Latent Space Approaches to Social Network Analysis

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Harvard University

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Motivation

Stochastic Block Model

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Only illustrative!

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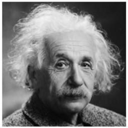

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Stochastic Block Model

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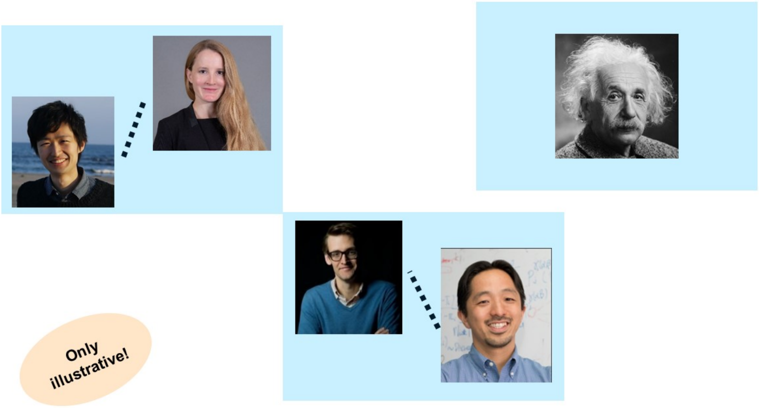

Stochastic Block Model

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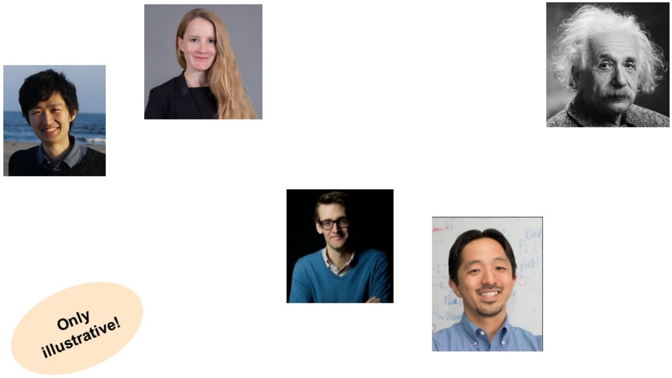

Latent Space Network Model

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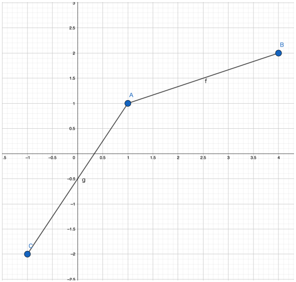
Latent Space Network Model

Distance vs Projection Model

- Goal of the model
  - We want to map actors on a latent space
  - If two actors are close in the space, they are more likely to have a tie

  $\Longrightarrow$ Distance model
- Incorporating how active actors are (asymmetric relation)

  $\Longrightarrow$ Projection model



Set Up for a Latent Space Network Model

Social network data set up

- n: number of actors
- $i, j = 1, ..., n$: actors
- y_{ij}: relational ties
- Y: $n \times n$ matrix representing these ties
- Direction of the path
    - Undirected: $i \sim j$, $y_{ij} = y_{ji}$
    - Directed: $i \rightarrow j$

Distance Models

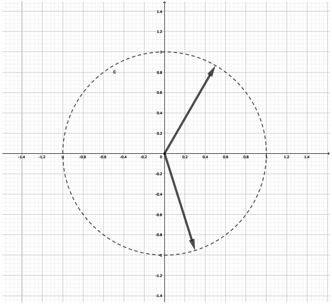
- i and j has a relationship $Y_{ij} \in \{0, \cdots, K-1\}$, typically $K = 2$
- We model $\pi_{ij} = p(Y_{ij} = 1 | z, x, \alpha, \beta)$
  - z: position in the latent space
  - x: covariates
  - α, β: parameters
- Same as the Logistic regression

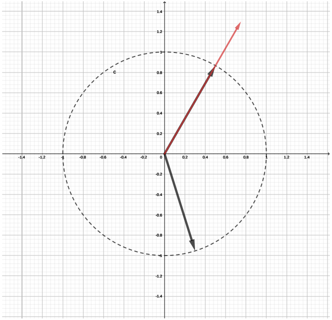
$$\begin{aligned} \text{logit}(\pi_{ij}) &= \log\left(\frac{\pi_{ij}}{1-\pi_{ij}}\right) \\ &= \alpha + \beta^\top x_{ij} - \underbrace{|z_i - z_j|}_{\text{distance in a latent space}} \end{aligned}$$

- All else equal, i and j are more likely to have a relationship if they are close in a latent space.

Projection Model

- We want to model asymmetric relationship
  - $|z_i - z_j|$ assumes symmetry
  - i and j have a tie but i is socially active
  - j is only a one of i's many ties
- Active level parameter $a_i > 0$
- A low-dimensional Euclidean space, e.g., $\vec{z} \in \mathbb{R}^2$
- Redefine: $\vec{z}_i = a_i \vec{e}_i$: $\color{red}{\vec{z}_i}$ $= 1.5\vec{e}_i$



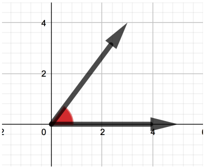


Projection Model

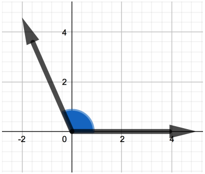
- Replace the distance term: $\operatorname{logit}(\pi_{ij}) = \alpha + \beta^\top x_{ij} + \frac{\vec{z}_i^\top \vec{z}_j}{|\vec{z}_j|}$

$$\frac{\vec{z}_i^\top \vec{z}_j}{|\vec{z}_j|} = \frac{|a_i \vec{e}_i||a_j \vec{e}_j| \cos\theta}{|a_j \vec{e}_j|} = a_i \cos\theta, \quad -a_i \le a_i \cos\theta \le a_i$$

- This is the model of homophily
- Given a_i, the smaller the angle between two vectors are, there more likely they have a tie



Prone to having a tie,
$\vec{e}_i^\top \vec{e}_j = \cos\theta > 0$



Averse to having a tie,
$\vec{e}_i^\top \vec{e}_j = \cos\theta < 0$

Great news: likelihood-based estimation methods are feasible...

Likelihood function:

$$\prod_{i\neq j} p(y_{ij}|z_i, z_j, x_{ij}, \alpha, \beta) = \prod_{i\neq j} (\pi_{ij})^{\mathrm{I}(y_{ij}=1)} (1-\pi_{ij})^{\mathrm{I}(y_{ij}=0)}, \ \ \pi_{ij} = p(Y_{ij} = 1|\cdot)$$

We define

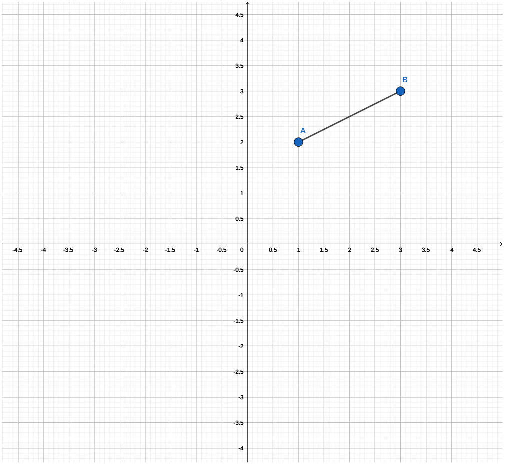
$$\eta_{ij} = \log \frac{\pi_{ij}}{1-\pi_{ij}} \ \Leftrightarrow \ \pi_{ij} = \frac{e^{\eta_{ij}}}{1+e^{\eta_{ij}}}$$

By taking log,

$$\begin{aligned} &\sum_{i\neq j} y_{ij}(\eta_{ij} - \log(1+e^{\eta_{ij}})) + (1-y_{ij})(-\log(1+e^{\eta_{ij}})) \\ &= \sum_{i\neq j} y_{ij}\eta_{ij} - \log(1+e^{\eta_{ij}}) \end{aligned}$$

Great news!

... but there is an identification problem



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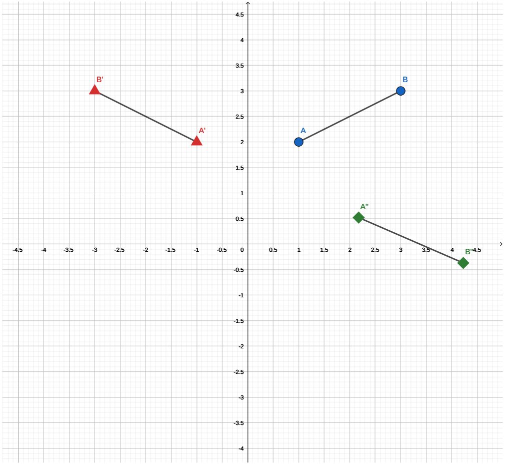


Distances between a set of points in Euclidean space are invariant under rotation, reflection, and translation

How to solve this

Idea:

- Define a class of positions equivalent to latent position Z under rotation, reflection, and translation ("configuration")
- Make inference on configurations via inference of particular elements of configurations that are comparable across configurations

Estimation

Procedure for sampling from posterior distribution:

0. Use prior information on α, β, and Z
1. Identify an MLE $\hat{Z}$ of Z, centered at origin, by direct maximization of likelihood
2. Construct Markov chain over model parameters:

   Start $Z_0 = \hat{Z}$

   a. Sample a proposal Z from a symmetric proposal distribution
   b. Accept that proposal for Z_{k+1} with some probability; otherwise, set $Z_{k+1} = Z_k$
   c. Store $\tilde{Z}_{k+1} = \arg\min_{TZ_{k+1}} tr(\hat{Z} - TZ_{k+1})' tr(\hat{Z} - TZ_{k+1})$
3. Update α and β with a Metropolis-Hastings algorithm

Applications

Actors	Tie (Y_{ij})	Distribution	Model
Monks	Affinity	$Y_{ij} \sim \text{Bern}(\pi_{ij})$	$\pi_{ij} = \text{logit}^{-1}(\alpha - \lVert Z_i - Z_j \rVert)$
Women	Contacts	$Y_{ij} \sim \text{Bern}(\pi_{ij})$	$\pi_{ij} = \text{logit}^{-1}(\alpha + \frac{Z_i Z_j}{\lVert Z_j \rVert})$
Senators	Bills co-sponsored	$Y_{ij} \sim \text{Pois}(\lambda_{ij})$	$\lambda_{ij} = \log^{-1}(\alpha - \lVert Z_i - Z_j \rVert)$

R package:
`latentnet`

Code repo:
`http://github.com/soubhikbarari/Gov2003-LatentNet`

Application 1: Monk Network (Sampson, 1986)

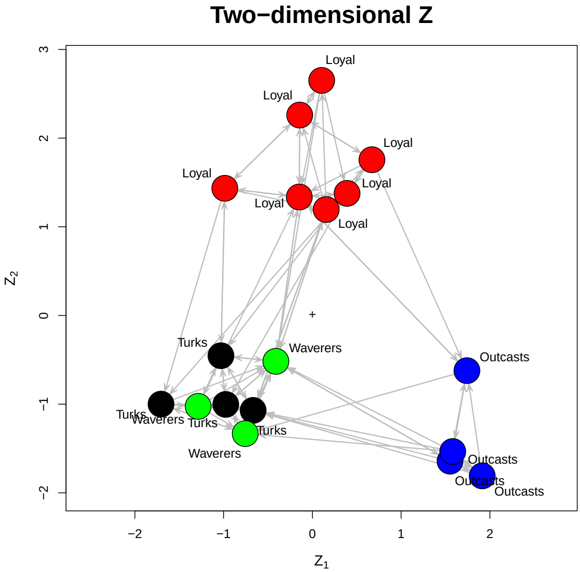
Political Preference and Social Relations between Novice Monks in New England

Actors	Tie (Y_{ij})	Distribution	Model
Monks	Affinity	$Y_{ij} \sim \text{Bern}(\pi_{ij})$	$\pi_{ij} = \text{logit}^{-1}(\underbrace{\alpha - \lVert Z_i - Z_j \rVert}_{\text{distance model}})$

Code:

```
ergmm(samplike ~ euclidean(d = 2))
```

Application 1: Monk Network (Sampson, 1986)



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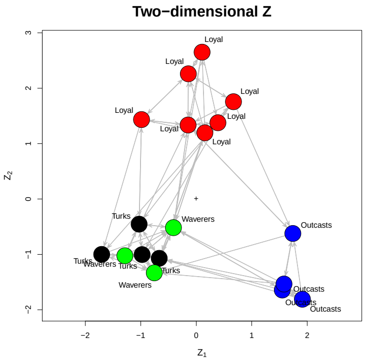
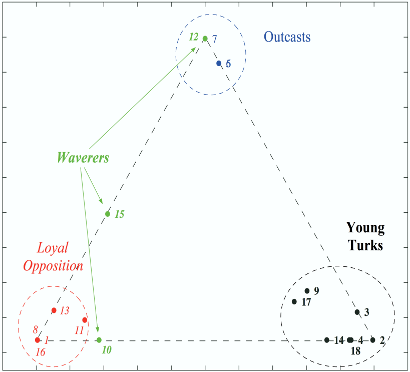


latent social position
(LSNM)



posterior mixed membership in a simplex
(MMSBM)

Application 1: Monk Network (Sampson, 1986)

Original distance model without clusters:

$$\begin{aligned} Y_{ij} &\sim \text{Bern}(\pi_{ij}) \\ \pi_{ij} &= \text{logit}^{-1}(\underbrace{\alpha - \|Z_i - Z_j\|}_{\text{distance model}}) \end{aligned}$$

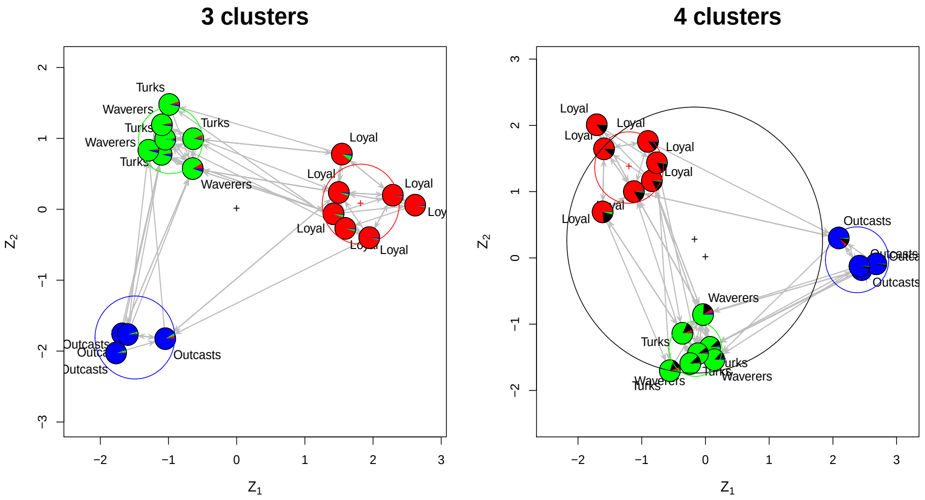
Extension of distance model to allow clustering:

$$\begin{aligned} Z_i &\sim \sum_{g=1}^{G} \lambda_g \text{MVN}_d(\mu_g, \sigma_g^2 I_d) \quad i = 1, \ldots, n, \\ \mu_g &\sim \text{MVN}_d(0, \omega^2 I_d) \quad g = 1, \ldots, G, \\ \sigma_g^2 &\sim \sigma_0^2 \text{Inv}\chi_\alpha^2 \quad g = 1, \ldots, G, \\ (\lambda_1, \ldots, \lambda_G) &\sim \text{Dirichlet}(\nu_1, \ldots, \nu_G) \end{aligned}$$

Code:

```
ergmm(samplike ~ euclidean(d = 2, G=3))
```

Application 1: Monk Network (Sampson, 1986)



Application 2: Women (Davis, Gardner, Gardner, 1941)

Social Stratification among Women in Mississippi

Actors	Tie (Y_{ij})	Distribution	Model
Women	Contacts	$Y_{ij} \sim \text{Bern}(\pi_{ij})$	$\pi_{ij} = \text{logit}^{-1}\Big(\underbrace{\alpha + Z_i Z_j}_{\text{projection model}}\Big)$

Code:

```
ergmm(davis ~ bilinear(d = 2, G = 2))
```

Application 2: Women (Davis, Gardner, Gardner, 1941)

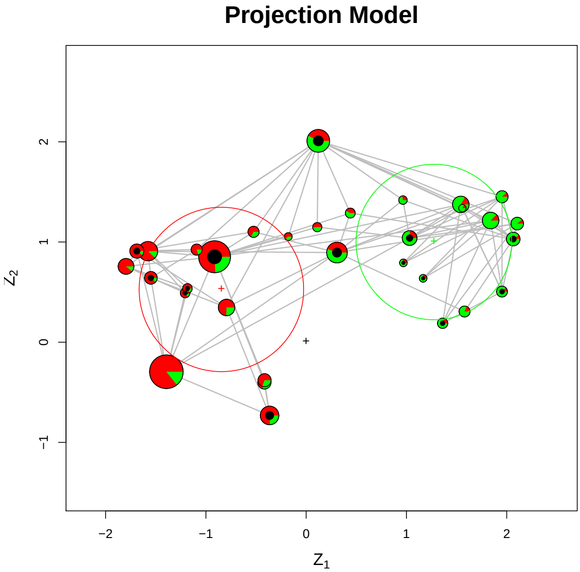
Social Stratification among Women in Mississippi

Actors	Tie (Y_{ij})	Distribution	Model
Women	Contacts	$Y_{ij} \sim \text{Bern}(\pi_{ij})$	$\pi_{ij} = \text{logit}^{-1}\Big(\underbrace{\alpha + Z_i Z_j}_{\text{projection model}} + \underbrace{\xi_i + \xi_j}_{\text{sociality effect}}\Big)$

Code:

```
ergmm(davis ~ bilinear(d = 2, G = 2) + rsociality)
```

Application 2: Women (Davis, Gardner, Gardner, 1941)



Application 3: Legislator Co-sponsorship (Fowler 2005)[1]

Hidden Social Network between U.S. Senators $\rightsquigarrow$ **Bill Co-sponsorship**. Can we discover the former using the latter?

Actors	Tie (Y_{ij})	Distribution	Model
Senators	Bills co-sponsored	$Y_{ij} \sim \text{Pois}(\lambda_{ij})$	$\lambda_{ij} = \log^{-1}(\alpha - \|Z_i - Z_j\|)$

Code:

```
cospons.latent <- ergmm(cospons.net ~ euclidean(d=2), family="Poisson")
```

[1] Fowler, James H. 'Legislative Cosponsorship Networks in the US House and Senate.' *Social Networks* 28.4 (2006): 454-465.

Application 3: Legislator Co-sponsorship (Fowler 2005)

Diagnostic tools

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- MCMC Convergence
  - Trace plot, autocorrelation plot, posterior density
  - `mcmc.diagnostics(cospons.latent)`

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- Posterior predictive checks
  - Do posterior-simulated networks resemble observed Y?
  - `plot(gof(cospons.latent))`

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Diagnostic tools

- MCMC Convergence
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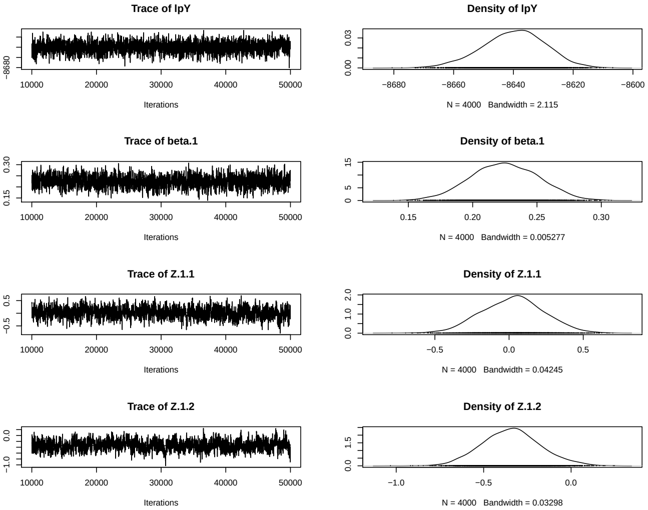
  - `mcmc.diagnostics(cospons.latent)`
- Posterior predictive checks
  - Do posterior-simulated networks resemble observed Y?
  - `plot(gof(cospons.latent))`
- Bayesian Information Criteria
  - $k\log(n) + 2\log\left(\frac{1}{L(\hat{Z}|Y)}\right)$ for n data points and k parameters
  - Model complexity additionally penalized by inverse likelihood
  - Can use for model comparison (pick lowest)
  - `summary(cospons.latent)`

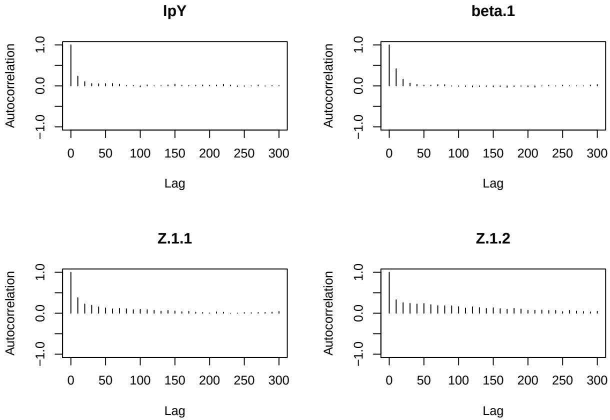
Application 3: Legislator Co-sponsorship (Fowler 2005)

MCMC Convergence – trace plot



Application 3: Legislator Co-sponsorship (Fowler 2005)

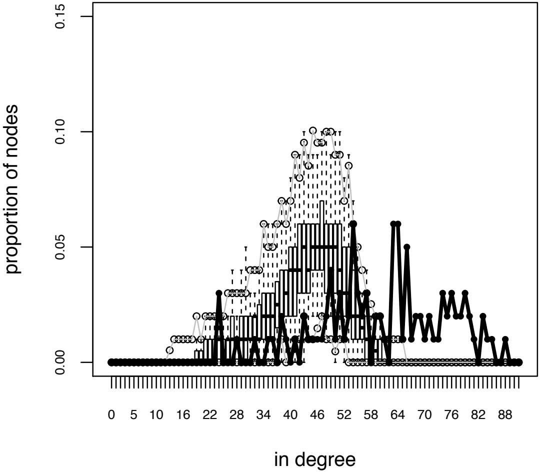
MCMC Convergence – autocorrelation plot



Application 3: Legislator Co-sponsorship (Fowler 2005)

Posterior predictive checks

network summary statistic: in-degree count



Application 3: Legislator Co-sponsorship (Fowler 2005)

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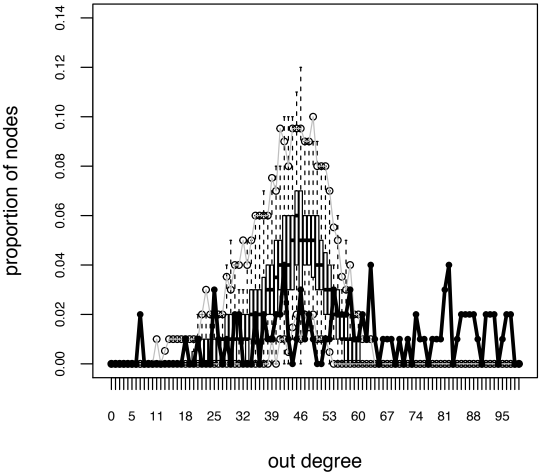
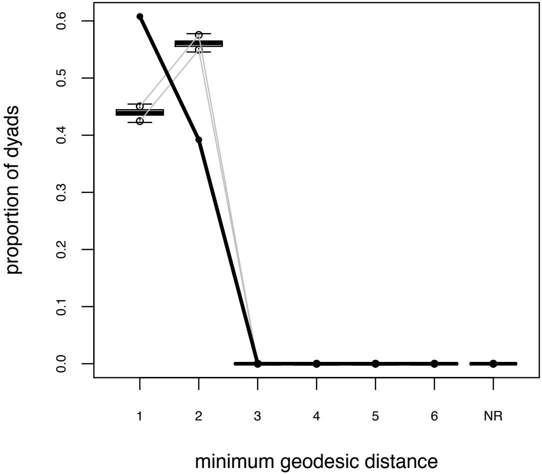


Application 3: Legislator Co-sponsorship (Fowler 2005)

Posterior predictive checks

network summary statistic: geodesic distance (min dist. b/t any node pair)



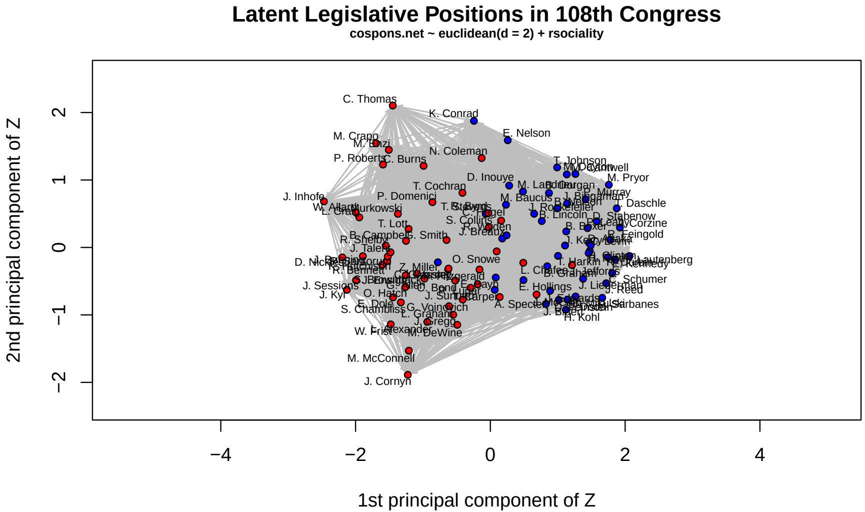
Application 3: Legislator Co-sponsorship (Fowler 2005)

Model comparison

Model	Overall BIC	Log lik.	Number of parameters
`cospons.net ~ euclidean(d=2)`	17325	-8640	$(102 \times 2) - 3 + 1$
`cospons.net ~ euclidean(d=3)`	11507	-5356	$(102 \times 3) - 5 + 1$
`cospons.net ~ euclidean(d=2) + rsociality`	11021	-5227	$(102 \times 3) - 3 + 1$
`cospons.net ~ euclidean(d=2, G=2)`	11486	-5531	$(102 \times 2) - 3 + 3$

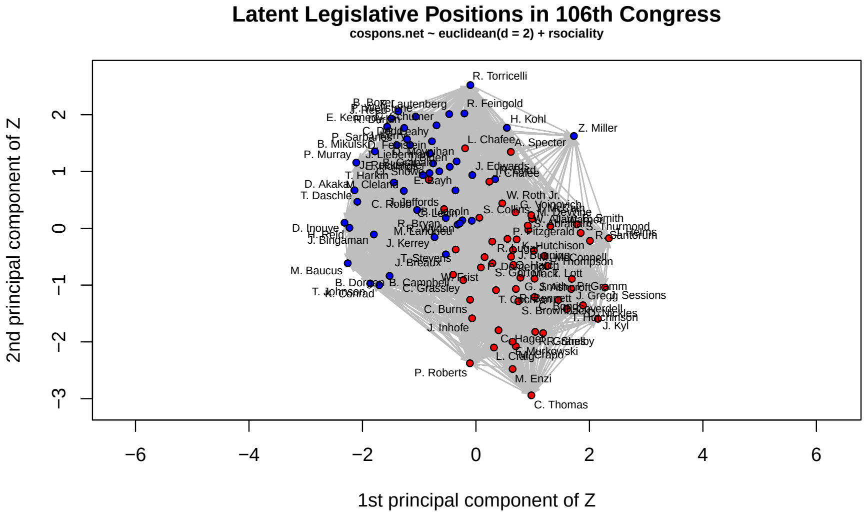
Application 3: Legislator Co-sponsorship (Fowler 2005)

2003-2005:



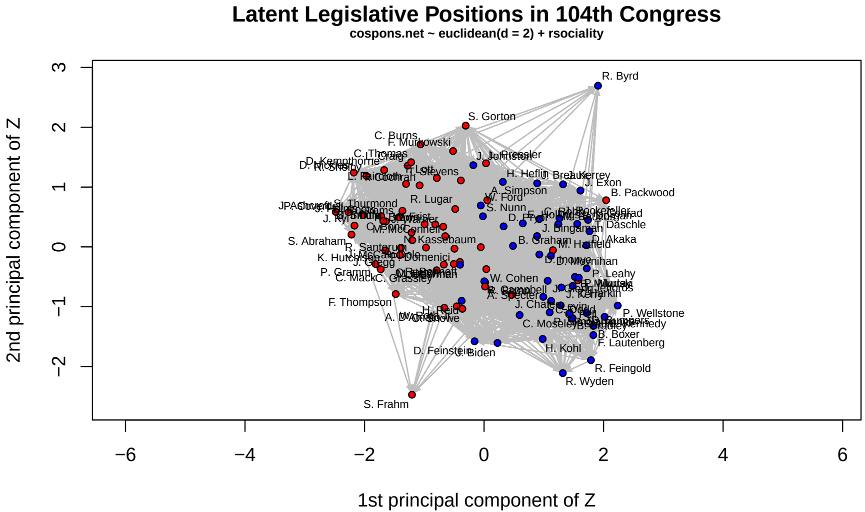
Application 3: Legislator Co-sponsorship (Fowler 2005)

1999-2000:



Application 3: Legislator Co-sponsorship (Fowler 2005)

1995-1997:



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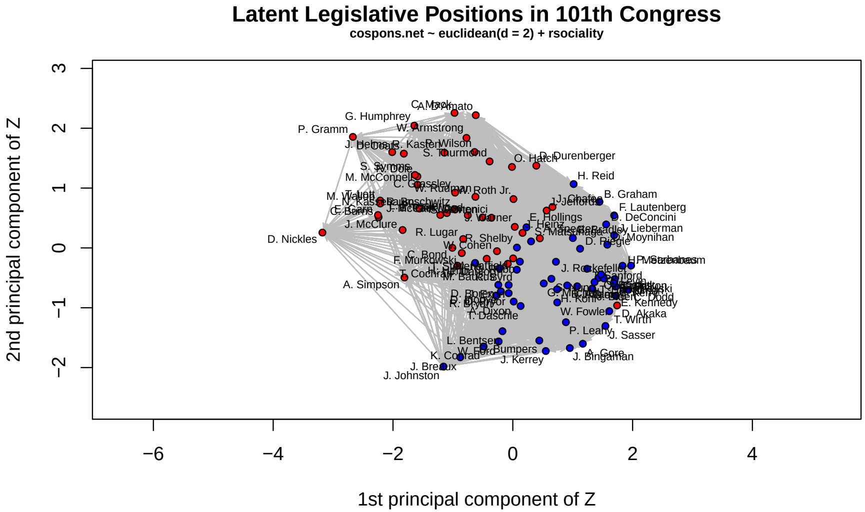
1989-1991:



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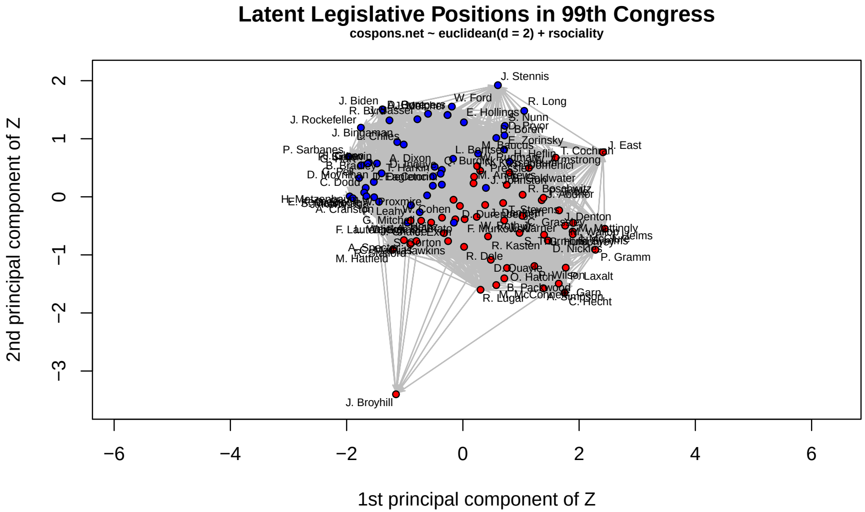
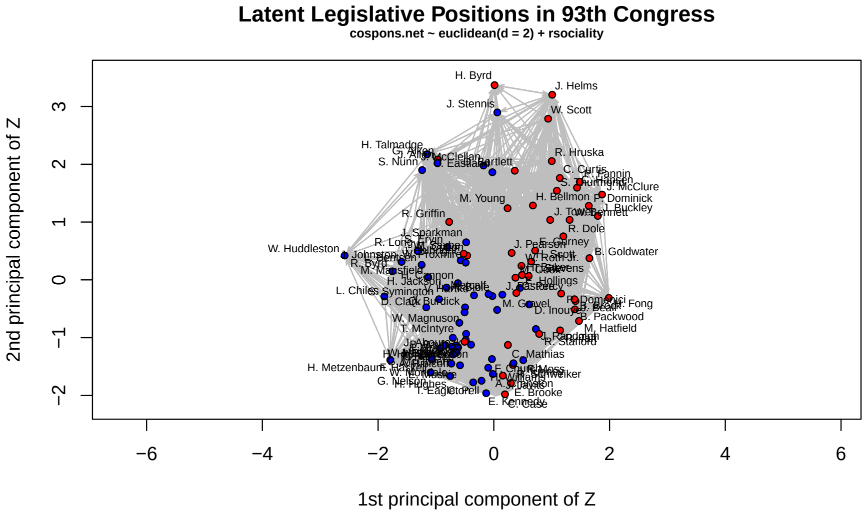
1985-1987:



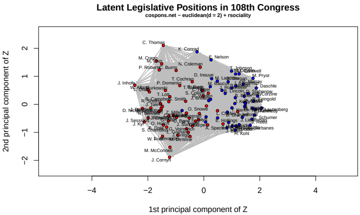
Application 3: Legislator Co-sponsorship (Fowler 2005)

1973-1975:

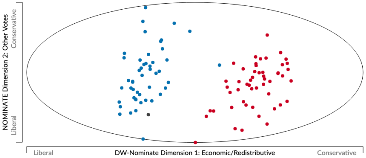


Application 3: Legislator Co-sponsorship (Fowler 2005)

Future extension: Connect **co-sponsorship** to **ideology**.



latent social position
(LSNM)



latent ideological position
(DW-nominate)

Concluding thoughts

- LSNM articulates an explicitly spatial model for network formation.
- Bayesian (e.g. posterior mean) and Frequentist methods (e.g. MLE) for estimating parameters.
- LSNM imposes fewer assumptions than MMSBM (conditionally independent ties).
  - freedom to specify the tie distribution, link function and priors.
- Choosing a larger $d \rightsquigarrow$ better fit (to a certain extent) but adds more parameters + exponentially increases run-time.
- Captures properties like transitivity[2] and reciprocity[3] in observed network.

[2] If $i \to j$ and $j \to k$ then probably $i \to k$.
[3] If $i \to j$ then probably ji.

Fun Fact

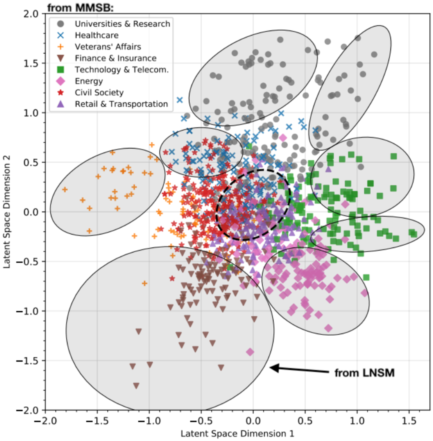
Even fictional researchers are using `ergmm` these days (from *The Wire*):

We hope you (real researchers) will too! ☺

Appendix

Comparing random graph models for validation

Scaling politicians and interest groups based on observed lobbying and sponsorship (Kim and Kunisky 2017)[4]:



[4] Kim, In Song, and Dmitriy Kunisky. 'Mapping Political Communities: A Statistical Analysis of Lobbying Networks in Legislative Politics'. Working Paper. `http://web.mit.edu/insong/www/pdf/network.pdf`, 2017.

LSNM vs. MMSBM

Supposing ties follow a Bernoulli distribution:

	Canonical LSNM	**Canonical MMSBM**
tie	$Y_{ij} \sim \text{Bern}(\text{logit}^{-1}(\alpha - \lVert Z_i - Z_j \rVert))$	$Y_{ij} \sim \text{Bern}(Z_{i\to j}^T \mathbf{B} W_{j\to i})$
structure	$Z_i \sim \underbrace{\text{MVN}_d([0],[100])}_{\text{diffuse prior}}$	$\text{Z}_{i\to j} \sim \text{Mult}(1,\pi_i)$ $\text{W}_{j\to i} \sim \text{Mult}(1,\pi_j)$ $\mathbf{B}_{sr} \sim \text{Beta}(\beta_{sr},\gamma_{sr})$
estimation	Metropolis-Hastings algorithm	Collapsed Gibbs sampling Variational Inference
process	position	community

Application: Tribes (Read, 1954)

Political Relationships between Tribes in New Guinea

Actors	Tie (Y_{ij})	Distribution	Model
Tribes	alliance neutral hostile	$Y_{ij} \sim \text{Mult}(1, \pi_{ij})$	?

Code:

```
ergmm(tribes ~ euclidean(d = 2, G = 3),
                         response = "sign.012",
                         family = "binomial.logit",
                         fam.par = list(trials = 2))
```

Application: Tribes (Read, 1954)