Optim Lett
DOI 10.1007/s11590-017-1122-0

ORIGINAL PAPER

Exact characterization for subdifferentials of a special optimal value function

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Received: 28 July 2016 / Accepted: 15 February 2017

Abstract For a closed set S and a bounded closed convex set U in a real normed vector space, we give exact subdifferential formulas of an optimal value function $\mathbb{\Gamma}_{S|U}$ whose definition is based on the Minkowski function of U. $\mathbb{\Gamma}_{S|U}$ covers distance function and indicator function as special cases. The main contribution is dropping two important assumptions of some main results in the literature.

Keywords Subdifferential · Normal cone · Optimal value function

1 Introduction

Let X be a real normed vector space and U be a bounded closed convex subset of X. The Minkowski function ρ_U of U is defined by

$$\rho_U(u) := \inf\{t > 0 : t^{-1}u \in U\}, \quad \text{for all } u \in X.$$

Let S be a closed subset of X. The function $\mathbb{\Gamma}_{S|U}$ is defined by

$$\mathbb{\Gamma}_{S|U}(x) := \inf_{s\in S} \rho_U(s-x), \quad \text{for all } x \in X. \tag{1.1}$$

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Published online: 21 February 2017

The definitions yield directly the following equality

$$\Gamma_{S|U}(x) = \inf\{t > 0 : S \cap (x + tU) \neq \emptyset\}. \tag{1.2}$$

$\Gamma_{S|U}$ is a minimal time function of a control system with constant dynamics; see [3]. In view of Proposition 2.1 of [7] and (1.2), one has for point $x \notin S$,

$$\Gamma_{S|U}(x) = \inf\{t \geq 0 : S \cap (x + tU) \neq \emptyset\}. \tag{1.3}$$

When U is the closed unit ball, $\rho_U(x) = \|x\|$, and hence $\Gamma_{S|U}(x)$ reduces to the usual distance function

$$\mathbf{d}_S(x) := \inf_{s \in S} \|s - x\|, \quad \text{for all } x \in X.$$

When $U = \{0\}$, $\Gamma_{S|U}$ reduces to the indicator function

$$I_S(x) := \begin{cases} 0, & x \in S \\ +\infty, & x \notin S, \end{cases}$$

which can be seen from (1.2).

Various properties of $\Gamma_{S|U}$ have been studied in the literature. If S is convex, the function $\Gamma_{S|U}$ is convex. However, if S is not convex, $\Gamma_{S|U}$ is not necessarily convex. Li and Ni [9] studied the relationships between the existence of minimizers of the minimization problem in (1.1) and directional derivatives of the function $\Gamma_{S|U}(x)$. De Blasi and Myjak [5] and Li [8] studied the well-posedness of the minimization problem in (1.1).

Observe that $\Gamma_{S|U}$ is an optimal value function. Thibault [13] gave upper estimate of subdifferential of optimal value function for an abstract optimization model

$$m(x) := \inf\{f(x, y) : y \in M(x) \cap A\}, \tag{1.4}$$

where M is a set-valued mapping from Banach space X to Banach space Y, $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$, and A is a subset of Y. Taking $A = S$, $M(x) \equiv Y$, and $f(x, y) = \rho_U(y - x)$, one has

$$m(x) = \Gamma_{S|U}(x).$$

Thus one can apply the main results in [13] to establish subdifferential formula of $\Gamma_{S|U}$. However, this way can only yield upper estimate of subdifferential of $\Gamma_{S|U}$.

Utilizing the special structure of $\Gamma_{S|U}$, one can establish an exact equality of subdifferential of $\Gamma_{S|U}$ in place of upper estimate, and hence improve the subdifferential formula in [13] from upper estimate to exact equality for a special optimal value function $\Gamma_{S|U}$. We review some related work in the following, one can refer to [1,3,6,7,10–12,14] for related discussions.

Assume that the origin is an interior point of U, [3,4] gave the formulas for the Fréchet and the proximal subdifferentials of $\Gamma_{S|U}$ in Hilbert spaces. It is proved that the Fréchet subdifferential of $\Gamma_{S|U}$ at $x \in S$ can be characterized in terms of Fréchet normal cone of S at x and a sublevel set of the support function of U, while the Fréchet subdifferential of $\Gamma_{S|U}$ for points outside S can be characterized in terms of Fréchet normal cone of an enlarged set of S and a level set of the support function of U. Similar results hold for proximal subdifferential.

Removing the key assumption that the origin is an interior point of U, [7] presented the same formulas for the Fréchet and the proximal subdifferentials of $\Gamma_{S|U}$. However, for subdifferential formulas at $x \notin S$, a calmness condition is required in [7], which plays an important role; see Theorems 3.2 and 4.2 therein. It is noted that calmness condition, also called center-Lipschitz condition in [12], is weaker than locally Lipschitz. When the origin is an interior point of U, $\Gamma_{S|U}$ is globally Lipschitz, and hence calmness condition holds naturally. Thus [7] improved the main results in [3,4].

So far, subdifferential formulas at points inside S are complete. This paper considers subdifferential formulas at points outside S. Our results improve known ones by removing both the assumptions that the origin is an interior point of U and the calmness condition, using a new argument.

We organize this paper as follows. Section 2 contains some related concepts and preliminary results. Sections 3 and 4 are devoted to give the formulas for the Fréchet and the proximal subdifferentials of $\Gamma_{S|U}$ at $x \notin S$ respectively.

2 Preliminaries

In this section, we recall some definitions and notations and most of them are derived from [2].

Unless otherwise stated, throughout this paper, X is a real normed vector space with norm denoted by $\|\cdot\|$ and X^* denotes the topological dual of X. The canonical paring $\langle\cdot,\cdot\rangle$ is between X^* and X. Let S be a nonempty closed subset of X and $U \subset X$ a bounded closed convex subset. $M > 0$ satisfies

$$M \geq \sup\{\|u\| : u \in U\}.$$

The support function $\Im_U : X^* \to (-\infty, \infty]$ of U is defined by

$$\Im_U(\xi) := \sup_{u\in U}\langle \xi, u\rangle, \quad \text{for all } \xi \in X^*.$$

Let $\mathbb{B}(x;\delta)$ denote the open ball centered at x with radius $\delta > 0$ and let $\mathbb{B}^\circ(x;\delta) := \mathbb{B}(x;\delta) \setminus \{x\}$. Suppose that $f : X \to (-\infty, \infty]$ is a lower semicontinuous function. For $x \in \operatorname{dom} f := \{x \in X : f(x) < \infty\}$, the proximal subdifferential of f at x is the set

$$\partial^P f(x) := \left\{\xi \in X^* : \liminf_{v\to 0} \frac{f(x+v) - f(x) - \langle \xi, v\rangle}{\|v\|^2} > -\infty\right\}.$$

Equivalently, there exist $\sigma, \delta > 0$ such that

$$f(y) - f(x) \geq \langle \xi, y - x \rangle - \sigma \|y - x\|^2, \text{for all } y \in \mathbb{B}(x; \delta).$$

Moreover, the Fréchet subdifferential of f at x is the set

$$\partial^F f(x) := \left\{ \xi \in X^* : \liminf_{v \to 0} \frac{f(x+v) - f(x) - \langle \xi, v \rangle}{\|v\|} \geq 0 \right\}.$$

The proximal normal cone N_S^P to S at $x \in S$ is the set of all $\xi \in X^*$ for which there exist $\sigma, \delta > 0$ such that

$$\langle \xi, y - x \rangle \leq \sigma \|y - x\|^2, \text{ for all } y \in S \cap \mathbb{B}(x; \delta),$$

and the Fréchet normal cone N_S^F to S at $x \in S$ is the set of all $\xi \in X^*$ for which for arbitrary $\sigma > 0$, there exists $\delta > 0$ such that

$$\langle \xi, y - x \rangle \leq \sigma \|y - x\|, \quad \text{for all } y \in S \cap \mathbb{B}(x; \delta).$$

The proximal normal cone and the Fréchet normal cone of S at x are actually the corresponding subdifferential of the indicator function of S at x; see [2].

Define

$$S(r) := \{x \in X : \mathbb{F}_{S|U}(x) \leq r\}, \quad \text{where } r \geq 0.$$

Lemma 2.1 *Let $v \in U$ and $t \geq 0$. Then*

$$\mathbb{F}_{S|U}(x - tv) \leq \mathbb{F}_{S|U}(x) + t, \quad \text{for all } x \in X. \tag{2.1}$$

Proof Fix any $x \in X$. If $\mathbb{F}_{S|U}(x) = \infty$, the inequality holds obviously. If $\mathbb{F}_{S|U}(x) < \infty$, it has been proved in [12, Lemma 5.2]. □

Lemma 2.2 *Let $\delta > 0$. Suppose $x \notin S$ and $y \in \mathbb{B}(x; \delta/2)$ with $r := \mathbb{F}_{S|U}(x) < \infty$ and $q := \mathbb{F}_{S|U}(y)$. If $q < r$, $v \in U$, and $0 < t < \min\left\{\frac{\delta}{2\|v\|+1}, r - q\right\}$, then*

$$y - tv \in S(r) \cap \mathbb{B}^\circ(x; \delta).$$

Proof Let $z_t := y - tv$. By virtue of (2.1) and the fact of $t < r - q$, we have

$$\mathbb{F}_{S|U}(z_t) \leq \mathbb{F}_{S|U}(y) + t < q + r - q = r, \tag{2.2}$$

and thus $z_t \in S(r)$. By the choice of t,

$$\|z_t - x\| = \|y - tv - x\| \leq \|y - x\| + t\|v\| < \delta,$$

it follows that $z_t \in \mathbb{B}(x; \delta)$. $z_t \neq x$ is a direct consequence of (2.2), as $\mathbb{F}_{S|U}(z_t) < r$ and $\mathbb{F}_{S|U}(x) = r$. □

3 Fréchet subdifferential of the minimal time function

In this section, we characterize the Fréchet subdifferential of $\mathbb{T}_{S|U}$ at points outside S without additional assumptions. Let us firstly establish a lemma.

Lemma 3.1 *Suppose that* $m, n, \sigma > 0$. *Let*

$$\begin{aligned} p &:= \frac{(n+\sigma)(1+mn) - \sqrt{(n+\sigma)^2(1+mn)^2 - 4n\sigma}}{2n}, \\ 0 &< \sigma_0 < \min\left\{\frac{1}{1+mn},\ p\right\}, \end{aligned} \tag{3.1}$$

and let

$$k := \frac{(\sigma_0+1)n}{1-\sigma_0(1+mn)}.$$

Then

$$\sigma_0 n(mk+1) < \sigma \text{ and } k > n.$$

Proof $k > n$ is obvious as $\frac{\sigma_0+1}{1-\sigma_0(1+mn)} > 1$. Now we show

$$\sigma_0 n(mk+1) - \sigma < 0.$$

Observe that

$$\begin{aligned} \sigma_0 n(mk+1) - \sigma &= \sigma_0 n\left(\frac{m(\sigma_0+1)n}{1-\sigma_0(1+mn)} + 1\right) - \sigma \\ &= a\sigma_0{}^2 + b\sigma_0 + c, \end{aligned}$$

where

$$\begin{aligned} a &:= -\frac{n}{1-\sigma_0(1+mn)}, \\ b &:= \frac{(1+mn)(n+\sigma)}{1-\sigma_0(1+mn)}, \\ c &:= -\frac{\sigma}{1-\sigma_0(1+mn)}. \end{aligned}$$

It remains to prove $a\sigma_0{}^2 + b\sigma_0 + c < 0$. By virtue of (3.1), $a < 0, b > 0, c < 0$. Moreover, it can be checked that $b^2 - 4ac > 0$. Since $p = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $0 < \sigma_0 < p$, it follows that $a\sigma_0{}^2 + b\sigma_0 + c < 0$. $\square$

Now we present a Fréchet subdifferential formula for $\mathbb{T}_{S|U}$ at points outside S, which improves main result in [7, Section 4] by removing an important assumption.

Theorem 3.1 *Let $x \notin S$ and $r := \Gamma_{S|U}(x) < \infty$. Then*

$$\partial^F \Gamma_{S|U}(x) = N^F_{S(r)}(x) \cap \{\xi \in X^* : \Im_U(-\xi) = 1\}.$$

Proof Theorem 4.2(a) of [7] shows that the left-hand side is contained in the right-hand side. Now we need only to prove the converse inclusion.

Let $\xi \in N^F_{S(r)}(x) \cap \{\xi \in X^* : \Im_U(-\xi) = 1\}$. Then

$$\xi \neq 0, \tag{3.2}$$

as ξ satisfies $\Im_U(-\xi) = 1$.

For any $\sigma > 0$, let

$$0 < \sigma_0 < \min\left\{\frac{1}{1 + M\|\xi\|},\ p\right\}, \tag{3.3}$$

where

$$p := \frac{(\|\xi\| + \sigma)(1 + M\|\xi\|) - \sqrt{(\|\xi\| + \sigma)^2(1 + M\|\xi\|)^2 - 4\|\xi\|\sigma}}{2\|\xi\|},$$

and let

$$k := \frac{(\sigma_0 + 1)\|\xi\|}{1 - \sigma_0(1 + M\|\xi\|)}.$$

Replacing m and n by M and $\|\xi\|$ respectively, one can apply Lemma 3.1 to obtain that

$$\sigma_0\|\xi\|(Mk + 1) < \sigma \text{ and } k > \|\xi\|. \tag{3.4}$$

Since $\xi \in N^F_{S(r)}(x)$ and $\xi \neq 0$, there exists $\delta > 0$ such that

$$\langle \xi, y - x\rangle < \sigma_0\|\xi\|\|y - x\|, \text{ for every } y \in S(r) \cap \mathbb{B}^\circ(x; \delta). \tag{3.5}$$

Let $\delta_1 := \frac{\delta}{2(1+Mk)}$. In order to prove the conclusion, we only need to verify that

$$\Gamma_{S|U}(y) - \Gamma_{S|U}(x) \geq \langle \xi, y - x\rangle - \sigma\|y - x\|, \text{ for every } y \in \mathbb{B}^\circ(x; \delta_1).$$

Fix any $y \in \mathbb{B}^\circ(x; \delta_1)$ and let $q := \Gamma_{S|U}(y)$. Since $\delta_1 < \delta$, we have

$$y \in \mathbb{B}^\circ(x; \delta). \tag{3.6}$$

In the following, we divide the discussion into three cases.

(i) $\Gamma_{S|U}(y) = r$. Then $y \in S(r)$. It follows from (3.5) that

$$\begin{aligned}\Gamma_{S|U}(y) - \Gamma_{S|U}(x) = 0 &> \langle \xi, y - x\rangle - \sigma_0 \|\xi\| \|y - x\| \\ &\geq \langle \xi, y - x\rangle - \sigma_0 \|\xi\|(1 + Mk)\|y - x\| \\ &> \langle \xi, y - x\rangle - \sigma \|y - x\|,\end{aligned}$$

where the last inequality follows from (3.4).

(ii) $\Gamma_{S|U}(y) > r$. Recall $q = \Gamma_{S|U}(y)$. By the definition of $\Gamma_{S|U}(y)$, for any $\varepsilon \in (0, \frac{\delta}{2M})$, there exists $q_\varepsilon \in [q, q + \varepsilon)$ such that

$$S \cap (y + q_\varepsilon U) \neq \emptyset.$$

Take $u_1 \in U$ satisfying $y + q_\varepsilon u_1 \in S$.

Define $z_1 := y + q_\varepsilon u_1 - r u_1$. Then $z_1 + r u_1 \in S$ and hence

$$\Gamma_{S|U}(z_1) \leq r.$$

If $q - r \leq k\|y - x\|$, then

$$\begin{aligned}\|z_1 - x\| &\leq \|z_1 - y\| + \|y - x\| \\ &= (q_\varepsilon - r)\|u_1\| + \|y - x\| \\ &\leq M(q_\varepsilon - r) + \|y - x\| \\ &\leq M(q - r + \varepsilon) + \|y - x\| \leq (Mk + 1)\|y - x\| + M\varepsilon \\ &< \delta.\end{aligned} \tag{3.7}$$

This verifies that

$$z_1 \in S(r) \cap \mathbb{B}(x; \delta),$$

and hence (3.5) implies

$$\langle \xi, z_1 - x\rangle \leq \sigma_0 \|\xi\| \|z_1 - x\|.$$

Therefore,

$$\begin{aligned}\Gamma_{S|U}(y) - \Gamma_{S|U}(x) - \langle \xi, y - x\rangle &= q - r - \langle \xi, y - z_1\rangle - \langle \xi, z_1 - x\rangle \\ &= q - r - (r - q_\varepsilon)\langle \xi, u_1\rangle - \langle \xi, z_1 - x\rangle \\ &\geq q - r + r - q_\varepsilon - \langle \xi, z_1 - x\rangle \\ &= q - q_\varepsilon - \langle \xi, z_1 - x\rangle \\ &\geq q - q_\varepsilon - \sigma_0 \|\xi\| \|z_1 - x\| \\ &\geq -\varepsilon - \sigma_0 \|\xi\|((Mk + 1)\|y - x\| + M\varepsilon),\end{aligned}$$

where the last inequality follows from (3.7).

As $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0+$ yields that

$$\Gamma_{S|U}(y) - \Gamma_{S|U}(x) - \langle \xi, y - x\rangle \geq -\sigma_0\|\xi\|(Mk+1)\|y - x\| > -\sigma\|y - x\|,$$

where the last inequality follows from (3.4).

If $q - r > k\|y - x\|$, it follows from $k > \|\xi\|$ that

$$\begin{aligned}\Gamma_{S|U}(y) - \Gamma_{S|U}(x) - \langle \xi, y - x\rangle &\geq q - r - \|\xi\|\|y - x\|\\ &> q - r - k\|y - x\|\\ &> 0 \geq -\sigma\|y - x\|.\end{aligned}$$

(iii) $\Gamma_{S|U}(y) < r$. Since $\Im_U(-\xi) = 1$ and $\sigma_0 < 1$, there exists $\overline{v} \in U$ such that

$$1 \geq \langle -\xi, \overline{v}\rangle > 1 - \sigma_0. \tag{3.8}$$

Take $z_t := y - t\overline{v}$ for $t > 0$. We claim that there exists $\widehat{t}$ such that

$$\Gamma_{S|U}(z_{\widehat{t}}) \geq r \text{ and } 0 < \widehat{t} \leq k\|y - x\|. \tag{3.9}$$

Indeed, if $0 < t < \min\left\{\frac{\delta}{2M+1}, r - q\right\}$, Lemma 2.2 implies that

$$z_t \in S(r) \cap \mathbb{B}^\circ(x;\delta),$$

and thus we obtain from (3.5) that

$$\langle \xi, z_t - x\rangle < \sigma_0\|\xi\|\|z_t - x\|. \tag{3.10}$$

On the other hand, noting that

$$\begin{aligned}\lim_{t\to+\infty}\frac{\langle \xi, z_t - x\rangle}{t} &= \langle -\xi, \overline{v}\rangle > 1 - \sigma_0 > \sigma_0\|\xi\|M \geq \sigma_0\|\xi\|\|\overline{v}\|\\ &= \lim_{t\to+\infty}\frac{\sigma_0\|\xi\|\|z_t - x\|}{t},\end{aligned}$$

where the second inequality follows from (3.3), we obtain for large enough t,

$$\langle \xi, z_t - x\rangle > \sigma_0\|\xi\|\|z_t - x\|. \tag{3.11}$$

Combing (3.11) with (3.10) yields that there exists $\widehat{t} > 0$ such that

$$\langle \xi, z_{\widehat{t}} - x\rangle = \sigma_0\|\xi\|\|z_{\widehat{t}} - x\|, \tag{3.12}$$

which, together with (3.10), implies that

$$z_{\widehat{t}} \notin S(r) \cap \mathbb{B}^\circ(x;\delta). \tag{3.13}$$

By (3.8) and (3.12), one has

$$\begin{aligned}-\|\xi\|\|y-x\|+\widehat{t}(1-\sigma_0) &\le \langle \xi, y-x\rangle+\widehat{t}(1-\sigma_0)\le \langle \xi, y-x\rangle+\widehat{t}\langle \xi,-\overline{v}\rangle\\ &= \langle \xi, z_{\widehat{t}}-x\rangle=\sigma_0\|\xi\|\|z_{\widehat{t}}-x\|\le\sigma_0\|\xi\|\left(\|y-x\|+\widehat{t}\|\overline{v}\|\right)\\ &\le \sigma_0\|\xi\|\|y-x\|+\sigma_0\|\xi\|\widehat{t}M,\end{aligned}$$

and hence

$$0<\widehat{t}\le\frac{(\sigma_0+1)\|\xi\|}{1-\sigma_0(1+M\|\xi\|)}\cdot\|y-x\|\equiv k\|y-x\|.$$

Since $y\in\mathbb{B}^\circ(x;\delta_1)$, that is, $\|y-x\|<\frac{\delta}{2(Mk+1)}$, then

$$\|z_{\widehat{t}}-x\|\le\|y-x\|+\widehat{t}M\le(kM+1)\|y-x\|<\delta/2<\delta.$$

and thus by virtue of (3.13),

$$z_{\widehat{t}}\notin S(r)\text{ or }z_{\widehat{t}}=x,$$

which implies $\Gamma_{S|U}(z_{\widehat{t}})\ge r$. We have therefore established the claim (3.9).

By Lemma 2.1, we have

$$\Gamma_{S|U}(z_{\widehat{t}})-\Gamma_{S|U}(y)=\Gamma_{S|U}(y-\widehat{t}\overline{v})-\Gamma_{S|U}(y)\le\widehat{t}.$$

Hence, it follows from (3.9) that

$$\Gamma_{S|U}(x)-\Gamma_{S|U}(y)=r-\Gamma_{S|U}(y)\le\Gamma_{S|U}(z_{\widehat{t}})-\Gamma_{S|U}(y)\le\widehat{t}\le k\|y-x\|. \tag{3.14}$$

For any $\varepsilon\in(0,r-q)$, the definition of $\Gamma_{S|U}$ implies the existence of $q_\varepsilon\in[q,q+\varepsilon)$ such that

$$S\cap(y+q_\varepsilon U)\ne\emptyset. \tag{3.15}$$

Since $\sup_{u\in U}\langle -\xi,u\rangle=1$, there exists $u_2\in U$ such that

$$\langle -\xi,u_2\rangle>1-\varepsilon. \tag{3.16}$$

Let $z_2:=y+q_\varepsilon u_2-ru_2$. By the convexity of U, $(r-q_\varepsilon)U+q_\varepsilon U\subset rU$, and then

$$\begin{aligned}y+q_\varepsilon U&=z_2-q_\varepsilon u_2+ru_2+q_\varepsilon U\\&\subset z_2+(r-q_\varepsilon)U+q_\varepsilon U\subset z_2+rU.\end{aligned}$$

From (3.15), we obtain $S\cap(z_2+rU)\ne\emptyset$ and hence

$$z_2\in S(r). \tag{3.17}$$

Moreover, (3.14) implies that

$$\begin{aligned}\|z_2 - x\| &\leq \|z_2 - y\| + \|y - x\| \\ &= (r - q_\varepsilon)\|u_2\| + \|y - x\| \\ &< M(r - q) + \|y - x\| \\ &\leq (Mk + 1)\|y - x\| < \delta. \end{aligned} \tag{3.18}$$

Combing (3.17) with (3.18), we have

$$z_2 \in S(r) \cap \mathbb{B}(x; \delta).$$

It follows from (3.5) that

$$\langle \xi, z_2 - x\rangle \leq \sigma_0\|\xi\|\|z_2 - x\|. \tag{3.19}$$

Applying (3.16), (3.18) and (3.19), we obtain

$$\begin{aligned}\mathrm{I\Gamma}_{S|U}(y) - \mathrm{I\Gamma}_{S|U}(x) - \langle \xi, y - x\rangle &= q - r - \langle \xi, y - z_2\rangle - \langle \xi, z_2 - x\rangle \\ &= q - r - \langle \xi, (r - q_\varepsilon)u_2\rangle - \langle \xi, z_2 - x\rangle \\ &\geq q - r + (r - q_\varepsilon)(1 - \varepsilon) - \langle \xi, z_2 - x\rangle \\ &= q - q_\varepsilon - (r - q_\varepsilon)\varepsilon - \langle \xi, z_2 - x\rangle \\ &\geq -\varepsilon(1 + r - q_\varepsilon) - \langle \xi, z_2 - x\rangle \\ &\geq -\varepsilon(1 + r - q_\varepsilon) - \sigma_0\|\xi\|\|z_2 - x\| \\ &\geq -\varepsilon(1 + r - q_\varepsilon) - \sigma_0\|\xi\|(Mk + 1)\|y - x\|. \end{aligned} \tag{3.20}$$

As $\varepsilon > 0$ is arbitrary in (3.20), we have

$$\mathrm{I\Gamma}_{S|U}(y) - \mathrm{I\Gamma}_{S|U}(x) - \langle \xi, y - x\rangle \geq -\sigma_0\|\xi\|(Mk + 1)\|y - x\| > -\sigma\|y - x\|,$$

where the last inequality is due to (3.4). □

4 Proximal subdifferential of the minimal time function

Now, we present a proximal subdifferential formula for $\mathrm{I\Gamma}_{S|U}$ at points outside S, which improves main result in [7, Section 3] by removing calmness assumption.

Theorem 4.1 *Let $x \notin S$ and $r := \mathrm{I\Gamma}_{S|U}(x) < \infty$. Then*

$$\partial^P \mathrm{I\Gamma}_{S|U}(x) = N^P_{S(r)}(x) \cap \{\xi \in X^* : \Im_U(-\xi) = 1\}.$$

Proof Note that Theorem 3.2(a) in [7] implies that

$$\partial^P \mathrm{I\Gamma}_{S|U}(x) \subset N^P_{S(r)}(x) \cap \{\xi \in X^* : \Im_U(-\xi) = 1\}.$$

Now we prove the converse inclusion.

Let $\xi \in N^P_{S(r)}(x) \cap \{\xi \in X^* : \Im_U(-\xi) = 1\}$. It follows from $\Im_U(-\xi) = 1$ that

$$\xi \neq 0, \tag{4.1}$$

which, together with $\xi \in N^P_{S(r)}(x)$, implies that there exist $\sigma,\ \delta > 0$ such that

$$\langle \xi, y - x\rangle \leq \sigma \|\xi\| \|y - x\|^2, \text{ for every } y \in S(r) \cap \mathbb{B}(x; \delta). \tag{4.2}$$

Let $k := 4(1 + \sigma)^2 \|\xi\|$. Then

$$\|\xi\| < k. \tag{4.3}$$

Let

$$\delta_1 := \min\left\{\frac{\delta}{2(1 + Mk)}, 1, \frac{1}{(4M\sigma\|\xi\| + 1)(1 + \sigma)}, \frac{1}{16\sigma\|\xi\|^2 M^2 (1 + \sigma)^3}\right\} \tag{4.4}$$

and

$$\sigma_1 := \sigma\|\xi\|(Mk + 1)^2. \tag{4.5}$$

It suffices to show that

$$\mathbb{\Gamma}_{S|U}(y) - \mathbb{\Gamma}_{S|U}(x) \geq \langle \xi, y - x\rangle - \sigma_1 \|y - x\|^2, \text{ for every } y \in \mathbb{B}(x; \delta_1).$$

Fix an arbitrary $y \in \mathbb{B}(x; \delta_1)$. Since $\delta_1 < \delta$, we have

$$y \in \mathbb{B}^\circ(x; \delta). \tag{4.6}$$

In the following, let $q := \mathbb{\Gamma}_{S|U}(y)$, and we divide the argument into three cases.

(i) $\mathbb{\Gamma}_{S|U}(y) = r$. Then $y \in S(r)$. It follows from (4.6) that

$$\begin{aligned}\mathbb{\Gamma}_{S|U}(y) - \mathbb{\Gamma}_{S|U}(x) = 0 &\geq \langle \xi, y - x\rangle - \sigma\|\xi\|\|y - x\|^2\\ &\geq \langle \xi, y - x\rangle - \sigma_1\|y - x\|^2,\end{aligned}$$

where the first inequality is due to (4.2) and the second inequality is due to (4.5).

(ii) $\mathbb{\Gamma}_{S|U}(y) > r$. If $q - r > \|\xi\|\|y - x\|$, then

$$\begin{aligned}\mathbb{\Gamma}_{S|U}(y) - \mathbb{\Gamma}_{S|U}(x) - \langle \xi, y - x\rangle &\geq \mathbb{\Gamma}_{S|U}(y) - \mathbb{\Gamma}_{S|U}(x) - \|\xi\|\|y - x\|\\ &= q - r - \|\xi\|\|y - x\|\\ &> 0 \geq -\sigma_1\|y - x\|^2.\end{aligned}$$

Now, we assume that $q - r \leq \|\xi\|\|y - x\|$. By the definition of $\mathbb{\Gamma}_{S|U}$, for any $\varepsilon \in (0, \frac{\delta}{2M})$, there exists $q_\varepsilon \in [q, q + \varepsilon)$ such that

$$S \cap (y + q_\varepsilon U) \neq \emptyset.$$

Take $u_1 \in U$ satisfing $y + q_\varepsilon u_1 \in S$.

Set $z_1 := y + q_\varepsilon u_1 - r u_1$. Then $z_1 + r u_1 = y + q_\varepsilon u_1 \in S$ and hence

$$\mathbb{\Gamma}_{S|U}(z_1) \le r.$$

As $q - r \le \|\xi\| \|y - x\|$, one has

$$\begin{aligned} \|z_1 - x\| &\le \|z_1 - y\| + \|y - x\| \\ &= (q_\varepsilon - r)\|u_1\| + \|y - x\| \\ &\le M(q_\varepsilon - r) + \|y - x\| \\ &\le M(q - r + \varepsilon) + \|y - x\| \le (M\|\xi\| + 1)\|y - x\| + M\varepsilon \\ &< \delta. \end{aligned} \tag{4.7}$$

This verifies that

$$z_1 \in S(r) \cap \mathbb{B}(x; \delta),$$

and thus (4.2) implies

$$\langle \xi, z_1 - x \rangle \le \sigma \|\xi\| \|z_1 - x\|^2.$$

Since $u_1 \in U$ and $\Im_U(-\xi) = 1$, it follows that

$$\begin{aligned} \mathbb{\Gamma}_{S|U}(y) - \mathbb{\Gamma}_{S|U}(x) - \langle \xi, y - x \rangle &= q - r - \langle \xi, y - z_1 \rangle - \langle \xi, z_1 - x \rangle \\ &= q - r - (r - q_\varepsilon)\langle \xi, u_1 \rangle - \langle \xi, z_1 - x \rangle \\ &\ge q - r + r - q_\varepsilon - \langle \xi, z_1 - x \rangle \\ &= q - q_\varepsilon - \langle \xi, z_1 - x \rangle \\ &\ge q - q_\varepsilon - \sigma \|\xi\| \|z_1 - x\|^2 \\ &\ge -\varepsilon - \sigma \|\xi\| ((M\|\xi\| + 1)\|y - x\| + M\varepsilon)^2, \end{aligned}$$

where the last inequality is due to (4.7).

Letting $\varepsilon \to 0+$ yields that

$$\mathbb{\Gamma}_{S|U}(y) - \mathbb{\Gamma}_{S|U}(x) - \langle \xi, y - x \rangle \ge -\sigma \|\xi\| (M\|\xi\| + 1)^2 \|y - x\|^2 \ge -\sigma_1 \|y - x\|^2,$$

where the last inequality follows from (4.3) and (4.5).

(iii) $\mathbb{\Gamma}_{S|U}(y) < r$. As $\Im_U(-\xi) = 1$, let $\overline{v} \in U$ be such that

$$1 \ge \langle -\xi, \overline{v} \rangle > \frac{1}{1 + \sigma}. \tag{4.8}$$

Take $z_t := y - t\overline{v}$ for $t > 0$. We claim that there exists $\widehat{t}$ such that

$$\Gamma_{S|U}(z_{\widehat{t}}) > r \text{ and } 0 < \widehat{t} \le k\|y - x\|. \tag{4.9}$$

Now, we find $t > 0$ such that

$$\langle \xi, z_t - x\rangle > \sigma\|\xi\|\|z_t - x\|^2. \tag{4.10}$$

Note that

$$\langle \xi, z_t - x\rangle = \langle \xi, y - x\rangle + t\langle \xi, -\overline{v}\rangle > \langle \xi, y - x\rangle + \frac{t}{1+\sigma}$$

and

$$\sigma\|\xi\|\|z_t - x\|^2 \le \sigma\|\xi\|(\|y - x\| + t\|\overline{v}\|)^2 \le \sigma\|\xi\|(\|y - x\| + tM)^2.$$

In order to verify (4.10), it remains to find $t > 0$ satisfying

$$\sigma||\xi||(||y - x|| + tM)^2 - \langle \xi, y - x\rangle - \frac{t}{1+\sigma} \le at^2 + bt + c \le 0, \tag{4.11}$$

where

$$\begin{aligned} a &:= \sigma M^2\|\xi\|, \\ b &:= 2M\sigma\|\xi\|\|y - x\| - \frac{1}{1+\sigma}, \\ c &:= \sigma\|\xi\|\|y - x\|^2 + \|\xi\|\|y - x\|. \end{aligned}$$

By virtue of (4.1), $a > 0$. It can be checked by (4.1) and (4.6) that $c > 0$. Since $y \in \mathbb{B}(x; \delta_1)$, we deduce from (4.4) that

$$\|y - x\| < \frac{1}{4M(1+\sigma)\sigma\|\xi\|}, \tag{4.12}$$

$$\|y - x\| < \min\left\{1, \ \frac{1}{16\sigma\|\xi\|^2M^2(1+\sigma)^3}\right\}. \tag{4.13}$$

(4.12) implies

$$b < -\frac{1}{2(1+\sigma)},$$

which, together with (4.13), implies

$$\begin{aligned} 4ac &= 4\sigma M^2\|\xi\|(\sigma\|\xi\|\|y - x\|^2 + \|\xi\|\|y - x\|) \\ &= 4\sigma M^2\|\xi\|^2(1+\sigma)\|y - x\| < \frac{1}{4(1+\sigma)^2} < b^2. \end{aligned}$$

Thus

$$\frac{-b-\sqrt{b^2-4ac}}{2a} > 0,$$

as $a > 0$, $b < 0$, and $c > 0$.

By the property of quadratic polynomial with respect to t, if t satisfies

$$\frac{-b-\sqrt{b^2-4ac}}{2a} \leq t \leq \frac{-b+\sqrt{b^2-4ac}}{2a}, \tag{4.14}$$

then (4.11) holds, so does (4.10). In particular, for $\hat{t} := \frac{-b-\sqrt{b^2-4ac}}{2a}$, (4.10) holds. As a consequence of (4.2),

$$z_{\widehat{t}} \notin S(r) \cap \mathbb{B}(x; \delta). \tag{4.15}$$

Note that

$$\begin{aligned} \widehat{t} &= \frac{2c}{-b+\sqrt{b^2-4ac}} \\ &\leq -\frac{2c}{b} \leq 4c(1+\sigma) \\ &\leq 4(1+\sigma)^2\|\xi\|\|y-x\| \\ &= k\|y-x\|. \end{aligned} \tag{4.16}$$

Since $y \in \mathbb{B}^\circ(x; \delta_1)$, it follows from (4.4) that $\|y-x\| < \frac{\delta}{2(Mk+1)}$. Then

$$\|z_{\widehat{t}} - x\| \leq \|y-x\| + \widehat{t}M \leq (1+kM)\|y-x\| \leq \delta,$$

where the second inequality is due to (4.16). Combing this with (4.15), we deduce

$$z_{\widehat{t}} \notin S(r),$$

which implies

$$\Gamma_{S|U}(z_{\widehat{t}}) > r, \tag{4.17}$$

and thus the claim (4.9) is verified.

By Lemma 2.1, we have

$$\Gamma_{S|U}(z_{\widehat{t}}) - \Gamma_{S|U}(y) = \Gamma_{S|U}(y - \widehat{t}\widehat{v}) - \Gamma_{S|U}(y) \leq \widehat{t}.$$

It follows from (4.16) and (4.17) that

$$\Gamma_{S|U}(x) - \Gamma_{S|U}(y) = r - \Gamma_{S|U}(y) \leq \Gamma_{S|U}(z_{\widehat{t}}) - \Gamma_{S|U}(y) \leq \widehat{t} \leq k\|y-x\|. \tag{4.18}$$

Note that $q < r$. For any $\varepsilon \in (0, r-q)$, the definition of $\Gamma_{S|U}$ implies the existence of $q_\varepsilon \in [q, q+\varepsilon)$ such that

$$S \cap (y + q_\varepsilon U) \neq \emptyset. \tag{4.19}$$

Since $\sup_{u\in U}\langle -\xi, u\rangle = 1$, there exists $u_2 \in U$ such that

$$\langle -\xi, u_2\rangle > 1-\varepsilon. \tag{4.20}$$

Let $z_2 := y-(r-q_\varepsilon)u_2$. By the convexity of U, $(r-q_\varepsilon)U+q_\varepsilon U \subset rU$. Therefore,

$$\begin{aligned} y+q_\varepsilon U &= z_2-(r-q_\varepsilon)u_2+q_\varepsilon U \\ &\subset z_2+(r-q_\varepsilon)U+q_\varepsilon U \subset z_2+rU. \end{aligned}$$

From (4.19), we obtain $S\cap(z_2+rU)\neq\emptyset$, and hence $\Gamma_{S|U}(z_2)\le r$, that is,

$$z_2 \in S(r). \tag{4.21}$$

Moreover, (4.18) and (4.4) imply that

$$\begin{aligned} \|z_2-x\| &\le \|z_2-y\|+\|y-x\| \\ &= (r-q_\varepsilon)\|u_2\|+\|y-x\| \\ &< M(r-q)+\|y-x\| \le (Mk+1)\|y-x\| \qquad (4.22)\\ &< \delta. \qquad (4.23) \end{aligned}$$

Therefore, (4.21) and (4.23) imply that

$$z_2 \in S(r)\cap\mathbb{B}(x;\delta).$$

By virtue of (4.2),

$$\langle \xi, z_2-x\rangle \le \sigma\|\xi\|\|z_2-x\|^2. \tag{4.24}$$

It follows that

$$\begin{aligned} \Gamma_{S|U}(y)-\Gamma_{S|U}(x)-\langle\xi, y-x\rangle &= q-r-\langle\xi, y-z_2\rangle-\langle\xi, z_2-x\rangle \\ &= q-r-\langle\xi,(r-q_\varepsilon)u_2\rangle-\langle\xi, z_2-x\rangle \\ &\ge q-r+(r-q_\varepsilon)(1-\varepsilon)-\langle\xi, z_2-x\rangle \\ &= q-q_\varepsilon-(r-q_\varepsilon)\varepsilon-\langle\xi, z_2-x\rangle \\ &\ge -\varepsilon(1+r-q_\varepsilon)-\langle\xi, z_2-x\rangle \\ &\ge -\varepsilon(1+r-q_\varepsilon)-\sigma\|\xi\|\|z_2-x\|^2 \\ &\ge -\varepsilon(1+r-q_\varepsilon)-\sigma\|\xi\|(Mk+1)^2\|y-x\|^2, \end{aligned} \tag{4.25}$$

where the first, the third, and the last inequality are respectively due to (4.20), (4.24), and (4.22).

Letting $\varepsilon\to 0+$ in (4.25), one has

$$\Gamma_{S|U}(y)-\Gamma_{S|U}(x)-\langle\xi, y-x\rangle \ge -\sigma\|\xi\|(Mk+1)^2\|y-x\|^2 = -\sigma_1\|y-x\|^2,$$

where the last equality follows from (4.5). □

Acknowledgements This work is supported by National Natural Science Foundation of China (Grant Nos. 11271274, 11461058), Scientific Research Fund of Sichuan Provincial Education Department (Grant Nos. 11ZB153, 11ZA180) and Scientific Research Fund of Sichuan Minzu College(Grant Nos. 13XYZB011, 12XYZB006).

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