DISCRETE MATHEMATICAL STRUCTURES

unit I - Mathematical logic

Propositional calculus

Statements & notations, connections, well formed formulas, truth tables, tatologies, equalance of formulaes, duality law, normal forms: Theory of inference consistancy of premises, predicate calculus

predicative logic, statement function, variables & Quantifiers, free and bound variables. Inference theory for predicate calculus.

* The rules of logic are used in design of computer circuit. The construction of computer programs, the verification of the correlatives of the programs and in many ways

Proposition

* A proposition is a declarative sentance i.e either true or false but not both
* Every proposition is associated with two values called truth value

True (T) & (1)

False (F) & (0).

1. True is an even integer → T
2. There is an integer b/w 1 and 2 → F
3. $1+1 = 2$ → T
4. $x+1 = 2$ if $x = 3$ → F
5. $x+y = x+z$ if $y = z$ → T
6. Read this carefully
7. will you go?

Logical operation

* Here we have five basic connectives or operation

Negation (~) :- The negation of a proposition (p) is denoted by '~p' which is a proposition

* It is not a case that p".

ex:- p: Today is Thursday

~p: Today is not thursday

P	~P
T	F
F	T

2. Conjunction (∧)

* Let p, q to be a two propositions the conjunction of P & q is denoted by P & Q is a statement/is a proposition which is true, when both P∧q are true & is false otherwise

3. Disjunction (∨)

* Let p, q to be a two proposition then disjunction of two proposition is denoted by P∨q. which is false when both P & q are false and is true otherwise.

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4. Conditional statement (Implecation)

* The Implication of two props P & q is denoted by $P \rightarrow q$ is a proposition " If P then q ", (q if P) ~~That is~~ which is a statement and it is false when P is true & q is false and it is true otherwise

P	q	$P \rightarrow q$
T	F	F
T	T	T
F	T	T
F	F	T

* The statement $\sim P \rightarrow \sim q$ is called inverse of $P \rightarrow q$

The statement $q \to p$ is called converse of $p \to q$

* $\sim q \to \sim p$ is called contraposition of the implication $p \to q$.

Ex:- If is read well then I will pass the exam.

p - If I read well

q - [redacted] I will pass the exam.

$p \to q$ If I read well then I will pass the exam

$q \to p$ I will pass the exam if I read well

$\sim p \to \sim q$ If I don't read well then I will not pass the exam.

$\sim q \to \sim p$ I wont pass the exam if I dont read well

4. Biconditional statement (Biimplication)

* The biconditional of the two statements is denoted by $p \leftrightarrow q$ which is a statement "p if and only if q" which is true when p & q has same truth values & false otherwise.

p	q	$p \leftrightarrow q$
T	T	T
F	T	F
T	F	F
F	F	T

5 Exclusive OR (XOR $\oplus$)

* The exclusive OR of the two propositions p & q is denoted by $p \oplus q$ and its ...

p	q	p ⊕ q
T	T	F
F	T	T
T	F	T
F	F	F

Translate the english sentences into connectives

1. You can access the internet from the campus if you are a computer science major & you are not a fresh man

p: You can access the internet from the campus

q: You are a computer science major

r: You are a fresh man

$(q \vee \sim r) \rightarrow p.$

2. The automated reply cannot be send when the file system is full.

p: The automated reply can be send

q: The file system is full.

$\sim q \rightarrow p.$

3. Jack and Jill went up the hill.

p: Jack went up the hill

q: Jill went up the hill

$p \wedge q$

Construct the truth tables for the following

① $[q \wedge (p \rightarrow q)] \rightarrow p$ ② $[p \vee (q \wedge r)] \leftrightarrow [(p \vee q) \wedge (p \vee r)]$

①

p	q	$p \rightarrow q$	$q \wedge (p \rightarrow q)$	$[q \wedge (p \rightarrow q)] \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

②

p	q	r	$p \vee q$	$p \vee r$	$q \wedge r$	(A) $p \vee (q \wedge r)$	(B)	$A \leftrightarrow B$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F	T
F	F	T	F	T	F	F	F	T
F	F	F	F	F	F	F	F	T

Tautology

* A statement formula which is always true is a tautology

Ex:- $p \vee \sim p$

Contradiction

* A statement formula which is always false is a contradiction

Ex:- $p \wedge \sim p$.

Note:- Neither tautology nor contradiction is called contingency.

Problem

1. verify which of the following formulas are tautology & contradiction.

a) $p \rightarrow (p \vee q)$

b) $(p \rightarrow \sim p) \rightarrow \sim p$

c) $(\sim q \wedge p) \wedge q$

d) $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$

e) $(\sim p \rightarrow q) \rightarrow (q \rightarrow p)$

f) $(p \wedge q) \leftrightarrow p$.

Well formed formula

* A well formed formula can be generated by the following rules

1. A statement variable standing alone is a well formed formula.
2. If A is well formend then $\sim A$ is also well formed
3. If A & B are well formed then $A \vee B$, $A \wedge B$

$A \rightarrow B$, $A \leftrightarrow B$ are well formed

4. A String of symbols containing the statement variables, connectives, parenthesis is well formed iff it can be obtained by finite applications of rules 1, 2 & 3.

Equivalence formulae

* Two propositions p & q are said to be logically equivalent if they have the same truth values

(or)

Two propositions p & q are said to be logically equivalent if $p \leftrightarrow q$ is a tautology.

Ex:- consider the two statements

p: I was born in 1980.

q: I has 20 yrs of age in 2000.

Here, p & q are logically equivalent.

Ex:- consider the statements

Good food is not cheap

cheap food is not good.

Let

p : food is good

q : food is ~~not~~ cheap.

Given statements will be

$p \rightarrow \sim q$

$q \rightarrow \sim p$

p	q	$\sim p$	$\sim q$	$p \to \sim q$ (A)	$q \to \sim p$ (B)	$A \leftrightarrow B$
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

$\therefore$ The given statements are logically equal.

Some other examples

*1. $p \to q \equiv \sim p \vee q$

*2 $p \to q \equiv \sim q \to \sim p$.

3 $p \vee q \equiv \sim p \to q$

4. $p \wedge q \equiv \sim (p \to \sim q)$

5. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

6. $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

* Suppose p, q, r be 3 propositions

1. Idempotent laws

$p \vee p \Leftrightarrow p$

$p \wedge p \Leftrightarrow p$.

2. Associative laws

$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$.

3. Distributive laws

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

4. Identity laws

$p \wedge T \Leftrightarrow p$

$p \vee F \Leftrightarrow p$.

5. Domination laws

$p \vee T \Leftrightarrow T$

$p \wedge F \Leftrightarrow F$

6. Double negation.

$\sim(\sim p) \Leftrightarrow p.$

7 Demorgan's laws

$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$

$\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

8. Absorption laws

$p \vee (p \wedge q) \Leftrightarrow p$

$p \wedge (p \vee q) \Leftrightarrow p.$

Problem

1. ST $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r.$

Sol LHS

$p \rightarrow (q \rightarrow r) \equiv p \rightarrow (\sim q \vee r)$

$\equiv \sim p \vee (\sim q \vee r)$

$\equiv (\sim p \vee \sim q) \vee r$

$\equiv \sim(p \wedge q) \vee r$

$\equiv (p \wedge q) \rightarrow r.$

Problem

1) $p \rightarrow (p \vee q)$

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

(Tautology)

ii) $(p \to \sim p) \to \sim p$

p	$\sim p$	$p \to \sim p$	$(p \to \sim p) \to \sim p$
T	F	F	T
F	T	T	T

(Tatology)

iii) $(\sim q \wedge p) \wedge q$

p	q	$\sim q$	$(\sim q \wedge p)$	$(\sim q \wedge p) \wedge q$
T	F	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	T	F	F

(contradiction)

iv) $[p \to (q \to r)] \to \{(p \to q) \to (p \to r)\}$

p	q	r	$p \to q$	$p \to r$	$q \to r$	$p \to (q \to r)$	$(p \to q) \to (p \to r)$	
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

(Tatology)

v) $(\sim p \overset{A}{\to} q) \to (q \overset{B}{\to} p)$.

p	q	$\sim p$	$\sim p \to q$	$q \to p$	$A \to B$
T	T	F	T	T	T
T	F	F	T	T	T
F	T	T	T	F	F
F	F	T	F	T	T

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vi) $(p \wedge q) \longleftrightarrow p$

p	q	$p \wedge q$	$(p \wedge q) \longleftrightarrow p$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T.

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Problem

2. S.T $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \Longleftrightarrow r.$

Sol

$\equiv (\sim p \wedge (\sim q \wedge r)) \vee ((q \vee p) \wedge r)$

$= (((\sim p \wedge \sim q) \wedge r) \vee ((q \vee p) \wedge r))$

$= ((\sim p \wedge \sim q) \vee (q \vee p)) \wedge r$

$= ((\sim p \vee (q \vee p)) \wedge (\sim q \vee (q \vee p))) \wedge r.$

$= ((T \vee q) \wedge (T \vee p)) \wedge r$

$= (T \wedge T) \wedge r$

$= r.$

3. S.T $\sim (p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equa

Sol $\sim (p \vee (\sim p \wedge q))$

$\sim p \wedge \sim (\sim p \wedge q)$

$\sim p \wedge (p \vee \sim q)$

$(\sim p \wedge p) \vee (\sim p \wedge \sim q)$

$F \vee (\sim p \wedge \sim q)$

$\sim p \wedge \sim q$

$\therefore$ logically equal.

4. S.T $(p\wedge q)\rightarrow(p\vee q)$ is Tautology.

Sol $(p\wedge q)\rightarrow(p\vee q) \equiv \sim(p\wedge q)\vee(p\vee q)$

$\equiv(\sim p\vee\sim q)\vee(p\vee q)$

$\equiv(p\vee\sim p)\vee(q\vee\sim q)$

$\equiv T\vee T$

$= T.$

1. $p\rightarrow(q\rightarrow p) \Leftrightarrow p\rightarrow(p\rightarrow q)$

2. $(p\rightarrow q)\wedge(r\rightarrow q) \Leftrightarrow (p\vee r)\rightarrow q$

3. $\sim(p\leftrightarrow q) \Leftrightarrow (p\vee q)\wedge\sim(p\wedge q)$.

<u>Unit law</u>

* Two formulas A & B are said to be duals of each other if one can be obtained from the other by replacing $\wedge$ by $\vee$ (and $\vee$ by $\wedge$

<u>Additional connectives</u>

* Two more connectives which are connective in logic gates which are NAND($\uparrow$) and NOR($\downarrow$)

* Let p, q be two propositions then p NAND q = $p\uparrow q$ and it is equal to $\sim(p\wedge q)$

p NAND q $\equiv p\uparrow q = \sim(p\wedge q)$

p NOR q $\equiv p\downarrow q = \sim(p\vee q)$

<u>Normal forms</u>

* <u>Elementary product</u> :- The product of the variables and their negations in a formula is elementary product. for example, for any propositions p, q

$p,\ \sim p\wedge\sim p,\ p\wedge q,\ p\wedge\sim q,\ p\wedge\sim p\wedge q$ etc. are elementary products

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2. <u>Elementary sum</u>:- The sum of the variables and their negations is called elementary sum

Ex:- P, $P \vee q$, $\sim P \vee q$, $P \vee \sim P \vee q$, $P \vee q \vee \sim q$ etc.

* There are two types of Normal forms

1. Disjunctive Normal form (DNF) (SOP)
2. conjunctive Normal form (CNF) (POS)

* A formula that is equivalent to the given formula which consists of sum of elimentary products is called DNF

* A formula which is equivalent to the given formula which consists of product of elementary sums is called CNF

<u>Problem</u>

1. obtain the conjunctive normal form of the statement formula $\sim(P \vee q) \leftrightarrow (P \wedge q)$

<u>Sol</u> $\sim(P \vee q) \leftrightarrow (P \wedge q)$

$[\sim(P \vee q) \rightarrow (P \wedge q)] \wedge [(P \wedge q) \rightarrow \sim(P \vee q)]$

$((P \vee q) \vee (P \wedge q)) \wedge (\sim(P \wedge q) \vee \sim(P \vee q))$

$(P \vee q \vee P) \wedge (P \vee q \vee q) \wedge ((\sim P \vee \sim q) \vee (\sim P \wedge \sim q))$

$(P \vee q \vee P) \wedge (P \vee q \vee q) \wedge (\sim P \vee \sim q \vee \sim P) \wedge (\sim P \vee \sim q \vee \sim q)$

2 obtain the conjunctive normal form of the statement formula $P \rightarrow ((P \rightarrow q) \wedge \sim(\sim q \vee \sim P))$

<u>Sol</u> $P \rightarrow (P \rightarrow q) \wedge \sim(\sim q \vee \sim P)$

$\sim P \vee (P \rightarrow q) \wedge \sim(\sim q \vee \sim P)$

$\sim p \vee (\sim p \vee q) \wedge (q \wedge p)$

$\sim p \vee (q \wedge p) \wedge (\sim p \vee q) \wedge (q \wedge p)$

$(\sim p \vee q) \wedge (\sim p \vee p)$

3. Obtain DNF of the statement formula

$\sim (p \vee q) \leftrightarrow (p \wedge q)$.

Sol $\sim (p \vee q) \leftrightarrow (p \wedge q)$

$[\sim (p \vee q) \wedge (p \wedge q)] \vee [(p \vee q) \wedge \sim (p \wedge q)]$

$[(\sim p \wedge \sim q) \wedge (p \wedge q)] \vee [(p \vee q) \wedge (\sim p \vee \sim q)]$

$[(p \wedge \sim p) \wedge (q \wedge \sim q)] \vee [((p \vee q) \wedge \sim p) \vee (p \vee q) \wedge \sim q]$

$[(p \wedge \sim p) \wedge (q \wedge \sim q)] \vee [(p \wedge \sim p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim q)]$

4. Obtain CNF & DNF for the statement formula

$(p \wedge (\sim q \vee r)) \vee (((p \wedge q) \vee \sim r) \wedge p)$.

PDNF & PCNF

Minterms :- for a given variables p, q, $p \wedge q$, $\sim p \wedge q$, $\sim p \wedge \sim q$, $p \wedge \sim q$ are called min terms

Max terms :- $p \vee q$, $\sim p \vee q$, $p \vee \sim q$, $\sim p \vee \sim q$ are max terms

for p, q, r

Minterms :- $p \wedge q \wedge r$, $\sim p \wedge q \wedge r$, $p \wedge \sim q \wedge r$, $p \wedge q \wedge \sim r$,
$\sim p \wedge \sim q \wedge r$, $p \wedge \sim q \wedge \sim r$, $\sim p \wedge q \wedge \sim r$,
$\sim p \wedge \sim q \wedge \sim r$.

Max terms :- $p \vee q \vee r$, $\sim p \vee q \vee r$, $p \vee \sim q \vee r$, $p \vee q \vee \sim r$
$\sim p \vee \sim q \vee r$, $p \vee \sim q \vee \sim r$, $\sim p \vee q \vee \sim r$
$\sim p \vee \sim q \vee \sim r$.

* For a given formula an equivalents consisting of ~~disjunction~~ disjunction of min terms is called PDNF (SOP canonical form)

* For a given formula an equivalent formula consisting of conjunction of max terms is called PCNF (POS canonical form).

Problem

1. Obtain PDNF of the statement formula $\sim p \vee q$.

Sol $\sim p \vee q \equiv (\sim p \wedge (q \vee \sim q)) \vee (q \wedge (p \vee \sim p))$

$\equiv (\sim p \wedge q) \vee (\sim p \wedge q) \vee (q \wedge p) \vee (q \wedge \sim p)$

$\equiv (\sim p \wedge q) \vee (\sim p \wedge q) \vee (p \wedge q)$

2. Obtain the PDNF of the statement formula $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

Sol $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

$= (p \wedge q \wedge (r \vee \sim r)) \vee (\sim p \wedge r \wedge (q \vee \sim q)) \vee (q \wedge r \wedge (p \vee \sim p))$

$= ((p \wedge q) \wedge r) \vee ((p \wedge q) \wedge \sim r) \vee (\sim p \wedge r \wedge q) \vee (\sim p \wedge r \wedge \sim q)$

$\vee (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$.

$= (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$

3. obtain PDNF of $\sim p \vee q$.

4. obtain PDNF & PCNF of the statement formula:

$(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

5. obtain PDNF & PCNF of $p \rightarrow ((p \rightarrow q) \wedge \sim(\sim q \vee \sim p))$

6. obtain PDNF & PCNF of $\sim(p \leftrightarrow q)$

7. obtain PDNF & PCNF of $(\sim p \rightarrow r) \wedge (q \leftrightarrow p)$

Theory of Inference for statement calculus

* The main function of logic is to provide rules of inference & principles of reasoning. the theory associated with such groups is known as inference theory, because it is concerned with the inference of a conclusion from certain premises. when a conclusion is derived from certain premises by using the accepted rules of reasoning, the process of derivation is called deduction or formal proof.

Argument

* A sequence of propositions $p_1, p_2, p_3, \ldots, p_n$ is called an argument in which the first $(n-1)$ propositions are called premises and p_n is called conclusion

validity of an argument:-

* If $[p_1 \wedge p_2 \wedge \ldots \ldots p_{n-1}] \rightarrow p_n$ is a tautology then the given argument is valid

* The following are rules of inference. (imp)

1. Rule P :- A premises may be introduced at any point in the derivation

2. Rule T :- A formula S may be introduced in the derivation if S is tautologically implied by one or more of the preceding formulae in the derivation

Rules of Inference

Rule of inference	Tautology	Name of the rule.
1. p, $p \to q$, $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
2. $p \to q$, $\sim q$, $\therefore \sim p$	$[(p \to q) \wedge \sim q] \to \sim p$	Modus Tollens
3 $p \to q$, $q \to r$, $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to r$	Hypothetical syllogism
4. $p \vee q$, $\sim p$, $\therefore q$	$[(p \vee q) \wedge (\sim p)] \to q$	Disjunctive Syllogism
5 p, $\therefore p \vee q$	$p \to (p \vee q)$	Addition.
6. $p \wedge q$, $\therefore p$	$(p \wedge q) \to p$	~~conjunction~~ simplification
7 p, q, $\therefore p \wedge q$	$(p \wedge q) \to (p \wedge q)$	conjunction.

8. $p \vee q$ $(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee r)$ Resolution.

$\sim p \vee r$

$\therefore q \vee r$

verification of argument

1. If socrates is a man then he is mortal.
Socrates is a man therefore socrates is mortal

p: socrates is a man

q: socrates is mortal.

$\therefore$ the given argument will be symbolized a

$$\left.\begin{array}{l} p \\ p \rightarrow q \end{array}\right\} \text{premises}$$

$\therefore q$ } conclusion.

verification

1. $p \rightarrow q$ premise
2. p premise
3. q conclusion. from ① & 2 by Modus ponens

$\therefore$ conclusion is verified

2. If it rains today then we will not have a barbecut today
If we do not have a barbecut today then we will have a barbecut tomorrow therefore if it rains today then we will have barbecue tomorrow

p: It is raining today

q: we will have a barbecue today

r: we will have a barbecue tomorrow

Now the given argument can be symbolized a.

$p \rightarrow \sim q$

$\sim q \rightarrow r$

$\therefore p \rightarrow r$.

verification.

1. $p \rightarrow \sim q$ premise
2. $\sim q \rightarrow r$ premise
3. $p \rightarrow r$ from ① & ② by hypothetical syllogism

$\therefore$ conclusion is verified and the given argument is valid

~~verify~~

Problem

1. verify the following argument valid or not

$p \rightarrow (r \rightarrow s)$

$\sim r \rightarrow \sim p$

p

$\therefore s$.

verification

① $p \rightarrow (r \rightarrow s)$ premise

② p premise

③ $r \rightarrow s$ from ① & ② by modus ponens

④ $\sim r \rightarrow \sim p$ premise

⑤ $p \rightarrow r$ contra position of ④

⑥ p. premise.

⑦ r from ⑤ & ⑥ by modus ponens

⑧ s from ③ & ⑦ by modus ponens

$\therefore$ conclusion is verified & argument is valid

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2. verify the arguments for its validity

* If Joe is a mathematician then he is ambitious
* If Joe is an early riser then he doesn't like oat meal
* If he is ambitious then he is an early riser

Now if Joe is a mathematician then he doesn't like oat meal

p: Joe is a mathematician

q: Joe is ambitious

r: Joe is an early riser

s: Joe likes oat meal

the given argument can be symbolized as

$$\begin{array}{l} p \rightarrow q \\ r \rightarrow \sim s \\ q \rightarrow r \\ \hline p \rightarrow \sim s \end{array}$$

Sol

① $p \rightarrow q$

② $q \rightarrow r$

③ $p \rightarrow r$

④ $r \rightarrow \sim s$

⑤ $p \rightarrow \sim s$

2. If clifton doesn't live in france then he doesn't speak french

* clifton doesn't dive a datson.

* If clifton lives in france then he rides a bicycle.

* Either clifton speaks french or he drives a datson

hence clifton drives a bicycle.

Sol

p: clifton lives in france

q: clifton speaks French

r: clifton drives a datson

s: clifton rides a bicycle

then the given argumen can be symbolized as

$$\begin{array}{l} \sim p \rightarrow \sim q \\ \sim r \\ p \rightarrow s \\ q \vee r \\ \hline \therefore s. \end{array}$$

1. $\sim p \rightarrow \sim q$ — premise
2. $q \rightarrow p.$ — ~~premise~~ contraposition of 1
3. $p \rightarrow s.$ — premise
4. $q \rightarrow s.$ — from 2 & 3 (hypothetical syllogism)
5. $r \vee q$ — premise
6. $\sim r$ — premise
7. q — from 5 & 6 (modus tollens)
8. s — from 4 & 7 (modus ponens

PREDICATE CALCULUS

Predicate :- In the statement x is greater than 3 the variable "x" is subject and "is greather than 3" is the predicate. So we denote such statements by

. $P(x)$: x is greater than 3 → propositional fn.

* Similarly, $T(x,y)$ denotes

$T(x,y)$: x is taller than y.

Quantifiers

* There are two types of quantifiers they are

1. Universal quantifier – $\forall$
2. Existential quantifier – $\exists$

* The area of logic deals with the predicate & quantifiers is called Predicate calculus.

* Many mathematical statements asserts that a property is true for all values of a variable in a particular domain called Domain of Discourse. or universe of discourse.

$$(\forall x)\, P(x) = P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \dots \wedge P(x_n)$$

$$(\exists x)\,(Px) = P(x_1) \vee P(x_2) \vee P(x_3) \vee \dots \vee P(x_n)$$

Examples

1. Write the each of the sentances into symbolic form

1. All the students in the class have studied logic

Let

$S(x)$: x is a student

$L(x)$: x studied logic.

$(\forall x)(S(x) \rightarrow L(x))$

2. Some students in the class who are good at talking or good at listening

$S(x)$: x is a student

$T(x)$: x is good at talking

$L(x)$: x is good at listening

$(\exists x)(S(x) \wedge (T(x) \vee L(x)))$

3. Every complete, bipartite graph is not planar

$G(x)$: x is a graph

$C(x)$: x is a complete graph.

$B(x)$: x is a bipartite graph

$P(x)$: x is a planar graph.

$(\forall x)((G(x) \wedge C(x) \wedge B(x)) \rightarrow \sim P(x))$

4. There is an american who doesn't eat cheese burger

$A(x)$: x is an American

$B(x)$: x eats cheese burger

$(\exists x)(A(x) \wedge \sim B(x))$

All americans eats cheese burges

$(\forall x)(A(x) \rightarrow B(x))$

5. Consider the statements?

1. All birds can fly

2. Some birds can fly

3. No birds can fly

4. Not all birds can fly

Sol Here we denote.

$B(x)$: x is a bird $\quad$ $F(x)$: x can fly

$(\forall x)\ (B(x) \rightarrow F(x))$

$(\exists x)\ ((B(x)) \wedge F(x))$.

$(\forall x)\ ((B(x)) \rightarrow \sim F(x))$

$(\exists x)\ ((B(x)) \wedge \sim F(x))$

6. a. All babies are illogical

b. Some babies are illogical

c. No baby is illogical

d. Not all babies are illogical.

Sol $B(x)$: x is a baby $\quad$ $L(x)$: x is logical

$(\forall x)\ (B(x) \rightarrow \sim L(x))$

$\exists x\ (B(x) \wedge \sim L(x))$

$(\forall x)\ (B(x) \rightarrow L(x))$

$(\exists x)\ (B(x) \wedge L(x))$

Note:- $\sim [(\forall x)\ P(x)] \equiv (\exists x)\ [\sim P(x)]$

7. There is a student who doesn't take a course in calculus.

$S(x)$: x is a student

$C(x)$: x takes a course in calculus

$(\exists x)\ [S(x) \wedge \sim C(x)]$

8. There is an honest politician

$P(x)$: x is honest politician

$(\exists x)(P(x))$

Observation

Universal specification

$(\forall x)(P(x))$ is true $\longrightarrow \exists$ 'a' $\ni P(a)$ is true.

universal generalization

~~(\forall x)(P(x)) is true~~

$P(a)$ is true for some a $\longrightarrow (\forall x)(P(x))$ is true.

existential Specification

$(\exists x)(P(x))$ is ture $\longrightarrow \exists$ a $P(a)$ is true

Existential generalization

~~(\exists x)(P(x)) is true \Rightarrow \exists~~

$P(a)$ is true for some a $\rightarrow (\exists x)(P(x))$ is true

verification of the validity of the arguments

Problem

1. Every living thing is a plant or animal.

David's dog is alive. and it is not a plant.

All animals have Harts

Therefore; Davids dog has a heart.

Sol We denote our domain of discourse is all living things.

$A(x)$: x is an animal

$P(x)$: x is a plant

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$H(x)$: x has heart.

d : david's dog.

Now the given sentences will be symbolized

$(\forall x)(P(x) \vee A(x))$

$\sim P(d)$

$(\forall x)(A(x) \rightarrow H(x))$

$\therefore \quad H(d)$

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verification

1. $(\forall x)(P(x) \vee A(x)) \longrightarrow$ Premise
2. $P(d) \vee A(d) \longrightarrow$ universal specification of ①
3. $\sim P(d) \longrightarrow$ premise
4. $A(d) \longrightarrow$ disjunctive Syllogism of ② & ③
5. $(\forall x)(A(x) \rightarrow H(x)) \rightarrow$ premise
6. $A(d) \rightarrow H(d) \longrightarrow$ U.S. of ⑤
7. $H(d) \longrightarrow$ from ④ & ⑥ using modus ponens.

$\therefore$ The given argument is truth

2. verify the validity of the argument
- A student in this class has not read the book
- Everyone in this class passed the 1st exam
- imply the conclusion someone who passed the first exam has not read the book.

Sol: let

$C(x)$: x is a student in the class

$R(x)$: x has read the book

$P(x)$: x passed the 1st exam

Now the sentences can be symbolized as:

$(\exists x)(C(x) \wedge \sim R(x))$

$(\forall x)(C(x) \rightarrow P(x))$

$\therefore (\exists x)(P(x) \wedge \sim R(x))$

verification

1. $(\exists x)(C(x) \wedge \sim R(x))$ → Premise
2. $C(a) \wedge \sim R(a)$ for some a → E.S of ①
3. $C(a)$ → simplification of ②
4. $(\forall x)(C(x) \rightarrow P(x))$ → premise
5. $C(a) \rightarrow P(a)$ → U.S of ④
6. $P(a)$ → from ③ & ⑤ Modus ponens
7. $\sim R(a)$ → simplification of ②
8. $P(a) \wedge \sim R(a)$ → conjunction ⑥ & ⑦
9. $(\exists x)(P(x) \wedge \sim R(x))$ → E.G of ⑧

∴ conclusion is verified & argument is valid

3. verify the validity of the argument

$(\forall x)(P(x) \vee Q(x))$

$(\forall x)(\sim Q(x) \vee S(x))$

$(\forall x)(R(x) \rightarrow \sim S(x))$

$(\exists x)(\sim P(x))$

$\therefore (\exists x)(\sim R(x))$

verification

1. $(\forall x)(P(x) \vee Q(x))$ → Premise
2. $(\forall x)(\sim Q(x) \vee S(x))$ → premise

3. $P(a) \vee Q(a)$ for some a → US of ①

4. $\sim Q(a) \vee S(a)$ → US of ②

5. $P(a) \vee S(a)$ → from ③ & ④ resolution

6. $(\exists x) \sim P(x)$ → Premise

7. $\sim P(a)$ → ES of ⑥

8. $S(a)$ → from ⑤ & ⑦ disjunctive syllogism

9. $(\forall x)(R(x) \to \sim S(x))$ → premise

10. $R(a) \to \sim S(a)$ → U.S of ⑨

11. $\sim R(a)$ → from ⑧ & ⑩

12. $(\exists x)(\sim R(x))$ → EG of ⑪

4. can be conclude $\sim P(a)$ from the statement $(\forall x)(P(x) \to Q(x))$ & $\sim Q(a)$ for some a in the taken domain.

Sol $(\forall x)(P(x) \to Q(x))$

$P(a) \to Q(a)$ for some a

$\sim Q(a)$

$\sim P(a)$

5. Babies are illogical.

- Nobody is despised who can manage a crocodile
- Illogical people as despised
- Hence babies cannot manage crocodiles

Sol

$(\forall x)(B(x) \to I(x))$

$\forall(x)(M(x) \to \sim D(x))$

$\forall(x)(I(x) \to D(x))$

$\therefore (\forall x)(B(x) \to \sim M(x))$

B(x) : x is a baby

I(x) : x is illogical

D(x) : x is despised

M(x) : x can manage crocodile.

1. $(\forall x)(B(x) \to I(x))$ → Premise.

2. $B(a) \to I(a)$ for some a → US of ①

3. $(\forall x)(M(x) \rightarrow \sim D(x))$ → premise

4. $M(a) \rightarrow \sim D(a)$ → US of ③

5. $(\forall x)(I(x) \rightarrow D(x))$ → premise

6. $I(a) \rightarrow D(a)$ → US of ⑤

7. $B(a) \rightarrow D(a)$ → from ② & ⑥ by Hypothetical syllogism

8. $D(a) \rightarrow \sim M(a)$ → contraposition of ④

9. $B(a) \rightarrow \sim M(a)$ → from ⑦ & ⑧ by Hypothetical syllogism

10. $(\forall x)(B(x) \rightarrow \sim M(x))$ → UG of ⑨

∴ Hence conclusion is verified & argument is valid

UNIT - 2

RELATIONS & ALGEBRAIC STRUCTURES

* Suppose A & B be two sets a binary relation from A to B is a subset of $A \times B$ (cartesian product of A & B)

* A relation of "A" means $R \subseteq A \times A$

Ex:- Define a relation R on A $\ni$ $(a,b) \in R$ iff $a | b$, where $a, b \in A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

Properties of relations

1. **Reflexive property**

* Suppose A be a given set and R be a relation on "A" we say that R is reflexive if $\forall a \in A$, $(a,a) \in R$. or aRa

Ex:- Define a relation R on integers (Z) $\ni (a,b) \in R$

$\left.\begin{array}{l} \Leftrightarrow a|b \\ \Leftrightarrow a=b \\ \Leftrightarrow a \leq b \end{array}\right\}$ Reflexive

Ex:- Define a relation R on L = {set of all st. lines in a plane} $\ni (l_1, l_2) \in R \Leftrightarrow l_1 \parallel^{el} l_2$

2. **Symmetric Relation property**

* A relation R on a set A is said to be symmetric if $(a,b) \in R$ then $(b,a) \in R$. $\forall a, b \in A$

Ex: 1. Define a relation R on Z $\ni (a,b) \in R \Leftrightarrow a=b$

2. Define a relation R on L {set of all st lines}

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$\ni (l_1, l_2) \in R \iff l_1 \parallel^{le} l_2$

$\iff l_1 \perp^{(or)} l_2$

3. <u>Anti symmetric property</u>

* A relation R on set A is said to be antisymmetric

if $(a,b) \in R$ & $(b,a) \in R$ then $a = b$.

(or)

if $(a,b) \in R$ & $(b,a) \notin R$ unless $a = b$.

Ex:- 1. Define a relation R on Z^+ such that $(a,b) \in R \iff a/b$

2. Define a relation R on Z^+ Such that $(a,b) \in R \iff a \le b$

3. Define a relation R on $P(A) \ni (A_1, A_2) \in R \iff A_1 \subseteq A_2$

4. <u>Transitive property</u>

* A relation R on a set A is said to be transitive if $(a,b) \in R$ & $(b,c) \in R$ then $(a,c) \in R$.

Ex:-

1. Define a relation R on $Z^+ \ni (a,b) \in R \iff a = b$

2. Define a relation R on $Z^+ \ni (a,b) \in R \iff a/b$

3. Define a relation R on $Z^+ \ni (a,b) \in R \iff a \le b$

4. Define a relation R on $Z^+ \ni (a,b) \in R \iff a < b$

Also set inclusion ($\subseteq$)

<u>Note</u>:-

* Irreflexive means $\forall a \in A$ $(a,a) \notin R$.

Given an example of a relation which is both symmetric and anti symmetric.

1. Define $R = \{(1,1), (2,2)\}$ on $A = \{1,2,3,4\}$

Composition of a Relation :-

* Suppose R be a relation from A to B, and S is a relation from B to C then the composition of R and S is denoted by $S \circ R = \{(a,c) \mid (a,b) \in R \ \& \ (b,c) \in S\}$

Also called product of two relations.

Ex: $R = \{(1,1) (1,4) (2,3) (3,1) (3,4)\}$

$S = \{(1,1) (1,2) (2,3) (3,4) (4,3)\}$

$S \circ R = \{(1,1) (1,4) (1,3) (2,1) (2,4) (4,1) (4,4)\}$

* Suppose R be a relation on A then, the powers of R. is defined recursively follows $R^{n+1} = R \circ R^n$

* Suppose R be relation on A then the inverse of R is denoted by R^{-1} and defined as

$R^{-1} = \{(b,a) \mid (a,b) \in R\}$

The compliment of a relation is denoted by $\bar{R}$ and defined as $\bar{R} = \{(a,b) \mid (a,b) \notin R\}$

* How many relations are possible on a set A on n elements $= 2^{n^2}$

$|A| = n$

$|A \times A| = n^2 \quad |P(A \times A)| = 2^{n^2}$

* How many reflexive relations possible $= 2^{n(n-1)}$

* How many symmetric relations possible $= 2^{n\left(\frac{n+1}{2}\right)}$

* Anti symmetric relations $= 2^n \cdot 3^{n\left(\frac{n-1}{2}\right)}$

* Asymmetric relations $= 3^{n\left(\frac{n-1}{2}\right)}$

* Irreflexive relations $= 2^{n(n-1)}$

* Reflexive & symmetric $= 2^{n(n-1)/2}$

Problems

1. Suppose $R = \{(x,y) \mid (x+y=10)\}$. where $x, y \in \{1, 2, \ldots\}$

represent R as ordered pairs and properties of R.

Sol

$R = \{ (1,9)\ (2,8)\ (3,7)\ (4,6)\ (5,5)\ (6,4)\ (7,3)\ (8,2)$
$(9,1) \}$

Properties

1. Not reflexive

2. ~~Not irreflexive~~

3. Symmetric

4. Not antisymmetric.

5. intransitive.

2. Consider the following relations on the set $A = \{1,2,3,4\}$

$R_1 = \{ (1,1)\ (1,2)\ (2,1) (2,2) (3,4)\ (4,1)\ (4,4) \}$

$R_2 = \{ (1,1),\ (1,2)\ (2,1) \}$

$R_3 = \{ (1,1)\ (1,2) (1,4)\ (2,1)\ (2,2)\ (3,3)\ (4,1)\ (4,4) \}$

$R_4 = \{ (2,1), (3,1)\ (3,2)\ (4,1)\ (4,2)\ (4,3) \}$.

Sol R_1:

1. Not reflexive
2. Not Symmetric
3. Not Antisymmetric
4. Not Transitive

R2

1. Not reflexive
2. Symmetric
3. Not antisymmetric
4. Not transitive.

R3

1. Reflexive

2. Symmetric

3. Antisymmetric

4. Transitive

2. Given an example of a relation which is irreflexive and transitive

Sol: Let "R" be the set of real numbers, the relations "<" and ">" are both irreflexive & transitive

3. Give an example of a relation which is reflexive Antisymmetric & transitive

Sol: The relations "$\leq$" on $\mathbb{Z}$ "$\subseteq$" (set inclusion)

4 Give an example of a relation which is irreflexive Antisymmetric and transitive,

Sol: A relation "$\subset$" proper inclusion on sets is irreflexive, Antisymmetric & transitive.

Representation of relations

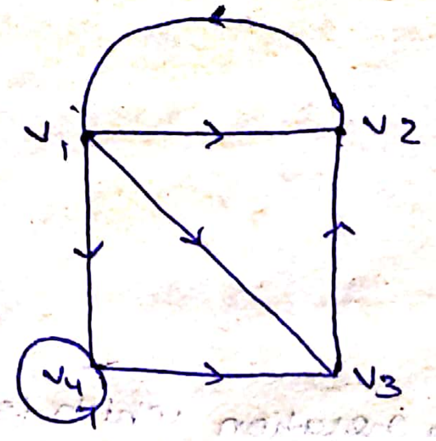
* We can represent the relations in many ways. Two of which are 1. Graphical representation (Diegraph)
   2. Relation matrix (MR)

* A Die graph is a pair $G = (V, E)$ in which V is the set of vertices and E is the set of edges

$\boxed{E \subset V \times V}$

For example

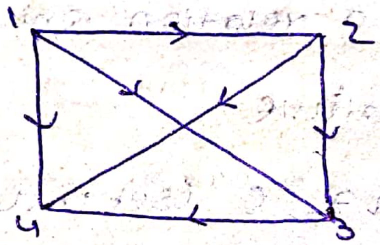
$V = \{v_1, v_2, v_3, v_4\}$



Suppose $A = \{1,2,3,4\}$ define a relation R

A ∋ $R = \{(x,y) \mid x<y\}$ draw the graph of R and give its matrix representation

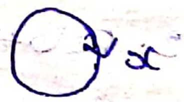
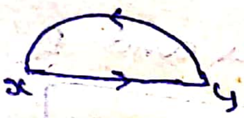
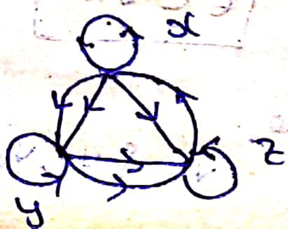
graph.



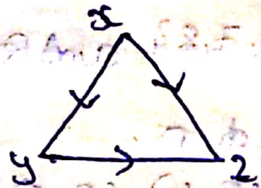
matrix representation:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

NOTE:- $x R x, \forall x \in A$

Relation	Graph
Reflexive	
Symmetric	
Equivalence	

Transitive



Problem

1. Let $A = \{1,2,3,4\}$ and $B = \{b_1, b_2, b_3\}$ consider a relation $R = \{(1,b_2)\ (1,b_3)\ (3,b_2)\ (4,b_1)\ (4,b_3)\}$ Determine the matrix of that relation.

Sol

$$\begin{array}{c|c} & \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

2. Find Relation R on a set $A = \{1,2,3,4\}$ given by

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Sol

$R = \{(1,1), (1,3)\ (2,3)\ (3,1)\ (4,1)\ (4,2)\ (4,4)\}$.

Note:-

* If a relation R is reflexive then all the diagnal entries must be "1"
* If a relation R is Symmetric then the relation matrix is M_R should be symmetric
* If a relation R is antisymmetric then its matrix is such that $a_{ij} = 1$ then $a_{ji} = 0$ for $i \neq j$

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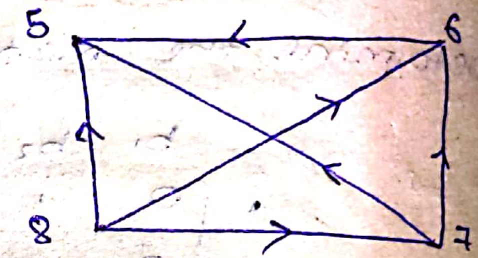
3. Suppose $A = \{5, 6, 7, 8\}$ and $R = \{(x,y) \mid x > y\}$ Draw the graph & matrix of R.

Sol $R = \{(8,7)\ (8,6)\ (8,5)\ (7,6)\ (7,5),\ (6,5)\}$

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$



Equivalence relation

* A relation R on a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive

ex:- "=" on integers, parallelism b/w lines in a plane

Problem

1. If R is symmetric on a set A then show that $R = R^{-1}$

R is symmetric $\Longleftrightarrow R = R^{-1}$

Sol suppose R is symmetric

let $(x,y) \in R \Rightarrow (y,x) \in R$ ($\because$ R is symmetric)

$\Rightarrow (y,x) \in R^{-1}$ (by def of R^{-1})

$\therefore R = R^{-1}$

converse, suppose that $R = R^{-1}$

let $(x,y) \in R \Rightarrow (x,y) \in R^{-1}$

$\Rightarrow (y,x) \in R$

$\therefore$ R is symmetric

2. If R is symmetric on A then P.T $R \cup R^2$ is also symmetric

Sol suppose R is symmetric

let $(x,y) \in R \cup R^2$

$\Rightarrow (x,y) \in R$ or $(x,y) \in R^2$

If $(x,y) \in R \Rightarrow (y,x) \in R$ ($\because R$ is symmetric)

$\Rightarrow (y,x) \in R \cup R^2$

otherwise, $(x,y) \in R^2$

then $\exists\ z \in A \Rightarrow (x,z) \in R$ & $(z,y) \in R$.

$\Rightarrow (z,x) \in R)$ & $(y,z) \in R$ ($\because R$ is symmetric)

$\Rightarrow (y,x) = R^2$

$\Rightarrow (y,x) \in R \cup R^2$

$\therefore R \cup R^2$ is symmetric.

3. Let "A" is the set of rational numbers. define a relation "R" on "A" such that $(a,b) \in R \Leftrightarrow (a-b)$ is an integer $\forall a,b \in A$. S.T 'R' is an equivalence relation.

Sol R is reflexive

Since $a-a=0$, which is an integer

$(a,a) \in R \quad \forall a \in A$

$\therefore$ R is Reflexive

R is symmetric

~~xxxxx xx~~ let $(x,y) \in R \Rightarrow x-y$ is an integer

$\Rightarrow y-x$ is also an integer

$\Rightarrow (y,x) \in R$

$\therefore$ R is symmetric.

R is transitive

let (x,y) & $(y,z) \in R$

$\Rightarrow x-y$ & $y-z$ are an integer

$\Rightarrow x-y+y-z$ is also an integer

$\Rightarrow (x,z) \in R$.

4. Suppose R_1 & R_2 be the two equivalence relations on "A" then (i) P.T $R_1 \cap R_2$ is an equivalence relation of A

(ii) $R_1 \cup R_2$ is reflexive, symmetric but not transitive.

Sol

(i) Suppose R_1 & R_2 be the equivalence relations on A

$R_1 \cap R_2$ is reflexive since $(a,a) \in R_1$ & $(a,a) \in R_2$ $\forall a \in A$

($\because R_1, R_2 \rightarrow$ reflexive)

$(a,a) \in R_1 \cap R_2$, $\forall a \in A$

$\therefore R_1 \cap R_2$ is reflexive

$R_1 \cap R_2$ is symmetric

Let $(x,y) \in R_1 \cap R_2$

$\Rightarrow (x,y) \in R_1$ & $(x,y) \in R_2$

$\Rightarrow (y,x) \in R_1$ & $(y,x) \in R_1$).

($\because R_1, R_2 \rightarrow$ ~~reflexive~~ symmetric)

$\Rightarrow (y,x) \in R_1 \cap R_2$

$R_1 \cap R_2$ is transitive

Let $(x,y) \in R_1 \cap R_2$ & $(y,z) \in R_1 \cap R_2$

$(x,y) \in R_1$ & $(x,y) \in R_2$ and

$(y,z) \in R_1$ & $(y,z) \in R_2$

$(x,z) \in R_1 \cap R_2$

$\therefore R_1 \cap R_2$ is transitive.

(ii) Suppose R_1 & R_2 be the equivalence relation on A

$R_1 \cup R_2$ is Reflexive

since $(a,a) \in R_1$ & $(a,a) \in R_2$ $\forall x \in A$

$(\because R_1, R_2$ are reflexive)

$(a,a) \in R_1 \cup R_2 \ , \forall a \in A)$

$\therefore R_1 \cup R_2$ is Reflexive.

- $R_1 \cup R_2$ is Symmetric

let $(x,y) \in R_1 \cup R_2$

$(x,y) \in R_1$ & $(x,y) \in R_2$

$(y,x) \in R_1$ & $(y,x) \in R_2$

$(y,x) \in R_1 \cup R_2$ ($\because R_1, R_2$ are symmetric)

$R_1 \cup R_2$ is not transitive

let $(x,y) \in R_1 \cup R_2$ & $(y,z) \in R_1 \cup R_2$

$\Rightarrow (x,y) \in R_1$ & $(x,y) \in R_2$ and

$(y,z) \in R_1$ & $(y,z) \in R_2$

$\therefore$ There is no evidence that $(x,y) \in R_1$ & $(y,z) \in R_1$

$\therefore R_1 \cup R_2$ is not transitive.

5. Suppose $M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ then find the matrix representing (i) $\bar{R}$ (ii) R^{-1} (iii) R^2.

sol:

Partition and covering

* Suppose S be a given set and $A = \{A_1, A_2, \dots A_n\}$ where A_i ; $i = 1, \dots n$ is a subset S and $\bigcup_{i=1}^{n} A_i = S$ then the set A is called a covering of S
* If $A_1, A_2, A_3 \dots A_n$ are mutually disjoint then the blocks in A are called partition to the S

For ex:- $S = \{a, b, c\}$ then consider the following subsets of S $A_1 = \{\{a,b\}, \{b,c\}\} \to C$

$A_2 = \{\{a\} \{a,c\}\}$

$A_3 = \{\{a\}, \{b,c\}\} \to C, P$

$A_4 = \{\{a,b,c\}\}$ $\downarrow$ C $\quad A_5 = \{\{a\}, \{b\}, \{c\}\}$ $\downarrow$ C, P $\quad A_6 = \{\{a\}, \{a,b\}, \{a,c\}\}$ $\downarrow$ C

Equivalence class

* Suppose R be an equivalence relation on a set A. Then the set A will be devided into some classes called equavalence classes
* These are denoted by $[a]$

Note:- Any two equivalence classes are either disjoint or identical

Proof

Suppose $[A]$ and $[B]$ be two equivalence classes on A

Assume $[a] \cap [b] \neq \phi$

Then $\exists$ an element $x \in [a] \cap [b]$

$\Rightarrow x \in [a]$ and $x \in [b]$

$\Rightarrow (a, x) \in R$ and $(b, x) \in R$

$\Rightarrow (a, b) \in R$.

let $c \in [a]$

$\Rightarrow (a,c) \in R$

~~[illegible]~~

$\left.\begin{array}{l}\Rightarrow (c,a) \in R \\ \Rightarrow (a,b) \in R\end{array}\right\} \Rightarrow (c,b) \in R.$

$\therefore c \in [b]$

$[a] \subseteq [b]$ —①

let $d \in [b]$

$\Rightarrow (b,d) \in R$

$\left.\begin{array}{l}(d,b) \in R \\ (b,a) \in R.\end{array}\right\} \Rightarrow (d,a) \in R$

$d \in [a]$

$[b] \subseteq [a]$ —②

From ① & ②

$[a] = [b]$.

<u>Congruence modulo 'm' (R_m)</u>

* Let A be set all the integers and m is fixed +ve integer, we define a relation $R_m \ni (a,b) \in R_m \Leftrightarrow a \equiv b \pmod m$

$\Leftrightarrow m \mid a-b$

$\Leftrightarrow a-b = km$

$k \in \mathbb{Z}$

<u>Problems</u>

1. Congruence modulo m is an equivalence relation on set of integers

(i) <u>R_m is reflexive</u> :-

W.K.T $\forall a \in \mathbb{Z}$; $m \mid a-a$

$\Rightarrow a \equiv a \pmod m$

$\forall a \in \mathbb{Z}$

$\therefore R_m$ is reflexive.

(ii) <u>R_m is symmetric</u>

Let $(a,b) \in R_m \Leftrightarrow a \equiv b$ (mod m

$\Leftrightarrow m | a-b$

$\Leftrightarrow m | b-a$

$\Leftrightarrow b \equiv a \pmod m$

$(b,a) \in R_m$

$\therefore R_m$ is symmetric.

(iii) R_m is transitive

Let $(a,b) \in R_m$ & $(b,c) \in R_m$

$\Rightarrow a \equiv b \pmod m$ & $b \equiv c \pmod m$

$m | a-b$ & $m | b-c$.

$a-b = mk_1$ —— ① $\qquad b-c = k_2 m$ —— ②

From ① + ② we get

$a-c = (k_1+k_2)m$.

i.e $m | a-c$.

$a \equiv c \pmod m$

$(a,c) \in R_m$

$\therefore R_m$ is Equivalence.

Note:- Since R_m is an equivalence relation on b then the observed equivalence from ① & ② we have classes are called congruence class $[0], [1] [-1]$.

Problem

1. Suppose $X = \{1,2,3,4\}$ and $R = \{(1,1)\ (1,4)\ (4,1)\ (4,4)\ (2,2)$ $(2,3)\ (3,2)\ (3,3)\}$. Write the matrix of the relation R and sketch its graph

2. Let $X = \{1,2,\dots 7\}$ and $R = \{(x,y) \mid x-y \text{ is divisible by } 3\}$ Show that R is an equivalence relation and draw the graph of R

Closures of the Relations

1. Reflexive closure

* The reflexive closure of the relation R on a set A is

$R \cup \Delta$; $\Delta = \{(x,x) \mid x \in A\}$.

2. Symmetric closure :-

* Suppose R be a relation on A the symmetric closure of R is denoted by $R \cup R^{-1}$

3. Transitive closure

* The transitive closure of a relation R on a set A is a relation defined as $R^{+} = R \cup R^2 \cup R^3 \cup \dots$

$$= \bigcup_{i=1}^{n} R^i$$

Problem

1. Find the transitive closure of the relation $R = \{(a,a),\ (a,b)\ (b,c),\ (c,d)\ (c,e),\ (d,e)\}$ on $A = \{a,b,c,d,e\}$

Sol

$R = \{(a,a)\ (a,b)\ (b,c)\ (c,d)\ (c,e)\ (d,e)\}$

$R^2 = R \circ R = \{(a,a)\ (a,b)\ (a,c)\ (b,d)\ (b,e)\ (c,e)\}$

$R^3 = R \circ R^2 = \{(a,a)\ (a,b)\ (a,c)\ (a,d)\ (b,e)\ (a,e)\}$

$R^4 = R^2 \cdot R^2 = \{(a,a)\ (a,b)\ (a,c)\ (a,d)\ (a,e)\}$

$R^5 = R \cdot R^4 = \{(a,a)\ (a,b)\ (a,c)\ (a,d)\ (a,e)\}$.

$\therefore R^+ = R \cup R^2 \cup R^3 \cup R^4$.

$= \{(a,a)\ (a,b)\ (a,c)\ (a,d)\ (a,e)\ (b,c)\ (b,d)\ (b,e)\ (c,d)$

$(c,e)\ (d,e)\}$.

2. Find the transitive closure of the relation

$R = \{(1,2)\ (2,1)\ (2,3)\ (3,4)\ (4,1)\}$.

Sol $R^2 = R \cdot R = \{(1,1)\ (1,3)\ (2,2)\ (2,4)\ (3,1)\ (4,2)\}$

$R^3 = R^2 \cdot R = \{(1,2)\ (1,4)\ (2,1)\ (2,3)\ (3,2)\ (4,1)\ (4,3)\}$

$R^4 = R^3 \cdot R^4 = \{(1,1)\ (1,3)\ (2,2)\ (2,4)\ (3,1)\ (3,3)\ (4,2)\ (4,4)\}$

$R^5 = R^4 \cdot R = \{(1,2)\ (1,4)\ (2,1)\ (2,3)\ (3,2)\ (3,4)\ (4,1)\ (4,3)\}$

$R^6 = R^5 \cdot R = \{(1,1)\ (1,3)\ (2,2)\ (2,4)\ (3,1)\ (3,3)\ (4,2)\ (4,4)\}$.

$\therefore R^+ = R \cup R^2 \cup R^3 \cup R^4 \cup R^5$

$= \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,1)\ (2,2)\ (2,3)\ (2,4)\ (3,1)\ (3,2)$
$(3,3)\ (3,4)\ (4,1)\ (4,2)\ (4,3)\ (4,4)\}$.

Warshall's Algorithm

* Aim:- To find the relation matrix of the transitive closure of the given relation R

Input :- M_R

Output :- M_{R^+}

Step 1:- Start with the relation matrix M_0

Step 2:- In the matrix $M_K(i,j) = 1$ if $M_{K-1}(i,j) = 1$

Step 3:- The matrix $M_K(i,j) = 1$ if $M_{K-1}(i,K) = 1$ & $M_{K-1}(K,j) = 1$

Problem

1. Using Warshall's Algorithm find transitive closure of the

relation $R = \{ (a,a)\ (a,b)\ (b,c)\ (c,d)\ (c,e)\ (d,e) \}$

Sol

$$M_0 = \begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \hline \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = M_R \qquad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = M_0$$

$$M_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

~~(d,e)~~
(d,e)
(c,d)
(c,e)

$$M_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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2. Using warshalls algorithm Find the Transitive closure of the relation $R = \{ (a,b)\ (b,c)\ (c,d)\ (d,e)\ (e,d) \}$

Sol

$$M_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

3\. Let R denote a relation on the set of ordered pairs of positive integers such that $(x,y) R (u,v) \iff xv = yu$ S.T R is an equivalence relation.

Sol

i) R is reflexive

Since $xy = yx$

$(x,y) R (x,y) \ \forall (x,y)$

$\therefore$ R is reflexive

ii) R is symmetric

Let $(x,y) R (u,v)$

$\Rightarrow xv = yu$

$\Rightarrow vx = uy \Rightarrow uy = vx$

$\therefore (u,v) R (x,y)$

$\therefore$ R is symmetric

iii) R is Transitive.

Let $(x_1,y_1) R (x_2,y_2)$ and $(x_2,y_2) R (x_3,y_3)$

$x_1y_2 = x_2y_1$ | $x_2y_3 = x_3y_2$

$x_1\cancel{y_2}\cancel{x_2}y_3 = \cancel{x_2}y_1x_3\cancel{y_2}$

$x_1y_3 = x_3y_1$

$(x_1,y_1) R (x_3,y_3)$

$\therefore$ R is Transitive.

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Compatibility and partial ordering

* A relation R on a set A is said to be compatible if it is reflexive and symmetric

obviously all equivalence relations are compatible

consider the example $A = \{ball, bed, dog, let, egg\}$

and Relation $R = \{(x,y) \mid x,y \in A$ and xRy if x and y contain some common letter$\}$

The compatibility is denoted by "$\approx$"

Partial ordering

* A relation R on a set A is said to partial ordering if

(i) R is Reflexive

(ii) R is antisymmetric

(iii) R is transitive.

Ex:- "$\leq$", "$\geq$" "$|$" "$\subseteq$" "$\supseteq$" are examples of partial ordering

* Partially ordered set (Poset)

* A non empty set A together with a partial ordering R is called Poset.

Ex:- $[Z^+, |]$, $[A, \leq]$, $[R, \leq]$ $[P(A), \subseteq]$ are examples of poset.

* Two elements x, y are said to be Compatable if $x \leq y$ or $y \leq x$

Totally ordered set (chain)

* A poset $[P, \leq]$ is said to be a totally ordered set if all elements in P are camparable under $\leq$

Well ordered set

* A poset $(P, \leq]$ is said to be a well Ordered set in which the total ordering $\leq$ is well ordered then that set is called well ordered set.

Hasse diagram

* A finite poset can be represented by a diagram called Hasse diagram.

construct the Hasse diagram for the following poset

(i) $[D_6; |]$ (ii) $[D_8; |]$ (iii) $[D_{12}; |]$ (iv) $[D_{27}; |]$

(v) $[D_{30}; |]$ (vi) $[D_{36}; |]$ (vii) $[D_{216}; |]$ (viii) $[\{2,3,6,12,24,36\}; |]$

(ix) $[P(A); \subseteq]$

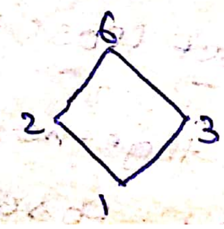
$A = \{1\}$ $A = \{1,2,3\}$

$A = \{1,2\}$ $A = \{1,2,3,4\}$

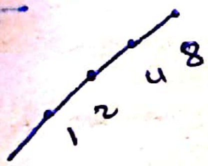
(x) $[\{2,3,6,30,60,120,180,360\}; |]$

Sol

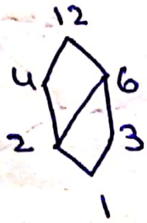
(i) $D_6 = \{1,2,3,6\}$



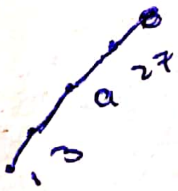
(ii) $D_8 = \{1, 2, 4, 8\}$



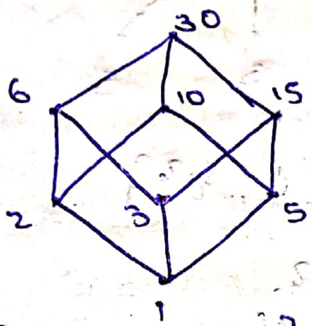
(iii) $D_{12} = \{1, 2, 3, 4, 6, 12\}$

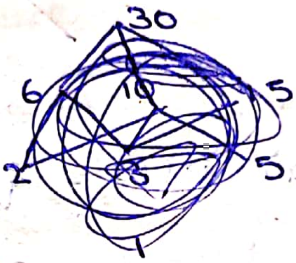


(iv) $D_{27} = \{1, 3, 9, 27\}$

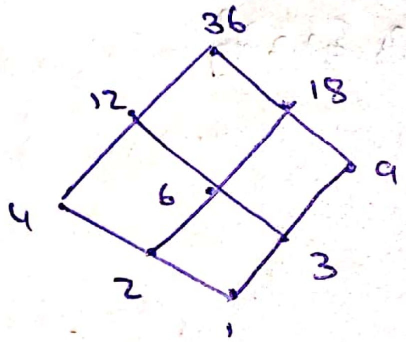


(v) $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

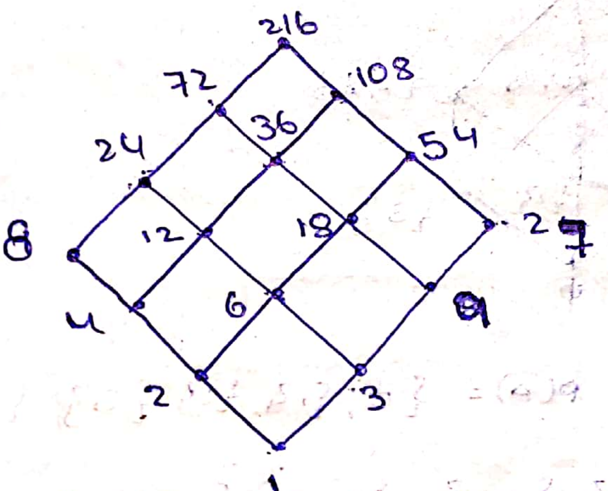




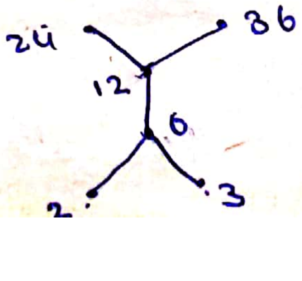
(vi) $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



(vii) $D_{216} = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 54, 72, 108, 216\}$

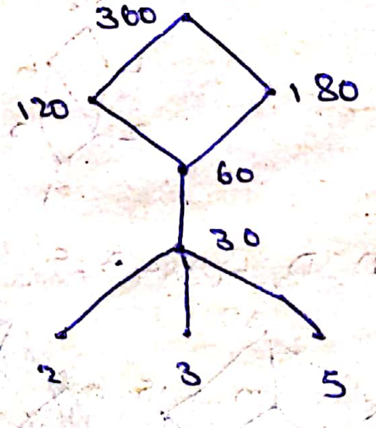


(viii) $[\{2, 3, 6, 12, 24, 36\}; |]$

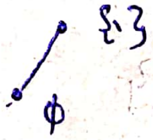


It is an example of poset having 2 minimal elements & maximal element

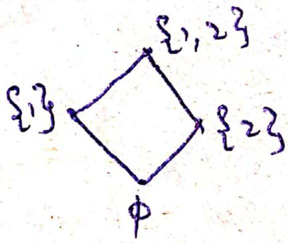
(x) ({2, 3, 5, 30, 60, 120, 180, 360})



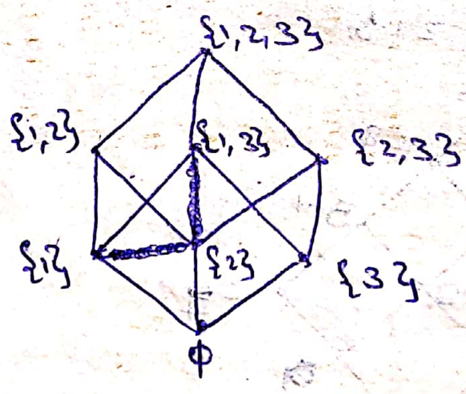
(ix) (a) $P(A) = \{\phi, \{1\}\}$



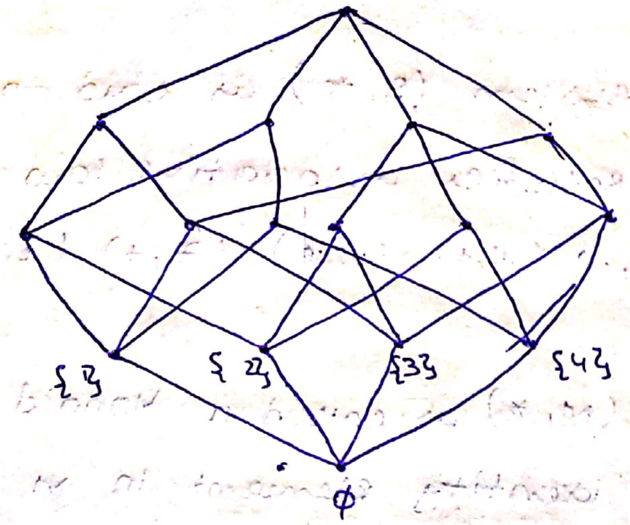
(b) $P(A) = \{\phi, \{1\}, \{2\} \{1,2\}\}$



(c) $P(A) = \{\phi, \{1\} \{2\} \{3\} \{1,2\} \{2,3\} \{1,2,3\}$ $\{1,3\}\}$



(d) $A = \{1,2,3,4\}$ $P(A) = \{\phi, \{1\}, \{2\} \{3\} \{4\} \{1,2\}$ $\{1,3\} \{1,4\} \{2,3\} \{2,4\} \{3,4\}, \{1,2,3\} \{1,2,4\} \{1,3,4\}$ $\{2,3,4\} \{1,2,3,4\}\}$



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Algebraic systems

* A binary operation * on a nonempty set A means * is maping from $A \times A$ to A i.e

$*: A \times A \rightarrow A \ni *(a,b)$ or $a*b \in A \quad \forall (a,b) \in A$

* A non-empty set A together with atleast one binary operation is a Algebraic system. It is denoted by $(A,*)$.

Ex:- $(N,+)$ $(N,\cdot)$ $(W,+)$ $(W,\cdot)$ $(Z,+)$ $(Z,\cdot)$

General properties of Algebraic systems

1. Closure property:- A binary operation $*: A \times A \rightarrow A$ is said to be closed if $\forall a,b \in A$, $a*b \rightarrow A$

2. Associativity:- $\forall$ $a,b,c \in A$, $a*(b*c) = (a*b)*c$.

3. Existence of identity:- There exists an element $e \in A$ Such that $a*e = e*a = a$, $\forall a \in A$ (and e is unique if it exists)

4. Existence of inverse:- For each $a \in A$ then $\exists$ an element $a^{-1} \in A \ni a*a^{-1} = a^{-1}*a = e \in A$.

5. Abelian property :- $\forall a,b \in A \quad a*b = b*a$.

Semigroup:-

* A algebraic system $(S, *)$ is said to be a semi group if $*$ satisfies associative law

Ex:- $(N,+)$ $(N,\cdot)$ $(W,+)$ $(W,\cdot)$ $(Z,+)$ $(Z,\cdot)$

Monoid :-

* A semigroup $(M, *)$ is called a Monoid if w.r.to $*$ thereexists an identity element in M

Ex:- $(W,+)$, $(Z,+)$ $(Z,\cdot)$

Group :

* A monoid $(G, *)$ is said to be a group if for each element in G has a unique inverse in G w.r.to $*$

Problem

1. Test whether the set of all non-zero real real numbers forms an Abelian group w.r.to $*$ defined by $a*b = ab/2$, $\forall a,b \in R-\{0\}$

Sol 1. Closure :- Let $a,b \in R-\{0\}$ $a \neq 0$ $b \neq 0$

Now, $a*b = ab/2 \in R$ & $ab/2 \neq 0$

$\therefore a*b = ab/2 \in R-\{0\}$

2. Associativity : Let $a,b,c \in R-\{0\}$ $a \neq 0$ $b \neq 0$ $c \neq 0$

Now consider $(a*b)*c = \left(\frac{ab}{2}\right)*c$

$$= \frac{\left(\frac{ab}{2}\right)\cdot(c)}{2} = \frac{abc}{4}$$

$$= \frac{a\cdot(bc/2)}{2} = a*\left(\frac{bc}{2}\right)$$

$\therefore$ It obeys Associativity

3. Existence of identity :

Let e be identity in $R-\{0\}$.

then $\forall a \in R-\{0\}$

$$a*e = e*a = a$$

$$\Rightarrow \frac{\cancel{a}e}{2} = \cancel{a}$$

$$\boxed{e=2}$$

$\therefore$ It has an identity element i.e $e=2$

4. Existence of inverse

Let $a \in R-\{0\}$ $a \neq 0$

Suppose b be the inverse of a

$\therefore$ $a*b = b*a = e = 2$

$a*b = 2$

$\frac{ab}{2} = 2$ $ab = 4$ $\boxed{b = 4/a}$

$\therefore b = 4/a \in R-\{0\}$

$\therefore (R-\{0\}, *)$ is a group.

5. Abelian property

Now $a*b = \frac{ab}{2} = \frac{ba}{2} = b*a$ $\forall a,b \in R-\{0\}$

$\therefore (R-\{0\}, *)$ is an abelian group

Q. PT the $(R-\{1\}, *)$ is an abelian group where $*$ is defined by $a*b = a+b-ab$.

Sol 1. Closure: Let $a,b \in R-\{1\}$ $a \neq 1$, $b \neq 1$, $ab \neq 1$

Now $a*b = a+b-ab \in R$

Let $a+b-ab=1$

$b-ab=1-a$

$b(1-a)=1-a$

$\therefore a=1$ or $b=1$

$\therefore a+b-ab$ is $\neq 1$

2. Associativity

Let $a, b, c \in R-\{1\}$ & $a\neq 1, b\neq 1, c\neq 1$.

Now consider $(a*b)*c = (a+b-ab)*c$.

$= a+b+c-ab-ac-bc+abc.$

Now $a*(b*c) = a*(b+c-bc)$

$= a+b+c-bc-ab-ac+abc$

$\therefore (a*b)*c = a*(b*c)$ is true for $\forall a,b,c \in R-\{1\}$

3. Existence of identity

Let e be an identity in $R-\{1\}$

then $\forall a \in R-\{1\}$

$a*e = e*a = a.$

$a*e = a.$

$\cancel{a}+e-ae = \cancel{a}$

$e(1-a)=0 \in R-\{1\}.$

$\boxed{e=0}$

4. Existence of inverse

Let $a \in R-\{1\}$

Let b be the inverse of a in $R-\{1\}$

$\therefore a*b=0$

$a*b = a+b-ab = 0$

$\therefore$ ~~a*b~~ $b - ab = -a$

$b(1-a) = -a.$

$b = \frac{-a}{1-a}$ (as $a \neq 1$)

$\therefore b = -a/_{1-a} = \frac{a}{a-1} \quad \forall a \in R-\{1\}$

5. <u>Abelian property</u>

Let $a*b = a+b-ab$

$= b+a-ba$

$= b*a. \quad \forall a,b \in R-\{1\}$

$\therefore (R-\{1\}, *)$ is an abelian group.

3. Prove that cube roots of unity $(\omega^2, \omega, 1)$ forms an abelian group w.r.to complex number multiplication (we use cayley's table to show it)

Sol Given that $G = \{1, \omega, \omega^2\}$

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

From the table it is clear that all operations are closed in G i.e(,) multiplication of complex numbers is closed in G

W.K.T multiplication of complex numbers satisfies associative law and commutative law on complex numbers, Here $\cdot$ is associative & commutative in G

Since the multiplicative is + Here 1 is the identity in G

From the table $(1)^{-1} = 1 \quad (\omega)^{-1} = \omega^2 \quad (\omega^2)^{-1} = \omega.$

$\therefore (\{1, \omega, \omega^2\}, \cdot)$ is an abelian group

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Note:- The inverse of identity element is itself i.e;

$e * e = e$.

4. P.T fouth roots of unity forms a group w.r.to complex multiplication.

Sol Given that $= \{1, -1, i, -i\}$

	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

From the table it is clear that all the operations are closed i.e; complex multiplication is closed in G

w.k.T complex multiplication satisfies associative law & commutative law. Here (.) satisfies associativity & commutative law. Since multiplicative is 1 then identity is one.

The inverse of $(1)^{-1} = 1$ $(-1)^{-1} = -1$ $(i)^{-1} = -i$ $(-i)^{-1} = i$

$\therefore$ $(\{1, -1, -i, i\}, \cdot)$ is group.

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* Set of all congruence classes modulo n is denoted by Z_n i.e; $Z_n = \{[0], [1], [2], \ldots\ldots [n-1]\}$ also called residue classes.

Addition modulo n $(+_n)$:-

$$[a] +_n [b] = \begin{cases} [a+b] ; & \text{if } a+b < n \\ [r] ; & \text{if } a+b \geq n \end{cases}$$

where $0 \leq r \leq n$

Multiplication modulo n $(\times_n)$

$$[a] \times_n [b] = \begin{cases} [a \times b] ; & \text{if } a \times b < n \\ [r] ; & \text{if } a \times b \geq n \end{cases}$$

where $0 \leq r \leq n$.

Problem

1. S.T $(Z_5, +_5)$ is an abelian group

Sol: From the calay's table.

Here $Z_5 = \{[0], [1], [2], [3], [4]\}$

$+_5$	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

(i) From the it is clear that all operations are closed Z_5. Therefore closure prop is statisfied

(ii) Clearly $+_5$ is associative ie $\forall$ [a] [b] [c]

wehave

$$([a] +_5 [b]) +_5 [c] = a +_5 ([b] +_5 [c])$$

(iii) clearly [0] is the identity

(iv) $\forall [a] \in Z_5 \; \exists [b] \ni$

$$[a] +_5 [b] = 0. \in Z_5$$

i.e; $[1] +_5 [4] = 0.$

$\therefore$ [1] -> [4], [2] -> [3], [3] -> [2], [4] -> [1], [0] -> [0].

are inverse elements.

(v) clearly from table

$$[a] +_5 [b] = [b] +_5 [a]$$

$\therefore (Z_5, +_5)$ is an abelian group)

Note:-

* The inverse of an element in a group is unique
* In a group cancellation laws hold.

$a*b = a*c \Rightarrow \boxed{b=c}$ (left cancellation law)

$b*a = c*a \Rightarrow \boxed{b=c}$ (right cancellation law)

Theorem

1. For any group $(G,*)$ if $a^2 = e$ with $a \neq e$ then prove that G is abelian

(or)

In any group $(G,*)$ if all elements posses their own inverses then that group is abelian

Proof.

Suppose $(G,*)$ be a group with

$a^2 = e \Rightarrow \boxed{a*a=e}$

Here $a^2=e$

$a*a=e$

$(a*a)*a^{-1} = e*a^{-1}$

$a*(a*a^{-1}) = a^{-1}$

$a*e = a^{-1}$

$\boxed{a=a^{-1}}$

$(G,*)$ is a abelian $\forall a \in G$

Let $a,b \in G \Rightarrow a^2 = e \rightarrow a^{-1} = a$

$b^2 = e \rightarrow b^{-1} = b$.

$a*b \in G$

$(a*b)^{-1} = (a*b)$

$b^{-1}*a^{-1} = a*b$.

$\boxed{b*a = a*b}$ $\forall a,b \in G$

converse need not to be true.

2. ~~Suppose~~ $(G,*)$ is an abelian group iff $(a*b)^2 = a^2*b^2$

Proof Part I

suppose $(G,*)$ be an abelian group i.e

$\forall a,b \in G$ we have

$a*b = b*a$ ——— (1)

consider

$$(a*b)^2 = (a*b)*(a*b)$$

$$= a*(b*a)*b$$

$$= a*(a*b)*b$$

$$= (a*a)*(b*b) = a^2*b^2$$

$$\therefore (a*b)^2 = a^2*b^2$$

Part 2 (converse)

In a Group $(G,*)$

$$(a*b)^2 = a^2*b^2 \quad \forall a,b \in G. \quad \text{——} (1)$$

let $a,b \in G$.

$$(a*b)^2 = a^2*b^2$$

$$(a*b)*(a*b) = (a*a)*(b*b)$$

$$a*(b*a)*b = a*(a*b)*b$$

By right cancel

$$a*(b*a) = a*(a*b)$$

By left cancel

$$(b*a) = (a*b) \quad \forall a,b \in G$$

$\therefore (G,*)$ is abelian.

3. S.T set of all invertible elements of a monoid forms a group w.r.to the same operation as that of a monoid.

Proof suppose $(M,*)$ be a monoid

and M' = set of all invertible elements in M

$= \{a \in M / a^{-1} \text{ exists}\}$

Claim $(M',*)$ is a group.

let $a,b \in M' \Rightarrow a^{-1}, b^{-1}$ exists

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Consider $(a*b)*(b^{-1}*a^{-1}) = a*(b*b^{-1})*a^{-1}$

$= a*e*a^{-1}$

$= a*a^{-1} = e.$

Illy

$(b^{-1}*a^{-1})*(a*b) = e$

i.e., $a*b$ is invertible & $*$ is closed in M'

Since $M' \subset M$, $*$ is associative in M

$*$ is associative in M' also

Since $e*e = e$ i.e., e is invertible

$\therefore e \in M'$

for all $a \in M'$, we have

$a*a^{-1} = a^{-1}*a = e \in M'$

$(a^{-1})^{-1} * a^{-1} = a^{-1} * (a^{-1})^{-1} = e \in M'$

$\therefore a^{-1}$ is invertible

$\therefore M'$ is group

4. For any commutative monoid $(M,*)$ S.T set of all idempotent elements of M forms a submonoid

Proof: An element is idempotent if $a^2 = a$.

Suppose $(M,*)$ is commutative monoid. &

$M' = \{a \in M \mid a^2 = a\}$.

Let $a,b \in M' \Rightarrow a^2 = a$; $b^2 = b$

consider $(a*b)^2 = (a*b)*(a*b)$

$= a*(b*a)*b$

$= a*(a*b)*b$

$= (a*a)*(b*b) = a^2*b^2$

$= a*b$

$(a*b)^2 = (a*b)$ & hence $a*b \in M'$

since $M' \subseteq M$ $*$ is associative in M

$*$ is associative in M' also

since $e * e = e$ $\boxed{e^2 = e}$

e is identity in M'

$\therefore (M', *)$ is a monoid and hence a

submonoid to $(M, *)$

Homomorphism & Isomorphism

semi group homomorphism:-

* consider two semi groups $(S, *)$ & (S', Δ) define amapping ϕ from S to S'

$\phi : S \to S' \ni \phi(a * b) = \phi(a) \Delta \phi(b) \quad \forall a, b \in S.$

then ϕ is called a semigroup homomorphism

Note:- If ϕ is one-one then it is called monomorphism.

* If ϕ is onto then ϕ is called Epimorphism
* If ϕ is bijective then ϕ is called an Isomorphism.
* suppose $(M, *)$ & (M', Δ) be two monoids with identities e & e' respec. Define a mapping ϕ from $M \longrightarrow M' \ni \phi(a * b) = \phi(a) \Delta \phi(b)$. Then ϕ is called a Monoid homomorphism
* Suppose $(G, *)$ & (G', Δ) be two groups. Define a mapping ϕ from $G \to G' \ni \phi(a * b) = \phi(a) \Delta \phi(b)$ & $\phi(e) = e'$ & $\phi(a^{-1}) = (\phi(a))^{-1}$
* Suppose $(N, +)$, $(Z_n, +_n)$ be two semi groups Define $\phi : N \longrightarrow Z_n \ni \phi(a) = [a] \ \forall a \in N$.

$$\phi(a+b) = [a+b]$$
$$= [a] +_n [b]$$
$$= \phi(a) \,\Delta\, \phi(b).$$

* Suppose $(N, +)$, $(\bar{N}, +)$ be two semigroups

Define $\phi : N \to \bar{N} \ni \phi(a) = a+1 \ \forall\, a \in N$

$$\phi(a+b) = (a+b)+1$$
$$\neq (a+1)+(b+1)$$
$$\neq \phi(a)+\phi(b)$$

ϕ is not a homomorphism.

Problem

1. Let $(S, *)$ be a semigroup and $(S^S, \circ)$ be also a semigroup where S^S is the set of all mappings from $S \to S = \{f \mid f : S \to S\}$. Then show that $\phi : S \to S^S$ is a semigroup $\ni \phi(a) = f(a)$ homomorphism.

Sol Given that

$$\phi : S \to S^2 \ni \phi(a) = f_a$$

where $f_a : S \to S \ni f_a(x) = a * x$.

Now, $\phi(a*b) = f_{a*b}$ —— ①

Now

$$f_{a*b}(x) = (a*b)*x = a*(b*x) + a* f_b(x)$$

$$f_a(f_b(x)) = (f_a \circ f_b)(x)$$

$$\boxed{f_{a*b} = f_a \circ f_b} \text{ —— ②}$$

from ① & ② $\Rightarrow \phi(a*b) = f_{a*b}$
$$= f_a \circ f_b$$
$$= \phi(a) \circ \phi(b)$$

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2. If $S = N \times N$ where N is natural number and $*$ is a binary operation defined by $(a,b) * (c,d) = (ad+bc, bd)$ & $f : (S,*) \to (Q,+)$ $\ni$ $f(a,b) = a/b$ then ST f is a semigroup homomorphism.

Sol Let $(a,b), (c,d) \in S$

consider $f((a,b)*(c,d)) = f(ad+bc, bd)$

$$= \frac{ad+bc}{bd}$$

$$= \frac{a}{b} + \frac{c}{d}$$

$\therefore$ $f(a,b) + f(c,d)$

3. Suppose $(\omega, +)$ and $(Z_n, +_n)$ be two monoids then show that there exists a homomorphism from $\omega \to Z_n$

Sol Define $\phi : \omega \to Z_n \ni \phi(a) = [n]$, $\forall a \in \omega$

(i) $\phi(a+b) = [a+b] = [a] +_n [b]$

$= \phi(a) +_n \phi(b)$

Now $\phi(0) = 0 \in Z_n$

4. Let $(G,+)$, $(\bar{G}, \cdot)$ be two groups then show that $\phi : G \to \bar{G} \ni \phi(a) = 2^a$, $\forall a \in G$ is a homomorphism

Sol Let $a,b \in G$; $\phi(a+b) = 2^{a+b}$

$= 2^a \cdot 2^b = \phi(a) \cdot \phi(b)$

Now $\phi(0) = 2^0 = 1 \in \bar{G}$

Let $a \in G$ Now $a^{-1} \in G$

consider $\phi(a^{-1}) = \phi(-a) = 2^{-a} = \frac{1}{2^a}$

$= (\phi(a))^{-1}$

$\therefore$ ϕ is a group homomorphism.

Sub Groups

* Suppose $(G, *)$ be a group and H be a non empty subset of G $((\phi \neq) H \subseteq G)$ then H is called subgroup of G if $(H, *)$ is a group

* Suppose $G = \{1, -1, i, -i\}$ & $H_1 = \{i, -i\}$

  $H_2 = \{1, -1\}$ $H_3 = \{i, 1\}$

  H_2 is a sub group.

* $(2\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$

Note: for any group $(G, *)$ the subgroups G & $\{e\}$ which are called trivial sub groups. All other subgroups are called non trivial.

Theorem 1:

* A Non-empty subset H of G is a subgroup of $(G, *)$ & H $\forall a, b \in H \Rightarrow a * b^{-1} \in H$

Proof

Suppose ~~that a~~ $(H, *)$ is a subgroup of $(G, *)$ then clearly $\forall a, b \in H$; $a * b^{-1} \in H$

conversely

Suppose that a non empty sub set H of G having that $\forall a, b \in H \Rightarrow a * b^{-1} \in H$

Claim $H \subset G$

H is subgroup of G

let $a \in H$, $a \in H \rightarrow a * a^{-1} \in H$

$\Rightarrow e \in H$

let $x \in H \subseteq G \Rightarrow \exists e \in G \ni x * e = e * x = x \in G$

Similarly

$e \in H$ is a identity of H.

Since $H \subseteq G$, $*$ is associative in G

let $a \in H$, $e \in H$ $\Rightarrow$ $e * a^{-1} \in H$

$a^{-1} \in H$ [$\because e \in H$]

let $a \in H \Rightarrow a^{-1} \in H \ \forall \ a \in H$

Now $a \in H$, $b \in H \Rightarrow a \in H$, $b^{-1} \in H$

$\Rightarrow a * (b^{-1})^{-1} \in H \Rightarrow a * b \in H$.

Thus $(H, *)$ is group & subgroup of $(G, *)$

<u>Note</u> :- For finite subset the condition for subgroup is $a, b \in H \Rightarrow a ** b \in H$.

1. Intersection of two subgroups is a subgroup.

<u>Sol</u> Let $(a, *)$ be a group & H, K be 2 subgroups

<u>Claim</u> $H \cap K$ is also a group

Since $e \in H$ & $e \in K$ $\Rightarrow e \in H \cap K$

$\Rightarrow H \cap K \neq \phi$

Let $a, b \in H \cap K \Rightarrow a, b \in H$ &

$a, b \in K$.

$a * b^{-1} \in H$ & $a * b^{-1} \in K$

$a * b^{-1} \in H \cap K$.

$\therefore H \cap K$ is a subgroup of $(G, *)$

3. Combinatories

* Basics of counting
* permutations, permutations with repeatation, circular permutations, restricted permutations, restricted combinations, combinations, generating function of permutations and combinations
* Binomial and Multinomial coefficients, theorem, principle of inclusion - exclusion,
* Pigeon hole principle

Elementary combinatories

* X be a non empty set and order of X denotes no. of elements in X. The basic principles of counting are

i) Sum rule - Principle of disjunctive counting
ii) Product rule - principle of sequential counting

Sum rule :-

* Let X be a set and $S_1, S_2, S_3, \ldots S_n$ be n subsets of X such that.

i) S_i is non empty
ii) $S_i \cap S_j = \phi$
iii) $\cup S_i = X$.

Then order of $|X| = |S_1| + |S_2| + \ldots + |S_n|$

Product rule :-

$|X| = |S_1||S_2||S_3| \ldots |S_n|$

problem:

1. How many ways are there to get a sum of 7 & 11 when two distinguishable die are rolled.

Sol: The outcomes are

sum of 7 => (1,6) (2,5) (3,4) (4,3) (5,2) (6,1)

sum of 11 => (5,6) (6,7)

By sum rule req. no. of ways = |S1|+|S2|

= 6+2

= 8.

2. How many ways we can set an even sum

Sol: the out comes are

Sum of 2 => {(1,1)}

sum of 4 => {(1,3) (2,2) (3,1)}

Sum of 6 => {(1,5) (2,4) (3,3) (4,2) (5,1)}

Sum of 8 => {(2,6) (3,5) (4,4) (5,3), (6,2)}

Sum of 10 -> {(4,6) (5,5) (6,4)}

Sum of 12 => {(6,6)

By sum rule req.no.of ways =

|S1| + |S2| + |S3| + |S4| + |S5| + |S6|

= 1+3 + 5 + 5 + 3 + 1

= 18.

2. If 2 distinguishable die are rolled in how many ways they can fall

Sol: 6x6 = 36 By sequential rule for n die 6^n ways

3. Suppose license plates of a certain state req 3 english letters & 4 digits.

a) How many plates if both letters & digits can be repeated

$26^3 \times 10^4$

b) How many plates if only letters can be repeated

$26^4 \times 10 \times 9 \times 8 \times 7$

c) How many plates if only digits can be repeated

$26 \times 25 \times 24 \times 10^4$

4. How many 3 digit numbers are there which are even & no repeated digits

Sol $9 \times 8 + 8 \times 8 \times 4 = 328$

If 3 digit number has last digit zero then 1st digit has 9 chances & 2nd digit has 8 chances => $9 \times 8 \times 1 = 72$

If last digit ends with 2, 4, 6, 8 the it has 4 chances & 1st digit has 8 chances (excluding 0) & 2nd digit has 8 chances => $8 \times 8 \times 4 = 256$

5. In how many ways 10 people be seated in a row such than a certain pair should not be next to each

Sol $10! - 9!\,2!$

6. There are 7 women & 3 men. Find no. of way of seating in a row if all men must stand next to each

Sol $8!\,3!$

7. How many bit strings of length 9

Sol 2 2 2 2 2 2 2 2 2 => 2^9

8. A student can choose a computer project from one of 5 lists. The 5 lists contains 15, 12, 9, 10, 20 projects repec. How many possible projects are there to choose from

Sol $15 + 12 + 9 + 10 + 20 = 66$

A new state flag is to design with 6 vertical stripes using the colours white blue yellow & red such that no two adjacent stripes have same colour

sol

$= 4 \times 3^5$

Permutations & combinations

* An unordered selection of "r" objects from "n" objects is a r-combination. it is denoted by $c(n, r)$ & ^nc_r.

* An ordered selection of "r" objects from "n" objects is a r-permutation. It is denoted by $p(n, r)$ & ^np_r.

For ex:- 1. 3 combinations of set {3.a's, 2 b's, 5 c's}

aaa, aab, aac, bba, bbc, ccc, cca, ccb, abc (combinations (selections))

2. 3combinations of {3.a, 2.b, 2.c, 1.d}

aaa, aab, aac, aad, bba, bbc, bbc, cca, ccb, ccd, abc, abd, bcd,

Enumeration of permutations & combinations

* enumerating r-permutations without repetation is equal to $\frac{n!}{(n-r)!} = {}^np_r$

* Enumerating r-combinations without repetation is equal to ${}^nc_r = \frac{n!}{(n-r)!\,r!}$

Problems

1. In how many ways can a hand of 5 cards be selected from 52 cards

Sol $^{52}C_5$

2. How many 5 cards contains only hearts

Sol $^{13}C_5$

3. How many 5 cards contains 2 club & 3 hearts

Sol $^{13}C_2 \cdot {}^{13}C_3$

4. How many 5 cards contains 3 kings & 2 aces

Sol $^{4}C_3 \cdot {}^{4}C_2$

5. How many commeties of 5 or more can be choosen from 9 people

Sol $^{9}C_5 + {}^{9}C_6 + {}^{9}C_7 + {}^{9}C_8 + {}^{9}C_9$

6. There are 30 FM & 35 M in junior class. There are 25 females & 20 males in senior class. In how many ways can a commity of 10 be choosen so that there are 5 females & 3 juniors

Sol

Juniors M	Juniors F	Seniors M	Seniors F	No. of ways
35	30	20	25	
3	0	2	5	$^{35}C_3 \cdot {}^{30}C_0 \cdot {}^{20}C_2 \cdot {}^{25}C_5$
2	1	3	4	$^{35}C_2 \cdot {}^{30}C_1 \cdot {}^{20}C_3 \cdot {}^{25}C_4$
1	2	4	3	$^{35}C_1 \cdot {}^{30}C_2 \cdot {}^{20}C_4 \cdot {}^{25}C_3$
0	3	5	2	$^{35}C_0 \cdot {}^{30}C_3 \cdot {}^{20}C_5 \cdot {}^{25}C_2$

Req. No. of ways : $^{35}C_3 \cdot {}^{30}C_0 \cdot {}^{20}C_2 \cdot {}^{25}C_5 + {}^{35}C_2 \cdot {}^{30}C_1 \cdot {}^{20}C_3 \cdot {}^{25}C_4 +$

$^{35}C_1 \cdot {}^{30}C_2 \cdot {}^{20}C_4 \cdot {}^{25}C_3 + {}^{35}C_0 \cdot {}^{30}C_3 \cdot {}^{20}C_5 \cdot {}^{25}C_2$

7. There are 21 consonents & 5 vowels in eng alphabet consider only 8 letter words using 3 diff vowels & 5 different consonents

i. how many words can be formed - $^{21}C_5 \cdot {}^5C_3 \cdot 8!$

ii. how many words contain a - $^{21}C_5 \cdot {}^4C_2 \cdot 8!$

iii. how many contain a & b - $^{20}C_4 \cdot {}^4C_2 \cdot 8!$

iv. How many contain a, b, c - $^{19}C_3 \cdot {}^4C_2 \cdot 8!$

v. How many begin with a & end with b - $^{20}C_4 \cdot {}^4C_2 \cdot 6!$

vi. How many begin with b & end c - $^{19}C_3 \cdot {}^5C_3 \cdot 6!$

8. A farmer buys 3 cows, 8 pigs, 12 chickens, from a man who have 9 cows, 25 pigs & 100 chickens how many choices does farmer have

Sol $^9C_3 \cdot {}^{25}C_8 \cdot {}^{100}C_{12}$.

Enumerating combinations & permutations with repetations

* Let U(n, r) denote r permutations of n objects with unlimited repeatations

$\boxed{U(n, r) = n^r}$

Problem

1. There are 25 T/F questions in an examination How many different ways can be choose if one can choose to leave question blank.

Sol 3^{25}

2. How many binary sequences are there of len 15

Sol 2^{15}

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3. The result of 50 football games are to be pred. how many diff forcast (win, loose, tie) can co.. exactly 28 correct results

Sol $^{50}C_{28} \cdot 2^{22}$

First choose 28 correct results in $^{50}C_{28}$ ways each of the remaining 22 games has 2 wrong for.. Thus there are $^{50}C_{28} \cdot 2^{22}$ forecasts with exactly 28 correct results.

* $v(n, r)$ → The no. of r combinations with unlimited repetation.

$v(n, r) = c(n-1+r, r)$ or $c(n-1+r, n-1)$

= The no. of combinations of n distinct objects with unlimited repetations.

= The no. of ways of placing r similar balls into n numbered boxes

= The no. of non negative integral solutions $x_1 + x_2 + x_3 + \dots x_n = r$

= The no. of binary numbers with n-1 1's & r 0's

Problem

1. Find the no. of 4 combinations of the set $\{\infty.a_1, \infty.a_2, \infty.a_3, \infty.a_4, \infty.a_5\}$

Sol Here $n = 5$ $r = 4$.

Req. no. of ways $= v(n, r)$

$= v(5, 4)$

$= c(5-1+4, 4)$

$= c(8, 4)$

2. The number of 3 combinations of 5 objects with unlimited repetitions

Sol $r = 3 \quad n = 5.$

Req. no. of ways $= v(5,3)$

$= C(5-1+3, 3)$

$= C(7, 3)$

$= {}^7C_3 = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$

3. Find the no. of non negative integral solutions of eq $x_1 + x_2 + x_3 + x_4 + x_5 = 50.$

Sol $r = 50, \quad n = 5.$

Req. no. of ways $= v(5, 50)$

$= C(5-1+50, 50)$

$= C(54, 50)$

$= {}^{54}C_{50}$

4. Find the no. of ways of placing 10 similar balls into 6 boxes

Sol $n = 6 \quad r = 10.$

Req. no. of ways $= v(6,10)$

$= C(15,10)$

$= {}^{15}C_{10}$

5/9/19

5. Find the no. of binary numbers with 10 1's & 5 0's

Sol here $n-1 = 10 \Rightarrow n = 11$

$r = 5.$

Req. no. of ways $= v(11, 5)$

$= C(15, 5)$

$= {}^{15}C_5.$

6. How many different outcomes are possible by tossing 10 similar coins

Sol Here $n = 2$

$r = 10$.

req no. of way $= v(2, 10)$

$= C(11, 10)$

$= {}^{11}C_{10}$

$= 11$.

H	T
10	0
9	1
8	2
7	3
6	4
5	5
4	6
3	7
2	8
1	9
0	10.

7. How many integral solutions are there to the eq $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ where each $x_i \geq 2$

Sol Let $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ — ①

where each $x_i \geq 2$

Take $y_i = x_i - 2 \geq 0$

$x_i = y_i + 2$

Eq ① $\Rightarrow$ $y_1 + 2 + y_2 + 2 + y_3 + 2 + y_4 + 2\ y_5 + 2 = 20$

$y_1 + y_2 + y_3 + y_4 + y_5 = 10$ — ②

where each $y_i \geq 0$

Here $n = 5$, $r = 10$

req. no. of ways $= v(5, 10)$

$= c(14, 10)$

$= {}^{14}C_{10}$

8. How many integral solutions are there for $x_1 + x_2 + x_3 + x_4 + x_5 = 20$. where $x_1 \geq -3$, $x_2 \geq 0$, $x_3 \geq 4$, $x_4, x_5 \geq 2$

Sol Here -3 balls given to 1st box means increasing the total no. of balls $= 23$.

let $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ — ①

$x_1 \geq -3$, $x_2 \geq 0$, $x_3 \geq 4$, $x_4, x_5 \geq 2$

$\therefore$ $y_1 = x_1 + 3 \geq 0$

$y_2 = x_2 \geq 0$

$y_3 = x_3 - 4 \geq 0$

$y_4 = x_4 - 2 \geq 0$ $y_5 = x_5 - 2 \geq 0$

then Eq ① $\Rightarrow$ $y_1 - 3 + y_2 + y_3 + 4 + y_4 + 2 + y_5 + 2 = 20$

$y_1 + y_2 + y_3 + y_4 + y_5 = 15$

where $y_i \geq 0$.

here $r = 15$ $n = 5$

req. no of solutions = $v(5, 15) = c(5-1+15, 15)$

$= c(19, 15)$

$= {}^{19}c_{15}$

** Q. Determine the no. of non-negative integral solution for the eq $x_1 + x_2 + x_3 + x_4 = 32$

where $x_1, x_2 \geq 5$; $x_3, x_4 \geq 7$

Sol

$x_1, x_2 \geq 5$ $x_3, x_4 \geq 7$.

$y_1 = x_1 - 5$ $y_3 = x_3 - 7$

$y_2 = x_2 - 5$ $y_4 = x_4 - 7$

$y_1 + 5 + y_2 + 5 + y_3 + 7 + y_4 + 7 = 32$

$y_1 + y_2 + y_3 + y_4 = 8$

$y_i \geq 0$.

$n = 4$ $r = 8$

No. of solutions = $v(4, 8)$

$= c(11, 8)$

${}^{11}c_8$

10. Enumerate the no. of non-negative integral solutions to the inequality $x_1+x_2+x_3+x_4+x_5 \le 19$ each $x_i \ge 0$.

Sol Let $x_1+x_2+x_3+x_4+x_5 = 1$.

Here $n=5, r=1$

No. of solutions $= V(5,1)$

$= C(5,1) = 5.$

Let $x_1+x_2+x_3+x_4+x_5 = 2$

Here $n=5, r=2$

No. of solutions $= V(5,2)$

$= C(6,2)$

⋮

Let $x_1+x_2+x_3+x_4+x_5 = 19$

Here $n=5, r=19$

No. of solutions $= V(5,19)$

$= C(23,19)$

Total no. of solutions $= C(5,1) + C(6,2) + C(7,3) + C(8,4)$

$+ \dots + C(23,19)$

Enumerating permutations with constrained repetitions

* Suppose that we are given a particular selection of n objects where there are some repetations then enumeration of n-permutations of those constrained repetations is given by.

$$P(n; q_1, q_2, \dots, q_t) = \frac{n!}{q_1!\,q_2!\dots q_t!}$$

$$= C(n, q_1)\cdot C(n-q_1, q_2)\cdot C(n-q_1-q_2, q_3)\dots\dots C(n-q_1-q_2-\dots-q_{t-1}, q_t)$$

Problem

where $q_1+q_2+\dots+q_t = n$.

1. Find no. of ways of arranging letters of the word TALLAHASSEE

Sol TALLAHASSEE

req. no. of ways $= \frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = \frac{11!}{3!\,(2!)^3}$

2. MISSISSIPPI

Sol req. no. of ways $= \frac{11!}{4!\,4!\,2!\,1!} = \frac{11!}{(4!)^2\,2!}$

3. In how many ways can 23 different books be given to 5 students so that two students have 4 books each & other 3 have 5 books each

Sol req no. of ways $= 5_{C_2} P(23, 4, 4, 5, 5, 5)$

$= 5_{C_2} \frac{23!}{4!\,4!\,5!\,5!}$

First we choose two students from 5 students who receive 4 books each. then we enumerate.

$= 5C_2\, C(23, 4)\, C(19, 4) \cdot C(15, 5)\, C(10, 5)\, C(5, 5)$

3* Suppose that FLORIDA university have 1 hall with 5 single rooms, 5 double rooms, and 3 rooms for 3 students each (3 triple) In how many ways 24 students can be assigned in 13 rooms

Sol req. no. of ways $= P(24; 1,1,1,1,1, 2,2,2,2,2, 3,3,3)$

$= \frac{24!}{(2!)^5\,(3!)^3}$

Binomial & Multinomial theorems

* Sum of 2 unlike things say (x, y); $x+y$ is called a Binomial
* Sum of 3 unlike things say (x, y, z); $x+y+z$ is Trinomial

Likewise sum of "n" unlike things

$x_1 + x_2 + x_3 + \dots + x_n$ is multinomial.

$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \cdots + {}^nC_r x^{n-r} y^r + \cdots + {}^nC_n y^n.$

Problem

1. obtain coeff of $x^{12}y^{13}$ in $(2x-3y)^{25}$

Sol coeff of $x^{12}y^{13} = {}^{25}C_{13} \cdot (2)^{12} (-3)^{13}$.

Multinomial theorm

* Let n be a positive integers then for all $x_1, x_2, \ldots x_t$ we have $(x_1+x_2+\cdots+x_t)^n$.

$$= \sum P(n; q_1, q_2, q_3, \ldots q_t) \cdot x_1^{q_1} \cdot x_2^{q_2} \cdots x_t^{q_t}$$

where $q_1+q_2+q_3+\cdots+q_t = n$.

Problem

1. Find the coeff of $x_1^4 x_2^5 x_3^6 x_4^3$ in $(x_1+x_2+x_3+x_4)^{18}$

Sol coeff of $x_1^4 x_2^5 x_3^6 x_4^3$

$$= \frac{18!}{4!\,5!\,6!\,3!}$$

2. Find the coeff of $x^3 y^3 z^2$ in $(2x-3y+5z)^8$.

Sol coeff of $x^3y^3z^2 = \frac{8!}{(3!)^2\,2!} (2)^3 (-3)^3 (5)^2$

3. find the coeff of $x^5 y^{10} z^5 w^5$ in $(x-7y+3z-w)^{25}$.

Sol coef $= -\frac{25!}{5!\,10!\,5!\,5!} (+7)^{10} (3)^5$

Principle of Inclusion & Exclusion

Suppose A & B be two non-empty sets then order

$|A \cup B| = |A| + |B| - |A \cap B|$

For the given sets A, B, C

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

Problem

1. Find the no. of binary number that start with 1 & ends with 00 length = 8

Sol Let A = set binary strings start with 1

B = set binary strings end with 00

$|A| = 2^7$ $|B| = 2^6$.

We wish to count $|A \cup B|$

$|A \cap B| = 2^5$.

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 2^7 + 2^6 - 2^5 = 2^5 (4+2-1)$

$= 5 \times 2^5 = 160$

2. Suppose that 200 faculty can speak french & 50 can speak russian, 100 speak spanish, 20 can speak both french & russian, 60 can speak french & spanish & 35 can speak russian & spanish only 10 can speak all three. How many speak either french, russian & spanish

Sol A - faculty speaking french.

B - faculty speaking russian

C - faculty speaking Spanish

$|A| = 200$ $|B| = 50$ $|C| = 100$

$|A \cap B \cap C| = 10$ $|A \cap B| = 20$ $|A \cap C| = 60$

$|B \cap C| = 35$.

$|A \cup B \cup C| = 200 + 50 + 100 - 20 + 60 - 35 + 10$

$= 360 - 115 = 245$

9/9/19

3. Count the no. of integral solutions to the equation

$x_1 + x_2 + x_3 = 20$ where $2 \le x_1 \le 5$, $4 \le x_2 \le 7$

$-2 \le x_3 \le 9$

Sol GT $x_1 + x_2 + x_3 = 20$ — ①

$2 \le x_1 \le 5$ $4 \le x_2 \le 7$ $-2 \le x_3 \le 9$.

Let $U = \{(x_1, x_2, x_3) / x_1 \ge 2, x_2 \ge 4, x_3 \ge -2\}$

$|U| = c(3-1+20-2-4+2, 3-1)$

$c(18, 2)$

Let $A = \{(x_1, x_2, x_3) / x_1 \ge 6, x_2 \ge 4, x_3 \ge -2\}$

$|A| = c(3-1+20-6-4+2, 3-1)$

$= c(14, 2)$

Let $B = \{(x_1, x_2, x_3) / x_1 \ge 2, x_2 \ge 8, x_3 \ge -2\}$

$|A| = c(3-1+20-2-8+2, 3-1)$

$= c(14, 2)$

Let $C = \{(x_1, x_2, x_3) / x_1 \ge 2, x_2 \ge 4, x_3 \ge 10\}$

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$|C| = C(3-1+20-2-4-10, 3-1) = C(6,2)$

$A \cap B = \{(x_1, x_2, x_3) \mid x_1 \geq 6, x_2 \geq 8, x_3 \geq -2\}$

$|A \cap B| = C(3-1+20-6-8+2, 3-1) = C(8,2)$

let $A \cap C = \{(x_1, x_2, x_3) \mid x_1 \geq 6, x_2 \geq 4, x_3 \geq 10\}$

$|A \cap C| = C(3-1+20-6-4-10, 3-1) = C(2,2)$.

let $B \cap C = \{(x_1, x_2, x_3) \mid x_1 \geq 2, x_2 \geq 8, x_3 \geq 10\}$

$|B \cap C| = C(3-1-2-8-10+20, 3-1\} = C(2,2)$

let $A \cap B \cap C = \{(x_1, x_2, x_3) \mid x_1 \geq 6, x_2 \geq 8, x_3 \geq 10\}$

Such solutions doesn't exist

$|A \cap B \cap C| = 0$.

we wish to count $|\bar{A} \cap \bar{B} \cap \bar{C}|$

$\bar{A} \cap \bar{B} \cap \bar{C} = \overline{A \cup B \cup C} = U - A \cup B \cup C$

$|\bar{A} \cap \bar{B} \cap \bar{C}| = |U| - \{|A \cup B \cup C|\}$

$= C(18,2) - \{C(14,2) + C(14,2) + C(6,2) - C(8,2)$

$- C(2,2) - C(2,2) + 0\}$.

4. How many integral solutions are there to the eq $x_1 + x_2 + x_3 + x_4 = 20$ where $1 \leq x_1 \leq 6$, $1 \leq x_2 \leq 7$, $1 \leq x_3 \leq 8$, $1 \leq x_4 \leq 9$.

Sol Gi $x_1 + x_2 + x_3 + x_4 = 20$

let $U = \{(x_1, x_2, x_3, x_4) \mid x_1 \geq 1, x_2 \geq 1, x_3 \geq 1, x_4 \geq 1\}$

$= C(4-1+20-1-1-1-1, 4-1) = C(19,3)$

let $U = \{(x_1, x_2, x_3, x_4) \mid x_1 \geq 7, x_2 \geq 1, x_3 \geq 1, x_4 \geq 1\}$

$|A| = C(4-1+20-7-1-1-1,\ 4-1) = C(13,3)$

$B = \{(x_1, x_2, x_3, x_4) \mid x_1 \geq 1,\ x_2 \geq 8,\ x_3 \geq 1,\ x_4 \geq 1\}$

$|B| = C(4-1+20-1-8-1-1,\ 4-1) = C(12,3)$

$C = \{(x_1, x_2, x_3, x_4) \mid x_1 \geq 1,\ x_2 \geq 1,\ x_3 \geq 9,\ x_4 \geq 1\}$

$|C| = C(4-1+20-1-9-1-1,\ 4-1) = C(11,3)$

$D = \{(x_1, x_2, x_3, x_4) \mid x_1 \geq 1,\ x_2 \geq 1,\ x_3 \geq 1,\ x_4 \geq 10\}$

$|D| = C(4-1+20-1-1-1-10,\ 4-1\} = C(10,3)$

$A \cap B = \{(x_1, x_2, x_3, x_4) \mid x_1 \geq 7,\ x_2 \geq 8,\ x_3, x_4 \geq 1\}$

$|A \cap B| = C(4-1+20-7-8-1-1,\ 4-1\} = C(6,3)$

$A \cap C = \{(x_1, x_2, x_3, x_4) \mid x_1 \geq 7,\ x_2 \geq 1,\ x_3 \geq 9,\ x_4 \geq 1\}$

$|A \cap C| = C(4-1+20-7-1-9-1,\ 4-1\} = C(5,3)$

$A \cap D = \{(x_1, x_2, x_3, x_4) \mid x_1 \geq 7,\ x_2 \geq 1,\ x_3 \geq 1,\ x_4 \geq 10\}$

$|A \cap D| = C(4-1+20-7-1-1-10,\ 4-1) = C(4,3)$

$B \cap C = \{(x_1, x_2, x_3, x_4) \mid x_1 \geq 1,\ x_2 \geq 8,\ x_3 \geq 9,\ x_4 \geq 1\}$

$|B \cap C| = C(4-1+20-1-8-9-1,\ 4-1) = C(4,3)$

$B \cap D = \{(x_1, x_2, x_3, x_4) \mid x_1 \geq 1,\ x_2 \geq 8,\ x_3 \geq 1,\ x_4 \geq 10\}$

$|B \cap D| = C(4-1+20-1-8-1-10,\ 3) = C(3,3)$

$|C \cap D| = C(4-1+20-1-1-9-10,\ 3) = 0.$

$|A \cap B \cap C| = 0 \quad |A \cap B \cap D| = 0 \quad |A \cap C \cap D| = 0$

$|B \cap C \cap D| = 0 \quad |A \cap B \cap C \cap D| = 0$

we wish to count

$|\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}| = |U| - |A \cup B \cup C \cup D|$

$= C(19,3) - \{C(13,3) + C(12,3) + C(11,3) +$

$C(10,3) - C(6,3) - C(5,3) -$

$C(4,3) - C(3,3)\}$

$= 2$

5 Determine the no. of integers b/w 1 & 500 which are not divisible by 2, 3 & 5.

Sol Let $U = \{x \mid 1 \le x \le 500\}$

suppose A = set of integers which are divisible by 2 b/w 1 & 500

$\therefore |A| = \lfloor 500/2 \rfloor = 250$

suppose B = set of integers b/w 1 & 500 which are divisible by 3

C = set of integers b/w 1 & 500 which divisible by 5

$|B| = \lfloor 500/3 \rfloor = 166$

$|A \cap B| = \lfloor 500/6 \rfloor = 83$

$|C| = \lfloor 500/5 \rfloor = 100$

$|A \cap C| = \lfloor 500/10 \rfloor = 50$

$|B \cap C| = \lfloor 500/15 \rfloor = 33$

$|A \cap B \cap C| = \lfloor 500/30 \rfloor = 16$

we require $|\bar{A} \cap \bar{B} \cap \bar{C}| = |U| - \{|A \cup B \cup C|\}$

$= 500 - \{250 + 166 + 100$

$- 83 - 50 - 33 + 16\}$

$= 500 - 366$

$= 134$

6. In how many way can the letters {4·a, 3·b, 2·c} be arranged so that all the letters of same kind are not in single block

Sol: Total no. of arrangments $(|U|) = \frac{9!}{4!3!2!}$

let A = set of arrangements letters in given set in which all a's are in single block

$|A| = \frac{6!}{3!2!}$

B = set of arrangement letters in given set in which all b's are in 1 block

$|B| = \frac{7!}{4!2!}$

C = set of arranglements of letters in the given set in which all c's are in 1 block

$|C| = \frac{8!}{4!3!}$

$|A\cap B| = \frac{4!}{2!}$ $\quad |A\cap C| = \frac{5!}{3!}$ $\quad |B\cap C| = \frac{6!}{4!}$

$|A\cap B\cap C| = 3!$

∴ No. of req. arrangements = $|U| - |A\cup B|$

$= \frac{9!}{4!3!2!} - \left(\frac{6!}{3!2!} + \frac{7!}{4!2!} + \frac{8!}{4!3!} - \frac{4!}{2!} - \frac{5!}{3!} - \frac{6!}{4!} + 3! \right) = 871$

Pigeonhole principle:

* If (n+1) pigeons are distributed among "n" no. of holes then some hole contains atleast two pigeons.

* If $(2n+1)$ pigeons are distributed in n holes then atleast one hole contains 3 [redacted] pigeons

In general $(kn+1)$ pigeons are distributed in n holes the one hole contains $(k+1)$ pigeons

* If $m_1, m_2, m_3, \dots m_n$ be the +ve integers, if $m_1 + m_2 + \dots + m_n - n + 1$ objects are put into n boxes the either 1^{st} box contains atleast m_1 objects, the 2^{nd} box contains atleast m_2 boxes n^{th} box contains atleast m_n objects.

__Proof__:

let $p : m_1 + m_2 + m_3 + \dots + m_n - n + 1$ = Total no. of obj

q_i : ith box contains atleast "m_i" objects

Now, we need to prove that $q_1 \vee q_2 \vee q_3 \vee \dots q_n$ is true.

If possible assume that $q_1 \vee q_2 \vee \dots q_n$ is false

$\sim(q_1 \vee q_2 \vee \dots q_n)$ is true.

$\sim q_1 \wedge \sim q_2 \wedge \sim q_3 \dots \sim q_n$ is true

$p = m_1 - 1 + m_2 - 1 + m_3 - 1 + \dots + m_n - 1$

$= m_1 + m_2 + m_3 + \dots + m_n - n < p$

$p < p$ ⊗ $\rightarrow$ contradiction

i.e our assumption is false.

$\therefore q_1 \vee q_2 \vee \dots q_n$ is true.

* In a group of 61 people, atleast 6 people were born in the same month.
* In a group of 367 people there must be atleast 1 pair with same birthday
* 401 letters were delivered to 50 appartments then some appartment receive atmost 8 letters

Unit - 4 Recurrence Relations

Generating function.

* suppose $\{a_n\}_{n=0}^{\infty}$ be a sequence, if we assign a symbol $A(x)$ i.e $A(x) = a_0 + a_1x + a_2x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series

* A generating function is a fn which generates all the coefficients of a formal power series.

* Generating functions can be used to solve many types of counting problems such as no. of ways to select & distribute the object of different types and finding the no. of non-negative integral solutions to the given equation and no. of ways to make the change for a dollar using the coins of different denominations.

Properties of Generating function.

* suppose $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$B(x) = \sum_{n=0}^{\infty} b_n x^n.$

then

(i) $A(x) = B(x)$ iff $a_n = b_n$

(ii) $c\,A(x) = \sum_{n=0}^{\infty} c a_n x^n$

(iii) $A(x) \pm B(x) = \sum_{n=0}^{\infty} (a_n x^n \pm b_n x^n)$

(iv) $A(x) \cdot B(x) = \sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n$

$= (a_0 + a_1x + a_2x^2 + \dots)(b_0 + b_1x + b_2x^2 + \dots)$

$= \underbrace{(a_0b_0)}_{P_0} + \underbrace{(a_0b_1 + a_1b_0)}_{P_1}x + \underbrace{(a_0b_2 + a_1b_1 + a_2b_0)}_{P_2}x^2 + \dots$

In general $P_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$

* **Note:** Suppose $A(x) = a_0 + a_3 x^3 + a_4 x^4 + a_8 x^8$

$$B(x) = b_0 + b_4 x^4 + b_5 x^5 + b_8 x^8$$

$$P_8 = a_0 b_8 + a_3 b_5 + a_4 b_4 + a_8 b_0.$$

Now the coefficient of x^8 in the above product $A(x) \cdot B(x)$ is P_8 since $(0,8)$ $(3,5)$ $(4,4)$ $(8,0)$ are the only pairs of exponents (e_1, e_2) of the product whose sum is 8 i.e $\boxed{e_1 + e_2 = 8}$

Therefore the coefficient of x^r in an expansion $A_1(x) \cdot A_2(x) \cdot A_3(x) \cdots A_n(x)$ is the no. of non-negative integral solutions for the equation $\boxed{e_1 + e_2 + e_3 + \cdots + e_n = r}$

Problem

1. Find the generating fn for a_r = non-negative integral solutions of the eq. $e_1 + e_2 + e_3 + e_4 + e_5 = r$ where $0 \le e_1, e_2 \le 3$, $2 \le e_3, e_4 \le 6$ & $1 \le e_5 \le 9$ (e_5 is odd)

Sol

Let $A_1(x) = 1 + x + x^2 + x^3$

$$A_2(x) = 1 + x + x^2 + x^3$$

$$A_3(x) = x^2 + x^3 + x^4 + x^5 + x^6$$

$$A_4(x) = x^2 + x^3 + x^4 + x^5 + x^6$$

$$A_5(x) = x + x^3 + x^5 + x^7 + x^9$$

Generating fn $= A_1(x) \cdot A_2(x) \cdot A_3(x) \cdot A_4(x) \cdot A_5(x)$

$$= (1 + x + x^2 + x^3)^2 (x^2 + x^3 + x^4 + x^5 + x^6)^2 (x + x^3 + x^5 + x^7 + x^9)$$

2. Find Gen fn for a_r = no. of non-negative integral solutions for the eq $e_1 + e_2 + e_3 + e_4 + e_5 \ldots e_n = r$ where $0 \le e_i$ for each i.

Sol for each i, $0 \le e_i$

let $A_1(x) = A_2(x) = A_3(x) \ldots = A_n(x)$

$$A_1(x) = x^0 + x^1 + x^2 + x^3 + \cdots + x^n$$

$$= 1 + x + x^2 + \cdots x^n$$

$$\text{Gen. fn} = A_1(x) \cdot A_2(x) \cdot A_3(x) \cdots A_n(x)$$

$$= (1 + x + x^2 + \cdots + x^n)^n.$$

3. Suppose a_r = no. of non-negative integral solutions for the eq $e_1 + e_2 + e_3 + \cdots + e_n = r$ where $0 \le e_i \le 1$

Sol for each i, $0 \le e_i \le 1$,

let $A_1(x) = 1 + x$

$A_2(x) = 1 + x$

$\vdots$

$A_n(x) = 1 + x.$

$$\text{Gen. fn} = A_1(x) \cdot A_2(x) \cdot A_3(x) \cdots A_n(x)$$

$$= (1+x)^n.$$

4. Find the gen. fn for a_r = no. of ways the sum "r" can be obtained a) when 2 distinguishable dice are rolled b) when 2 distinguishable dice are rolled 1st shows even & 2nd shows odd c) 10 distinguishable dice are ~~rolled~~ rolled, 6 specified dice shows an even number & remaining shows an odd number.

Sol a) $e_1 + e_2 = r;$ $1 \le e_1, e_2 \le 6$

$A_1(x) = A_2(x)$

$A_1(x) = x + x^2 + x^3 + x^4 + x^5 + x^6 = A_2(x)$

Gen fn $= A_1(x) \cdot A_2(x)$

$= (x + x^2 + x^3 + x^4 + x^5 + x^6)^2$

b) $e_1 + e_2 = r$ $1 \le e_1, e_2 \le 6$ $e_1 \to$ even $e_2 \to$ odd

$A_1(x) = x^2 + x^4 + x^6$

$A_2(x) = x + x^3 + x^5$

Gen fn $= A_1(x) \cdot A_2(x)$

$= (x^2 + x^4 + x^6)(x + x^3 + x^5)$

c) $e_1 + e_2 + e_3 + e_4 + \dots e_{10} = r$ $1 \le e_i \le 6$ $e_1, \dots e_6 =$ even

$\Rightarrow A_1(x) = A_2(x) = A_3(x) = A_4(x) = A_5(x) = A_6(x)$

$\Rightarrow A_7(x) = A_8(x) = A_9(x) = A_{10}(x)$

$A_1(x) = x^2 + x^4 + x^6$

$A_7(x) = x + x^3 + x^5$

Gen. fn $= A_1(x) \cdot A_2(x) \cdot \dots \cdot A_{10}(x)$

$= (x^2 + x^4 + x^6)^6 (x + x^3 + x^5)^4$

9/9/1

Problem

1. Solve the eq $e_1+e_2+e_3 = 20$ where $1 \le e_1 \le 5$, $1 \le e_2 \le 6$, $1 \le e_3 \le 7$ using generating fn.

Sol given

$$e_1+e_2+e_3 = 20 \quad \text{---} \; ①$$

using generating function

let $A_1(x) = x+x^2+x^3+x^4+x^5$

$A_2(x) = x+x^2+x^3+x^4+x^5+x^6$

$A_3(x) = x+x^2+x^3+x^4+x^5+x^6+x^7$

Then no. of non-negative sol for eq ① is same as the coeff of x^{20} in $A_1(x) \cdot A_2(x) \cdot A_3(x)$

i.e coeff of x^{20} in $(x+x^2+x^3+x^4+x^5)(x+x^2+x^3+x^4+x^5+x^6)(x+x^2+x^3+x^4+x^5+x^6+x^7)$

$\Rightarrow x^3[(1+x+x^2+x^3+x^4)(1+x+x^2+x^3+x^4+x^5)(1+x+x^2+x^3+x^4+x^5+x^6)]$

Now we req coeff of x^{17} in

$(1+x+x^2+x^3+x^4)(1+x+x^2+x^3+x^4+x^5)(1+x+x^2+x^3+x^4+x^5+x^6)]$

$$\Rightarrow \left(\frac{1-x^5}{1-x}\right)\left(\frac{1-x^6}{1-x}\right)\left(\frac{1-x^7}{1-x}\right)$$

$$\Rightarrow (1-x^5)(1-x^6)(1-x^7)\left(\frac{1}{1-x}\right)^3$$

$$\Rightarrow (1-x^5-x^6-x^7+x^{11}+x^{12}+x^{13}-x^{18})\sum_{r=0}^{\infty} c(2+r, r)\, x^r$$

$$\Rightarrow (1-x^5-x^6+x^{11}-x^7+x^{12}+x^{13}-x^{18}) \cdot \sum_{r=0}^{\infty} \frac{(r+1)(r+2)}{2}\, x^r$$

coeff of x^{17} is ${}^{19}c_{17} - {}^{14}c_{12} - {}^{13}c_{11} + {}^{8}c_{6} - {}^{12}c_{10}$

$+ {}^{7}c_{5} + {}^{6}c_{4} = 0$

$\therefore$ the coeff of x^{20} is 0.

Recurrence Relation

* It is a formula that relates for every $n \geq 1$ the term of the sequence $A = \{a_r\}_{r=0}^{\infty}$ to one of more of the terms $a_0, a_1, a_2, \dots a_{n-1}$

Ex: 1. $a_n = 5a_{n-1} + 6a_{n-2}$; $\forall n \geq 2$

2. $a_n = 2a_{n-1}^2$

3. $a_n + 5a_{n-2} + 7a_{n-3} = n^2$

How to form a recurrance relation?

Ex:- In how many ways can a person climb up a flight of n steps. if the person can skip atmost one step at a time.

Sol Suppose a_n = no. of ways of climbing a flight of n steps

$a_1 = 1$ means one can proceed 1 step at a time.

$a_2 = 2$ means one can proceed 2 steps at a time

Suppose the person takes only 1 step at one stair then there are a_{n-1} ways to climb the remaining $(n-1)$ steps. If a person took two steps in the 1^{st} stair then there are $n-2$ steps for which there are a_{n-2} ways

$\therefore a_n = a_{n-1} + a_{n-2}$

* $H_n = 2H_{n-1} + 1$ (Recurrance relation for TOH)

* Suppose the no. of bacteria in a colony triples in every hour setput a recurrance relation for no. of bacteria after n hours elapsed.

Sol: suppose a_n = no. of bacteria at the end of n hours

then a_{n-1} = no. of bacteria at the end of (n-1) hours

∴ the suitable recurrance relation is

$a_n = 3a_{n-1}, \forall\, n \geq 1$

Problems

1. Find 1st 5 terms of the seq. defined by following relations and initial conditions

a) $a_n = 6a_{n-1}$; $a_0 = 2$ $\forall, n \geq 1$.

b) $a_n = a_{n-1} + 3a_{n-2}$, $\forall\, n \geq 2$. $a_0 = 1, a_1 = 2$.

Sol a) $n=1$ $a_1 = 6a_0 = 6(2) = 12$

$n=2$ $a_2 = 6(a_1) = 6(12) = 72$

$n=3$ $a_3 = 6(a_2) = 6(72) = 432$

$n=4$ $a_4 = 6(a_3) = 6(432) = 2592$

$n=5$ $a_4 = 6(a_4) = 36(432) = 15552$

Solving linear Recurrance relations

* A linear recurrance relation of const. coeff is of the form $a_n + c_1 a_{n-1} + c_2 a_{n-1} + \cdots c_k a_{n-k} = f(x)$ — (1)

$c_k \neq 0$ $n \geq k$.

1. By substitution (by iteration)

2. using characteristic roots

3. using generating functions

4. Method of undefined coefficients.

Problem (By substitution)

1. Solve the recurrance relation

$a_n = a_{n-1} + n$, $\forall\, n \geq 1$ & $a_0 = 1$.

Sol

$a_0 = 1$

$n=1$ $a_1 = a_0 + 1 = 2$

$n=2$ $a_2 = a_0 + 1 + 2 = 4$.

$n=3 \quad a_3 = a_0 + 1+2+3 = 7$

$\vdots$

$a_n = a_0 + 1 + 2 + \cdots n$

$= 1 + \sum n = 1 + \frac{(n+1)n}{2}.$

$$\boxed{a_n = 1 + \frac{n(n+1)}{2}}$$

2. solve the linear recurrance relation

$a_n = a_{n-1} + n^2, \quad n \geq 1, \; a_0 = 2.$

sol

$a_0 = 2$

$n=1 \quad a_1 = a_0 + 1^2 = 3$

$n=2 \quad a_2 = a_0 + 1^2 + 2^2 = 7$

$n=3 \quad a_3 = a_0 + 1^2 + 2^2 + 3^2 = 16.$

$\vdots$

$\therefore a_n = a_0 + 1^2 + 2^2 + \cdots + n^2$

$= a_0 + \sum n^2 =$

$$\boxed{a_n = 2 + \frac{n(n+1)(2n+1)}{6}.}$$

Problem (using characteristic roots)

1. The characteristic polynomial for the eq ① is

sol $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(x)$

$c(t) = t^k + c_1 t^{k-1} + \cdots + c_k$

The characteristic eq is $c(t) = 0$

$t^k + c_1 t^{k-1} + \cdots + c_k = 0.$

then we get $t = m_1, m_2, m_3, \ldots m_k$

· case i :- If $m_1, m_2, m_3, \ldots m_k$ are distinct.

then $a_n = c_1 m_1^n + c_2 m_2^n + c_3 m_3^n + \cdots + c_k m_k^n$

case ii: If $m_1 = m_2, m_3, m_4, \ldots \ldots m_n$

$$a_n = (c_1 n + c_2)(m_1)^n + c_3 m_3^n + \cdots + c_k m_k^n$$

2. Solve the recurrance relation.

$$a_n = 2a_{n-1} \quad \forall n \geq 1 \quad \& \quad a_0 = 3.$$

Sol Here $a_n = 2a_{n-1} \Rightarrow a_n - 2a_{n-1} = 0$

$c(t) = t^1 - 2t^0$

$= t - 2$

The charac eq $\Rightarrow c(t) = 0$

$t - 2 = 0$

$t = 2.$

$a_n = c_1 2^n$ Given $a_0 = 3$

$3 = c_1$ $\therefore \boxed{c_1 = 3}$

$\therefore a_n = 3(2^n)$

2. Solve the recurrance relation

$a_n = 4a_{n-1} - 4a_{n-2}$ $\forall n \geq 2$ $a_0 = 6$ $a_1 = 8.$

Sol

$a_n - 4a_{n-1} + 4a_{n-2} = 0$ — ①

$c(t) = t^2 - 4t + 4.$

chara eq $\Rightarrow c(t) = 0$

$t^2 - 4t + 4 = 0.$

$t = 2, 2.$

$\therefore a_n = (c_1 n + c_2) 2^n.$

$a_0 = 6.$

$6 = c_2(1)$ $\boxed{c_2 = 6}$

$a_1 = 8$

$8 = (c_1 + 6) 2$ $c_1 + 6 = 4$

$\boxed{c_1 = -2}$

3. Solve $a_n = 7a_{n-1} - 10a_{n-2}$ $\forall\ n \geq 2$ $a_0 = 2$ $a_1 = 1$

Sol $a_n - 7a_{n-1} + 10a_{n-2} = 0$

- $c(t) = t^2 - 7t + 10$.

$c(t) = 0$

$t^2 - 7t + 10 = 0$

$t = 5, 2.$

$a_n = c_1 5^n + c_2 2^n.$

$a_0 = c_1 + c_2 = 2$

$5c_1 + 2c_2 = 1$

$2c_1 + 2c_2 = 4$

$3c_1 = -3$ $\boxed{c_1 = -1}$ $\boxed{c_2 = 3}$

$$\boxed{a_n = -5^n + 3(2^n)}$$

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4. Solve RR $a_n - a_{n-1} - 9a_{n-2} + 9a_{n-3} = 0$ $\forall\ n \geq 3$

$a_0 = 0,\ a_1 = 1,\ a_2 = 2$

Sol $a_n - a_{n-1} - 9a_{n-2} + 9a_{n-3} = 0$

$t^3 - t^2 - 9t + 9 = 0$

$t = 1, -3, 3$

$a_n = c_1(1)^n + c_2(-3)^n + c_3(3)^n.$

$n = 0$

$0 = c_1 + c_2 + c_3$

$n = 1$ $1 = c_1 - 3c_2 + 3c_3$

$n = 2$ $2 = c_1 + 9c_2 + 9c_3$

$c_1 = -1/4$ $c_2 = -1/12$ $c_3 = 1/3.$

$$\therefore\ a_n = \frac{-(1)^n}{4} - \frac{(-3)^n}{12} + \frac{(3)^n}{3}.$$

5. Solve the RR $a_n + 5a_{n-1} + 5a_{n-2} = 0 \ \forall n \geq 2$

$a_0 = 0, \ a_1 = 2\sqrt{5}$

Sol

$a_n + 5a_{n-1} + 5a_{n-2} = 0$ —— (1)

$t^2 + 5t + 5 = 0$

$t = \dfrac{-5 \pm \sqrt{25-20}}{2} = \dfrac{-5 \pm \sqrt{5}}{2}$

$m_1 = \dfrac{-5+\sqrt{5}}{2} \qquad m_2 = \dfrac{-5-\sqrt{5}}{2}$

$a_n = c_1 (m_1)^n + c_2 (m_2)^n$ —— (2)

$a_0 = c_1 + c_2 = 0$

$a_1 = m_1 c_1 + m_2 c_2 = 2\sqrt{5}$

$m_1 c_1 + m_1 c_2 = 0$

$- \ m_1 c_1 + m_2 c_2 = 2\sqrt{5}$

$c_2(m_1 - m_2) = -2\sqrt{5}$

$c_2 \sqrt{5} = -2\sqrt{5}$

$\boxed{c_2 = -2}$

$\boxed{c_1 = 2}$

$\boxed{a_n = 2(m_1)^n - 2(m_2)^n}$

6. Solve the R.R $a_n - 2a_{n-1} + a_{n-2} = 0, \ \forall n \geq 2$

Sol

$a_n - 2a_{n-1} + a_{n-2} = 0$

$t^2 - 2t + 1 = 0$

$(t-1)^2 = 0$

$t = 1, 1$

$\boxed{a_n = c_1 n + c_2}$

7. Find the explicit formula for fibinocci recurrance relation.

Sol W.K.T.

$$f_n = f_{n-1} + f_{n-2} \quad \forall\, n \geq 2$$

$$f_0 = f_1 = 1$$

$$f_n - f_{n-1} - f_{n-2} = 0 \quad \text{--- (1)}$$

charac eq of (1) is

$$c(t) \Rightarrow t^2 - t - 1 = 0$$

$$\Rightarrow \quad t = \frac{1 \pm \sqrt{1+4}}{2}$$

$$m_1 = \frac{1+\sqrt{5}}{2}, \quad m_2 = \frac{1-\sqrt{5}}{2}.$$

$$f_n = c_1 (m_1)^n + c_2 (m_2)^n.$$

$$f_0 = 1 = c_1 + c_2.$$

$$f_1 = 1 = c_1 m_1 + c_2 m_2$$

$$m_1 c_1 + m_1 c_2 = m_1$$

$$m_1 c_1 + m_2 c_2 = 1$$

$$c_2(m_1 - m_2) = (m_1 - 1)$$

$$c_2 \sqrt{5} = \frac{\sqrt{5}-1}{2}$$

$$c_2 = \frac{1}{2} - \frac{1}{2\sqrt{5}}$$

$$c_1 = \frac{1}{2} + \frac{1}{2\sqrt{5}}. = \frac{\sqrt{5}+1}{2\sqrt{5}}$$

$$f_n = \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{\sqrt{5}+1}{2} \right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]$$

Solutions of Recurrence Relations by Generating fn

Equivalent expressions for the generating functions

* Let us take $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series

1. $\sum_{n=k}^{\infty} a_n x^n = A(x) - a_0 - a_1 x - a_2 x^2 \ldots\ldots a_{k-1} x^{k-1}$

2. $\sum_{n=k}^{\infty} a_{n-1} x^n = x[A(x) - a_0 - a_1 x \ldots\ldots a_{k-2} x^{k-2}]$

3. $\sum_{n=k}^{\infty} a_{n-2} x^n = x^2[A(x) - a_0 - a_1 x \ldots\ldots a_{k-3} x^{k-3}]$

4. $\sum_{n=k}^{\infty} a_{n-3} x^n = x^3[A(x) - a_0 - a_1 x \ldots\ldots a_{k-4} x^{k-4}]$

5. $\sum_{n=k}^{\infty} a_{n-k} x^n = x^k [A(x)]$

Problem

1. Solve the R.R $a_n - 2a_{n-1} + a_{n-2} = 0 \quad \forall n \geq 2$ — ① using generating functions.

Sol Let us take

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$ be the formal power series

Now multiplying eq ① with x^n on b.s & take the summation from $n=2$ to ∞

$$\sum_{n=2}^{\infty} a_n x^n - 2\sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$(A(x) - a_0 - a_1(x)) - 2(x(A(x) - a_0)) + (x^2 A(x)) = 0$$

$$A(x)[1 - 2x + x^2] - a_0 - a_1 x + 2x a_0 = 0$$

$$A(x) = \frac{a_0(1-2x)+a_1 x}{1-2x+x^2} = \frac{P(x)}{(1-x)(1-x)}.$$

$$= \frac{A}{1-x} + \frac{B}{(1-x)^2}$$

$$= A\sum_{n=0}^{\infty} x^n + B\sum_{n=0}^{\infty}(n+1)x^n.$$

$$a_n = A + B(n+1)$$

2. Solve the linear recurrence relation.

$a_n - 9a_{n-1} + 20a_{n-2} = 0 \quad \forall n \geq 2; \quad a_0 = -3 \quad a_1 = -10.$

Sol Given

$$a_n - 9a_{n-1} + 20a_{n-2} = 0$$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series

Multiply the given eq bs with x^n and take the summation. from $n=2$ to ∞.

$$\sum_{n=2}^{\infty} a_n x^n - 9\sum_{n=2}^{\infty} a_{n-1}x^n + 20\sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$(A(x) - a_0 - a_1 x) - 9(x(A(x) - a_0)) + 20(x^2 A(x)) = 0$$

$$A(x)[1 - 9x + 20x^2] - a_0 - a_1 x + 9x a_0 = 0$$

$$A(x)[1 - 9x + 20x^2] - a_0(1 - 9x) - a_1 x = 0$$

$$A(x) = \frac{a_0(1-9x) + a_1 x}{(1 - 9x + 20x^2)} = \frac{-3(1-9x) - 10x}{20x^2 - 9x + 1}$$

$$= \frac{17x - 3}{20x^2 - 9x + 1} = \frac{17x - 3}{(1-4x)(1-5x)}$$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{c_1}{1-4x} + \frac{c_2}{1-5x} = c_1\sum 4^n x^n + c_2\sum 5^n x^n$$

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$$= \sum_{n=0}^{\infty} (c_1 4^n + c_2 5^n) x^n.$$

$$a_n = c_1 4^n + c_2 5^n$$

$$-3 = c_1 + c_2$$

$$-10 = 4c_1 + 5c_2$$

$$12 = -4c_1 - 4c_2$$

$$\boxed{2 = c_2} \qquad \boxed{c_1 = -5}$$

$$\boxed{a_n = (-5)4^n + (2)5^n.}$$

3 $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0 \quad \forall n \geq 3.$

$a_0 = 1; \quad a_1 = 0; \quad a_2 = -1$

sol $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$ — ①

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series

Multiply the given eq's with x^n and take the summation from $n=3$ to ∞.

$$\sum_{n=3}^{\infty} a_n x^n - 7\sum_{n=3}^{\infty} a_{n-1} x^n + 16\sum_{n=3} a_{n-2} x^n - 12\sum_{n=3} a_{n-3} x^n = 0.$$

$$(A(x) - a_0 - a_1 x - a_2 x^2) - 7(x(A(x) - a_0 - a_1 x))$$

$$+16(x^2(A(x) - a_0)) - 12x^3 A(x) = 0.$$

$$A(x)[1 - 7x + 16x^2 - 12x^3] - a_0 - a_1 x - a_2 x^2 + 7x a_0 + 7x^2 a_1$$

$$-16x^2 a_0 = 0.$$

$$A(x)[1 - 7x + 16x^2 - 12x^3] - a_0[1 - 7x + 16x^2]$$

$$-a_1[x - 7x^2] - a_2 x^2 = 0$$

$$A(x) = \frac{a_0[1 - 7x + 16x^2] - a_1[x - 7x^2] - a_2 x^2}{1 - 7x + 16x^2 - 12x^3}$$

$$A(x) = \frac{16x^2 - 7x + 1 + x^2}{1 - 7x + 16x^2 - 12x^3} = \frac{17x^2 - 7x + 1}{1 - 7x + 16x^2 - 12x^3}$$

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= \frac{A}{(1-2x)} + \frac{B}{(1-2x)^2} + \frac{C}{1-3x}$$

$$= A\sum (2)^n (x)^n + B\sum (n+1) 2^n x^n + C\sum (3)^n x^n$$

$$a_n = (A)2^n + B(n+1)(2^n) + C(3)^n.$$

$n=0$ $\quad 1 = A + B + C.$

$n=1$ $\quad 0 = 2A + 4B + 3C$

$n=2$ $\quad -1 = 4A + 12B + 9C.$

4. Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 4^n.$

Sol $\quad a_n - 7a_{n-1} + 10a_{n-2} = 4^n$ — ①

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the formal power series

$$\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1}x^n + 10\sum_{n=2}^{\infty} a_{n-2}x^n = \sum_{n=2}^{\infty} 4^n x^n$$

$$(A(x) - a_0 - a_1 x) - 7(x(A(x) - a_0)) + 10x^2 A(x) = \frac{4^2x^2}{1-4x}$$

$$A(x)[1 - 7x + 10x^2] - a_0 - a_1 x + 7x a_0 = \frac{4^2x^2}{1-4x}$$

$$A(x) = \frac{\cancel{a_0 + a_1 x} - P(x)}{1-7x+10x^2} + \frac{4^2x^2}{(1-4x)(1-7x+10x^2)}$$

$$= \frac{(\quad\quad)}{(1-2x)(1-5x)(1-4x)}.$$

$$= \frac{c_1}{1-2x} + \frac{c_2}{1-5x} + \frac{c_3}{1-4x}.$$

$$= \sum_{n=0}^{\infty} c_1 2^n x^n + \sum_{n=0}^{\infty} c_2 5^n x^n + \sum_{n=0}^{\infty} c_3 4^n x^n.$$

$$a_n = c_1(2^n) + c_2(5^n) + c_3(4^n).$$

Put $a_n = c_3 4^n$. in ①

$$c_3 4^n - 7c_3 4^{n-1} + 10c_3 4^{n-2} = 4^n$$

$$c_3 - \frac{7c_3}{4} + \frac{10c_3}{16} = 1$$

$$16c_3 - 28c_3 + 10c_3 = 16.$$

$$-2c_3 = 16$$

$$\boxed{c_3 = -8}$$

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Unit 5 :- Graph Theory

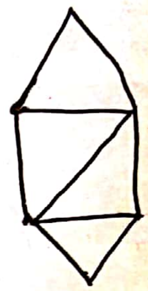
Graph :- A graph G is a pair $G = (V, E)$ where V is set of vertices and E is the set of edges

* Here order of V = order of the graph G

order of E = size of graph G

~~Note:- of a graph of~~

Degree of a Graph

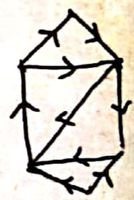


* consider a graph $G = (V, E)$

$V = \{a, b, c, d, e, f\}$

$E = \{\{a,b\}, \{a,c\}, \ldots\}$

* The degree of a vertex in G is the no. of edges incident to or incident from that vertex



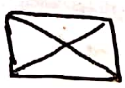
* In a directed graph, degree of a vertex $v = deg^{+}(v) + deg^{-}(v)$

indegree + outdegree

* The minimum of degrees of all vertices of a graph G is denoted by $\delta(G)$ and the maximum of degrees of all vertices in G is denoted by $\Delta(G)$

Note:- $\delta(G) = \Delta(G) = K$ then that graph G is called K-regular graph.

2-regular - 

3-regular - 

4-regular

Sum of degrees Theorem

Suppose $V = \{v_1, v_2, v_3, \ldots v_n\}$ be the vertex set of a non directed graph G then

$$\sum_{i=1}^{n} (\deg(v_i)) = 2|E|$$

* In a directed graph G

$$\sum_{i=1}^{n} \deg^+(v_i) = \sum_{i=1}^{n} \deg^-(v_i) = |E|$$

Proof :-

Suppose G be a directed graph in which

$V = \{v_1, v_2, v_3, \ldots v_n\}$

If the indegree of v_1 is n_1 i.e: there are n_1 edges which are insident to v_1, If the indegree of v_2 is n_2 means there are n_2 edges which are incident to v_2 and so on.

Then, $\sum_{i=1}^{n} \deg^+(v_i) = \deg^+(v_1) + \deg^+(v_2) + \ldots + \deg^+(v_n)$

$= n_1 + n_2 + \ldots + n_n$

$= |E|$

Similarly $\sum_{i=1}^{n} \deg^-(v_i) = |E|$

But in a non-directed graph G

$\sum \deg(v_i) = \sum \deg^+(v_i) + \sum \deg^-(v_i)$

$= |E| + |E|$

$= 2|E|$

Corollary

* In any non-directed graph G there is an even no. of vertices of odd degree

Proof:- Suppose G be a non-directed graph with n vertices. We shall prove that the no. of odd degree vertices in G is even. If possible assume that there is an odd number "k" of vertices of odd degree in G then $(n-k)$ vertices are of even degree. Therefore the sum of k vertices is equal to odd number and the sum of $(n-k)$ vertices is equal to even number. Therefore sum of degrees of all vertices = odd + even

= odd. ⊗

a contradiction to above theorem.

∴ Therefore for any non-directed graph there is even no. of vertices of odd degree.

Problem

1. How many ~~degree~~ ~~vertices~~ edges are there in a graph with 10 vertices each of degree 6.

Sol Since the sum of all the degrees of vertices $= 6 \times 10 = 60$

but we know that $\Sigma \deg(v_i) = 2|E|$

$60 = 2|E|$

$|E| = 30$.

2. Is there any graph with degree sequence $(2, 2, 3, 3, 3, 4, 6)$

Sol NO.

Isomorphisms & Subgraph

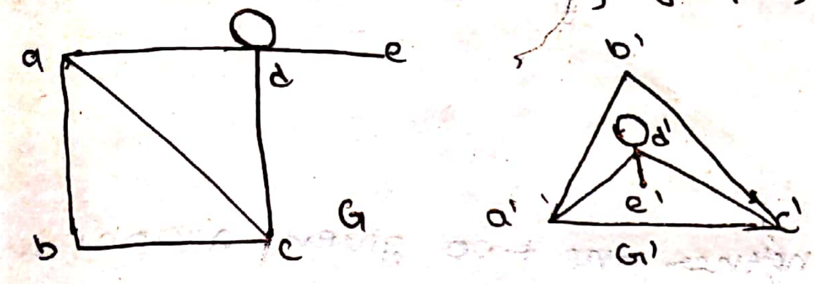
Isomorphism:- Two graphs G, G' are said to be isomorphic to each other if there is a function $f: V(G) \longrightarrow V(G')$ such that

(i) f is one-one

(ii) f is on-to
(iii) for each pair of vertices u, v of G $\{u,v\} \in E(G)$ iff $\{f(u), f(v)\} \in E(G')$

Problem

1. Consider the following graphs



The no. of vertices in G = No. of vertices in G'

The no. of edges in G = No. of edges in G'

The degree sequence in G is (3, 2, 3, 5, 1)

degree sequence in G' is (3, 2, 3, 5, 1)

Now we define a mapping f ~~from vertex~~

$f : V(G) \to V(G')$ $\ni$ $f(a) = a'$

$f(b) = b'$

$f(c) = c'$

$f(d) = d'$

$f(e) = e'$

Now we can redraw G as G' or G' as G

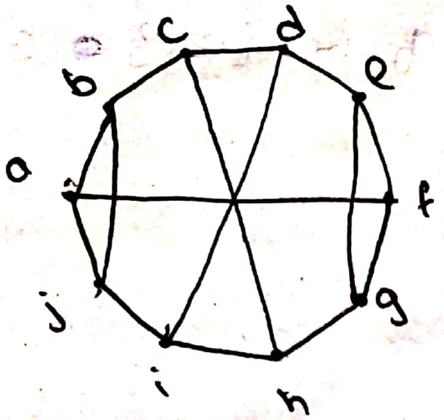
A_G = Adjacency matrix of G.

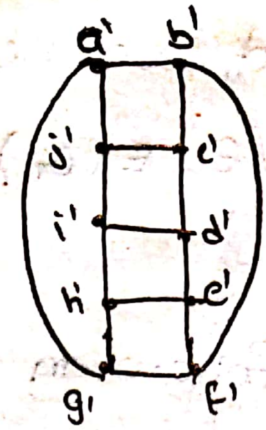
$$A_G = \begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad A_{G'} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$A_G = A_{G'}$

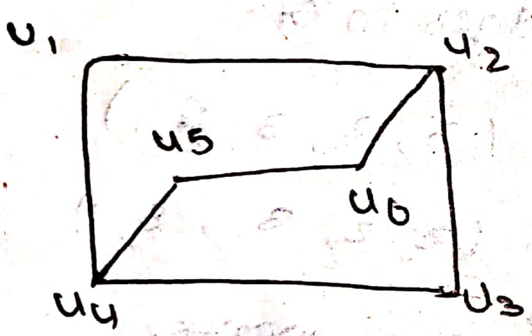
Hence the given ~~matrices~~ graphs are isomorphic to each other

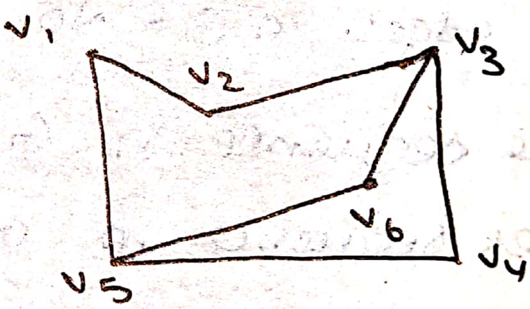
2.





3. verify whether the two given graphs are isomorphic or not





10/10/19

* Suppose $|R|=1$, i.e; the graph is a tree

In a tree $|E| = |V|-1$

$$\therefore |V|-|E|+|R|$$
$$= |V|-|V|+1+1$$
$$= 2.$$

* Let us take a planar graph G with K+1 region. Now we remove an edge which is common to the two regions of G Then we have a graph G' in which $|V| = |V'|$ $|E'| = |E|-1$ $|R'| = |R|-1$

Now the theorm is applicable for the graph G' (by our hypothesis) i.e

$$|V'| - |E'| + |R'| = 2.$$

Now,

$$|V| - |E| + |R|$$
$$= |V'| - (|E'|+1) + (|R'|+1)$$
$$= |V'| - |E'| \cancel{-1} + |R'| \cancel{+1}$$
$$= |V'| - |E'| + |R'|$$
$$= 2.$$

Thus the theorm is applicable

Corollary

* If G be a simple connected planar graph with $|E|>1$ then

i) $|E| \leq 3|V| - 6.$

ii) $\exists$ a vertex "v" $\ni$ degree(v) ≤ 5.

Note:- For any simple planar graph we also have $3|R| \leq 2|E|$

For a complete bipartiete graph $K_{m,n}$ if it is planar $4|R| \le 2|E|$

proof $(|E| \le 3|V| - 6)$:

* Let G be a simple planar graph with $|E|>1$

Then by Eulers formula

$$|V| - |E| + |R| = 2 \quad \text{—— (1)}$$

since. $3|R| \le 2|E|$

$$|R| \le \frac{2|E|}{3}$$

Now, $|V| + |R| \le |V| + \frac{2|E|}{3}$ —— (2)

From (1) $|V| + |R| = |E| + 2$ —— (3)

From (2) & (3)

$$|E| + 2 \le |V| + \frac{2|E|}{3}$$

$$\Rightarrow 3|E| + 6 \le 3|V| + 2|E|$$

$$|E| \le 3|V| - 6.$$

Proof

let us assume ~~xxxxx xxxx~~ allvertexes in G have degree atleast '6'

& $\deg(v_i) = 2|E|$

$$6|V| \le 2|E| \quad \text{—— (1)}$$

since G is simple, planar $|R| \le \frac{2|E|}{3}$.

From. (1) $|V| \le \frac{|E|}{3}$.

$$|V| + |R| \le E \quad \text{—— (3)}$$

By Eulers Formula.

$$|V| + |R| = |E| + 2 \quad \text{—— (4)}.$$

from (3) & (4).

$$|E| + 2 \le E$$

$$2 \le 0 \quad (\times)$$

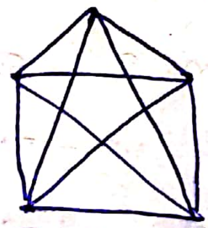
$\therefore$ Our assumption is wrong

* A complete graph K_n is planar iff $n \le 4$

proof

It is enough to show that K_5 is non-planar

Suppose K_5 is planar.



$\therefore$ by Eulers Formula

$|R| = 7$.

Now, for planar graph

$3|R| \le 2|E|$

$3(7) \le 2(10)$

$21 \le 20$ $(\times)$

$\therefore$ our assumption is wrong

$\therefore$ K_5 is non-planar

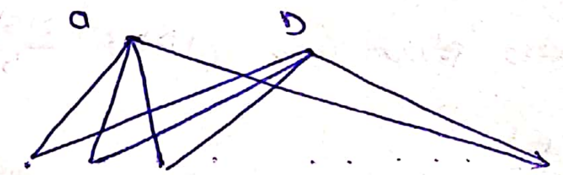
Hence, K_n is planar iff $n = 1, 2, 3, 4$.

* $K_{m,n}$ is planar iff $m \le 2$ & $n \le 2$

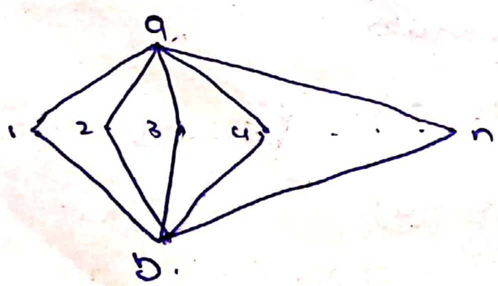
proof

It is sufficient to show that $K_{2,n}$ is planar & $K_{3,3}$ is non-planar.

The graph of $K_{2,n}$ is



we re draw the graph as planar



$\therefore$ $K_{2,n}$ is planar.

suppose $K_{3,3}$ is planar

by Eulers formula

$$|R| = 2 + |E| - |V|$$
$$= 2 + 9 - 6$$
$$= 5.$$

For a complete Bipartite graph we also have

$$4|R| \le 2|E|$$
$$4(5) \le 2(9)$$
$$20 \le 18 \quad (\times)$$

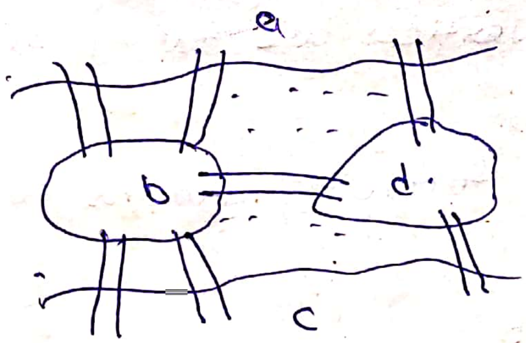
$\therefore$ our assumption is wrong and $K_{3,3}$ is non-planar

$\therefore$ For a complete bipartite graph $K_{m,n}$ is planar iff $m \le 2$ & $n \le 2$

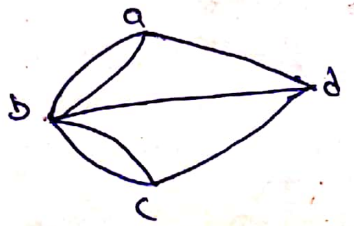
Multigraphs & Euler Circuits

Konigsberg Bridge problem

* It is the unsolved problem of Euler, the problem was to begin at any of 4 landareas, denoted by the letters a, b, c, d, to walk across a root that cross each bridge exactly once and to return to the starting point.



* Now Euler prepared a graph for this problem as

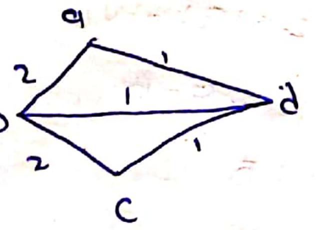


Here all vertices have odd degree and this graph has multiple edges.

Euler prepared the multiplicity matrix as.

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 2 & 0 & 1 \\ b & 2 & 0 & 2 & 1 \\ c & 0 & 2 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{array}$$

and the multiplicity graph is



Finally Euler concluded that such type of route cannot be traced.

<u>Euler path</u>:

* A Euler path in a graph is a path that includes each edge exactly one. and intersects each vertex atleast once.

<u>Eulers Circuit</u>

* A Euler circuit in a multigraph is a ~~path~~ euler path whose end points are identical.

<u>Note</u>:-

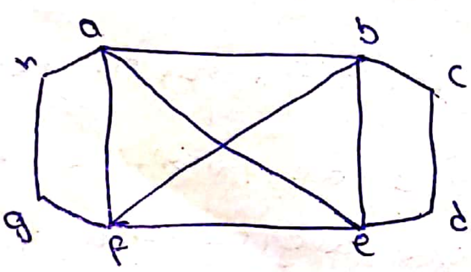
* An undirected graph posses Euler path | Eulerian path iff it is connected and has either 0 or 2 vertices of add degree vertices

* An undirected graph posses Euler circuit iff it is connected and its vertices are all of even degree.

<u>Problem</u>

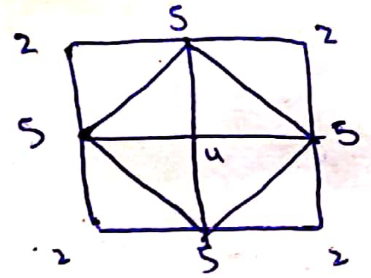
1. check whether the following graphs posses euler path or not.

i)



→ (yes)

ii)



It has 4 odd degree vertices

$\therefore$ It has no Euler path & Euler circuit.

Hamiltonian graph :

* ~~Hamil~~ A graph is said to be hamiltonian if there exists a cycle containing every vertex of graph 'G' such a cycle is called hamiltonian cycle