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Order preservation of solution correspondence to single-parameter generalized variational inequalities on Hilbert lattices

Shu-qin Sun

Abstract. We establish order preservation of solution correspondence provided that the set-valued mapping has bounded order-closed values for each parameter of single-parameter generalized variational inequalities on Hilbert lattices. This work is different from the earlier results which assume that set-valued mapping has compact values. An example is used to show their difference. We also investigate order preservation of solution correspondence on the Tikhonov Regularization method for generalized variational inequality problem. In addition, more details about the connection between the structure of norm and order in Hilbert lattices are listed.

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1. Introduction

Given a real Banach space X with dual X^*, let K be a nonempty closed and convex subset of X and $\Gamma : K \to 2^{X^*}$ a set-valued mapping. We consider the problem of finding $x \in K$ and $x^* \in \Gamma(x)$, such that

$$\langle x^*, y - x\rangle \geq 0, \text{ for all } y \in K.$$

This problem, denoted by $\mathrm{GVI}(K, \Gamma)$, is called generalized variational inequality problem. If there exists at least one solution to this problem, we say that $\mathrm{GVI}(K, \Gamma)$ is solvable and the solution set is denoted by $S(K, \Gamma)$. If Γ is a single-valued mapping, then this problem is reduced to variational inequality problem and denoted by $\mathrm{VI}(K, \Gamma)$.

As we know, by Ky Fan Minimax Theorem, the existence of solution of $\mathrm{GVI}(K, \Gamma)$ is ensured if K is compact and Γ is upper semicontinuous with

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nonempty compact convex values. By KKM Theorem, if X is reflexive, K is bounded, Γ is quasi-monotone and upper hemicontinuous with nonempty weakly compact convex values, and $\mathrm{GVI}(K,\Gamma)$ has a solution. Without compactness of K, a coercivity condition is usually needed.

Recently, there is a new way to deal with solvability of generalized variational inequalities in Hilbert spaces requiring that Γ be neither hemicontinuous nor monotone. This way is based on partial order theory in Hilbert spaces; see [1–4].

Nishimura and Ok [1] consider the solvability of generalized variational inequalities on separable Hilbert lattice, assuming that Γ has compact values, such that $I-\lambda\Gamma$ (I is the identity operator on K) is upper order-preserving for some function $\lambda : X \to R_{++}$. They even establish the existence of the so-called order-maximal solution. Li [2] prove the existence of order-optimum solution of generalized variational inequalities on Hilbert lattices when $I-\lambda\Gamma$ is order-preserving with bounded order-closed values for some function $\lambda : X \to R_{++}$. Under the same assumption on $J_X-\lambda\Gamma$, Li and Ok [3] explore the optimal solutions of generalized variational inequalities on reflexive, smooth, strictly convex Banach lattices, where $J_X : X \to 2^{X^*}$ is normalized duality map defined by $J_X(x) = \{j(x) \in X^* : \langle j(x), x\rangle = \|j(x)\|\|x\| = \|x\|^2 = \|j(x)\|^2\}$, and J_X is equal to the identity operator on X when X is a Hilbert space, see [3].

Among the above results, Li and Ok [3] have extended the results of Li [2] from Hilbert lattice to Banach lattice. However, the assumptions on $I-\lambda\Gamma$ in Li [2] are different from Nishimura and Ok [1] as we give an example to show that a set-valued map having compact values is not followed by $I-\lambda\Gamma$ having upper bounded order-closed values.

Besides $\mathrm{GVI}(K,\Gamma)$ itself, many known results focus on parametric generalized variational inequality problem, see [1,4–7], including boundedness, convexity, and, especially, the dependence of solutions on parameter.

Let $(X,\succeq)$ be a Banach lattice, $(\Theta,\succeq_\Theta)$ a poset, and K a nonempty bounded closed and convex set of X. Let $\Gamma : K\times\Theta \to 2^X$ be a set-valued mapping. Then, the parametric generalized variational inequalities for each parameter θ is to find $x^* \in K$, such that x^* is a solution to $\mathrm{GVI}(K,\Gamma(\cdot,\theta))$. Define the solution correspondence $\Lambda : \Theta \to 2^K$ by

$$\Lambda(\theta) := \{x^* \text{ is a solution to } \mathrm{GVI}(K,\Gamma(\cdot,\theta))\}.$$

As a set-valued mapping, we say that Λ is order preservation if for every $\theta,\sigma \in \Theta$, $\theta \succeq \sigma$ implies that for every $b \in \Gamma(\sigma)$, there is an $a \in \Gamma(\theta)$, such that $a \succeq b$, and meanwhile, for every $c \in \Gamma(\theta)$, there is a $d \in \Gamma(\sigma)$, such that $c \succeq d$.

Applying partial order theory, Theorem 3.4 in [1] proves that solution correspondence is order-preserving.

Under the assumption that Γ has compact values and $I-\lambda\Gamma(\cdot,\theta)$ is upper $\succeq$-preserving for each $\theta \in \Theta$ on separable Hilbert lattice, Theorem 3.4 of Hiroki [1] proves the upper order-preserving of solution correspondence $\mathrm{GVI}(K,\Gamma(\cdot,\theta))$. Moreover, Sect. 4 of Wang and Zhang [4] extends the result from single-parameter generalized variational inequalities on Hilbert lattices to two-parameter generalized variational inequalities on Banach lattices, with not only the set-valued mapping Γ, but also the considered nonempty subset K being perturbed. Still, the assumptions on $\Gamma(\cdot,\theta)$ are that $J_X - \lambda\Gamma(\cdot,\theta)$ is upper $\succeq$-preserving and has compact values for each $\theta \in \Theta$, which is the same as Theorem 3.4 [1]

In this paper, Sect. 4 exactly, we discuss order preservation of solution correspondence to single-parameter generalized variational inequalities on Hilbert lattices, provided that the set-valued mapping $\Gamma(\cdot,\theta)$ has bounded order-closed values for each $\theta \in \Theta$ rather than Γ is compact-valued and $I - \lambda\Gamma(\cdot,\theta)$ is upper $\succeq$-preserving for each $\theta \in \Theta$. Moreover, we assume that the set-valued mapping Γ is weakly Lipschitz, but not $I - \lambda\Gamma$ is upper $\succeq$-preserving. Therefore, we assume the conditions only on Γ.

As one kind of perturbed generalized variational inequality problem, to the best of our knowledge, Tikhonov regularization method for generalized variational inequality problem (GVI$(K, \Gamma + \varepsilon I)$ for short) has established by He [6]. This problem is just to replace the set-valued mapping Γ of generalized variational inequality problem by $\Gamma + \varepsilon I$, where $\varepsilon > 0$.

He [6] considers this problem by assuming a coercivity condition beyond monotonicity and that parallel some known results for regularization methods for variational inequality problem; see [8–10].

Motivated by the above work of He [6], we establish order preservation of solution correspondence of GVI$(K, \Gamma + \varepsilon I)$ by partial order theory without any monotonicity, hemicontinuity, and coercivity conditions in Sect. 5.

In addition, to get more details about the connection between the structure of norm and the structure of order in Hilbert lattices, we investigate the existence of order-minimal solutions to generalize variational inequalities under the set-valued map with compact values on Hilbert lattices in Sect. 3.

2. Preliminaries

In this section, we recall some definitions and notations, and most of them are derived from [1,11].

A nonempty set M with a partial order $\succeq$ is said to be a poset. Let A be a subset of M. $x \in M$ is called an upper bound of A if $x \succeq A$, that is, $x \succeq y$ for every $y \in A$. $z \in M$ is a lower bound of A, if $A \succeq z$, that is, $y \succeq z$ for every $y \in A$. If there is an upper(a lower) bound of A, then A is called bounded from above(below). If A is bounded from above and below, then A is ordered bounded. For a nonempty subset A of M, if either $x \succeq y$ or $y \succeq x$ holds for each $x, y \in A$, then A is called a $\succeq$-chain in M.

Given any poset $M, \succeq$, if $x \in M$ and $y \succeq x$ does not hold for any $y \in M \setminus \{x\}$, then x is called a $\succeq$-maximal element of M. If $x \in M$ and x is a $\succeq$-upper bound for M, then x is the $\succeq$-maximum in M. (The $\succeq$-minimal and $\succeq$-maximum elements of M are defined similarly.)

A poset $(M, \succeq)$ is called a lattice if any two elements $x, y \in M$ have a least upper bound denoted by $x \vee y = \sup\{x, y\}$ and a greatest lower bound denoted by $x \wedge y = \inf\{x, y\}$.

Similarly, we denote the supremum and the infimum for arbitrary subset $A \subset M$. The least upper (greatest lower) bound of A is denoted by $\vee_M A(\wedge_M A)$. That is

$$\vee_M A = \sup A = \sup\{x : x \in A\} \ (\wedge_M A = \inf A = \inf\{x : x \in A\}).$$

For every $x \in M$, let

$$x^+ = x \vee 0, x^- = (-x) \vee 0, \text{ and } |x| = x \vee -x$$

be the positive part, the negative part, and the absolute value of x, respectively.

In what follows, $(X, \succeq)$ is a Hilbert lattice, that is, X is a Hilbert space with inner product $\langle , \rangle$, and the induced norm $\| .\|$, $\succeq$ is a partial order on X, such that $(X, \succeq)$ is a poset and has the following properties:

(i) $(X, \succeq)$ is a lattice;
(ii) the norm on X is compatible with the partial order, in the sense that $| x| \succeq | y |$ implies $\| x\| \geq \| y\|$ for every $x, y \in X$;
(iii) the linear structure on X is compatible with the partial order, that is, $\alpha I + z$ is a $\succeq$-preserving self-map on X for every $z \in X$ and real number $\alpha > 0$ where I is the identical mapping on X.

For any $x \in X$, we denote $x^\uparrow = \{y \in X : y \succeq x\}$, $x^\downarrow = \{y \in X : x \succeq y\}$. In turn, for any nonempty subset K of X, $K^\uparrow = \cup\{x^\uparrow : x \in K\}$, $K^\downarrow = \cup\{x^\downarrow : x \in K\}$. A nonempty subset K of $(X, \succeq)$ which contains $x \vee y$ and $x \wedge y$ for every $x, y \in K$ is said to be a $\succeq$-sublattice of $(X, \succeq)$. K is said to be a subcomplete $\succeq$-sublattice of $(X, \succeq)$, if for any nonempty subset B of K, $\vee_X B$, and $\wedge_X B \in K$. By the continuity of the lattice operations $\vee$ and $\wedge$, the positive cone $X_+ := 0^\uparrow$ is a closed and convex cone of X. From Lemma 2.2 in [1], we get that every $\succeq$-bounded, closed, and convex $\succeq$-sublattice of $(X, \succeq)$ is subcomplete whenever X is separable.

Order interval $[x, y]$ of $(X, \succeq)$ with $x, y \in X$, is the set $\{z \in X \mid y \succeq z \succeq x\}$, and every $\succeq$-bounded set of $(X, \succeq)$ is contained in some order interval of $(X, \succeq)$; see [11].

Let K be a nonempty subset of $(X, \succeq)$. We say a set-valued map $\Gamma : K \to 2^X \backslash \{\emptyset\}$ is upper(lower) order-preserving if $x \succeq y$ implies that for every $b \in \Gamma(y)$($a \in \Gamma(x)$), there is an $a \in \Gamma(x)$(a $b \in \Gamma(y)$), such that $a \succeq b$. Γ is order-preserving if it is both upper and lower order-preserving. Γ is called upper(lower) order-reserving if $x \succeq y$ implies that for every $b \in \Gamma(y)$(a$\in \Gamma(x)$), there is an $a \in \Gamma(x)$(a $b \in \Gamma(y)$), such that $b \succeq a$. And obviously, Γ is order-reserving if it is both upper and lower order-reserving. Furthermore, if Γ is upper(lower) order-preserving, the single-valued mapping $\vee_X \Gamma(\wedge_X \Gamma)$ is order-preserving; see [2].

Γ is said to have upper(lower) bounded $\succeq$-closed values if for every $x \in K$, $\vee_X \Gamma(x) \in \Gamma(x)(\wedge_X \Gamma(x) \in \Gamma(x))$. Γ has bounded $\succeq$-closed values if it has both upper and lower bounded $\succeq$-closed values. It can be seen that every single-valued mapping has bounded $\succeq$-closed values.

For further details about above kinds of set-valued mapping, we refer the reader to [1,2,11–13].

Mostly, the problem of GVI(K,Γ) is transformed to a fixed point problem: Let X be a Hilbert space, $\lambda : X \to R_{++}$ any function, C a nonempty closed and convex subset of X, and $\Gamma : C \to 2^X$ any set-valued mapping. Then, x^* is a solution to GVI(C,Γ) if and only if $x^* \in \mathrm{Fix}(\Pi_C \circ (id_C - \lambda\Gamma))$, where $\Pi_C : X \to C$ is a metric projection operator onto C.

In the next, I is always supposed to be an identity operator on a nonempty subset K of $(X, \succeq)$.

3. Order-minimal solutions to generalized variational inequalities

Theorem 3.1 [1] gives the existence of order-maximal solutions to generalized variational inequalities on separable Hilbert lattice. Dually, the existence of order-minimal solutions to generalized variational inequalities on separable Hilbert lattice is studied bellow.

Theorem 3.1. *Let $(X, \succeq)$ be a separable Hilbert lattice and K a weakly compact and convex $\succeq$-sublattice of X. Let $\Gamma : K \to 2^X$ be of compact values, such that $I - \lambda\Gamma$ is lower $\succeq$-preserving for some function $\lambda : X \to R_{++}$. Then, there is a $\succeq$-minimal solution to* GVI(K,Γ).

Proof. Define set-valued mappings $\Psi : K \to 2^X \backslash \{\emptyset\}$ and $f : K \to 2^K \backslash \{\emptyset\}$ by

$$\Psi := I - \lambda\Gamma \text{ and } f := \Pi_K \circ \Psi.$$

We claim that f is lower $\succeq$-preserving. Indeed, take any $x, y \in K$ with $x \succeq y$. For arbitrary $a \in f(x)$, there exists an $a^{'} \in \Psi(x)$, such that $a = \Pi_K(a^{'})$. As Ψ is lower $\succeq$-preserving, there is a $b^{'} \in \Psi(y)$, such that $a^{'} \succeq b^{'}$. Let $b = \Pi_K(b^{'})$. Then, $b \in f(y)$ and $a \succeq b$ as Π_K is $\succeq$-preserving by Lemma 2.4 in [1].

Consider the following set:

$$Y := \{x \in K : x \in f(x)^{\uparrow}\}.$$

Obviously, $S(K,\Gamma) \subseteq Y$. By Corollary 2.3 [1], K is a subcomplete $\succeq$-sublattice of X. In particular, $\vee_X K \in K$ and $\vee_X K \in Y$. Let S be any chain in Y. For any $x \in S \subseteq Y$, there exists an $w(x) \in f(x)$, such that $x \succeq w(x)$. As $x \succeq \vee_X S$ for every $x \in S$ and f is lower $\succeq$-preserving, we can get for each $x \in S$, there exists a $u(x) \in f(\wedge_X S)$, such that $x \succeq w(x) \succeq u(x)$, thus

$$S \subseteq f(\wedge_X S)^{\uparrow}.$$

Let

$$\mathcal{F} := \{x^{\downarrow} \cap f(\wedge_X S) : x \in S\}.$$

Then, $\mathcal{F}$ has the finite intersection property. Indeed, take T as a nonempty finite subset of S, then T is also a $\succeq$-chain. There exists an $\bar{x}$, such that $T \succeq \bar{x}$. Since $\bar{x} \in f(\wedge_X S)^{\uparrow}$, we have $y \in f(\wedge_X S)$, such that $\bar{x} \succeq y$. By transitivity of $\succeq$,

$$y \in \cap\{f(\wedge_X S) \cap x^{\downarrow}; \quad x \in T\}.$$

It follows that $\mathcal{F}$ has the finite intersection property.

By virtue of $x^{\downarrow} = x - X_+$, we get that $x^{\downarrow}$ is closed as X_+ is closed. Due to Γ is of compact values and Π_K is continues, it follows that f is of compact values either. Therefore, $f(\wedge_X S) \cap x^{\downarrow}$ is compact for every $x \in S$. As a result,

$$\cap\{x^{\downarrow} \cap f(\wedge_X S) : x \in S\} \neq \emptyset.$$

Take

$$w := \cap\{x^{\downarrow} \cap f(\wedge_X S) : x \in S\}.$$

Obviously, $\wedge_X S \succeq w$ and $w \in f(\wedge_X S)$. That means $\wedge_X S \in f(\wedge_X S)\uparrow$, $\wedge_X S \in Y$. Consequently, every $\succeq$-infimum (in X) of every $\succeq$-chain in Y belongs to Y. Apply Zorn's Lemma to the poset $(Y, \succeq)$, we find a $\succeq$-minimal element x^* in Y. By definition of Y, there exists a $y^* \in f(x^*)$, such that $x^* \succeq y^*$. Furthermore, as f is lower $\succeq$-preserving and $y^* \in f(x^*)$, there exists a $z^* \in f(y^*)$, such that $y^* \succeq z^*$, $y^* \in Y$. It concludes that $y^* \succeq x^*$. Therefore, $x^* = y^*$, and hence, $x^* \in f(x^*)$. As a fixed point of f, $x^* \in S(K, \Gamma)$. Due to $S(K, \Gamma) \subseteq Y$ and x^* is a $\succeq$-minimal element in Y, we complete the proof. $\square$

Combing this theorem with Theorem 3.1 in [1], we get the following corollary.

Corollary 3.2. *Let $(X, \succeq)$ be a separable Hilbert lattice, K a weakly compact and convex $\succeq$-sublattice of X. Let $\Gamma : K \to X$ be a single-valued mapping, such that $I - \lambda\Gamma$ is $\succeq$-preserving for some function $\lambda : X \to R_{++}$. Then, there are both $\succeq$-maximal and $\succeq$-minimal solutions to* GVI(K, Γ).

Remark 3.3. Let $(X, \succeq)$ be a Hilbert lattice, and $[0, x_0]$ be any order interval with $x_0 \in X$ and $x_0 \succeq 0$, then the set $[0, x_0]$ is a subcomplete $\succeq$-sublattice of X. This implies that when K is an order interval, the assumption of separability of $(X, \succeq)$ in above theorem can be delete.

Remark 3.4. As for the set $Y := \{x \in K : x \in f(x)^{\uparrow}\}$ in the proof of above theorem, we can get if $S(K, \Gamma) \neq Y$, then for every $x \in Y$, there exists a $w \in Y$, such that $w \neq x$ and $x \succeq w$. To find this, we can take any $x \in Y$ but $x \notin S(K, \Gamma)$, there exists an $w \in f(x)$ but $w \neq x$, such that $x \succeq w$. Since f is lower $\succeq$-preserving, there exists a $b_w \in f(w)$, such that $w \succeq b_w$. It equals $w \in Y$.

For example, take $X = R^2$ and define the partial order $\succeq$ as follows:

$$(x_1, y_1) \succeq (x_2, y_2), \text{ iff } x_1 \succeq x_2 \text{ and } x_2 \succeq y_2.$$

Let

$$K = \bar{\text{co}}\{(0, 0), (1, 2), (2, 1), (2, 2)\}$$

$$\Gamma(x, y) = \{(x, -x), (y, -y)\}$$

$$\lambda = 1.$$

We can observe that K is a compact and convex closed $\succeq$-sublattice of X, and Γ is a set-valued mapping with compact values. Furthermore,

$$(I-\lambda\Gamma)(x,y)=\{(0,x+y),(x+y,0)\},\ \text{for every } (x,y)\in K.$$

And easily, $I-\lambda\Gamma$ is lower $\succeq$-preserving. In addition, the mapping $f := \Pi_K\circ(I-\lambda\Gamma)$ has the set of fixed points $\{(0,0),(1,2),(2,1)\}$. Without generality, we choose $(2,2)\in K$. For $f((2,2))=\{(2,1),(1,2)\}$, we have $(2,2)\in Y$ and $(2,2)\succeq(2,1)\in Y$, but $(2,2)\notin S(K,\Gamma)$.

Remark 3.5. Consider the relationship of a subset in Hilbert lattice between norm bounded and $\succeq$-bounded.

It is easily to check that a subset in Hilbert lattice which is norm bounded is certainly $\succeq$-bounded by the compatibility not only between the linear structure and the partial order, but also between the norm and the partial order. However, the converse is not always true. For example, let (S,Σ,μ) be an arbitrary measure space. Consider the space $X = L^2(S,\Sigma,\mu)$, equipped with the norm $\| f\| = \{\int| f(t) |^2\mathrm{d}\mu(t)\}^{1/2}$, and the positive cone of functions in X that are non-negative on S μ-almost everywhere. Then, $L^2(S,\Sigma,\mu)$ is a Dedekind complete Banach lattice; however, the unite ball $B=\{f\in X; \| f \|\le 1\}$ is norm bounded but not $\succeq$-bounded.

4. Order preservation of solution correspondence to single-parameter generalized variational inequalities

This section is devoted to the study of order preservation of solution correspondence to $\mathrm{GVI}(K,\Gamma(\cdot,\theta))$ if $\Gamma(\cdot,\theta)$ has bounded order-closed values for each $\theta\in\Theta$.

Definition 4.1. Let K be a nonempty subset of $(X,\succeq)$, we say that a set-valued map $\Gamma: K\to 2^X$ is upper weakly $\succeq$-Lipschitz if there is an $\alpha> 0$, such that for arbitrary $b\in F(y)$, there is an $a\in F(x)$, such that $\alpha(x-y)\succeq a-b$ for every $x,y\in K$ with $x\succeq y$. Γ is called lower weakly $\succeq$-Lipschitz if there is a $\alpha\geq 0$, such that for arbitrary $a\in F(x)$, there is a $b\in F(y)$, such that $\alpha(x-y)\succeq a-b$ for every $x,y\in K$ with $x\succeq y$. Furthermore, Γ is weakly $\succeq$-Lipschitz if it is both upper and lower weakly $\succeq$-Lipschitz. The real constant α is called Lipschitz constant.

Obviously, the identity operator in a Hilbert lattice is weakly $\succeq$-Lipschitz and the Lipschitz constant is 1.

Now, we give an example concerning when Γ is a set-valued mapping.

Example 4.2. Let $X=R^2$ and define the partial order $\succeq$ as follows:

$$(x_1,y_1)\succeq(x_2,y_2),\ \text{iff } x_1\succeq x_2 \text{ and } y_1\succeq y_2.$$

Let K be a nonempty subset of X and

$$\Gamma(x,y)=\{(x,0),(0,y)\}.$$

Then, Γ is weakly $\succeq$-Lipschitz and of Lipschitz constant 1.

Remark 4.3. When Γ is a single-valued mapping and K a closed convex cone of $(X, \succeq)$, then Γ is weakly $\succeq$-Lipschitz which is introduced by [14] is equivalent to Γ is upper or lower weakly $\succeq$-Lipschitz defined here.

Lemma 4.4. *If Γ is upper weakly $\succeq$-Lipschitz and the Lipschitz constant is $1/\lambda$. That is, for every $x, y \in K$ with $x \succeq y$, for arbitrary $b \in F(y)$, there is an $a \in F(x)$, such that $(x-y)/\lambda \succeq a - b$. Then, $I - \lambda\Gamma$ is upper $\succeq$-preserving. In addition, the converse is true.*

Proof. Take any $x, y \in K$ with $x \succeq y$.

For arbitrary $b \in (I - \lambda\Gamma)(y)$, there exists a $b' \in \Gamma(y)$, such that $b := y - \lambda b'$, $b' = (y-b)/\lambda$. Since Γ is upper weakly $\succeq$-Lipschitz and the Lipschitz constant is $1/\lambda$, then $(x-y)/\lambda \succeq a' - b'$ for an $a' \in \Gamma(x)$. Therefore, $x - \lambda a' \succeq b$. Let $a := x - \lambda a'$, then $a \in (I - \lambda\Gamma)(x)$ and

$$a \succeq b.$$

For the converse, we take arbitrary $d' \in \Gamma(y)$, then $d := y - \lambda d' \in (I - \lambda\Gamma)(y)$. As $I - \lambda\Gamma$ is upper $\succeq$-preserving, there exists a $c' \in \Gamma(x)$ with $c := x - \lambda c' \in (I - \lambda\Gamma)(x)$, such that $c \succeq d$. That is $x - \lambda c' \succeq y - \lambda d'$ which equals $x - y \succeq \lambda(c' - d')$, and then

$$1/\lambda(x-y) \succeq c' - d'.$$

Corollary 4.5. *Γ is lower weakly $\succeq$-Lipschitz and the Lipschitz constant is $1/\lambda$ iff $I - \lambda\Gamma$ is lower $\succeq$-preserving.*

Lemma 4.6. *Suppose X is a complete Riesz space, K a nonempty subset of X. Let $\Gamma : K \to 2^X$ be a set-valued mapping, $\lambda : K \to R_{++}$ any function. Then, $\vee_X(I - \lambda\Gamma) = I - \lambda\wedge_X\Gamma$.*

Proof. Aim to prove $\vee_X(I - \lambda\Gamma)(x) \succeq (I - \lambda\wedge_X\Gamma)(x)$ and $(I - \lambda\wedge_X\Gamma)(x) \succeq \vee_X(I - \lambda\Gamma)(x)$ for arbitrary $x \in K$.

Let $y := \vee_X(I - \lambda\Gamma)(x)$. Then, $y \succeq (I - \lambda\Gamma)(x)$, i.e., $y - x \succeq -\lambda\wedge_X\Gamma(x)$, and hence, $\vee_X(I - \lambda\Gamma)(x) \succeq (I - \lambda\wedge_X\Gamma)(x)$.

Let $y' := \wedge_X\Gamma(x)$. Then, $\Gamma(x) \succeq y'$. From a serious computation

$$-y' \succeq -\Gamma(x),$$

$$-y' \succeq \vee_X(-\Gamma(x)),$$

$$-\lambda y' \succeq \vee_X(-\lambda\Gamma(x)),$$

$$x - \lambda y' \succeq x + \vee_X(-\lambda\Gamma(x)) = \vee_X(x - \Gamma(x)).$$

We have$(I - \lambda\wedge_X\Gamma)(x) \succeq \vee_X(I - \lambda\Gamma)(x)$.

Corollary 4.7. *Suppose, X is a complete Riesz space and K a nonempty subset of X. Let $\Gamma : K \to 2^X$ be a set-valued mapping, $\lambda : K \to R_{++}$ any function. Then, $\wedge_X(I - \lambda\Gamma) = I - \lambda\vee_X\Gamma$.*

An example is now given to show that $I - \lambda\Gamma$ has upper bounded $\succeq$-closed values does not imply that Γ has compact values. This fact shows that Theorem 4.9 in the next is different from Theorem 3.4 [1].

Example 4.8. Let $(X, \succeq) = (R^2, \succeq)$ be a Hilbert lattice with the coordinate lattice order $\succeq$ on R^2 and let K be a nonempty $\succeq$-bounded closed and convex $\succeq$-sublattice of X. Take $\lambda = 1$ and define Γ as follows:

$$\Gamma(x, y) = \{(u, v) \in R^2 : (u, v) \succeq (x, y)\}, \quad (x, y) \in K.$$

Γ is $\succeq$ preserving and $(I - \lambda\Gamma)(x, y) = \{(u, v) : (0, 0) \succeq (u, v)\}$, which implies that $I - \lambda\Gamma$ has upper bounded $\succeq$ closed values, but Γ and also $I - \lambda\Gamma$ does not have compact values.

Theorem 4.9. *Let $(X, \succeq)$ be a separable Hilbert lattice, $(\Theta, \succeq_\Theta)$ a poset, and K a nonempty bounded closed and convex $\succeq$-sublattice of X. Let $\Gamma : K \times \Theta \to 2^X$ be a set-valued mapping that satisfies the following conditions:*

- (i) *There exists a constant map $\lambda : K \to R_{++}$ with range $\{\lambda\}$, such that $\Gamma(\cdot, \theta)$ is weakly $\succeq$-Lipschitz and the Lipschitz constant is $1/\lambda$ for each $\theta \in \Theta$;*
- (ii) *$\Gamma(x, \cdot)$ is $\succeq$-reserving for each $x \in K$;*
- (iii) *$\Gamma(\cdot, \theta)$ has bounded $\succeq$-closed values for each $\theta \in \Theta$.*

Then, $\mathrm{GVI}(K, \Gamma(\cdot, \theta))$ *is solvable for each $\theta \in \Theta$, and the solution correspondence $\Lambda : \Theta \to 2^K \backslash \{\emptyset\}$, defined by*

$$\Lambda(\theta) := \{x^* \text{ is a solution to } \mathrm{GVI}(K, \Gamma(\cdot, \theta))\}$$

is $\succeq$-preserving.

Proof. Define $f : K \times \Theta \to 2^X \backslash \{\emptyset\}$ by

$$f(x, \theta) := \Pi_K(x - \lambda\Gamma(x, \theta))$$

where $x \in K$ and $\theta \in \Theta$. In view of condition (i) and Lemma 4.4, $I - \lambda\Gamma(\cdot, \theta)$ is upper $\succeq$-preserving for each $\theta \in \Theta$. This and the $\succeq$-preservation of Π_K imply that $f(\cdot, \theta)$ is upper $\succeq$-preserving for each $\theta \in \Theta$.

In addition, $f(x, \cdot)$ is upper $\succeq$-preserving for each $x \in K$. Take $u, v \in \Theta$ with $u \succeq v$. For arbitrary $b \in f(x, v) = \Pi_K(x - \lambda\Gamma(x, v))$, there is a $b' \in \Gamma(x, v)$, such that $b = \Pi_K(x - \lambda b')$. Since $\Gamma(x, \cdot)$ is upper $\succeq$-reserving for each $x \in K$, there exists an $a' \in \Gamma(x, u)$, such that $b' \succeq a'$. Let $a = \Pi_K(x - \lambda a')$, then $a \succeq b$ which means $f(x, \cdot)$ is upper $\succeq$-preserving for each $x \in K$.

As $I - \lambda\Gamma(\cdot, \theta)$ is upper $\succeq$-preserving for each $\theta \in \Theta$, then if we prove that $I - \lambda\Gamma(\cdot, \theta)$ is also with upper bound $\succeq$-closed values, $\mathrm{GVI}(K, \Gamma(\cdot, \theta))$ is solvable for each $\theta \in \Theta$ which is proved by Theorem 3.1 [2].

Indeed, by virtue of condition (iii), $\Gamma(\cdot, \theta)$ has lower bounded $\succeq$-closed values for each $\theta \in \Theta$ which means $\wedge_X\Gamma(x, \theta) \in \Gamma(x, \theta)$ for every $x \in K$. Combing this with Lemma 4.6, we have

$$\vee_X(I - \Gamma(\cdot, \theta))(x) = (I - \wedge_X\Gamma(\cdot, \theta))(x) \in (I - \Gamma(\cdot, \theta))(x)$$

and therefore, $I - \lambda\Gamma(\cdot,\theta)$ has upper bound $\succeq$-closed values for each $\theta \in \Theta$.

Choose any $x_v \in \Lambda(v)$. Define

$$g(x) := K_v \cap f(x,u)$$

where $x \in K_v$ and $K_v := K \cap {x_v}^{\uparrow}$. As $x_v \in K_v$, we have $K_v \neq \emptyset$. Now, we confirm that the set-valued mapping g is well defined because for arbitrary $x \in K_v$, $g(x) \neq \emptyset$. In fact, as $x_v \in f(x_v, v)$, there is a $y \in f(x_v, u)$, such that $y \succeq x_v$, and thus $y \in K_v$. Since $f(\cdot, u)$ is upper $\succeq$-preserving, then for any $x \in K_v$, there is a $z \in f(x,u)$, such that $z \succeq y$, and hence, $z \in K_v \cap f(x,u)$. Therefore, $g(x) \neq \emptyset$ for every $x \in K_v$.

Next, we will show that g is upper $\succeq$-preserving and has upper bounded $\succeq$-closed values.

Take $x_1, x_2 \in K_v$ with $x_1 \succeq x_2$. As $f(\cdot, u)$ is upper $\succeq$-preserving, there exists an $a \in g(x_1)$, such that $a \succeq b$ for arbitrary $b \in g(x_2)$, which means that g is upper $\succeq$-preserving.

As for g has upper bounded $\succeq$-closed values, that is $\vee_X g(x) = \vee_X(K_v \cap f(x,u)) \in g(x)$, we can get from the following facts.

Since $K_v \cap f(x,u) \in K$, $K_v \cap f(x,u) \neq \emptyset$ and K is a $\succeq$-sublattice,

$$\vee_X g(x) = \vee_X(K_v \cap f(x,u)) \in K.$$

From $\vee_X(K_v \cap f(x,u)) = \vee_X(K \cap {x_v}^{\uparrow} \cap f(x,u)) \succeq x_v$,

$$\vee_X g(x) = \vee_X(K_v \cap f(x,u)) \in {x_v}^{\uparrow}$$

is obtained. Since ${x_v}^{\uparrow} \cap f(x,u) \neq \emptyset$, $\vee_X(K_v \cap f(x,u)) = \vee_X({x_v}^{\uparrow} \cap f(x,u)) = \vee_X f(x,u)$. And, moreover

$$\begin{aligned} \vee_X f(x,u) &= \vee_X(\Pi_K(x - \lambda\Gamma(x,u))) = \Pi_K \vee_X (x - \lambda\Gamma(x,u)) \\ &= \Pi_K(x - \lambda \wedge_X \Gamma(x,u)) \end{aligned}$$

where the second equality is due to that Π_K is $\succeq$-preserving, and the last one is followed by Lemma 4.6.

Since $\Gamma(\cdot,\theta)$ has lower bounded $\succeq$-closed values for each $\theta \in \Theta$, $\vee_X f(x,u) \in \Pi_K(x - \lambda\Gamma(x,u))$, and then

$$\vee_X g(x) \in f(x,u).$$

As a result,

$$\vee_X g(x) = \vee_X(K_v \cap f(x,u)) \in K_v \cap f(x,u) = g(x)$$

which means that g has upper bounded $\succeq$-closed values.

By Corollary 1.8 [15], there is an $x_u \in K$, such that $x_u \in g(x_u) = K_v \cap f(x_u, u)$. That means $x_u \in f(x_u, u)$ which implies that $\mathrm{GVI}(K, \Gamma(\cdot,\theta))$ is solvable for each $\theta \in \Theta$ by Lemma 2.5 [1]. Therefore, $x_u \in K_v \cap \Lambda(u)$ which verifies that $\Lambda(\theta) := \{x^* \text{ is a solution to } \mathrm{GVI}(K, \Gamma(\cdot,\theta))\}$ is upper $\succeq$-preserving.

The lower $\succeq$-preserving of solution correspondence is similarly proved.

Remark 4.10. In contrast with Theorem 3.4 in [1], Theorem 4.9 assumes some conditions only on the set-valued mapping Γ but not on $I - \lambda\Gamma$. Moreover, Theorem 4.9 requires $\Gamma(\cdot,\theta)$ has upper bounded $\succeq$-closed values for each

$\theta \in \Theta$ rather than $\Gamma : K \times \Theta \to 2^X$ is a compact-valued set-valued mapping and $I - \lambda\Gamma$ is upper $\succeq$-preserving for each $\theta \in \Theta$.

5. Order preservation of solution correspondence to Tikhonov regularization for generalized variational inequalities

Order preservation of solution correspondence to Tikhonov regularization for generalized variational inequalities on Hilbert lattice is established in this part.

Although Tikhonov regularization for generalized variational inequality is a specific case of parametric generalized variational inequality, but as it has a special structure $F + \varepsilon I$, many conclusions on it are more particular.

Lemma 5.1. *If Γ is upper weakly $\succeq$-Lipschitz and the Lipschitz constant is $\alpha = 1/\lambda - \varepsilon$ for some $\lambda \in (0, 1]$, then $\Gamma + \varepsilon I$ is upper weakly $\succeq$-Lipschitz for any $\varepsilon \in (0, 1)$ with the Lipschitz constant $1/\lambda$.*

Proof. Take any $x, y \in K$ with $x \succeq y$. For arbitrary $b \in (F + \varepsilon I)(y)$, $b - \varepsilon y \in F(y)$. In view of the definition of upper weakly $\succeq$-Lipschitz, there exists an $a' \in F(x)$, such that $\alpha(x - y) \succeq a' - (b - \varepsilon y)$. It follows that $(x - y)/\lambda = (\alpha + \varepsilon)(x - y) \succeq a' + \varepsilon x - b$. Take $a := a' + \varepsilon x$ and we complete the proof.

Remark 5.2. From the conclusion above, we get that if Γ is upper weakly $\succeq$-Lipschitz and the Lipschitz constant is $\alpha = 1/\lambda - \varepsilon$ for some $\lambda \in (0, 1]$, then $I - \lambda(\Gamma + \varepsilon I)$ is upper $\succeq$-preserving for any $\varepsilon \in (0, 1)$. Similarly, if Γ is lower weakly $\succeq$-Lipschitz and the Lipschitz constant is $\alpha = 1/\lambda - \varepsilon$ for some $\lambda \in (0, 1]$, then $I - \lambda(\Gamma + \varepsilon I)$ is lower $\succeq$-preserving for any $\varepsilon \in (0, 1)$.

Lemma 5.3. *If Γ has lower(upper) bounded $\succeq$ closed values, then $I - \lambda(\Gamma + \varepsilon I)$ has upper(lower) bounded $\succeq$- closed values for any $\lambda \in R_{++}$ and $\varepsilon \in R_{++}$.*

Proof. For arbitrary $x \in K$, $\lambda \in R_{++}$ and $\varepsilon \in R_{++}$,

$\vee_X(I - \lambda(\Gamma + \varepsilon I))(x) = (I - \lambda \wedge_X (\Gamma + \varepsilon I))(x) = x - \lambda \wedge_X (\Gamma + \varepsilon I)(x)$
$= x - \lambda(\wedge_X \Gamma(x) + \varepsilon x) \in (I - \lambda(\Gamma + \varepsilon I))(x)$

where the first equality follows from Lemma 4.6.

This verifies that $I - \lambda(\Gamma + \varepsilon I)$ has upper bounded $\succeq$- closed values for any $\lambda \in R_{++}$ and $\varepsilon \in R_{++}$.

$I - \lambda(\Gamma + \varepsilon I)$ has lower bounded $\succeq$- closed values for any $\lambda \in R_{++}$ and $\varepsilon \in R_{++}$ is similarly proved.

Theorem 5.4. *Let $(X, \succeq)$ be a separable Hilbert lattice, K a nonempty bounded closed and convex $\succeq$-sublattice of X_-. $\Gamma : K \to 2^X$ is weakly $\succeq$-Lipschitz and the Lipschitz constant is $\alpha = 1/\lambda - \varepsilon$ for some $\lambda \in (0, 1]$ and has bounded $\succeq$-closed values for any $\varepsilon \in (0, 1)$. Then,* GVI$(K, \Gamma + \varepsilon I)$ *is solvable for each $\varepsilon \in (0, 1)$, and the solution correspondence $\Lambda : (0, 1) \to 2^K$, defined by*

$$\Lambda(r) := \{x^* \text{ is a solution to } \mathrm{GVI}(K, \Gamma + rI)\}$$

is $\succeq$-preserving.

Proof. Because that $\Gamma : K \to 2^X$ is weakly $\succeq$-Lipschitz and the Lipschitz constant is $\alpha = 1/\lambda - \varepsilon$ for some $\lambda \in (0,1]$ and has bounded $\succeq$-closed values for any $\varepsilon \in (0,1)$, one has $I - \lambda(\Gamma + \varepsilon I)$ is upper $\succeq$-preserving and has upper bounded $\succeq$-closed values by Remark 5.2 and Lemma 5.3. As a result, GVI$(K, \Gamma + \varepsilon I)$ is solvable for each $\varepsilon \in (0,1)$ which is ensured by Theorem 3.1 [2].

Let $f_\varepsilon(x) := \Pi_K(I + \lambda(\Gamma - \varepsilon I))(x)$ for some $\lambda \in (0,1]$, where $x \in K$ and $\varepsilon \in (0,1)$. Then, $f_\varepsilon(x)$ is upper $\succeq$-preserving for each $\varepsilon \in (0,1)$ by virtue of Remark 5.2 and Lemma 2.4 in [1].

Take $u, v \in R_{++}$ with $u \geq v$ and take any $x \in K$. Then

$$f_u(x) = \Pi_K(I - \lambda\Gamma - \lambda u I)(x).$$

For arbitrary $b \in f_v(x) = \Pi_K(I - \lambda\Gamma - \lambda v I)(x)$, there exists a $b^{'} \in \Gamma(x)$, such that

$$b = \Pi_K((1 - \lambda v)x - \lambda b^{'}).$$

Let

$$a := \Pi_K((1 - \lambda u)x - \lambda b^{'}).$$

Then, $a \succeq b$ and $f_.(x)$ is upper $\succeq$-preserving.

Take $x_v \in f_v(x_v)$. Let $g(x) := K_v \cap f_u(x)$ and $K_v := K \cap {x_v}^{\uparrow}$. First, the set-valued mapping g is well defined, because we confirm that for arbitrary $x \in K_v \cap X_-$, $g(x) \neq \emptyset$.

In fact, because $x_v \in f_v(x_v)$ and $u \geq v$, there is a $y \in f_u(x_v)$, such that $y \succeq_X x_v$, and thus $y \in K_v$. As $f_u(\cdot)$ is upper $\succeq$-preserving, for any $x \in K_v$, there exists a $z \in f_u(x)$, such that $z \succeq y$, and thus $z \in K_v \cap f_u(x) = g(x) \neq \emptyset$.

Next, we turn to prove that the set-valued mapping g is upper $\succeq$-preserving and has upper bounded $\succeq$-closed values.

Take $x_1, x_2 \in K_v$ with $x_1 \succeq x_2$. For arbitrary $b \in g(x_2)$, that is $b \in K_v \cap f_u(x_2)$, as f_u is upper $\succeq$-preserving, then there exists an $a \in g(x_1)$, such that $a \succeq b$ which means that g is upper $\succeq$-preserving.

As for g has upper bounded $\succeq$-closed values, that is $\vee_X g(x) = \vee_X(K_v \cap f_u(x)) \in g(x)$, we can get from the following facts.

Since $K_v \cap f_u(x) \in K$, $K_v \cap f_u(x) \neq \emptyset$ and K is a $\succeq$-sublattice,

$$\vee_X g(x) = \vee_X(K_v \cap f_u(x)) \in K.$$

From $\vee_X(K_v \cap f_u(x))$=$\vee_X(K \cap {x_v}^{\uparrow} \cap f_u(x)) \succeq x_v$,

$$\vee_X g(x) = \vee_X(K_v \cap f_u(x)) \in {x_v}^{\uparrow}$$

is obtained. Since ${x_v}^{\uparrow} \cap f_u(x) \neq \emptyset$, $\vee_X(K_v \cap f_u(x))$=$\vee_X({x_v}^{\uparrow} \cap f_u(x))$ =$\vee_X f_u(x)$. And, moreover

$$\begin{aligned}\vee_X f_u(x) &= \vee_X(\Pi_K(I - \lambda(\Gamma + uI))(x)) = \Pi_K \vee_X((I - \lambda(\Gamma + uI))(x)) \\ &= \Pi_K(x - \lambda \wedge_X (\Gamma + uI)(x))\end{aligned}$$

where the second equality is due to Π_K is $\succeq$-preserving, and the third one is followed by Lemma 4.6 and 5.3.

By virtue of Proposition 1.1.2 in [12] and Γ has lower bounded $\succeq$-closed values, we get

$$\vee_X f_u(x) = \Pi_K(x - \lambda(\wedge_X \Gamma(x) + ux)) \in f_u(x)$$

and thus

$$\vee_X g(x) \in f(x, u).$$

Therefore

$$\vee_X g(x) = \vee_X (K_v \cap f(x, u)) \in K_v \cap f(x, u) = g(x)$$

which means that g has upper bounded $\succeq$-closed values.

By Corollary 1.8 [15], there is an $x_u \in K$, such that $x_u \in g(x_u) = K_v \cap f_u(x_u)$. That means $x_u \in f_u(x_u)$ which implies that $\mathrm{GVI}(K, \Gamma + \varepsilon I)$ is solvable for every $\varepsilon \in (0, 1)$ by Lemma 2.5 [1]. From $x_u \in K_v \cap \Lambda(u)$, we have $\Lambda(r) := \{x^* \text{ is a solution to } \mathrm{GVI}(K, \Gamma + rI)\}$ is upper $\succeq$-preserving.

Be analogous to the proof above, the solution correspondence $\Lambda : (0, 1) \to 2^K \backslash \{\emptyset\}$ is lower $\succeq$-preserving is proved.

Remark 5.5. Compare with the assumptions on Theorem 4.9, we restrict our study on $X_- := \{x \in X : 0 \succeq x\}$, the negative cone of $(X, \succeq)$. And, furthermore, we ask for Lipschitz constant which is connected with the parameter ε.

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Shu-qin Sun
College of Mathematics and Software Science
Sichuan Normal University
Chengdu 610066
China
e-mail: sunshuqin8361@163.com

and

College of Mathematics and Information
China West Normal University
Nanchong 637009
China