EX. 2

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Consider the partition

$$P = \{t_0, t_1, t_2, \dots, t_n\} = \{0, \frac{a}{n}, \frac{2a}{n}, \dots, a\}$$

for the interval $[0,a]$.

Let $f : [a,b] \to \mathbb{R}$ be $f(x) = \exp(x)$.

By definition

$$U(P,f) = \sum_{i=1}^{n} \inf\{f(t) : t_{i-1} \leq t \leq t_i\} \cdot (t_i - t_{i-1})$$

$$O(P,f) = \sum_{i=1}^{n} \sup\{f(t) : t_{i-1} \leq t \leq t_i\} \cdot (t_i - t_{i-1})$$

We observe that $f$ is monotone increasing, therefore $\inf\{f(t) : t_{i-1} \leq t \leq t_i\} = \exp(t_{i-1})$

$$\sup\{f(t) : t_{i-1} \leq t \leq t_i\} = \exp(t_i) \quad .$$

Further $(t_i - t_{i-1}) = \frac{a}{n}$.

As a consequence $U(P,f) = \sum_{i=1}^{n} \exp(t_{i-1}) \cdot \frac{a}{n}$

and $O(P,f) = \sum_{i=1}^{n} \exp(t_i) \cdot \frac{a}{n}$ .

Then

$$U(P,f)=\left(\sum_{i=1}^{n}\exp\left((i-1)\frac{a}{n}\right)\right)\cdot\frac{a}{n}=\left(\sum_{i=0}^{n-1}\exp\left(i\frac{a}{n}\right)\right)\cdot\frac{a}{n}=\frac{a}{n}\cdot\frac{1-\exp(a)}{1-\exp\left(\frac{a}{n}\right)}$$

$$O(P,f)=\left(\sum_{i=1}^{n}\exp\left(\frac{ia}{n}\right)\right)\cdot\frac{a}{n}=\left(\sum_{i=0}^{n-1}\exp\left(\frac{(i+1)a}{n}\right)\right)\cdot\frac{a}{n}=\frac{a}{n}\cdot\exp\left(\frac{a}{n}\right)\cdot\frac{1-\exp(a)}{1-\exp\left(\frac{a}{n}\right)}$$

Now recall that:

$$\lim_{n\to\infty}\frac{a}{n}\cdot\frac{1}{1-\exp\left(\frac{a}{n}\right)}=-1\ .$$

We can conclude that:

$$\int_0^a \exp(x)\,dx=\exp(a)-1\ .$$

EX. 3

a) $\displaystyle\int_0^{\pi} x\sin(x)\,dx=$ By parts $f=x \quad f'=1$

$\qquad g=-\cos x \quad g'=\sin x$

$$=\left[-x\cos x\right]_0^{\pi}+\int_0^{\pi}\cos x\,dx=\pi+\left[\sin x\right]_0^{\pi}=\pi$$

b) $\int_0^1 x\exp(x^2)\,dx = \frac{1}{2}\int_0^1 2x\exp(x^2)\,dx$ Substitution rule $\begin{cases} g(x) = x^2 \\ g'(x) = 2x \\ g(0) = 0 \\ g(1) = 1 \end{cases}$

$= \frac{1}{2}\int_0^1 \exp(x)\,dx = \frac{e-1}{2}$

c) $\int_1^e \ln(x)\,dx =$ By parts $\begin{cases} f = x, & f' = 1 \\ g = \ln(x), & g' = \frac{1}{x} \end{cases}$

$= \left[x\ln(x)\right]_1^e - \int_1^e dx =$

$= e - (e-1) = 1$

d) $\int_{-1}^1 \frac{x+1}{(x^2+2x+2)^3}\,dx = \frac{1}{2}\int_{-1}^1 \frac{2x+2}{(x^2+2x+2)^3}\,dx$

Substitution rule

$\begin{cases} g(x) = x^2+2x+2 \\ g'(x) = 2x+2 \\ g(-1) = 1 \\ g(1) = 5 \end{cases}$

$= \frac{1}{2}\int_1^5 \frac{1}{x^3}\,dx$

$$= \frac{1}{2}\left(\left[-\frac{1}{2x^2}\right]_1^5\right) = \frac{1}{2}\left(-\frac{1}{50}+\frac{1}{2}\right) = \frac{6}{25}$$

EX. 4

$$\int_0^1 x\sqrt{1+x}\,dx$$

By Substitution Rule: $\begin{cases} g(x) = \frac{2}{3}(1+x)^{\frac{3}{2}} \\ g'(x) = \sqrt{1+x} \\ g(0) = \frac{2}{3},\ g(1) = \frac{2}{3}\cdot 2^{\frac{3}{2}} \end{cases}$

$$\int_0^1 \underbrace{x}_{\parallel}\underbrace{\sqrt{1+x}}_{g'(x)}\,dx = \int_{\frac{2}{3}}^{\frac{2}{3}\cdot 2^{\frac{3}{2}}} \left(\frac{3}{2}x\right)^{\frac{2}{3}} - 1\,dx =$$

$$f(g(x)) \Rightarrow f(x) = \left(\frac{3}{2}x\right)^{\frac{2}{3}} - 1$$

$$= \left(\frac{3}{2}\right)^{\frac{2}{3}} \int_{\frac{2}{3}}^{\frac{2}{3}\cdot 2^{\frac{3}{2}}} (x)^{\frac{2}{3}}\,dx - \int_{\frac{2}{3}}^{\frac{2}{3}\cdot 2^{\frac{3}{2}}} dx =$$

$$= \left(\frac{3}{2}\right)^{\frac{2}{3}} \frac{3}{5}\left[x^{\frac{5}{3}}\right]_{\frac{2}{3}}^{\frac{2}{3}\cdot 2^{\frac{3}{2}}} - \left[\,x\,\right]_{\frac{2}{3}}^{\frac{2}{3}\cdot 2^{\frac{3}{2}}} \approx 0.64 \, .$$

By parts:

$$\begin{cases} f = x \quad , \; f' = 1 \\ g = \frac{2}{3}(1+x)^{\frac{3}{2}} , \; g' = \sqrt{1+x} \end{cases}$$

$$\Rightarrow \int_0^1 x\sqrt{1+x}\,dx = \frac{2}{3}\left[x(1+x)^{\frac{3}{2}}\right]_0^1 - \frac{2}{3}\int_0^1 (1+x)^{\frac{3}{2}} dx$$

$$= \frac{2}{3}\left[x(1+x)^{\frac{3}{2}}\right]_0^1 - \frac{2}{3}\cdot\frac{2}{5}\left[(1+x)^{\frac{5}{2}}\right]_0^1 \approx 0.64 \, .$$

Other suggestions when using integration by parts:

1) $\displaystyle\int_0^1 x \tan^{-1}(x)\,dx =$ $\qquad \begin{cases} f = \frac{x^2}{2} \quad , \; f' = x \\ g = \tan^{-1}(x) , \; g' = \frac{1}{1+x^2} \end{cases}$

$$= \left[\frac{x^2}{2}\tan^{-1}(x)\right]_0^1 - \underbrace{\int_0^1 \frac{x^2}{2}\cdot\frac{1}{1+x^2}\,dx}_{\text{this is still a complicated integral ...}}$$

Let's use a more clever partial integration:

$$\begin{cases} f = \frac{x^2+1}{2}, & f' = x \\ g = \tan^{-1}x, & g' = \frac{1}{1+x^2} \end{cases}$$

then $$\int_0^1 x\tan^{-1}x\,dx = \left[\frac{x^2+1}{2}\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x^2+1}{2}\cdot\frac{1}{x^2+1}\,dx$$

$$= \left[\frac{x^2+1}{2}\tan^{-1}x\right]_0^1 - \frac{1}{2}[x]_0^1$$

2) $$\int_0^1 \ln(x+2)\,dx =$$

$$\begin{cases} f = x, & f' = 1 \\ g = \ln(x+2), & g' = \frac{1}{x+2} \end{cases}$$

$$= \Big[x \ln(x+2)\Big]_0^1 - \underbrace{\int_0^1 \frac{x}{x+2}\,dx}_{\text{still a complicated integral ...}}$$

Let's find a smarter partial integration:

$$\begin{cases} f = x+2, & f' = 1 \\ g = \ln(x+2), & g' = \dfrac{1}{x+2} \end{cases}$$

then $$\int_0^1 \ln(x+2)\,dx = \Big[(x+2)\ln(x+2)\Big]_0^1 - \int_0^1 x+2 \cdot \frac{1}{x+2}\,dx$$

$$= \Big[(x+2)\ln(x+2)\Big]_0^1 - \Big[x\Big]_0^1$$