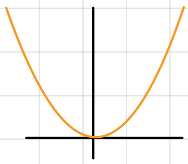
Convex Functions

Raphael Joost
Chris Rüttimann

Simple functions

1. $f(x) = x^2, \quad x \in \mathbb{R}$



$\text{dom}(f(x)) = \mathbb{R} \rightarrow$ convex

$f'(x) = 2x, \quad f''(x) = 2 \geq 0 \rightarrow$ convex

2. $f(x) = e^{x^2}, \quad x \in \mathbb{R}$

$\text{dom}(f(x)) = \mathbb{R} \rightarrow$ convex

$f'(x) = 2x \cdot e^{x^2}, \quad f''(x) = \underbrace{4x^2}_{\geq 0} \cdot \underbrace{e^{x^2}}_{\geq 0}$ for all x

3. $f(x,y) = x^2 + 3xy + 2y^2, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}$

$\text{dom}(f(x,y)) = \mathbb{R}^2 \rightarrow$ convex

$$\nabla f(x,y) = \left[\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}\right]^T = \left[2x + 3y, 3x + 4y\right]^T$$

$$\nabla^2 f(x,y) = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x^2} & \frac{\partial f(x,y)}{\partial x \partial y} \\ \frac{\partial f(x,y)}{\partial x \partial y} & \frac{\partial f(x,y)}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \succeq 0 \rightarrow \text{convex}$$

Log-sum-exp

$$f(\vec{x}) = \log\left(e^{x_1} + \dots + e^{x_n}\right) \qquad x \in \mathbb{R}^n$$

$$e^{x} > 0 \quad \forall \; x \in \mathbb{R}$$

$$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f(\vec{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\vec{x})}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{e^{x_1}}{\sum_{i=1}^{n} e^{x_i}} \\ \vdots \\ \frac{e^{x_n}}{\sum_{i=1}^{n} e^{x_i}} \end{bmatrix}$$

Side calculetion : $\frac{\partial \left(\log\left(\sum_{i=1}^{n} e^{x_i}\right)\right)}{\partial x_k} = \frac{\partial \log(u)}{\partial u} \cdot \frac{\partial u}{\partial x_k}$ with $u = \sum_{i=1}^{n} e^{x_i}$

$$= \frac{1}{u} \cdot e^{x_k} = \frac{e^{x_k}}{\sum_{i=1}^{n} e^{x_i}}$$

$$\nabla^2 f = \begin{bmatrix} \frac{\partial f(\vec{x})}{\partial x_1^2} & \cdots & \frac{\partial f(\vec{x})}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f(\vec{x})}{\partial x_1 \partial x_n} & \cdots & \frac{\partial f(\vec{x})}{\partial x_n^2} \end{bmatrix} =$$

Quotient rule : $f(x) = \frac{g(x)}{h(x)}$, $f'(x) = \frac{g'(x)\,h(x) - g(x)\,h'(x)}{h(x)^2}$

Side calculation : $\frac{\partial}{\partial x_k} \frac{e^{x_k}}{\sum_{i=1}^{n} e^{x_i}} = \frac{e^{k} \cdot \left(\sum_{i=1}^{n} e^{x_i}\right) - e^{x_k} \cdot e^{x_k}}{\left(\sum_{i=1}^{n} e^{x_i}\right)^2} = \frac{e^{x_k} \cdot \left(\left(\sum_{i=1}^{n} e^{x_i}\right) - e^{x_k}\right)}{\left(\sum_{i=1}^{n} e^{x_i}\right)^2}$

$$= \frac{e^{x_k} \cdot \left(\sum_{i=1 \wedge i \neq k}^{n} e^{x_i}\right)}{\left(\sum_{i=1}^{n} e^{x_i}\right)^2}$$

$$\frac{\partial}{\partial x_p} \frac{e^{x_k}}{\sum_{i=1}^{n} e^{x_i}} = 0 - e^{x_k} \cdot e^{x_p}$$