Exact line search for the convex quadratic function

Consider the convex quadratic function

$$f(x) = \frac{1}{2} x^T Q x + q^T x + c \quad , x \in \mathbb{R}^n \; , Q \text{ symmetric positive definite matrix } Q \in \mathbb{R}^{n \times n}$$

$$, q \in \mathbb{R}^n \; , c \in \mathbb{R}$$

Compute the exact line search parameter for an arbitrary point x in the domain of f given the search direction Δx.

$$t := \arg\min_{s \geq 0} \underbrace{f(x + s\Delta x)}$$

$$\frac{1}{2}(x + s\Delta x)^T Q (x + s\Delta x) + q^T (x + s\Delta x) + c$$

$$y(s) = (x + s\Delta x)$$

$$\nabla_s \frac{1}{2} y(s)^T Q y(s) + q^T y(s) + c$$

$$= \frac{1}{2} \Delta x^T Q y(s) + \frac{1}{2} y(s)^T Q \Delta x + q^T \Delta x$$

$$= \frac{1}{2}\left(\Delta x^T Q (x + s\Delta x) + (x + s\Delta x)^T Q \Delta x\right) + q^T \Delta x$$

$$= \frac{1}{2}\left(\underbrace{\Delta x^T Q x}_{(x^T Q^T \Delta x)^T} + s \Delta x^T Q \Delta x + x^T Q \Delta x + s \Delta x^T Q \Delta x\right) + q^T \Delta x$$

$$(x^T \underset{Q}{\underset{\shortparallel}{Q^T}} \Delta x)^T \in \mathbb{R} \Rightarrow (x^T Q \Delta x)^T = x^T Q \Delta x$$

$$= x^T Q \Delta x + s \Delta x^T Q \Delta x + q^T \Delta x = 0$$

$$\Rightarrow t = \frac{-x^T Q \Delta x - q^T \Delta x}{\Delta x^T Q \Delta x}$$

Gredient descent with exect line search

minimize $f(x) = \frac{1}{4} x_1^2 + x_2^2$

$$Q = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \qquad \nabla f = \begin{pmatrix} \frac{1}{2} x_1 \\ 2 x_2 \end{pmatrix}$$

$$\Delta x = -\nabla f\left(x^{(0)}\right) = -\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Take equation from exercise 1

$$q = 0$$

$$t^{(0)} = -\frac{x^{(0)T} Q \,\Delta x^{(0)}}{\Delta x^{(0)T} Q \,\Delta x^{(0)}} = \frac{(2,1)\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{(1,2)\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \frac{1+4}{\frac{1}{2}+8} = \frac{10}{17}$$

$$x^{(1)} = x^{(0)} + t^{(0)} \cdot \Delta x^{(x)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{10}{17}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 24/17 \\ -3/17 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 24 \\ -3 \end{pmatrix}$$