Matrix and Vector

Contents

1 Calculation of the squared Euclidean norm

From StackExchange.

1.1 Question

$$
\begin{aligned}
&\|\mathbf{x} - \alpha\|^2 - \|\mathbf{x} - \beta\|^2 \\
&= \|\mathbf{x}\|\|\mathbf{x}\| - 2\|\alpha\|\|\mathbf{x}\| + \|\alpha\|\|\alpha\| - \|\mathbf{x}\|\|\mathbf{x}\| + 2\|\beta\|\|\mathbf{x}\| - \|\beta\|\|\beta\| \\
&= \alpha^T\alpha - \beta^T\beta + 2(\sqrt{\beta \cdot \beta} - \sqrt{\alpha \cdot \alpha})\|\mathbf{x}\| \\
&= \alpha^T\alpha - \beta^T\beta + 2(\sqrt{\beta \cdot \beta} - \sqrt{\alpha \cdot \alpha})(\sqrt{\mathbf{x} \cdot \mathbf{x}}),
\end{aligned}
$$

where I used the fact that

$$\|a\|\|a\| = \sqrt{a \cdot a}\sqrt{a \cdot a} = \sqrt{a^T a}\sqrt{a^T a} = a^T a.$$

However, the article gives

$$2(\beta - \alpha)^T\mathbf{x} + \alpha^T\alpha - \beta^T\beta$$

1.2 Answer

Your transition from the first line to the second is incorrect. We should have

$$
\begin{aligned}
&\|x - \alpha\|^2 - \|x - \beta\|^2 \\
&(x - \alpha)^T(x - \alpha) - (x - \beta)^T(x - \beta) \\
&= \|x\|^2 + \|\alpha\|^2 - \|x\|^2 - \|\beta\|^2 - x^T\alpha - \alpha^T x + x^T\beta + \beta^T x \\
&= \|\alpha\|^2 - \|\beta\|^2 - 2\alpha^T x + 2\beta^T x \\
&= \alpha^T\alpha - \beta^T\beta + 2(\beta - \alpha)^T x
\end{aligned}
$$

which is the desired result.

1.3 Caution

1.3.1

For any vectors u, v, $u^T v = v^T u$.

1.3.2

$$(x-\alpha)^T(x-\alpha) = x^T x - x^T\alpha - \alpha^T x + \alpha^T\alpha$$

2 General answer of $(\mathbf{x}_n - \boldsymbol{\mu}_k)^\top \mathbf{\Lambda}_k (\mathbf{x}_n - \boldsymbol{\mu}_k)$

From StackExchange.

2.1 Question

Can we say this generally?

$$(\mathbf{x}_n - \boldsymbol{\mu}_k)^\top \mathbf{\Lambda}_k (\mathbf{x}_n - \boldsymbol{\mu}_k) = \mathbf{x}_n^\top \mathbf{\Lambda}_k \mathbf{x}_n - 2\boldsymbol{\mu}_k^\top \mathbf{\Lambda}_k \mathbf{x}_n + \boldsymbol{\mu}_k^\top \mathbf{\Lambda}_k \boldsymbol{\mu}_k$$

Or is this the case when $\mathbf{x}_n$ comes from normal distribution, $\mathcal{N}(\mathbf{x}_n|\boldsymbol{\mu}_k, \Lambda_k)$?

I'm a bit confused because I know following equation is generally correct.

$$(\mathbf{x}_n - \boldsymbol{\alpha})^T(\mathbf{x}_n - \boldsymbol{\alpha}) = \mathbf{x}_n^T \mathbf{x}_n - \mathbf{x}_n^T \boldsymbol{\alpha} - \boldsymbol{\alpha}^T \mathbf{x}_n + \boldsymbol{\alpha}^T \boldsymbol{\alpha}$$

2.2 Answer

Your statement is true as long as the $\mathbf{\Lambda}_n$ matrix is symmetric.

Expand the product and you'll get:

$$(\mathbf{x}_n - \mu_n)^T \mathbf{\Lambda}_n (\mathbf{x}_n - \mu_n) = \mathbf{x}_n^T \mathbf{\Lambda}_n \mathbf{x}_n - \mathbf{x}_n^T \mathbf{\Lambda}_n \mu_n - \mu_n^T \mathbf{\Lambda}_n \mathbf{x}_n + \mu_n^T \mathbf{\Lambda}_n \mu_n$$

The cross terms are equal only if the lambda matrix is symmetric.