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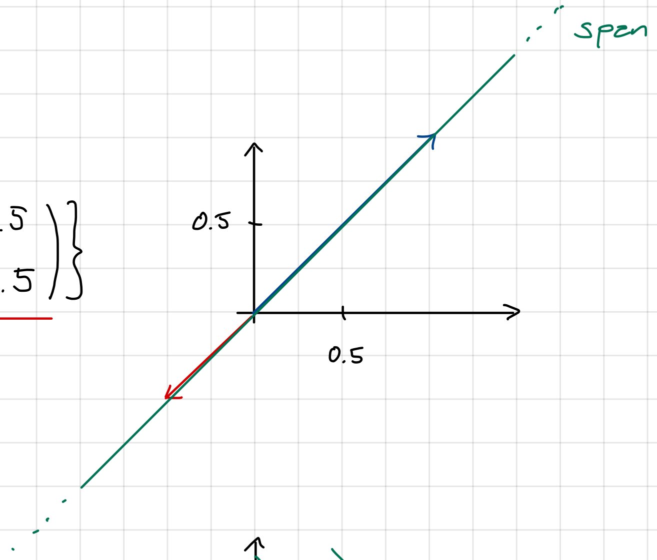
Applied Optimization

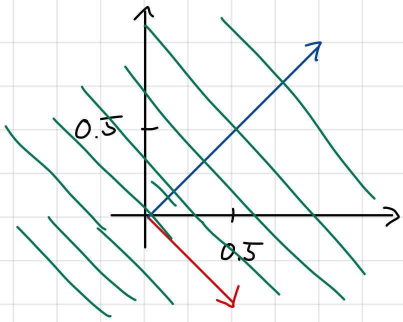
Exercise 1 - Convex Sets

Convex Sets
Example Sets

1. $\text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-0.5\\-0.5\end{pmatrix}\right\}$

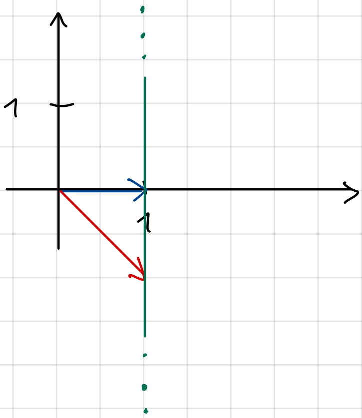


2. $\text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0.5\\-0.5\end{pmatrix}\right\}$

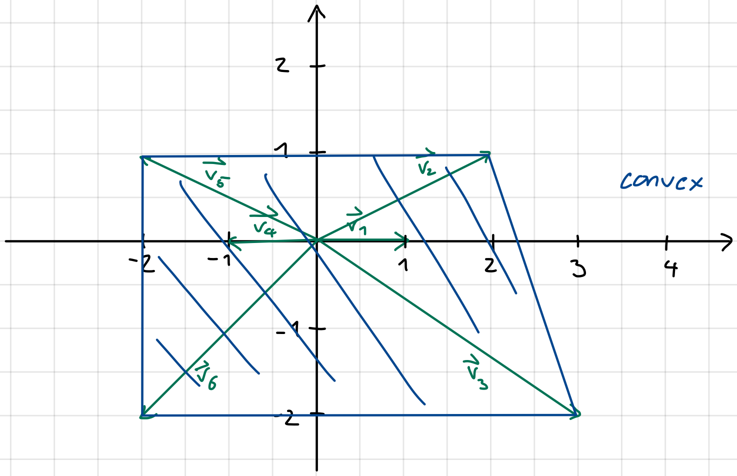


span $= \mathbb{R}^2$

3. $\text{aff}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$



4. $\operatorname{conv}\left\{\underset{\vec{v}_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{\vec{v}_2}{\begin{pmatrix}2\\1\end{pmatrix}}, \underset{\vec{v}_3}{\begin{pmatrix}3\\-2\end{pmatrix}}, \underset{\vec{v}_4}{\begin{pmatrix}-1\\0\end{pmatrix}}, \underset{\vec{v}_5}{\begin{pmatrix}-2\\1\end{pmatrix}}, \underset{\vec{v}_6}{\begin{pmatrix}-2\\-2\end{pmatrix}}\right\}$



convex Set

Convexity

$$\operatorname{conv}(S) = \left\{ \sum_i a_i v_i \;\middle|\; v_i \in S,\ a_i \in \mathbb{R}_+,\ \sum_i a_i = 1 \right\}$$

Induction assumption:

$$w_k = x_1 + \sum_{i=2}^{k} \theta_i (x_i - x_1) \in \text{convex set}$$

Induction step: $k \to k+1$:

$$w_{k+1} = x_1 + \sum_{i=2}^{k+1} \theta_i (x_i - x_1) = \underbrace{x_1 + \sum_{i=2}^{k} \theta_i (x_i - x_1)}_{\text{assumption}} + \underbrace{\theta_{k+1} (x_{k+1} - x_1)}_{\text{same as case for } k=2}$$

$\Rightarrow w_{k+1} \in$ convex set

Linear Equations

$$\{x \mid Ax = b\} \qquad x \in \mathbb{R}^n \quad A \in \mathbb{R}^{m \times n} \quad b \in \mathbb{R}^m$$

$$\begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

x_1, x_2 fullfill $Ax = b$

$$\alpha \cdot x_1 + \beta x_2 \quad , \alpha + \beta = 1$$

$$= \alpha x_1 + (1-\alpha) \cdot x_2$$

$$A(\alpha x_1 + (1-\alpha) x_2) = \alpha \underbrace{A x_1}_{b} + (1-\alpha) \underbrace{A x_2}_{b} = b$$

$$= \alpha b + (1-\alpha) b$$

$$= \cancel{\alpha b} + 1b - \cancel{\alpha b} = b \quad \checkmark \qquad (1)$$

$x_1, \dots, x_k$
$a_1, \dots, a_k \qquad a_1 + \dots + a_k = 1$

$$A(a_1 x_1 + (1 - a_1 - a_3 \dots a_k) x_2 + \dots + (1 - a_1 \dots - a_{k-1}) x_k)$$

$$= a_1 \underbrace{A x_1}_{b} + (1 - a_1 - a_3 \dots a_k) \underbrace{A x_2}_{b} + \dots + (1 - a_1 \dots - a_{k-1}) \underbrace{A x_k}_{b}$$

$$= a_1 b + (1 - a_1 - a_3 \dots a_k) b + \dots + (1 - a_1 \dots - a_{k-1}) b = b$$

all coefficient cancel each other out as seen in (1)

Linear Inequations

$$\{x \mid Ax \preceq b, Cx = d\} \quad x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, C \in \mathbb{R}^{k \times n}, d \in \mathbb{R}^k$$

$x_1, \dots, x_k$

$a_1, \dots, a_k \qquad a_1 + \dots + a_k = 1$

$$A\left(a_1 x_1 + (1 - a_1 - a_3 \dots a_k) x_2 + \dots + (1 - a_1 \dots - a_{k-1}) x_k\right)$$

$$= a_1 A x_1 + (1 - a_1 - a_3 \dots a_k) A x_2 + \dots + (1 - a_1 \dots - a_{k-1}) A x_k$$

$$\preceq a_1 b + (1 - a_1 - a_3 \dots a_k) b + \dots + (1 - a_1 \dots - a_{k-1}) b = b \checkmark$$

all coefficient cancel each other out as seen in (1)

It also satisfies the secound condition ($Cx = d$) as shown in the previous exercise.

2.

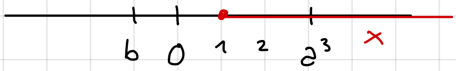
Yes it is an affine set because for the proof that it is a convex set, the condition, that $a_1, \dots, a_k \geq 0$ is not used.

Voronoi description of halfspace

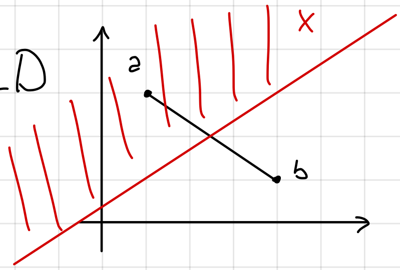
$a, b \in \mathbb{R}^n$ $\quad \{x \mid \|x-a\|^2 \leq \|x-b\|^2\}$

$c^T x \leq d$ $\qquad\qquad$ $c^T x - d \leq 0$

1D



2D



$$x^2 - 2a^T x + a^2 \leq x^2 - 2b^T x + b^2$$

$$a^2 - 2a^T x \leq b^2 - 2b^T x$$

$$2b^T x - 2a^T x \leq b^2 - a^2$$

$$\underbrace{2(b^T - a^T)}_{c^T} x \leq \underbrace{b^2 - a^2}_{d}$$

Convex Illumination problem

minimize $\max h(I_k/I_{des})$,

with $h(I_k/I_{des}) = \max\{I_k/I_{des}, I_{des}/I_k\}$

Is equivalent to:

minimize $\max\left[\log(\max\{I_k/I_{des}, I_{des}/I_k\})\right]$

$=$ minimize $\max\left[\max\{\log(I_k/I_{des}), \log(I_{des}/I_k)\}\right]$

$\boxed{\text{log rule: } \log(a/b) = \log(a) - \log(b)}$

$=$ minimize $\max\left[\max\{\log(I_k)-\log(I_{des}), \underbrace{\log(I_{des})-\log(I_k)}_{-(\log(I_k)-\log(I_{des}))}\}\right]$

$=$ minimize $\max\left|\log(I_k) - \log(I_{des})\right|$

$\Rightarrow$ equal ✓