Variational Bayes Derivation of Latent Dirichlet Allocation

Simple LDA, not smoothed LDA

September 7, 2016

1 Model

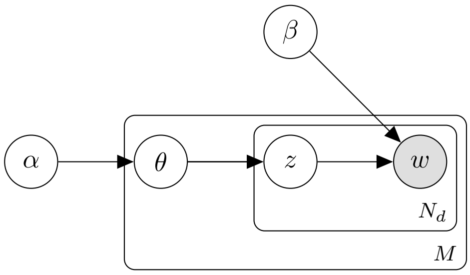

Figure 1: LDA Model

Variables:

- M: a number of document
- N_d: a number of words in document d
- $w_{d,i}$: a word
- β: (number of topic K) $\times$ (number of unique words V)
  - $\beta_{k,v} = p(w_{d,i} = v | z_{d,i} = k)$ is the probability of the word v occuring given the topic k
  - Caution: At least in the Python code, β is $V \times K$ (maybe in C as well)
- $z_{d,i}$: latent topic

2 Derivation with Code

2.1 Evidence lower bound

$$
\begin{aligned}
\log p(\boldsymbol{w}|\boldsymbol{\alpha}, \boldsymbol{\beta}) &= \log \int \sum_{\boldsymbol{z}} p(\boldsymbol{w}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{\alpha}, \boldsymbol{\beta}) d\boldsymbol{\theta} \qquad \text{(intdotuce latent variables)} && (1) \\
&= \log \int \sum_{\boldsymbol{w}} q(\boldsymbol{z}, \boldsymbol{\theta}) \frac{p(\boldsymbol{w}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{\alpha}, \boldsymbol{\beta})}{q(\boldsymbol{z}, \boldsymbol{\theta})} d\boldsymbol{\theta} && (2) \\
&\leq \int \sum_{\boldsymbol{z}} q(\boldsymbol{z}, \boldsymbol{\theta}) \log \frac{p(\boldsymbol{w}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{\alpha}, \boldsymbol{\beta})}{q(\boldsymbol{z}, \boldsymbol{\theta})} d\boldsymbol{\theta} \qquad \because \text{Jensen's Inequality} && (3) \\
&\equiv F[q(\boldsymbol{z}, \boldsymbol{\theta})] && (4)
\end{aligned}
$$

From factorization assumption:

$$q(\boldsymbol{z}, \boldsymbol{\theta}) = \left[\prod_{d=1}^{M}\prod_{i=1}^{N_d} q(z_{d,i})\right]\left[\prod_{d=1}^{M} q(\boldsymbol{\theta}_d)\right] \tag{5}$$

Expand the joint distribution using Bayes' Theorem:

$$\begin{aligned}
p(\boldsymbol{w}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{\alpha}, \boldsymbol{\beta}) &= p(\boldsymbol{w}|\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z}, \boldsymbol{\theta}) \quad \underbrace{p(\boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{\alpha}, \boldsymbol{\beta})}_{p(\boldsymbol{z}|\boldsymbol{\theta},\boldsymbol{\alpha},\boldsymbol{\beta})p(\boldsymbol{\theta}|\boldsymbol{\alpha},\boldsymbol{\beta})} && (6)\\
&= p(\boldsymbol{w}|\boldsymbol{z}, \boldsymbol{\beta})p(\boldsymbol{z}|\boldsymbol{\theta})p(\boldsymbol{\theta}|\boldsymbol{\alpha}) \qquad \because \text{Graphical Model} && (7)\\
&= \left[\prod_{d=1}^{M}\prod_{i=1}^{N_d} p(w_{d,i}|\beta_{z_{d,i}})p(z_{d,i}|\boldsymbol{\theta}_d)\right]\left[\prod_{d=1}^{M} p(\boldsymbol{\theta}_d|\boldsymbol{\alpha})\right] && (8)
\end{aligned}$$

Evidence lower bound (ELBO) is:

$$\begin{aligned}
F[q(\boldsymbol{z}, \boldsymbol{\theta})] &= \int \sum_{\boldsymbol{z}} q(\boldsymbol{z})q(\boldsymbol{\theta}) \log \frac{p(\boldsymbol{w}|\boldsymbol{z}, \boldsymbol{\beta})p(\boldsymbol{z}|\boldsymbol{\theta})p(\boldsymbol{\theta}|\boldsymbol{\alpha})}{q(\boldsymbol{z})q(\boldsymbol{\theta})} d\boldsymbol{\theta} && (9)\\
&= \int \sum_{\boldsymbol{z}} q(\boldsymbol{z})q(\boldsymbol{\theta}) \log \frac{p(\boldsymbol{w}|\boldsymbol{z}, \boldsymbol{\beta})p(\boldsymbol{z}|\boldsymbol{\theta})}{q(\boldsymbol{z})} d\boldsymbol{\theta} + \int \sum_{\boldsymbol{z}} q(\boldsymbol{z})q(\boldsymbol{\theta}) \log \frac{p(\boldsymbol{\theta}|\boldsymbol{\alpha})}{q(\boldsymbol{\theta})} d\boldsymbol{\theta} && (10)\\
&= \int \sum_{\boldsymbol{z}} q(\boldsymbol{z})q(\boldsymbol{\theta}) \log p(\boldsymbol{w}|\boldsymbol{z}, \boldsymbol{\beta})p(\boldsymbol{z}|\boldsymbol{\theta}) d\boldsymbol{\theta} - \sum_{\boldsymbol{z}} q(\boldsymbol{z}) \log q(\boldsymbol{z}) + \int q(\boldsymbol{\theta}) \log \frac{p(\boldsymbol{\theta}|\boldsymbol{\alpha})}{q(\boldsymbol{\theta})} d\boldsymbol{\theta} && (11)\\
&\qquad \text{(integrate out unrelated variables)}\\
&= \int \sum_{d=1}^{M}\sum_{i=1}^{N_d} q(z_{d,i})q(\boldsymbol{\theta}_d) \log p(w_{d,i}|z_{d,i}, \beta)p(z_{d,i}|\boldsymbol{\theta}_d) d\boldsymbol{\theta}_d\\
&\quad - \sum_{d=1}^{M}\sum_{i=1}^{N_d}\sum_{k=1}^{K} q(z_{d,i} = k) \log q(z_{d,i} = k) && (12)\\
&\quad - \sum_{d=1}^{M} \underbrace{\int q(\boldsymbol{\theta}_d) \log \frac{q(\boldsymbol{\theta}_d)}{p(\boldsymbol{\theta}_d|\boldsymbol{\alpha})} d\boldsymbol{\theta}_d}_{\mathrm{KL}[q(\boldsymbol{\theta}_d)||p(\boldsymbol{\theta}_d|\boldsymbol{\alpha})]}
\end{aligned}$$

2.2 Update Equation of $q(z_{d,i})$

2.2.1 Derivation

$$\widetilde{F}[q(z_{d,i})] = \sum_{k=1}^{K} q(z_{d,i} = k) \int q(\boldsymbol{\theta}_d) \log\{p(w_{d,i}|z_{d,i}, \beta_{k,i})p(z_{d,i} = k|\boldsymbol{\theta}_d)\} d\boldsymbol{\theta}_d - \sum_{k=1}^{K} q(z_{d,i} = k) \log q(z_{d,i} = k) \tag{13}$$

Use variational inference:

$$\frac{\delta\widetilde{F}[q(z_{d,i})]}{\delta q(z_{d,i} = k)} = \frac{\partial\widetilde{F}[q(z_{d,i})]}{\partial q(z_{d,i} = k)} = \int q(\boldsymbol{\theta}_d) \log(\beta_{k,w_{d,i}}, \theta_{d,k}) d\boldsymbol{\theta}_d - \log q(z_{d,i} = k) - 1 = 0 \tag{14}$$

Hence,

$$
\begin{aligned}
q(z_{d,i}=k) &\propto \exp\left[\int q(\boldsymbol{\theta}_d)\log(\beta_{k,w_{d,i}}\theta_{d,k})d\boldsymbol{\theta}_d\right] & (15)\\
&= \exp\left[\int q(\boldsymbol{\theta}_d)\log(\beta_{k,w_{d,i}})d\boldsymbol{\theta}_d\right]\exp\left[\int q(\boldsymbol{\theta}_d)\log(\theta_{d,k})d\boldsymbol{\theta}_d\right] & (16)\\
&= \beta_{k,w_{d,i}}\ \exp\left[\mathbb{E}_{q(\boldsymbol{\theta}_d)}\log(\theta_{d,k})\right] & (17)\\
&\propto \beta_{k,w_{d,i}}\ \frac{\exp\left[\Psi(\xi^{\theta}_{d,k})\right]}{\exp\left[\Psi(\sum_{k'=1}^{K}\xi^{\theta}_{d,k'})\right]} \qquad \xi^{\theta}_{d,k}=\mathbb{E}_{q(\boldsymbol{z}_d)}[N_{d,k}]+\alpha_k & (18)
\end{aligned}
$$

Note $\Psi(\ )$ is a digamma function. If $p(\boldsymbol{\theta}|\boldsymbol{\alpha})$ is a K-dimensional Dirichlet distribution,

$$
\mathbb{E}_{p(\boldsymbol{\theta}|\boldsymbol{\alpha})}[\log\theta_k] = \Psi(\alpha_k) - \Psi\left(\sum_{k=1}^{K}\alpha_k\right)
$$

2.2.2 Code

Caution: At least in the Python code, β is $V \times K$ (maybe in C as well)

In Python,

```python
q = lda.mnormalize(matrix(beta[d[0],:]) * matrix(diag(exp(digamma(alpha0 +
    nt))[0])), 1)
```

In C,

```c
/* In vbem.c */
/* vb-estep */
for (k = 0; k < K; k++)
        ap[k] = exp(psi(alpha[k] + nt[k]));
/* accumulate q */
for (l = 0; l < L; l++)
        for (k = 0; k < K; k++)
                q[l][k] = beta[d->id[l]][k] * ap[k];
/* normalize q */
for (l = 0; l < L; l++) {
        z = 0;
        for (k = 0; k < K; k++)
                z += q[l][k];
        for (k = 0; k < K; k++)
                q[l][k] /= z;
}
```

`ap[k]` is the second term in (18).

2.3 Update Equation of $q(\boldsymbol{\theta}_d)$

2.3.1 Derivation

Again, ELBO is

$$F[q(\boldsymbol{z},\boldsymbol{\theta})] = \int \sum_{d=1}^{M}\sum_{i=1}^{N_d} q(z_{d,i})q(\boldsymbol{\theta}_d)\log p(w_{d,i}|z_{d,i},\beta)p(z_{d,i}|\theta_d)d\boldsymbol{\theta}_d - \sum_{d=1}^{M}\sum_{i=1}^{N_d}\sum_{k=1}^{K} q(z_{d,i}=k)\log q(z_{d,i}=k) \quad (19)$$
$$- \sum_{d=1}^{M} \underbrace{\int q(\boldsymbol{\theta}_d)\log\frac{q(\boldsymbol{\theta}_d)}{p(\boldsymbol{\theta}_d)|\boldsymbol{\alpha})}d\boldsymbol{\theta}_d}_{\mathrm{KL}[q(\boldsymbol{\theta}_d)||p(\boldsymbol{\theta}_d|\boldsymbol{\alpha})]}$$

We use terms only related to $\boldsymbol{\theta}$.

$$\widetilde{F}[q(\boldsymbol{\theta})] = \int q(\boldsymbol{\theta})\sum_{\boldsymbol{z}} q(\boldsymbol{z})\log p(\boldsymbol{z}|\boldsymbol{\theta})d\boldsymbol{\theta} - \int q(\boldsymbol{\theta})\log\frac{q(\boldsymbol{\theta})}{p(\boldsymbol{\theta}|\boldsymbol{\alpha})}d\boldsymbol{\theta} \quad (20)$$

$$\widetilde{F}[q(\boldsymbol{\theta}_d)] = \int q(\boldsymbol{\theta}_d)\sum_{\boldsymbol{z}} q(\boldsymbol{z})\sum_{i=1}^{N_d}\log p(z_{d,i}|\boldsymbol{\theta}_d)d\boldsymbol{\theta}_d - \int q(\boldsymbol{\theta}_d)\log\frac{q(\boldsymbol{\theta}_d)}{p(\boldsymbol{\theta}_d|\boldsymbol{\alpha})}d\boldsymbol{\theta}_d \quad (21)$$

Using variational inference,

$$\frac{\delta\widetilde{F}[q(\boldsymbol{\theta}_d)]}{\delta q(\boldsymbol{\theta}_d)} = \frac{\partial\widetilde{F}[q(\boldsymbol{\theta}_d)]}{\partial q(\boldsymbol{\theta}_d)} = \sum_{\boldsymbol{z}} q(\boldsymbol{z})\sum_{i=1}^{N_d}\log p(z_{d,i}|\boldsymbol{\theta}_d) - \log\frac{q(\boldsymbol{\theta}_d)}{p(\boldsymbol{\theta}_d|\boldsymbol{\alpha})} - 1 = 0 \quad (22)$$

Before we move on, let's check some deformations:

- Dirichlet distribution

$$\mathrm{Dir}(\boldsymbol{\pi}|\boldsymbol{\alpha}) \equiv \frac{\Gamma(\sum_{k=1}^{K}\alpha_k)}{\prod_{k=1}^{K}\Gamma(\alpha_k)}\prod_{k=1}^{K}\pi_k^{\alpha_k-1} \quad (23)$$

- If we consider a category k in a document d, the average number of words that belong to the category k under certain latent variables is

$$\mathbb{E}_{q(\boldsymbol{z}_d)}[N_{d,k}] = \sum_{i=1}^{N_d} q(z_{d,i}=k) \quad (24)$$

- Remember $z_{d,i} \sim \mathrm{Multi}(\boldsymbol{\theta}_d)$ (Sato pp.26-27, Equation 2.1). Be careful that a word belongs to a category or not, so we can use $\delta(z_{d,i}=k)$ here.

$$p(z_{d,i}|\boldsymbol{\theta}_d) = \prod_{k=1}^{K}\theta_{d,k}^{\delta(z_{d,i}=k)} \quad (25)$$

Now, we can back to the variational inference

$$
\begin{aligned}
p(\boldsymbol{\theta}_d) &\propto p(\boldsymbol{\theta}_d|\boldsymbol{\alpha}) \exp\left[\sum_{\boldsymbol{z}} q(\boldsymbol{z}) \sum_{i=1}^{N_d} \log p(z_{d,i}|\boldsymbol{\theta}_d)\right] && (26)\\
&\propto \prod_{k=1}^{K} \theta_{d,k}^{\alpha_k - 1} \exp\left[\sum_{\boldsymbol{z}} q(\boldsymbol{z}) \sum_{i=1}^{N_d} \sum_{k=1}^{K} \delta(z_{d,i}=k) \log \theta_{d,k}\right] && (27)\\
&= \exp\left[\sum_{k=1}^{K} (\alpha_k - 1) \log \theta_{d,k}\right] \exp\left[\sum_{k=1}^{K} \sum_{i=1}^{N_d} q(z_{d,i}=k) \log \theta_{d,k}\right] && (28)\\
&= \exp\left[\sum_{k=1}^{K} (\alpha_k - 1) \log \theta_{d,k}\right] \exp\left[\sum_{k=1}^{K} \mathbb{E}_{q(\boldsymbol{z}_d)}[N_{d,k}] \log \theta_{d,k}\right] && (29)\\
&= \exp\left[\sum_{k=1}^{K} (\mathbb{E}_{q(\boldsymbol{z}_d)}[N_{d,k}] + \alpha_k - 1) \log \theta_{d,k}\right] && (30)\\
&= \prod_{k=1}^{K} \theta_{d,k}^{\mathbb{E}_{q(\boldsymbol{z}_d)}[N_{d,k}] + \alpha_k - 1} && (31)
\end{aligned}
$$

From (27) to (28), we marginalize the equation with respect to $q(\boldsymbol{z})$. For various $z_{d,i}$ in $\boldsymbol{z}$, some take $\delta(z_{d,i}=k) = 0$ and other take $\delta(z_{d,i}=k) = 1$. We only need to consider those that are $\delta(z_{d,i}=k) = 1$.

If we define $\xi_{d,k}^{\theta} = \mathbb{E}_{q(\boldsymbol{z}_d)}[N_{d,k}] + \alpha_k$, $q(\boldsymbol{\theta}_d)$ is a Dirichlet distribution whose parameters are $\boldsymbol{\xi}_d^{\theta} = (\xi_{d,1}^{\theta}, \xi_{d,2}^{\theta}, \cdots, \xi_{d,K}^{\theta})$. We can easily normalize it:

$$
q(\boldsymbol{\theta}_d|\boldsymbol{\xi}_d^{\theta}) = \frac{\Gamma(\sum_{k=1}^{K} \xi_{d,k}^{\theta})}{\prod_{k=1}^{K} \Gamma(\xi_{d,k}^{\theta})} \prod_{k=1}^{K} \theta_{d,k}^{\xi_{d,k}^{\theta} - 1} \tag{32}
$$

2.3.2 Code

2.3.2.1 $\mathbb{E}_{q(\boldsymbol{z}_d)}[N_{d,k}]$ in Equation (24)

In the dataset, we only have how many times each word appears, so we calculate (number of times a word appears) $\times\ q(z_{d,i} = k)$

In Python,

```python
nt = matrix(di[1]) * q
```

In C,

```c
/* In vbem.c */
for (k = 0; k < K; k++) {
        z = 0;
        for (l = 0; l < L; l++)
                z += q[l][k] * d->cnt[l];
        nt[k] = z;
}
```

What are stored in `di[1]` and `d->cnt[1]` is the word count for each word in each document.

3 More notes on code

3.1 `accum_beta()`

This makes matrix β. Maybe it corresponds to §A.4.1 in the original article.

In Python (mpre details are in `LDA-explanation.ipynb`),

```
t = d[1]
betas[t[0],:] += matrix(diag(t[1])) * q
```

Matrix `betas[t[0],:]` is (number of unique words in document no1) × (number of class (category). `t[0]` is word ID.

In C,

```
/* in learn.c */
int i, k;
int n = dp->len;

for (i = 0; i < n; i++)
        for (k = 0; k < K; k++)
                betas[dp->id[i]][k] += q[i][k] * dp->cnt[i];
```

Reference

1. Blei et al., "Latent Dirichlet Allocation", The Journal of Machine Learning Research, 2003.

2. Mochihashi, Daichi. "lda, a Latent Dirichlet Allocation package" at http://chasen.org/ daiti-m/dist/lda/ for C code

3. Sato, Makoto. 「PythonでLDAを実装してみる」at http://satomacoto.blogspot.jp/2009/12/pythonlda.html for Python code