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Syllabus

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2.	**Logic :** • Propositions and logical operations, Truth tables • Equivalence, Implications • Laws of logic, Normal Forms • Predicates and Quantifiers • Mathematical Induction
3.	**Relations, Digraph and Lattices :** • Relations, paths and digraphs; • Properties and types of binary relations; • Manipulation of relations, closures, Warshall's algorithm; • Equivalence and Partial ordered relations; • Posets and Hasse diagram; • Lattice.
4.	**Functions and Pigeon Hole Principle :** • Definition and types of functions : Injective, Surjective and Bijective; • Composition, Identity and Inverse; • Pigeon-hole principle.
5.	**Generating Functions and Recurrence Relations :** • Series and Sequences • Generating functions • Recurrence relations • Recursive Functions: Applications of recurrence relations e.g. Factorial, Fibonacci, Binary search, Quick Sort etc.

Module	Contents
6.	**Graphs and Subgraphs :** • Definitions, Paths and circuits: Eulerian and Hamiltonian • Planer graphs, Graph coloring • Isomorphism of graphs • Subgraphs and Subgraph isomorphism
7.	**Trees :** • Trees and weighted trees • Spanning trees and minimum spanning tree • Isomorphism of trees and sub trees • Prefix codes
8.	**Algebraic Structures :** • Algebraic structures with one binary operation: semigroup, monoids and groups • Product and quotient of algebraic structures • Isomorphism, Homomorphism and Automorphism • Cyclic groups, Normal subgroups • Codes and group codes

Discrete Structures

Statistical Analysis

Chapter No.	Dec. 13	May 14	Dec. 14	May 15	Dec. 15	May 16
Chapter. 1	–	04 Marks	12 Marks	12 Marks	-	06 Marks
Chapter 2	24 Marks	11 Marks	09 Marks	09 Marks	10 Marks	12 Marks
Chapter 3	27 Marks	32 Marks	37 Marks	25 Marks	18 Marks	26 Marks
Chapter 4	09 Marks	13 Marks	16 Marks	12 Marks	19 Marks	14 Marks
Chapter 5	16 Marks	15 Marks	12 Marks	13 Marks	07 Marks	14 Marks
Chapter 6	13 Marks	06 Marks	13 Marks	04 Marks	11 Marks	20 Marks
Chapter. 7	04 Marks	13 Marks	09 Marks	05 Marks	07 Marks	20 Marks
Chapter 8	07 Marks	26 Marks	16 Marks	40 Marks	44 Marks	28 Marks
Repeated Questions	–	12 Marks	25 Marks	43 Marks	07 Marks	20 Marks

Dec. 2013

Chapter 2 : Logic [Total Marks - 24]

Q. 1(b) Let G be the set of rational numbers other than 1. Let define an operation * on G by $a * b = a + b - ab$ for all $a, b \in G$. Prove that (G, *) is a group. **(6 Marks)**

Ans. : Get set G is set of rational numbers

(i) For any $a, b \in G$, $a * b \in G$,

So set Z is closed under binary operation $*$.

(ii) Now check for associativity

$$(a * b) * c = a * (b * c)$$
$$(a + b - ab) * c = a * (b + c - bc)$$
$$a + b - ab + c - (a + b - ab)\, c = a + b + c - bc - a\,(b + c - bc)$$
$$= a + b - ab + c - (ac + bc - abc) = a + b + c - bc - (ab + ac - abc)$$
$$\cancel{a} + \cancel{b} - \cancel{ab} + \cancel{c} - ac - bc + \cancel{ab}c = \cancel{a} + \cancel{b} + \cancel{c} - bc - \cancel{ab} - ac + a\cancel{bc}$$
$$- ac - bc = - bc - ac$$
$$bc + ac = bc + ac$$

$\therefore$ $*$ is associative operation. $\therefore$ Algebraic system $(G, *)$ is semigroup

(iii) Now check for identity $a * 0 = a + 0 - a * 0 = a$

$\therefore$ '0' is identity element

$\therefore$ Algebraic system $(G, *)$ is monoid

(iv) and
$$a * b = a + b - ab$$
$$b * a = b + a - ba \qquad \therefore a * b = b * a$$

$\therefore$ It is commutative operation. From all above we can say that $(G,*)$ is group.

Q. 1(c) Find the number of integers between 1 and 1000 which are **(5 Marks)**

(i) Divisible by 2, 3 or 5. (ii) Divisible by 3 only but not by 2 nor by 5.

Ans. :

(i) Let A_1 be the set of integers between 1 and 100 divisible by 2.

Let A_2 be the set of integers between 1 and 100 divisible by 3.

Let A_3 be the set of integers between 1 and 100 divisible by 5.

Then $|A_1| = |1000/2| = 500$ $\qquad |A_2| = |1000/3| = 333$

$|A_3| = |1000/5| = 200$

$|A_1 \leftrightarrow A_2| = |1000/2*3| = 167$ $\qquad |A_1 \leftrightarrow A_3| = |1000/2*5| = 100$

$|A_2 \leftrightarrow A_3| = |1000/3*5| = 67$

and $|A_1 \leftrightarrow A_2 \leftrightarrow A_3| = |1000/2*3*5| = 33$

Number of integers between 1 and 1000 which are divisible by 2 or 3 or 5 i.e.

$$|A_1 \approx A_2 \approx A_3| = |A_1| + |A_2| + |A_3| - |A_1 \leftrightarrow A_2| - |A_1 \leftrightarrow A_3| - |A_2 \leftrightarrow A_3| + |A_1 \leftrightarrow A_2 \leftrightarrow A_3|$$

$$= 500 + 333 + 200 - 167 - 100 - 67 + 33 = 732$$

Hence, number of integer between 1 and 1000 are not divisible by 2, 3, or 5 i.e.

$$|A_1 \approx A_2 \approx A_3| = 1000 - 732 = 268$$

(ii) Number of integers between 1 and 1000 which are divisible by 3 Not by 2 nor by 5

$$= |A_2| - |A_1 \leftrightarrow A_2| - |A_2 \leftrightarrow A_3| + |A_1 \leftrightarrow A_2 \leftrightarrow A_3|$$

$$= 333 - 167 - 67 + 33 = 132$$

Q. 2(a) Prove by mathematical induction $x^n - y^n$ is divisible by $x - y$. **(4 Marks)**

Ans. : **Prove by mathematical induction that x-y is a factor of $x^n - y^n$:**

1. Test for n = 1....$(X^1 - Y^1) = X - Y$ is divisible by $(X - Y)$
2. Assume it is true for N = K, SO $(X^K - Y^K)$ is divisible by $(X - Y)$... let $X^K - Y^K = A(X - Y)$... where A is an integer.
3. Test for X = K + 1 ..... The statement is true.

$$X^{(K+1)} - Y^{(K+1)} = X^{(K+1)} - Y^{(K+1)} + X^K - Y^K - (X^K Y^K) = X^K(X + 1) - Y^K(Y + 1) - (X^K - Y^K)$$

$$= (X^K - Y^K)(X + 1 - Y - 1) - (X^K - Y^K) = (X^K - Y^K)(X - Y) - (X^K - Y^K)$$

An integer since every term. Is divisible by X – Y. So it is true for X = K + 1.

4. But this is true for X =1. So it is true for 1 + 1 = 2 ..... 2 + 1 = 3..... etc. For all integral values of N.
5. Hence the proof.

Q. 4(b) If (G, *) is an abelian group, then for all a, b ∈ G, prove that by mathematical induction $(a * b)^n = a^n * b^n$. **(5 Marks)**

Ans. : We resort to induction to prove that the result holds for positive integers. For n = 1, we have $(a \cdot b)^1 = a \cdot b = a^1 \cdot b^1$. So that result is valid for the base case. Suppose result holds for n = k –1, i.e. $(a \cdot b)^{k-1} = a^{k-1} b^{k-1}$. We need to show result also holds good for n = k. We have

$$(a \cdot b)^k = (a.b)^{k-1} \cdot (a.b) = (a^{k-1} \cdot b^{k-1}) \cdot (a \cdot b)$$
$$= (a^{k-1} \cdot b^{k-1}) \cdot (b \cdot a) = (a^{k-1} \cdot b^k) \cdot a = a \cdot (a^{k-1} \cdot b^k) = a^k \cdot b^k$$

So the result holds for $n = k$ too. Therefore, result holds for all $n \in \mathbb{N}$. Next suppose $n \in \mathbb{Z}$. If $n = 0$, then $(a \cdot b)^0 = e$ where e the identity element. Therefore $(a \cdot b)^0 = e = e \cdot e = a^0 \cdot b^0$. So the result is valid for $n = 0$ too. Next suppose n is a negative integer. So $n = -m$, where m is some positive integer. We have

$$(a \cdot b)^n = (a \cdot b)^{-m} = ((a \cdot b)^{-1})^m \text{ by definition of the notation}$$
$$= (b^{-1} \cdot a^{-1})^m = ((a^{-1}) \cdot (b^{-1}))^m$$
$$= (a^{-1})^m \cdot (b^{-1})^m \text{ as the result is valid for positive integers} = (a^{-m}) \cdot (b^{-m}) = a^n \cdot b^n$$

So the result is valid for negative integers too. Hence the result that $(a \cdot b)^n = a^n \cdot b^n$ holds in an abelian group for all $n \in \mathbb{Z}$.

Q. 5(d) Use the laws of logic to show that $[(p \Rightarrow q) \wedge \neg p] \Rightarrow \neg p$ is a tautology. **(4 Marks)**

Ans. : $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg P$ Putting $A \rightarrow B \Leftrightarrow \neg A \vee B$,

$\therefore [(\neg p \vee q) \wedge \neg q] \rightarrow \neg p$

$$= [(\neg p \wedge \neg q) \vee \frac{(q \wedge \neg q)}{F}] \rightarrow \neg p \quad \text{... (Applying distributive Law)}$$
$$= [(\neg p \wedge \neg q)] \rightarrow \neg p \; (\because q \wedge \neg p \Leftrightarrow F) = \neg(\neg p \wedge \neg q) \vee \neg p$$
$$= (p \vee q) \vee \neg p = p \vee q \vee \neg p$$
$$= T \vee q \quad \text{... } (\because p \vee \neg p \Leftrightarrow T)$$
$$= T \quad \text{... } (\because T \vee q \Leftrightarrow T)$$

$\therefore$ It is tautology, Hence Proved.

Chapter 3 : Relations, Diagraphs and Lattices [Total Marks - 27]

Q. 2(b) Let m be the positive integers greater than 1. Show that the relation $R = \{(a, b) \mid a \equiv b \pmod m\}$, i.e. aRb if and only if m divides a-b, is an equivalance relation on the set of integers. **(6 Marks)**

Ans. : $R = \{(a, b) \mid a - b \pmod m\}$

The relation is, aRb if m divides $(a - b)$ where m is positive integer greater than 1.

1) For any a, $(a - a)$ is always divisible by m.
Hence, aRa
$\therefore$ R is reflexive.

2) For any a, b, if $(a - b)$ is divisible by m, then $(b - a)$ is also divisible by m.
i.e. if $aRb \Rightarrow bRa$ $\therefore$ R is symmetric.

3) For any a, b, c, if aRb and bRc, then both $(a - b)$ and $(b - c)$ are divisible by m.
Here,
$a - c = (a - b) + (b - c)$ is also divisible by m.
Hence, aRc. Thus R is transitive.
As, R is reflexive, symmetric and transitive. Hence by definition of equivalence relation.
R is equivalence relation.

Q. 2(c) Let s = {1, 2, 3, 4} and A= S × S. Define the following relation : R on A : (a, b) R (a′, b′) if and only if a + b = a′ + b′. **(6 Marks)**

(i) Show that R is an equivalence relation, (ii) Compute A/R.

Ans. : $S = \{1, 2, 3, 4\}, A = S \times S$ ... given

$\therefore A = \{<1, 1>, <1, 2>, <1, 3>, <1, 4>, <2, 1> <2, 2> <2, 3>, <2, 4>, <3, 1>, <3, 2>, <3, 3>, <3, 4> <4, 1>, <4, 2>, <4, 3>, <4, 4>\}$

R is given on A, $(a + b)\ R\ (a^1 + b^1)$, iff $a + b = a^1 + b^1$

Reflexivity : $a + b\ R\ a + b$ i.e. $a + b = a + b$ ∴ Relation is reflexive.

Symmetric : We know that,

$$a + b = b + a \therefore (a + b)\ R\ (b + a) \therefore \text{The relation is symmetric.}$$

Transitive : Suppose,

$$a + b = p + q \text{ i.e. } (a + b)\ R\ (p + q) \text{ and}$$
$$p + q = x + y \text{ i.e. } (p + q)\ R\ (x + y)$$
$$\therefore a + b = x + y. \text{ Thus, } (a + b)\ R\ (x + y)$$

∴ The relation is Transitive. As the given relation is reflexive, symmetric and transitive. ∴ By define of equivalence relation. The above relation is equivalence relation.

Q. 2(d) If $f : A \rightarrow B$ be both one-to-one and onto, then prove that $f^{-1} : B \rightarrow A$ is also both one-to-one and onto. **(4 Marks)**

Ans. : Suppose $f : A \rightarrow B$ is one to one correspondence. Then there is a function $f^{-1} : B \rightarrow A$ called the inverse of f defined

As follows : $f^{-1}(B) \rightarrow A \Leftrightarrow f(A) = B$.

Inverse function are very important both in mathematics and in real world applications (e.g. Population modeling, nuclear physics etc). Determining inverse function is generally an easy problem in algebra.

For example : Given f, how do we find f^{-1}?

Let $f : R \rightarrow R$ be given by $f(x) = 4x - 1 = y$. Now, swap x and y and solve for y

$$4y - 1 = x \qquad 4y = x + 1$$
$$y = \frac{x+1}{4} \qquad \text{Thus } f^{-1}(x) = \frac{x+1}{4}$$

Q. 3(b) Let L_1 and L_2 be lettices shown below : **(7 Marks)**

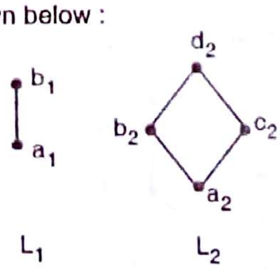


Fig. 1

Draw the Hasse diagram of $L_1 \times L_2$ with product partial order.

Ans. :

For Lattice $L_1\ R = \{(a_1, a_1), (b_1, b_1), (a_1, b_1)\}$

For Lattice L_2 R $= \{(a_2, a_2), (b_2, b_2), (c_2, c_2), (d_2, d_2), (a_2, d_2), (a_2, b_2), (a_2, b_2), (a_2, c_2), (b_2, d_2), (c_2, d_2)\}$

For $L_1 \times L_2$ with product partial order

$$R = \{((a_1, a_2), (a_1, a_2)), ((a_1, a_2), (a_1, b_2)), ((a_1, a_2), (a_1, c_2)), ((a_1, a_2), (a_1, d_2)), ((a_1, a_2), (b_1, a_2)), ((a_1, a_2), (b_1, b_2)), ((a_1, a_2), (b_1, c_2)), ((a_1, a_2), ((b_1, d_2)), ((a_1, b_2), (a_1, b_2)), ((a_1, b_2), (a_1, d_2)), ((a_1, b_2), (b_1, d_2)), ((a_1, b_2), (b_1, b_2)), ((a_1, c_2), (a_1, c_2)), ((a_1, c_2), (a_1, d_2)), ((a_1, c_2), (b_1, c_2)), ((a_1, c_2), (b_1, d_2)), ((a_1, d_2), (a_1, d_2)), ((a_1, d_2), (b_1, d_2)), ((b_1, a_2), (b_1, a_2)), ((b_1, a_2), (b_1, b_2)), ((b_1, a_2), (b_1, c_2)), ((b_1, a_2), (b_1, d_2)), ((b_1, b_2), (b_1, b_2)), ((b_1, b_2), (b_1, d_2)), ((b_1, c_2), (b_1, c_2)), ((b_1, c_2), (b_1, d_2)), ((b_1, d_2), (b_1, d_2))\}$$

Digraph for the above relation

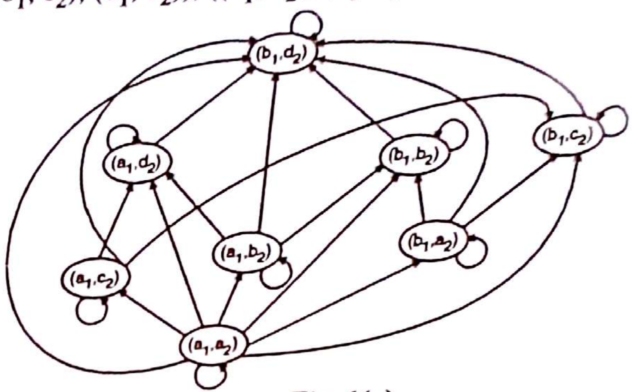


Fig. 1(a)

To convert this digraph into Hasse diagram

Step 1 : Remove cycles

Step 2 : Remove transitive edges

$((a_1, a_2), (a_1, d_2)), ((a_1, a_2), (b_1, b_2)), ((a_1, b_2), (b_1, d_2)), ((a_1, a_2), (b_1, d_2)), ((a_1, a_2), (b_1, c_2)), ((a_1, c_2), (b_1, d_2)), ((b_1, a_2), (b_1, d_2))$

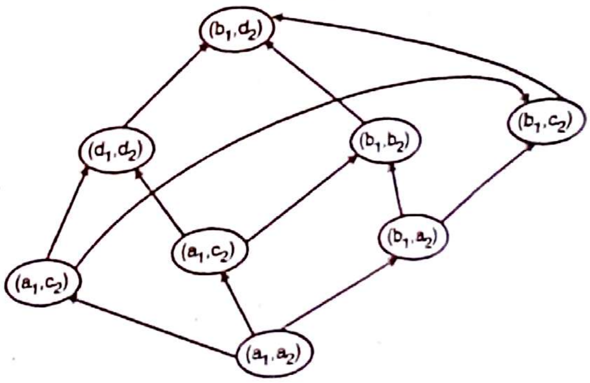


Fig. 1(b)

Step 3 : All edges are pointing upwards. Now remove arrows from edges, replace circles by dots.

Hasse diagram :

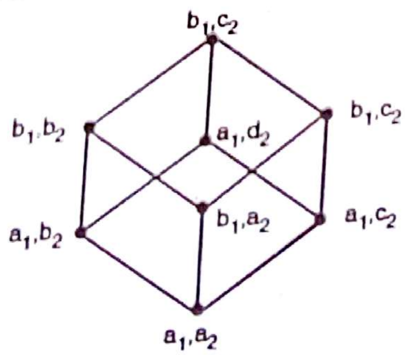


Fig. 2

Q. 3(c) Let A = {a, b, c}. Show that (P(A), ⊆) is a poset. Draw its Hasse diagram. P(A) is the power set of A. **(4 Marks)**

Ans. :

$$A = \{a, b, c\}$$

$$\text{Power set } P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

The relation given is subset relation (⊆) on P (A).

(1) For any $x \in P(A)$, $x \subseteq x$ i.e. x is a subset of x. Hence xRx. ∴ R is reflexive.

(2) For any $x, y \in P(A)$ $x \subseteq y$ and $y \subseteq x$ iff $x = y$

i.e. xRy and yRx iff $x = y$, ∴ R is antisymmetric.

(3) For any x, y and $z \in P(A)$

if $x \subseteq y$ and $y \subseteq z$ then it is obvious that $x \subseteq z$. i.e. if xRy and yRz implies xRz

∴ R is transitive.

As R is Reflexive, Antisymmetric and transitive.

Thus, $< P(A), E >$ is partially ordered set (Poset)

Hasse diagram :

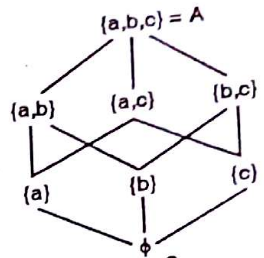


Fig. 3

Chapter 4 : Functions and Pigeon Hole Principle [Total Marks - 09]

Q. 3(a) Consider an equilateral triangle whose sides are of length 3 units. If ten points are chosen lying on or inside the triangle, then show that at least two of them are no more than 1 unit apart. **(5 Marks)**

Ans. :

M1, M2, M3, M4, M5, M6 are midpoints of sides AC, AB and BC, respectively. Let the nine small triangles created be the pigeonholes. For any ten points in or on triangle ABC, at least two must be in or on the small triangle and thus are no more than 1 unit apart.

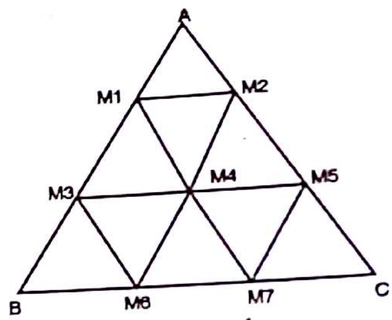


Fig. 4

Q. 4(c) If f is a homorphism from a commutative group (S, *) to another group (T, *'), then prove that (T, *') is also commutative. **(4 Marks)**

Ans. : Let t_1 and t_2 any elements of T

Then there exist s_1 and s_2 in S.

Such that $t_1 = f(s_1)$ and $t_2 = f(s_2)$

$$\therefore t_1 *' t_2 = f(s_1) *' f(s_2) = f(s_1 * s_2) = f(s_2 * s_1) = f(s_2) *' f(s_1)$$

$$= t_2 *' t_1$$

Hence (T, *') is commutative

Chapter 5 : Generating Functions and Recurrence Relations [Total Marks - 16]

Q. 1(d) Find all solutions of the recurrence relation $a_n = 5a_{n-1} + 6a_{n-2} + 7^n$ **(5 Marks)**

Ans. : Let us guess $a_n^{(p)} = c \cdot 7^n$ Then $c \cdot 7^n = 5c \cdot 7^{n-1} - 6c \cdot 7^{n-2} + 7^n$

Hence $49c = 35c - 6c + 49, \quad c = \frac{49}{20}$

$$a_n^{(p)} = \frac{49}{20} \cdot 7^n \text{ is a particular solution.}$$

By Theorem 9, we have $a_n^{(n)} = \alpha_1 2^n + \alpha_2 3^n$ as solutions of the associated homogeneous recurrence relation. Therefore

$$a_n = \alpha_1 2^n + \alpha_2 3^n + \frac{49}{20} 7^n \text{ are all solutions.}$$

Q. 5(a) Find the generating function for the following sequence 1,2,3,4,5,6,............ **(5 Marks)**

Ans. : The generating function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

∴ The generating function for sequence $\{1, 2, 3, 4, 5, 6 \ldots\}$ is $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5$

Q. 5(b) Solve the recurrence relation $a_r = 3a_{r-1} + 2$, $r \geq 1$ with $a_0 = 1$, using generating function. **(6 Marks)**

Ans. : Given recurrence relation is

$$a_r = 3a_{r-1} + 2 \text{ is converted as follows.}$$

$$a_r - 3a_{r-1} = 2$$

Multiplying both sides by z^r, we obtain $a_r z^r - 3a_{r-1} z^r = 2z^r$

Since $r \geq 1$, summing for all r, we get, $\sum_{r=1}^{\infty} a_r z^r - 3 \sum_{r=1}^{\infty} a_{r-1} z^r = 2 \sum_{r=1}^{\infty} z^r$

Consider the first term, $\sum_{r=1}^{\infty} a_r z^r = a_1 z + a_2 z^2 + a_3 z^3 + \ldots$

Since the generating function

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

$$z \sum_{r=1}^{\infty} a_r z^r = A(z) - a_0$$

For the second term, $\sum_{r=1}^{\infty} a_{r-1} z^r = z \sum_{r=1}^{\infty} a_{r-1} z^{r-1} = z\,A(z)$

Also the third term gives

$$\sum_{r=1}^{\infty} z^r = \frac{z}{(1-z)}$$

Hence we obtain

$$[A(z) - a_0] - 3z\,A(z) = \frac{2z}{1-z} \quad \text{or} \quad (1-3z)\,A(z) = \frac{2z}{1-z} + a_0$$

But, $a_0 = 1$

$$\therefore \quad (1-3z)\,A(z) = \frac{2z}{1-z} + 1 = \frac{1+z}{1-z}$$

or

$$A(z) = \frac{1+z}{(1-z)(1-3z)} = \frac{2}{(1-3z)} - \frac{1}{(1-z)}$$

Consequently we have, $a_r = 2(3)^r - (1)^r,\ r \geq 0$

which is the solution of the given recurrence relation.

Chapter 6 : Graphs [Total Marks - 13]

Q. 3(d) How many vertices are necessary to construct a graph with exactly 6 edges in which each vertex is of degree 2. **(4 Marks)**

Ans. :

We know that, Sum of degrees of all the nodes is twice of the no. of edges in the graph.

i.e. Sum of degrees of all nodes $= 2 * E$.

Here, degree of each node is given as 2 and number of edges are given as 6.

$$\therefore 2 * N = 2 * 6 \qquad \therefore N = 6$$

$\therefore$ 6 vertices are necessary to construct a graph with exactly 6 edges and each vertex having degree = 2.

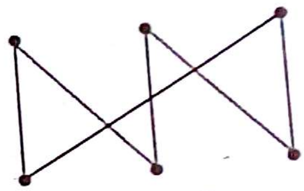


Fig. 5

Q. 4(a) Show that if every element in a group is its own inverse, then the group must be abelian. **(4 Marks)**

Ans. : Suppose $x = x^{-1}$ for all $x \in G$. this means that $x^2 = x(x) = x(x^1) = e$, for ALL $x \in G$. now if $x, y \in G$, then so is xy, so it must also be true that :

$$(xy)^2 = e, \text{ so } (xy) \quad (xy) = xyxy = e.$$

$$\text{then, } xy = (xe) \quad y = x(xyxy)$$

$$y = (xx)yx(yy) = eyxe = yx, \text{ so G is abelian.}$$

Q. 5(c) Show that the following graphs are isomorphic **(5 Marks)**

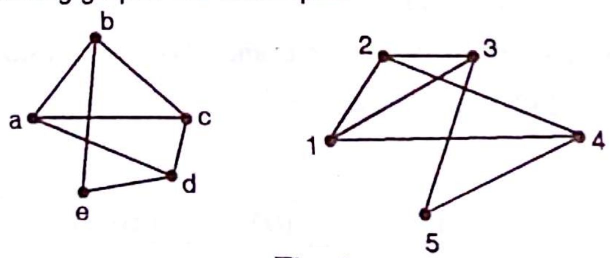


Fig. 6

Ans. : There are 5 nodes in both are graphs. $\therefore$ No. of nodes in graph 1 = No. of nodes in graph 2. Similarly No. of edges in G_1 = No. of edges in G_2.

Here, the mappings are given below,

$a \leftrightarrow 1$, $b \leftrightarrow 2$, $c \leftrightarrow 3$, $d \leftrightarrow 4$, $e \leftrightarrow 5$

The degree of node 'a' in a G_1 and degree of node '1' in G_2 are equal. Similarly for other nodes we can show that, their degrees are equal. $\therefore$ The graphs are Isomorphic.

Chapter 7 : Trees [Total Marks - 04]

Q. 1(a) Prove that in a full binary tree with n vertices, the number of pendant vertices is $(n + 1)/2$.

(4 Marks)

Ans. : If a full binary tree, only one vertex, namely, the root is of even degree (namely 2) and each of the other $(n - 1)$ vertices is of odd degree (namely 1 or 3). Since the number of vertices of odd degree in an undirected graph is even, $(n - 1)$ is even.

$\therefore$ n is odd

Now let p be the number of pendant vertices of the full binary tree.

$\therefore$ The number of vertices of degree 3 = $n - p - 1$.

The sum of the degrees of all the vertices of the tree

$$= 1 \times 2 + p \times 1 + (n - p - 1) \times 3 = 3n - 2p - 1.$$

Number of edges of the tree $= \frac{1}{2}(3n - 2p - 1)$ ($\because$ each edge contributes 2 degrees)

But the number of edges of a tree with n vertices = $n - 1$(by an earlier property)

$\therefore$ $\frac{1}{2}(3n - 2p - 1) = n - 1$ i.e. $3n - 2p - 1 = 2n - 2$

i.e. $2p = n + 1$ or $p = \frac{n+1}{2}$

Chapter 8 : Algebraic Structures [Total Marks - 07]

Q. 4(d) Consider the (3, 5) group encoding function :

$e\ B^3 \rightarrow B^5$ defined by **(7 Marks)**

e(000) = 00000 e(100) = 10011

e(001) = 00110 e(101) = 10101

e(010) = 01001 e(110)= 11010

e(011) = 01111 e(111) = 11100

Decode the following words relative to a maximum likelyhood decoding function.

(i) 11001 (ii) 01010 (iii) 00111

Ans. : Prepare decoding table :

00000	00110	01001	01111	10011	10101	11010	11100
00001	00111	01000	01110	10010	10100	11011	11101
00010	00100	01011	01101	10001	10111	11000	11110
00100	00010	01101	01011	10111	10001	11110	11000

01000	01110	00001	00111	11011	11101	10010	10100
10000	10110	11001	11111	00011	00101	01010	01100
10001	10111	11000	11110	00010	00100	01011	01101
10010	10100	11011	11101	00001	00111	01000	01110

(i) If we receive the word 11001, we first locate it in the 3rd column of the decoding table. Where it is underlined once. The word at the top of the 3rd column is 01001. Since e (010) = 01001. We decode 11001 as 010.

(ii) If we receive the word 01010, we first locate it in the 7th column of the decoding table. Where it is underlined twice. The word at the top of the 7th column is 11010. Since e (110) = 11010. We decode 01010 as 110.

(iii) Similarly 00111 is located in 2nd column of the decoding table where it is underlined thrice. The word at the top of the 2nd column is 00110. Since e (001) = 00110, we decode 00111 as 001.

❑❑❑

May 2014

Chapter 1 : Set Theory [Total Marks : 04]

Q. 2(c) Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$ **(4 Marks)**

Ans. : Let $(a, b) \in A \times (B \cap C)$ ...(1)

By the definition of Cartesian product, this means

$$a \in A \text{ and } b \in c$$

$$(a, b) \in A \times B \text{ and } (a, b) \in A \times C$$

$$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C) \quad ...(2)$$

Again Let $(a, b) \in (A \times B) \cap (A \times C)$

$$a \in A, b \in (B \cap C)$$

$$(a, b) \in A \times (B \cap C)$$

$$\therefore (A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$$

From Equation (1), (2), (3) we have, ...(3)

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Chapter 2 : Logic [Total Marks : 11]

Q. 1(a) Prove that $8^n - 3^n$ is a multiple of 5 by mathematical induction, $n \geq 1$. **(5 Marks)**

Ans. : Let $P(n) : 8^n - 3^n$ is a multiple of 5

(i) Basis of induction : For $n = 1$

$P(1) : 8^1 - 3^1 = 5$ which is divisible by 5

$\therefore$ P(1) is true.

(ii) Induction step : Assume P(k) is true i.e. $(8^k - 3^k)$ is a multiple of 5, and prove $P(k + 1)$ is also true.

$$P(k+1) : 8^{k+1} - 3^{k+1} = 8^k \cdot 8 - 3^k \cdot 3$$

$$= 8^k \cdot 8 - 3^k \cdot 8 + 3^k \cdot 5 = 8 (8^k - 3^k) + 3^k \cdot 5$$

Now $8(8^k - 3^k)$ is a multiple of 5 by induction hypothesis and $3^k . 5$ is already a multiple of 5.

Hence $8^n - 3^n$ is a multiple of 5, for $n \geq 1$.

Q. 6(b) Use the laws of logic to determine the following expression as tautology or contradiction.

Ans. : **(6 Marks)**

$$(p \wedge (p \Rightarrow q)) \Rightarrow q \equiv \sim (p \wedge (\sim p \vee q)) \vee q$$

$$\equiv (\sim p \vee \sim (\sim p \vee q)) \vee q$$

$$\equiv (\sim p \vee (p \wedge \sim q)) \vee q$$

$$\equiv (\sim p \vee \sim q) \vee q$$

$$\equiv \sim p \vee \text{True} \quad \text{- Complement law}$$

$$\equiv \text{True} \quad \text{- Identity law}$$

Chapter 3 : Relations, Digraphs and Lattice [Total Marks : 32]

Q. 1(b) Show that if a relation on set A is transitive and irreflexive, then it is asymmetric. **(5 Marks)**

Ans. :

Assume that a relation is transitive and irreflexive and let it be symmetric.

Let (a, b) and (b, c) belong to R and

let (b, a) and (c, b) belong to R

$\therefore$ $(a, c) \in R$ ... As R is transitive

Also $(c, b) \in R$

We have $(b, c) \in R$ and $(c, b) \in R$ ...As R is transitive.

$\therefore$ $(b, b) \in R$

But our relation is irreflexive $\rightarrow$ Assumption

$\therefore$ Our Assumption is false

$\therefore$ The relation is not symmetric i.e. it is asymmetric.

$\therefore$ If a relation on a set A is transitive and irreflexive, then it is asymmetric.

Q. 2(b) Let R be a relation on the set of integers Z defined by a R b if and only if $a \equiv m \pmod 5$. Prove that R is an equivalence relation. Find Z/R. **(8 Marks)**

Ans. :

Let A = Z and Let $R = \{(a, b) | a \equiv m \pmod 5 \text{ and } b \equiv m \pmod 5\}$

Equivalence Relation :

(a) **Reflexive :** We know a R b if $a \equiv m \pmod 5$ and $b = m \pmod 5$. Congruence module m is reflexive. This is reflexive relation.

(b) **Symmetric :** This relation is symmetric.

As a R b, $a \equiv m \pmod 5$ and $b \equiv m \pmod 5$

Hence b R a, $b \equiv m \pmod 5$ and $a \equiv m \pmod 5$.

(c) **Transitive :** This relation is Transitive.

if a R c, $a \equiv m \pmod 5$ and $b \equiv m \pmod 5$

b R c, $b \equiv m \pmod 5$ and $c \equiv m \pmod 5$

then

a R c, $a \equiv m \pmod 5$ and $c = m \pmod 5$

by transitivity a – m is divisible by 5 and c – m is divisible by 5

Z|R

Now a R b if a mod 5 = m and b mod 5 = m

Here, m = 0 or m = 1

a R b if a mod 5 = 1 and b mod 5 = 1

a R b if a mod 5 = 0 and b mod 5 = 0

We can represent integers either as Z_n or Z_{n+1} partition of Z i.e.

$$Z | R = \{\{5n\}, \{5n + 1\}\}$$

Q. 3(a) Let A = {1, 2, 3, 4} and R = {(1, 2), (2, 3), (3, 4), (2, 1)}. Find the transitive closure using Warshall's algorithm. **(6 Marks)**

Ans. :

$$W_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

and $n = 4$

First we find W_1, so that $k = 1$. W_0 has 1's in location 2 of column 1 i.e. (2, 1) and location 2 of row 1 i.e. (1, 2)

$$\begin{array}{ll} \text{i j} & p_1 : (2, 1) \\ \text{i j} & q_1 : (1, 2) \end{array}$$

$\therefore$ add (p_i, q_j) i.e. (2, 2) in W_k

Thus W_1 is just W_0 with a new 1 in position (2, 2)

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Now we compute W_2 so that $k = 2$. We must consult column 2 and row 2 of W_1. Matrix W_1 has 1's locations 1 and 2 of column 2 and locations 1, 2, and 3 of row 2. i.e.

$$\begin{array}{llllll} & \text{i j} & & \text{i j} & & \\ p_1 : & (1, 2) & p_2 : & (2, 2) & & \\ & \text{i j} & & \text{i j} & & \text{i 1} \\ q_1 : & (2, 1) & q_2 : & (2, 2) & q_3 : & (2, 3) \end{array}$$

We must put is in positions (p_i, q_j) i.e. (1, 1), (1, 2), (1, 3), (2, 1), (2,2) and (2, 3) of matrix W_1 (if 1's are not already there). We see that

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Proceeding, we see that column 3 of W_2 has 1's in locations 1 and 2, and row 3 of W_2 has a 1 in location 4. To obtain W_3 we must put 1's in position 1, 4 and 2, 4 of W_2 i.e. $k = 3$,

$$\begin{array}{llll} & \text{i j} & & \text{i j} \\ p_1 : & (1, 3) & p_2 : & (2, 3) \\ & \text{i j} & & \\ q_1 : & (3, 4) & & \end{array}$$

Thus, to obtain W_3 we must put 1's in position (1, 4) and (2, 4) so

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, W_3 has 1's in locations 1, 2, 3 of column 4 and no 1's in row 4, so no new 1's are added and $MR_{\infty} = W_4 = W_3$. The procedure illustrated in above examples yields the following algorithm for computing the matrix, Closure, of the transitive closure of a relation R represented by the $N \times N$ matrix MAT.

Algorithm Warshall

1. CLOSURE ← MAT
2. FOR K = 1 THRU N
   a. FOR I = I THRU N
      1. FOR J = 1 THRU N
         a. CLOSURE [I, J] ← CLOSURE [I,J]
            V(CLOSURE[I,K]∧
            CLOSURE [K, J])

END OF ALGORITHM WARSHALL.

Q. 3(b) Consider the lattices $L_1 = \{1, 2, 4\}$, $L_2 = \{1, 3, 9\}$ under divisibility. Draw the lattice $L_1 \times L_2$. **(7 Marks)**

Ans. : Partial ordering relation of division on L_1 is

$$R_1 = \{(1, 1), (1, 2), (1, 4), (2, 2), (2, 4), (4, 4)\}$$

Partial ordering relation of division on L_2 is

$$R_2 = \{(1, 1), (1, 3), (1, 9), (3, 3), (3, 9), (9, 9)\}$$

Product partial order relation on $L_1 \cdot L_2$ is

$$\begin{aligned} R_3 = \{&((1, 1), (1, 1))\ ((1, 1), (1, 3)), ((1, 1), (1, 9)),\\ &((1, 1), (2, 1)), ((1, 1), (2, 3)), ((1, 1), (2, 9)),\\ &((1, 1), (4, 1)), ((1, 1), (4, 3)), ((1, 1), (4, 9)),\\ &((1, 3), (1, 3)), ((1, 3), (1, 9)), ((1, 3), (2, 3)), ((1, 3), (2, 9)),\\ &((1, 3), (4, 3)), ((1, 3), (4, 9)),\\ &((1, 9), (1, 9)), ((1, 9), (2, 9)), ((1, 9), (4, 9)),\\ &((2, 1), (2, 1)), ((2, 1), (2, 3)), ((2, 1), (2, 9)),\\ &((2, 1), (4, 1)), ((2, 1), (4, 3)), ((2, 1), (4, 9)),\\ &((2, 3), (2, 3)), ((2, 3), (2, 9)), ((2, 3), (4, 3)), ((2, 3), (4, 9)),\\ &((2, 9), (2, 9)), ((2, 9), (4, 9)),\\ &((4, 1), (4,1)), ((4, 1), (4, 3)), ((4, 1), (4, 9)),\\ &((4, 3), (4, 3)), ((4, 3), (4, 9)), ((4, 9), (4, 9))\} \end{aligned}$$

Digraph of this relation set is shown in Fig. 1,

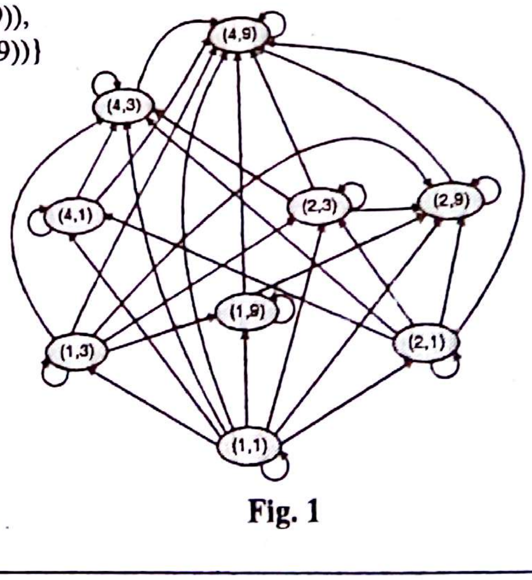


Fig. 1

To convert this digraph into Hasse diagram

Step 1 : Remove cycles

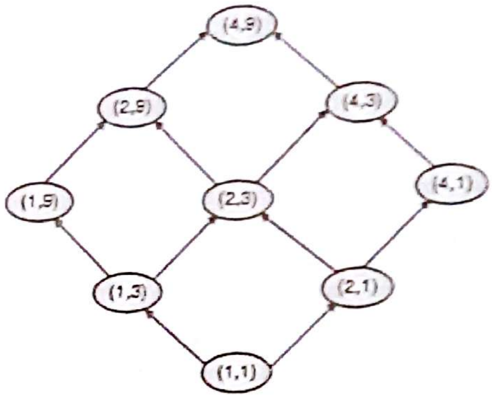


Fig. 2 (a)

Step 2 : Remove transitive edges. ((1, 1), (1, 9)), ((1, 1), (2, 3)), ((1, 1), (2, 9)), ((1, 1), (4, 1)), ((1, 1), (4, 3)), ((1, 1), (4, 9)), ((1, 3), (2, 9)), ((1, 3), (4, 3)), ((1, 3), (4, 9)), ((1, 9), (4, 9)), ((2, 1), (2, 9)), ((2, 1), (4, 3)), ((2, 1), (4, 9)), ((2, 3), (4, 9)), ((4, 1), (4, 9)).

Step 3 : All edges are pointing upwards.

Now remove arrows from edges and replace circles by dots. Hasse diagram is shown in Fig. 2 (b)

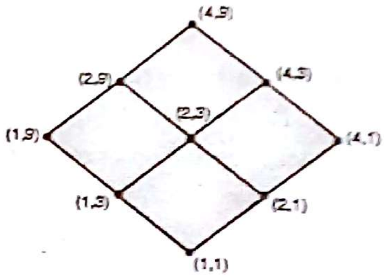


Fig. 2 (b)

Q. 6(c) Draw the Hasse Diagram of the following :

(a) D_{105} (b) D_{72} **(6 Marks)**

Ans. :

$$D_{72} = L = \{1, 2, 3, 6, 4, 9, 8, 12, 18, 24, 36, 72\}$$

Hasse Diagram :

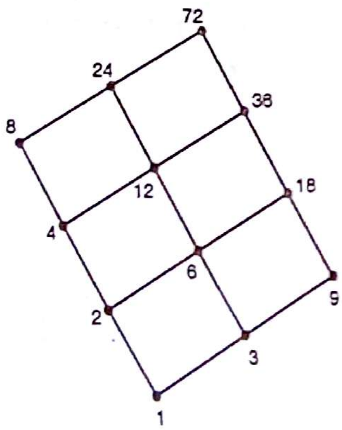


Fig. 3

$$D_{105} = L = \{1, 3, 5, 7, 15, 21, 35, 105\}$$

Diagram :

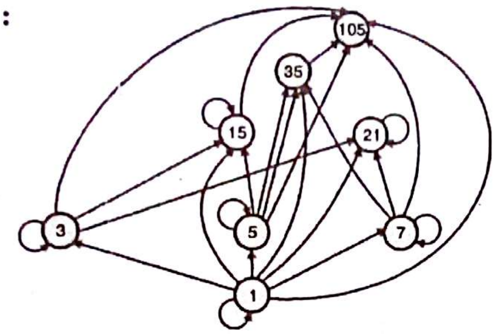


Fig. 4

Hasse Diagram :

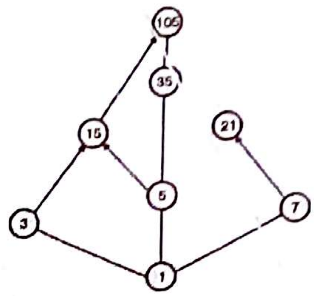


Fig. 5

Chapter 4 : Functions and Pigeon Hole Principle [Total Marks : 13]

Q. 1(c) Function $f(x) = \frac{(4x+3)}{(5x-2)}$. Find f^{-1} **(5 Marks)**

Ans. : $f(x) = \frac{(4x+3)}{(5x-2)}$

interchange x and y to get :

$$x = \frac{(4y+3)}{(5y-2)}$$

solve for "y" $5xy - 2x = 4y + 3$

$$5xy - 4y = 2x + 3$$

$$(5x-4)\,y = 2x + 3$$

$$y = \frac{(2x+3)}{(5x-4)}$$

$$f^{-1} = \frac{(2x+3)}{(5x-4)}$$

Q. 2(a) Let $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$ for all $x \in$ R. (R is the set of real number) **(8 Marks)**

Find (i) $f \circ g \circ h$ (ii) $h \circ g \circ f$ (iii) $f \circ f \circ f$

Ans. :

(i) $f \circ g \circ h(x) = f(g \circ (h(x))) = f(g \circ (3x)) = f((3x) - 2) = (3x - 2) + 2 = 3x$

(ii) $h \circ g \circ f(x) = h(g(f(x))) = h(g(x+2)) = h((x+2) - 2) = h(x) = 3x$

(iii) $f \circ f \circ f = f(f(f(x))) = f(f(x+2)) = f((x+2)+2) = f(x+4) = (x+4) + 2 = x + 6$

Chapter 5 : Generating Functions & Recurrence Relations [Total Marks : 15]

Q. 3(c) Solve the recurrence relation $a_n = -3(a_{n-1} + a_{n-2}) - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$ and $a_2 = 15$. **(7 Marks)**

Ans. :

Given : $a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$

The characteristic equation is

$$\alpha^3 + 3\alpha^2 + 3\alpha + 1 = 0$$

$$(\alpha + 1)_3 = 0$$

$$\therefore \quad \alpha = -1, -1, -1$$

Thus, the roots are $-1, -1, -1$ (triple or 3-fold roots). Therefore the solution of the given recurrence relation is,

$$a_n = (A_1 n^2 + A_2 n + A_3)(-1)^n$$

To find A_1, A_2 and A_3 putting $n = 0, 1, 2$ in above equation we get,

For $n = 0$, $a_0 = 5$ $\therefore\ 5 = A_3$

For $n = 1$, $a_1 = -9$ $\therefore\ -9 = (A_1 + A_2 + A_3)(-1)^1$

$$= -A_1 - A_2 - A_3 = -A_1 - A_2 - 5$$

$$-9 + 5 = -A_1 - A_2 - 4 = -A_1 - A_2$$

For $n = 2$, $a_2 = 15$ $\therefore\ 15 = (4A_1 + 2A_2 + A_3)(-1)^2$

$$= 4A_1 + 2A_2 + 1 \qquad 14 = 4A_1 + 2A_2$$

Soling we get, $A_1 = 3$, $A_2 = 1$, $A_3 = 5$.

Hence the homogeneous solution of the given recurrence relation is,

$$a_n = (3n^2 + n + 5)\ (-1)^n$$

Q. 4(c) Find the generating function for the following series. **(8 Marks)**

(i) {0, 1, 2, 3, 4,.................} (ii) {1, 2, 3 ,4, 5.....................}

(iii) {2, 2, 2, 2, 2,.....................} (iv) {0, 0, 0, 1, 1, 1, 1,...................}

Ans. :

(i) {0, 1, 2, 3, 4,.................}
(ii) {1, 2, 3 ,4, 5.....................}
(iii) {2, 2, 2, 2, 2,.....................}

Please refer Q. 7(c) of May 2013.

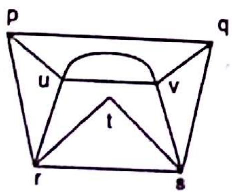
(iv)
$$\sum_{n=3}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots.$$

$\therefore$ The generating function for sequence

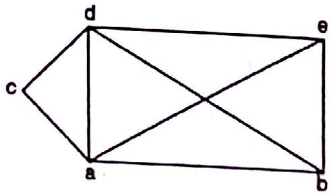
$$\sum_{n=3}^{\infty} (1)^n x^n = \{0, 0, 0, 1, 1, 1, 1,\ldots\} \text{ is } 0 + 0x + x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6$$

Chapter 6 : Graphs [Total Marks : 06]

Q. 5(b) Determine the Eulerian and Hamiltonian path, if exists, in the following graphs : **(6 Marks)**



(a)



(b)

Fig. 6

Ans. :

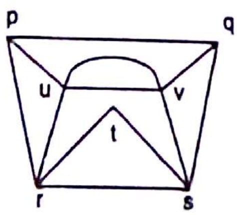


Fig. 6(a)

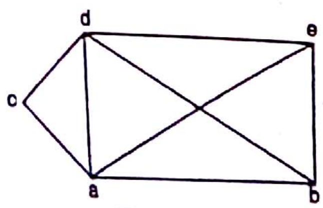


Fig. 6(b)

Hamiltonian path : p, u, v, q, s, t, r

Hamiltonian circuit : r, p, u, v, q, s, j, r

Eulerian path : (p, u, v, q, s, v, u, r, t, s, r, p, q)

Hamiltonian path : c, d, e, b, a

Hamiltonian circuit : c, d, b, a, c

Eulerian path : (e, d, b, a, d, c, a, e)

Chapter 7 : Trees [Total Marks : 13]

Q. 1 (d) What is the total number of vertices in a full binary tree with 20 leaves ? **(5 Marks)**

Ans. : Let n represent the total number of nodes in a full binary tree. Then numbers of internal nodes $i = (n - 20)$.

The total number of nodes in a full m-ary tree is given by $n = m_i + 1$. In full binary tree $m = 2$.

$$n = 2i + 1 \qquad n = 2(n - 20) + 1$$

$$n = 2n - 40 + 1 \qquad n = 39$$

Hence a full binary tree with 20 leaves has total 39 nodes.

Q. 6(a) Determine whether following graphs are isomorphic or not. **(8 Marks)**

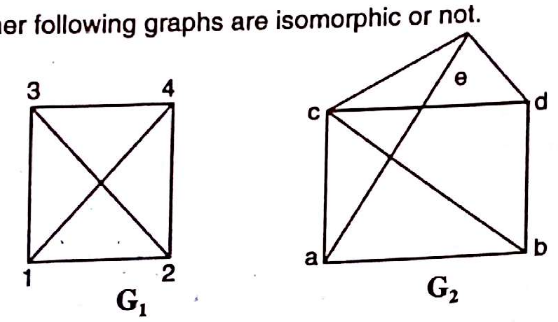


Fig. 7

Ans. : Graph G_1 have 4 vertices and 6 edges ; where Graph G_2 have 5 vertices and 8 edges. Adjacency is not available between G_1 and G_2. Hence given two graphs are not isomorphic.

Chapter 8 : Algebraic Structures [Total Marks : 26]

Q. 4(a) Show that a group G is abelian if and only if $(ab)^2 = a^2b^2$ for all $a, b \in G$. **(6 Marks)**

Ans. :

Proof : Part I :

$$(a\,b)^2 = a^2 b^2 \qquad \text{...Given}$$

$$\therefore\ a * b * a * b = a * a * b * b$$

Premultiply by a^{-1} and post multiply by b^{-1} on both sides

$$\therefore a^{-1} * a * b * a * b * b^{-1} = a^{-1} * a * a * b * b * b^{-1} \qquad \therefore\ b * a = a * b$$

As $\quad a * a^{-1} = b * b^{-1} = e \qquad$ and $\quad e * x = x$

Now since $\quad b * a = a * b$

G is an Abelian group.

Part II :

Now assume G is Abelian

Therefore $a * b = b * a$

$$(a * b) * (a * b) = (ab)^2 = a * (b * a) * b \qquad \because \text{associative}$$

$= a * (a * b) * b$ $\because$ Abelian

$= (a * a) * (b * b)$ $\because$ associative

$= a^2 b^2$

Hence proved.

Q. 4(b) Prove that the set G = {1, 2, 3, 4, 5, 6} is an abelian group under multiplication modulo 7. **(6 Marks)**

Ans. : **Please refer Q. 7(b) of May 2012.**

Q. 5 (a) Let $H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be parity check matrix.

Decode the following words relative to maximum likelyhood decoding function

(i) 011001 (ii) 101011 (iii) 111010 (iv) 110110 **(8 Marks)**

Ans. : First compute encoding function $e_H : B^3 \rightarrow B^6$.

$$B^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$$

$e(000) = 000\, x_1\, x_2\, x_3$

$x_1 = 0 \cdot 1 + 0 \cdot 1 + 0 \cdot 0 = 0$

$x_2 = 0 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 = 0$

$x_3 = 0 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 = 0$

$\therefore e(000) = 000000$

$e(001) = 001\, x_1\, x_2\, x_3$

$x_1 = 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 0$

$x_2 = 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1$

$x_3 = 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1 = 1$

$\therefore e(001) = 001011$

$e(010) = 010\, x_1\, x_2\, x_3$

$x_1 = 0 \cdot 1 + 1 \cdot 1 + 0 \cdot 0 = 1$

$x_2 = 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 = 1$

$x_3 = 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0$

$\therefore e(010) = 010110$

$e(011) = 011\, x_1\, x_2\, x_3$

$x_1 = 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 = 1$

$x_2 = 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 0$

$x_3 = 0 \cdot 0 + 1 \cdot 0 + 1 \cdot 1 = 1$

$\therefore e(011) = 011101$

$e(100) = 100\, x_1\, x_2\, x_3$

$x_1 = 1 \cdot 1 + 0 \cdot 1 + 0 \cdot 0 = 1$

$x_2 = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 = 0$

$x_3 = 1 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 = 0$

$\therefore e(100) = 100100$

$e(101) = 101\, x_1\, x_2\, x_3$

$x_1 = 1 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 1$

$x_2 = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1$

$x_3 = 1 \cdot 0 + 0 \cdot 0 + 1 \cdot 1 = 1$

$\therefore e(101) = 101111$

$e(110) = 110\, x_1\, x_2\, x_3$

$x_1 = 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0 = 0$ $\qquad x_2 = 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 = 1$

$x_3 = 1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0$

$\therefore$ e (110) = 110010 $\qquad$ e (111) $= 111\, x_1 x_2 x_3$

$x_1 = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 = 0$ $\qquad x_2 = 1 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 0$

$x_3 = 1 \cdot 0 + 1 \cdot 0 + 1 \cdot 1 = 1$

$\therefore$ e (111) = 111001

Now construct decoding table.

000000	001011	010110	011101	100100	101111	110010	111001
000001	001010	010111	011100	100101	101110	110011	111000
000010	001001	010100	011111	100110	101101	110000	111011
000100	001111	010010	011001	100000	101011	110110	111101
001000	000011	011110	010101	101100	100111	111010	110001
010000	011011	000110	001101	110100	111111	100010	101001
011000	010011	001110	000101	111100	110111	101010	100001
001100	000111	011010	010001	101000	100011	111110	110101

(i) The received word 011001 is located in 4th column. The word at top of the 4th column is 011101. Since e (011) = 011101, we decode 011001 as 011.

(ii) 101011 :
The received word "101011" is located in 6th column. The word at the top of 6th column is 101111. Since e(101) = 101111, we decode "101011" as 101.

(iii) 110110 :
The received word "110110" is located in 7th column. The word at the top of 7th column is 110010. Since e(110) = 110010, we decode "110110" as 110.

(iv) The received word 111010 is located in 7th column. The word at the top of the 7th column is 110010. Since e (110) = 110010, we decode 111010 as 110.

Q. 5(c) Let G be the set of real numbers and let G be the set of real numbers and let a * b = ab/2. Show that (G,*) is a abelian group. **(6 Marks)**

Ans. :

(i) We first verify that * is a binary operation. If a and b are elements of G then ab/2 is a non zero real number and hence is in G.

(ii) We next verify associativity since

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{(ab)\, c}{4}$$

$$\text{and since } a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a(bc)}{4} = \frac{(ab)\, c}{4}$$

the operation * is associative.

(iii) The number 2 is the identity in G, for if $a \in G$, then

$$a * 2 = \frac{(a)\,(2)}{2} = a = \frac{(2)\,(a)}{2} = 2 * a$$

(iv) Finally, if $a \in G$, then $a' = 4/a$ is an inverse of a, since

$$a * a' = a * \frac{4}{a} = \frac{a(4/a)}{2} = 2 = \frac{(4/a)\,(a)}{2} = \frac{4}{a} * a = a' * a$$

(v) Since a * b = b * a for all a and b in G, we conclude that G is an Abelian group.

❑❑❑

Dec. 2014

Chapter 1 : Set Theory [Total Marks : 12]

Q. 2(b) Find how many integers between 1 to 60 are :

(i) not divisible by 2 nor by 3 and nor by 5 (ii) Divisible by 2 not by 3 and nor by 5. **(8 Marks)**

Ans. :

(i) not divisible by 2 nor by 3 and nor by 5 :

Let A_1, A_2 and A_3 be the set of integers between 1 and 60 divisible by 2, 3 and 5 respectively.

$$\therefore \quad |A_1| = \left|\frac{60}{2}\right| = 30 \qquad |A_2| = \left|\frac{60}{3}\right| = 20$$

$$|A_3| = \left|\frac{60}{5}\right| = 12 \quad \text{and} \quad |A_1 \cap A_2| = \left|\frac{60}{2\times 3}\right| = 10$$

$$|A_1 \cap A_3| = \left|\frac{60}{2\times 5}\right| = 6$$

$$|A_2 \cap A_3| = \left|\frac{60}{3\times 5}\right| = 4 \quad \text{and} \quad |A_1 \cap A_2 \cap A_3| = \left|\frac{60}{2\times 3\times 5}\right| = 2$$

Number of integers between 1 and 60 which are divisible by 2, 3 or 5 are

$$= |A_1 \cup A_2 \cup A_3|$$

$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$= 30 + 20 + 12 - 10 - 6 - 4 + 2 = 44$$

Hence the number of integers between 1 and 60 are not divisible by 2, 3 or 5 = 60 – 44 = 16

(ii) Divisible by 2 not by 3 and nor by 5 :

Let A_1 denote the set of integers between 1 and 60 divisible by 2. Similarly A_2 and A_3 be the sets of integers divisible by 3 and 5 respectively.

$$\text{Then } |A_1| = \left|\frac{60}{2}\right| = 30, \qquad |A_2| = \left|\frac{60}{3}\right| = 20,$$

$$|A_3| = \left|\frac{60}{5}\right| = 12$$

Number of integers between 1 and 60 which are divisible by 2 but not by 3 and nor by 5.

$$= |A_1|\,|A_1 \cap A_2| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3| = 30 - 10 - 6 + 2 = 16$$

Q. 3(a) Show that $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$ **(4 Marks)**

Ans. :

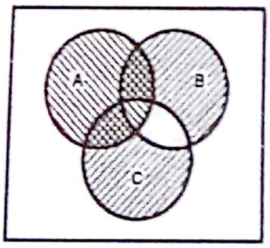


Fig. 1 (a) $B \oplus C$ is shaded with ///
A is shaded with \\\

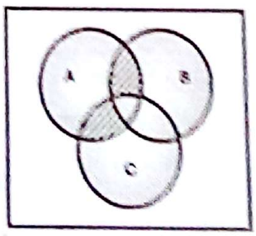
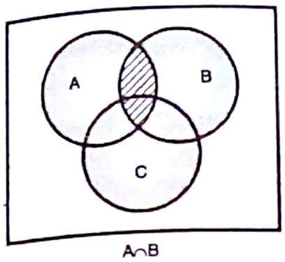
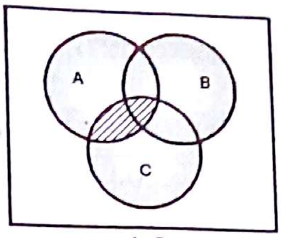


Fig. 1 (b) $A \cap (B \oplus C)$ is shaded
$(A \cap B) \oplus (A \cap C)$



A∩B

Fig. 1 (c)



A∩C

Fig. 1 (d)

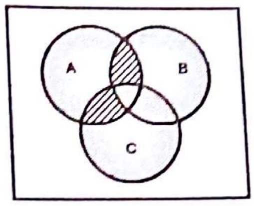


Fig. 1 (e)

Fig. 1 (b) and (e) are same. $\therefore$ $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$.

Chapter 2 : Logic [Total Marks : 09]

Q. 1(a) Prove by mathematical induction $x^n - y^n$ is divisible by $x - y$. **(5 Marks)**

Ans. : **Please refer Q. 2(a) of Dec. 2013.**

Q. 6(a) Determine if $[(p \Rightarrow q) \wedge \neg q] \Rightarrow \neg p$ is a tautology. **(4 Marks)**

Ans. : We need only to show that $p \rightarrow q$ and $\sim q$ both true imply $\sim p$ is true.

Since truth value of $\sim q$ is T, truth value of q is F. Since $p \rightarrow q$ is true, this means that p is false, i.e. truth value of $\sim p$ is T. Hence the proof.

Chapter 3 : Relations, Digraphs and Lattice [Total Marks : 37]

Q. 1(c) Show that a relation is reflexive and circular if and only if it is an equivalence relation. **(5 Marks)**

Ans. : Let R be reflexive and circular.

If a R b then a R b and b R b, so b R a. Hence R is symmetric.

If a R b and b R c, then c R a. But R is symmetric, so a R c and R is transitive. Let R be an equivalence relation.

Then R is reflexive

If a R b and b R c, then a R c (transitivity) and c R a (symmetry), so R is also circular.

Q. 3(c) Let R be a relation on set A = {1, 2, 3, 4}, given as

R = {(1, 1), (1, 4), (2, 2), (2, 3), (3, 2), (3, 3), (4, 1), (4, 4)}

Find transitive closure using Warshall's Algorithm. **(8 Marks)**

Ans. :

$n = 4$

Let $M_R = W_0$

$$W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Now compute W_1, so $k = 1$ $\therefore$ check column 1 and row 1 of W_0

$p_1 : (1, 1), p_2 : (4, 1)$ $\quad$ $q_1 : (1, 1), q_2 : (1, 4)$

To obtain W_2, we must put 1's in positions (1, 1), (1, 4), (4, 1) and (4, 4). Thus

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$W_2 = W_0$

Now compute W_2, so $k = 2$ $\therefore$ check column 2 and row 2 of W_0

$p_1 : (2, 2), p_2 : (3, 2)$ $\quad$ $q_1 : (2, 2), q_2 : (2, 3)$

To obtain W_2, we must put 1's in positions (2, 2), (2, 3), (3, 2) and (3, 3). But no new additional ordered pair thus $W_1 = W_2$. Hence

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Now compute W_3, so $k = 3$ $\therefore$ check column 3 and row 3 of W_2

$p_1 : (2, 3), p_2 : (3, 3)$ $\quad$ $q_1 : (3, 2), q_2 : (3, 3)$

To obtain W_3, we must put 1's in positions (2, 2), (2, 3), (3, 2) and (3, 3). Thus

$$W_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$W_3 = W_2$

Now compute W_4, so $k = 4$ $\therefore$ check column 4 and row 4 of W_3

$p_1 : (1, 4), p_2 : (4, 4)$ $\quad$ $q_1 : (4, 1), q_2 : (4, 4)$

To obtain W_4, we must put 1's in positions (1, 1), (1, 4), (4, 1) and (4, 4). But these ordered pair are already exist in W_3. $\therefore$ $W_3 = W_4$. Thus

$$W_4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Here $\quad W_0 = W_1 = W_2 = W_3 = W_4 \quad \therefore \; WR_\infty = W_4 =$ Transitive closure.

Q. 4(c) Draw Hasse Diagram of D_{42}. Find the complement of each element in D_{42}. **(8 Marks)**

Ans. : First we have to draw Hasse diagram of D_{42}.

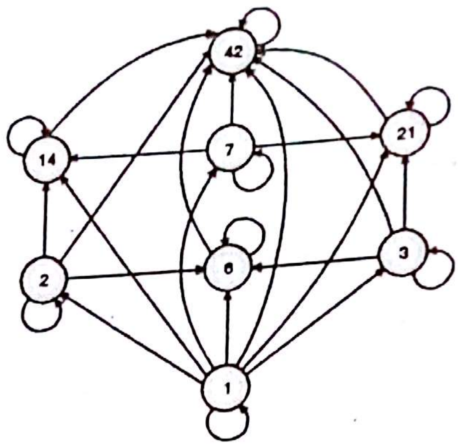
The set of numbers which divide 42 is {1, 2, 3, 6, 7, 14, 21, 42}

Partial order relation of divisibility on the above set.

$$R = \{(1, 1), (1, 2), (1, 3), (1, 6), (1, 7), (1, 14), (1, 21)$$
$$(1, 42), (2, 2), (2, 6), (2, 14), (2, 42), (3, 6), (3, 6),$$
$$(3, 21), (3, 42), (6, 6), (6, 42), (7, 7), (7, 14), (7, 21),$$
$$(7, 42), (14, 14), (14, 42), (21, 21), (21, 42), (42, 42)\}$$

Matrix for the above relation.

$$M_R = \begin{array}{c|c} & \begin{array}{cccccccc} 1 & 2 & 3 & 6 & 7 & 14 & 21 & 42 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 6 \\ 7 \\ 14 \\ 21 \\ 42 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



Digraph :

Fig. 2

To convert this digraph into Hasse diagram.

Step 1 : Remove cycles

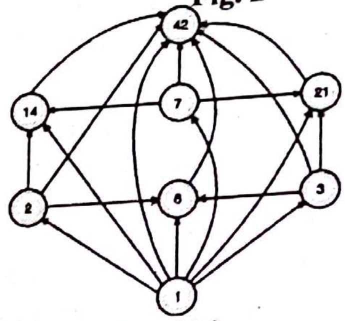


Fig. 3(a)

Step 2 : Remove transitive edges : (1, 14), (1, 6), (1, 42), (1, 21), (2, 42), (3, 42), (7, 42),

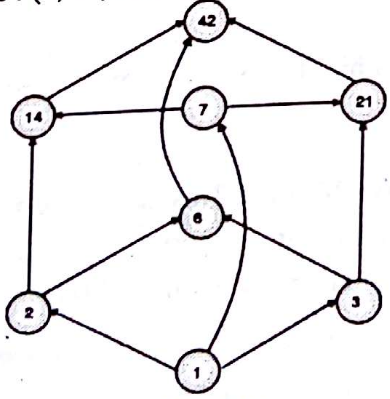


Fig. 3(b)

Step 3 : All edges are pointing upward

Remove all arrows from edges

Replace circles by dots.

In the above Hasse diagram greatest element is 42 and least element is 1.

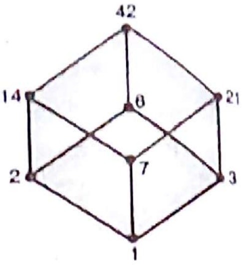


Fig. 3(c)

$1 \vee 42 = 42,$	$1 \wedge 42 = 1$	∴ Complement of 1 is 42
$42 \vee 1 = 42$	$42 \wedge 1 = 1$	∴ Complement of 42 is 1
$2 \vee 21 = 42$	$2 \wedge 11 = 1$	∴ Complement of 2 is 21
$21 \vee 2 = 42$	$21 \wedge 2 = 1$	∴ Complement of 21 is 2
$3 \vee 14 = 42$	$3 \wedge 14 = 1$	∴ Complement of 3 is 14
$14 \vee 3 = 42$	$14 \wedge 3 = 1$	∴ Complement of 14 is 3
$7 \vee 6 = 42$	$7 \wedge 6 = 1$	∴ Complement of 7 is 6
$6 \vee 7 = 42$	$6 \wedge 7 = 1$	∴ Complement of 6 is 7

Q. 5(a) Define Distributive Lattice along with one appropriate example. **(8 Marks)**

Ans. : A lattice L is called distributive if for any elements a, b and c in L we have the following distributive properties.

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ 2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

If L is not distributive, we say that L is non-distributive.

$$a \wedge (d \vee c) = a \wedge d = a$$

$$(a \wedge d) \vee (a \wedge c) = a \vee 0 = a$$

$$\therefore \quad a \wedge (d \vee c) = (a \wedge d) \vee (a \wedge c)$$

$$a \wedge (b \vee c) = a \wedge I = a$$

$$(a \wedge b) \vee (a \wedge c) = a \vee 0 = a$$

$$\therefore \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

The lattice shown in above Fig. 4 is distributive, as can be seen by verifying the distributive properties for all ordered triples chosen from the elements a, b, c and d.

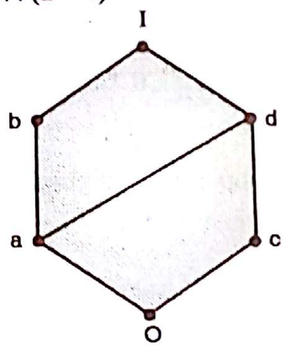


Fig. 4

Q. 6(c) R be a relation on set of integers Z defined by

R = {(x, y) | x – y is divisible by 3}

Show that R is an equivalence relation and describe the equivalence classes. **(8 Marks)**

Ans. : If x is any integer, then $x - x = 0$ which is divisible by 3.

∴ R is reflexive.

Suppose that $x - y$ is divisible by 3. This means that $x - y = 3n$, n is integer. Then $y - z = -3n$, which is divisible by 3.

∴ R is symmetric.

Suppose $x - y$ and $y - z$ are divisible by 3. This means that $x - y = 3m$

$y - z = 3n$, where m and n are integers.

$$\therefore \quad x - z = (x - y) + (y - z) = 3m + 3n = 3(m + n)$$

∴ $x - z$ is divisible by 3.

R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.

Equivalence class : Now we develop the equivalence classes.

Suppose we choose θ. The equivalence class is R (θ).

i.e.R (θ) = { x ∈ A | x R 0 }

It is the set of all integers x such that x – θ is divisible by 3. This is the set of all multiples of 3. (positive, negative, zero).

$$R(\theta) = \{ \ldots\ldots, -6, -3, 0, 3, 6, \ldots\ldots \}$$

We choose an element 1 in A. Then R (1) will contain all the integers x such that x – 1 is divisible by 3.

$$R(1) = \{ \ldots\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\ldots \}$$

$$R(2) = \{ \ldots\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\ldots \}$$

$$R(3) = R(0)$$

Chapter 4 : Functions and Pigeon Hole Principle [Total Marks : 16]

Q. 3(b) State and explain Pigeonhole principle, extended Pigeonhole principle. How many numbers must be selected from the set {1, 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add upto 7 ? **(8 Marks)**

Ans. : Pigeonhole Principle :

If n pigenons are assigned to m pigeonholes, and m < n, then at least one pigeonhole contains two or more pigeons.

Proof : Consider labeling the m pigeonholes with the numbers 1 through m and the n pigeons with the numbers 1 through n. Now, beginning with pigeon 1, assign each pigeon in order to the pigeonhole with the same number. This assigns as many pigeons as possible to individual pigeon holes, but because m < n, there are n – m pigeons that have not yet been assigned to a pigeonhole. At least one pigeonhole will be assigned a second pigeon.

Extended Pigeonhole principle :

If there are m pigeonholes and more than 2m pigeons, then three or more pigeons will have to be assigned to at least. One of the pigeonholes. In general if the number at pigeons is much larger than the number at pigeonholes, previous theorm can be restated to give a stronger conclusion. First , a word about notation. If n and m are positive integers, then $\lfloor n/m \rfloor$ stands for the largest integer less than or equal to the rational number n/m. Thus $\lfloor 3/2 \rfloor$ is 1, $\lfloor 9/4 \rfloor$ is 2 and $\lfloor 6/3 \rfloor$ is 2.

Theorem : (The extended pigeonhole principle)

If n pigeons are assigned to m, pigeonholes, then one of the pigeonholes must obtain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

Proof : (by contradiction) :

If each pigeonhole contains no more than $\lfloor (n-1)/m \rfloor$ pigeons, then there are at most m. $\lfloor (n-1)/m \rfloor \leq$ m. (n – 1) / m = n – 1 pigeons in all. This contradicts our assumption, so one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

Solve by extended Pigeonhole principle 4 numbers must be selected from the set {1, 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add upto 7.

Q. 5(b) Let the functions f, g and h defined as follows :

$f : R \to R,\ f(x) = 2x + 3$ $g : R \to R,\ g(x) = 3x + 4$

$h : R \to R,\ h(x) = 4x$

Find $g \circ f$, $f \circ g$, $f \circ h$, $h \circ f$, $g \circ f \circ h$ **(8 Marks)**

Ans. : $g \circ f \circ h = g(f(h(x))) = g(f(4x)) = g(8x + 3) = 3(8x + 3) + 4 = 24x + 9 + 4 = 24x + 13$

(i) $g \circ f(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 4 = 6x + 13$

(ii) $f \circ g(x) = f(g(x)) = f(3x + 4) = 2(3x + 4) + 3 = 6x + 11$

(iii) $f \circ h(x) = f(h(x)) = f(4x) = 2(4x) + 3 = 8x + 3$

(iv) $h \circ f(x) = h(2x + 3) = 4(2x + 3) = 8x + 12$

(v) $g \circ h(x) = g(4x) = 3(4x) + 4 = 12x + 4$

Chapter 5 : Generating Functions & Recurrence Relations [Total Marks : 12]

Q. 2(c) Solve the recurrence relation $a_{r+2} - a_{r+1} - 6a_r = 4$ **(8 Marks)**

Ans. : The corresponding homogeneous recurrence relation of the given recurrence relation is given by

$a_{r+2} - a_{r+1} - 6a_r = 0$

The characteristic equation is

$\alpha^2 - \alpha - 6 = 0$ $\therefore \alpha = 3, -2$

Therefore, the homogeneous solution of the given recurrence relation is

$$a_r^{(h)} = A_1 (3)^r + A_2 (-2)^r$$

To find the particular solution, we consider the term f(r) (right hand side of the given equation). Since f (r) is a constant, the particular solution will also be a constant P.

i.e. $a_r = P$, for all r $\Rightarrow a_{r+1} = a_{r+2} = P$ (constant)

Substituting the values of a_r, a_{r+1}, and a_{r+2} in the given recurrence relation, we get,

$P - P - 6P = 4$ $\therefore P = -\frac{2}{3}$

Hence, the particular solution is

$$a_r^{(p)} = -\frac{2}{3}$$

Thus the total solution or the general solution of the given recurrence relation is

$$a_r = a_r^{(h)} + a_r^{(p)} \Rightarrow a_r = A_1 (3)^r + A_2 (-2)^r - \frac{2}{3}$$

Q. 4(a) Find the generating function for the following sequence.

(i) 1, 2, 3, 4, 5, 6.... (ii) 3, 3, 3, 3, 3... **(4 Marks)**

Ans. :

(I) **1, 2, 3, 4, 5, 6.... : Please refer Q. 3(d) of Dec. 2013.**

(II) **{3, 3, 3, 3, 3} :** The generating function for the sequence {3, 3, 3, 3, 3} is $3 + 3x + 3x^2 + 3x^3 + 3x^4 + \ldots$

Chapter 6 : Graphs [Total Marks : 13]

Q. 1(b) How many vertices are necessary to construct a graph with exactly 6 edges in which each vertex is of degree 2. **(5 Marks)**

Ans. : **Please refer Q. 3(d) of Dec. 2013.**

Q. 6(b) Define isomorphic graphs. Show that following graphs are isomorphic. **(8 Marks)**

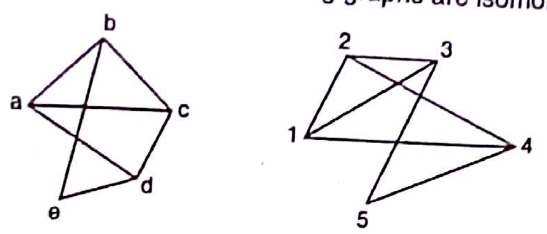


Fig. 5

Ans. : **Please refer Q. 5(c) of Dec. 2013.**

Chapter 7 : Trees [Total Marks : 09]

Q. 1(d) Prove that the set G = {1, 2, 3, 4, 5, 6} is abelian group under multiplication modulo 7. **(5 Marks)**

Ans. : **Please refer Q. 7(b) of May 2012 or May 2014.**

Q. 2(a) It is possible to draw a tree with five vertices having degrees 1, 1, 2, 2, 4 ? **(4 Marks)**

Ans. : Since the tree has 5 vertices hence, it has 4 edges. Now given the vertices of tree are having degrees 1, 1, 2, 2, 4 i.e. the total degree of the tree = 10.

By handshaking lemma

$$2e = \sum_{i=1}^{5} d(v_i)$$

where e is the number of edges in the graph.

$$2e = 10 \; ; \qquad e = 5$$

which is contradiction to the statement that the tree has 4 edges. Hence the tree with given degrees of vertices does not exist.

Chapter 8 : Algebraic Structures [Total Marks : 16]

Q. 4(b) Show that the (2, 5) encoding function $e : B^2 \rightarrow B^5$ defined by

e(00) = 00000 e(01) = 01110

e(10) = 10101 e(11) = 11011 is a group code.

How many errors will it detect and correct. **(8 Marks)**

Ans. : Let N = {00000, 01110, 10101, 11011} be the set of all code words.

$\oplus$	00000	01110	10101	11011
00000	00000	01110	10101	11011
01110	01110	00000	11011	10101

10101	10101	11011	00000	01110
11011	11011	10101	01110	00000

(i) For any a, b $\in$ N, a $\oplus$ b $\in$ N

$\therefore$ Set N is closed under $\oplus$ operation.

(ii) Identity element of B^5 i.e. 00000 also belongs to N.

$00000 \oplus 00000 = 00000 \oplus 00000$ $\quad$ $01110 \oplus 00000 = 00000 \oplus 01110$

$10101 \oplus 00000 = 00000 \oplus 10101$ $\quad$ $11011 \oplus 00000 = 00000 \oplus 11011$

(iii) $\oplus$ is associative operation.

(iv) Each element of N is its own inverse.

$$00000 \oplus 00000 = 00000 \oplus 00000 = 00000$$
$$01110 \oplus 01110 = 01110 \oplus 01110 = 00000$$
$$10101 \oplus 10101 = 10101 \oplus 10101 = 00000$$
$$11011 \oplus 11011 = 11011 \oplus 11011 = 00000$$

$\therefore$ N is subgroup of B^5 and the given encoding function is a group code.

d (00000, 01110) = 3 $\quad$ d (00000, 10101) = 3

d (00000, 11011) = 4 $\quad$ d (01110, 10101) = 4

d (01110, 11011) = 3 $\quad$ d (10101, 11011) = 3

$\therefore$ Minimum distance is 3.

The code will detect k or fewer errors if and only if its minimum distance is atleast k + 1. Since the minimum distance is 3, we have $3 \geq k + 1$ or $k \leq 2$. The code will detect 2 or fewer errors. The code will correct k or fewer errors if and only if its minimum distance is atleast 2 k + 1. Since the minimum distance is 3 we have $3 \geq 2k + 1$ or $k \leq 1$. So the code will correct 1 or fewer errors.

Q. 5(c) Let $H = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ Be a parity check matrix. Determine the group code e_H. $B^3 \rightarrow B^6$

(8 Marks)

Ans. : We have $B^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$. Then

$e(000) = 000\, x_1\, x_2\, x_3$.

where $x_1\, x_2\, x_3$ are determined using equations.

$$x_1 = b_1 \cdot h_{11} + b_2 h_{21} + \ldots + b_m \cdot h_{m1}$$
$$x_2 = b_1 \cdot h_{12} + b_2 h_{22} + \ldots + b_m \cdot h_{m2}$$
$$\vdots$$
$$x_r = b_1 \cdot h_{1r} + b_2 h_{2r} + \ldots + b_m \cdot h_{mr}$$

$\therefore \quad x_1 = 0\cdot 1 + 0\cdot 0 + 0\cdot 1 = 0$

$x_2 = 0\cdot 0 + 0\cdot 1 + 0\cdot 1 = 0$

$x_3 = 0\cdot 0 + 0\cdot 1 + 0\cdot 1 = 0$

$\therefore \quad e(000) = 000000$

$e(001) = 001\, x_1 x_2 x_3$

$x_2 = 0\cdot 0 + 0\cdot 1 + 1\cdot 1 = 1$

$x_1 = 0\cdot 1 + 0\cdot 0 + 1\cdot 1 = 1$

$x_3 = 0\cdot 0 + 0\cdot 1 + 1\cdot 1 = 1$

$\therefore \quad e(001) = 001111$

$e(010) = 010\, x_1 x_2 x_3$

$x_2 = 0\cdot 0 + 1\cdot 1 + 0\cdot 1 = 1$

$x_1 = 0\cdot 1 + 1\cdot 0 + 0\cdot 1 = 0$

$x_3 = 0\cdot 0 + 1\cdot 1 + 0\cdot 1 = 1$

$\therefore \quad e(010) = 010011$

$e(011) = 011\, x_1 x_2 x_3$

$x_2 = 0\cdot 0 + 1\cdot 1 + 1\cdot 1 = 0$

$x_1 = 0\cdot 1 + 1\cdot 0 + 1\cdot 1 = 1$

$x_3 = 0\cdot 0 + 1\cdot 1 + 1\cdot 1 = 0$

$\therefore \quad e(011) = 011100$

$e(100) = 100\, x_1 x_2 x_3$

$x_2 = 1\cdot 0 + 0\cdot 1 + 0\cdot 1 = 0$

$x_1 = 1\cdot 1 + 0\cdot 0 + 0\cdot 1 = 1$

$x_3 = 1\cdot 0 + 0\cdot 1 + 0\cdot 1 = 0$

$\therefore \quad e(100) = 100100$

$e(101) = 101\, x_1 x_2 x_3$

$x_2 = 1\cdot 0 + 0\cdot 1 + 1\cdot 1 = 1$

$x_1 = 1\cdot 1 + 0\cdot 0 + 1\cdot 1 = 0$

$x_3 = 1\cdot 0 + 0\cdot 1 + 1\cdot 1 = 1$

$\therefore \quad e(101) = 101011$

$e(110) = 110\, x_1 x_2 x_3$

$x_2 = 1\cdot 0 + 1\cdot 1 + 0\cdot 1 = 1$

$x_1 = 1\cdot 1 + 1\cdot 0 + 0\cdot 1 = 1$

$x_3 = 1\cdot 0 + 1\cdot 1 + 0\cdot 1 = 1$

$\therefore \quad e(110) = 110111$

$e(111) = 111\, x_1 x_2 x_3$

$x_2 = 1\cdot 0 + 1\cdot 1 + 1\cdot 1 = 0$

$x_1 = 1\cdot 1 + 1\cdot 0 + 1\cdot 1 = 0$

$x_3 = 1\cdot 0 + 1\cdot 1 + 1\cdot 1 = 0$

$\therefore \quad e(111) = 111000$

$\therefore \quad e_H : B^3 \to B^6$ is

$e(000) = 000000 \qquad e(001) = 001111$

$e(010) = 010011 \qquad e(011) = 011100$

$e(100) = 100100 \qquad e(101) = 101011$

$e(110) = 110111 \qquad e(111) = 111000$

❑❑❑

May 2015

Chapter 1 : Set Theory [Total Marks : 12]

Q. 4(b) A survey of 500 television watchers produced the following information: 285 watch football games, 195 watch hockey games, 115 watch basket ball games, 45 watch football and basketball games, 70 watch football and hockey games, 50 watch basketball and hockey games. 50 do not watch any three kinds of games. Find :

(i) How many in the survey watch all 3 kinds of games ?

(ii) How many watch exactly one of the sports languages ?

(iii) Draw Venn diagram showing results of the survey. (8 Marks)

Ans. : Let F be the set of people who watch football

Let H be the set of people who watch hockey.

Let B be the set of people who watch basketball.

Given : $|F| = 285$; $|H| = 195$; $|B| = 115$

$|F \cap B| = 45$; $|F \cap H| = 70$; $|H \cap B| = 50$

50 people do not watch any of the 3 kinds of games.

(i) People in survey who watch all 3 kinds of games i.e. football or hockey or basketball,

$$|F \cup H \cup B| = 500 - 50 = 450$$

450 people watch atleast one of the 3 games.

Using the formula of inclusion and exclusion, we have,

$$|F \cup H \cup B| = |F| + |H| + |B| - |F \cap H| - |F \cap B| - |H \cap B| + |F \cap H \cap B|$$

$$\therefore \quad 450 = 285 + 195 + 115 - 45 - 70 - 50 + |F \cap H \cap B|$$

$$\therefore \quad |F \cap H \cap B| = 20$$

20 people in survey watch all 3 kinds of games.

(ii) Exactly one of the sports :

People who watch only football.

$$= |F| - |F \cap H| - |F \cap B| + |F \cap H \cap B| = 285 - 70 - 45 + 20 = 190$$

People who watch only hockey

$$= |H| - |H \cap F| - |H \cap B| + |F \cap H \cap B| = 195 - 70 - 50 + 20 = 95$$

People who watch only basketball.

$$= |B| - |F \cap B| - |B \cap H| + |F \cap H \cap B| = 115 - 45 - 50 + 20 = 40$$

Number of people watching exactly one kind of game

$$= 190 + 95 + 40 = 325$$

$\therefore$ 325 people watch exactly one kind of game.

(iii) Venn diagram

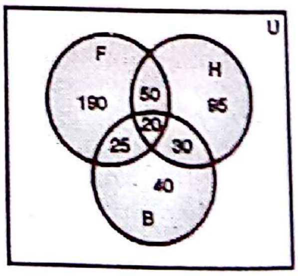


Fig. 1-Q. 4(b)

Q. 6(a) Show that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$ **(4 Marks)**

Ans. : By definition

$$
\begin{aligned}
A - (B - C) &= \{x \mid x \in A \text{ and } x \notin (B - C)\} = \{x \mid x \in A \text{ and } x \notin B \text{ or } x \in C\} \\
&= \{x \mid x \in A \text{ and } x \notin B\} \quad \text{or} \quad \{x \mid x \in A \text{ and } x \in C\} \\
&= \{x \mid x \in A \text{ and } x \notin B\} \text{ or } \{x \mid x \in A \text{ and } x \in B \cap C\} \\
&= \{x \mid x \in (A - B)\} \text{ or } \{x \mid x \in (A \cap B \cap C)\} = \{x \mid x \in (A - B) \cup (A \cap B \cap C)\} \\
A - (B - C) &= (A - B) \cup (A \cap B \cap C)
\end{aligned}
$$

Chapter 2 : Logic [Total Marks : 09]

Q. 1(d) Prove $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are logically equivalent. **(5 Marks)**

Ans. : Consider the truth tables.

(i)

p	q	r	$(q \vee r)$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

(ii)

p	q	r	$(p \wedge q)$	$(p \wedge r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

The last columns in both (i) and (ii) are identical. Hence the two forms are logically equivalent.

Q. 5(c)(I) Prove by mathematical induction $x^n - y^n$ is divisible by $x - y$. **(4 Marks)**

Ans.: **Please refer Q. 2(a) of Dec. 2013.**

Chapter 3 : Relations, Digraphs and Lattice [Total Marks : 25]

Q. 1(a) Let $A = \{a, b, c\}$. Show that $(P(A), \subseteq)$ is a poset. Draw its Hasse Diagram. P (A) is the power set of A. **(5 Marks)**

Ans. : **Please refer Q. 3(c) of Dec. 2013.**

Q. 3(a) Show that the set of all divisors of 70 forms a lattice. **(4 Marks)**

Ans. : Let A be the set of all integers of all divisors of 70.

$$A = \{1, 2, 5, 7, 10, 14, 35, 70\}$$

Partial order relation of divisibility on set A is

$$R = \{(1, 1), (1, 2), (1, 5), (1, 7), (1, 10), (1, 14), (1, 35), (1, 70), (2, 2), (2, 10), (2, 14), (2, 70), (5, 5), (5, 10), (5, 35), (5, 70), (7, 7), (7, 14), (7, 35), (7, 70), (10, 10), (10, 70), (14, 14), (14, 70), (35, 35), (35, 70), (70, 70)\}$$

Matrix for the above relation is,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 7 \\ 10 \\ 14 \\ 35 \\ 70 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 5 & 7 & 10 & 14 & 35 & 70 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

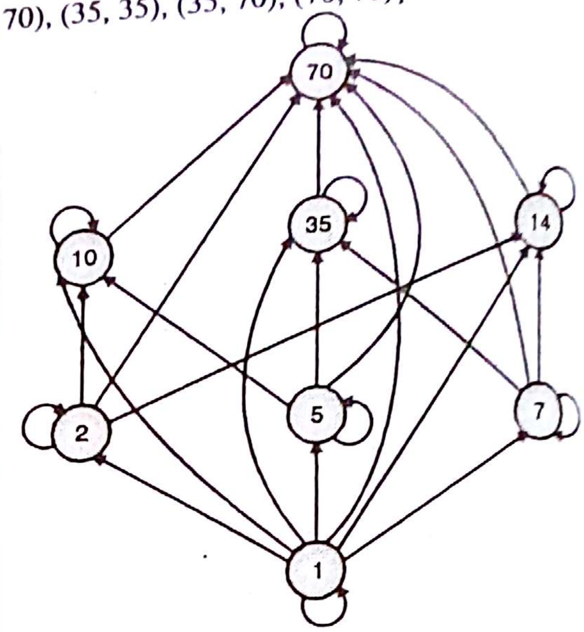


Fig. 1-Q. 3(a)

Diagraph for the above matrix is,

To convert this digraph into Hasse diagram

Step 1 : Remove cycles

Step 2 : Remove transitive edges (1, 10), (1, 70), (1, 35), (1, 14), (2, 70), (5, 70), (7, 70),

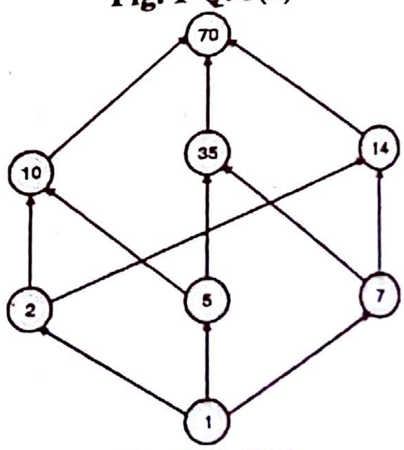


Fig. 2-Q. 3(a)

Step 3 : All edges are pointing upwards. Now remove arrows from edges, and replace circles by dots.

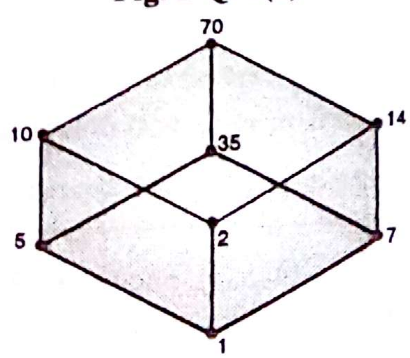


Fig. 3-Q. 3(a)

LUB

∨	1	2	5	7	10	14	35	70
1	1	2	5	7	10	14	35	70
2	2	2	10	14	10	14	35	70
5	5	10	5	35	10	70	35	70
7	7	14	35	7	70	14	35	70
10	10	10	10	70	10	70	70	70
14	14	14	70	14	70	14	70	70
35	35	35	35	35	70	70	35	70
70	70	70	70	70	70	70	70	70

GLB

∧	1	2	5	7	10	14	35	70
1	1	1	1	1	1	1	1	1
2	1	2	1	1	2	2	1	2
5	1	1	5	1	5	1	5	5
7	1	1	1	7	1	7	7	7
10	1	2	5	1	10	2	5	10
14	1	2	1	7	2	14	1	14
35	1	1	5	7	5	1	35	35
70	1	2	5	7	10	14	35	70

Every pair of elements has a least upper bound and a greatest lower bound. So this Hasse diagram or poset is a lattice.

Q. 2(c) Define Equivalence relation with an example. Let m be a positive integer other than 1. Show that the relation R = {(a, b) | a = b (mod m)} i.e. m divides a-b is an equivalence relation on the set of integers. **(8 Marks)**

Ans. : Definition :

A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive. The following are some of the common but important examples of equivalence relations.

Examples :

(i) Let A = I R and R be 'equality' of numbers.

(ii) Consider all subsets of a universal set and R be the relation, 'equality' of sets.

(iii) A is the set of triangles and R is 'similarity' of triangles.

(iv) A is a set of students and R is the relation of being in the same class or division.

(v) Let A be set of statement forms and R be the relation of 'Logical equivalence.'

(vi) A is set of lines in a plane and R is the relation of lines being 'parallel'.

The digraph of an equivalence relation will have the following characteristics. (i) Every vertex will have a loop.

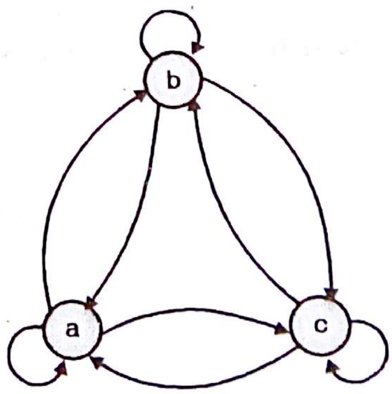


Fig. 1-Q. 2(c)

(ii) If there is an arc from a to b, there should be an arc from b to a.

(iii) If there is an arc from a to b and arc from b to c, there should be an arc from a to c

Example :

In short, the following is a typical digraph of an equivalence relation.

(a) **Reflexive :** We know $a \equiv b \pmod m$ if and only if m divides $a - b$.

Now $a - a = 0$ is divisible by m.

So $a \equiv a \pmod m$

$\therefore$ Congruence module m is reflexive.

Hence given relation R is reflexive.

(b) **Symmetric :** Suppose $a \equiv b \pmod m$ then $(a - b)$ is divisible by m. So that $a - b = km$, whenever k is an integer.

If follows that $(b - a) = (-km)$

So that $b \equiv a \pmod m$

So congruence module m is symmetric. Hence given relation R is symmertic.

(c) **Transitive :** Suppose $a \equiv b \pmod m$ and $b \equiv c \pmod m$ then m divides both $a - b$ and $b - c$.

Therefore there are integers k and l with

$a - b = km$ and $b - c = lm$

$(a - b) + (b - c) = (k + l)\,m$

$a - c = (k + l)\,m$

$\therefore \quad a \equiv c \pmod m$

$\therefore$ Congruence modulo m is transitive.

Hence given relation R is transitive.

$\therefore$ Give relation R is Equivalence relation.

Q. 3(c) Define Reflexive closure, Symmetric closure along with a suitable example. Let R be a relation on set S = {a, b, c, d, e}, given as

R = { (a, a) ,(a, d) , (b, b), (c, d), (c, e), (d, a), (e,b), (e, e)}

Find transitive closure using Warshall's Algorithm. **(8 Marks)**

Ans. : Reflexive closure : A relation R* is the reflexive closure of R if R* is the smallest relation containing R which is reflexive.

Symmetric closure : A relation R* is the symmetric closure of R if R* is the smallest relation containing R which is symmetric.

Transitive closure using Warshall's Algorithm :

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Chapter 4 : Functions and Pigeon Hole Principle [Total Marks : 12]

Q. 2(a) If $f : A \rightarrow B$ be both one-to-one and onto, then prove that $f^{-1} : B \rightarrow A$ is also both one-to-one and onto. **(4 Marks)**

Ans. :

Theorem : If $f : A \rightarrow B$ be both one to one and onto, then $f^{-1} : B \rightarrow A$ is both one to one and onto

Proof :

Let $f : A \rightarrow B$ be both one to one and onto then there exists elements.

$a_1, a_2 \in A$ and elements

$b_1, b_2 \in B$

Such that $f(a_1) = b_1$ and $f(a_2) = b_2$

$a_1 = f^{-1}(b_1)$ and $a_2 = f^{-1}(b_2)$

Now let $f^{-1}(b_1) = f^{-1}(b_2)$

$\Rightarrow a_1 = a_2$ $\Rightarrow f(a_1) = f(a_2)$

$b_1 = b_2$

$\therefore f^{-1}$ is one to one, again since f is onto, for $b \in B$, there is some element $a \in A$, such that

$f(a) = b$ $\Rightarrow a = f^{-1}(b)$

$\Rightarrow f^{-1}$ is onto

Hence f^{-1} is both one-one and onto.

Q. 5(b) Explain Pigeonhole principle and Extended Pigeonhole principle. Show that if 7 colors are used to paint 50 bicycles, at least 8 bicycles will be of same color. **(8 Marks)**

Ans. :

Pigeonhole principle and extended pigeonhole principle: **Please refer Q. 3(b) of Dec. 2014.**

Example :

If n pigeons are assigned to m pigeonholes, and $m < n$, then at least one pigeonhole contains two or more pigeons. By the extended pigeonhole principle at least. $\lfloor (50 - 1) / 7 \rfloor + 1 = 8$ will be of the same color.

Chapter 5 : Generating Functions & Recurrence Relations [Total Marks : 13]

Q. 1(b) Find the generating function for the following finite sequences. **(5 Marks)**

(i) 1, 2, 3, 4, ... (ii) 1, 1, 1, 1, 1, 1

Ans. :

(i) 1,2,3,4,.....

The generating function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$\therefore$ The generating function for sequence $\{1, 2, 3, 4, \dots\}$ is $1 + 2x + 3x^2 + 4x^3 + \dots$

(ii) 1,1,1,1,1,1

We know $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

$$= \sum_{n=0}^{\infty} (1)^n x^n$$

$\therefore$ Generating function for sequence 1, 1, 1, 1, 1, 1 is

$$\frac{1}{1-x} = \sum_{n=0}^{5} (1)^n x^n$$

Q. 4(c) Find the solution to the recurrence relation

$a_n = 6a_{n-1} + 11a_{n-2} - 6a_{n-3}$ given $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$. **(8 Marks)**

Ans. :

Given $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

$\therefore$ The characteristic equation is

$$\alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0$$

$$\therefore \quad (\alpha - 1)(\alpha - 2)(\alpha - 3) = 0$$

$$\therefore \quad \alpha = 1, 2, 3$$

which are the characteristic roots of the equation.

Therefore the solution of the given recurrence relation is

$$a_n = A_1 (1)^n + A_2 (2)^n + A_3 (3)^n \quad \ldots(1)$$

To find A_1, A_2 and A_3, putting $r = 0$, $r = 1$ and $r = 2$ we get

Given $a_0 = 2$ $\quad \therefore \quad 2 = A_1 + A_2 + A_3$

$a_1 = 5$ $\quad \therefore \quad 5 = A_1 + 2A_2 + 3A_3$

$a_2 = 15$ $\quad \therefore \quad 15 = A_1 + 4A_2 + 9A_3$

On solving we get

$A_1 = 1$, $A_2 = -1$ and $A_3 = 2$

Hence the homogeneous solution of the given recurrence relation is

$$a_n = 1 - 2^n + 2.3^n$$

Chapter 6 : Graphs [Total Marks : 04]

Q. 4(a) Determine Euler Cycle and path in graph shown below. **(4 Marks)**

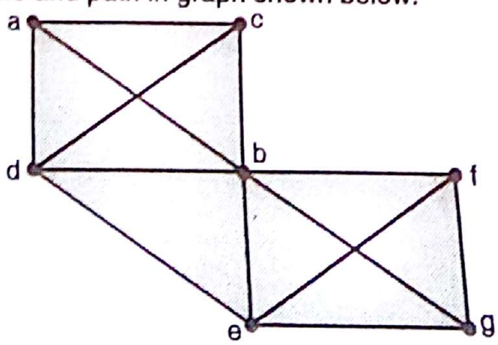


Fig. 1- Q.4(a)

Ans. :

Vertex a – degree 3; Vertex b – degree 6; Vertex c – degree 3

Vertex d – degree 4; Vertex e – degree 4; Vertex f – degree 3

Vertex g – degree 3

Number of vertices with odd degree = 4.

If a graph G has a vertex of odd degree, there can be no Euler circuit in G.

If a graph G has more than two vertices of odd degree then there can be no Euler path in G.

Hence given graph has no Euler path and no Euler circuit.

Chapter 7 : Trees [Total Marks : 05]

Q. 1(c) Is it possible to draw a tree with five vertices having degrees 1, 1, 2,2,4 ? **(5 Marks)**

Ans. : **Please refer Q. 2(a) of Dec. 2014**

Chapter 8 : Algebraic Structures [Total Marks : 40]

Q. 2(b) Let G be a set of rational numbers other than 1. Let * be an operation on G defined by $a * b = a + b - ab$ for all $a, b \in G$. Prove that (G,*) is a group. **(8 Marks)**

Ans. : **Please refer Q. 1(b) of Dec. 2013.**

Q. 3(b) Consider the (3, 5) group encoding function defined by **(8 Marks)**

e (000) = 00000 e (001) = 00110 .

e (010) = 01001 e (011) = 01111

e (100) = 10011 e (101) = 10101

e (110) = 11010 e (111) = 11000

Decode the following words relative to a maximum likelyhood decoding function.

(i) 11001 (ii) 01010 (iii) 00111

Ans. : **Please refer Q. 4(d) of Dec. 2013.**

Q. 5(a) Show that if every element in a group is its own inverse, then the group must be abelian. **(4 Marks)**

Ans. : The inverse exists for all $x \in A$ as $x . x = e$ i.e. every element is its own inverse

$\therefore$ Now, the identity is e

$\therefore x . e = e . x = x$ $\because$ by definition of identity

$\therefore x . e = e . x$

$\therefore$ But for all $y \in A$. $y . y = e$

$\therefore x . y . y = y . y . x$

Now, we see that y . y is another element of A say b

$x . b = b . x$

For all $x \in A$

$\therefore$ (A, .) is an Abelian group.

Q. 5(c)(II) Consider the group G = {1, 2, 3, 4, 5, 6} under multiplication modulo 7.

(a) Find multiplication table of G. (b) Find inverse of every element. (4 Marks)

Ans. :

(a) Multiplication modulo 7 table for set A is

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	1	3	2	1

(b) Note that 1 is an identity element of the algebraic system $(A, \times_7)$

Since for any $a \in A$,

$$a \times_7 1 = a = 1 \times_7 a$$

that is, $1 \times_7 1 = 1 \times_7 1 = 1$

$$2 \times_7 1 = 1 \times_7 2 = 2$$

$$3 \times_7 1 = 1 \times_7 3 = 3$$

$$4 \times_7 1 = 1 \times_7 4 = 4$$

$$5 \times_7 1 = 1 \times_7 5 = 5$$

$$6 \times_7 1 = 1 \times_7 6 = 6$$

Recall that a^{-1} is that element of G such that $a * a^{-1} = 1$

$$2 \times_7 4 = 1 = 4 \times_7 2 = 1$$

$$3 \times_7 5 = 1 = 5 \times_7 3 = 1$$

$$6 \times_7 6 = 1 = 6 \times_7 6 = 1$$

$\therefore$ Inverse of 2 is 4

Inverse of 4 is 2

Inverse of 3 is 5

Inverse of 5 is 3

Inverse of 6 is 6

(c) We have $2^1 = 2,$

$$2^2 = 2 \times_7 2 = 4$$

$$2^3 = 2^2 \times_7 2 = 4 \times_7 2 = 1$$

$$2^4 = 2^3 \times_7 2 = 1 \times_7 2 = 2$$

$\therefore$ Hence $|2| = 3$

$\therefore$ 2 is not generator of this group

We have $3^1 = 3$

$3^2 = 3 \times_7 3 = 2$

$3^3 = 3^2 \times_7 3 = 2 \times_7 3 = 6$

$3^4 = 3^3 \times_7 3 = 6 \times_7 3 = 4$

$3^5 = 3^4 \times_7 3 = 4 \times_7 3 = 5$

$3^6 = 3^5 \times_7 3 = 5 \times_7 3 = 1$

Hence $|3| = 6$

$\therefore$ 3 is generator of this group and this group is cyclic.

(d) Subgroup generated by $\{3, 4\}$ is denoted by $< \{3, 4\} >$ since 3, 4 are elements of this set they have to be there in $< \{3, 4\} >$

Inverse of 3 is 5, inverse of 4 is 2

$\therefore$ $3, 4, 5, 2, \in < \{3, 4\} >$

$3 \times_7 4 = 5$	$5 \times_7 4 = 6$	$3 \times_7 3 = 2$	$6 \times_7 6 = 1$
$3 \times_7 5 = 1$	$5 \times_7 1 = 5$	$4 \times_7 4 = 2$	$1 \times_7 1 = 1$
$3 \times_7 2 = 6$	$5 \times_7 2 = 3$	$5 \times_7 5 = 4$	
$3 \times_7 6 = 4$	$5 \times_7 6 = 2$	$2 \times_7 2 = 4$	

$\therefore$ $< \{3, 4\} > = < 1, 2, 3, 4, 5, 6 >$

$\therefore$ Subgroup generated by $\{3, 4\}$ is the set A itself whose order is 6.

Subgroup generated by $\{2, 3\}$ is denoted by $< \{2, 3\} >$.

Since 2, 3 are elements of this set they have to be there in $< \{2, 3\} >$.

Inverse of 2 is 4.

Inverse of 3 is 5

$\therefore$ $2, 3, 4, 5 \in < \{2, 3\} >$

$2 \times_7 3 = 6$	$3 \times_7 4 = 5$	$4 \times_7 4 = 2$	$5 \times_7 5 = 4$
$2 \times_7 4 = 1$	$3 \times_7 5 = 1$	$4 \times_7 1 = 4$	$5 \times_7 6 = 2$
$2 \times_7 5 = 3$	$3 \times_7 6 = 4$	$4 \times_7 5 = 6$	$5 \times_7 1 = 5$
$2 \times_7 6 = 5$	$3 \times_7 1 = 3$	$6 \times_7 6 = 1$	
$2 \times_7 1 = 2$	$3 \times_7 3 = 2$	$6 \times_7 1 = 6$	
$2 \times_7 2 = 4$			

$\therefore$ $< \{2, 3\} > = < 1, 2, 3, 4, 5, 6 >$

$\therefore$ Subgroup generated by $< \{2, 3\} >$ is the set A and is of order 6.

Q. 6(b) Let $H = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ **(8 Marks)**

Be a parity check matrix. Determine the group code $e_H: B^3 \rightarrow B^6$.

Ans. : **Please refer Q. 5(c) of Dec. 2014.**

(8 Marks)

Q. 6(c) Determine if following graphs g and G are isomorphic or not.

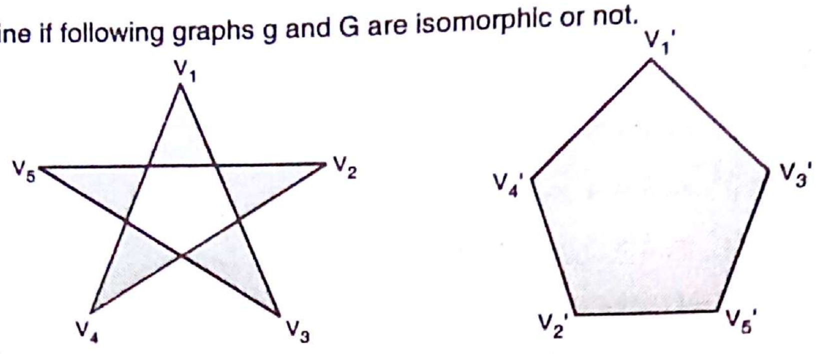


Fig. 1-Q. 6(c)

Ans. :

Number of vertices = both 5

Number of edges = both 5

Degree of corresponding vertices = all degree 2

connectedness : each is fully connected.

Number of connected components both 1

Pails of connected vertices : all correspond

Number of loops : 0

Number of parallel edges = 0

Everything is equal, so the graphs are isomorphic more formally,

$G = (V_1, E)$ where $V = \{ V_1, V_2, V_3, V_4, V_5\}$ and

$E = \{(V_1, V_2), (V_2, V_3), (V_3, V_4), (V_4, V_5), (V_5, V_1)\} = \{ e_1, e_2, e_3, e_4, e_5\}$

$G' = \{ V', E'\}$ where $V' = \{V_1', V_2', V_3', V_4', V_5'\}$ and

$E' = \{(V_1', V_2'), (V_2', V_3'), (V_3', V_4') (V_4', V_5'), (V_5', V_1')\}$

$= \{ e_1', e_2', e_3', e_4', e_5'\}$

Construct 2 functions $f : V \to V'$ and $g : E \to E'$

$f : V \to V'$		$g : E \to E'$	
V	V′	E	E′
V_1	V_1'	e_1	e_1'
V_2	V_2'	e_2	e_2'
V_3	V_3'	e_3	e_3'
V_4	V_4'	e_4	e_4'
V_5	V_5'	e_5	e_5'

□□□

Dec. 2015

Chapter 2 : Logic [Total Marks : 10]

Q. 1(c) $6^{n+2} + 7^{2n+1}$ is divisible by 43. **(5 Marks)**

Ans. :

Given : Let $P(n) : (6)^{n+2} + (7)^{2n+1}$

(1) Basis of Induction :

For n = 1, $(6)^{n+2} + (7)^{2n+1} = (6)^3 + (7)^3 = 216 + 343 = 559$ divisible by 43

Since $43 \times 13 = 559$

(2) Induction step :

Assume the result for n = k i.e. P(k) is true ie. $(6)^{k+2} + (7)^{2k+1}$ is divisible by 43

To prove that P(k + 1) is true,

$$(6)^{(k+1)+2} + (7)^{2(k+1)+1} = 6 . (6)^{k+2} + (7)^{(2k+1)} \cdot 49 = 6 \cdot (6)^{k+2} + (43 + 6) \cdot (7)^{2k+1}$$

$$= 6 \cdot (6)^{k+2} + (7)^{2k+1} + 43 \cdot 7^{(2k+1)}$$

$6 \cdot 6^{(k+2)} + (7)^{2k+1}$ is divisibly by 43 by induction hypothesis.

Since both these terms are divisible by 43 the result following.

Q. 5(c) Show that $(\neg q \wedge (p \Rightarrow q)) \Rightarrow \neg p$ is a tautology. **(5 Marks)**

Ans. : We need only to show that $p \rightarrow q$ and $\sim q$ both true imply $\sim p$ is true.

Since truth value of $\sim q$ is T, truth value of q is F. Since $p \rightarrow q$ is true, this means that p is false, i.e. truth value of $\sim p$ is T. Hence the proof.

Chapter 3 : Relations, Digraphs and Lattice [Total Marks : 18]

Q. 1(d) Draw the Hasse diagram of D_{60}. Also find whether it is a lattice. **(5 Marks)**

Ans. : The set of numbers that divides 60 is {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}

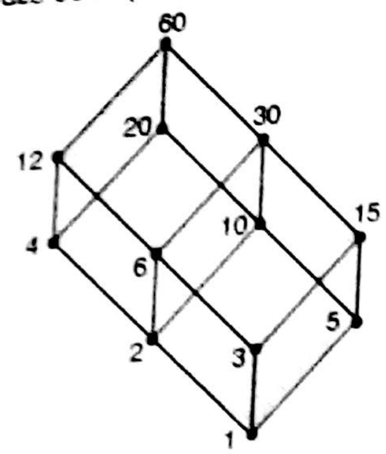


Fig. 1-Q. 1(d) : Hasse diagram.

Resultant Hasse diagram is a Lattice as every pair of elements has a GLB and a LUB.

Q. 3(c) Determine whether the below Hasse diagram represents a lattice. **(6 Marks)**

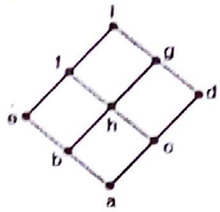


Fig. 1- Q. 3(c)

Ans. :

(I)

LUB :

$\vee$	a	b	c	d	e	f	g	h	i
a	a	b	c	d	e	f	g	h	i
b	b	b	e	d	e	h	g	h	i
c	c	e	c	g	e	f	g	h	i
d	d	d	g	d	g	i	g	i	i
e	e	e	e	g	e	h	g	h	i
f	f	h	f	i	h	f	i	h	i
g	g	g	g	g	g	i	g	i	i
h	h	h	h	i	h	h	i	h	i
i	i	i	i	i	i	i	i	i	i

GLB :

$\wedge$	a	b	c	d	e	f	g	h	i
a	a	a	a	a	a	a	a	a	a
b	a	b	a	b	b	a	b	b	b
c	a	a	c	a	c	c	c	c	c
d	a	b	a	d	b	a	d	b	d
e	a	b	c	b	e	c	e	e	e
f	a	a	c	a	c	f	c	f	f
g	a	b	c	d	e	c	g	e	g
h	a	b	c	b	e	f	e	h	h
i	a	b	c	d	e	f	g	h	i

Yes, it is a lattice as every pair of elements has a GLB and LUB.

Q. 5(b) Consider the chain of divisors of 4 and 9, i.e. $L_1 = \{1, 2, 4\}$ and $L_2 = \{1, 3, 9\}$ **(7 Marks)**

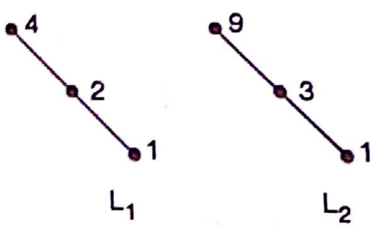


Fig. 1-Q. 5(b)

Find the Hasse diagram of $L_1 \times L_2$

Ans. : Partial ordering relation of division on L_1 is

$$R_1 = \{(1, 1), (1, 2), (1, 4), (2, 2), (2, 4), (4, 4)\}$$

Partial ordering relation of division on L_2 is

$$R_2 = \{(1, 1), (1, 3), (1, 9), (3, 3), (3, 9), (9, 9)\}$$

Product partial order relation on $L_1 \cdot L_2$ is

R_3 = {((1, 1), (1, 1)) ((1, 1), (1, 3)), ((1, 1), (1, 9)),
((1, 1), (2, 1)), ((1, 1), (2, 3)), ((1, 1), (2, 9)),
((1, 1), (4, 1)), ((1, 1), (4, 3)), ((1, 1), (4, 9)),
((1, 3), (1, 3)), ((1, 3), (1, 9)), ((1, 3), (2, 3)),
((1, 3), (2, 9)), ((1, 3), (4, 3)), ((1, 3), (4, 9)),
((1, 9), (1, 9)), ((1, 9), (2, 9)), ((1, 9), (4, 9)),
((2, 1), (2, 1)), ((2, 1), (2, 3)), ((2, 1), (2, 9)),
((2, 1), (4, 1)), ((2, 1), (4, 3)), ((2, 1), (4, 9)) ,
((2, 3), (2, 3)), ((2, 3), (2, 9)), ((2, 3), (4, 3)),
((2, 3), (4, 9)), ((2, 9), (2, 9)), ((2, 9), (4, 9)),
((4, 1), (4,1)), ((4, 1), (4, 3)), ((4, 1), (4, 9)),
((4, 3), (4, 3)), ((4, 3), (4, 9)), ((4, 9), (4, 9))}

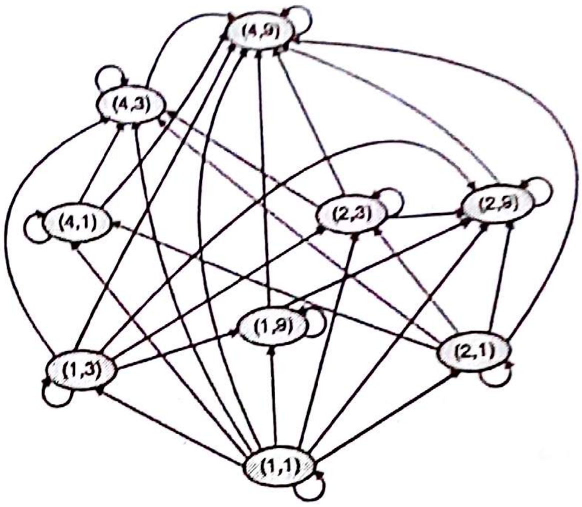


Fig. 2- Q. 5(b)

Digraph of this relation set is shown in Fig. 2- Q. 5(b),

To convert this digraph into Hasse diagram

Step 1 : Remove cycles

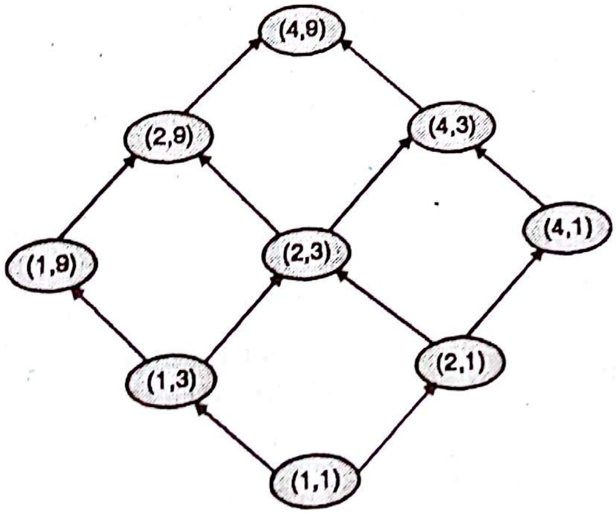


Fig. 3- Q. 5(b)

Step 2 : Remove transitive edges. ((1, 1), (1, 9)), ((1, 1), (2, 3)), ((1, 1), (2, 9)), ((1, 1), (4, 1)), ((1, 1), (4, 3)), ((1, 1), (4, 9)), ((1, 3), (2, 9)), ((1, 3), (4, 3)), ((1, 3), (4, 9)), ((1, 9), (4, 9)), ((2, 1), (2, 9)), ((2, 1), (4, 3)), ((2, 1), (4, 9)), ((2, 3), (4, 9)), ((4, 1), (4, 9)),

Step 3 : All edges are pointing upwards.

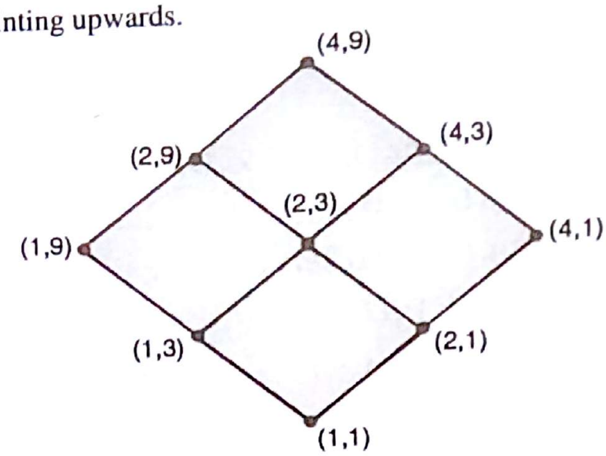


Fig. 4- Q. 5(b)

Now remove arrows from edges and replace circles by dots. Hasse diagram is shown in Fig. 4- Q. 5(b).

Chapter 4 : Functions and Pigeon Hole Principle [Total Marks : 19]

Q. 1(a) Show that if any seven points are chosen in a regular hexagon whose sides are of 1 unit, then two of them must be no further apart than 1 unit. **(5 Marks)**

Ans. : Divide the region into six equilateral triangles as shown in Fig. 1- Q. 1(a)

If seven points are chosen in the region, we can assign each of them to a triangle that contains it. If the point belongs to several triangles arbitrarily assign to one at them. Then the seven points are assigned to six triangular regions. So by the pigeonhole principle at least two points must belong to the same region. These two cannot be more than 1 unit apart.

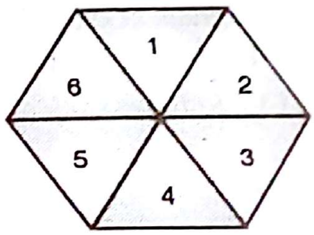


Fig. 1-Q. 1(a)

Q. 2(a) Define injective, surjective and bijective functions. If f : R → R defined by f(x) = x + 2 and $g(x) = x^2$. Find : i) f o g o f (ii) g o f o g **(6 Marks)**

Ans. : Definition of injective :

A function from A to B is said to be a **one to one function** if no two elements of A have the same image. One to one function is also called as **injective function.**

Definition of surjective :

A function from A to B is said to be an **onto** function if every element of B is the image of one or more elements of A. **Onto function** is also called **surjective** or 'f' is **ONTO** if Ran (f) = B.

Definition of bijective :

A function from A to B is said to be a **one to one onto function** if it is both an onto and one to one function. One to one onto function is also called as **bijective function.**

Example :

Let $f(x) = x + 2$ and $g(x) = x^2$

(i)

$$\begin{aligned} f \circ g \circ f &= f(g(f(x))) \\ &= f(g(x+2)) \\ &= f((x+2)^2) \\ &= (x+2)^2 + 2 \\ &= x^2 + 4x + 6 \end{aligned}$$

(ii)

$$\begin{aligned} g(f(g(x)) &= g(f(x^2)) \\ &= g(x^2+2) = (x^2+2)^2 \\ &= (x^2+2)(x^2+2) \\ &= x^4 + 2x^2 + 2x^2 + 4 \\ &= x^4 + 4x^2 + 4 \end{aligned}$$

Q. 3(a) If 11 people are chosen from a set of A = (1, 2, 3 ......20), then one of them is multiple of other. **(4 Marks)**

Ans. :

Every positive integer n can be written as $n = 2^k$, where m is odd and $k \geq 0$. This can be seen by simply factoring all powers of 2 (if any) out of n. In this case let us call m the odd part of n. If 11 numbers are chosen from the set { 1, 2, ...20 }, then two of them must have the same odd parts. This follows from the pigeonhole principle since there are 11 numbers (pigeons), but only 10 odd numbers between 1 and 20 (pigeonholes) that can be odd parts of these numbers. Let n_1 and n_2 be two chosen numbers with the same odd part we must have $n_1 = 2^{k_1} m$ and $n_2 = 2^{k_2} m$, for some k_1 and k_2. If $k_1 \geq k_2$, then n_1 is a multiple at n_2; otherwise n_2 is a multiple of n_1.

Q. 3(b) If F : A → B and If g : B → C are both one-one and onto, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. **(4 Marks)**

Ans. :

Proof :

$f : A \rightarrow B$ is one to one and onto

$g : B \rightarrow C$ is one to one and onto

$\therefore \quad g \circ f : A \rightarrow C$ is one to one and onto

$\Rightarrow \quad (g \circ f)^{-1} : C \rightarrow A$ is one to one and onto.

Let $a \in A$, then there exists an element $b \in B$ such that $f(a) = b$

$\Rightarrow \quad a = f^{-1}(b)$

Now $b \in B \Rightarrow$ there exists an element $c \in C$ such that $g(b) = C$

$\Rightarrow \quad b = g^{-1}(c)$

Then $(g \circ f)(a) = g[f(a)] = g[b] = c$ ...(i)

$\Rightarrow \quad a = (g \circ f)^{-1}(c)$

$(f^{-1} \circ g^{-1})(c) = f^{-1}[g^{-1}(c)] = f^{-1}(b) = a$ ...(ii)

$\Rightarrow \quad a = (f^{-1} \circ g^{-1})(c)$

Combining (i) and (ii) we have,

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Chapter 5 : Generating Functions & Recurrence Relations [Total Marks : 07]

Q. 6(b) Solve the following recurrence relation : **(7 Marks)**

$a_n - 5a_{n-1} + 6a_{n-2} = 2^n$ with initial conditions $a_0 = -1$ and $a_1 = 1$

Ans. : **Given :** $a_n = 5a_{n-1} - 6a_{n-2} + 2^n$

$\therefore \quad a_n - 5a_{n-1} + 6a_{n-2} = 2^n$

The characteristic equation is

$\alpha^2 - 5\alpha + 6 = 0 \qquad \therefore \quad \alpha = 3, 2$

The homogeneous solution is

$$a_n^{(h)} = A_1 (3)^n + A_2 (2)^n$$

Now, consider the form of particular solution. Here f (n) is 2^n, the form of particular solution is, $P7^n$.

Subsituting in the given recurrence relation, we get,

$$P.2^n - 5P.2^{n-1} + 6P.2^{n-2} = 2^n$$

which simplifies to

$$P = 2.44$$

Therefore, the particular solution is,

$$a_n^{(P)} = 2.442^n$$

Thus the general solution is

$$a_n = A_1 (3)^n + A_2 (2)^n + 2.442^n$$

Chapter 6 : Graphs [Total Marks : 11]

Q. 1(b) Determine the number of edges in a graph with 6 nodes, 2 nodes of degree 4 and 4 nodes of degree 2. Draw two such graphs. **(5 Marks)**

Ans. : Suppose the graph with 6 vertices has e number of edges. Therefore, by handshaking lemma

$$\sum_{i=1}^{6} d(v_i) = 2e$$

$$\Rightarrow d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 2e$$

Now, given 2 vertices are of degree 4 and 4 vertices are of degree 2.

Hence from the above equation

$$\Rightarrow \quad (4 + 4) + (2 + 2 + 2 + 2) = 2e$$

$$\Rightarrow \quad 16 = 2e$$

$$\Rightarrow \quad e = 8$$

Hence the number of edges in a graph with 6 vertices with given conditions is 8.

Two such graphs are shown in Figs. 1-Q. 1(b) and 2-Q. 1(b).

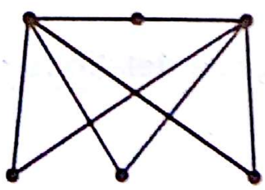


Fig. 1-Q. 1(b)

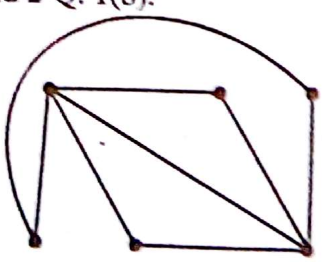


Fig. 2-Q. 1(b)

Q. 2(c) Let S = {1,2,3,4,5} and A– S × S. Define the following relation R on A : (a,b) R (c,d) if and only if ad = bc. Show that R is an equivalence relation and compute A/R. **(6 Marks)**

Ans. :

Lets $S = \{1, 2, 3, 4, 5\}$, $A = 5 \times 5$

$R = (a, b)\ R\ (c, d)$ if $ad = bc$

Let (a, b) R (a, b) $\therefore$ ab = ba

This expression is true, hence (a, b) R (a, b)

$\therefore$ Given relation is reflective.

If (a, b) R (c, d) $\therefore$ ad = bc ...(i)

Then check for (c, d) R (a, b) $\therefore$ cb = da` ...(ii)

Above two expression are similar

$\therefore$ If (a, b) R (c, d) then (c, d) R (a, b)

$\therefore$ The given relation is symmetric

Let (a, b) R (c, d) and (c, d) R (e, f)

(a, b) R (c, d) $\therefore$ ad = bc

(c, d) R (e, f) $\therefore$ cf = ed

Check for (a, b) R (e, f)

$$ad = bc \qquad \therefore a = \frac{bc}{d}$$

$$cf = ed \qquad \therefore f = \frac{ed}{c}$$

$$af = \frac{b\cancel{c}}{\cancel{d}} + \frac{e\cancel{d}}{\cancel{c}} \qquad \therefore af = be$$

$\therefore$ (a, b) R (e, f)

Hence R is transitive.

Thus R is an equivalence relation.

Chapter 7 : Trees [Total Marks : 07]

Q. 4(c) Find the generating functions for the following sequence **(7 Marks)**

(i) 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, ... (ii) 6, – 6, 6, – 6, 6, – 6, 6 – ....

Ans. : Find the generating function.

The generating function is,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

(i) The generating function for sequence {0, 0, 0, 1, 2, 3, 4, 5, 6, 7} is,

$0 + 0x + 0x^2 + 1x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + \ldots$

$x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + \ldots$

(ii) The generating function for sequence {6, – 6, 6, – 6, 6, – 6,...}

$= 6 + - 6x + 6x^2 - 6x^3 + 6x^4 - 6x^5$

Chapter 8 : Algebraic Structures [Total Marks : 44]

Q. 2(b) It was found that in a class, 80 students are passed in English, 60 in Science and 50 in Mathematics. If was also found that 30 students passed in both English and Science, 15 students passed in both English and Mathematics and 20 students passed in both Mathematics and Science, 10 students passed in all three subjects. If there are 150 students in the class, find.

(i) How many students passed in at least one subject ?

(ii) How many students passed in English only ?

(iii) How many students failed in all three subjects ?

(8 Marks)

Ans. : 80 students are passed in English.

60 students are passed in Science.

50 students are passed in maths.

30 students are passed in both English and Science.

15 students are passed in both English and Mathematics.

20 students are passed in both Mathematics and Science.

10 students are passed in all 3 subjects.

Let 'E' denote students passed in English.

Let 'S' denote students passed in Science.

Let 'M' denote students passed in Mathematics.

$\therefore \quad |E| = 80 \qquad |S| = 60 \qquad |M| = 50$

$|E \cap S| = 30 \qquad |E \cap M| = 15 \qquad |M \cap S| = 20$

$|E \cap S \cap M| = 10$

Using principle of inclusion and exclusion.

$$|E \cup S \cup M| = |E| + |S| + |M| - |E \cap S| - |E \cap M| - |M \cap S| + |E \cap S \cap M|$$
$$= 80 + 60 + 50 + 30 - 15 - 20 + 10 = 200 - 65 = 135$$

(i) $\therefore$ 135 students passed in atleast one subject.

(ii) Students passed in only English =

$|E| - |E \cap S| - |E \cap M| + |E \cap S \cap M| = 80 - 30 - 15 + 10 = 90 - 45 = 45$

(iii) Students failed in all three subjects. $= 150 - |E \cup S \cup M| = 150 - 135 = 15$

Q. 3(d) If (G, *) is an Abelian group, then for all a, b $\in$ G show that $(a * b)^n = a^n + b^n$.(use mathematical induction). **(6 Marks)**

Ans. :

(i) Basis step : For n = 1

$(a * b)' = a' * b'$

$\therefore \; a * b = a * b$

$\therefore$ Statement is true for n = 1

(ii) Induction step :

Let the given statement be true for some n = k where k $\in$ N.

$\therefore \quad (a * b)^k = a^k * b^k$ ...(1)

To prove the statement for $n = k + 1$

$$(a * b)^{k+1} = (a * b)^k * (a * b) \quad \text{... definition of } a^n$$

$$\therefore \quad (a * b)^{k+1} = a^k * b^k * a * b \quad \text{... from Equation (1)}$$

$$\therefore \quad (a * b)^{k+1} = a^k * a * b^k * b \quad \because (G, *) \text{ is an Abelian group}$$

$$(a * b)^{k+1} = a^{k+1} * b^{k+1} \quad \because \text{ definition of } a^n$$

Using the principle of mathematical induction we state that

$$(a * b)^n = a^n * b^n \text{ for all } n \in N$$

Q. 4(a) Let G = {1, 2, 3, 4, 5, 6}. Prove that (G, X_7) is a finite Abelian group with respect to multiplication modulo 7. **(10 Marks)**

Ans. :

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

$$\left.\begin{aligned} 2 \times 1 \bmod 7 &= 2 \\ 2 \times 2 \bmod 7 &= 4 \\ 2 \times 3 \bmod 7 &= 6 \\ 2 \times 4 \bmod 7 &= 1 \\ 2 \times 5 \bmod 7 &= 3 \\ 2 \times 6 \bmod 7 &= 5 \end{aligned}\right\} 2^{nd} \text{ row}$$

Similarly row 1, row 3, row 4, row 5, and row 6 are calculated.

(i) All the entries in the composition table are elements of G. Hence G is closed under multiplication modulo 7 ($\times_7$)

(ii) The composition of $\times_7$ is associative. Let a, b, c are any three elements of G, then

$$a \times_7 (b \times_7 c) = (a \times_7 b) \times_7 c$$

Let $a = 1,\ b = 2,\ c = 3$

$$1 \times_7 (2 \times_7 3) = (1 \times_7 2) \times_7 3$$

$$1 \times_7 6 = 2 \times_7 3$$

$$6 = 6$$

Let $a = 4,\ b = 5,\ c = 6$

$$4 \times_7 (5 \times_7 6) = (4 \times_7 5) \times_7 6$$

$$4 \times_7 2 = 6 \times_7 6$$

$$1 = 1$$

Hence, $\times_7$ is an associative operation. Since it is satisfying for all $a, b, c \in G$.

(iii) We have $1 \in G$

If a is any element of G, then from the composition table, we can see that

$$1 \times_7 a = a = a \times_7 1$$

that is, $1 \times_7 0 = 0 \times_7 0 = 0$

$$1 \times_7 1 = 1 \times_7 1 = 1$$

$$1 \times_7 2 = 2 \times_7 1 = 2$$
$$1 \times_7 3 = 3 \times_7 1 = 3$$
$$1 \times_7 4 = 4 \times_7 1 = 4$$
$$1 \times_7 5 = 5 \times_7 1 = 5$$
$$1 \times_7 6 = 6 \times_7 1 = 6$$

$\therefore$ 1 is an identity element

(iv) From the composition table, we can see that the left inverses of 1, 2, 3, 4, 5, 6 are 1, 4, 5, 2, 3, 6 respectively. Since,

e.g. $3 \times_7 5 = 1 = 5 \times_7 3$ i.e. inverse of 3 is 5.

(v) The composition $\times_7$ is commutative as the corresponding rows and columns in the table are identical.

(vi) The set has 6 elements hence group $(G, \times_7)$ is a finite Abelian group of order 6.

Q. 4(b) Let $H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be parity check matrix

Determine the group code $e_H : B^3 \rightarrow B^6$ **(7 Marks)**

Ans. : **Please refer Q. 5(c) of Dec. 2014.**

Q. 5(a) If function f is an isomorphism from semigroup (S, *) to (T, *') then prove that f^{-1} is an isomorphism (T, *') to (S, *). **(5 Marks)**

Ans. :

Given : $f : S \rightarrow T$ is isomorphism

$\therefore$ It is one to one from S to T

$\therefore f(a * b) = f(a) *' f(b)$

for all a and b in S

$\therefore$ f^{-1} exists and is one to one correspondence from T to S.

We now show that f^{-1} is an isomorphism from (T, *') to (S, *)

Let a' b' be the elements of T

Since f is onto, we can find elements a and b in S such that

$$f(a) = a' \text{ and } f(b) = b'$$

Then $\quad a = f^{-1}(a')$ and $b = f^{-1}(b')$

Now $f^{-1}(a' *' b') = f^{-1}[f(a) *' f(b)] = f^{-1}[f(a * b)]$

$$= a * b$$

$$= f^{-1}(a') * f^{-1}(b')$$

Hence f^{-1} is **isomorphism.**

Q. 6(a) Determine whether following graphs are isomorphic **(8 Marks)**

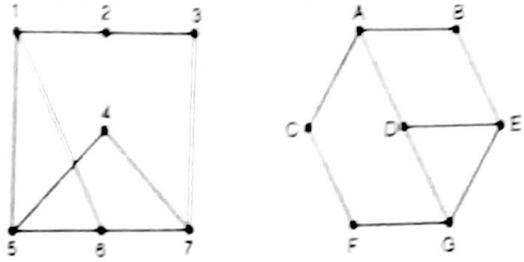


Fig. 1- Q. 6(a)

Ans. :

1. **Condition :**

1. Both the graphs G1 and G2 contain 7 vertices and 9 edges.
2. The number of vertices of degree 3 in the both the graphs is four and the number of vertices of degree 2 in both the graphs is 3.
3. For adjacency.

Isomorphic graph

$A \rightarrow 7, B \rightarrow 4, C \rightarrow 3, D \rightarrow 6, E \rightarrow 5, F \rightarrow 2, G \rightarrow 1$

$\therefore$ Given two graphs are isomorphic

Show that the set $G = \{ f_1, f_2, f_3, f_4, f_5, f_6 \}$

Where the functions are defined by

$$f_1(x) = x \qquad \therefore \quad f_2(x) = 1$$

$$f_3(x) = \frac{x}{x-1} \qquad \therefore \quad f_4(x) = \frac{1}{x}$$

$$f_5(x) = \frac{1}{1-x}$$

$f_6(6) = 1 - \dfrac{1}{x}$ is a group under composition of function frame the composition table

$(G_1 *)$ is group under operation composition.

$$(f_2 * f_3)(x) = f_2 * (f_3(X)) = f_2 * \left(\frac{x}{x-1}\right) = \frac{x}{(x-1)}$$

$$f_1 * (f_2 * f_3)(X) = (f_2 * f_2) * f_3(x)$$

2. **Condition :** 1. It associative. 2. f_1 is identify element as $f_1(X) = X$

$f_1 * f_2(X) = X * (1) = X$

Each of $f_1, f_2, f_3, f_4, f_5, f_6$ has its inverse under composition.

$\therefore (G, *)$ is group.

X	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_6	f_5	f_4	f_3
f_3	f_3	f_4	f_2	f_6	f_1	f_5
f_4	f_4	f_3	f_5	f_6	f_2	f_1
f_5	f_5	f_6	f_4	f_3	f_1	f_2
f_6	f_6	f_5	f_2	f_2	f_3	f_4

It is closed operation

$\therefore$ (G, X) is a group.

May 2016

Chapter 1 : Set Theory [Total Marks - 06]

Q. 1(a) Find how many integers between 1 and 60 are not divisible by 2 nor by 3 and nor by 5 respectively. **(6 Marks)**

Ans. : Let A_1, A_2 and A_3 be the set of integers between 1 and 60 divisible by 2, 3 and 5 respectively.

$$\therefore \quad |A_1| = \left|\frac{60}{2}\right| = 30$$

$$|A_2| = \left|\frac{60}{3}\right| = 20$$

$$|A_3| = \left|\frac{60}{5}\right| = 12$$

and $$|A_1 \cap A_2| = \left|\frac{60}{2 \times 3}\right| = 10$$

$$|A_1 \cap A_3| = \left|\frac{60}{2 \times 5}\right| = 6$$

$$|A_2 \cap A_3| = \left|\frac{60}{3 \times 5}\right| = 4$$

and $$|A_1 \cap A_2 \cap A_3| = \left|\frac{60}{2 \times 3 \times 5}\right| = 2$$

Number of integers between 1 and 60 which are divisible by 2, 3 or 5 are

$$= |A_1 \cup A_2 \cup A_3|$$

$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$= 30 + 20 + 12 - 10 - 6 - 4 + 2 = 44$$

Hence the number of integers between 1 and 60 are not divisible by 2, 3 or 5 = 60 – 44 = 16

Chapter 2 : Logic [Total Marks : 12]

Q. 1(b) By using mathematical induction prove that $1 + a + a^2 + \ldots a^n = \frac{1 - a^{n+1}}{1 - a}$ where $n \geq 0$. **(6 Marks)**

Ans. :

Let $$P(n) : 1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

(i) Basis of induction :

For $n = 0$, $P(0) = \frac{1 - a}{1 - a} = 1$

$\therefore$ It is true for $n = 0$

For $n = 1$, $P(1) = \frac{1 - a^{1+1}}{1 - a} = \frac{1 - a^2}{1 - a} = 1 + a$

$\therefore$ True for $n = 1$

(ii) Induction step :

Assume P(k) is true i.e.

$$P(k) : 1 + a + a^2 + \ldots + a^k = \frac{1 - a^{k+1}}{1 - a} \quad \ldots(i)$$

To prove that $P(k + 1)$ is true.

$$P(k + 1) : 1 + a + a^2 + \ldots + a^{k+1} = \frac{1 - a^{(k+1)+1}}{1 - a}$$

$$\text{L.H.S.} = 1 + a + a^2 + \ldots + a^k + a^{k+1}$$

$$= \frac{1 - a^{k+1}}{1 - a} + a^{k+1} \quad \ldots\text{from Equation (i)}$$

$$= \frac{1 - a^{k+1} + a^{k+1} - a^{k+2}}{1 - a} = \frac{1 - a^{k+2}}{1 - a}$$

$$= \frac{1 - a^{(k+1)+1}}{1 - a}$$

$\therefore$ It is true for $n = k + 1$. Hence, $P(n)$ is true for all n.

Q. 5(a) What is an universal and existential quantifiers ?Prove the distribution law.
$(p \vee q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ **(6 Marks)**

Ans. : Universal Quantifier :

If $P(x)$ is a predicate with the individual variable x as an argument, then the assertion "For all x, P(x)" which is interpreted as "For all values of x, the assertion P(x) is true," is a statement in which the variable x is said to be universally quantified. We denote the phrase "For all" by $\forall$, called the **universal quantifier**. The meaning of $\forall$ is "for all" or "for every" or "for each". If P(x) is true for every possible value of x, then $\forall$ x P(x) is true; otherwise $\forall$ P(x) is false.

Example : Let P(x) be the predicate $x \geq 0$; where x is any positive integer. Then the proposition $\forall$ x P(x) is true. However, if x is any real number, then $\forall$ x P(x) is a false proposition.

Existential Quantifier :

Suppose for the predicate P(x), $\forall$x P(x) is false, but there exists atleast one value of x for which P(x) is true, then we say that in this proposition, x is bound by existential quantification. We denote the words "there exists" by the symbol $\exists$. Then the notation $\exists$ x P(x) means "there exists a value of x (in the universe of discourse) for which P(x) is true".

Distribution Law :

p	q	r	$q \wedge r$	$p \vee q \wedge r$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Since truth tables are identical, (that is 5th and 8th column of the Truth table are same) the propositions are equivalent.

Chapter 3 : Relations, Digraphs and Lattice [Total Marks - 26]

Q. 1(c) Let A={1,2,3,4,5} and R be the relation defined by a R b if and only if a < b compute R, R^2 and R^3. Draw digraph of R, R^2 and R^3. **(8 Marks)**

Ans. : R = {(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}

Diagraph of R is shown in Fig. 1-Q. 1(c).

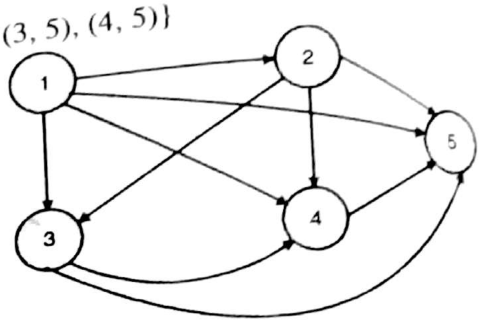


Fig. 1-Q. 1(c)

R^2 :

$1 R^2 3$ Since $1 R 2$ and $2 R 3$

$1 R^2 4$ Since $1 R 2$ and $2 R 4$

$1 R^2 5$ Since $1 R 2$ and $2 R 5$

$2 R^2 4$ Since $2 R 3$ and $3 R 4$

$2 R^2 5$ Since $2 R 4$ and $4 R 5$

$3 R^2 5$ Since $3 R 4$ and $4 R 5$

R^2 = {(1, 3), (1, 4), (1, 5), (2, 4), (2, 5), (3, 5)}

Diagraph of R is shown in Fig. 2-Q. 1(c).

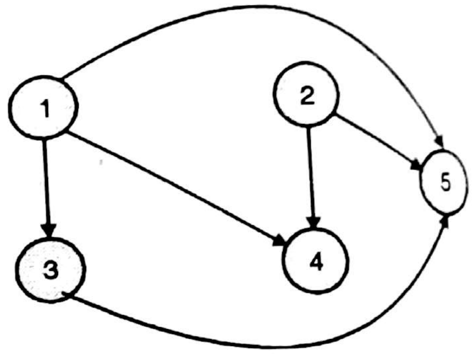


Fig. 2-Q. 1(c)

R^3 :

$1 R^3 4$ Since $1 R 2$, $2 R 3$ and $3 R 4$

$1 R^3 5$ Since $1 R 2$, $2 R 4$ and $4 R 5$

$2 R^3 5$ Since $2 R 3$, $3 R 4$ and $4 R 5$

$\therefore$ R^3 = {(1, 4), (1, 5), (2, 5)}

Digraph of R^3 in shown in Fig. 3-Q. 1(c)

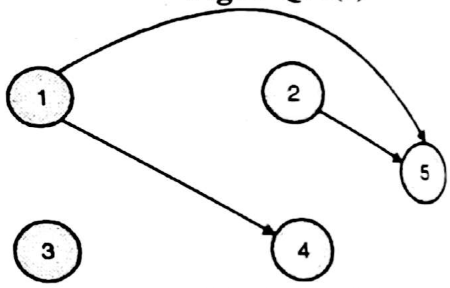


Fig. 3-Q. 1(c)

In Example 12 through 14, let R be the relation whose digraph is given in Fig. 4-Q. 1(c).

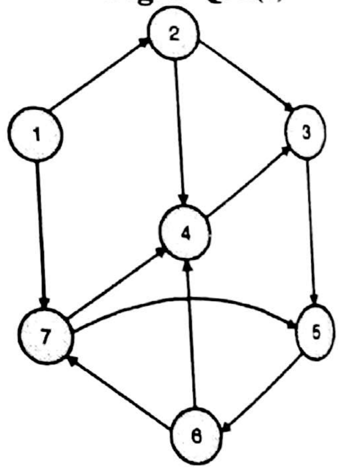


Fig. 4-Q. 1(c)

Q. 2(b) Let A = {1,2, 3,4,6} = B, a R b if and only if a is multiple of b. Find R. Find each of the following (i) R(4) (ii) R(G) (iii) R({2,4,6}). **(6 Marks)**

Ans. :

$$R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4,1), (4, 2), (4, 4), (6, 1), (6, 2), (6, 3), (6, 6)\}$$

$$\text{Dom}(R) = \{1, 2, 3, 4, 6\}$$

$$\text{Ran}(R) = \{1, 2, 3, 4, 6\}$$

$$M_R = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

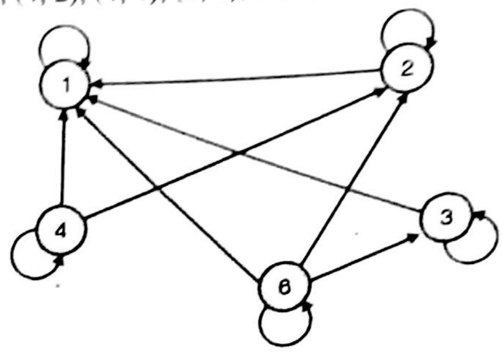


Fig. 1-Q. 2(b)

Q. 3(b) Define distributive lattice. Show that in a bounded distributive lattice, if a complement exists, its unique. **(6 Marks)**

Ans. : Definition of Distributive lattice :

A lattice L is called distributive if for any elements a, b and c in L we have the following distributive properties.

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

If L is not distributive, we say that L is non-distributive.

Let a′ and a″ be complements of the element. a ∈ L. Then

$$a \vee a' = I \qquad a \vee a'' = I$$

$$a \wedge a' = 0 \qquad a \wedge a'' = 0$$

Using the distributive laws, we obtain

$$\begin{aligned} a' = a' \vee 0 &= a' \vee (a \wedge a'') \\ &= (a' \vee a) \wedge (a' \vee a'') \\ &= (a \vee a') \wedge (a' \vee a'') \\ &= I \wedge (a' \vee a'') \\ &= a' \vee a'' \end{aligned}$$

Also,

$$\begin{aligned} a'' &= a'' \vee 0 \\ &= a'' \vee (a \wedge a') \\ &= (a'' \vee a) \wedge (a'' \vee a') \\ &= (a \vee a'') \wedge (a' \vee a'') \\ &= I \wedge (a' \vee a'') = a' \vee a'' \end{aligned}$$

Hence, $a' = a''$

Q. 5(b) Let A={1,2,3,4}and let R={(1, 2) (2, 3) (3, 4) (2, 1)}.Find transitive closure of R by using Warshall's algorithm. **(6 Marks)**

Ans.: **Please refer Q. 3(a) of May 2014.**

Chapter 4 : Functions and Pigeon Hole Principle [Total Marks : 14]

Q. 3(a) State pigeon hole and extended pigeon hole principle. Show that 7 colors are used to paint 50 bicycles, at least 8 bicycle will be of same color. **(6 Marks)**

Ans.: Pigeon hole and extended hole : Please refer Q. 3(b) of Dec. 2014.

For example : Please refer Q. 5(b) of May 2015.

Q. 3(c) Functions f, g, h are defined on a set X = {1,2,3} as f = {(1,2) (2,3) (3,1)}, g = {(1,2) (2,1) (3,3)}, h = {(1, 1) (2,2) (3,1)}. (i) Find f o g, g o f are they equal ? (ii) Find f o g o h and f o h o g. **M(18)-6** **(8 Marks)**

Ans. : We may depict f, g , h graphically as shown in Fig. 1-Q. 3(c)

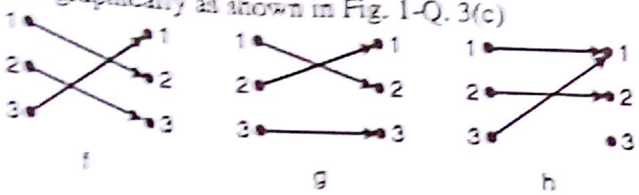


Fig. 1-Q. 3(c)

(i)

$$f(1) = 2, \quad f(2) = 3, \quad f(3) = 1.$$
$$g(1) = 2, \quad g(2) = 1, \quad g(3) = -3.$$
$$f\,o\,g(1) = f(g(1)) = f(2) = 3.$$
$$f\,o\,g(2) = f(g(2)) = f(1) = 2.$$
$$f\,o\,g(3) = f(g(3)) = f(3) = 1.$$
$$\therefore \quad f\,o\,g = \{(1, 3), (1, 2), (1, 1)\}.$$

f o g is depicted as shown in Fig. 2-Q. 3(c)

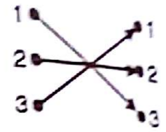


Fig. 2-Q. 3(c)

$$g\,o\,f(1) = g(f(1)) = g(2) = 1$$
$$g\,o\,f(2) = g(f(2)) = g(3) = 3$$
$$g\,o\,f(3) = g(f(3)) = g(1) = 2$$
$$\therefore \quad g\,o\,f = \{(1, 1), (2, 3), (3, 2)\}.$$

g o f is depicted as shown in Fig. 3-Q. 3(c)

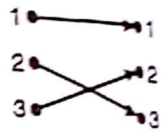


Fig. 3-Q. 3(c)

$\therefore f\,o\,g \neq g\,o\,f$

(ii)

$$h(1) = 1, \quad h(2) = 2, \quad h(3) = 1$$
$$f\,o\,g\,o\,h(1) = f(g(h(1))) = f(g(1)) = f(2) = 3.$$
$$f\,o\,g\,o\,h(2) = f(g(h(2))) = f(g(2)) = f(1) = 2.$$
$$f\,o\,g\,o\,h(3) = f(g(h(3))) = f(g(1)) = f(2) = 3.$$
$$\therefore \quad f\,o\,g\,o\,h = \{(1, 3), (2, 2), (3, 3)\}.$$

f o g o h can be depicted as shown in Fig. 4-Q. 3(c)

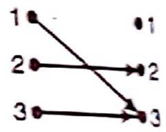


Fig. 4-Q. 3(c)

$$f\,o\,h\,o\,g(1) = f(h(g(1))) = f(h(2)) = f(2) = 3.$$
$$f\,o\,h\,o\,g(2) = f(h(g(2))) = f(h(1)) = f(1) = 2.$$
$$f\,o\,h\,o\,g(3) = f(h(g(3))) = f(h(3)) = f(1) = 2.$$
$$\therefore \quad f\,o\,h\,o\,g = \{(1, 3), (2, 2), (3, 2)\}.$$

f o h o g can be depicted as shown in Fig. 5-Q. 3(c).

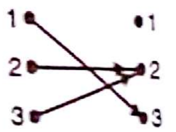


Fig. 5-Q. 3(c)

Chapter 5 : Generating Functions & Recurrence Relations [Total Marks - 14]

Q. 6(c) Solve the following recurrence relation: $a_n - 7a_{n-1} + 10a_{n-2} = 0$ with initial condition $a_1 = 1$, $a_2 = 6$ **(8 Marks)**

Ans. :

Given $\quad a_n - 7a_{n-1} + 10a_{n-2} = 0$

Initial condition $a_1 = 1$, $a_2 = 6$

The characteristic equation is

$$\alpha^2 - 7\alpha + 10 = 0$$

$$\therefore (\alpha - 5)(\alpha - 2) = 0$$

$$\text{or} \quad \therefore \quad \alpha = 5, 2$$

Which are the characteristic roots of the equation,

$\therefore$ The solution of the given recurrence relation is

$$A_n = A_1(5)^n + A_2(2)^n$$

Where A_1, A_2 are constants to find A_1 and A_2,

Solving two equations we get

$$A_1 = \frac{4}{3} \quad , \quad A_2 = \frac{-1}{3}$$

$\therefore$ Solution of given recurrence relation is

$$A_n = \frac{4}{3}(5)^n \frac{-1}{3}(2)^n$$

Q. 6(a) Find the ordinary generating functions for the given sequences: **(6 Marks)**

(i) {1, 2,3,4,5--} (ii) {2,2, 2, --} (iii) {1,1,1,1---}

Ans. :

(i) The generating function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$\therefore$ The generating function for sequence $\{0, 1, 2, 3, 4, \ldots\}$ is $0 + x + 2x^2 + 3x^3 + 4x^3 + \ldots$

(ii) The generating function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$\therefore$ The generating function for sequence $\{1, 2, 3, 4, \ldots\}$ is $1 + 2x + 3x^2 + 4x^3 + \ldots$

(iii) The generation function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$\therefore$ The generating function for sequence $\{0, 3, 3^2, 3^3, \ldots\}$ is $0 + 3x + 3^2x^2 + 3^3x^3 + \ldots$

i.e. $3x + 3^2x^2 + 3^3x^3 + \ldots$

(iv) The generating function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$\therefore$ The generating function for the sequence $\{2, 2, 2, 2, \ldots\}$ is $2 + 2x + 2x^2 + 2x^3 + \ldots$

(i) We know $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

$$= \sum_{n=0}^{\infty} (1)^n x^n$$

∴ Generating function for sequence 1, 1, 1, 1, 1, 1 is

$$\frac{1}{1-x} = \sum_{n=0}^{5} (1)^n x^n$$

(ii) Generating function for sequence 2, 2, 2, 2, 2

$$\sum_{n=0}^{5} (2)^n x^n = 2^0 x^0 + 2^1 x^1 + 2^2 x^2 + 2^3 x^3 + 2^4 x^4 + 2^5 x^5$$

(iii) Similarly generating function for sequence 1, 1, 1, 1,........

$$\sum_{n=0}^{\infty} (1)^n x^n = \frac{1}{1-x}$$

Chapter 6 : Graphs [Total Marks : 20]

Q. 4(a) Define Eular path and Eular circuit, determine whether the given graph has Eular path and Eular circuit. **(6 Marks)**

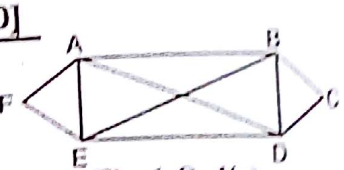


Fig. 1-Q. 4(a)

Ans. : Definition of Eular :

A path in a graph G is called an **Euler path** if it includes every edge exactly once. An **Euler circuit** is an Euler path that is a circuit.

Examples :

1. An Euler path in Fig. 2-Q. 4(a) is π : E, D, B, A, C, D. There is no Euler circuit.

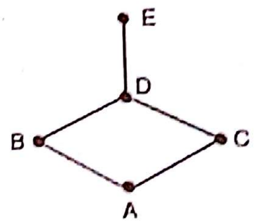


Fig. 2-Q. 4(a)

2. One Euler circuit in the graph of Fig. 3-Q. 4(a)is π : 5, 3, 2, 1, 3, 4, 5 Euler path is not possible for this graph.

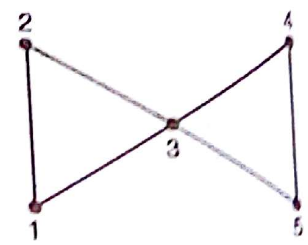


Fig. 3-Q. 4(a)

Two questions arise naturally at this point. Is it possible to determine whether an Euler path or Euler circuit exists without actually finding it ? If there must be an Euler path or Euler circuit, is there an efficient way to find one ? See following theorems. In Fig. 4-Q. 4(a), all vertices have even degree. So this graph contains an Euler circuit, by theorem 1 (b), and that Euler circuit is,

π : A, B, C, D, E, F, A, E, B, D, A.

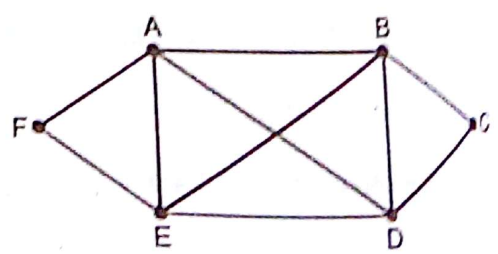


Fig. 4-Q. 4(a)

Q. 4(b) Define Hamiltonian path and Hamiltonian circuit, determine whether the given graph has Hamiltonian path and Hamiltonian circuit. **(6 Marks)**

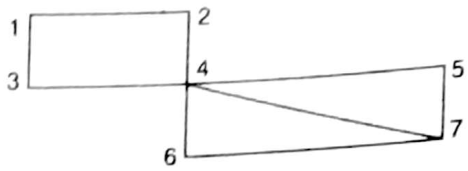


Fig. 1-Q. 4(b)

Ans. : Definition : A **Hamiltonian path** is a path that contains each vertex exactly once. A **Hamiltonian circuit** is a circuit that contains each vertex exactly once except for the first vertex, which is also the last.

$\therefore$ Degree of 1 + degree of 7 > number of vertices

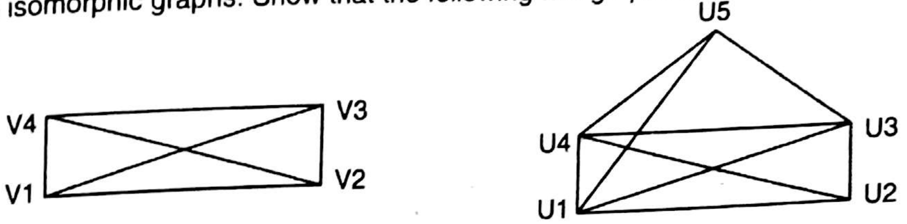
$$2 + 3 > 7$$

$$5 \not> 7$$

$\therefore$ There is no Hamiltonian circuit.

But there is an Hamiltonian path π : 3, 1, 2, 4, 6, 7, 5.

Q. 4(c) Define isomorphic graphs. Show that the following two graphs are isomorphic. **(8 Marks)**



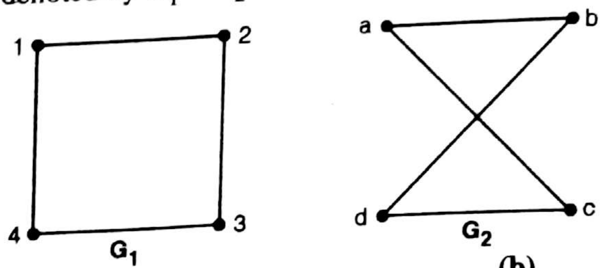
(a) (b)

Fig. 1-Q. 4(c)

Ans. : Definition :

In geometry, two Figures are thought of as equivalent if they have identical behaviour in terms of geometric properties. Likewise two graphs are thought of as equivalent or **isomorphic** if they have identical behaviour in terms of graph properties. The problem of graph isomorphism arises in many fields, such as chemistry, switching theory, information retrieval etc. Now we define isomorphism of graph as follows :

Two graphs G_1 (V_1, E_1) and $G'(V', E')$ are said to be isomorphic to each other if there is a one–one correspondence between their vertices and between their edges such that the incidence relationship is preserved. In other words, suppose that edge e is incident on vertices v_1 and v_2 of G_1 then the corresponding edge e' in G' must be incident on the vertices v_1' and v_2' that corresponds to V_1 and V_2. In particular, adjacency between vertices is preserved. More generally two graphs G_1 and G_2 are isomorphic if we can find the bijections $f_v : V_1 \rightarrow V_2$ and $f_e : E_1 \rightarrow E_2$ such that if $e \in E_1$, given by $e = (a, b)$ where a and b are vertices in V_1 then $f_e(e) = (f_v(a), f_v(b))$, where $f_v(a)$ and $f_v(b)$ are vertices in V_2. Isomorphic graphs are denoted by $G_1 \cong G_2$. In following Fig. 2-Q. 4(c), G_1 is isomorphic to G_2.



(a) (b)

Fig. 2-Q. 4(c)

The one-one correspondence between the vertices are :

$1 \rightarrow a$, $2 \rightarrow b$,

$3 \rightarrow d$, $4 \rightarrow c$

It is immediately apparent by definition of isomorphism that two isomorphic graphs must have

1. The same number of vertices
2. The same number of edges
3. An equal number of vertices with a given degree.

However, these conditions are by no means sufficient, for instance, the two graphs shown in Fig. 3-Q. 4(c), satisfy all the three conditions given above, yet they are not isomorphic.

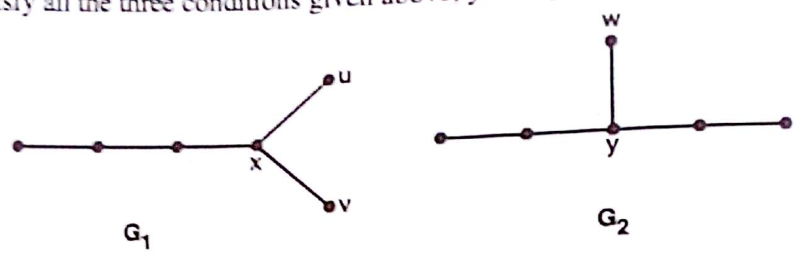


Fig. 3-Q. 4(c)

In Fig. 3-Q. 4(c), the graph G_1 has a vertex X of degree 3 which is adjacent to two pendant vertices u and v and one vertex of degree 2. But in G_2, the vertex y of degree 3 is adjacent to only one pendant vertex y and two vertices of degree 2. Hence adjacency is not preserved.

$$\text{Therefore } G_1 \not\cong G_2$$

Both the graphs have 5 vertices and 8 edges. Both the graphs have 4 vertices of degree 3 and 1 vertex of degree 4. And adjacency is also preserved. The one–one correspondance between the vertices is given by, function $f : v \rightarrow v^*$,

where $v = \{v_1, v_2, v_3, v_4, v_5\}$,

$v^* = \{u_1, u_2, u_3, u_4, u_5\}$

Such that $\{a, b\}$ is an edge in G_1 if and only, if $\{f(a), f(b)\}$ is an edge in G_2.

$$\therefore \quad f = \{(v_1, u_1), (v_2, u_2), (v_3, u_3), (v_4, u_5), (v_5, u_4)\}$$

Hence G_1 and G_2 are isomorphic graphs.

Chapter 8 : Algebraic Structures [Total Marks : 28]

Q. 2(a) Show that a group G is Ablian,if and only if $(ab)^2 = a^2 b^2$ for all elements a and band G. **(6 Marks)**

Ans. :

Part I :

$$(a\,b)^2 = a^2 b^2 \quad \text{...Given}$$

$$\therefore \quad a * b * a * b = a * a * b * b$$

Pre-multiply by a^{-1} and post multiply by b^{-1} on both sides

$$\therefore \quad a^{-1} * a * b * a * b * b^{-1} = a^{-1} * a * a * b * b * b^{-1}$$

$$\therefore \quad b * a = a * b$$

As $a * a^{-1} = b * b^{-1} = e$

and $e * x = x$

Now since $b * a = a * b$

G is an Abelian group.

Part II :

Now assume G is Abelian

Therefore $a * b = b * a$

$$
\begin{aligned}
(a * b) * (a * b) &= (ab)^2 \\
&= a * (b * a) * b && \because \text{associative} \\
&= a * (a * b) * b && \because \text{Abelian} \\
&= (a * a) * (b * b) && \because \text{associative} \\
&= a^2 b^2
\end{aligned}
$$

Hence proved.

Q. 2(c) Show that the (2,5)encoding function e: $B^2 \rightarrow B^5$ defined by e (00) = 00000, e (01) =01110, e (10) = 10101, e (11) = 11011 is a group code. How many errors will it detect and correct ? **(8 Marks)**

Ans. : Please refer Q. 4(b) of Dec. 2014

Q. 5 (c) Prove that the set A = {0,1,2,3,4,5} is a finite Abelian group under addition modulo 6. **(8 Marks)**

Ans. :

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$$
\left.
\begin{aligned}
0 + 0 \bmod 6 &= 0 \\
0 + 1 \bmod 6 &= 1 \\
0 + 2 \bmod 6 &= 2 \\
0 + 3 \bmod 6 &= 3 \\
0 + 4 \bmod 6 &= 4 \\
0 + 5 \bmod 6 &= 5
\end{aligned}
\right\} 1^{st} \text{ row}
$$

Similarly other rows are calculated.

(i) All the entries in the composition table are elements of the set G. Hence G is closed with respect to addition modulo 6 ($+_6$).

(ii) The composition $+_6$ is associative. If a, b, c are any three elements of G, then

$a +_6 (b +_6 c) = (a +_6 b) +_6 c$

Let $a = 1,\ b = 2,\ c = 3.$

$1 +_6 (2 +_6 3) = (1 +_6 2) +_6 3$

$1 +_6 5 = 3 +_6 3$

$0 = 0$

Let $a = 3,\ b = 4,\ c = 5$

$3 +_6 (4 +_6 5) = (3 +_6 4) +_6 5$

$3 +_6 3 = 1 +_6 5$

$0 = 0$

Hence, $+_6$ is an associative operation. Since it is satisfying for all a, b, c, $\in$ G.

(iii) If a is any element of G, then from the composition table we see that

$$0 +_6 a = a = a +_6 0 = 0$$

that is, $0 +_6 0 = 0 +_6 0 = 0$

$$0 +_6 1 = 1 +_6 0 = 1$$
$$0 +_6 2 = 2 +_6 0 = 2$$
$$0 +_6 3 = 3 +_6 0 = 3$$
$$0 +_6 4 = 4 +_6 0 = 4$$
$$0 +_6 5 = 5 +_6 0 = 5$$

$\therefore$ 0 is identity element.

(iv) From the composition table we can also see the left inverses of 0, 1, 2, 3, 4, 5 are 0, 5, 4, 3, 2, 1 respectively. Since,

$$0 +_6 0 = 0 \quad 1 +_6 5 = 0 \quad 2 +_6 4 = 0$$
$$3 +_6 3 = 0 \quad 4 +_6 2 = 0 \quad 5 +_6 1 = 0$$

e.g. $4 +_6 2 = 0 = 2 +_6 4$ implies 4 is the inverse of 2.

(v) The composition is commutative as the corresponding rows and columns in the position are *identical*.

(vi) The number of elements in the set G = 6

$\therefore$ (G, $+_6$) is a finite Abelian group of order 6.

Q. 6 (b) Define group, monoid, semigroup. **(6 Marks)**

Ans. : Definition of semigroup :

Let (A, $*$) be an algebraic system, where $*$ is a binary operation on A. (A, $*$) is called a **semigroup** if the following conditions are satisfied.

1. $*$ is a closed operation.
2. $*$ is an associative operation.

The semigroup (A, $*$) is said to be commutative if $*$ is a commutative operation.

Definition of Monoid:

Let (A, $*$) be an algebraic system, where $*$ is a binary operation on A. (A, $*$) is called a **monoid** if the following conditions are satisfied.

1. $*$ is a closed operation
2. $*$ is an associative operation
3. There is an identity

Definition of Group :

Let (A, $*$) be an algebraic system, where $*$ is a binary operation. (A, $*$) is called a **group** if the following conditions are satisfied.

1. $*$ is a closed operation.
2. $*$ is an associative operation
3. There is an identity.
4. Every element in A has a left inverse.

Because of associativity, a left inverse, of an element is also a right inverse of the element in a group.

Find the Hasse diagram of $L_1 \times L_2$

c) Show that the set $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ where the functions are defined by $f_1(x) = x$ $\quad f_2(x) = 1 - x$ $\quad f_3(x) = \frac{x}{x-1}$ $\quad f_4(x) = \frac{1}{x}$ $\quad f_5(x) = \frac{1}{1-x}$ $\quad f_6(x) = 1 - \frac{1}{x}$

is a group under composition of functions. Frame the composition table. **(8 Marks)**

Q.6 a) Determine whether following graphs are isomorphic

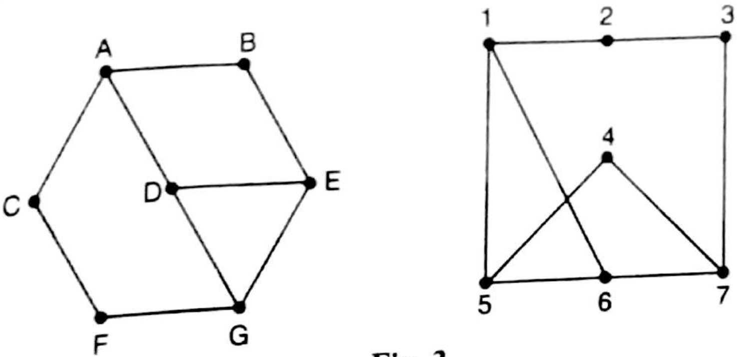


Fig. 3

(7 Marks)

b) Solve the following recurrence relation :
$a_n - 5a_{n-1} + 6a_{n-2} = 2^n$ with initial conditions $a_0 = -1$ and $a_1 = 1$ **(5 Marks)**

c) Show that $(\neg q \wedge (p \Rightarrow q)) \Rightarrow \neg p$ is a tautology.

May 2016

Q.1. (a) Find how many integers between 1 and 60 are not divisible by 2 nor by 3 and nor by 5 respectively. **(6 Marks)**

(b) By using mathematical induction prove that $1 + a + a^2 + \ldots a^n = \frac{1 - a^{n+1}}{1 - a}$ where $n \geq 0$. **(6 Marks)**

(c) Let A={1,2,3,4,5}and Rbe the relation defined by a R b if and only if a < b compute R, R^2 and R^3.Draw digraph of R, R^2 and R^3. **(8 Marks)**

Q.2. (a) Show that a group G is Ablian, if and only if $(ab)^2 = a^2 b^2$ for all elements a and band G . **(6 Marks)**

(b) Let A = {1,2, 3,4,6} = B, a R b if and only if a is multiple of b. Find R. Find each of the following (i) R(4) (ii) R(G) (iii) R({2,4,6}). **(6 Marks)**

(c) Show that the (2,5) encoding function e: $B^2 \rightarrow B^5$ defined by e (00) = 00000, e (01) =01110 e (10) = 10101, e (11) = 11011is a group code. How many errors will it detect and correct ? **(8 Marks)**

Q.3. (a) State pigeon hole and extended pigeon hole principle. Show that 7 colors are used to paint 50 bicycles, at least 8 bicycle will be of same color. **(6 Marks)**

(b) Define distributive lattice. Show that in a bounded distributive lattice, if a complement exists ,its unique. **(6 Marks)**

(c) Functions f, g, h are defined on a set X= {1,2,3} as f = {(1,2) (2,3) (3,1)} g = {(1,2) (2,1) (3,3)} h =(1, 1) (2,2) (3,1)}. (i) Find f o g, g o f are they equal ? (ii) Find f o g o h and f o h o g. **(8 Marks)**

Q. 4 (a) Define Eular path and Eular circuit. Determine whether the given graph has Eular path and Eular circuit. **(6 Marks)**

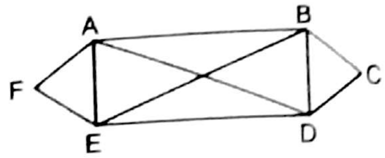


Fig. 1

(b) Define Hamiltonian path and Hamiltonian circuit, determine whether the given graph has Hamiltonian path and Hamiltonian circuit. **(6 Marks)**

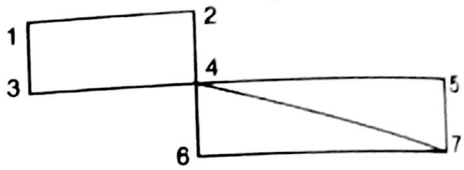
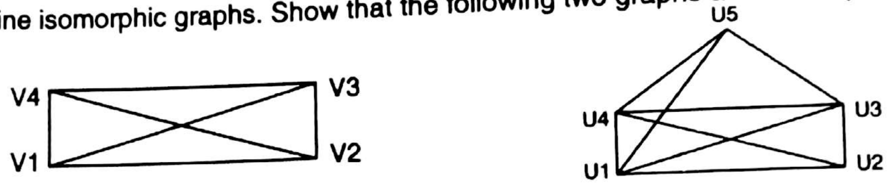


Fig. 2

(c) Define isomorphic graphs. Show that the following two graphs are isomorphic. **(8 Marks)**



(a) **(b)**

Fig. 3

Q. 5. (a) What is an universal and existential quantifiers ?Prove the distribution law.
$(p \vee q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ **(6 Marks)**

(b) Let A = {l,2,3,4}and let R = {(1,2) (2,3)(3,4)(2, l)}.Find transitive closure of R by using Warshall's algorithm. **(6 Marks)**

(c) Prove that the set A = {0,1,2,3,4,5} is a finite Abelian group under addition moduio 6. **(8 Marks)**

Q. 6. (a) Find the ordinary generating functions for the given sequences: **(6 Marks)**

(i) {1, 2,3,4,5 --} (ii) {2,2, 2, -- } (iii) {1,1,1,1---} **(6 Marks)**

(b) Define group, monoid, semigroup. **(6 Marks)**

(c) Solve the following recurrence relation: $a_n - 7a_{n-1}, + 10\ a_{n-2} = 0$ with initial condition $a_0 = 1$, $a_2 = 6$ **(8 Marks)**

❑❑❑

Discrete Mathematics

Statistical Analysis

Chapter No.	Dec. 2017	May 2018
Chapter 1	10 Marks	17 Marks
Chapter 2	05 Marks	09 Marks
Chapter 3	33 Marks	25 Marks
Chapter 4	19 Marks	12 Marks
Chapter 5	16 Marks	21 Marks
Chapter 6	09 Marks	12 Marks
Chapter 7	22 Marks	24 Marks
Repeated Questions	–	–

Dec. 2017

Chapter 1 : Set Theory [Total Marks – 10]

Q. 4(a) **(i)** **Among 50 students in a class, 26 got an A in the first examination and 21 got an A in the second examination. If 17 students did not get an A in either examination, how many students got an A in both examinations ?**

(ii) **If the number of students who got an A in the first examination is equal to that in the second examination, if the total number of students who got an A in exactly one examination is 40 and if 4 students did not get an A in either examination then determine the number of students who got an A in the first examination only, who got an A in the second examination only and who got an A in both the examination.** **(6 Marks)**

Ans. :

(i) Let 'T' be the number of students.

Let 'F' be the students who got A in first examination

Let 'S' be the students who got A in second examination

$$|T| = 50$$
$$|F| = 26$$
$$|S| = 21$$

Number of students who did not get an A in either examination=17

Number of students got an A in at least one examination is 50-17 = 33

Number of students got an A in both examinations is $|F \cap S|$

$$33 = |F| + |S| - |F \cap S|$$
$$|F \cap S| = (26 + 21) - 33$$
$$|F \cap S| = 47 - 33 = 14$$

(ii) Number of students who got an A in the first examination is equal to that in the second examination, so $|F| = |S|$

Total number of students who got an A in exactly one examination is 40

$$|F| + |S| - 2|F \cap S| = 40 \quad \text{...(1)}$$

4 students did not get an A in either examination

So the number of students who got A in at least one examination is $50 - 4 = 46$

$$|F| + |S| - |F \cap S| = 46 \quad \text{...(2)}$$

Using Equation (1)

$$|F| + |S| - 2|F \cap S| = 40$$
$$|F| + |S| - |F \cap S| - |F \cap S| = 40$$
$$46 - |F \cap S| = 40 \quad \text{...using Equation (2)}$$
$$|F \cap S| = 6$$

6 Students got an A in both examinations.

Using Equation (1)

$$|F| + |S| - 2|F \cap S| = 40$$
$$|F|+|S| - (2 * 6) = 40 \quad \text{...(3)}$$
$$|F| + |S| = 52$$
$$\text{So, } |F| = |S| = 26$$

$|F| - |F \cap S| = 26 - 6 = 20$ students got an 'A' in first examination only.

$|S| - |F \cap S| = 26 - 6 = 20$ students got an 'A' in second examination only.

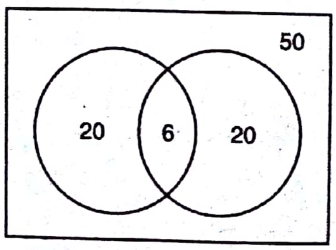


Fig. 1-Q. 4(a)

Q. 6(d) Prove the following (use laws of set theory)
$A \times (X \cap Y) = (A \times X) \cap (A \times Y)$ (4 Marks)

Ans. :

Let $(a, x) \in A \times (X \cap Y)$ ...(1)

By the definition of the Cartesian product, this means that

$a \in A$ and $x \in X \cap Y$

Since $x \in X \cap Y$

$\therefore$ $x \in X$ and $x \in Y$

$\therefore$ $(a, x) \in A \times X$ and $(a, x) \in A \times Y$

$\therefore$ $(a, x) \in (A \times X) \cap (A \times Y)$

$\therefore A \times (X \cap Y) \subseteq (A \times X) \cap (A \times Y)$ ...(2)

Again, Let$(a, x) \in (A \times X) \cap (A \times Y)$

$\therefore$ $(a, x) \in (A \times X)$ and $(a, x) \in A \times Y$

$\therefore$ $a \in A, x \in X$ and $x \in Y$

$\therefore$ $a \in A$ and $x \in X \cap Y$

$(a, x) \in A \times (X \cap Y)$

$\therefore (A \times X) \cap (A \times Y) \subseteq A \times (X \cap Y)$ ...(3)

From Equations (1) and (2) we have,

$A \times (X \cap Y) = (A \times X) \cap (A \times Y)$

Chapter 2 : Logic [Total Marks – 05]

Q. 1(a) Prove that $1.1! + 2.2! + 3.3! + \dots + n \cdot n! = (n + 1)! - 1$, where n is a positive integer. (5 Marks)

Ans. :

Let $P(n) = 1.1! + 2.2! + 3.3! + \dots n.n! = (n + 1)! - 1$

(i) **Basis of induction :**

For $n = 1$, $1.1! = 1$ and $(1 + 1)! - 1 = 2! - 1 = 1$

$\therefore$ P(1) is true.

(ii) **Induction step :**

Assume P(k) is true, i.e,

$1.1! + 2.2! + 3.3! + \dots k.k! = (k + 1)! - 1$ ...(i)

To prove that P(k + 1) is true,

$P(k + 1) : 1.1! + 2.2! + 3.3! + \dots (k + 1) \cdot (k + 1)! = (k + 2)! - 1$

$$\begin{aligned} \text{L.H.S} &= 1.1! + 2.2! + 3.3! + \dots k\,k! + (k + 1) \cdot (k + 1)! \\ &= (k + 1)! - 1 + (k + 1) \cdot (k + 1)! \\ &\text{(by induction hypothesis).} \\ &= (k + 1)! + (k + 1) \cdot (k + 1)! - 1 \\ &= (k + 1)!\,(1 + k + 1) - 1 \\ &= (k + 1)!\,(k + 2) - 1 = (k + 2)! - 1 \end{aligned}$$

L.H.S. = R.H.S.

Hence the result is proved.

Chapter 3 : Relations, Digraphs and Lattice [Total Marks – 33]

Q. 1(b) Let A = {a, b, c}. Show that (P(A), $\subseteq$) is a poset and draw its Hasse diagram. (5 Marks)

Ans. : Set containment $\subseteq$ is always a partial order since for any subset B of A, $B \subseteq B$ i.e. $\subseteq$ is reflexive.

If $B \subseteq C$ and $C \subseteq B$, $B = C$. So $\subseteq$ is anti-symmetry.

If $B \subseteq C$, $C \subseteq D$ then $B \subseteq D$. So $\subseteq$ is transitive.

Partial ordered relation of set containment on set P(A) is as follows

R = {{ϕ}, {ϕ}}, {{ϕ}, {a}}, {{ϕ}, {b}}, {{ϕ}, {c}}, {{ϕ}, {a, b}}, {{ϕ}, {a, c}}, {{ϕ}, {b, c}}, {{ϕ}, {a, b, c}}, {{a}, {a, b}}, {{a}, {a, c}}, {{b}, {a, b}}, {{b}, {b, c}}, {{c}, {a, c}}, {{c}, {b, c}}, {{a, b}, {a, b, c}}, {{a, c}, {a, b, c}}, {{b, c}, {a, b, c}}, {{a, b, c,}, {a, b, c}}

Matrix of the above relation is as follows :

$$M_R = \begin{array}{c} \\ \phi \\ \{a\} \\ \{b\} \\ \{c\} \\ \{a,b\} \\ \{a,c\} \\ \{b,c\} \end{array} \begin{array}{c} \begin{array}{cccccccc} \phi & \{a\} & \{b\} & \{c\} & \{a,b\} & \{a,c\} & \{b,c\} & \{a,b,c\} \end{array} \\ \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

Digraph of the abve matrix is

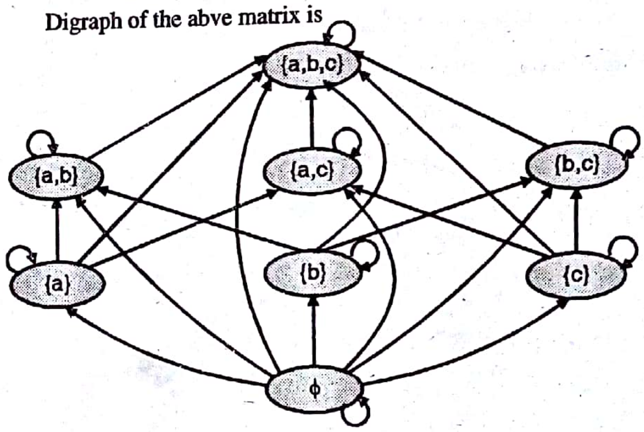


Fig. 1 - Q. 1(b)

To convert this digraph into Hasse Digraph

Step 1 : Remove Cycles.

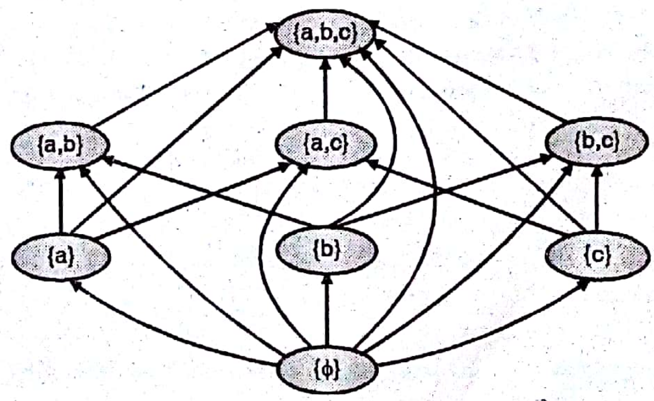


Fig. 2- Q. 1(b)

Step 2 : Remove transitive edges.

{{ϕ}, {a, b}}, {{ϕ},
{a,c}} {{ϕ}, {b, c}},
{{ϕ}, {a, b, c}},
{{a}, {a, b, c}},
{{b}, {a, b, c}},
{{c}, {a, b, c}}

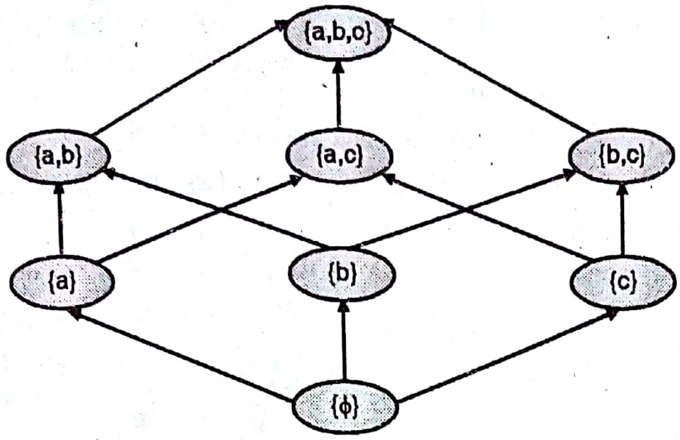


Fig. 3 - Q. 1(b)

Step 3 : All edges are pointing upwords. Now replace circles by dots and remove arrows from edges.

Hasse Diagram :

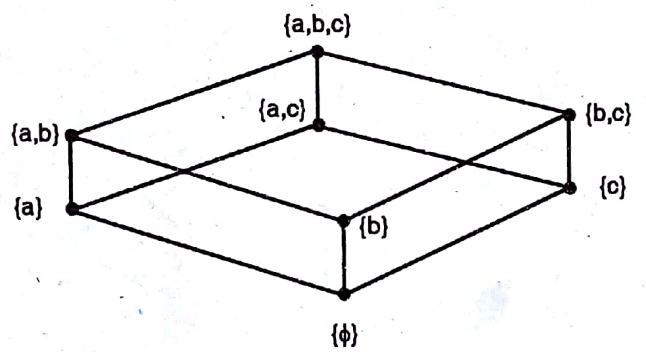


Fig. 4 - Q. 1(b)

Q. 1(c)(i) Explain the terms : Lattice (1 Mark)

Ans. : A lattice is a poset (L, ≤) in which every subset {a, b} consisting of two elements has a least upper bound and a greatest lower bound. We denote LUB ({a, b}) by $a \vee b$, and call it the join of a and b. Similarly, we denote GLB ({a, b}) by $a \wedge b$ and call it the meet of a and b.

Q. 1(c)(ii) Explain the terms : Poset (1 Mark)

Ans. : Partially ordered relation : A relation R on a set A is called **partial order** if R is reflexive, anti-symmetric and transitive poset.

The set A together with the partial order R is called a **partially ordered set** or simply a poset. It is denoted by (A, R).

Q. 2(b) Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and let R be a relation on A whose matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find M_R * by Warshall's algorithm. (6 Marks)

Ans. :

Let $M_R = W_0, \; n = 5$

$$W_0 = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

First we compute W_1 so that $k = 1$ ∴ check for column 1 and row 1 of W_0

$p_1 : (a_1, a_1), p_2 : (a_4, a_1)$

$q_1 : (a_1, a_1), q_2 : (a_1, a_4)$

To obtain W_1 we must put 1's in positions (a_1, a_1), (a_1, a_4), (a_4, a_1) and (a_4, a_4). Thus

$$W_1 = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

To obtain W_2 so that $k = 2$ ∴ check column 2 and row 2 of W_1

$p_1 : (a_2, a_2), p_2 : (a_5, a_2)$

$q_1 : (a_2, a_2)$

To obtain W_2 we must put 1's in positions (a_2, a_2), (a_5, a_2). Thus

$$W_2 = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Now compute W_3 so that $k = 3$ ∴ check column 3 and row 3 of W_2

$p_1 :$

$q_1 : (a_3, a_4), (a_3, a_5)$

Hence no new ordered pair. Thus $W_2 = W_3$

$$W_3 = \begin{array}{cc} & \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Now compute W_4 so that $k = 4$ $\therefore$ check column 4 and row 4 of W_3

$p_1 : (a_1, a_4), p_2 : (a_3, a_4), p_3 : (a_4, a_4)$

$q_1 : (a_4, a_1), q_2 : (a_4, a_4)$

To obtain W_4, we must put 1's in positions (a_1, a_1), (a_1, a_4), (a_3, a_1), (a_3, a_4), (a_4, a_1), (a_4, a_4).

Thus

$$W_4 = \begin{array}{cc} & \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Now we compute W_5 so that $k = 5$ $\therefore$ check column 5 and row 5 of W_4

$p_1 : (a_3, a_5), p_2 : (a_5, a_5),$

$q_1 : (a_5, a_2), q_2 : (a_5, a_5)$

To obtain W_5, we must put all 1's in positions (a_3, a_2), (a_3, a_5), (a_5, a_2), and (a_5, a_2). Thus

$$W_4 = \begin{array}{cc} & \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\therefore M_{P^\infty} = W_5 = \text{Transitive closure}$$

Q. 3(b) Define equivalence relation with example. Let 'T' be a set of triangles in a plane and define R as the set R = {(a, b) | a, b ∈ T and a is congruent to b} then show that R is an equivalence relation. (6 Marks)

Ans. :

A relation R on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

The following are some of the common but important examples of equivalence relations.

For Example :

(i) Let A = IR and R be 'equality' of numbers.

(ii) Consider all subsets of a universal set and R be the relation, 'equality' of sets.

(iii) A is the set of triangles and R is 'similarity' of triangles.

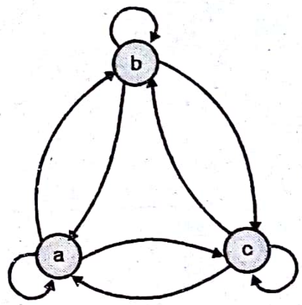


Fig. 1 - Q. 3(b)

(iv) A is a set of students and R is the relation of being in the same class or division.

(v) Let A be set of statement forms and R be the relation of 'Logical equivalence.'

(vi) A is set of lines in a plane and R is the relation of lines being 'parallel'.

The digraph of an equivalence relation will have the following characteristics.

(i) Every vertex will have a loop.

(ii) If there is an arc from a to b, there should be an arc from b to a.

(iii) If there is an arc from a to b and arc from b to c, there should be an arc from a to c

In short, the following is a typical digraph of an equivalence relation.

R is reflexive since, a triangle is congruent to itself.

R is symmetric since, if a is congruent to b, then b is congruent to a.

R is transitive since, if a is congruent to b, b is congruent c, then a is congruent to c.

The relation satisfies all the three properties of an equivalence relation.

Q. 5(b) Determine whether the Poset with the following Hasse diagrams are lattices or not. Justify your answer. (6 Marks)

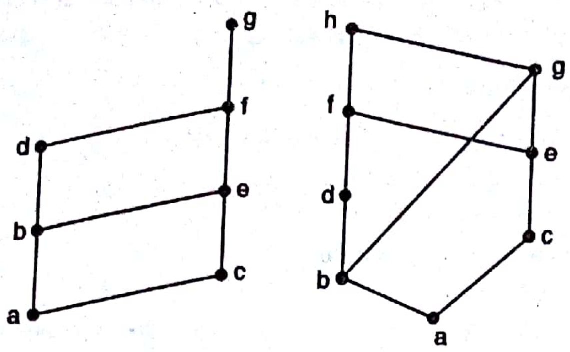


Fig. 1-Q. 5(b)

Ans. :

(i)

LUB :

∨	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	e	d	e	f	g
c	c	e	c	f	e	f	g
d	d	d	f	d	f	f	g
e	e	e	e	f	e	f	g
f	f	f	f	f	f	f	g
g	g	g	g	g	g	g	g

GLB :

∧	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	a	c	c	c
d	a	b	a	d	b	d	d
e	a	b	c	b	e	e	e
f	a	b	c	d	e	f	f
g	a	b	c	d	e	f	g

Fig. 1(a) - Q. 5(b) is a lattice since every pair of elements has a GLB and a LUB.

(ii)

LUB :

∨	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	b	f	d	f	f	g	h
c	c	f	c	f	e	f	g	h
d	d	d	f	d	f	f	h	h
e	e	f	e	f	e	f	g	h
f	f	f	f	f	f	f	h	h
g	g	g	g	h	g	h	g	h
h	h	h	h	h	h	h	h	h

GLB :

∧	a	b	c	d	e	f	g	h
a	a	a	a	a	a	a	a	a
b	a	b	a	b	a	b	b	b
c	a	a	c	a	c	a	c	a
d	a	b	a	d	a	d	b	d
e	a	a	c	a	e	e	e	e
f	a	b	a	d	e	f	-	f
g	a	b	c	b	e	-	g	g
h	a	b	a	d	e	f	g	h

Fig. 1(b) - Q. 5(b) is not a lattice because GLB (f, g) does not exist.

Q. 5(c) From the following diagraphs, write the relation a set of ordered pairs. Are the relations equivalence relations ? (4 Marks)

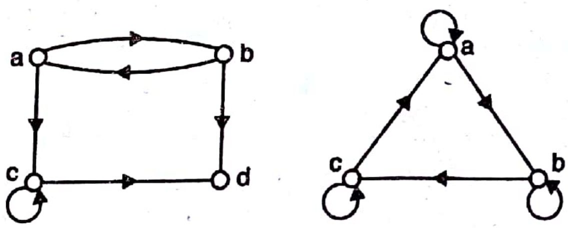


Fig. 1-Q. 5(c)

Ans. :

An equivalence relation is reflexive, symmetric and transitive.

R1 = {(a, b), (a, c), (b, a), (b, d), (c, c), (c, d)}

Relation R1 is not reflexive because (a,a) , (b,b), (d,d) do not belong to the relation set.

Hence R1 is not an equivalence relation.

R2 = (a, a), (a, b), (b, b), (b, c), (c, c), (c, a)}

Relation R2 is reflexive and transitive but not symmetric because (b, a), (c, b), (a, c) do not belong to the relation set. Hence R2 is not an equivalence relation.

Q. 5(d) For the set X = {2, 3, 6, 12, 24, 36}, a relation ≤ is defined as x ≤ y if x divides y. Draw the Hasse diagram for (X, ≤). Answer the following.

(i) What are the maximal and minimal elements ?

(ii) Give one example of chain and antichain.

(iii) Is the poset a lattice ? (4 Marks)

Ans. :

R = {(2,2), (2,6), (2,12), (2,24), (2,36), (6,6), (6,12), (6,24), (6,36), (12,12), (12,24), (12,36), (24,24), (3,3), (3,6), (3,12), (3,24), (3,36)}

Final hasse diagram :

Maximal Elements : 24,36

Minimal Elements : 2,3

Chain = {2,6,12,24}

Antichain = {2,3}

This poset is not a lattice.

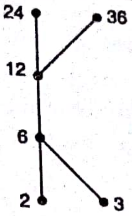


Fig. 1 - Q. 5(d)

Chapter 4 : Functions and Pigeon Hole Principle [Total Marks – 19]

Q. 1(d) **Comment whether the function f is one to one or onto.**

Consider function : f : N → N where N is set of natural numbers including zero.

$f(j) = j^2 + 2$ **(5 Marks)**

Ans. :

$$N = \{0, 1, 2, 3, 4………………\}$$

$$f(0) = 2$$

$$f(1) = 3$$

$$f(2) = 6$$

$$f(3) = 11$$

$$f(4) = 18$$

..and so on.

For every number 'n' from 'N' we can find another 'n' .So the given function is one to one.

But not every element 'n' of set 'N' is image of some element 'n'. So the given function is not onto.

Q. 2(d) **Let f : R → R defined as $f(x) = x^3$ and $g : R \to R$ defined as $g(x) = 4x^2 + 1$**

Find out g o f, f o g, f^2, g^2. **(4 Marks)**

Ans. :

$$g \circ f = g(f(x)) = g(x^3) = [4x^6+1]$$

$$f \circ g = f(g(x)) = f(4x^2 + 1) = (4x^2 + 1)^3$$

$$f \circ f = f(f(x)) = f(x^3) = x^9$$

$$g \circ g = g(g(x)) = g(4x^2 + 1) = 4(4x^2 + 1)^2+1$$

Q. 3(c) **Let A = B = R, the set of real numbers**

Let f : A → B be given by the formula $f(x) = 2x^3 - 1$ and Let g : B → A be given by

$$g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}$$

Show that f is a bijection between A and B and g is a bijection between B and A. **(4 Marks)**

Ans.:

A function from A to B is Bijection if it is one to one and onto

∴ for $f(x) = 2x^3 - 1$ to be one to one and onto.

If a, b, ∈ A

such that $f(a) = f(b)$.

$$\Rightarrow \quad 2a^3 - 1 = 2b^3 - 1.$$

$$a = b.$$

∴ f is one to one

Now for

$$y = 2x^3 - 1.$$

$$1 + y = 2x^3.$$

$$x^3 = \frac{1}{2} + \frac{y}{2}.$$

$$\therefore \quad x = \sqrt[3]{\frac{1}{2} + \frac{y}{2}}.$$

∴ for each y ∈ B. There is a unique x in A such that f(x) = y.

∴ f is onto.

∴ f is bijective function between A and B.

Similarly for g : B → A to be one to one and onto

$$g(a) = g(b) = \sqrt[3]{\frac{1}{2} + \frac{a}{2}} = \sqrt[3]{\frac{1}{2} + \frac{b}{2}}$$

$$\therefore \quad \frac{1}{2} + \frac{a}{2} = \frac{1}{2} + \frac{b}{a}$$

$$\Rightarrow \quad a = b$$

∴ g is one to one.

Also for

$$x = \sqrt[3]{\frac{1}{2} + \frac{y}{2}}.$$

$$x^3 = \frac{1}{2} + \frac{y}{2}$$

$$2x^3 = 1 + y$$

$$y = 2x^3 - 1$$

for each x in A. There is a corresponding y in B. Such that g(y) = x

∴ g is onto function

so g is bijective function between B and A.

Q. 5(a) **Explain Pigeonhole principle and Extended Pigeonhole principle. Show that in any room of people who have been doing some handshaking there will always be atleast two people who have shaken hands the same number of times.** **(6 Marks)**

Ans. :

Theorem of Pigeonhole Principle :

If n pigenons are assigned to m pigeonholes, and m < n, then at least one pigeonhole contains two or more pigeons.

Proof : Consider labeling the m pigeonholes with the numbers 1 through m and the n pigeons with the numbers 1 through n. Now, beginning with pigeon 1, assign each pigeon in order to the pigeonhole with the same number. This assigns as many pigeons as possible to individual pigeon holes, but because m < n, there are n – m pigeons that have not yet been assigned to a pigeonhole. At least one pigeonhole will be assigned a second pigeon.

The Extended Pigeonhole Principle :

If there are m pigeonholes and more than 2m pigeons, then three or more pigeons will have to be assigned to at least. One of the pigeonholes. In general if the number at pigeons is much larger than the number at pigeonholes, given theorem can be restated to give a stronger conclusion.

First, a word about notation. If n and m are positive integers, then $\lfloor n / m \rfloor$ stands for the largest integer less than or equal to the rational number n/m. Thus $\lfloor 3/2 \rfloor$ is 1, $\lfloor 9/4 \rfloor$ is 2 and $\lfloor 6/3 \rfloor$ is 2.

Theorem : (The extended pigeonhole principle)

If n pigeons are assigned to m, pigeonholes, then one of the pigeonholes must obtain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

Proof : (by contradiction) :

If each pigeonhole contains no more than $\lfloor (n-1)/m \rfloor$ pigeons, then there are at most $m . \lfloor (n-1)/m \rfloor \leq m . (n-1)/m = n - 1$ pigeons in all. This contradicts our assumption, so one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

There are n people attending the party. Obviously, this problem makes sense only when $n \geq 2$. If no two people have shaken hands with equal number of people then there handshake count must differ at least by 1. So the possible choices for hand shake count would be 0,1,…,n – 1. There are exactly n choices and n people. If there exist a person with (n-1) handshake count, there can't be a person with 0 handshake count. Thus reducing the possible choices to (n-1). Now, due to pigeon hole principle, we have that at least two person will have the same number of handshake count.

Chapter 5 : Counting [Total Marks – 16]

Q. 2(c) Find the complete solution of the recurrence relation :

$a_n + 2a_{n-1} = n + 3$ for $n \geq 1$ and with $a_0 = 3$.

(4 Marks)

Ans. : The characteristic equation is

$$\alpha + 2 = 0$$

$$\therefore \quad \alpha = -2$$

Hence, Homogeneous solution is

$$a_n^{(h)} = A_1 (-2)^n$$

For particular solution, since f (r) (Right hand side) is a linear polynomial, therefore, the particular solution will be of the form $(P_0 + P_{1n})$.

i.e.
$$a_n = P_0 + P_{1n}$$
$$a_{n-1} = P_0 + P_1 (n-1)$$

Substituting these values in the given recurrence relation, we get,

$$(P_0 + P_{1n}) + 2 [P_0 + P_1 (n-1)] = n + 3$$
$$P_0 + P_1 n + 2P_0 + 2P_1 n - 2P_1 = n + 3$$
$$(P_0 + 2P_0 - 2P_1) + n (P_1 + 2P_1) = n + 3$$
$$(3 P_0 - 2P_1) + n (3P_1) = n + 3$$

On comparing the coefficients of polynomials, we get

$$\text{and } 3 P_0 - 2P_1 = 3$$
$$3P_1 = 1$$

Which on solving give $P_1 = 1/3$, $P_0 = 11/9$

Hence, the particular solution is

$$a_n^{(p)} = \frac{11}{9} + \frac{1}{3} n$$

Thus the general solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$\therefore \quad a_n = A_1 (-2)^n + \frac{11}{9} + \frac{1}{3}$$

Using the initial conditions $a_0 = 3$ we get

$$a_0 = A_1 + \frac{11}{9}$$

Given $a_0 = 3$

$$3 = A_1 + \frac{11}{9}$$

$$3 - \frac{11}{9} = A_1$$

$$A_1 = 1.78$$

Therefore
$$a_n = 1.78 (-2)^n + \frac{11}{9} + \frac{1}{3} n$$

Q. 3(a) Given that a student had prepared, the probability of passing a certain entrance exam is 0.99. Given that a student did not prepare, the probability of passing the entrance exam is 0.05. Assume that the probability of preparing is 0.7. The student fails in the exam. What is the probability that he or she did not prepare ?

(6 Marks)

Ans. : Refer Fig. 1-Q. 3(a) probability tree diagram,

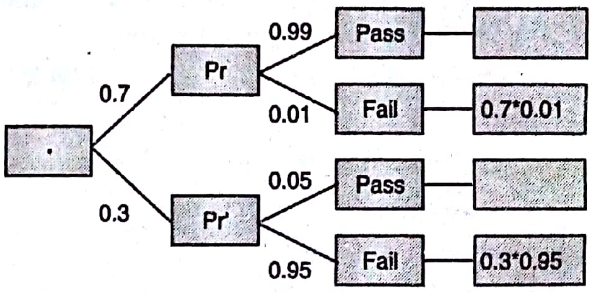


Fig. 1-Q. 3(a)

We will find :

$$P (\text{Fail} \mid Pr') = \frac{P (\text{Fail} \cap Pr')}{P (\text{Fail} \cap Pr') + P (\text{Fail} \cap Pr)}$$

$$\frac{0.3 \cdot 0.95}{0.3 \cdot 0.95 + 0.7 \cdot 0.01} = 0.976$$

Q. 6(b) Given a generating function, find out corresponding sequence.

(i) $\frac{1}{3 - 6x}$ (ii) $\frac{x}{1 - 5x + 6x^2}$ (6 Marks)

Ans. :

(i) $\dfrac{1}{(3-6x)} = \dfrac{1}{3(1-2x)}$

The simple generating function that gives the sum of a geometric series :

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Replace x by 2x

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n$$

Multiply this by $\frac{1}{3}$

$$\frac{1}{3(1-2x)} = \frac{1}{3}\sum_{n=0}^{\infty} 2^n x^n$$

$$\frac{1}{3(1-2x)} = \sum_{n=0}^{\infty} \frac{2^n x^n}{3}$$

Thus, the associated sequence is

$$\left(0, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \ldots\ldots\right)$$

(ii) Given

$$\frac{x}{1-5x+6x^2}$$

We know that $1-5x+6x^2 = (1-2x)(1-3x)$

Therefore

$$\frac{x}{1-5x+6x^2} = \frac{A}{(1-2x)} + \frac{B}{(1-3x)}$$

for some suitable A and B. By multiplying this equation by x^2 we obtain

$$x = A(1-3x) + B(1-2x)$$

By equating the coefficients or by substituting $x = \frac{1}{2}$ and $x = \frac{1}{3}$, we find the solution $A = -1$ and $B = 1$, so

$$f(x) = -\frac{1}{1-2x} + \frac{1}{1-3x} = -\sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} 3^n x^n$$

$$= \sum_{n=0}^{\infty} (3^n - 2^n)\, x^n,$$

therefore $a_n = 3^n - 2^n$ for all integers $n \geq 0$.

Thus, the associated sequence is (0, 1, 5, 19, 65, ...)

Chapter 6 : Graphs[Total Marks – 09]

Q. 1(c)(v) Explain the terms : Plannar Graph (1 Mark)

Ans. :

Plannar Graph :

A graph is said to be **planar** if it can be drawn on a plane in such a way that no edges cross one another, except of course at common vertices.

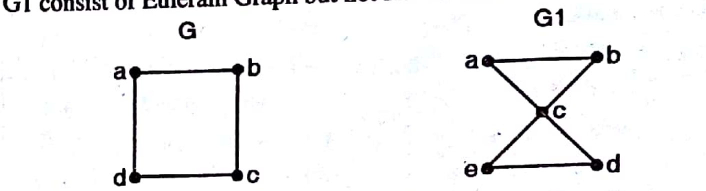
Q. 4(c) (I) Is every Eulerian graph a Hamiltonian ?
(II) Is every Hamiltonian graph a Eulerian ?
Explain with the necessary graph. (4 Marks)

Ans. :

(i) Let G = (V, E) be a graph. A Eulerian Graph is a graph which passes through every edge exactly once.

Let G1 = (V1, E1) be a graph. A Hamiltonian circuit is a circuit which passes through every vertex exactly once (with only the first and last vertex being a repeat). A graph is called Hamiltonian if it possesses a Hamiltonian circuit.

In following Fig. 1(a) - Q. 4(c) Graph G consists of both Eulerian and Hamiltonian Graph. But in Fig. 1(b) - Q. 4(c) Graph G1 consist of Eulerain Graph but not Hamiltonian.



(a) Eulerian Circuit : a, b, c, d, a (covers all edges of Graph G) Hamltonian Graph : a, b, c, d, a (covers all vertices of Graph G)

(b)Eulerian Circuit : c, a, b, c, e, d, c No Hamiltonian Graph

Fig. 1 - Q. 4(c)

Hence every Eulerian Graph is not necessarily Hamiltonian

(ii) In Hamiltonian Graph, we need to visit each and every vertex only once except the last vertex which is also the first vertex. In this type of graph we don't require to visit each and every edge of the graph (which is necessary in case of Eulerian Graph) i.e. need to cover all vertices only once. Since every Hamiltonian graph not necessarily be Eulerian. Refer Fig. 2 - Q. 4(c) to better understand this statement. In following Graph G2 solid lines indicate Hamiltonian Graph, but it is not Eulerian Graph, since it is not possible to cover all edges exactly once.

G2

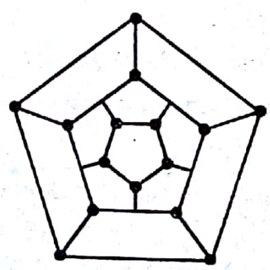


Fig. 2

Q. 6(c) Determine whether following graphs are isomorphic or not. (4 Marks)

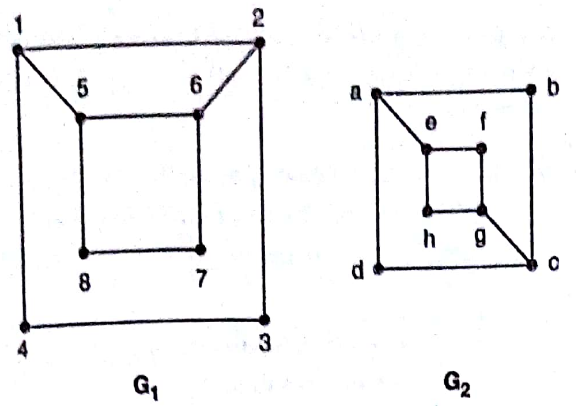


Fig. 1-Q. 6(c)

Ans. :

Hence both the graphs G_1 and G_2 contain 8 vertices and 10 edges. The number of vertices of degree 2 in both the graphs are four. Also the number of vertices of degree 3 in both the graphs are 4.

For adjacency, consider the vertex 1 of degree 3 in G_1. It is adjacent to two vertices of degree 3 and 1 vertex of degree 2. But in G_2 there does not exist any vertex of degree 3 which is adjacent to two vertices of degree. 3 and 1 vertex of degree 2. Hence adjacency is not preserved. Hence given graphs are not isomorphic.

Chapter 7 : Algebraic Structures [Total Marks – 22]

Q. 1(c)(iii) Explain the terms : Normal Subgroup. (1 Mark)

Ans. :

A subgroup H of G is said to be a **normal subgroup** of G if for every $a \in G$, $aH = Ha$.

A subgroup of an Abelian group is normal.

Q. 1(c)(iv) Explain the terms : Group. (1 Mark)

Ans. :

Let (A, *) be an algebraic system, where * is a binary operation. (A, *) is called a **group** if the following conditions are satisfied.

1. * is a closed operation.
2. * is an associative operation
3. There is an identity.
4. Every element in A has a left inverse.

Because of associativity, a left inverse, of an element is also a right inverse of the element in a group.

Q. 3(d) Let Z_n denote the set of integers {0, 1, 2, ... n – 1}. Let O be binary operation on z_n denote such that a O b = the remainder of ab divided by n.

(i) Construct the table for the operation O for n = 4.

(ii) Show that (Z_n, O) is a semigroup for any n. (4 Marks)

Ans. : assuming O as binary operation such as *.

(i) The table for the operation * for n = 4

$*_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

(ii) The set Z_n is closed under the operation * because for any a, $b \in Z_n$, $a * b \in Z_n$ ...(1)

Now check for associativity for any a, b, $c \in Z_n$.

$$(a *_4 b) *_4 c = a *_4 (b *_4 c)$$

Let a = 1, b = 2, c = 3

$$(1 *_4 2) *_4 3 = 1 *_4 (2 *_4 3)$$

$$2 *_4 3 = 1 *_4 2$$

$$2 = 2$$

$\therefore$ '*' is an associative operation ...(2)

From Equations (1) and (2) we conclude that (Z_n, *) is a semigroup for any 'n'.

Q. 4(b) Consider the (2, 5) group encoding function $e : B^2 \to B^5$ defined by,

e (00) = 00000 e (01) = 01110

e (10) = 10101 e (11) = 11011

Decode the following words relative to a maximum likelihoods decoding function.

(i) 11110 (ii) 10011 (iii) 10100 (6 Marks)

Ans. :

Prepare decoding table :

00000	01110	10101	11011
00001	01111	10100	11010
00010	01100	10111	11001
00100	01010	10001	11111
01000	00110	11101	10011
10000	11110	00101	01011

(i) If we receive the word 1 1 1 1 0 we first locate it in 2^{nd} column of the decoding table. Where it is underlined once. The word at the top of the 2^{nd} column is 0 1 1 1 0. Since e (01) = 0 1 1 1 0. We decode 1 1 1 1 0 as 0 1.

(ii) If we receive the word 1 0 0 1 1, we first locate it in 4^{th} column. Where it is underlined twice. The word at the top of the 4^{th} column is 1 1 0 1 1. Since e (11) = 11011. We denote 1 0 0 1 1 as 11.

(iii) Similarly 1 0 1 0 0 is located in 3^{rd} column of decoding table it is underlined thrice. The word at the top of the 3rd column is 1 0 1 0 1. Since e (1 0) = 1 0 1 0 1. We decode 1 0 1 0 0 as 1 0.

Q. 4(d) Given the parity check matrix.

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find the minimum distance of the code generated by H. How many errors it can detect and correct ? (4 Marks)

Ans. : In the given parity check matrix all columns are distinct and non-zero, so d> = 3.

We can use the property that the minimum distance of a binary linear code is equal to the smallest number of columns of the parity-check matrix H that sum up to zero.

We can see that sum of first three columns is zero. So minimum distance $d_{min} = 3$.

It can correct $(d_{min} - 1)/2 = 1$ error.

It can detect $d_{min} - 1 = 2$ errors.

Q. 6(a) Prove that the set {1, 2, 3, 4, 5, 6} is group under multiplication modulo 7. (6 Marks)

Ans. :

(a) Multiplication modulo 7 table for set A is

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	1	3	2	1

(b) Note that 1 is an identity element of the algebraic system $(A, \times_7)$

Since for any $a \in A$,

$$a \times_7 1 = a = 1 \times_7 a$$

that is,

$$1 \times_7 1 = 1 \times_7 1 = 1$$
$$2 \times_7 1 = 1 \times_7 2 = 2$$
$$3 \times_7 1 = 1 \times_7 3 = 3$$
$$4 \times_7 1 = 1 \times_7 4 = 4$$
$$5 \times_7 1 = 1 \times_7 5 = 5$$
$$6 \times_7 1 = 1 \times_7 6 = 6$$

Recall that a^{-1} is that element of G such that $a * a^{-1} = 1$

$$2 \times_7 4 = 1 = 4 \times_7 2 = 1$$
$$3 \times_7 5 = 1 = 5 \times_7 3 = 1$$
$$6 \times_7 6 = 1 = 6 \times_7 6 = 1$$

∴ Inverse of 2 is 4

Inverse of 4 is 2

Inverse of 3 is 5

Inverse of 5 is 3

Inverse of 6 is 6

(c) We have

$$2^1 = 2,$$
$$2^2 = 2 \times_7 2 = 4$$
$$2^3 = 2^2 \times_7 2 = 4 \times_7 2 = 1$$
$$2^4 = 2^3 \times_7 2 = 1 \times_7 2 = 2$$

∴ Hence $|2| = 3$

∴ 2 is not generator of this group

We have

$$3^1 = 3$$
$$3^2 = 3 \times_7 3 = 2$$
$$3^3 = 3^2 \times_7 3 = 2 \times_7 3 = 6$$
$$3^4 = 3^3 \times_7 3 = 6 \times_7 3 = 4$$
$$3^5 = 3^4 \times_7 3 = 4 \times_7 3 = 5$$
$$3^6 = 3^5 \times_7 3 = 5 \times_7 3 = 1$$

Hence $|3| = 6$

∴ 3 is generator of this group and this group is cyclic.

(d) Subgroup generated by {3, 4} is denoted by < {3, 4} > since 3, 4 are elements of this set they have to be there in < {3, 4} >

Inverse of 3 is 5, inverse of 4 is 2

∴ 3, 4, 5, 2, ∈ < {3, 4} >

$3 \times_7 4 = 5$	$5 \times_7 4 = 6$
$3 \times_7 3 = 2$	$6 \times_7 6 = 1$
$3 \times_7 5 = 1$	$5 \times_7 1 = 5$
$4 \times_7 4 = 2$	$1 \times_7 1 = 1$
$3 \times_7 2 = 6$	$5 \times_7 2 = 3$
$5 \times_7 5 = 4$	$3 \times_7 6 = 4$
$5 \times_7 6 = 2$	$2 \times_7 2 = 4$

∴ < {3, 4} > = < 1, 2, 3, 4, 5, 6 >

∴ Subgroup generated by {3, 4} is the set A itself whose order is 6.

Subgroup generated by {2, 3} is denoted by < {2, 3} >.

Since 2, 3 are elements of this set they have to be there in < {2, 3} >.

Inverse of 2 is 4.

Inverse of 3 is 5

∴ 2, 3, 4, 5 ∈ < {2, 3} >

$2 \times_7 3 = 6$	$3 \times_7 4 = 5$
$4 \times_7 4 = 2$	$5 \times_7 5 = 4$
$2 \times_7 4 = 1$	$3 \times_7 5 = 1$
$4 \times_7 1 = 4$	$5 \times_7 6 = 2$

$2 \times_7 5 = 3$ $\quad$ $3 \times_7 6 = 4$

$4 \times_7 5 = 6$ $\quad$ $5 \times_7 1 = 5$

$2 \times_7 6 = 5$ $\quad$ $3 \times_7 1 = 3$

$6 \times_7 6 = 1$ $\quad$ $2 \times_7 1 = 2$

$3 \times_7 3 = 2$ $\quad$ $6 \times_7 1 = 6$

$2 \times_7 2 = 4$

$\therefore \; <\{2, 3\}> \; = \; <1, 2, 3, 4, 5, 6>$

$\therefore$ Subgroup generated by $<\{2, 3\}>$ is the set A and is of order 6.

May 2018

Chapter 1 : Set Theory [Total Marks – 17]

Q. 1(b) Find the generating function for the following finite sequences. (5 Marks)

(i) 2, 2, 2, 2, 2, 2

(ii) 1, 1, 1, 1, 1, 1

Ans. :

i) 2, 2, 2, 2, 2, 2

Here 2 comes 6 times $\rightarrow$ 0 to 5

i.e. $a_n = 2$ for $n \leq 5$

$$\therefore \; G(x) = \sum_{n=0}^{5} 2\, x^n = \frac{2(1-x^6)}{1-x}$$

ii) 1, 1, 1, 1, 1, 1

Here 1 comes 6 times $\rightarrow$ 0 to 5

i.e. $a_n = 1$ for $n \leq 5$

$$\therefore \; G(x) = \sum_{n=0}^{5} x^n = \frac{1-x^6}{1-x}$$

Q. 4(c) How many friends must you have to guarantee that at least five of them will have birthdays in the same month ? (4 Marks)

Ans. :

If there are 4 friends who have birthday in the same month, then

Total friends $= 12 * 4 = 48$

The 5^{th} friend will have birthday to be the same month as one of the others.

$\therefore$ Total friends $= 48 + 1 = 49$

Q. 5(a) Let G be a set of rational numbers other than 1. Let * be an operation on G defined by a * b = a + b – ab for all a, b € G. Prove that (G, *) is a group. (8 Marks)

Ans. :

(i) as $a, b \in G$. hence G is closed with respect to *

(ii)
$$\begin{aligned} a * (b * c) &= a * (b + c - bc) = a + (b + c - bc) - a(b + c - bc) \\ &= a + b + c - bc - ab - ac + abc \\ (a * b) * c &= (a + b - ab) * c = a + b - ab + c - (a + b - ab)\, c \\ &= a + b - ab + c - ac - bc + abc \\ &= a + b + c - ab - ac - bc + abc \end{aligned}$$

Hence $a * (b * c) = (a * b) * c$. hence G is associative under operation *.

(iii) We know that for every $a \in G$

$$\begin{aligned} a * e &= a \\ \therefore \; a + e - ae &= a \\ \therefore \; a + e - a &= a \qquad \therefore \; e = a \end{aligned}$$

(iv)
$$\begin{aligned} a * a^{-1} &= e \\ \therefore \; a + a^{-1} - aa^{-1} &= e \\ \therefore \; a + a^{-1} - e &= e \\ \therefore \; a + a^{-1} &= 2e \qquad \therefore \; a^{-1} = 2e - a = 2a - a = a \end{aligned}$$

$\therefore$ a is the inverse of G

Hence (G, *) satisfies above conditions

Hence (G, *) is a group.

Chapter 2 : Logic [Total Marks – 09]

Q. 1(a) Prove by induction that the sum of the cubes of three consecutive numbers is divisible by 9. (5 Marks)

Ans. :

Step 1 : Basic of induction

$P(1) = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$

36 is divisible by 9

Hence P(1) is ture

Step 2 : Induction Hypothesis

P(k) : For n = k is true

$P(k) = k^3 + (k+1)^3 + (k+2)^3$ is divisible by 9

$\therefore \; k^3 + (k+1)^3 + (k+2)^3 = 9\lambda$ is true where λ is a number multiple of 9.

Step 3 : Induction

P (k + 1)

$$\begin{aligned} & (k+1)^3 + (k+2)^3 + (k+3)^3 \\ &= (k+1)^3 + (k+2)^3 + (k+3)(k^2 + 6k + 9) \\ &= (k+1)^3 + (k+2)^3 + k^3 + 6k^2 + 9k + 3k^2 + 18k + 27 \\ &= (k+1)^3 + (k+2)^3 + k^3 + 9k^2 + 27k + 27 \\ &= (k+1)^3 + (k+2)^3 + k^3 + 9(k^2 + 3k + 3) \\ &= 9\lambda + 9(k^2 + 3k + 3) \\ &= 9(\lambda + k^2 + 3k + 3) \end{aligned}$$

Hence P (k + 1) is multiple of 3.

Hence P (n) is true.

Q. 2(c) **Prove ¬ (p ∨ (¬ p ∧ q)) and ¬ p ∧ ¬ q are logically equivalent by developing a series of logical equivalences.** **(4 Marks)**

Ans. :

		Reason
$\neg(p \vee (\neg p \wedge q))$	$\equiv \neg p \wedge \neg(\neg p \wedge q)$	1[st] Demorgan law
	$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$	1[st] Demorgan law
	$\equiv \neg p \wedge (p \vee \neg q)$	Double Negation law
	$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	2[nd] distributive law
	$\equiv F \vee (\neg p \wedge \neg q)$	$\neg p \wedge p \equiv F$
	$\equiv (\neg p \wedge \neg q) \vee F$	Commutative law
	$\equiv \neg p \wedge \neg q$	Identity law for F

Hence ¬ (p ∨ (¬ p ∧ q)) and ¬ p ∧ ¬ q are logically equivalent.

Chapter 3 : Relations, Digraphs and Lattice [Total Marks – 25]

Q. 1(d) **Find the complement of each element in D_{30}.** **(5 Marks)**

Ans. :

(i) D_{30} means all nos. by which 30 is divisible

$\therefore$ $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Let's draw the Hasse Diagram.

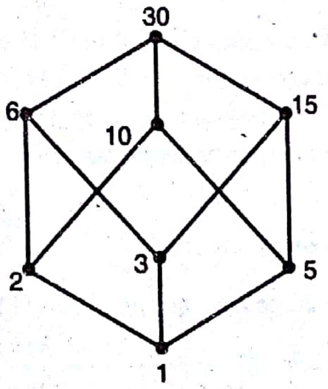


Fig. 1-Q. 1(d)

(ii)

a	b	GLB	LUB
1	30	1	30
2	15	1	30
3	10	1	30
5	6	1	30

Hence $1' = 30$ and $30' = 1$, $2' = 15$ and $15' = 2$,
$3' = 10$ and $10' = 3$, $5' = 6$ and $6' = 5$

Q. 3(b) **Let A = {1, 2, 3, 4, 5}, let R = {(1 ,1) , (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)} and S = {(1, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (4, 5)} be the relations on A. Find the smallest equivalence relation containing the relation R and S.** **(8 Marks)**

Ans. :

Let's consider Q is relation $R \cup S$

$\therefore$ Q = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (4, 5) (5, 4), (5, 5)}

(i) Q is reflexive since it has {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)}

(ii) Q is symmetric since it has (1, 2), (2, 1), (3, 4), (4, 3), (4, 5), (5, 4)

(iii) By adding two pairs as follows, it becomes transitive

1. As (3, 4), (4, 5) → Add (3, 5)
2. As (5, 4), (4, 3) → Add (5, 3)

Hence smallest equivalence relation containing R and S is

$\therefore$ Q = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4) (4, 5), (5, 3), (5, 4), (5, 5)}

Q. 5(c) **Let A = {a, b, c ,d, e, f, g, h}. Consider the following subsets of A**

A_1 = {a, b, c, d} **A_2 = {a, c, e, g, h}**

A_3 = {a, c, e, g} **A_4 = {b, d}**

A_5 = {f, h}

Determine whether following is partition of A or not. Justify your answer.

(i) {A_1, A_2} **(ii) {A_3, A_4, A_5}** **(4 Marks)**

Ans. :

(i) {A_1, A_2} → It is not a partition. Since a is common element between A_1 and A_2. Hence A_1 and A_2 are not disjoint sets.

(ii) {A_3, A_4, A_5} → It is a partion. Since A_3, A_4 and A_5 are disjoint sets and $A_1 \cup A_2 \cup A_3 = A$.

Q. 6(a) **Draw the Hasse Diagram of the following sets under the partial order relation divides and indicate which are chains. Justify your answers.**

I. A = {2, 4,12, 24}

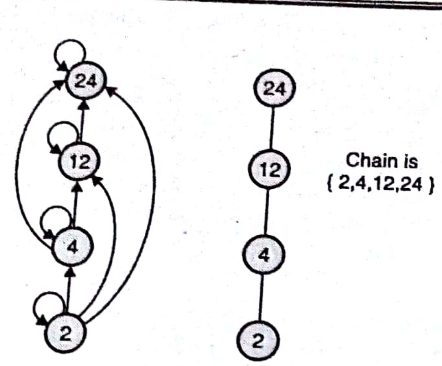
II. A = {1, 3, 5, 15, 30} **(8 Marks)**

Ans. :

(i) A = {2, 4, 12, 24}

First find out relation R and it is,

R = {(2, 2), (2, 4), (2, 12), (2, 24), (4, 4), (4, 12), (4, 24), (12, 12), (12, 24), (24, 24)}



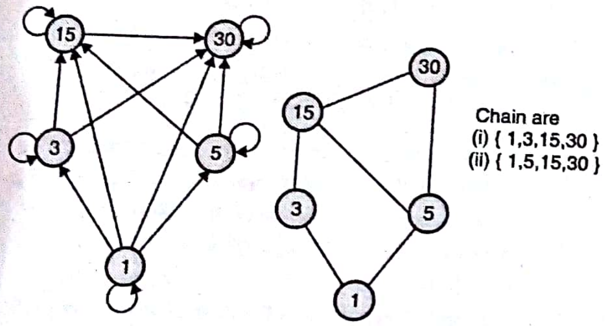
(a) : Diagraph (b) : Hasse Diagram

Fig. 1 - Q. 6(a)

(ii) A = {1, 3, 5, 15, 30}

First find out relation R and it is

R = {(1, 1), (1, 3), (1, 5), (1, 15), (1, 30), (3, 3), (3, 15), (3, 30), (5, 5), (5, 15), (5, 30), (15, 15), (15, 30), (30, 30)}



Diagraph

Fig. 2 - Q. 6(a)

Chapter 4 : Functions and Pigeon Hole Principle [Total Marks – 12]

Q. 3(c) **Test whether the following function is one-to-one, onto or both. $F : Z \to Z$, $f(x) = x^2 + x + 1$.** **(4 Marks)**

Ans. :

(i) For $x = 1$

$f(x) = 1^2 + 1 + 1 = 3$

(ii) for $x = 2$

$f(x) = 2^2 + 2 + 1 = 7$

(ii) for $x = -2$

$f(x) = (-2)^2 - 2 + 1 = 3$

(ii) for $x = -3$

$f(x) = (-3)^2 - 3 + 1 = 7$

i.e.

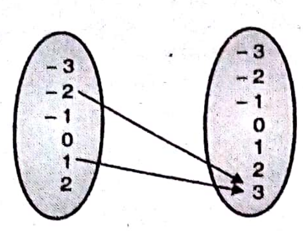


Fig. 1-Q. 3(c)

Hence it not one - to - one negative value of second set are also unassigned as f(x) is always + ve. Hence it not onto also.

Q. 6(b) **Let the functions f, g, and h defined as follows :**

$f : R \to R$, $f(x) = 2x + 3$

$g : R \to R$, $g(x) = 3x + 4$

$h : R \to R$, $h(x) = 4x$

Find gof, fog, foh, gofoh **(8 Marks)**

Ans. :

(i) $(g \circ f)(x) = g\,f(x) = g(2x + 3) = 3(2x + 3) + 4$

$= 6x + 9 + 4 = 6x + 13$

(ii) $(f \circ g)(x) = f(g(x)) = f(3x + 4) = 2(3x + 4) + 3$

$= 6x + 8 + 3 = 6x + 11$

(iii) $(f \circ h)(x) = f(h(x)) = f(4x) = 2(4x) + 3 = 8x + 3$

(iv) $(g \circ f \circ h)(x) = g(f(h(x)))$ As $(f \circ h)(x) = 8x + 3$

$g(8x + 3) = 3(8x + 3) + 4 = 24x + 9 + 4$

$= 24x + 13$

Chapter 5 : Counting [Total Marks – 21]

Q. 1(c) **A box contains 6 white balls and 5 red balls. In how many ways 4 balls can be drawn from the box if, (I) they are to be of any color (II) all the balls to be of the same color.** **(5 Marks)**

Ans. :

1. They are to be of any color

4 balls can be drawn randomly from total 11 balls.

$\therefore$ No. of ways $= {}^{11}C_4 = \dfrac{11\,!}{4\,! \times 7\,!} = \dfrac{11 \times 10 \times 9 \times 8 \times \cancel{7!}}{4 \times 3 \times 2 \times \cancel{7!}}$

$= 330$

2. All the balls to be of the same color

Either 4 balls are white i.e. No. of ways $= {}^6C_4$

OR 4 balls are red i.e. No. of ways $= {}^5C_4$

$\therefore$ Total no. of ways $= {}^6C_4 + {}^5C_4 = \dfrac{6\,!}{4\,! \times 2\,!} + \dfrac{5\,!}{4\,! \times 1\,!}$

$= \dfrac{6 \times 5 \times 4\,!}{4\,! \times 2} + \dfrac{5 \times 4\,!}{4\,!} = 15 + 5 = 20$

Q. 2(b) In a certain college 4% of the boys and 1% of the girls are taller than 1.8 mts. Furthermore 60% of the students are girls. If a student selected at random is taller than 1.8 mts, what is the probability that the student was a boy ? Justify your answer (8 Marks)

Ans. :

Since 60% of students are girls, 40% of students are boys.

We use Bayes theorem,

$$P(B \mid h > 1.8) = \frac{P(B \text{ and } h > 1.8)}{P(B \text{ and } h > 1.8) + P(G \text{ and } h > 1.8)}$$

Since 40% of boys and of those 4% are taller than 1.8 mts.

$P(B \text{ and } h > 1.8) = 0.4 * (0.04)$

Now,

$P(h > 1.8) = P(G \text{ and } h > 1.8) + P(B \text{ and } h > 1.6)$

$\therefore\ P(G \text{ and } h > 1.8) = 0.6 * 0.01$

$$\therefore\ P(B \mid h > 1.8) = \frac{0.4 * 0.04}{(0.4 * 0.04) + 0.6 * 0.01} = \frac{.016}{0.016 + 0.006} = 0.7273$$

Hence probability that a student was boy and if a student selected random is taller than 1.8 mts. = 72.73%.

Q. 5(b) Solve $a_r - 7a_{r-1} + 10\, a_{r-2} = 6 + 8\, r$ given $a_0 = 1, a_1 = 2$. (8 Marks)

Ans. :

The corrosponding homogeneous equation is

$$a_r - 7a_{r-1} + 10\, a_{r-2} = 0$$

here degree = 2

thus,

$$\begin{aligned} \alpha^2 - 7\alpha - 10 &= 0 \\ \alpha^2 - 5\alpha - 2\alpha - 10 &= 0 \\ \therefore\ (\alpha - 5)(\alpha - 2) &= 0 \\ \therefore\ \alpha &= 2, 5 \end{aligned}$$

The homogeneous solution is

$$a_r^{(h)} = A_1 2^r + A_2 5^r$$

Particular solution

Since R.H.S. is polynomial, the particular solution will be of the form $(P_0 + P_1 r)$

$$\begin{aligned} \text{i.e.}\quad a_r &= P_0 + P_1 r \\ a_{r-1} &= P_0 + P_1 (r-1) \\ a_{r-2} &= P_0 + P_1 (r-2) \end{aligned}$$

Substituting these values in the given relation,

We get,

$$\begin{aligned} (P_0 + P_1 r) - 7[P_0 + P_1 (r-1)] + 10\,[P_0 + P_1 (r-2)] &= 6 + 8r \\ \text{i.e. } P_0 + P_1 r - 7P_0 - 7\,P_1 (r-1) + 10\,P_0 + 10\,P_1 (r-2) &= 6 + 8r \\ \therefore\ 4P_0 + P_1 [r - 7r + 7 + 10r - 20] &= 6 + 8r \\ 4\,P_0 - P_1 (4r - 13) &= 6 + 8r \end{aligned}$$

$\therefore\ (4\,P_0 - 13\,P_1) + 4\,P_1\, r = 6 + 8r$

On comparing the coefficients and polynomials,

We get

$$\begin{aligned} 4\,P_0 - 13\,P_1 &= 6 \quad \text{and} \quad 4\,P_1 = 8 \\ \therefore\ P_1 &= 2 \\ 4\,P_0 - 13\,P_1 &= 6 \\ \therefore\ 4\,P_0 - 23 \times 2 &= 6 \\ \therefore\ 4\,P_0 - 26 &= 6 \\ \therefore\ P_0 &= 8 \end{aligned}$$

Here the particular solution is

$$a_r(p) = 8 + 2r$$

Thus general solution is

$$\begin{aligned} a_r &= a_r(h) + a_r(p) \\ \therefore\ a_r &= A_1 2^r + A_2 5^r + 8 + 2r \end{aligned}$$

Using the initial conditions $a_0 = 1$ and $a_1 = 2$

We get,

$$A_1 = -9 \quad \text{and} \quad A_2 = 2$$

Therefore,

$$a_r = -9\,(2)^r + 2\,(5)^r\, 8 + 2r$$

Chapter 6 : Graphs [Total Marks – 12]

Q. 2(a) Define Isomorphism of graphs. Find if the following two graphs are isomorphic . If yes, find the one-to-one correspondence between the vertices. (8 Marks)

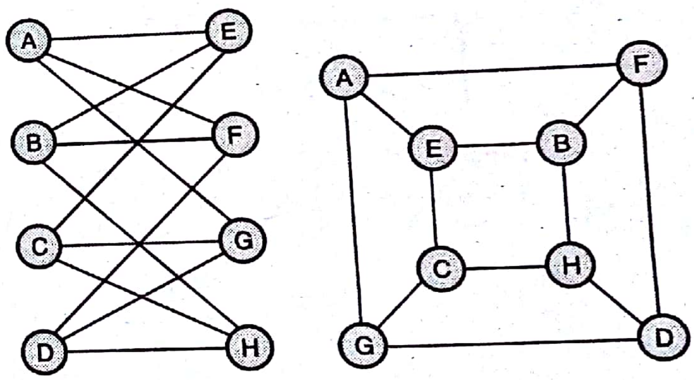


Fig. 1-Q. 2(a)

Ans. :

Isomorphism of graphs

Two graphs have exactly the same form, in the sense that there is one - to - one correspondence between their vertex sets that preserve edges. In such a case, two graphs are isomorphic.

Degree of each vertex is 3 in both graphs.

	A	B	C	D	E	F	G	H
A	0	0	0	0	1	1	1	0
B	0	0	0	0	1	1	0	1
C	0	0	0	0	1	0	1	1
D	0	0	0	0	0	1	1	1
E	1	1	1	0	0	0	0	0
F	1	1	0	1	0	0	0	0
G	1	0	1	1	0	0	0	0
H	0	1	1	1	0	0	0	0

Adjancy matrix of G_1

	A	B	C	D	E	F	G	H
A	0	0	0	0	1	1	1	0
B	0	0	0	0	1	1	0	1
C	0	0	0	0	1	0	1	1
D	0	0	0	0	0	1	1	1
E	1	1	1	0	0	0	0	0
F	1	1	0	1	0	0	0	0
G	1	0	1	1	0	0	0	0
H	0	1	1	1	0	0	0	0

Adjancy matrix of G_2

By observing G1 and G2 (Adjancy matirx), mapping is as follows :

Graph G_1	Graph G_2
A	A
B	B
C	C
D	D
E	E
F	F
G	G
H	H

Hence graph G_1 and G_2 are isomorphic

Q. 6(c) Determine Euler Cycle and path in graph shown below. (4 Marks)

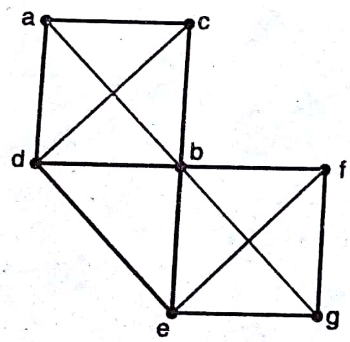


Fig. 1 - Q. 6(c)

Ans. :

Euler Cycle

Theorem : A connected multigraph with at least two vertices has an Euler cycle if and only if each of its vertices has an even degree.

Here deg (a) = 3.

Hence it doesn't have Euler cycle

Euler Path

Since given graph has not Euler Cycle.

Theorem

A Connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree. deg (a) = 3, deg (c) = 3, deg (f) = 3, deg (g) = 3.

Hence it doesn't have Euler path.

Chapter 7 : Algebraic Structures [Total Marks – 24]

Q. 3(a) Prove that set G = {1, 2, 3, 4, 5, 6} is a finite abelian group of order 6 with respect to multiplication module 7. (8 Marks)

Ans. : First draw a matrix for multiplication modulo 7.

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

(i) As $\{1, 2, 3, 4, 5, 6\} \in G$. Hence G is closed with respect to multiplication modulo 7.

(ii) $1 * (2 * 3) = 1 * 6 = 6$

$(1 * 2) * 3 = 2 * 3 = 6$

OR

$3 * (4 * 5) = 3 * 6 = 4$

$(3 * 4) * 5 = 5 * 5 = 4$

Hence G is associative with respect to multiplication modulo 7.

(iii) 1 is the identity element

(iv) 1 is inverse of 1, 2 is inverse of 4, 3 is inverse of 5, 6 is inverse of 6 and vice versa.

(v) $4 * 2 = 1$ and $2 * 4 = 1$

$4 * 6 = 3$ and $6 * 4 = 3$

Hence G is commutative under operation multiplication modulo 7.

As G satisfies above (i) to (v) conditions, hence G is a finite abelian group.

Q. 4(a) Show that the (2, 5) encoding function $e : B^2 \rightarrow B^5$ defined by e(00) = 00000 e(01) = 01110 e(10) = 10101 e(11) = 11011 is a group code. How many errors will it detect and correct? (8 Marks)

Ans. :

(i) $x, y \in S$, then $x \oplus y \in S$. Hence it is closed.

(ii) Identity element

$$00000 \oplus 00000 = 00000$$
$$00000 \oplus 01110 = 01110$$
$$00000 \oplus 10101 = 10101$$
$$00000 \oplus 11011 = 11011$$

Here (0, 0, 0, 0, 0)

(iii) $(B_6, \oplus)$ is associative

$$00000 \oplus (01110 \oplus 10101) = (00000 \oplus 01110) \oplus 10101$$

(iv) Every code is in inverse of itself.

Hence it is a group code.

Error detection :

Minimum distance is the minimum weight of non-zero codewords of a group code.

$\therefore$ Minimum distance $= 3$

$\therefore$ Number of errors can be detect $= 3 - 1 = 2$

Number of errors can be correct $= 3 - 2 = 1$

Q. 4(b) **Let** $H = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Be a parity check matrix. Determine the group code $e_H : B^3 \to B^6$ **(8 Marks)**

Ans. :

(i) $H = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$$\text{As } H = [P^T \mid I_K]$$

$$P_T = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\therefore \quad P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(ii) Find parity bits p_1, p_2, p_3 as

$$(p_1\, p_2\, p_3) = [x_1\, x_2\, x_3] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\therefore \quad p_1 = x_1, p_2 = x_2 \oplus x_3, p_3 = x_1 \oplus x_2 \oplus x_3$

(iii) As I_K is 3×3 matrix No. of msg. bits = 3

$\therefore$ No. of codewords $= 2^3 = 8$

x_1	x_2	x_3	p_1	p_2	p_3	
0	0	0	0	0	0	e(000) = 000000
0	0	1	0	1	1	e(001) = 001011
0	1	0	0	1	1	e(010) = 010011
0	1	1	0	0	0	e(011) = 011000
1	0	0	1	0	1	e(100) = 100101
1	0	1	1	1	0	e(101) = 101110
1	1	0	1	1	0	e(110) = 110110
1	1	1	1	0	1	e(111) = 111101

Question Paper

Dec. 2017

Q. 1 (a) Prove that $1.1\,! + 2.2\,! + 3.3\,! + \dots\dots + n \cdot n\,! = (n + 1)\,! - 1$, where n is a positive integer. **(5 Marks)**

(b) Let A = {a, b, c}. Show that (P(A), $\subseteq$) is a poset and draw its Hasse diagram. **(5 Marks)**

(c) Explain the terms : **(5 Marks)**

(i) Lattice

(ii) Poset

(iii) Normal Subgroup

(iv) Group

(v) Plannar Graph

(d) Comment whether the function f is one to one or onto.

Consider function : $f : N \to N$ where N is set of natural numbers including zero.

$f(j) = j^2 + 2$ **(5 Marks)**

Q. 2 (a) Find the number of the ways a person can be distributed Rs. 601 as pocket money to his three sons, so that no son should receive more than the combined total of the other two. (Assume no fraction of a rupee is allowed). **(6 Marks)**

(b) Let A = $\{a_1, a_2, a_3, a_4, a_5\}$ and let R be a relation on A whose matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find M_R * by Warshall's algorithm. **(6 Marks)**

(c) Find the complete solution of the recurrence relation :

$a_n + 2\,a_{n-1} = n + 3$ for $n \geq 1$ and with $a_0 = 3$.

(4 Marks)

(d) Let $f : R \to R$ defined as $f(x) = x^3$ and $g : R \to R$ defined as $g(x) = 4x^2 + 1$

Find out $g \circ f$, $f \circ g$, f^2, g^2. **(4 Marks)**

Q. 3 (a) Given that a student had prepared, the probability of passing a certain entrance exam is 0.99. Given that a student did not prepare, the probability of passing the entrance exam is 0.05. Assume that the probability of preparing is 0.7. The student fails in the exam. What is the probability that he or she did not prepare ? **(6 Marks)**

(b) Define equivalence relation with example. Let 'T' be a set of triangles in a plane and define R as the set R = {(a, b) | a, b ∈ T and a is congruent to b} then show that R is an equivalence relation.

(6 Marks)

(c) Let A = B = R, the set of real numbers

Let $f : A \to B$ be given by the formula $f(x) = 2x^3 - 1$ and Let $g : B \to A$ be given by -

$$g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}$$

Show that f is a bijection between A and B and g is a bijection between B and A. **(4 Marks)**

(d) Let Z_n denote the set of integers {0, 1, 2, ... n − 1}. Let O be binary operation on z_n denote such that a O b = the remainder of ab divided by n.

(i) Construct the table for the operation O for n = 4.

(ii) Show that (Z_n, O) is a semigroup for any n.

(4 Marks)

Q. 4 (a) (i) Among 50 students in a class, 26 got an A in the first examination and 21 got an A in the second examination. If 17 students did not get an A in either examination, how many students got an A in both examinations ?

(ii) If the number of students who got an A in the first examination is equal to that in the second examination, if the total number of students who got an A in exactly one examination is 40 and if 4 students did not get an A in either examination then determine the number of students who got an A in the first examination only, who got an A in the second examination only and who got an A in both the examination. **(6 Marks)**

(b) Consider the (2, 5) group encoding function $e : B^2 \to B^5$ defined by,

e (00) = 00000 e (01) = 01110

e (10) = 10101 e (11) = 11011

Decode the following words relative to a maximum likelihoods decoding function.

(i) 11110 (ii) 10011 (iii) 10100 **(6 Marks)**

(c) (i) Is every Eulerian graph a Hamiltonian ?

(ii) Is every Hamiltonian graph a Eulerian ?

Explain with the necessary graph. **(4 Marks)**

(d) Given the parity check matrix. **(4 Marks)**

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find the minimum distance of the code generated by H. How many errors it can detect and correct ?

Q. 5 (a) Explain Pigeonhole principle and Extended Pigeonhole principle. Show that in any room of people who have been doing some handshaking there will always be atleast two people who have shaken hands the same number of times.

(6 Marks)

(b) Determine whether the Poset with the following Hasse diagrams are lattices or not. Justify your answer. **(6 Marks)**

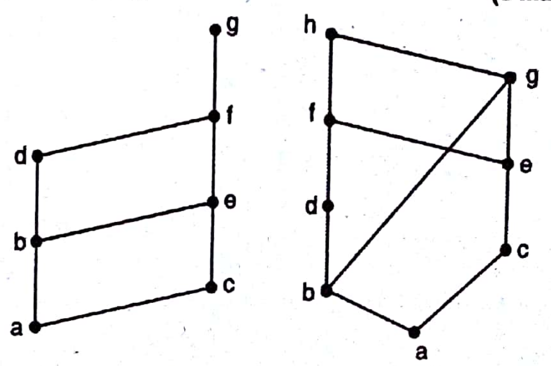


Fig. 1-Q. 5(b)

Q. 5(c) From the following diagraphs, write the relation a set of ordered pairs. Are the relations equivalence relations ? **(4 Marks)**

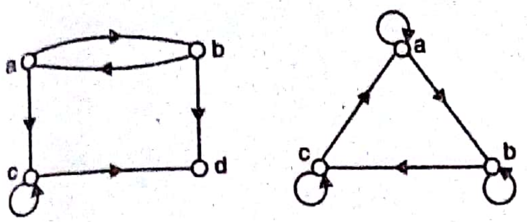


Fig. 1-Q. 5(c)

(d) For the set X = {2, 3, 6, 12, 24, 36}, a relation $\leq$ is defined as $x \leq y$ if x divides y. Draw the Hasse diagram for (X, $\leq$). Answer the following .

(i) What are the maximal and minimal elements ?

(ii) Give one example of chain and antichain.

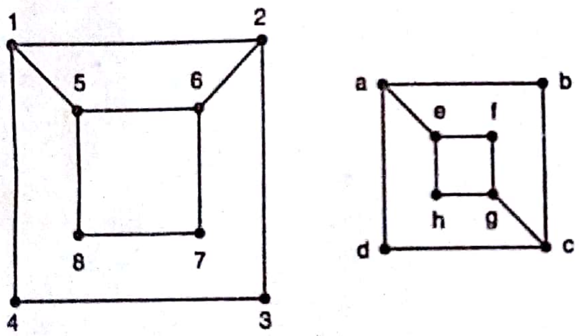
(iii) Is the poset a lattice ? **(4 Marks)**

Q. 6 (a) Prove that the set {1, 2, 3, 4, 5, 6} is group under multiplication modulo 7. **(6 Marks)**

(b) Given a generating function, find out corresponding sequence. **(6 Marks)**

(i) $\frac{1}{3-6x}$ (ii) $\frac{x}{1-5x+6x^2}$

(c) Determine whether following graphs are isomorphic or not. **(4 Marks)**



G_1 G_2

Fig. 1-Q. 6(c)

Q. 6(d) Prove the following (use laws of set theory)

$A \times (X \cap Y) = (A \times X) \cap (A \times Y)$ **(4 Marks)**

May 2018

Q. 1 (a) Prove by induction that the sum of the cubes of three consecutive numbers is divisible by 9. **(5 Marks)**

(b) Find the generating function for the following finite sequences. **(5 Marks)**

(i) 2, 2, 2, 2, 2, 2

(ii) 1, 1, 1, 1, 1, 1

(c) A box contains 6 white balls and 5 red balls. In how many ways 4 balls can be drawn from the box if, (i) they are to be of any color (ii) all the balls to be of the same color. **(5 Marks)**

(d) Find the complement of each element in D_{30}. **(5 Marks)**

Q. 2 (a) Define Isomorphism of graphs. Find if the following two graphs are isomorphic . If yes, find the one-to-one correspondence between the vertices. **(8 Marks)**

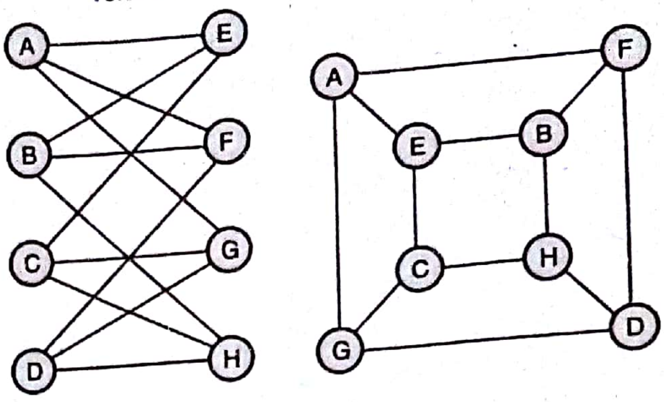


Fig. 1-Q. 2(a)

(b) In a certain college 4% of the boys and 1% of the girls are taller than 1.8 mts. Furthermore 60% of the students are girls. If a student selected at random is taller than 1.8 mts, what is the probability that the student was a boy ? Justify your answer **(8 Marks)**

(c) Prove $\neg (p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences . **(4 Marks)**

Q. 3 (a) Prove that set G = {1,2,3,4,5,6} is a finite abelian group of order 6 with respect to multiplication module 7. **(8 Marks)**

(b) Let A = {1,2,3,4,5}, let R = {(1 ,1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)} and S = {(1, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (4,5)} be the relations on A. Find the smallest equivalence relation containing the relation R and S. **(8 Marks)**

(c) Test whether the following function is one-to-one, onto or both. $F : Z \to Z$, $f(x) = x^2 + x + 1$. **(4 Marks)**

Q. 4 (a) Show that the (2, 5) encoding function $e : B^2 \to B^5$ defined by

e(00) = 00000 e(01) = 01110

e(10) = 10101 e(11) = 11011 is a group code. How many errors will it detect and correct ? **(8 Marks)**

(b) Let H =

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Be a parity check matrix. Determine the group code $e_H : B^3 \rightarrow B^6$ **(8 Marks)**

(c) How many friends must you have to guarantee that at least five of them will have birthdays in the same month? **(4 Marks)**

Q. 5 (a) Let G be a set of rational numbers other than 1. Let * be an operation on G defined by $a * b = a + b - ab$ for all a, b € G. Prove that (G,*) is a group. **(8 Marks)**

(b) Solve $a_r - 7a_{r-1} + 10\, a_{r-2} = 6 + 8\, r$ given $a_0 = 1$, $a_1 = 2$. **(8 Marks)**

(c) Let A = {a, b, c ,d, e, f, g, h}. Consider the following subsets of A

A_1 = {a, b, c, d} A_2 = {a, c, e, g, h}

A_3 = {a, c, e, g} A_4 = {b, d}

A_5 = {f, h}

Determine whether following is partition of A or not. Justify your answer.

(i) {A_1, A_2} (ii) {A_3, A_4, A_5} **(4 Marks)**

Q. 6 (a) Draw the Hasse Diagram of the following sets under the partial order relation divides and indicate which are chains. Justify your answers.

i. A = {2, 4,12, 24}

ii. A = {1, 3, 5, 15, 30} **(8 Marks)**

(b) Let the functions f, g, and h defined as follows :

$f : R \rightarrow R,\ f(x) = 2x + 3$

$g : R \rightarrow R,\ g(x) = 3x + 4$

$h : R \rightarrow R,\ h(x) = 4x$

Find gof, fog, foh, gofoh **(8 Marks)**

(c) Determine Euler Cycle and path in graph shown below. **(4 Marks)**

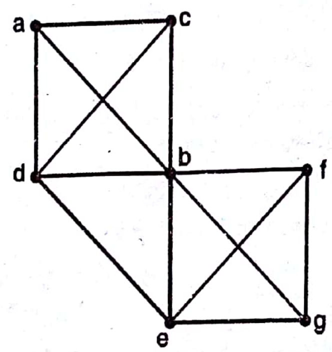


Fig. 1 – Q. 6(c)

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