Lagrange Duality

minimize $x^2 + 1$
subject to $(x-2)(x-4) \leq 0$

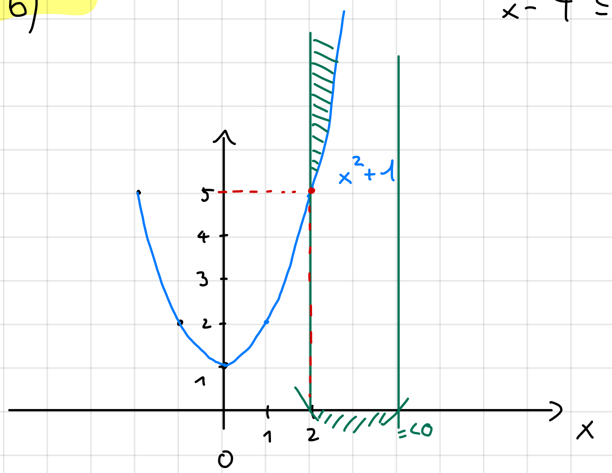
(a) feasible set $x \in [2,4]$

optimal value $x = 2$

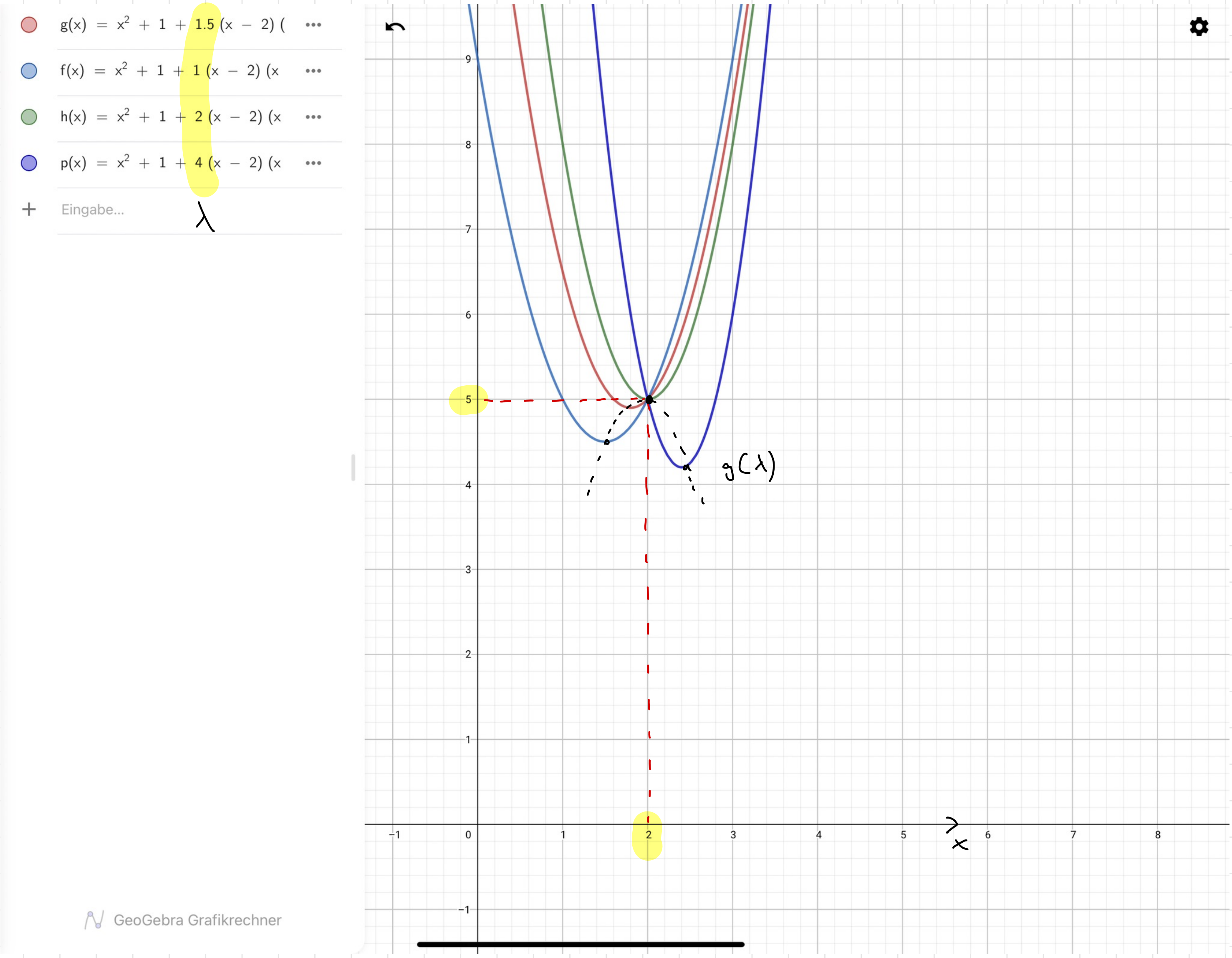
optimal solution 5

b)

$2 - x \leq 0$
$x - 4 \leq 0$



$$f_0(x) + \sum_{i=1}^{2} \lambda_i f_i(x) = x^2 + 1 + \lambda\left((x-2)(x-4)\right)$$



$\tilde{x}$ feasible point

$$L(\tilde{x}, \lambda) = f_0(\tilde{x}) + \lambda f_1(\tilde{x}) = \tilde{x}^2 + 1 + \underbrace{\underbrace{\lambda}_{\geq 0} \underbrace{(\tilde{x} - 2)(\tilde{x} - 4)}_{\leq 0}}_{\leq 0}$$

$$\Rightarrow f_0(\tilde{x}) \geq L(\tilde{x}, \lambda) \geq \inf_{x \in D} L(x, \lambda) = g(\lambda)$$

$\Rightarrow g(\lambda) \leq f_0(\tilde{x})$ for all feasiable $\tilde{x}$

$$\Rightarrow g(\lambda) \leq f_0(x^*) = p^*$$

$$x^2 + 1 + \lambda \cdot (x^2 - 6x + 8)$$

$$x^2 + 1 + \lambda x^2 - \lambda 6x + \lambda 8$$

$$2x + 2\lambda x - \lambda 6 \overset{!}{=} 0$$

$$(2x - 6)\lambda + 2x = 0$$

$$\lambda = \frac{-2x}{x-6} = \frac{2x}{6-2x} = \frac{x}{3-x} \geq 0$$

$$g(\lambda(x)) = x^2 + 1 + \frac{x}{3-x} \cdot (x-2)(x-4) = \frac{x^3}{3-x} - \frac{6x^2}{3-x} + x^2 + \frac{8x}{3-x} + 1$$

asymptotic at $x=3$

c)

$$\frac{d^2}{dx^2} g(\lambda(x)) = \frac{6}{(x-3)^3}$$ (calculated with wolfram alpha)

$$\frac{6}{(x-3)} \leq 0 \quad \forall x \leq 3$$

Secound order derivative condition ≤ 0 => concave ✓

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maximize $g(\lambda)$
subject to $\lambda \geq 0$

It is a maximization problem because
$g(\lambda) \leq p^*$

=> max $g(\lambda)$ is the best Approximation

Optimal value: $\frac{d}{dx} g(\lambda(x)) = \frac{3(x^2 - 6x + 8)}{(3-x)^2} = 0$

=> $x = 2$, $\cancel{x = 4}$ > 3

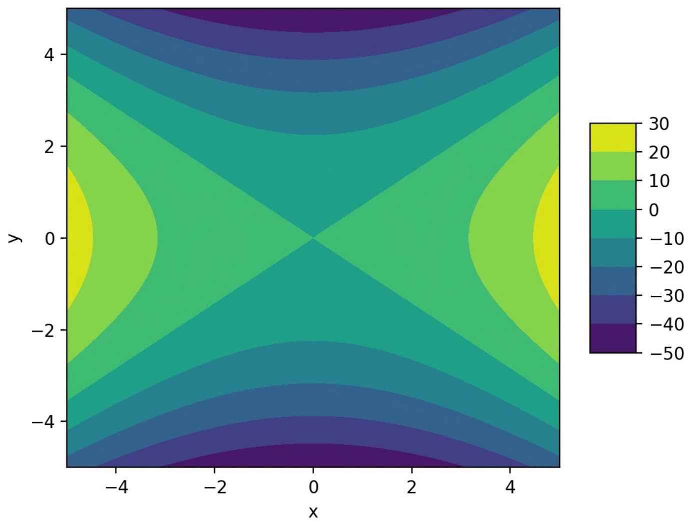
=> optimal value $x = 2 = p^*$

=> strong duality holds

Exercise KKT Condition

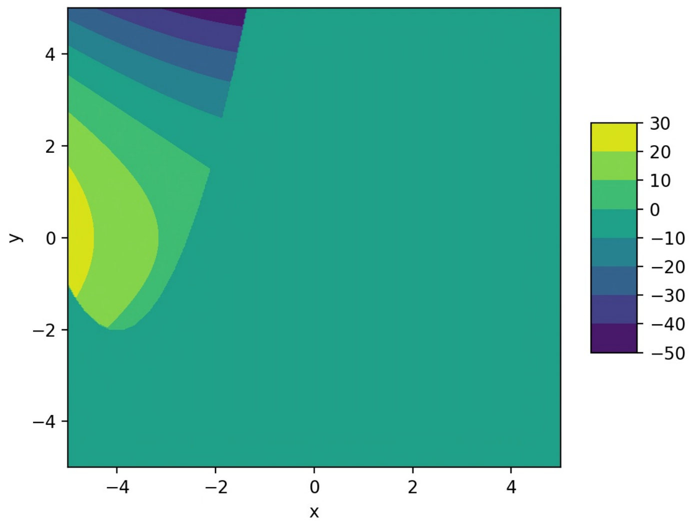
$x_1 \hat{=} x \quad , x_2 \hat{=} y$

$x^2 - 2y^2$



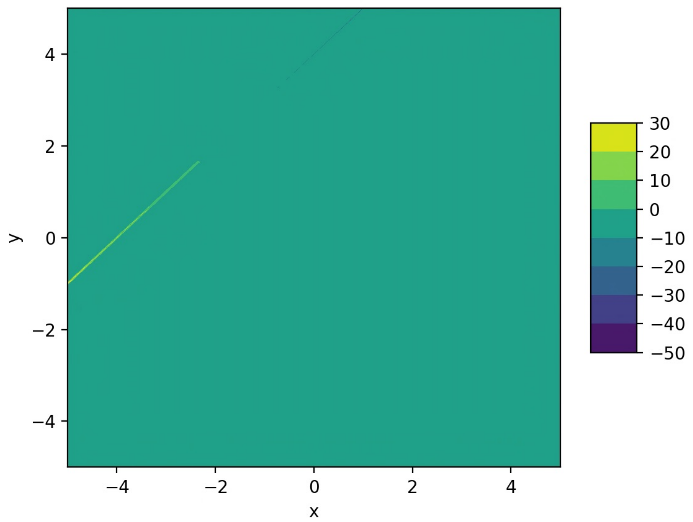
$x^2 - 2y^2$

$(x+4)^2 - 2 \leq y$



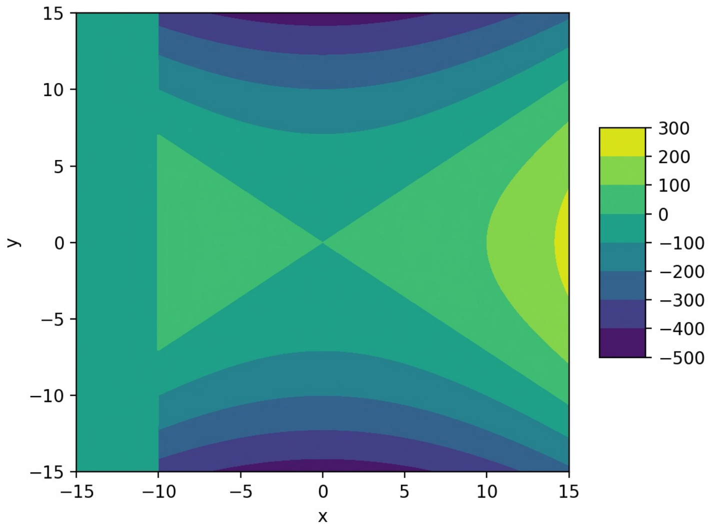
$x^2 - 2y^2$

$x - y + 4 = 0$



$$x^2 - 2y^2$$

$$x \geq -10$$



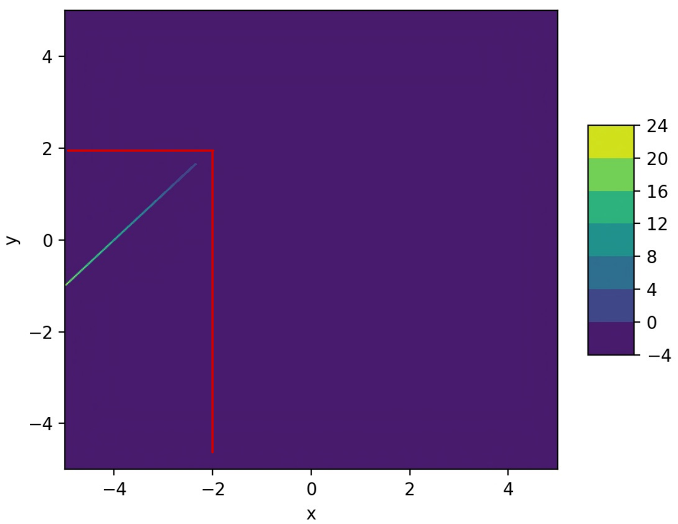
$$x^2 - 2y^2$$

$$(x+4)^2 - 2 \leq y$$

$$x - y + 4 = 0$$

$$x \geq -10$$

minimum at $[-2, 2]$
it's difficult to plot



(2)

$x^* = (-2, 2)$

$f_1(x^*) = (\underset{x_1}{\underset{\nabla}{-2}} + 4)^2 - 2 - \underset{x_2}{\underset{\vee}{2}} = 0$

$h_1(x^*) = -2 - 2 + 4 = 0$

$f_2(x^*) = -10 - 2 = -12 \overset{\text{complementary slackness}}{\Rightarrow} \lambda_2^* = 0$

$$\nabla_x \mathcal{L}(x^*, \lambda^*, v^*) = \begin{pmatrix} 2x_1^* + \lambda_1^* \, 2(x_1^* + 4) + \lambda_2^* (-1) + v^* \\ -4x_2^* - \lambda_1^* - v^* \end{pmatrix} = 0$$

$$= \begin{pmatrix} -4 + \lambda_1^* (4) - \cancel{\lambda_2^*}^{0} + v^* \\ -8 - \lambda_1^* - v^* \end{pmatrix} = 0$$

$\Rightarrow -12 + 3\lambda_1^* = 0$

$\Rightarrow \lambda_1^* = 4 \geq 0 \checkmark$

$\Rightarrow -8 - 4 - v^* = 0$

$\Rightarrow v^* = -12$

$\Rightarrow \lambda^* = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \; v^* = -12$

λ^*, v^* satisfy all KKT conditions
at $x^* = (-2, 2)$
$\Rightarrow x^*$ is globel minimum (if x^* is local minimum)