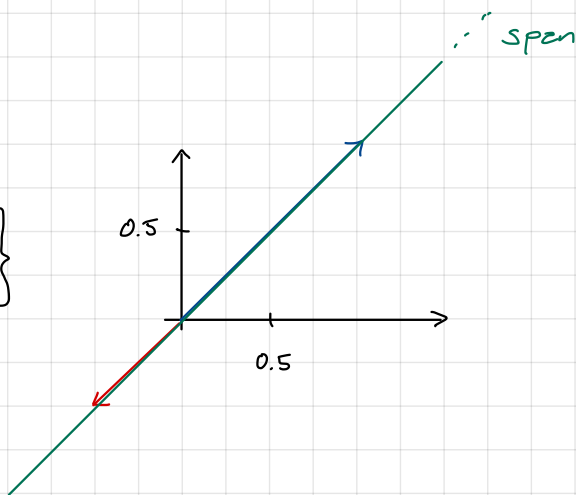


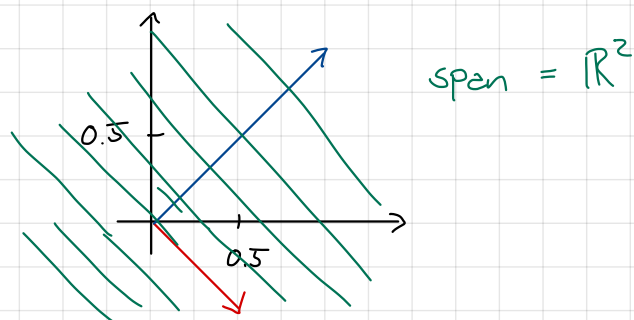
Exercise 1 - Convex Sets

Convex Sets
Example Sets

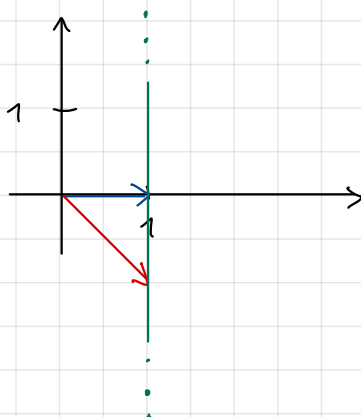
1. $\text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix} \right\}$



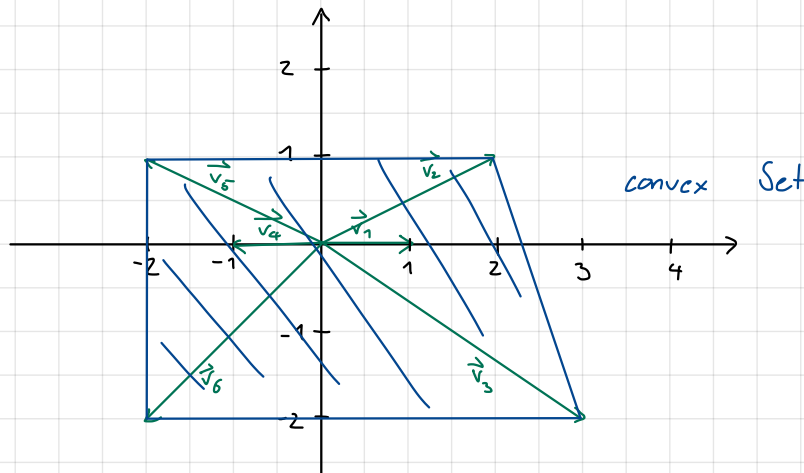
2. $\text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} \right\}$



3. $\text{aff} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$



$$4. \text{conv} \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\vec{v}_2}, \underbrace{\begin{pmatrix} 3 \\ -2 \end{pmatrix}}_{\vec{v}_3}, \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{\vec{v}_4}, \underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{\vec{v}_5}, \underbrace{\begin{pmatrix} -2 \\ -2 \end{pmatrix}}_{\vec{v}_6} \right\}$$



Convexity

$$\text{conv}(S) = \left\{ \sum_i a_i v_i \mid v_i \in S, a_i \in \mathbb{R}_+, \sum_i a_i = 1 \right\}$$

Induction assumption:

$$w_k = x_1 + \sum_{i=2}^k \theta_i (x_i - x_1) \in \text{convex set}$$

Induction step: $k \rightarrow k+1$:

$$w_{k+1} = x_1 + \sum_{i=2}^{k+1} \theta_i (x_i - x_1) = x_1 + \underbrace{\sum_{i=2}^k \theta_i (x_i - x_1)}_{\text{assumption}} + \underbrace{\theta_{k+1} (x_{k+1} - x_1)}_{\text{same as case for } k=2}$$

$\Rightarrow w_{k+1} \in \text{convex set}$

Linear Equations

$$\{x \mid Ax = b\} \quad x \in \mathbb{R}^n \quad A \in \mathbb{R}^{m \times n} \quad b \in \mathbb{R}^m$$

$$\begin{bmatrix} -a_1^T \\ \vdots \\ -a_m^T \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$x_1, x_2 \text{ fulfill } Ax = b$$

$$\alpha \cdot x_1 + \beta x_2, \quad \alpha + \beta = 1$$

$$= \alpha x_1 + (1 - \alpha) \cdot x_2$$

$$\begin{aligned} A(\alpha x_1 + (1 - \alpha) x_2) &= \alpha \underbrace{Ax_1}_b + (1 - \alpha) \underbrace{Ax_2}_b = b \\ &= \alpha b + (1 - \alpha) b \\ &= \cancel{\alpha b} + 1b - \cancel{\alpha b} = b \quad \checkmark \end{aligned} \quad (1)$$

$$\begin{matrix} x_1, \dots, x_k \\ a_1, \dots, a_k \end{matrix} \quad a_1 + \dots + a_k = 1$$

$$\begin{aligned} &A(a_1 x_1 + (1 - a_1 - a_3 \dots a_k) x_2 + \dots + (1 - a_1 \dots - a_{k-1}) x_k) \\ &= \underbrace{a_1 Ax_1}_b + (1 - a_1 - a_3 \dots a_k) \underbrace{Ax_2}_b + \dots + (1 - a_1 \dots - a_{k-1}) \underbrace{Ax_k}_b \\ &= a_1 b + (1 - a_1 - a_3 \dots a_k) b + \dots + (1 - a_1 \dots - a_{k-1}) b = b \end{aligned}$$

all coefficient cancel each other out as seen in (1)

Linear Inequations

$$\{x \mid Ax \leq b, Cx = d\} \quad x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, C \in \mathbb{R}^{k \times n}, d \in \mathbb{R}^k$$

$$\begin{array}{c} x_1, \dots, x_k \\ a_1, \dots, a_k \end{array} \quad a_1 + \dots + a_k = 1$$

$$\begin{aligned} & A(a_1 x_1 + (1 - a_1 - a_3 \dots a_k) x_2 + \dots + (1 - a_1 \dots - a_{k-1}) x_k) \\ &= a_1 A x_1 + (1 - a_1 - a_3 \dots a_k) A x_2 + \dots + (1 - a_1 \dots - a_{k-1}) A x_k \end{aligned}$$

$$\leq a_1 b + (1 - a_1 - a_3 \dots a_k) b + \dots + (1 - a_1 \dots - a_{k-1}) b = b \quad \checkmark$$

all coefficient cancel each other out as seen in (1)

It also satisfies the second condition ($Cx = d$) as shown in the previous exercise.

2.

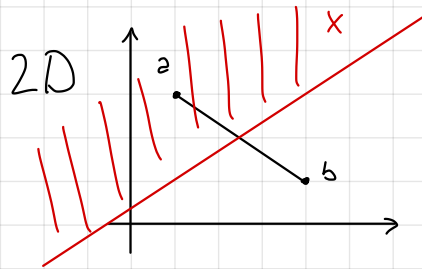
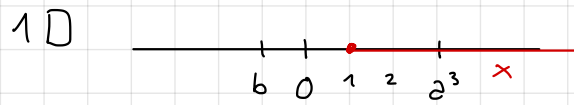
Yes it is an affine set because for the proof that it is a convex set, the condition that $a_1, \dots, a_k \geq 0$ is not used.

Voronoi description of halfspace

$$a, b \in \mathbb{R}^n \quad \{x \mid \|x-a\|^2 \leq \|x-b\|^2\}$$

$$c^T x \leq d$$

$$c^T x - d \leq 0$$



$$x^2 - 2a^T x + a^2 \leq x^2 - 2b^T x + b^2$$

$$a^2 - 2a^T x \leq b^2 - 2b^T x$$

$$2b^T x - 2a^T x \leq b^2 - a^2$$

$$\underbrace{2(b^T - a^T)}_{c^T} x \leq \underbrace{b^2 - a^2}_d$$