The κ-connected Arborescence Star Problem: Formulation and Branch-and-Cut Algorithm

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1 Introduction

To fulfill this commitment of achieving net-zero greenhouse gas emissions by 2050, the adoption of cleaner energy sources is crucial. However, the distribution of renewable energy to consumers poses significant financial challenges. Therefore, it is crucial to optimize these investments. Optimization and operational research play a pivotal role in determining the most efficient strategies for designing and implementing renewable energy systems.

The customers and the candidates refuelling stations can be modelled as a set of vertices $\mathcal{V}$ and a set of vertices $\mathcal{R}$, respectively. The hydrogen pipelines networks can be viewed as the set of directed edges $\mathcal{A}$ to connect the sources $\mathcal{R}$ and customers $\mathcal{V}$. In addition, the networks $\mathcal{A}$ can be denoted as a set of arborescences T_r rooted at $r \in \mathcal{R}$, which can be viewed as a forest topology. Previous researchers adopted the forest-stars network [1]. In this design approach, the "forest"consists of a set of trees, which are used to model the pipeline routes used to distribute the renewable energy [2]. These trees originate from the renewable energy production facilities and extend outward to reach the refueling stations and the customers, which resembles stars. However, the limitation lies in its vulnerability to disruptions. Specifically, the removal of a single component in the distribution network could lead to a complete system failure. This vulnerability is a significant concern since it makes the renewable energy infrastructure susceptible to unforeseen disruptions and maintenance needs.

The focus of this research is to first present the arborescence star problem, and then the develop a rigorous model and solution algorithms for determining a robust and financially optimal renewable energy refueling network by introducing an additional constraint, and thus ensures uninterrupted functionality of the whole system.

2 Preliminaries

Definition 1. A *directed graph* (or *digraph*) is an ordered pair $G = (V, E)$, where V is (finite)set of objects called vertices, and E is a set of ordered pairs of distinct vertices, called directed edges. A *weighted digraph* is a digraph together with a weight function $w : E \to \mathbb{R}^+$. A *rooted directed graph* (or *flow graph*) is a digraph in which a vertex r has been distinguished as the root. Digraphs with a set of vertices $R \subset V$ designated as roots are called *multiply rooted digraphs.*

Definition 2. A path P in a digraph $G = (V, E)$ is a sequence of vertices $v_1, v_2, \cdots, v_n$ with $n \geq 2$ and a corresponding sequence of $n-1$ edges such that the i^{th} edge in the sequence is (v_i, v_{i+1}). Vertices v_1 and v_n are called the origin and the destination of P, respectively. A set of paths from are called *edge-disjoint* if no two have an edge in common.

Definition 3. The *edge-connectivity* of s and t, denoted as $\kappa(s, t)$ is the minimal number κ such that there exists $X \subseteq E$, $|X| = \kappa$ and the subgraph $G' = (V, E \setminus X)$ contains no s-t path. The maximal number of edge-disjoint s-t paths is denoted as $\kappa'(s, t)$.

Theorem 4 (Edge Connectivity Menger's Theorem for Digraph).
In a digraph $G = (V, E)$, for any $s, t \in V$, and $s \neq t$, we have

$$\kappa(s, t) = \kappa'(s, t).$$

Definition 5. A multiply rooted digraph $G = (V, E)$ with roots $R \subset V$ is called *κ-connected* if for every vertices $v \in V \setminus R$, there exists κ edge-disjoint directed paths from distinct roots to v.

Definition 6. A cut $C = (S, T)$ is a partition of V of a digraph $G = (V, E)$ into two subsets S and T. The cut-set of a cut $C = (S, T)$ is the set of edges

$$\delta^+(S) = \{(u, v) \in E | u \in S, v \in T\}$$

that have one endpoint in S and the other endpoint in T. In a weighted digraph, the value or weight is defined by the sum of the weights of the edges crossing the cut, i.e.

$$c(S, T) = \sum_{e \in \delta^+(S)} w(e).$$

3 The arborescence star problem

Consider a set of vertices $\mathcal{V}$, a set of roots $\mathcal{R}$ and a di-graph $\mathcal{G} = (\mathcal{R} \cup \mathcal{V}, \mathcal{A})$, where $\mathcal{A} = \{(\mathcal{R} \cup \mathcal{V}) \times \mathcal{V}\}$. Additionally, each arc $(i, j) \in \mathcal{A}$ is associated with a construction cost $c_{i,j}$ and an allocation cost $a_{i,j}$. The the arborescence star problem is defined as follows: Find a set of vertices $\mathcal{W} \subset \mathcal{V}$ and edges $\mathcal{E} \subset \mathcal{A}$ such that

- in the induced subgraph $\mathcal{G}' = (\mathcal{R} \cup \mathcal{W}, \mathcal{E})$, each weakly connected component is an arborescence rooted in some $r \in \mathcal{R}$
- any $v \in \mathcal{V} \setminus \mathcal{W}$ is assigned to a vertex in $\mathcal{G}'$

that minimises the total edge weights and assignment cost.

3.1 Desicion Variables

To formulate the problem, we introduce the following decision variables: Let

$$x_{i,j} = \begin{cases} 1 & \text{if } (i, j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and let

$$y_{i,j} = \begin{cases} 1 & \text{if vertex } j \in \mathcal{V} \text{ is assigned to vertex } i \in \mathcal{W} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where we employ the notation that if $y_{i,i} = 1$ then vertex $i \in \mathcal{W}$, i.e.,

$$y_{i,i} = \begin{cases} 1 & \text{if } i \in W \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

3.2 Objective function

The objective function is

$$\sum_{(i,j) \in \mathcal{A}} c_{i,j} x_{i,j} + a_{i,j} y_{i,j}.$$

3.3 Model constraints

$$y_{r,r} = 1 \quad \forall\, r \in \mathcal{R} \tag{4}$$

$$\sum_{i \in \mathcal{V}} x_{i,j} = y_{j,j} \quad \forall\, j \in \mathcal{V} \tag{5}$$

$$\sum_{i \in \mathcal{V}} y_{i,j} = 1 \quad \forall\, j \in \mathcal{V} \tag{6}$$

$$\sum_{(p,q) \in \delta^+(S)} x_{p,q} \geq \sum_{t \in S} y_{k,t} \quad \forall\, S \subset \mathcal{V}, k \in \mathcal{V} \tag{7}$$

$$x_{i,j} \in \{0, 1\} \quad \forall\, (i, j) \in \mathcal{A} \tag{8}$$

$$y_{i,j} \in \{0, 1\} \quad \forall\, (i, j) \in \mathcal{A} \tag{9}$$

The constrant (4) is labeling the roots $r \in \mathcal{R}$ as tree nodes. The second constraint (5) is that any node $v \in \mathcal{V}$ is on one arborescence if and only if the node v has one predeseccor on one arborescence. The constraint (6) is that for any node $v \in \mathcal{V}$, it is either on one arborescence or is assigned to one node on one arborescence. The constraint (7) ensures the arboresence topology.

Proposition 7.
The inequality

$$y_{i,i} \geq y_{i,j} + x_{i,j} + x_{j,i} \quad \forall\ i,\ j \in \mathcal{V} \tag{10}$$

is valid.

Proof. The inequality (10) is indeed a special case of the cycle elimination constraint (7). Letting $S = \{i, j\}$, and $k = j$ yields

$$\sum_{(p,q)\in\delta^+(\{i,j\})} x_{p,q} \geq \sum_{t\in\{i,j\}} y_{j,t},$$

that is

$$\sum_{(o,i)\in\delta^+(\{i\})} x_{o,i} - x_{j,i} + \sum_{(o,j)\in\delta^+(\{j\})} x_{o,j} - x_{i,j} \geq y_{j,j} + y_{j,i}$$

Replacing by constraint (5) yields

$$y_{i,i} - x_{j,i} + y_{j,j} - x_{i,j} \geq y_{j,j} + y_{j,i}$$

that is

$$y_{i,i} \geq x_{j,i} + x_{i,j} + y_{i,j}.$$

□

We then apply the branch and bound method to solve the arboresence star model [3]. We first solve the linear model with constraints (4)(5)(6)(10), and then solve the seperation problem of the constraint (7)

4 The κ-connected Arborescence Star Problem

Then, the κ connected arborescence star problem is defined as follows: Find a set of vertices $\mathcal{W} \subset \mathcal{V}$ and edges $\mathcal{E} \subset \mathcal{A}$ such that

- in induced subgraph $\mathcal{G}' = (\mathcal{R} \cup \mathcal{W}, \mathcal{E})$, for each $w \in \mathcal{W}$, there is κ edge-disjoint paths from κ different roots in $\mathcal{R}$ to w.
- any $v \in \mathcal{V} \setminus \mathcal{W}$ is assigned to a vertex in $\mathcal{G}'$

that minimises the total edge weights and assignment cost.

4.1 Model Constarints

$$y_{r,r} = 1 \quad \forall\ r \in \mathcal{R} \tag{11}$$

$$\sum_{i\in\mathcal{V}} x_{i,j} = \kappa y_{j,j} \quad \forall\ j \in \mathcal{V} \tag{12}$$

$$\sum_{i\in\mathcal{V}} y_{i,j} = 1 \quad \forall\ j \in \mathcal{V} \tag{13}$$

$$x_{i,j} + x_{j,i} \leq 1 \quad \forall\ i, j \in \mathcal{V} \tag{14}$$

$$\sum_{(p,q)\in\delta^+(S)} x_{o,i} \geq (\kappa - |S \cap \mathcal{R}|) \sum_{t\in S} y_{k,t} \quad \forall\ S \subset \mathcal{V}, k \in \mathcal{V} \tag{15}$$

$$x_{i,j} \in \{0, 1\} \quad \forall\ (i, j) \in \mathcal{A} \tag{16}$$

$$y_{i,j} \in \{0, 1\} \quad \forall\ (i, j) \in \mathcal{A} \tag{17}$$

$$\tag{18}$$

Proposition 8.
The inequality

$$y_{i,i} \geq y_{i,j} + x_{i,j} + x_{j,i} \quad \forall\ i,\ j \in \mathcal{V} \tag{19}$$

is valid.

Proof. By the constraint 15, letting $S = \{i, j\}$ and $k = j$ where $i, j \in \mathcal{V}$ yields

$$\sum_{(p,q)\in\delta^+(\{i,j\})} x_{p,q} \geq \kappa \sum_{t\in\{i,j\}} y_{j,t}, \tag{20}$$

that is

$$\sum_{(o,i)\in\delta^+(\{i\})} x_{o,i} - x_{j,i} + \sum_{(o,j)\in\delta^+(\{j\})} x_{o,j} - x_{i,j} \geq \kappa(y_{j,j} + y_{j,i})$$

Replacing by constraint (12) yields

$$\kappa y_{i,i} - x_{j,i} + \kappa y_{j,j} - x_{i,j} \geq \kappa(y_{j,j} + y_{j,i})$$

that is

$$\kappa(y_{i,i} - y_{i,j}) \geq x_{j,i} + x_{i,j}. \tag{21}$$

Clearly, the equation 19 is stronger than the equality (21). By the Non-negativity of $x_{i,j}, x_{j,i}$ and (21), we get

$$y_{i,i} \geq y_{i,j}, \quad \forall i, j \in \mathcal{V}.$$

We consider (21) in two different cases: $y_{i,i} - y_{i,j} = 0$ and $y_{i,i} - y_{i,j} = 1$.

- If $y_{i,i} - y_{i,j} = 0$, we get $x_{i,j} = x_{j,i} = 0$, and thus (19) is valid.
- If $y_{i,i} - y_{i,j} = 1$, by the constraint 14, we have $y_{i,i} - y_{i,j} = 1 \geq x_{i,j} + x_{j,i}$ as well.

Hence, the claim follows. □

4.2 Seperation Method

We apply the branch and bound method to solve the κ-connected arboresence star model. We first solve the linear model with constraints (11)(12)(13)(19), and then solve the seperation problem of the constraint (15). Sadly, no algorithm is found to solve the separation problem (15). However, there is a algorithm to solve the seperation problem of the weaker inequality:

$$\sum_{(p,q)\in\delta^+(S)} x_{o,i} \geq \kappa \sum_{t\in S} y_{k,t} - |S \cap \mathcal{R}| \tag{22}$$

Algorithm 1: A separation procedure for SEC's

Input: $\bar{x}_{i,j}$, $\bar{y}_{i,j}$ a solution to $\mathcal{P}$ and $\mathcal{G}^*$
$s \leftarrow -1$ defines a dummy source
$\mathcal{V}^* \leftarrow \mathcal{R} \cup \{i \in \mathcal{V} : \bar{y}_{i,i} > 0\} \cup \{s\}$
$\mathcal{A}^* \leftarrow \{(i,j) \in \mathcal{A} : \bar{x}_{i,j} > 0\}$
foreach $(i,j) \in \mathcal{A}$ **do**
  $c_{i,j} \leftarrow \bar{x}_{i,j}$
$\mathcal{A}^* \leftarrow \mathcal{A}^* \cup \{(s,r) : r \in \mathcal{R}\}$
foreach $r \in \mathcal{R}$ **do**
  $c_{s,r} = 1$
$\mathcal{G}^* \leftarrow (\mathcal{V}^*, \mathcal{A}^*)$
foreach $i \in \mathcal{V}^*$ **do**
  $\mathcal{V}^{**} \leftarrow \mathcal{V}^* \cup \{t\}$;
  $\mathcal{A}^{**} \leftarrow \mathcal{A}^* \cup \{(j,t)\ \forall j \in \mathcal{V}^*\}$;
  $c_{j,t} \leftarrow \kappa \bar{y}_{i,j}\ \forall j \in \mathcal{V}^*$;
  Δ, $\langle \mathcal{S}, \mathcal{T} \rangle \leftarrow$ MIN_ST_CUT$(\mathcal{G}^{**} = (\mathcal{V}^{**}, \mathcal{A}^{**}), s, t)$;
  if $\Delta < \kappa$ *and* $i \in \mathcal{T}$ **then**
    $\mathcal{T} \setminus \{t\}$ defines the most violated cut involving i;

We then prove the validation of the seperation procedure. Suppose $\mathcal{S}, \mathcal{T}$ defines the most violated cut involving i and $\Delta < \kappa$, we have

$$\sum_{(p,q):p\in\mathcal{S},q\in\mathcal{T}} c_{p,q} < \kappa$$

that is

$$\sum_{q\in\mathcal{T}\cap\mathcal{R}} c_{s,q} + \sum_{(p,q)\in\delta^+(\mathcal{T})} c_{p,q} + \sum_{p\in\mathcal{S}} c_{p,t} < \kappa$$

$$|\mathcal{T}\cap\mathcal{R}| + \sum_{(p,q)\in\delta^+(\mathcal{T})} \bar{x}_{p,q} + \kappa \sum_{p\in\mathcal{S}} \bar{y}_{i,p} < \kappa$$

By constranint 13, we get

$$|\mathcal{T}\cap\mathcal{R}| + \sum_{(p,q)\in\delta^+(\mathcal{T})} \bar{x}_{p,q} + \kappa(1 - \sum_{p\in\mathcal{T}} \bar{y}_{i,p}) < \kappa$$

that is

$$\sum_{(p,q)\in\delta^+(\mathcal{T})} \bar{x}_{p,q} < \kappa \sum_{p\in\mathcal{T}} \bar{y}_{i,p} - |\mathcal{T}\cap\mathcal{R}| \tag{23}$$

Hence, the algorithm find a violation of the constraint (15). We can apply the same algorithm with $\kappa = 1$ for the arborescence star problem.

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