$\begin{aligned} &\frac{1}{s_n^{2+\delta}} \sum_{i=1}^n \mathbb{E} \mid G_{n,i} \mid^{2+\delta} \\ =&\frac{\mathbb{E} \left| \boldsymbol{a}^\intercal \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) - \mu_n \right|^{2+\delta}}{n^{\frac{\delta}{2}} \left\{ \boldsymbol{a}^\intercal V \left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) \right] \boldsymbol{a} \right\}^{1+\frac{\delta}{2}}} \\ =&\frac{C_1}{n^{\frac{\delta}{2}} C_2}. \end{aligned}$$

As long as $\delta > 0$, for finite C_1 and finite C_2, we have $C_1/n^{\frac{\delta}{2}} C_2 \to 0$, as $n \to \infty$. This means that the Lyapunov condition is satisfied, if $\mathbb{E}\,|G_{ni}|^{2+\delta}$ and $s_n^{2+\delta}$ are finite. Then, by Lyapunov Central Limit Theorem, we have

$$\frac{T_n}{s_n} \xrightarrow{d} N(0,1).$$

As this hold for any arbitary non-random vector $\boldsymbol{a} \in \mathbb{R}^p$, we have, by Cramer-Wold Theorem, that

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial \boldsymbol{\tau}}\int y\, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i}=\boldsymbol{h}_{1,i}) - \mathbb{E}\left\{\frac{\partial}{\partial \boldsymbol{\tau}}\int y\, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i}=\boldsymbol{h}_{1,i})\right\}\right] \xrightarrow{d} N\left(0, V\left[\frac{\partial}{\partial \boldsymbol{\tau}}\int y\, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i}=\boldsymbol{h}_{1,i})\right]\right),$$

as $n \to \infty$. We denote $\boldsymbol{L}_{ni} = \frac{\partial}{\partial \boldsymbol{\tau}}\int y\, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i}=\boldsymbol{h}_{1,i})$, then this is written as

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{L}_{ni} - \mathbb{E}\boldsymbol{L}_{n1}\right] \xrightarrow{d} N\left(0, V\left[\boldsymbol{L}_{n1}\right]\right).$$

Then, we have

$$\frac{1/n\sum_{i=1}^{n}\boldsymbol{L}_{ni} - \mathbb{E}\boldsymbol{L}_{n1}}{[V(\boldsymbol{L}_{n1})/n]^{1/2}}\,\frac{[V(\boldsymbol{L}_{n1})/n]^{1/2}}{[AV(\boldsymbol{L}_{n1})/n]^{1/2}} \xrightarrow{d} N(0,1).$$

As $n \to \infty$,

$$\frac{V(\boldsymbol{L}_{n1})^{1/2}}{AV(\boldsymbol{L}_{n1})^{1/2}} \to 1,$$

then we have

$$\frac{1/n\sum_{i=1}^{n}\boldsymbol{L}_{ni} - \mathbb{E}\boldsymbol{L}_{n1}}{[AV(\boldsymbol{L}_{n1})/n]^{1/2}} \xrightarrow{d} N(0,1),$$

i.e.,

$$\sqrt{n}\left[1/n\sum_{i=1}^{n}\boldsymbol{L}_{ni} - \mathbb{E}\boldsymbol{L}_{n1}\right] \xrightarrow{d} N\left(0, AV(\boldsymbol{L}_{n1})\right).$$

As $\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{L}_{ni} = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\boldsymbol{\tau}}\int y\, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) = \nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau})$, we have

$$\sqrt{n}\left[\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \mathbb{E}\left\{\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau})\right\}\right] \xrightarrow{d} N\left(0, AV\left[\frac{\partial}{\partial\boldsymbol{\tau}}\int y\, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right]\right)$$

■

Assume $\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$, the following lemma shows that the estimations do not effect the limiting distribution obtained above.

Corollary 1. *Suppose all the assumptions in Lemma 3 hold, and $\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$ are consistent estimators of $F_{Y^*(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$. Then, we have*

$$\sqrt{n}\left[\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \nabla\mathbb{E}Y_n^*(\boldsymbol{\tau})\right] \xrightarrow{d} N\left(0, AV\left[\frac{\partial}{\partial\boldsymbol{\tau}}\int y\, dF_{Y^*(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right]\right)$$

Proof. We write

$$\begin{aligned}&\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \nabla\mathbb{E}Y^*(\boldsymbol{\tau})\\ =&\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \mathbb{E}\left\{\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau})\right\} + \mathbb{E}\left\{\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau})\right\} - \nabla\mathbb{E}Y^*(\boldsymbol{\tau}),\end{aligned}$$

where $\mathbb{E}\left\{\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau})\right\} - \nabla\mathbb{E}Y^*(\boldsymbol{\tau}) = \mathbb{E}\left\{\frac{\partial}{\partial\boldsymbol{\tau}}\int y\, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right\} - \mathbb{E}\frac{\partial}{\partial\boldsymbol{\tau}}\int y\, dF_{Y^*x(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i}) = o_p(1)$ is due to the consistency of $\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$ and dominated convergence theorem.
By lemma 1, we have

$$\sqrt{n}\left[\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \mathbb{E}\left\{\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau})\right\}\right] \xrightarrow{d} N\left(0, AV\left[\frac{\partial}{\partial\boldsymbol{\tau}}\int y\, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right]\right)$$

As $\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})$ is consistent, we have

$$\frac{AV\left[\frac{\partial}{\partial\boldsymbol{\tau}}\int y\, d\widehat{F}_{Y_n(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right]}{AV\left[\frac{\partial}{\partial\boldsymbol{\tau}}\int y\, dF_{Y^*(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right]} \xrightarrow{p} 1.$$

Then, we have

$$\sqrt{n}\left[\nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}) - \nabla\mathbb{E}Y^*(\boldsymbol{\tau})\right] \xrightarrow{d} N\left(0, AV\left[\frac{\partial}{\partial\boldsymbol{\tau}}\int y\, dF_{Y_n^*(\boldsymbol{\tau})}(y|\boldsymbol{H}_{1,i} = \boldsymbol{h}_{1,i})\right]\right)$$

$\blacksquare$

Limiting distribution of $\widehat{\boldsymbol{\tau}}_{\kappa,\mu}$

Now, we investigate the limiting distribution of $\widehat{\boldsymbol{\tau}}_{\kappa,\mu}$.

Theorem 1. *Suppose all the assumptions above hold. Then we have, as $n \to \infty$*

$$\sqrt{n}(\widehat{\boldsymbol{\tau}}_{\kappa,\mu} - \boldsymbol{\tau}^*_{\kappa,\mu}) \xrightarrow{d} N\left(\mathbf{0}, \mathbf{\Sigma}^*\right),$$

where $\mathbf{\Sigma}^* = \boldsymbol{D}^{*-1}\boldsymbol{C}^*\boldsymbol{D}^{*-1}$,

$$\boldsymbol{C}^* := AV\left[\frac{\partial}{\partial \boldsymbol{\tau}}\int y\, dF_{Y^*(\boldsymbol{\tau})}(y|\boldsymbol{H}_1 = \boldsymbol{h}_1)\right],$$

and

$$\begin{aligned}\boldsymbol{D}^* :=& \nabla^2\mathbb{E}Y^*(\boldsymbol{\tau}^*_{\kappa,\mu}) - \mu\frac{\nabla^2\mathbb{E}Z^*(\boldsymbol{\tau}^*_{\kappa,\mu})\left[\kappa - \mathbb{E}Z^*(\boldsymbol{\tau}^*_{\kappa,\mu})\right] + \{\nabla\mathbb{E}Z^*(\boldsymbol{\tau}^*_{\kappa,\mu})\}^2}{\left\{\kappa - \mathbb{E}Z^*(\boldsymbol{\tau}^*_{\kappa,\mu})\right\}^2}\\ =& \nabla^2 S^*(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu).\end{aligned}$$

with $\boldsymbol{A}_0'(\boldsymbol{\tau}^*_{\kappa,\mu}, \boldsymbol{\beta}^*_{Y1}) = \mathbb{E}\{2\boldsymbol{X}_1^\intercal\boldsymbol{\beta}^*_{Y1}\delta\left(\boldsymbol{X}^\intercal\boldsymbol{\tau}^*_{\kappa,\mu}\right)\boldsymbol{X}\}$ *and* $\boldsymbol{A}_0''(\boldsymbol{\tau}^*_{\kappa,\mu}, \boldsymbol{\beta}^*_{Y1}) = \mathbb{E}\left[2\boldsymbol{X}_1^\intercal\boldsymbol{\beta}^*_{Y1}\delta'\left(\boldsymbol{X}^\intercal\boldsymbol{\tau}^*_{\kappa,\mu}\right)\boldsymbol{X}\boldsymbol{X}^\intercal\right]$.

Proof. Taylor expansion, for each μ

$$\nabla\widehat{S}(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu) = \nabla\widehat{S}(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) - \nabla^2\widehat{S}(\tilde{\boldsymbol{\tau}}_{\kappa,\mu}, \mu)(\widehat{\boldsymbol{\tau}}_{\kappa,\mu} - \boldsymbol{\tau}^*_{\kappa,\mu}),$$

where $\tilde{\boldsymbol{\tau}}_{\kappa,\mu}$ is a vector in between $\widehat{\boldsymbol{\tau}}_{\kappa,\mu}$ and $\boldsymbol{\tau}^*_{\kappa,\mu}$. As $\widehat{\boldsymbol{\tau}}_{\kappa}(\mu)$ is the maximizer of $\widehat{S}(\boldsymbol{\tau}, \mu)$, it satifies the first order condtion such that $\nabla\widehat{S}(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) = 0$. Then,

$$\begin{aligned}&\sqrt{n}\nabla\widehat{S}(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu) = -\sqrt{n}\nabla^2\widehat{S}(\tilde{\boldsymbol{\tau}}_{\kappa,\mu}, \mu)(\widehat{\boldsymbol{\tau}}_{\kappa,\mu} - \boldsymbol{\tau}^*_{\kappa,\mu})\\ &\nabla\widehat{S}(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu) = \nabla\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}^*_{\kappa,\mu}) - \mu\frac{\nabla\widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})}{\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})} + +\frac{2}{\mu}(\boldsymbol{\tau}^{*\intercal}_{\kappa,\mu}\boldsymbol{\tau}^*_{\kappa,\mu} - 1)\boldsymbol{\tau}^*_{\kappa,\mu}, \text{where } \boldsymbol{\tau}^{*\intercal}_{\kappa,\mu}\boldsymbol{\tau}^*_{\kappa,\mu} - 1 = 0.\\ &\nabla^2\widehat{S}(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu) = \nabla^2\widehat{\mathbb{E}}Y_n(\boldsymbol{\tau}^*_{\kappa,\mu}) - \mu\frac{\nabla^2\widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})\left[\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})\right] + \{\nabla\widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})\}^2}{\left\{\kappa - \widehat{\mathbb{E}}Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})\right\}^2}\end{aligned}$$

Then, we have

$$\nabla \widehat{S}(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu) = \nabla \widehat{\mathbb{E}} Y_n(\boldsymbol{\tau}^*_{\kappa,\mu}) - \mu \frac{\nabla \widehat{\mathbb{E}} Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})}{\kappa - \widehat{\mathbb{E}} Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})},$$

By Lemma 2 , we have the first term on the right hand side as

$$\nabla \widehat{\mathbb{E}} Y_n(\boldsymbol{\tau}) \xrightarrow{d} N\left\{\nabla \mathbb{E} Y^*(\boldsymbol{\tau}), \frac{1}{n} AV\left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y\, dF_{Y^*(\boldsymbol{\tau})}(y|\boldsymbol{H}_1 = \boldsymbol{h}_1)\right]\right\}$$

For the second term on the right hand side, we also have that

$$\mu \frac{\nabla \widehat{\mathbb{E}} Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})}{\kappa - \widehat{\mathbb{E}} Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})} \xrightarrow{p} \mu \frac{\nabla \mathbb{E} Z(\boldsymbol{\tau}^*_{\kappa,\mu})}{\kappa - \mathbb{E} Z(\boldsymbol{\tau}^*_{\kappa,\mu})},,$$

where we assume both $\kappa - \widehat{\mathbb{E}} Z_n(\boldsymbol{\tau}^*_{\kappa,\mu}) > 0$ and $\kappa - \mathbb{E} Z(\boldsymbol{\tau}^*_{\kappa,\mu}) > 0$. This convergence in probablity is due to the consitency of $\widehat{F}_{Z_n(\boldsymbol{\tau})}(z|\boldsymbol{H}_1 = \boldsymbol{h}_1)$ and dominated convergence theorem. Together, by Sluskty's theorem and the stationarity of $\boldsymbol{\tau}^*_{\kappa,\mu}$ of $S^*(\boldsymbol{\tau}, \mu)$, we have

$$\sqrt{n} \nabla \widehat{S}(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu) \xrightarrow{d} N\left(0, \boldsymbol{C}^*\right),$$

where

$$\boldsymbol{C}^* := AV\left[\frac{\partial}{\partial \boldsymbol{\tau}} \int y\, dF_{Y^*(\boldsymbol{\tau})}(y|\boldsymbol{H}_1 = \boldsymbol{h}_1)\right]$$

We have that $\nabla^2 \widehat{S}(\tilde{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) = \nabla^2 \widehat{S}(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu) + o_p(1)$, as $\tilde{\boldsymbol{\tau}}_{\kappa,\mu}$ is in between $\widehat{\boldsymbol{\tau}}_{\kappa,\mu}$ and $\boldsymbol{\tau}^*_{\kappa,\mu}$ and $\widehat{\boldsymbol{\tau}}_{\kappa,\mu} - \boldsymbol{\tau}^*_{\kappa,\mu} =$

$o_p(1)$. Therefore, we have

$$\begin{aligned}
\boldsymbol{D}^* \triangleq & p \lim_{n\to\infty} \nabla^2_{\boldsymbol{\tau}} \widehat{S}_\kappa(\tilde{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) \\
= & p \lim_{n\to\infty} \left[\nabla^2 \widehat{\mathbb{E}} Y_n(\boldsymbol{\tau}^*_{\kappa,\mu}) - \mu \frac{\nabla^2 \widehat{\mathbb{E}} Z_n(\boldsymbol{\tau}^*_{\kappa,\mu}) \left[\kappa - \widehat{\mathbb{E}} Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})\right] + \{\nabla \widehat{\mathbb{E}} Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})\}^2}{\left\{\kappa - \widehat{\mathbb{E}} Z_n(\boldsymbol{\tau}^*_{\kappa,\mu})\right\}^2} \right] \\
= & \nabla^2 \mathbb{E} Y^*(\boldsymbol{\tau}^*_{\kappa,\mu}) - \mu \frac{\nabla^2 \mathbb{E} Z^*(\boldsymbol{\tau}^*_{\kappa,\mu}) \left[\kappa - \mathbb{E} Z^*(\boldsymbol{\tau}^*_{\kappa,\mu})\right] + \{\nabla \mathbb{E} Z^*(\boldsymbol{\tau}^*_{\kappa,\mu})\}^2}{\left\{\kappa - \mathbb{E} Z^*(\boldsymbol{\tau}^*_{\kappa,\mu})\right\}^2} \\
= & \nabla^2 S^*(\boldsymbol{\tau}^*_{\kappa,\mu}, \mu).
\end{aligned}$$

■

We can estimate $\boldsymbol{\Sigma}^*$ by plug in the corresponding estimators stated, and denote the estimator $\widehat{\boldsymbol{\Sigma}}$, that is, $\widehat{\boldsymbol{\Sigma}} = \widehat{\boldsymbol{D}}^{-1} \widehat{\boldsymbol{C}} \widehat{\boldsymbol{D}}^{-1}$,

$$\begin{aligned}
\widehat{\boldsymbol{C}} = & \widehat{V} \left\{ \frac{\partial}{\partial \boldsymbol{\tau}} \int y \, d\widehat{F}_{Y_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu})}(y | \boldsymbol{H}_1 = \boldsymbol{h}_1) \right\} \\
= & \widehat{V} \left[\nabla \widehat{\mathbb{E}} \left\{ Y_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}) \mid \boldsymbol{H}_1 = \boldsymbol{h}_{\mathbf{1}} \right\} \right]
\end{aligned}$$

and

$$\widehat{\boldsymbol{D}} = \nabla^2 \widehat{S}(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}, \mu) = \nabla^2 \widehat{\mathbb{E}} Y_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}) - \mu \frac{\nabla^2 \widehat{\mathbb{E}} Z_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu}) \left[\kappa - \widehat{\mathbb{E}} Z_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu})\right] + \{\nabla \widehat{\mathbb{E}} Z_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu})\}^2}{\left\{\kappa - \widehat{\mathbb{E}} Z_n(\widehat{\boldsymbol{\tau}}_{\kappa,\mu})\right\}^2},$$

Step 1: $\sup_{\boldsymbol{\tau}: \boldsymbol{\tau}_1^\intercal \boldsymbol{\tau}_1 = 1, \boldsymbol{\tau}_2^\intercal \boldsymbol{\tau}_2 = 1} \left| \widehat{S}_\kappa(\boldsymbol{\tau}, \mu) - S^*_\kappa(\boldsymbol{\tau}, \mu) \right| = o_p(1).$

Step 2: conditions for $\widehat{F}_{Y_n}(y; \boldsymbol{\tau}) \to F_Y(y; \boldsymbol{\tau})$, as $n \to \infty$

Step 3: Consistency of $\widehat{\boldsymbol{\tau}}_{\kappa,\mu} \to \boldsymbol{\tau}^*_{\kappa,\mu}$. It can be proven by following similar proof in Lemma 2 at one-stage.

Step 4: Asymptotic Normality of $\widehat{\boldsymbol{\tau}}_{\kappa,\mu}$

Step 5: Projected CI for $\widehat{\mathbb{E}}Y_n(\widehat{\boldsymbol{\tau}}_{n,\kappa})$

Step 6: Bootstrap Monte Carlo Sampling