Game Theory 1

Operations Research (University of Mumbai) chapter 24

THEORY OF GAMES

INTRODUCTION

Life is full of struggle and competitions. A great variety of competitive situations is commonly seen in everyday life. For example, candidates fighting an *election* have their conflicting interests, because each candidate is interested to secure more votes than those secured by all others. Besides such pleasurable activities in competitive situations, we come across much more earnest competitive situations, of military battles, advertising and marketing compaigns by competing business firms, etc.

What should be the bid to win a big Government contract in the pace of competition from several contractors ? Game must be thought of, in a broad sense, not as a kind of sport but as competitive situation, a kind of conflict in which somebody must *win* and somebody must *lose*.

Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game rather than just by the possibilities of several outcomes.

The mathematical analysis of competitive problems is fundamentally based upon the ***'minimax (maximin)*** criterion' of ***J. Von Neumann*** (*called the father of game theory*). This criterion implies the assumption of rationality from which it is argued that each player will act so as **to *'maximize his minimum gain'*** or ***'minimize his maximum loss'***. The difficulty lies in the deduction from the assumption of 'rationality' that the other player will maximize his minimum gain. There is no agreement even among game theorists that rational players should so act. In fact, rational players do not act apparently in this way or in any consistent way. Therefore, game theory is generally interpreted as an "as if" theory, that is, as if rational decision maker (player) behaved in some well defined (but arbitrarily selected) way, such as *maximizing the minimum gain.*

The game theory has only been capable of analysing very simple competitive situations. Thus, there has been a great gap between what the theory can handle and most actual competitive situations in industry and elsewhere. So the primary contribution of game theory has been its concepts rather than its formal application to solving real problems.

Definition 1. *Game is defined as an activity between two or more persons involving activities by each person according to a set of rules, at the end of which each person receives some benefit or satisfaction or suffers loss (negative benefit).*

Definition 2. *The set of rules defines the* ***game****. Going through the set of rules once by the participants defines a* ***play.***

CHARACTERISTICS OF GAME THEORY

There can be various types of games that can be classified on the basis of the following characteristics :

(i) *Chance of strategy* : If in a game, activities are determined by skill, it is said to be a ***game of strategy;*** if they are determined by chance, it is a *game of chance.* In general, a game may involve game of strategy as well as a game of chance. In this chapter, simplest models of games of strategy will be considered.

(ii) *Number of persons* : A game is called an n-person game if the number of persons playing is n. The person means an *individual* or a *group* aiming at a particular objective.

(iii) *Number of activities* : These may be *finite* or *infinite.*

(iv) *Number of alternatives (choices) available to each person* in a particular activity may also be finite or *infinite*. A *finite* game has a finite number of activities each involving a finite number of alternatives, otherwise the game is said to be *infinite.*

(v) *Information to the players about the past activities of other players* is completely available, partly available, or not available at all.

(vi) *Payoff* : A quantitative measure of satisfaction a person gets at the end of each play is called a *payoff.* It is a real-valued function of variables in the game. Let v_i be the payoff to the player P_i, $1 \le i \le n$, in an n-person game. If $\Sigma^n_{i=1} \mathbf{v}_i = 0$, then the game is said to be a *zero-sum game.*

In this chapter, we shall discuss *rectangular games (also called two-person zero-sum games)* only.

Q. 1. Write a short note on characteristics of game theory.
[Rewa M.Sc. (Math) 1993]

2. What is game theory ? List out the assumptions made in the theory of games.
[MTU (MBA) 2012; GNDU 2007; JNTU 2003, 02]

BASIC DEFINITIONS

1. Competitive Game. A competitive situation is called a ***competitive game*** if it has the following four properties :
[JNTU (B. Tech.) 2004, 03; Meerut 2002]

(i) There are finite number (n) of competitors (called players) such that $n \ge 2$. In case $n = 2$, it is called a

two-person game and in case $n > 2$, it is referred to as an ***n*-person game.**

(ii) Each player has a list of finite number of possible activities (the list may not be the same for each player).

(iii) A play is said to *occur* when each player chooses one of his activities. The choices are assumed to be made simultaneously, *i.e.* no player knows the choice of the other until he has decided on his own.

(iv) Every combination of activities determines an outcome (which may be *points, money* or *any thing else* whatsoever) which results in a gain of payments (+ ve, – ve or zero) to each player, provided each player is playing uncompromisingly to get as much as possible. *Negative gain* implies the *loss* of same amount.

2. Zero-sum and Non-zero-sum Games. Competitive games are classified according to the number of players involved, *i.e.* as a *two person game, three person game,* etc. Another important distinction is between *zero-sum games* and *non-zero-sum* games. If the players make payments only to each other, *i.e.* the loss of one is the gain of others, and nothing comes from outside, the competitive game is said to be *zero-sum.*

Mathematically, suppose an *n*-person game is played by *n* players $P_1, P_2, \ldots, P_n$ whose respective pay-offs at the end of a play of the game are $v_1, v_2, \ldots v_n$ then, the game will be called zero-sum if $\sum_{i=1}^{n} v_i = 0$ at each play of the game.
[JNTU (Mech. & Prod.) 2004]

A game which is not zero-sum is called a *nonzero-sum game.* Most of the competitive games are zero-sum games. An example of a non-zero-sum game is the 'poker' game in which a certain part of the pot is removed from the 'house' before the final payoff.

3. Strategy. A strategy of a player has been loosely defined as a rule for decision-making in advance of all the plays by which he decides the activities he should adopt. In other words, a strategy for a given player is a set of rules (programmes) that specifies which of the available course of action he should make at each play. This strategy may be of two kinds :
[JNTU (B. Tech.) 2004, 03; Meerut 2002; IGNOU 2001, 2000, 98, 97]

(i) Pure Strategy : If a player knows exactly what the other player is going to do, a *deterministic* situation is obtained and objective function is to maximize the gain. *Therefore, the pure strategy is a decision rule always to select a particular course of action.* **[Meerut 2002; Madras ME (Struct.) 2000]**

A pure strategy is usually represented by a number with which the course of action is associated.

(ii) Mixed Strategy : If a player is guessing as to which activity is to be selected by the other on any particular occasion, a *probabilistic* situation is obtained and objective function is to maximize the *expected gain.*
[Meerut 2003, 02; Madras ME (Struct.) 2000]

Thus, the mixed strategy is a selection among pure strategies with fixed probabilities.

Mathematically, a mixed strategy for a player with $m\ (\geq 2)$ possible courses of action, is denoted by the set S of m non-negative real numbers whose sum is unity, representing probabilities with which each course of action is chosen. If $x_i\ (i = 1, 2, 3, \ldots, m)$ is the probability of choosing the course i, *then*

$$S = (x_1, x_2, x_3, \ldots, x_m) \quad \ldots(1)$$

subject to : $$x_1 + x_2 + x_3 + \ldots + x_m = 1 \quad \ldots(2)$$

and $$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \ldots, x_m \geq 0. \quad \ldots(3)$$

NOTE It should be noted that if some $x_i = 1$, $(i = 1, 2, \ldots, m)$ and all others are zero, the player is said to use a pure strategy. Thus, the pure strategy is a particular case of mixed strategy.

4. Two-Person, Zero-Sum (or *Rectangular*) Games. A game with only two players (say, *Player A* and *Player B*) is called a *'two-person, zero-sum game'* if the losses of one player are equivalent to the gains of the other, so that the sum of their net gains is zero. **[Delhi B.Sc. (Stat.) 2006; JNTU (B. Tech.) 2003; Madras (MBA) 2000]**

Two-person, zero-sum games, are also called *rectangular games* as these are usually represented by a payoff matrix in rectangular form. **[Meerut (OR) 2003]**

5. Payoff Matrix. Suppose the player *A* has *m* activities and the player *B* has *n* activities. Then a payoff matrix can be formed by adopting the following rules :
[Meerut (OR) 2003; JNTU (B. Tech) 97]

(i) Row designations for each matrix are activities available to player *A*.

(ii) Column designations for each matrix are activities available to player *B*.

(iii) Cell entry 'v_{ij}' is the payment to player *A* in *A*'s payoff matrix when *A* chooses the activity *i* and *B* chooses the activity *j*.

(iv) With a 'zero-sum, two-person game', the cell entry in the player *B*'s payoff matrix will be negative of the corresponding cell entry 'v_{ij}' in the player *A*'s payoff matrix so that sum of payoff matrices for player *A* and player *B* is ultimately zero.

Table 1 The player *A*'s payoff matrix

		Player *B*					
		1	2	...	j	...	n
	1	v_{11}	v_{12}	...	v_{1j}	...	v_{1n}
	2	v_{21}	v_{22}	...	v_{2j}	...	v_{2n}
Player *A*	:	:	:		:		:
	i	v_{i1}	v_{i2}	...	v_{ij}	...	v_{in}
	:	:	:		:		:
	m	v_{m1}	v_{m2}	...	v_{mj}	...	v_{mn}

Table 2 The player *B*'s payoff matrix

		Player *B*					
		1	2	...	j	...	n
	1	$-v_{11}$	$-v_{12}$	...	$-v_{1j}$	...	$-v_{1n}$
	2	$-v_{21}$	$-v_{22}$	...	$-v_{2j}$	...	$-v_{2n}$
Player *A*	:	:	:		:		:
	i	$-v_{i1}$	$-v_{i2}$	...	$-v_{ij}$	...	$-v_{in}$
	:	:	:		:		:
	m	$-v_{m1}$	$-v_{m2}$	...	$-v_{mj}$	...	$-v_{mn}$

NOTE Further, there is no need to write the B's payoff matrix as it is just the –ve of A's payoff matrix in a *zero-sum two-person game*. Thus, if 'v_{ij}' is the gain to A, then '$-v_{ij}$' will be the gain to B.

In order to make the above concepts clear, consider the coin matching game involving two players only. Each player selects either a head H or a tail T. If the outcomes match (H, H or T, T), A wins Re 1 from B, otherwise B wins Re 1 from A. This game is a two-person zero-sum game, since the winning of one player is taken as the losses for the other. Each has his choices between two *pure* strategies (H or T). This yields the following (2×2) payoff matrix to player A.

Table 3

B

A	H	+1	–1
	T	–1	+1

It will be shown later that the optimal solution to such games requires each player to play one pure strategy or a mixture of pure strategies.

Q. 1. State the four properties which a competitive situation should have, if it is to be called a competitive game.

2. What is the problem studied in game theory? **[Delhi (MBA) 2001]**

3. Define :

(*i*) Competitive game

(*ii*) Pure strategies

(*iii*) Mixed strategies

(*iv*) Two-person, Zero-sum (or Rectangular games)

(*v*) Payoff matrix **[IGNOU 2001, 2000]**

4. (*i*) Explain zero-sum two-person game giving suitable example.

(*ii*) What is a zero-sum two-person game ? **[IGNOU 2001, 2000, 98, 97; Agra 92]**

(*iii*) Explain the difference between pure strategy and mixed strategy. **[GBTU (MBA) II Sem. 2012]**

5. Write a note on zero-sum games. **[JNTU (IV B. Tech., I Sem.) Feb. 2007 (Set 1)]**

6. Distinguish between games with iadelle points and games without saddle points. **[JNTU (IV B. Tech., III CS & Engg.) I Sem. 2011]**

MINIMAX (MAXIMIN) CRITERION AND OPTIMAL STRATEGY

The 'minimax criterion of optimality' *states that if a player lists the worst possible outcomes of all his potential strategies, he will choose that strategy to be the most suitable for him which corresponds to the best of these worst outcomes. Such a strategy is called an* ***optimal strategy.***

Example 1. *Consider (two-person, zero-sum) game matrix which represents payoff to the player A. Find the optimal strategy, if any.* [*see* ***Table***]

		B: I	II	III
	I	–3	–2	6
A	II	2	0	2
	III	5	–2	–4

Solution The player A wishes to obtain the largest possible 'v_{ij}' by choosing one of his activities (I, II, III), while the player B is determined to make A's gain the minimum possible by choice of activities from his list (I, II, III). Then the player A is called the ***maximizing player*** and B the ***minimizing player.***

Saddle Point

		B: I	II	III	Row minimum	
	I	–3	–2	+6	–3	
A	II	2	0	2	0	→Maximin Value ($\underline{v}$)
	III	5	–2	–4	–4	
Column maximum		5	0	6		
			Minimax Value ($\bar{v}$)			

If the player A chooses the Ist activity, then it could happen that the player B also chooses his Ist activity. In this case the player B can guarantee a gain of at least – 3 to player A, *i.e.*

$$\min\{-3, -2, 6\} = -3$$

Similarly, for other choices of the player A, *i.e.* II and III activities, B can force the player A to get only 0 and – 4, respectively, by his proper choices from (I, II, III), *i.e.*

$$\min\{2, 0, 2\} = 0 \text{ and } \min\{5, -2, -4\} = -4.$$

The minimum value in each row guaranteed by the player A is indicated by 'row minimum' as shown in the table. The best choice for the player A is to maximize his least gains –3, 0, – 4 and opt II strategy which assures at most the gain 0, *i.e.*

$$\max\{-3, 0, -4\} = \boxed{0}.$$

In general, the player A should try to maximize his least gains or to find out "$\max_i \min_j v_{ij}$".

Player B, on the other hand, can argue similarly to keep A's gain the minimum. He realizes that if he plays his Ist pure strategy, he can loose no more than $5 = \max\{-3, 2, 5\}$ regardless of A's selections. Similar arguments can be applied for remaining strategies II and III. Corresponding results are indicated in the table. by *'column maximum'*. The player B will then select the strategy that minimizes his maximum losses. This is given by the strategy II and his corresponding loss is given by

$$\min\left\{\boxed{5}, \boxed{0}, \boxed{6}\right\} = 0.$$

The player A's selection is called the ***maximin strategy*** and his corresponding gain is called the ***maximin value*** or ***lower value*** ($\underline{v}$) of the game. The player B's selection is called the ***minimax value*** or ***upper value*** ($\bar{v}$) of the game. The selections made by player A and B are based on the so called ***minimax*** (or ***maximin***) ***criterion***. It is seen from the governing conditions that the minimax (upper) value $\bar{v}$ is greater than or equal to the maximin (lower) value $\underline{v}$ (see the *Theorem on p. GMS/4*). In the case where equality holds, *i.e.*

$$\max_i \min_j v_{ij} = \min_j \max_i v_{ij} \quad \text{or} \quad \underline{v} = \bar{v}, \qquad \ldots(1)$$

the corresponding pure strategies are called the *'optimal'* strategies and the game is said to have a ***saddle point.*** It may not always happen as it is clear from Example 2.

NOTE For convenience, the minimum values are shown by 'O' and maximum values by '□' in Table of *Example 1.*

Q. 1. When the competitive situation is called a game ? What is the maximin criterion of optimality ?

2. What is a game in the game theory ? What are the properties of a game ? Explain the best (optimal) strategy on the basis of minimax criterion of optimality.

3. What are the assumptions made in the theory of games?

4. Explain Maxi-Min and Mini-Max principle used in Game Theory.

[JNTU (B. Tech) 2000; Agra 1994]

Example 2. *Consider the following game :*

	$j \backslash i$	B 1	2	3
	1	3	−4	8
A	2	−8	5	−6
	3	6	−7	6.

As discussed in *Example* 1,

$\max_i \min_j v_{ij} = 4, \quad \min_j \max_i v_{ij} = 5$.

Also, $\max_i \min_j v_{ij} < \min_j \max_i v_{ij}$.

Such games are said to be the games ***without saddle point.***

Example 3. *Find the range of values of p and q which will render the entry (2, 2) a saddle point for the following game.*

	Player B		
	2	4	5
Player A	10	7	q
	4	p	6

[Meerut 2007; JNTU (B. Tech.) 2003]

Solution First ignoring the values of p and q determine the *maximin* and *minimax* values of the payoff matrix as given below :

Since the entry (2, 2) is a saddle point, maximin value $\underline{v} = 7$, minimax value $\bar{v} = 7$.

This imposes the condition on p as $p \le 7$ and on q as $q \ge 7$. Hence the range of p and q will be $p \le 7, q \ge 7$.

Player B	B_1	B_2	B_3	Row Min.
Player A A_1	2	$\boxed{4}$	5	2
A_2	$\boxed{10}$	$\boxed{7}$	q	7
A_3	4	p	$\boxed{6}$	4
Column Max.	$\boxed{10}$	$\boxed{7}$	$\boxed{6}$	

Theorem : *Let $\{v_{ij}\}$ be the payoff matrix for a two-person zero-sum game. If $\underline{v}$ denotes the maximin value and $\bar{v}$ the minimax value of the game, then $\bar{v} \ge \underline{v}$. That is,*

$\min_j [\max_i \{v_{ij}\}] \ge \max_i [\min_j \{v_{ij}\}]$. **[Meerut (Stat.) 1990]**

Proof. We have, $\max_i \{v_{ij}\} \ge v_{ij}$ for any j, and $\min_j \{v_{ij}\} \le v_{ij}$ for any i.

Let the above maximum be attained at $i = i^*$ and the minimum be attained at $j = j^*$. So

$$v_{i^*j} \ge v_{ij} \ge v_{ij^*} \qquad \text{for any } i \text{ and } j.$$

This implies that

$$\min_j \{v_{i^*j}\} \ge v_{ij} \ge \max_i \{v_{ij^*}\}, \qquad \text{for any } i \text{ and } j.$$

Hence, $\min_j [\max_i \{v_{ij}\}] \ge \max_i [\min_j \{v_{ij}\}]$ or $\bar{v} \ge \underline{v}$.

REMARK

1. A game is said to be fair, if $\underline{v} = 0 = \bar{v}$.
2. A game is said to be strictly determinable, if $\underline{v} = \bar{v} = v$.

Illustrative Examples

Example 1 *Determine which of the following two-person zero-sum games are strictly determinable and fair. Give optimum strategies for each player in the case of strictly determinable games.*

$(a)\ A \overset{B}{\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}} \qquad (b)\ A \overset{B}{\begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix}}$ **[Madurai (M.Com.) 1997]**

Solution (*a*) The payoff matrix for player *A* is,

Palyer B	B_1	B_2	*Row minima*
Player A A_1	$\boxed{5}$	0	0
A_2	0	$\boxed{2}$	0
Column maxima	$\boxed{5}$	$\boxed{2}$	

The payoffs marked with '0' represent the *minimum* payoff in each row and those marked with '□' represent the *maximum* payoff in each column of the payoff matrix. The largest component of row minima represents the maximin value ($\underline{v}$) and the smallest component of column maxima represents *minimax value* ($\bar{v}$).

Thus, we have

$$\underline{v} = 0 \quad \text{and} \quad \bar{v} = 2.$$

Since, $\underline{v} \ne \bar{v}$, the game is *not* strictly determinable.

(*b*) In this problem, the payoff matrix for player *A* is,

Player B			Row Minima
Player A	$\boxed{0}$	2	$\boxed{0}$ ← Maximin ($\underline{v}$)
	−1	$\boxed{4}$	
Column Maxima	$\boxed{0}$	$\boxed{4}$	

↑ Minimax ($\bar{v}$)

Here, as discussed in (*a*), we have

$$\underline{v} = \bar{v} = 0.$$

Therefore, the game is *determinable* and *fair*. Optimum strategies for player *A* and *B* are given by (A_1, B_1).

Examination PROBLEMS

1. Determine which of the following two-person zero-sum games are strictly determinable and fan. Give the optimum strategies for each player in the case of strictly determinable games :

(a)

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \text{Player } B \\ \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} -5 & 2 \\ -7 & -4 \end{bmatrix} \end{array}$$

(b)

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \text{Player } B \\ \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 10 & 6 \\ 8 & 2 \end{bmatrix} \end{array}$$

[GBTU (MBA) II Sem. 2012; Madurai (M.Com.) 1993]

[**Ans.** (a) Not fair, (A_1, B_1), $v = -5$; *(b)* Not fair, (A_1, B_2), $v = 6$.]

2. Consider the game G with the following pay off matrix :

Player B

Player A	B_1	B_2
A_1	2	6
A_2	−2	μ

(a) Show that G is strictly determinable whatever μ may be.

(b) Determine the value of G. **[Amravati BE(Rur) 1994]**

[**Ans.** (A_1, B_1), $v = 2$.]

SADDLE POINT, OPTIMAL STRATEGIES AND VALUE OF THE GAME

Definitions

Saddle Point. *A* ***saddle point*** *of a payoff matrix is the position of such an element in the payoff matrix which is minimum in its row and maximum in its column.*

[JNTU (MCA III) 2004; Madars (MBA) 2004; Meerut 2003, 02]

Mathematically, *if a payoff matrix* $\{v_{ij}\}$ *is such that*

$$\max_i [\min_j \{v_{ij}\}] = \min_j [\max_i \{v_{ij}\}] = v_{rs} \text{ (say)},$$

then the matrix is said to have a saddle point (r, s).

Optimal Strategies. If the payoff matrix $\{v_{ij}\}$ has the saddle point (r, s), then the players (A and B) are said to have ***rth*** *and* ***sth optimal strategies***, *respectively.* **[Meerut (OR) 2003; JNTU 1997]**

3. Value of Game. *The payoff* (v_{rs}) *at the saddle point* (r, s) *is called the* ***value of game*** *and it is obviously equal to the maximin* $(\underline{v})$ *and minimax value* $(\bar{v})$ *of the game.*

[Madurai (M.Com.) Nov. 2002]

A game is said to be ***fair*** if $\bar{v} = \underline{v} = 0$. A game is said to be ***strictly determinable*** if $\bar{v} = v = \underline{v}$.

> **NOTE** A saddle point of a payoff matrix is, sometimes, called the equilibrium point of the payoff matrix.

In *Example 1*, $\underline{v} = \bar{v} = 0$. This implies that the game has a saddle point given by the entry (2, 2) of payoff matrix. The value of the game is thus equal to zero and both players select their strategy as the optimal strategy. In this example, it is also seen that no player can improve his position by other strategy.

In general, a matrix need not have a saddle point as defined above. Thus, these definitions of optimal strategy and value of the game are not adequate to cover all cases so need to be generalized. The definition of a saddle point of a function of several variables and some theorems connected with it form the basis of such generalization. These theorems are presented on page GMS/45.

Rules for Determining a Saddle Point

Step 1. ***Select the minimum element of each row of the payoff matrix and mark them by 'O'.***

Step 2. ***Select the greatest element of each column of the payoff matrix and mark them by '□'.***

Step 3. ***If there appears an element in the payoff matrix marked by 'O' and '□' both, the position of that element is a 'saddle point' of the payoff matrix.***

Q. 1. Define : (i) Competitive Game, (ii) Payoff matrix, (iii) Pure and mixed strategies, (iv) Saddle point, (v) Optimal strategies, and (vi) Rectangular game **[JNTU 2000; Kanpur M.Sc. (Maths.) 93]**

2. Explain "best strategy" on the basis of minimax criterion of optimalities. **[Delhi (MBA) 2001]**

3. Describe the maximin principle of game theory. What do you understand by pure strategies and saddle point. **[SJMIT (BE Mech.) 2002; Punjabi (M.B.A.) 1990]**

4. Define saddle point and the value of game with examples. **[Meerut 2002; GNDU (B. Com.) 1991]**

5. Define saddle point. Is it necessary that a said game should always possess a saddle point ?

6. State the rules for detecting a saddle point.

7. What is 'strictly determined game' ? When a game is said to be determinable ?

8. Write short notes on the following : (i) Pure Strategy, (ii) Mixed Strategy, (iii) Max-Min Criterion.

9. Let $A = \{a_{ij}\}$ be an $m \times n$ payoff matrix for a zero-sum, two-person game. Define a Saddle point for matrix A and show that the value of the game is equal to the saddle value.

10. Differentiate between strictly determinable game and non-determinable games. **[JNTU (Mech. & Prod.) 2004]**

Examinations PROBLEMS

1. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give the optimum strategies for each player in the case of strictly determinable games : **[JNTU (B. Tech.) 2003]**

   (i) Player B
   $$\text{Player } A\begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix}$$
   [Ans. Not fair, $v = 1$**]**

   (ii) Player B
   $$\text{Player } A\begin{bmatrix} -5 & 2 \\ -7 & -4 \end{bmatrix}$$
   [Ans. Not fair, (I, II), $v = -5$.**]**

2. Consider the game G with the following payoff matrix :
   Player B
   $$\text{Player } A\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$$
   (i) Show that G is strictly determinable whatever μ may be.
   (ii) Determine the value of G. **[Jodhpur M.Sc. (Maths.) 1992]**
   [Ans. (I, I), $v = 2$.**]**

3. Find out whether there is any saddle point in the following problem :
   Player B
   $$\text{Player } A\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}.$$
   [Ans. Saddle point does not exist.**]**

4. For the game with payoff matrix :
   Player B
   $$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player } \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix}, \end{array}$$
   determine the best strategies for players A and B and also the values of the game for them. Is this game : (*i*) fair, (*ii*) strictly determinable ?
   [Ans. (A_1, B_3), $v = -2$. Game is strictly determinable and not fair.**]**

SOLUTION OF GAMES WITH SADDLE POINT(S)

To obtain a solution of a rectangular game, it is feasible to find out,

(i) the best strategy for player A, *(ii)* the best strategy for player B, and *(iii)* the value of the game (v_{rs}).

It is already seen that the best strategies for players A and B will be those which correspond to the row and column, respectively, through the saddle point. The value of the game to the player A is the element at the saddle point, and the value to the player B will be its negative.

Illustrative Examples

Example 1. *Player A can choose his strategies from $\{A_1, A_2, A_3\}$ only, while B can choose from the set (B_1, B_2) only. The rules of the game state that the payments should be made in accordance with the selection of strategies :*

Strategy Pair Selected	*Payments to be Made*	*Strategy Pair Selected*	**Payments to be Made**
(A_1, B_1)	*Payer A Pays Re. 1 to player B*	(A_2, B_2)	*Player B pays Rs 4 to player A*
(A_1, B_2)	*Player B pays Rs. 6 to player A*	(A_3, B_1)	*Player A pays Rs 2 to player B*
(A_2, B_1)	*Player B pays Rs 2 to player A*	(A_3, B_2)	*Player A pays Rs. 6 to player B*

What strategies should A and B play in order to get the optimum benefit of the play ?

Solution With the help of above rules the following payoff matrix is constructed :

The payoffs marked 'O' represent the minimum payoff in each row and those marked ' ' represent the maximum payoff in each column of the payoff matrix.

Obviously, the matrix has a saddle point at position (2, 1) and the value of the game is 2.

Thus, the optimum solution to the game is given by

(i) the optimum strategy for player A is A_2 ; *(ii)* the optimum strategy for player B is B_1 ; and

(iii) the value of the game is Rs. 2 for player A and Rs. (– 2) for player B.

Player B

		B_1	B_2
	A_1	–1	$\boxed{6}$
Player A	A_2	$\boxed{2}$	4
	A_3	–2	–6

Also, since $v \neq 0$, the game is not fair, although it is strictly determinable.

Example 2. *The payoff matrix of a game is given. Find the solution of the game to the player A and B.*

$$\begin{array}{cc} & B \\ & \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ A \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} & \begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix} \end{array}$$

[JNTU (MCA III) 2004, (B. Tech.) 2000, 1999]

Solution First find out the saddle point by encircling each row minima and putting squares around each column maxima.

The saddle point thus obtained is shown by having a circle and square both (*see the table*) below :

Hence, the solution to this game is given by : *(i)* the best strategy for player A is 2nd; *(ii)* the best strategy for player B is 3rd; and *(iii)* the value of the game is 1 to player A and – 1 to player B.

Optimum Strategy for B

	I	II	III	IV	V	Row Minimum	
I	−2	0	0	5	3	−2	
Optimum Strategy for A ← II	3	2	1	2	2	1	Maximin Value ($\underline{v}$)
III	−4	−3	0	−2	6	−4	
IV	5	3	−4	2	−6	−6	
Column Maximum	5	3	1	5	6		

Minimax Value ($\bar{v}$)

Example 3. *Solve the game whose payoff matrix is given by*

$$\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{bmatrix} -2 & 15 & -2 \\ -5 & -6 & -4 \\ -5 & 20 & -8 \end{bmatrix} \end{array}.$$

[Kanpur M.Sc. (Math.) 96; Rewa (M.P.) 93]

Solution *Following table* may be formed as explained earlier.

		Opt. St. I	B II	III (Opt. St.)	Row Minimum	
Opt. St. ←	I	−2	15	−2	−2	−2 Maximin Value ($\underline{v}$)
A	II	−5	−6	−4	−6	
	III	−5	20	−2	−8	
Column Max		−2	20	−2		

Minimax Value ($\bar{v}$)

This game has two saddle points in positions (1, 1) and (1, 3). Thus, the solution to this game is given by,

(i) the best strategy for the player A is I, *(ii)* the best strategy for the player B is either I or III, *i.e.* the player B can use either of the two strategies (I, III), and *(iii)* the value of the game is – 2 for player A and + 2 for player B.

Example 4 *Solve the game whose payoff matrix is given by*

$$\text{Player } A \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \text{Player } B \\ \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}$$

[Bharthidasan (B.Com.) 1999]

Solution *Step* **1.** Consider $\{A_1, A_2, A_3\}$ for A and $\{B_1, B_2, B_3\}$ for B.

Step **2.** If player B starts the game, A will counter with a strategy to minimize the payoff to B. That is, if B selects B_1, then will reply by choosing A_1 or A_2 because, this is corresponding to the minimum payoff to B in the first row corresponding to B_1.

Step **3.** Similarly, if B selects the strategy B_2 he will loose 4 or 3 or will neither loose nor gain depending upon the strategy selected by A . But B is sure about a gain of at least min. {0, – 4, – 33} = – 4 irrespective of the stragety selected by A. Meaning thereby, whatever strategy B adops, he can be indicated by obtaining columns {1, – 4, –1} of the row minima. Obvously, B would like to maximize his minimum gain, that will be just the largest of column {1, – 4, –1}, *i.e.* 1 which is the *maximin* strategy.

Step **4.** On the other side, player A wants to minimize his losses. In case he plays strategy A_1, his loss is at the most maximum of {1, 0, 1}, *i.e.* 1 irrespective of what strategy B has selected. He loses not more than max. {3, – 4, 9} if he plays A_2 and not more than max. {1, – 3, –1} if he adops A_3. The maximum losses corresponding to each A_1, A_2, A_3 are forming a row (1, 5, 1) of the *column maxima*. The smallest element of this gives the minimum possible loss to A whatever strategy B may select. Thus, the minimum value of the game is min. {1, 5, 1} = 1, which corresponds to A_1 and A_3, the *minimax* strategy.

Step **5.** The *minimum value* is here marked by circle 'O' and the maximum value by a square '□' as shown below.

	Player B			Row Minima	
	1	3	1	1	← *Maximin value*
Player A	0	−4	−3	−4	
	1	5	−1	−1	
Column Maxima	1	5	1		

Minimax value

It is seen from above table that there are *two* saddle points (having 'O' and '□' both) at the positions (1, 1) and (1, 3). Thus, the solutions to the game is given by

(i) The optimum strategy for palyer B is B_1.

(ii) The optimum strategy for player A are A_1 and A_3.

(iii) The value of the game is 1 for B and –1 for A.

Examination PROBLEMS

Find the saddle point (or points) and hence solve the following games :

1. Player B

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{bmatrix} \end{array}.$$

[JNTU (IV B. Tech., I Sem.) Feb. 2007 (Set 1).]

[Ans. (A_2, B_2), $v = 5$.]

2. B

$$A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 1 & 7 & 3 & 4 \\ 5 & 6 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{bmatrix} \end{array}.$$

[Ans. (A_2, B_2), $v = 4$.]

3. B

$$A \quad \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{array}{c} \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{bmatrix} -5 & 2 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ 4 & -2 & 0 & -5 \end{bmatrix} \end{array}.$$

[Ans. (II, III), $v = 4$.]

4. B

$$A \quad \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ \begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 4 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix} \end{array}.$$

[Madras BE (Mech.) 2000; Delhi B.Sc. (Stat.) 1995]

[Ans. (II, III), $v = 4$.]

5. B

$$A \quad \begin{array}{c} \\ \text{I} \\ \text{II} \end{array} \begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{bmatrix} 6 & 8 & 6 \\ 4 & 12 & 2 \end{bmatrix} \end{array}.$$

[Meerut M.Sc. (OR) 2011]

[Ans. (I, I), (I, III), $v = 6$.]

6. B

$$A \quad \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} 3 & 0 & -3 \\ 2 & 3 & 1 \\ -4 & 2 & -1 \end{bmatrix} \end{array}.$$

[Ans. (R_2, C_3), $v = 1$]

7. Solve the games whose payoff matrices are given by

(*a*) Player B

$$\text{Player } A \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

(*b*) B

$$A \begin{bmatrix} 7 & 8 & 4 & 6 & 8 \\ -8 & 6 & 1 & 9 & 6 \end{bmatrix}$$

[Madras (MCA) 2002]

[Ans. (*a*) (I, I) or (I, III), $v = 1$ for A, $v = -1$ for B.
(*b*) (I, III), $v = 4$ for A, $v = -4$ for B.]

8. Solve the games whose payoff matrices are given below :

(*a*) Player B

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} -3 & -1 & 6 \\ 2 & 0 & 2 \\ 5 & -2 & -4 \end{bmatrix} \end{array}$$

[Kanpur 2000]

(*b*) Player B

$$\text{Player } A \begin{bmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{bmatrix}$$

[Madurai (M.Com.) 2003]

[Ans. (*a*) (A_2, B_2), $v = 0$; (*b*) (A_2, B_2), $v = 5$.]

9. For what values of 'a' the game with the following pay-off matrix is strictly determinable ?

B

$$A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{array}{|ccc|} \hline a & 6 & 2 \\ -1 & a & -7 \\ -2 & 4 & a \\ \hline \end{array} \end{array}$$

[JNTU (Mech. & Prod.) 2004]

RECTANGULAR GAMES WITHOUT SADDLE POINTS—MIXED STATEGIES

As discussed earlier, if the payoff matrix $\{v_{ij}\}$ has a saddle point (r, s), then $i = r$, $j = s$ are the optimal strategies of the game and the payoff v_{rs} $(= v)$ is the value of the game. On the other hand, if the given matrix has no saddle point, the game has no optimal strategies. The concept of optimal strategies can be extended to all matrix games by introducing a probability with choice and mathematical expectation with payoff.

Let player A choose a particular activity i such that $1 \le i \le m$ with probability x_i. This can also be interpreted as the relative frequency for which A chooses activity i from number of activities of the game. Then set $\mathbf{x} = \{x_i, 1 \le i \le m\}$ of probabilities constitute the strategy of A. Similarly, $\mathbf{y} = \{y_j, 1 \le j \le n\}$ defines the strategy of the player B.

Thus, the vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ of non-negative numbers satisfying $x_1 + x_2 + \ldots + x_m = 1$ is called the *mixed strategy of the player A*. Similarly, the vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ of non-negative numbers satisfying $y_1 + y_2 + \ldots + y_n = 1$ is called the *mixed strategy of the player B*.

Consider the symbol S_m which denotes the set of ordered m-tuples of non-negative numbers whose sum is unity and $\mathbf{x} \in S_m$. Similarly, $\mathbf{y} \in S_n$. Unless otherwise stated, assume that $\mathbf{x} \in S_m$ and $\mathbf{y} \in S_n$, where $\mathbf{x}$ and $\mathbf{y}$ are mixed strategies of player A and B, respectively.

The mathematical expectation of the payoff function $E(\mathbf{x}, \mathbf{y})$ in a game whose payoff matrix is $\{v_{ij}\}$ is defined by

$$E(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i\, v_{ij})\, y_j = \mathbf{x}^T \mathbf{v}\, \mathbf{y} \quad (\textit{in matrix form})$$

where $\mathbf{x}$ *and* $\mathbf{y}$ *are the mixed strategies of players A and B,* respectively.

Thus the player A should choose $\mathbf{x}$ so as to maximize his minimum expectation and the player B should choose $\mathbf{y}$ so as to minimize the player A's greatest expectation. In other words, the player A tries for $\max\limits_{x} \min\limits_{y} E(\mathbf{x}, \mathbf{y})$ and B tries for $\min\limits_{y} \max\limits_{x} E(\mathbf{x}, \mathbf{y})$.

At this stage it is possible to define the *strategic saddle point* of the game with mixed strategies.

Strategic Saddle Point. ***Definition.*** *If* $\min\limits_{y} \max\limits_{x} E(\mathbf{x}, \mathbf{y}) = E(\mathbf{x}_0, \mathbf{y}_0) = \max \min E(\mathbf{x}, \mathbf{y})$, *then* $(\mathbf{x}_0, \mathbf{y}_0)$ *is called the* ***strategic saddle point*** *of the game where* $\mathbf{x}_0$ *and* $\mathbf{y}_0$ *define the optimal strategies, and* $v = E(\mathbf{x}_0, \mathbf{y}_0)$ *is the value of the game.*

According to the *minimax theorem* (p. GMS/12), a strategic saddle point will always exist.

Example *In a game of matching coins with two players, suppose one player wins Rs. 2 when there are two heads and wins nothing when there are two tails; and losses Re. 1 when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game.*

Solution The payoff matrix (for the player A) is as follows :

Here, maximin value ($\underline{v}$) = – 1 ≠ minimax value ($\bar{v}$) = 2.

So the matrix is *whithout* saddle point.

Player B

		H	T	Row Min.
Player A	H	2	– 1	– 1
	T	– 1	0	– 1
Column Max :		2	0	

Maximin (lower) value ($\underline{v}$)

Maximin (upper) value ($\bar{v}$)

Now, let us outline here how one finds the best strategies for such games and the expected amounts to be gained or lost by the players.

Let the player A plays H with probability x and T with probability $1 - x$ so that $x + (1 - x) = 1$. Then, if the player B plays H all the time, A's expected gain will be

$$E(A, H) = x.2 + (1 - x)(-1) = 3x - 1. \qquad ...(1)$$

Similarly, if the player B plays T all the time, A's expected gain will be

$$E(A, T) = x(-1) + (1 - x)\,0 = -x. \qquad ...(2)$$

It can be shown mathematically that if the player A chooses x such that

$$E(A, H) = E(A, T) = E(A), \text{ say}, \qquad ...(3)$$

then this will determine the best strategy for him.

Thus, $\quad 3x - 1 = -x$

or $\quad x = 1/4 \qquad ...(4)$

Therefore, best strategy for the player A is to play H and T with probability 1/4 and 3/4 , respectively. Since this is a mixed strategy, it is usually denoted by the set {1/4 , 3/4}. So expected gain for the player A is given by

$$E(A) = \frac{1}{4}.2 + \frac{3}{4}(-1) = -\frac{1}{4}$$

Now, whatever be the set $\{y, 1 - y\}$ of probabilities with which the player B plays either H or T, A's expected gain will always remain equal to – 1/4. To verify this,

$$E(A, y, 1 - y) = y\left[\frac{1}{4}.2 + \frac{3}{4}(-1)\right] + (1 - y)\left[\frac{1}{4}(-1) + \frac{3}{4}\cdot 0\right]$$

$$= y\left(-\frac{1}{4}\right) + (1 - y)\left(-\frac{1}{4}\right) = -\frac{1}{4}. \qquad ...(5)$$

The same procedure can be applied for the player B . Let the probability of the choice of H be denoted by y and that of T by $(1 - y)$. For best strategy of the player B ,

$$E(B, H) = E(B, T) = E(B), \text{ say} \qquad ...(6)$$

or $\quad y.2 + (1 - y)(-1) = y(-1) + (1 - y)\,0$

or $\quad 4y = 1$

or $\quad y = 1/4$ and therefore $1 - y = 3/4$.

Therfore, $E(B) = \frac{1}{4}2 + \frac{3}{4}(-1) = -\frac{1}{4}$.

Here, $E(A) = E(B) = -1/4$. Thus, the complete solution of the game is :

(*i*) The player A should play H and T with probabilities 1/4 and 3/4, respectively. Thus, A's optimal strategy is $\mathbf{x}_0 = (1/4, 3/4)$.

(*ii*) The player B should play H and T with probabilities 1/4 and 3/4, respectively. Thus, B's optimal strategy is $\mathbf{y}_0 = (1/4, 3/4)$

(*iii*) The expected value of the game is – 1/4 to the player A. Here $(\mathbf{x}_0, \mathbf{y}_0)$ is the strategic saddle point of this game.

REMARK

Although this example can be easily solved by using the formula for solving 2×2 games withqut saldle point as given on page GMS/15, the present discussion will be of great help in understanding the further discussion.

Examination PROBLEMS

1. Find the optimal strategies for the games for which the pay off matrices are given below. Also, find the value of the game.

(*a*)

$$\begin{array}{cc} & P_2 \\ & \begin{array}{cc} \text{I} & \text{II} \end{array} \\ P_1 \begin{array}{c} \text{I} \\ \text{II} \end{array} & \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \end{array}.$$

[Ans. (1/2, 1/2), (1/4, 3/4) ; v = – 3/4.]

(*b*)

$$\begin{array}{cc} & P_2 \\ & \begin{array}{cc} \text{I} & \text{II} \end{array} \\ P_1 \begin{array}{c} \text{I} \\ \text{II} \end{array} & \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix} \end{array}.$$

[Ans. (1/3, 2/3), (3/5, 2/5); v = 0.]

(*c*)

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}.$$

[Madural (M. Com.) Nov. 2002]

[Ans. (1/5, 4/5), (3/5, 2/5); v = 17/5.]

2. For the game with the following payoff matrix for the row player, determine the optimal strategies for both the players and the value of the game :

(*a*) $\begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix}$.

[Ans. (1/4, 3/4), (1/4, 3/4); v = – 3/4.]

(*b*)

$$\begin{array}{cc} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 10 & -3 \\ -3 & 0 \end{bmatrix} \end{array}.$$

[GBTU (MBA) 2012]

(*c*) $\begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix}$.

[Ans. (2/5, 3/5), (1/2, 1/2); v = 4.]

3. A game has the payoff matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$. Show that $E(x, y) = 1 - 2x\left(y - \frac{1}{2}\right)$ and deduce that in the solution of the game the first player follows a pure strategy while the second has infinite number of mixed strategies.

4. State the fundamental theorem of rectangular games. Show that $\max_i \min_j a_{ij} \le \min_j \max_i a_{ij}$ in the arbitrary matrix :

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

5. Two players A and B match coins. If the coins match, then A wins two coins of value. If the coins do not match, then B wins 2 units of value. Determine the optimum strategies for the players and the value of the game. **[Madras (NBA) Nov. 2006]**
[Ans. (1/2, 1/2) for A and B both, Value of game = 0.]

6. In a game of matching coins with two players, suppose A wins one unit of value, when there are two heads, wins nothing when there are two tails and loses 1/2 unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game to A. **[Amravati BE(Rural) 1994]**
[Ans. (1/4, 3/4) A and B both, value of the game = – 1/8].

7. A and B each take out one or two matches and guess how many matches opponent has taken. If one of the players guesses correctly, then the loser has to pay him as many rupees as the sum of the number held by both players. Otherwise, the payout is zero. Write down the payoff matrix and obtain the optimal strategies of both players. **[Jodhpur M.Sc. (Math.) 1994]**
[Ans. (2/3, 1/3) for A and B both, Value of the game for A = 4/3.]

MINIMAX-MAXIMIN PRINCIPLE FOR MIXED STRATEGY GAMES

It has been observed earlier that if a game does not have a saddle point, two players cannot use the maximin-minimax (pure) strategies as their optimal strategies. This failure of the minimax-maximum (pure) strategies, in general, give an optimal solution to the game and led to the idea of using mixed strategies. Each player, instead of selecting pure strategies only, may play all his strategies according to a predetermined set of probabilities.

Let $x_1, x_2, \ldots, x_m$ and $y_1, y_2, y_3, \ldots y_n$ be the probabilities of two players A and B, respectively, to select their pure strategies.

Then, $$x_1 + x_2 + x_3 + \ldots + x_m = 1 \qquad \ldots(1)$$
and $$y_1 + y_2 + y_3 + \ldots + y_n = 1, \qquad \ldots(2)$$
where $x_i \ge 0$ and $y_j \ge 0$ for all i and j. Thus if v_{ij} represents the (i, j)th entry of the game matrix, probabilities x_i and y_j will appear (as shown below) :

Player B

	Prob. ↓ i → j		y_1	y_2	...	y_j	...	y_n
			1	2	...	j	...	n
	x_1	1	v_{11}	v_{12}	...	v_{1j}	...	v_{1n}
	x_2	2	v_{21}	v_{22}	...	v_{2j}	...	v_{2n}
Player A	:	:	:	:		:		:
	x_i	i	v_{i1}	v_{i2}	...	v_{ij}	...	v_{in}
	:	:	:	:		:		:
	x_m	m	v_{m1}	v_{m2}	...	v_{mj}	...	v_{mn}

The solution of mixed strategy problem is also based on the *minimax criterion* given on page GMS/3.

The only difference is that the player A selects probabilities x_i which *maximize his minimum 'expected' gain* in a column, while the player B selects the probabilities y_j which *minimize his maximum 'expected' loss* in a row.

Mathematically, the minimax criterion for a mixed strategy is as follows :

The player A selects x_i $(x_i \ge 0, \sum_{i=1}^{m} x_i = 1)$ which gives the lower value of the game,

$$\max_{x_1, x_2, \ldots, x_m} [\min \{(v_{11}x_1 + v_{21}x_2 + \ldots + v_{m1}x_m), (v_{12}x_1 + v_{22}x_2 + \ldots + v_{m2}x_m), \ldots, (v_{1n}x_1 + v_{2n}x_2 + \ldots + v_{mn}x_m)\}] \qquad \ldots(3a)$$

or more precisely,

$$\underline{v} = \max_{x_i}\left[\min\left\{\sum_{i=1}^{m} v_{i1}\, x_i,\ \sum_{i=1}^{m} v_{i2} x_i, \ldots, \sum_{i=1}^{m} v_{in}\, x_i\right\}\right] \qquad \ldots(3b)$$

Similarly, the player B chooses y_j $(y_j \ge 0, \sum_{j=1}^{n} y_j = 1)$ which gives the upper value of the game

$$\bar{v} = \min_{y_j}\left[\max\left\{\sum_{j=1}^{n} v_{1j}\, y_j,\ \sum_{j=1}^{n} v_{2j}\, y_j, \ldots, \sum_{j=1}^{n} v_{mj}\, y_j\right\}\right] \ldots(4)$$

These values are referred to the *maximin* ($\underline{v}$) and the *minimax* ($\bar{v}$) expected values, respectively.

In pure strategies, the relationship, $\bar{v} \ge \underline{v}$, holds in general. When x_i and y_j correspond to the optimal solution, this relation holds in 'equality' sense and the 'expected' values thus obtained become equal to the (optimal) expected values of the game. This result follows from the *minimax theorem* (called the *fundamental theorem of rectangular games*) which is derived on page GMS/12.

We shall require the following *Lemma* in proving the equivalence of *rectangular game* and *linear programming* (p.GMS/11).

Lemma. *Let $A = (v_{ij})$ be the payoff matrix of an $m \times n$ game. If $B = (v'_{ij})$ is obtained from A by adding a constant c to every element of A, then an optimal strategy for B is also an optimal strategy for A.*

Proof. Let v' be the value of the game with payoff matrix **B**. Then for the strategies **x, y,**

$$\sum_{i=1}^{m}\sum_{j=1}^{n} v'_{ij}\, x_i y_j = \sum_{i=1}^{m}\sum_{j=1}^{n} v_{ij} x_i y_j + c.$$

If $\mathbf{x}^*, \mathbf{y}^*$ are optimal strategies for the game $\boldsymbol{B}$, then

$$\sum_i \sum_j v'_{ij}\, x_i\, y_j^* \le v' \le \sum_i \sum_j v'_{ij}\, x_i^*\, y_j$$

$$\Rightarrow \quad \sum_i \sum_j v_{ij} x_i y_j^* + c \le v' \le \sum_i \sum_j v_{ij} x_i^* y_j + c.$$

$$\therefore \quad \sum_i \sum_j v_{ij} x_i y_j^* \le v' - c \le \sum_i \sum_j v_{ij} x_i^* y_j .$$

Thus $\mathbf{x}^*, \mathbf{y}^*$ are optimal for game A with the value of game $v = v' - c$.

Hence arbitrarily chosen constant c can be added to each element of A and then we can solve the resulting game B. The value v of the original game is then obtained simply by subtracting the constant c from the value of the game B. Constant c is chosen so large that $v_{ij} + c$ is positive (> 0) for all i and j, so that the value of the game is certainly positive.

Q. 1. Define the terms "maximin element, minimax element and saddle point" of the payoff matrix of a two-person zero-sum games.

2. Explain 'minimax criterion' as applied to the theory of games. **[Delhi B.Sc (Stat.) 1999, 96]**

3. Let (v_{ij}) be the payoff matrix for a two-person zero-sum game. If $\underline{v}$ denotes the maximin value and $\bar{v}$ the minimax value of the game, then prove that $\bar{v} \geq \underline{v}$. That is, **[Bhubneshwar (IT) 2004]**

$$\min_j [\max_i \{v_{ij}\}] \geq \max_i [\min_j \{v_{ij}\}].$$ **[Delhi B.Sc. (Stat.) 1995]**

Examination PROBLEMS

Find the minimax and maximin value of the following games :

(i) $\begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 3 \\ 6 & 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 7 & -1 & 3 \\ 4 & 8 & 0 & -6 \\ 6 & -9 & -2 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 9 & 6 & 0 \\ 2 & 3 & 8 & 4 \\ -5 & -2 & 10 & -3 \\ 7 & 4 & -2 & -5 \end{bmatrix}$ (iv) $\begin{bmatrix} -1 & 9 & 6 & 8 \\ -2 & 10 & 4 & 6 \\ 5 & 3 & 0 & 7 \\ 7 & -2 & 8 & 4 \end{bmatrix}$

[Ans. (*i*) minimax = 3, maximin = 1 (*ii*) minimax = 0, maximin = − 1 (*iii*) $2 < v < 4$, (*iv*) $4 \leq v < 7$.]

EQUIVALENCE OF RECTANGULAR GAME AND LINEAR PROGRAMMING

It has been shown that the player A chooses his optimum mixed stratgies in order to *maximize his minimum 'expected' gain, i.e.*

$$\max_{x_i} \left[\min \left\{ \sum_{i=1}^{m} v_{i1} x_i \, , \; \sum_{i=1}^{m} v_{i2} x_i \, , \ldots , \; \sum_{i=1}^{m} v_{in} x_i \right\} \right] \quad ...(1)$$

subject to the constraints :

$$x_1 + x_2 + x_3 + \ldots + x_m = 1 \quad ...(2)$$

$$x_i \geq 0, i = 1, 2, \ldots, m . \quad ...(3)$$

Now, in order to express this problem in linear programming form, let

$$\min \left[\sum_{i=1}^{m} v_{i1} x_i \, , \; \sum_{i=1}^{m} v_{i2} x_i \, , \ldots , \; \sum_{i=1}^{m} v_{in} x_i \right] = v \quad ...(4)$$

which immediately implies that

$$\sum_{i=1}^{m} v_{i1} x_i \geq v \, , \; \sum_{i=1}^{m} v_{i2} x_i \geq v \, , \ldots , \; \sum_{i=1}^{m} v_{in} \, x_i \geq v \, . \quad ...(5)$$

Thus, the problem now becomes :

Maximize $\mathbf{x_0} = v$, subject to the constraints :

$$\left. \begin{array}{l} v_{11} \, x_1 + v_{21} \, x_2 + v_{31} \, x_3 + \ldots + v_{m1} \, x_m \geq v \\ v_{12} \, x_1 + v_{22} \, x_2 + v_{32} \, x_3 + \ldots + x_{m2} \, x_m \geq v \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ v_{1n} \, x_1 + v_{2n} \, x_2 + v_{3n} \, x_3 + \ldots + v_{mn} \, x_m \geq v \\ \qquad x_1 + x_2 + x_3 + \ldots + x_m = 1 \\ \qquad x_1 \, , x_2 \, , x_3 \, , \ldots , x_m \geq 0 \, . \end{array} \right\} . \quad ...(6a)$$

and

Here v represents the value of the game. This linear programming formulation can be simplified by dividing all $(n+1)$ constraints by v ; the division is valid so long as $v > 0$[†]. In case, $v < 0$, the direction of the inequality constraints must be reversed, and if $v = 0$, division would be meaningless. The later point creates no special difficulty since a constant c can be added to all entries of the matrix ensuring that the value (v) of the game for the 'revised' matrix becomes greater than zero. After the optimal solution is obtained, the true value of the game is obtained by subtracting the same amount c.

In general, if the maximum value of the game is non-negative, the value of the game is greater than zero (provided the game does not have a saddle point). Thus, assuming $v > 0$, the constraints become :

$$\left. \begin{array}{l} v_{11} \dfrac{x_1}{v} + v_{21} \dfrac{x_2}{v} + \ldots + v_{m1} \dfrac{x_m}{v} \geq 1 \\ v_{12} \dfrac{x_1}{v} + v_{22} \dfrac{x_2}{v} + \ldots + v_{m2} \dfrac{x_m}{v} \geq 1 \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ v_{1n} \dfrac{x_1}{v} + v_{2n} \dfrac{x_2}{v} + , \ldots + v_{mn} \dfrac{x_m}{v} \geq 1 \\ \qquad \dfrac{x_1}{v} + \dfrac{x_2}{v} + \ldots + \dfrac{x_m}{v} = \dfrac{1}{v} . \end{array} \right\} . \quad ...(6b)$$

Now, suppose $\dfrac{x_1}{v} = X_1, \dfrac{x_2}{v} = X_2, \ldots, \dfrac{x_m}{v} = X_m$,

and $\dfrac{1}{v} = \mathbf{x_0}$, then

$$\max v = \min \left(\frac{1}{v} \right) = \min \left\{ \frac{x_1}{v} + \frac{x_2}{v} + \ldots + \frac{x_m}{v} \right\} \quad ...(7)$$

$$= \min \{ X_1 + X_2 + X_3 + \ldots + X_m \}$$

(which is justified by the last constraint),

Now, finally, the equivalent LP problem for player A becomes :

Minimize $\mathbf{x_0} = X_1 + X_2 + \ldots + X_m$, subject to the constraints : ...(8)

$$\left. \begin{array}{l} v_{11} X_1 + v_{21} X_2 + \ldots + v_{m1} X_m \geq 1 \\ v_{12} X_1 + v_{22} X_2 + \ldots + v_{m2} X_m \geq 1 \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ v_{1n} X_1 + v_{2n} X_2 + \ldots + v_{mn} X_m \geq 1 \\ X_1 \geq 0, X_2 \geq 0, \ldots, X_m \geq 0. \end{array} \right\} \quad ...(6c)$$

After an optimal solution is obtained by the *simplex method*, original optimal values can be obtained from the given transformation formulae.

On the other hand, player B chooses his mixed strategies in order to *minimize his maximum 'expected' loss, i.e.*

† For convenience, in order to convert a matrix game into a linear programming problem, first make all entries of the matrix positive by adding a positive constant c to all elements of the matrix game. Of course, c will be subtracted later on from the value of the game v.

$$\min_{y_j}\left[\max\left\{\sum_{j=1}^{n} v_{1j}\,y_j,\ \sum_{j=1}^{n} v_{2j}\,y_j,\ \ldots,\ \sum_{j=1}^{n} v_{mj}\,y_j\right\}\right] \quad \ldots(9)$$

subject to the constraints :

$$y_1 + y_2 + \ldots + y_n = 1 \quad \ldots(10)$$

$$y_1 \ge 0,\ y_2 \ge 0,\ \ldots,\ y_n \ge 0. \quad \ldots(11)$$

Proceeding in the like manner, linear programming form of the B's problem becomes :

Maximize $\mathbf{y}_0 = Y_1 + Y_2 + \ldots + Y_n$ subject to the constraints : ...(12)

$$\left.\begin{array}{l} v_{11}\,Y_1 + v_{12}\,Y_2 + \ldots + v_{1n}\,Y_n \le 1 \\ v_{21}\,Y_1 + v_{22}\,Y_2 + \ldots + v_{2n}\,Y_n \le 1 \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \\ v_{m1}\,Y_1 + v_{m2}\,Y_2 + \ldots + v_{mn}\,Y_n \le 1 \\ \quad Y_1 \ge 0,\ Y_2 \ge 0,\ \ldots,\ Y_n \ge 0 \end{array}\right\} \quad \ldots(13)$$

where $\quad \mathbf{y}_0 = \frac{1}{v},\ Y_1 = \frac{y_1}{v},\ Y_2 = \frac{y_2}{v},\ \ldots,\ Y_n = \frac{y_n}{v}$.

Further, it has been observed that the player B's problem is exactly the dual of the player A's problem. The optimal solution of one problem will automatically give the optimal solution to the other and that min $\mathbf{x}_0$ = max $\mathbf{y}_0$. The player B's problem can be solved by *regular simplex method* while player A's problem can be solved by the *dual simplex method*.

The choice of either method will depend on which problem has a smaller number of constraints. This in turn depends on the number of pure strategies for either player.

Q. 1. Show how a 'game' can be formulated as a linear programming problem. **[IAS (Maths.) 1999]**

2. With the help of an appropriate example establish the relationship between 'Game theory' and 'Linear Programming'.

3. Establish the relation between a linear programming problem and a two-person zero–sum game. **[Meerut (OR) 2003]**

4. Discuss equivalence of matrix game and the problem of linear programming. **[Kanpur M.Sc. (Maths.) 1997; Delhi (OR.) 95; Banasthali (M.Sc.) 93]**

5. Explain the method of solving a zero-sum two person game as a linear programming problem. **[Meerut 2005]**

6. Establish the equivalence of matrix game and the problem of linear programming. **[Delhi B.Sc. (Math.) 1995]**

MINIMAX THEOREM : FUNDAMENTAL THEOREM

Theorem *Fundamental Theorem of Rectangular Games. If mixed strategies are allowed, there always exists a value of the game, i.e.* $\bar{v} = \underline{v} = v$.

Alternative Statement. If $\Sigma x_i = \Sigma y_j = 1,\ x_i \ge 0,\ y_j \ge 0$, *then*

$$\max_{x}\ \min_{y\,|\,x}\ \sum_{i}\sum_{j} v_{ij}\,(x_i\,y_j) = \min_{y}\ \max_{x\,|\,y}\ \sum_{i}\sum_{j} v_{ij}\,(x_i\,y_j),$$

where the symbol $\mathbf{y} \mid \mathbf{x}$ *means "*$\mathbf{y}$ *given* $\mathbf{x}$*". The left side relates that for some fixed (given)* $\mathbf{x}$*, minimize the sum with respect to* $\mathbf{y}$*. This results in a value showing it is a function of* $\mathbf{x}$ *select* $\mathbf{x}$ *so that this value is maximum.*

Proof The player A's problem (see p. GMS/11) is :

$$\text{Min. } \mathbf{x}_0 = X_1 + X_2 + X_3 + \ldots + X_m \text{ subject to :}$$

$$v_{11}X_1 + v_{21}X_2 + \ldots + v_{m1}X_m \ge 1$$

$$v_{12}X_1 + v_{22}X_2 + \ldots + v_{m2}X_m \ge 1$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$v_{1n}X_1 + v_{2n}X_2 + \ldots + v_{mn}X_m \ge 1$$

$$X_1 \ge 0,\ X_2 \ge 0,\ \ldots,\ X_m \ge 0.$$

The dual problem corresponding to above linear programming problem (called the primal problem) is :

$$\text{Max. } \mathbf{y}_0 = Y_1 + Y_2 + Y_3 + \ldots + Y_n \text{ subject to :}$$

$$v_{11}Y_1 + v_{12}Y_2 + \ldots + v_{1n}Y_n \le 1$$

$$v_{21}Y_1 + v_{22}Y_2 + \ldots + v_{2n}Y_n \le 1$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$v_{m1}Y_1 + v_{m2}Y_2 + \ldots + v_{mn}Y_n \le 1$$

$$Y_1 \ge 0,\ Y_2 \ge 0,\ \ldots,\ Y_n \ge 0.$$

It has been seen that this dual problem is similar to the problem obtained for the player B (see (13)).

But, the duality theorem states that :

If either the primal or the dual problem has a finite optimum solution, then the other problem has a finite optimum solution, and optimum absolute values of the objective function are equal, i.e.

max $\mathbf{y}_0$ = min $\mathbf{x}_0$ $\quad$ or $\quad$ $\underline{v} = \bar{v} = v$ *(value of the game)*

This completes the proof of the theorem.

Q. 1. State, explain and prove the '*minimax theorem*' (fundamental theorem) for two-person zero-sum finite games. **[Delhi B.Sc. (Stat.) 1999, 96]**

2. Let v be the value of a rectangular game with payoff matrix $B = (p_{ij})$.

Show that $\min p_{ij} \le v \le \max (p_{ij})$ and

$$\max_{i}\ \min_{j}\ p_{ij} \le v \le \min_{j}\ \max_{i}\ p_{ij}.$$

3. Let $E(p,\ q)$ be expectation function in an $m \times n$ matrix (rectangular) game between player A and B, such that $p \in R^n$, $q \in R^m$. If $E\ (p,\ q)$ be such that both $\max_{p}\min_{q} E\ (p,\ q)$ and $\min_{q}\max_{p} E\ (p,\ q)$ exist, then show that

$\min_{q}\max_{p} E\ (p,\ q) \ge \max_{p}\min_{q} E\ (p,\ q)$ (p and q are probability vectors).

SOLUTION OF (m × n) GAMES BY SIMPLEX METHOD

Following example of (3×3) game will make the computational procedure clear.

Example : *Solve* (3×3) *game by the simplex method of linear programming whose payoff matrix is given below.*

		Player B		
		1	2	3
	1	3	–1	–3
Player A	2	–3	3	–1
	3	–4	–3	3

[JNTU (B. Tech.) 2004; Meerut (MCA) 2000]

Solution. First apply *minimax (maximin)* criterion to find the minimax ($\bar{v}$) and maximin ($\underline{v}$) value of the game. Thus, the following matrix is obtained.

		B 1	2	3	Row Minimum.
	1	$\boxed{3}$	−1	−3	−3
A	2	−3	$\boxed{3}$	−1	−3
	3	−4	−3	$\boxed{3}$	−4
Column Maximum →		3	3	3	

Maximin Value ($\underline{v}$)

Minimax Value ($\bar{v}$)

Since, maximin value is – 3, it is possible that the value of the game ($\mathbf{v}$) may be *negative* or *zero* because $-3 < v < 3$.

		B 1	2	3
	1	8	4	2
A	2	2	8	4
	3	1	2	8

Thus, a constant c is added to all elements of the matrix which is *at least* equal to the – ve of the maximin value, *i.e.* $c > 3$. Let $c = 5$. The modified matrix is shown here in the table form. Now, following the reasoning (***see***. on page GMS/12), the player B's linear programming problem is :

$$\text{Maximize } \mathbf{y}_0 = Y_1 + Y_2 + Y_3 \qquad ...(1)$$

subject to the constraints :

$$8Y_1 + 4Y_2 + 2Y_3 \le 1,\ 2Y_1 + 8Y_2 + 4Y_3 \le 1,$$
$$1Y_1 + 2Y_2 + 8Y_3 \le 1,\ Y_1 \ge 0,\ Y_2 \ge 0,\ Y_3 \ge 0. \qquad ...(2)$$

Introducing slack variables, the constraint equations become :

$$\left.\begin{aligned} 8Y_1 + 4Y_2 + 2Y_3 + Y_4 \qquad\qquad &= 1 \\ 2Y_1 + 8Y_2 + 4Y_3 \qquad + Y_5 \qquad &= 1 \\ 1Y_1 + 2Y_2 + 8Y_3 \qquad\qquad + Y_6 &= 1 \end{aligned}\right\} \qquad ...(3)$$

$$Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 \ge 0.$$

Now the following simplex table is formed.

Simplex Table

B	C_B	Y_B	α_1	α_2	α_3	α_4 (β_1)	α_5 (β_2)	α_6 (β_3)	Min. Ratio $(Y_B / \alpha_k), \alpha_k > 0$
		$c_j \rightarrow$	1	1	1	0	0	0	
α_4	0	1	$\boxed{8}$	4	2	1	0	0	1/8 ←
α_5	0	1	2	8	4	0	1	0	1/2
α_6	0	1	1	2	8	0	0	1	1/1
	$\mathbf{y}_0 = C_B Y_B = 0$		(−1)* ↑	−1	−1	0 ↓	0	0	$\leftarrow \Delta_j = C_B \alpha_j - c_j$
α_1	1	1/8	1	1/2	1/4	1/8	0	0	1/2
α_5	0	3/4	0	7	$\boxed{7/2}$	− 1/4	1	0	3/14
α_6	0	7/8	0	3/2	31/4	− 1/8	0	1	7/62 ←
	$\mathbf{y}_0 = 1/8$		0	− 1/2	(− 3/4)* ↑	1/8	0	0 ↓	$\leftarrow \Delta_j$
α_1	1	3/31	1	14/31	0	4/31	0	− 1/31	3/14
α_5	0	11/31	0	$\boxed{196/31}$	0	− 6/31	1	− 14/31	11/196 ←
α_3	1	7/62	0	6/31	1	− 1/62	0	4/31	7/12
	$\mathbf{y}_0 = 13/62$		0	(− 11/31)* ↑	0	7/62	0 ↓	3/31	$\leftarrow \Delta_j$
α_1	1	1/14	1	0	0	1/7	1/14	0	
α_2	1	11/196	0	1	0	− 3/98	31/196	− 1/14	
α_3	1	5/49	0	0	1	− 1/98	− 3/98	1/7	

Thus, the solution for B's original problem is obtained as :

$$y_1^* = \frac{Y_1}{\mathbf{y}_0} = \frac{1/14}{45/196} = \frac{14}{45},\ y_2^* = \frac{Y_2}{\mathbf{y}_0} = \frac{11/196}{45/196} = \frac{11}{45},$$

$$y_3^* = \frac{Y_3}{\mathbf{y}_0} = \frac{5/49}{45/196} = \frac{20}{45},\ v^* = \frac{1}{\mathbf{y}_0} - c = \frac{196}{45} - 5 = -\frac{29}{45}.$$

The optimal strategies for the player A are obtained from the final table of the above problem. This is given by *duality rules* :

$$\mathbf{x}_0 = \mathbf{y}_0 = \frac{45}{196},\ X_1 = \Delta_4 = \frac{5}{49},\ X_2 = \Delta_5 = \frac{11}{196},\ X_3 = \Delta_6 = \frac{1}{14}.$$

Hence,

$$x_1^* = \frac{X_1}{\mathbf{x}_0} = \frac{20}{45},\ x_2^* = \frac{X_2}{\mathbf{x}_0} = \frac{11}{45},\ x_3^* = \frac{X_3}{\mathbf{x}_0} = \frac{14}{45},\ v^* = \frac{29}{45}.$$

Examination PROBLEMS

1. Two companies A and B are competing for the same product. Their different strategies are given in the following payoff matrix :

$$\begin{array}{cc|ccc|} & & & A & \\ & & A_1 & A_2 & A_3 \\ \hline B & B_1 & 2 & -2 & 3 \\ & B_2 & -3 & 5 & -1 \\ \hline \end{array}$$

Use linear programming to determine the best strategies for both the players. **[Meerut M.Sc.(Math.) 2001]**

[Hint. First, make the payoff's positive by adding a constant quantity $c = 4$ (say). The modiefied payoff matrix becomes

$$\begin{array}{cc} & A \\ B & \begin{bmatrix} 6 & 2 & 7 \\ 1 & 9 & 3 \end{bmatrix} \end{array}.$$

Then, formulate the problem for player B by usual transformation as :
Maximize $y_0 = Y_1 + Y_2$, subject to :
$6Y_1 + Y_2 \le 1,\ 2Y_1 + 9Y_2 \le 1,\ 7Y_1 + 3Y_2 \le 1$ and $Y_1 \ge 0, Y_2 \ge 0$.
Now apply simplex method to find the following solution for B :

$$v = \frac{1}{y_0} - 4 = \frac{13}{3} - 4 = \frac{1}{3}.$$

$$y_1 = \frac{Y_1}{y_0} = \frac{7}{52} \times \frac{13}{3} = \frac{7}{12},\ y_2 = \frac{Y_2}{y_0} = \frac{5}{52} \times \frac{13}{3} = \frac{5}{12}.$$

For player A, read the solution to the dual of above problem

$$v = \frac{1}{\mathbf{x}_0} - 4 = \frac{1}{y_0} - 4 = \frac{13}{3} - 4 = \frac{1}{3}.$$

$$x_1 = \frac{X_1}{\mathbf{x}_0} = \frac{2}{13} \times \frac{13}{3} = \frac{2}{3}, x_2 = \frac{X_2}{\mathbf{x}_0} = \frac{1}{13} \times \frac{13}{3} = \frac{1}{3}, x_3 = 0.]$$

[Ans. (2/3 , 1/3 , 0) ; (7/12 , 5/12) ; v = 1/3.]

2. For the following payoff table, transform the zero-sum game into an equivalent linear programming problem and solve it by simplex method :

		Player Q		
		Q_1	Q_2	Q_3
	P_1	9	1	4
Player P	P_2	0	6	3
	P_3	5	2	8

[Hint. Payoff's are already non-negative. Formulation of L.P. problem for Q in usual notations is :

Max. $y_0 = Y_1 + Y_2 + Y_3$ subject to the constrants :
$9Y_1 + 1Y_2 + 4Y_3 \le 1,\ 0Y_1 + 6Y_2 + 3Y_3 \le 1,$
$5Y_1 + 2Y_2 + 8Y_3 \le 1,$ and $Y_1, Y_2, Y_3 \ge 0.$

Its dual is the formulation for player P. Proceeding exactly as in solved example, apply simplex method.

[Ans. (3/8 , 13/24 , 1/12) ; (7/24 , 5/9 , 11/72) ; v = 91/24 .]

3. Solve the following games by linear programming :

(i)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 2 \\ 3 & 4 & -3 \end{bmatrix} \end{array}$$

[Ans. A (17/46 , 20/46 , 9/46).]
B (7/23 , 6/23 , 10/23) , v = 15/23.]

(ii)

$$\begin{array}{ccc} & & B \\ & 1 & \\ A & 2 & \begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & 2 \\ 3 & 3 & -3 \end{bmatrix} \\ & 3 & \end{array}$$

[Ans. (6/11, 3/11, 2/11) ; (5/22 , 8/22 , 9/22) ; v = 6/11]

(iii)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix} \end{array}$$

[Ans. (2/3, 1/3, 0), (0, 1/2, 1/2), v = 1]

(iv)

$$\begin{array}{ccc} & & B \\ & 1 & \\ A & 2 & \begin{bmatrix} 2 & 4 \\ 2 & 3 \\ 3 & 2 \end{bmatrix} \\ & 3 & \end{array}.$$

4. Solve the following 3 × 3 games by linear programming :

(i)

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix} \end{array}$$ **[Agra 1998, 93, 92]**

[Ans. A (6/13, 3/13 , 4/13), B (6/13, 4/13, 3/13), v^* = − 1/13.]

(ii)

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} \end{array}$$ **[Meerut 1993]**

[Ans. (0, 0, 1), (4/5, 1/5 , 0), v = 2 .]

5. A and B play a game in which each has three coins : a penny, a nickel and a dime. Each selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, A wins B's coins; *If the sum is even*, B wins A's coin. Find the best strategies for each player and the value of game.

[Ans. A $\begin{pmatrix} \text{Penny} & \text{NIckel} & \text{Dime} \\ 1/2 & 1/2 & 0 \end{pmatrix}$,

B $\begin{pmatrix} \text{Penny} & \text{Nickel} & \text{Dime} \\ 2/3 & 1/3 & 0 \end{pmatrix}$, v = 0.]

6. A and B play a game as follows :
They simultaneously and independently write one of the three numbers 1, 2 and 3. If the sum of the numbers written is even, B pays to A this sum in Rupees. If it is odd, A pays the sum to B in Rupees. Form the payoff matrix of player A and solve the game to find out the value of the game and probabilities of mixed strategies of A and B.

[Ans. $\begin{pmatrix} 2 & -3 & 4 \\ -3 & 4 & -5 \\ 4 & -5 & 6 \end{pmatrix}$, A (1/4, 1/2, 1/4), B (1/4, 1/2, 1/4), v = 0.]

7. Transform the following matrix games into their corresponding primal and dual linear programming problems. Hence solve them.

(a) $\begin{pmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \end{pmatrix}$ **[Delhi B.Sc (Math.) 1991]**

[Ans. (1/2, 1/2), (3/5, 2/5, 0), v = 1]

(b) $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ **[Delhi B.Sc. (Math.) 1991]**

[Ans. (1/2 , 1/2 , 0), (0, 1/2 , 1/2), v = 0.]

8. Use simplex method to solve the following games :

(a) $\begin{bmatrix} 5 & 3 & 7 \\ 7 & 9 & 1 \\ 10 & 6 & 2 \end{bmatrix}$

[Ans. (0, 0, 1), (4/5, 1/5, 0), v = 2.]

(b) $\begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix}$

[Ans. (0, 0, 1), (4/5, 1/5, 0), v = 2.]

9. Transform the following matrix game into its corresponding primal and dual linear programming problems :

$$\begin{pmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{pmatrix}$$

Solve one of these linear programming problems to obtain the value and the optimal strategies for the two players.

[Ans. ***Primal.*** Min. $\mathbf{x}_0 = X_1 + X_2$ subject to the constraints :
$5X_1 + 4X_2 \ge 1,\ 4X_1 + 3X_2 \ge 1,\ 3X_1 + 6X_2 \ge 1,\ X_1 + 5X_2 \ge 1$
and $X_1 \ge 0, X_2 \ge 0.$

Dual : Max. $\mathbf{y}_0 = Y_1 + Y_2 + Y_3$ subject to the constraints :
$5Y_1 + 4Y_2 + 3Y_3 + Y_4 \le 1,\ 4Y_1 + 3Y_2 + 6Y_3 + 5Y_4 \le 1$
$Y_i \ge 0, i = 1, 2, 3, 4,$ and $c = 3.$]

10. In a two person game each player simultaneously shows either one or two fingers. If the number of fingers match, player A wins a rupee from player B, otherwise A pays a rupee to B. Show that the payoff matrix for this game is $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ Solve this game by an L.P.P.

[**Ans.** (1/2, 1/2), (1/2, 1/2), $v = 0$.]

11. Two players independently select one of 'mouse', 'cat', 'tiger'. and 'elephant' and simultaneously reveal their choices. It is known that the cat chases the mouse (for score 1), the tiger chases the cat (for score 2), the elephant chases the tiger (for score 3) and the mouse chases the elephant (for a score 4). All other combinations yield a zero score. Formulate the payoff matrix and determine the optimal strategies of the two players.

[**Hint.** The payoff matrix is skew-symmetric :

$$\begin{array}{l} \\ \textit{Mouse} \\ \textit{Cat} \\ \textit{Tiger} \\ \textit{Elephant} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Mouse} & \textit{Cat} & \textit{Tiger} & \textit{Elephant} \end{array} \\ \begin{bmatrix} 0 & -1 & 0 & 4 \\ 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -3 \\ -4 & 0 & 3 & 0 \end{bmatrix} \end{array}.$$

12. Solve by using L.P. process, whose pay-off matrix is :

$$\textit{Player A} \begin{array}{c} \textit{Player B} \\ \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 6 & 2 \end{bmatrix} \end{array}$$

[Delhi B.Sc. (Stat.) 2000]

[**Ans.** $A(2/3, 1/3, 0)$, $B(0, 1/2, 1/2)$ and $v = 1$.]

13. For the following pay-off matrix, find the value of the game and the strategies of players A and B by using linear programming :

		Player B 1	2	3
Player A	**1**	3	−1	4
	2	6	7	−2

[Delhi (M.B.A.) 1996]

[**Ans.** The solution to the problem, therefore, is :
$S_A = (9/14, 5/14)$, $S_B = (0, 3/7, 4/7)$, value of game = 13/7.]

14. Distinguish between deterministic and probabilistic games. Children A and B play a game who have some 25 paise coins and 50 paise coins. Each draw a coin from their bags wihout knowing other's choice. If the sum of coins drawn by both is even A wins them, otherwise B wins. Find the best strategy for each player and also find the value of the game. **[JNTU B.Tech. III (CS & Engg.) 2011]**

TWO-BY-TWO (2× 2) GAMES WITHOUT SADDLE POINT

There are several methods for determining the optimal strategies and the value of the game. But, in most of the situations, the matrix game can be reduced to a 2×2 game (to be discussed later on page GMS/17&26). It is therefore worthwhile to determine the solution of 2×2 game in the following theorem.

Theorem : *Show that for any zero-sum two-person game where optimal strategies are not pure strategies (i.e. there is no saddle point) and for which the player A's payoff matrix is*

		B y_1	y_2
A	x_1	v_{11}	v_{12}
	x_2	v_{21}	v_{22}

and optimal strategies (x_1, x_2) *and* (y_1, y_2) *are determined by*

$$\frac{x_1}{x_2} = \frac{v_{22} - v_{21}}{v_{11} - v_{12}}, \quad \frac{y_1}{y_2} = \frac{v_{22} - v_{12}}{v_{11} - v_{21}},$$

and the value (v) *of the game to the player A is given by*

$$v = \frac{v_{11} v_{22} - v_{12} v_{21}}{v_{11} + v_{22} - (v_{12} + v_{21})}.$$ **[Meerut 2002]**

Proof. Let a mixed strategy for player A be given by (x_1, x_2) where $x_1 + x_2 = 1$. Thus if player B moves his first strategy, the net expected gain of A will be $E_1(x) = v_{11}x_1 + v_{21}x_2$;
and if B moves his second strategy, the net expected gain of A will be $E_2(x) = v_{12}x_1 + v_{22}x_2$.

But, player A wants to maximize his minimum expected gain. So the value of the game (v) must be minimum of $E_1(x)$ and $E_2(x)$, *i.e.* $E_1(x) \geq v$, $E_2(x) \geq v$.

Thus for the player A, we have to find $x_1 \geq 0$, $x_2 \geq 0$, and v to satisfy the following three relationships (***See p.GMS/11***) :

$$v_{11}x_1 + v_{21}x_2 \geq v, \quad ...(1)$$

$$v_{12}x_1 + v_{22}x_2 \geq v, \quad ...(2)$$

$$x_1 + x_2 = 1. \quad ...(3)$$

For optimum strategies, inequalities (1) and (2) become strict equations, *i.e.*

$$v_{11}x_1 + v_{21}x_2 = v, \quad ...(4)$$

$$v_{12}x_1 + v_{22}x_2 = v. \quad ...(5)$$

Subtracting equation (5) from the equation (4) we get

$$(v_{11} - v_{12})\, x_1 + (v_{21} - v_{22})\, x_2 = 0. \quad ...(6)$$

which gives

$$\frac{x_1}{x_2} = \frac{v_{22} - v_{21}}{v_{11} - v_{12}}. \quad ...(7)$$

Hence, we evaluate x_1 and x_2 separately by using the equation (3)

$$x_1 = \frac{v_{22} - v_{21}}{v_{11} + v_{22} - (v_{12} + v_{21})} \quad ...(8)$$

$$x_2 = 1 - x_1 = \frac{v_{11} - v_{12}}{(v_{11} + v_{22}) - (v_{12} + v_{21})}. \quad ...(9)$$

The value of the game can be obtained by substituting the values of x_1 and x_2 in either of the equations (4) or (5) to obtain

$$v = \frac{v_{11}(v_{22} - v_{21})}{v_{11} + v_{22} - (v_{12} + v_{21})} + \frac{v_{21}(v_{11} - v_{12})}{v_{11} + v_{22} - (v_{12} + v_{21})}$$

or $$v = \frac{v_{11}v_{22} - v_{21}v_{12}}{v_{11} + v_{22} - (v_{12} + v_{21})}. \quad ...(10)$$

In the same manner for the player B, find $y_1 \geq 0$, $y_2 \geq 0$, and v to satisfy the following three relations :

$$v_{11}y_1 + v_{12}y_2 \leq v, \quad ...(11)$$

$$v_{21}y_1 + v_{22}y_2 \leq v, \quad ...(12)$$

$$y_1 + y_2 = 1. \quad ...(13)$$

Here it should be remembered that the player B wants to minimize his maximum loss.

Again for optimum strategies of player B, consider the inequalities (11) and (12) as strict equations and obtain

$$\frac{y_1}{y_2} = \frac{v_{22} - v_{12}}{v_{11} - v_{21}}. \quad ...(14)$$

Using the equation (13)

$$y_1 = \frac{v_{22} - v_{12}}{v_{11} + v_{22} - (v_{21} + v_{12})} \quad ...(15)$$

$$y_2 = 1 - y_1 = \frac{v_{11} - v_{21}}{v_{11} + v_{22} - (v_{21} + v_{12})}. \qquad ...(16)$$

Substituting values of y_1 and y_2 in either of the equation (11) or (12), to obtain the value

$$v = \frac{v_{11}v_{22} - v_{21}v_{12}}{v_{11} + v_{22} - (v_{12} + v_{21})}, \qquad ...(17)$$

which is the same as desired by the *minimax theorem.*

If ratios x_1/x_2 and y_1/y_2 are both positive, these will give acceptable values of x_1 , x_2 , y_1 and y_2 . A solution satisfying all constraints including non-negativity, may be obtained.

This proves the requied results.

Further for such games, in a payoff matrix *the largest and second largest elements must lie on one of the diagonals.* This implies that there are only 8 possible orderings (instead of 24) of entries v_{11} , v_{12} , v_{21} , v_{22} without saddle point.

These possibilities are :

$$\begin{Bmatrix} v_{11} \ge v_{22} \ge v_{12} \ge v_{21} \\ v_{11} \ge v_{22} \ge v_{21} \ge v_{12} \end{Bmatrix}, \begin{Bmatrix} v_{22} \ge v_{11} \ge v_{21} \ge v_{12} \\ v_{22} \ge v_{11} \ge v_{12} \ge v_{21} \end{Bmatrix},$$

$$\begin{Bmatrix} v_{12} \ge v_{21} \ge v_{11} \ge v_{22} \\ v_{12} \ge v_{21} \ge v_{22} \ge v_{11} \end{Bmatrix}, \begin{Bmatrix} v_{21} \ge v_{12} \ge v_{11} \ge v_{22} \\ v_{21} \ge v_{12} \ge v_{22} \ge v_{11} \end{Bmatrix}.$$

It can be easily verified that, with all above orderings, ratios x_1/x_2 and y_1/y_2 are non-negative.

Illustrative Examples

Example 1 *Consider a 'modified' form of "matching biased coins" game problem. The matching player is paid Rs.* 8 *if the two coins turn both heads and Re* 1 *if the coins turn both tails. The non-matching player is paid Rs.* 3 *when the two coins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your stretagy?* **[Delhi (MBA) 1999]**

Solution For the matching player, the payoff matrix is given by

$$\begin{array}{c|c} & \text{Non-matching player} \\ \text{Matching Player} \begin{matrix} \\ H \\ T \end{matrix} & \begin{bmatrix} H & T \\ 8 & -3 \\ -3 & 1 \end{bmatrix} \end{array}$$

Here, $v_1 1 = 8, v_{12} = -3, v_{21} = -3, v_{22} = 1$

Clearly, the payoff matrix does not possess any *saddle point.* The players will use the mixed strategies. The optimum mixed strategy for matching player is determined by

$$x_1 = \frac{v_{22} - v_{21}}{v_{11} + v_{22} - (v_{12} + v_{21})} = \frac{1 - (-3)}{8 + 1 - [-3 + (-3)]} = \frac{4}{15}$$

$$\therefore \qquad x_2 = 1 - \left(\frac{4}{15}\right) = \frac{11}{15}.$$

And, for the non-matching player, by

$$y_1 = \frac{v_{22} - v_{12}}{v_{11} + v_{22} - (v_{21} + v_{12})} = \frac{1 - (-3)}{8 + 1 - [-3 + (-3)]} = \frac{4}{15}$$

$$\therefore \qquad y_2 = 1 - \left(\frac{4}{15}\right) = \frac{11}{15}$$

The expected value of the game is given by

$$v = \frac{v_{11}v_{22} - v_{21}v_{12}}{v_{11} + v_{22} - (v_{12} + v_{21})} = \frac{8 \times 1 - (-3) \times (-3)}{8 + 1 - [-3 + (-3)]} = \frac{-1}{15}.$$

Thus, the optimum mixed strategies for matching player and non-matching player are given by (4/15, 11/15) and (4/15, 11/15). Thus, we would like to be non-matching player.

Example 2 *For the game with the following payoff matrix, determine the optimum strategies and the value of the game :*

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

[Madurai (M.Com.) 2002]

Solution Obviously the given matrix is without saddle point. Thus, the mixed strategies of A and B are (x_1, x_2) and (y_1, y_2) respectively. Clearly, $x_1 + x_2 = 1$ and $y_1 + y_2 = 1$. If $E(x, y)$ denotes the expected payoff functions, then

$$E(x, y) = 5x_1y_1 + 3(1 - x_1)y_1 + x_1(1 - y_1) + 4(1 - x_1)(1 - y_1)$$

$$= 5x_1y_1 - 3x_1 - y_1 + 4 = 5(x_1 - 1/5)(y_1 - 3/5)$$

If A closes $x_1 = \frac{1}{5}$, he enswres that his expectation is at least 17/5. He cannot be sure of more than 17/5. Because, by selecting $y_1 = 3/5$, B can keep $E(x_1, y_1)$ down to 17/5. Therefore, A might as well settle for 17/5 and B reconcile to 17/5. Thus, the optinum strategies for A and B are (1/5, 4/5) and (3/5, 2/5), respectively and the value of the game is $v = 17/5$.

REMARK

If these formulae for x_1, x_2, y_1, y_2 and v are applied to a 2×2 games with saddle point, these may give an incorrect solution.

Q. 1. For the game $\begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$, where a, b, c, d are all non-negative ≥ 0, prove that the optimal strategies are :

$$A : \left(\frac{c+d}{a+b+c+d}, \frac{a+b}{a+b+c+d}\right),$$

$$B : \left(\frac{b+d}{a+b+c+d}, \frac{a+c}{a+b+c+d}\right)$$

and $v = \dfrac{ad - bc}{a+b+c+d}$.

[Delhi B.Sc. (Stat.) 1998]

2. Given the 2×2 payoff matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose player A adopts the strategy (x, y); while B adopts the strategy (u, v) where x, y, u, v are all ≥ 0, such that

$x + y = u + v = 1$

(i) Express A's expected gain z in terms of x, y, u, v and a, b, c, d.

(ii) What is the effect on z of adding the same constant k to each element of the payoff matrix ?

(iii) What is the effect on z of multiplying each element of payoff matrix by the same constant k ?

(iv) How are the optimal strategies affected by these operations on payoff matrix.

3. If $G = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a non-strictly determined matrix game, then show that either :

(i) $a < b, a < c, d < c, d < b$ or (ii) $a > b, a > c, d > c, d > b$.

[Kanpur 2000]

4. Prove that 2×2 matrix game is strictly determined only if its principal diagonal elements are either strictly greater or strictly smaller than the other elements.

5. If all the elements of the payoff matrix of a game are non-negative and every column of this matrix has at least one positive element, then the value of the corresponding game is positive.

6. What do you mean by saddle point of a two-person zero-sum game ? In a 2×2 game, if the largest and second largest elements lie along a diagonal, then prove that the game has no saddle point.

7. Let (a_{ij}) be the payoff matrix for a two-person zero-sum game. Examine the game for saddle point under the following orderings of its elements :

(*i*) $a_{21} \geq a_{22} \geq a_{11} \geq a_{12}$ (*ii*) $a_{11} \leq a_{12} \leq a_{21} \leq a_{22}$
(*iii*) $a_{12} \leq a_{22} \leq a_{11} \leq a_{21}$ (*iv*) $a_{22} \leq a_{11} \leq a_{12} \leq a_{21}$.

8. For a two-person zero-sum game, the payoff matrix for player A is $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with no saddle point. Obtain the optimal strategies (x_1, x_2) and (y_1, y_2), respectively.

NOTE Students are advised to solve 2×2 games without saddle point by originally constructing the relationship for both the players instead of using the formulae (8), (9), (15), (16) and (17) directly.

Examination PROBLEMS

Solve the following 2×2 games without saddle points :

1. $A \overset{B}{\begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix}}$. [**Ans.** (1/4, 3/4) for both player, $v = -3/4$.]

2. $A \overset{B}{\begin{bmatrix} 2 & 5 \\ 7 & 3 \end{bmatrix}}$. [**Ans.** (1/3, 2/3), (3/5, 2/5), $v = 0$.]

3. $A \overset{B}{\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}}$. [**Ans.** (1/3, 2/3), (3/5, 2/5), $v = 0$.]

4. $A \overset{B}{\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}}$. [**Ans.** A (1/2, 1/2), B (1/2, 1/2), $v = 0$.]

5. $A \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$. [**Ans.** A (1/2, 1/2), B (1/2, 1/2), $v = 0$.]

6. Two players A and B match coins. If the coins match, then A wins one unit of value, if the coins do not match, then B wins one unit of value. Determine optimum strategies for the players and the value of the game.
[**Hint.** Formulation of the game is :

$$\begin{array}{cc} & B \\ & \begin{array}{cc} H & T \end{array} \\ A \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \end{array}.]$$

[**Ans.** (1/2, 1/2), (1/2, 1/2), $v = 0$]

7. A and B each take out one or two matches and guess how many matches opponent has taken. If one of the players guess correctly then the looser has to pay him as many rupees as the sum of the numbers held by both players. Otherwise, the payout is zero. Write down the payoff matrix and obtain the optimal strategies of both players.
[**Hint.** Formulation of the game is :

$$\begin{array}{cc} & B \\ & \begin{array}{cc} 1 & 2 \end{array} \\ A \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \end{array}.$$

[**Ans.** (2/3, 1/3), (2/3, 1/3), $v = 4/3$.]

8. In a game of matching coins with two players, suppose A wins one unit of value when there are two heads, wins nothing when there are two tails, and loses 1/2 unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies for each player, and the value of the game to A.
[**Hint.** Formulation of the game is :

$$\begin{array}{cc} & B \\ & \begin{array}{cc} H & T \end{array} \\ A \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 1 & -1/2 \\ -1/2 & 0 \end{bmatrix} \end{array}.$$

[**Ans.** (1/4, 3/4), (1/4, 3/4), $v = -1/8$.]

9. Consider a modified form of 'matching biased wins' game problem. The matching player is paid eight rupees if the two wins turn both heads and one rupee if the two wins turn both tails. The non-matching player is paid three rupees when the two wins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy ?
[Delhi (MBA) 1999; IAS (Maths.) 97]

10. Solve the following game and determine the value of the game :

		Player Y	
		Strategy 1	**Strategy 2**
Player X	**Strategy 1**	4	1
	Strategy 2	2	3

[Allahabad (M.B.A.) 1998]

[**Ans.** The optimum strategies for the two players are :
$S_X = (1/4, 3/4)$ and $S_Y = (1/2, 1/2)$ and the value of game = 10/4.]

ARITHMETIC METHOD FOR (2 × 2) GAMES

Arithmetic method provides an easy technique for obtaining the optimum strategies for each player in (2×2) games without saddle point. This method consists of the following steps :

Step 1. Find the difference of two numbers in column I, and put it under the column II, neglecting the negative sign if occurs.

Step 2. Find the difference of two numbers in column II, and put it under the column I, neglecting the negative sign if occurs.

Step 3. Repeat the above two steps for the two rows also.

The values thus obtained are called the ***oddments.*** These are the frequencies with which the players must use their courses of action in their optimum strategies.

The above steps can be explained by solving the following example.

Illustrative Examples

Example *Two players A and B without showing each other, put on a table a coin, with head or tail up. A wins Rs. 8 when both the coins show head and Re. 1 when both are tails. B wins Rs. 3 when the coins do not match. Given the choice of being matching player (A) or non-matching player (B), which one would you choose and what would be your strategy ?* **[Delhi (MBA) 1999]**

Solution. The payoff matrix for A is found to be as follows :

		Player B			
		H	T		
Player A	H	8	−3	4	$\frac{4}{11+4}=\frac{4}{15}$
	T	−3	1	11	$\frac{11}{11+4}=\frac{11}{15}$
		4/15	11/15		

Since no saddle point is found, the optimal strategies will be the mixed strategies.

Step 1. Taking the difference of two numbers in column I, we find $8-(-3)=11$, and put it under column II.

Step 2. Taking the difference of two numbers in column II, we find $(-3-1)=-4$, and put the number 4 (neglecting the − ve sign) under column I.

Step 3. Repeat the above two steps for the two rows also.

Thus for optimum gains, player A must use strategy H with probability 4/15 and strategy T with probability 11/15, while player B must use strategy H with probability 4/15 and strategy T with probabiltiy 11/15.

Step 4. To obtain the value of the game any of the following expressions may be used.

Using B's oddments **:**

B plays H, value of the game,

$$v = \text{Rs.}\ \frac{4\times 8+11\times(-3)}{11+4} = \text{Rs.}\left(-\frac{1}{15}\right)$$

B plays T, value of the game $v = \text{Rs.}\ \frac{4\times(-3)+11\times 1}{11+4}$

$$= \text{Rs.}\left(-\frac{1}{15}\right).$$

Using A's oddments **:**

A plays H, value of the game, $v = \text{Rs.}\ \frac{4\times 8+11\times(-3)}{4+11}$

$$= \text{Rs.}\left(-\frac{1}{15}\right)$$

A plays T, value of the game, $v = \text{Rs.}\ \frac{4\times(-3)+11\times 1}{4+11}$

$$= \text{Rs.}\left(-\frac{1}{15}\right).$$

The above values of v are equal only if the sum of the oddments vertically and horizontally are equal. Cases in which it is not so, will be discussed later.

Thus the complete solution of the game is :

(i) optimum strategy for A is $(4/15, 11/15)$. and for B is $(4/15, 11/15)$.

(ii) value of the game to A is $v=\text{Rs.}\ (-1/15)$ and to B is 1/15.

Thus, player A gains Rs. (−1/15), *i. e.*, he loses Rs. (1/15) which B, in turn, gets.

> **NOTE** Arithmetic method is easier than the algebraic method but it cannot be applied to larger games.

PRINCIPLE OF DOMINANCE TO REDUCE THE SIZE OF THE GAME

For easiness of solutions, it is always convenient to deal with smaller payoff matrices. Fortunately, the size of the payoff matrix can be considerably reduced by using the so called *principle of dominance*. Before stating this principle, let us define a few important terms.

Inferior and Superior Strategies. *Consider two n-tuples* $\mathbf{a}=(a_1, a_2, \ldots, a_n)$ *and* $\mathbf{b}=(b_1, b_2, \ldots, b_n)$. *If* $a_i \geq b_i$ *for all* $i=1, 2, \ldots, n$, *then for player A the strategy corresponding to* **b** *is said to be* ***inferior*** *to the strategy corresponding to* **a**; *and equivalently, the strategy corresponding to* **a** *is said to* ***be superior*** *to the strategy corresponding to* **b**.

For player B, the above situation will be reversed, because player A's gain-matrix is player B's loss-matrix.

Dominance. *An n-tuple* $\mathbf{a}=(a_1, a_2, \ldots, a_n)$ *is said to* ***dominate*** *the n-tuple* $\mathbf{b}=(b_1, b_2, \ldots, b_n)$ *if* $a_i \geq b_i$ *for all* $i=1, 2, \ldots, n$. *The superior strategies are said to dominate the inferior ones.*

Thus a player would not like to use inferior strategies which are dominated by other's. Now we are able to state the principle of dominance as follows :

Principle of Dominance. *If one pure strategy of a player is better or superior than another one (irrespective of the strategy emplayed by his opponent), then the inferior strategy may be simply ignored by assigning a zero probability while searching for optimal strategies.*

Theorem 1. Dominance Property. *Let* $\mathbf{A}=[v_{ij}]$ *be the payoff matrix of an* $m\times n$ *rectangular game. If the ith row of* **A** *is dominated by the rth row of* **A**, *then the deletion of ith row of* **A** *does not change the set of optimal strategies for the row player (player* **A**).

Further, if the jth column of **A** *dominates the kth column of* **A**, *then the deletion of jth column of* **A** *does not change the set of optimal strategies for the column player (player B).*

Proof. Given that

$$v_{ij} \leq v_{rj}, \text{ for all } j=1, 2, \ldots, n \text{ and } v_{ij} \neq v_{rj} \text{ for at least one } j. \quad \ldots(1)$$

Let $\mathbf{y}^*=(y_1{}^*, y_2{}^*, \ldots, y_n{}^*)$ be an optimal strategy for the column player B. It follows from (1) that

$$\sum_{j=1}^{n} v_{ij}\, y_j{}^* < \sum_{j=1}^{n} v_{rj}\, y_j{}^*$$

or $$E(\mathbf{e_i}, \mathbf{y^*}) < E(\mathbf{e_r}, \mathbf{y^*})$$

∴ $$v \geq E(\mathbf{e_r}, \mathbf{y^*}) > E(\mathbf{e_i}, \mathbf{y^*}). \quad \ldots(2)$$

Now let $\mathbf{x}^* = (x_1{}^*, x_2{}^*, \ldots, x_m{}^*)$ be an optimal strategy for the row player. If possible, let us suppose $x_i{}^* > 0$, then from (2),

$$x_i^* \, v > x_i^* \, E(\mathbf{e_i}, \mathbf{y}^*).$$

Also, we have

$$v = E(\mathbf{x}^*, \mathbf{y}^*) = \sum_{i=1}^{m} x_i^* \, E(\mathbf{e_i}, \mathbf{y}^*)$$

$$= x_r^* \, E(\mathbf{e_r}, \mathbf{y}^*) + \sum_{i \neq r}^{m} x_i^* \, E(\mathbf{e_i}, \mathbf{y}^*)$$

$$< x_r^* v + \sum_{i \neq r}^{m} x_i^* v = v \sum_{i=1}^{m} x_i^* = v \quad (\because \sum_{i=1}^{m} x_i^* = 1)$$

which is a contradiction, and hence $x_i^* = 0$.

Second part can also be proved similarly.

Generalized Dominance Property

The dominance property is not only based on the superiority of pure strategies only, but on the superiority of some *convex linear combination* of two or more pure strategies also. A given strategy can also be said to be dominated if it is inferior to some convex linear combination of two or more strategies. This concept generalizes the above dominance principle in the following theorem.

Theorem 2. Generlized Dominance. Let $\mathbf{A} = [v_{ij}]$ be the pay-off matrix of an $m \times n$ rectangular game. If the ith row of A is strictly dominated by a convex combination of the other rows of A, then the deletion of the ith row of A does not effect the set of optimal strategies for the row player (the player A).

Further, if the jth column of $\mathbf{A}$ *strictly dominates a convex combination of the other columns, then the deletion of the jth column of* $\mathbf{A}$ *does not effect the optimal strategies for the column player (the player B).*

Proof. Let $A = [v_{ij}]$ be the payoff matrix considering the first part, we are given that there exist scalars (probabilities) $x_1, x_2, \ldots, x_m$ $(0 \le x_i \le 1, x_r = 0, \Sigma x_i = 1)$ such that

$$\sum_{i=1,\, i \neq r}^{m} x_i v_{ij} \ge v_{rj}, \text{ for } j = 1, 2, \ldots, n$$

or $$\sum_{i=1}^{m} x_i v_{ij} \ge v_{rj}, \quad \text{for } j = 1, 2, \ldots n \qquad (\because x_r = 0) \quad \ldots(1)$$

where strict inequality holds for at least one j.

Let $y^* = (y_1{}^*, y_2{}^*, \ldots, y_n{}^*)$ be an optimal strategy for player B. Then it follows from (1) that

$$\sum_{j=1}^{n} v_{rj} \, y_j{}^* < \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} \, x_i \, y_j{}^*,$$

or $$E(\boldsymbol{e_r}, \mathbf{y}^*) < \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} \, x_i \, y_j^* \le v. \quad \ldots(2)$$

Let $\mathbf{x}^* = (x_1{}^*, x_2{}^*, \ldots, x_m{}^*)$ be an optimal strategy for player A.

If possible, let us suppose that $x_r{}^* \neq 0$. From (2), we know that $E(\mathbf{e_r}, \mathbf{y}^*) < v$.

Then since $x_r{}^* \neq 0$, we must have $x_r^* E(\mathbf{e_r}, \mathbf{y}^*) < x_i^* \, v$.

Thus $$E(\boldsymbol{x}^*, \mathbf{y}^*) = \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} \, x_i^* \, y_j^*$$

$$= x_r{}^* \, E(\mathbf{e_r}, \mathbf{y}^*) + \sum_{i \neq r} x_i^* \, E(\mathbf{e_i}, \mathbf{y}^*).$$

This implies $v < x_r^* \, v + v \sum_{i \neq r} x_i^* = v \sum_{i=1}^{m} x_i^* = v,$

which is a contradiction.

Hence we must have $x_r^* = 0$. This completes the proof for the *first part.*

Similarly, we can prove the *second part.*

REMARKS

1. If $v_{rj} = \sum_{1}^{m} x_i \, v_{ij}$ for all $j = 1, 2, \ldots, n$; the result follows trivially, for then any probability assigned to the rth row can be easily distributed over the other rows, and rth row itself is ignored.
2. It should also be noted here that the dominating column is deleted whereas the row dominated by a convex combination of other rows is deleted.

Summary of Dominance Rules

The *"dominance property"* can be summerized in the following rules :

***Rule* 1.** If each element in one row, say rth of the payoff matrix $[v_{ij}]$, is less than or equal to the corresponding element in the other row, say sth, then the player A will never choose the rth strategy. In other words, if for all $j = 1, 2, \ldots, n$, and $v_{rj} \le v_{sj}$, then the probability x_r of choosing rth strategy will be zero. The value of the game and the non-zero choice of probabilities remain unaltered even if rth row is deleted from the payoff matrix. Such rth row is said to be dominated by the sth row.

***Rule* 2.** Following the similar arguments, if each element in one column, say C_r, is greater than or equal to the corresponding element in the other column, say C_s, then the player B will never use the stategy corresponding to column C_r. In this case, the column C_s dominates the column C_r.

***Rule* 3.** Dominance need not be based on the superiority of pure strategies only. A given strategy can be dominated if it is inferior to an average of two or more other pure strategies. In general, if some convex linear combination of some rows dominates the ith row, then the ith row will be deleted. If the ith row dominates the convex linear combination of some other rows, then one of the rows involving in the combination may be deleted. Similar arguments follow for columns also.

***Rule* 4.** If (x_1, x_2) be the optimal strategy for the player A for the reduced game and (w_1, x_2) be the optimal strategy for the original game, then w_1 is the ith place extension of x_1.

***Rule* 5.** If (y_1, y_2) be the optimal strategy for the player B for the reduced game and (y_1, w_2) be the optimal strategy for the original game, then w_2 is the jth place extension of y_2.

Rule 6. If the dominance holds strictly, then values of optimal strategies do coincide, and when the dominance does not hold strictly, then optimal strategies may not coincide.

NOTE Using dominance properties, try to reduce the size of payoff matrix.

Demonstration of Dominance Properties

1. *To illustrate first and second properties,* consider the example of (3×3) game [*Table* 1].

Table 1

		B I	B II	B III
A	I	−4	6	3
A	II	−3	−3	4
A	III	2	−3	4

It is clear that this game has no saddle point. However, consider Ist and IIIrd columns from player B's point of view. It is seen that each payoff (element) in IIIrd column is greater than the corresponding element in Ist column regardless of the player A's strategy. Evidently, the choice of IIIrd strategy by the player B will always result in the greater loss as compared to that of selecting the Ist strategy. The column III is *inferior* to I as never to be used. Hence, deleting the IIIrd column which is dominated by I, the reduced-size payoff matrix (*Table 2*) is obtained.

Table 2

		B I	B II
A	I	−4	6
A	II	−3	−3
A	III	2	−3

Again, if the reduced matrix (*Table 2*) is looked from player A's point of view, it is seen that the player A will never use the II strategy which is dominated by III. Hence, the size of matrix can be reduced further by deleting the II row (*Table 2*). This reduced matrix can be further reduced by deleting II row as shown in *Table 3*. The solution of the reduced 2×2 matrix game without saddle point can be easily obtained by solving the following simultaneous equations in usual notations :

Table 3

		B I	B II
A	I	−4	6
A	III	2	−3

$$-4x_1 + 2x_3 = v, \; 6x_1 - 3x_3 = v, \; x_3 + x_1 = 1 \qquad (For\ A)$$

and $$-4y_1 + 6y_2 = v, \; 2y_1 - 3y_2 = v, \; y_1 + y_2 = 1 \qquad (For\ B)$$

It is advisable to verify the solution :

(*i*) The player A chooses mixed strategy $(x_1, x_2, x_3) = (1/3, 0, 2/3)$.

(*ii*) The player B chooses mixed strategy $(y_1, y_2, y_3) = (3/5, 2/5, 0)$.

(*iii*) The value of the game is zero, *i.e.* the game is *fair*.

Q. 1. Explain the term 'saddle point' and 'dominance' used in game theory.

2. Write short note on 'concept of dominance'.

3. Explain the concept of generalized dominance in the context of game theory. **[Kanpur M.Sc. (Maths.) 1997]**

4. Briefly explain the general rules for dominance. **[JNTU (IV B. Tech., I Sem., Mech.) Feb. 2007 (Set 3), March 2006 (Set 1), May 2004]**

5. Explain the theory of dominance in the solution of rectangular games.

6. Explain the principle of derminance to reluce the size of the game. **[Meerut M.Sc (OR) 2012]**

2. *To illustrate the third property,* consider the following game matrix [*Table 4*]

Table 4

		B 1	B 2	B 3
A	1	5	0	2
A	2	−1	8	6
A	3	1	2	3

None of the *pure* strategies of the player A is inferior to any of his other pure strategies. However, the average of the player A's first and second pure strategies gives

$$\left\{\frac{5-1}{2}, \frac{0+8}{2}, \frac{2+6}{2}\right\} \quad \text{or} \quad (2, 4, 4)$$

Obviously, this is superior to the player A's third pure strategy. So the third strategy may be deleted from the matrix. The reduced matrix is shown in *Table* 5.

Table 5

	1	2	3
1	5	0	2
2	−1	8	6

Q. 1. Explain the following terms :

(i) Two-person zero-sum game,

(ii) Principle of dominance,

(iii) Pure strategy in game theory. **[Meerut 2001]**

2. How is the concept of dominance used in simplifying the solution of a rectangular game ? **[Meerut (OR) 2003]**

3. Explain the principle and rules of dominance to reduce the size of payoff matrix. **[VTU (BE Mech.) 2002]**

4. State the general rules of dominance for two-person, zero-sum games. **[Annamalai 2009]**

5. Let R_1, R_2 be the subsets of the rows of an $m \times n$ pay-off matrix A. Likewise, let C_1, C_2 be the subsets of the columns of A. Show that if a convex combination of the rows (columns) in R_1 (C_1) dominates a convex combination of the rows (columns) in R_2 (C_2), then there exists a row (column) in R_2 (C_1) which, if deleted, does not change the set of optimal strategies for player A (player B).

6. Show that the existence of a saddle point in 2×2 game implies the existence of a dominating pure strategy for at least one of the players, and conversely.

Illustrative Examples

Example 1. *Solve the game whose payoff matrix to the player A is given in the table :*

		B: I	II	III
A	I	1	7	2
	II	6	2	7
	III	5	2	6

[JNTU (IV B. Tech,, I Sem.) 2004, 03 (type)]

Solution Since the row III is inferior to the row II, row III can be deleted from the payoff matrix. Thus the reduced matrix (*Table 1*) is obtained.

Table 1

		B: I	II	III
A	I	1	7	2
	II	6	2	7

Again, column III is dominated by column I, therefore column III can also be deleted from the above matrix. The reduced matrix is given in (*Table 2*).

Table 2

		B: (y_1) I	(y_2) II
A	(x_1) I	1	7
	(x_2) II	6	2

This 2×2 game without saddle point can be solved either by putting $v_{11} = 1, v_{12} = 7, v_{21} = 6, v_{22} = 2$ in the formulae of (8), (9), (15), (16), and (17), or by solving the simultaneous equations :

$$1x_1 + 6x_2 = v,\ 7x_1 + 2x_2 = v,\ x_1 + x_2 = 1 \quad (\textbf{For player } A)$$

$$1y_1 + 7y_2 = v,\ 6y_1 + 2y_2 = v,\ y_1 + y_2 = 1 (\textbf{For player } B).$$

Thus the following solution is obtained :

(i) The player A chooses optimal strategy $(x_1, x_2, x_3) = (2/5, 3/5, 0)$.

(ii) The player B chooses optimal strategy $(y_1, y_2, y_3) = (1/2, 1/2, 0)$.

(iii) The value of the game to the player A is $v = 4$.

Example 2. *Use the relation of dominance to solve the rectangular game whose payoff matrix to A is given in Table below :*

		B: I	II	III	IV	V	VI
A	I	0	0	0	0	0	0
	II	4	2	0	2	1	1
	III	4	3	1	3	2	2
	IV	4	3	7	−5	1	2
	V	4	3	4	−1	2	2
	VI	4	3	3	−2	2	2

[Delhi B.Sc. (Stat.) 2005; Meerut M.Sc. (Math.) 1998; Gujarat (MBA) 1997]

Solution. In the payoff matrix from player A's point of view, rows I and II are dominated by the row III. Hence the player A will never use strategies I and II in comparison to the strategy III. Thus, deleting I and II rows we obtain the reduced matrix (*Table 1*).

Table 1

		B: I	II	III	IV	V	VI
A	III	4	3	1	3	2	2
	IV	4	3	7	−5	1	2
	V	4	3	4	−1	2	2
	VI	4	3	3	−2	2	2

Again, from the player B's point of view, columns I, II and VI are dominated by the column V. Therefore, the player B will never use strategies I, II and VI in comparison to the strategy V. Now, delete columns I, II and VI from the matrix to obtain the new matrix (*Table 2*).

Table 2

		B: III	IV	V
A	III	1	3	2
	IV	7	−5	1
	V	4	−1	2
	VI	3	−2	2

Again the row VI is dominated by the row V from the player A's point of view. Hence, deleting VIth row, obtain the next reduced matrix.

Table 3

	III	IV	V
III	1	3	2
IV	7	−5	1
V	4	−1	2

None of the pure strategies of the player B is inferior to any of his other strategies. However, the average of player B's III and IV pure strategies gives,

$$\left\{\frac{1+3}{2}, \frac{7-5}{2}, \frac{4-1}{2}\right\} \quad \text{or} \quad (2, 1\ 3/2)$$

which is obviously superior to the player B's Vth pure strategy, because Vth strategy will result much more losses to B. Thus deleting the Vth strategy from the matrix, the revised matrix (*Table 4*). is obtained :

Table 4

		B: III	IV
A	III	1	3
	IV	7	−5
	V	4	−1

Also, the average of the player A's III and IV pure strategies give

$$\left\{\frac{1+7}{2}, \frac{3-5}{2}\right\} \quad \text{or} \quad (4, -1).$$

This is obviously the same as the player A's Vth strategy.

In this case, the Vth strategy may be deleted from the matrix. Finally, (2×2) reduced matrix (*Table 5*) is obtained.

Table 5

		B (y_3) III	(y_4) IV
A	(x_3) III	1	3
	(x_4) IV	7	−5

Now, for (2×2) game, having no saddle point, solve the following simultaneous equations :

$1.x_3 + 7x_4 = v$, $3x_3 - 5x_4 = v$, $x_3 + x_4 = 1$ (*For A*)

$1.y_3 + 3y_4 = v$, $7y_3 - 5y_4 = v$, $y_3 + y_4 = 1$ (*For B*).

The solution is :

(i) The player A chooses the optimal strategy (0, 0, 6/7, 1/7, 0, 0),

(ii) The player B chooses the optimal strategy (0, 0, 4/7, 3/7, 0, 0),

(iii) The value of the game to player A is 13/7 .

Example 3. *Two competitors A and B are competing for the same product. Their different strategies are given in the following payoff matrix:*

Company B

		I	II	III	IV
Company A	I	3	2	4	0
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

Use dominance principle to find the optimal strategies.

[Punjab (MCA) 2007; Madurai (M. Com.) 2003; JNTU (B. Tech.) 2003 Type; Meerut 02; Delhi (Stat.) 95, B.Sc. (Maths.) 90; Rohilkhand 94, 93; Banasthali 93]

Solution. First, we can find that this game does not have a saddle point. Now try to reduce the size of the given payoff matrix by using the principle of dominance.

From the player A's point of view, Ist row is dominated by the IIIrd row. So delete Ist row from the given pay-off matrix.

Again, from the player B's point of view, 1st column is dominated by the IIIrd column. Hence, Ist column may also be deleted from the matrix. Thus, the reduced payoff matrix [*Table 1*] is obtained.

Table 1

		B II	III	IV
A	II	4	2	4
	III	2	4	0
	IV	4	0	8

In order to check the further reduction of this reduced matrix, the average of the player B's III and IV pure strategies give

$$\left\{\frac{2+4}{2}, \frac{4+0}{2}, \frac{0+8}{2}\right\} \text{ or } (3, 2, 4)$$

which is obviously superior to the player B's II pure strategy. Under this condition, the player B will not use II strategy. Hence, II column may be deleted from the matrix. Thus, new matrix (*Table 2*), is obtained.

Table 2

		B III	IV
A	II	2	4
	III	4	0
	IV	0	8

Again, in the new matrix, the average of the player A's III and IV pure strategies give

$$\left\{\frac{4+0}{2}, \frac{0+8}{2}\right\} \text{ or } (2, 4)$$

which is obviously the same as the player A's II strategy. Therefore, the player A will gain the same amount even if the II strategy is never used by him. Hence deleting the player A's II strategy from the matrix to obtain the reduced (2×2) matrix (*Table 3*).

Table 3

			B (y_3) III	(y_4) IV
A	x_3	III	4	0
	x_4	IV	0	8

Since this (2×2) payoff matrix has no saddle point, solve the simultaneous equations :

$4.x_3 + 0x_4 = v$, $0x_3 + 8x_3 = v$, $x_3 + x_4 = 1$ (*For player A*)

$4.y_3 + 0y_4 = v$, $0y_3 + 8y_4 = v$, $y_3 + y_4 = 1$ (*For player B*)

to get the solution :

(*i*) Optimal strategy for the player A

$= (x_1, x_2, x_3, x_4) = (0, 0, 2/3, 1/3)$.

(*ii*) Optimal strategy for the player B $= (y_1, y_2, y_3, y_4) = (0, 0, 2/3, 1/3)$.

(*iii*) The value of the game to the player A is $v = 8/3$.

Example 4. *Solve the following game using dominance principle :*

$$\text{Player } A \quad \begin{array}{c|ccccc} & \multicolumn{5}{c}{\text{Player } B} \\ & \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \\ \hline \text{I} & 3 & 5 & 4 & 9 & 6 \\ \text{II} & 5 & 6 & 3 & 7 & 8 \\ \text{III} & 8 & 7 & 9 & 8 & 7 \\ \text{IV} & 4 & 2 & 8 & 5 & 3 \end{array}$$

Solution. In the given payoff matrix, IVth column dominates the Ist column and also Vth column dominates the IInd column. So IVth and Vth columns can be deleted without affecting the optimal strategies of B. Thus we get the reduced payoff matrix as shown in (*a*).

		B I	II	III
A	I	3	5	4
	II	5	6	3
	III	8	7	9

(*a*)

	I	II	III
III	8	7	9

(*b*)

	II
III	7

(*c*)

Again, we observe that IIIrd row of the reduced matrix dominates all the other rows. Thus the payoff matrix (*b*) is obtained.

Again, the IInd column of (*b*) is dominated by both the Ist and IIIrd columns. Thus the reduced payoff matrix (*c*) is obtained.

Thus the solution to the game is :

(*i*) best strategy for player *A* is III;

(*ii*) best strategy for player *B* is II; and

(*iii*) value of the game for player *A* is 7, and for player *B* is –7.

Example 5 *The firms A and B have for years been selling a competitive product which forms a part of both firms' total sales. The marketing executive of firm A raised the question. "What should be the firm's strategies in terms of advertising for the product in question? The market research team of firm A developed the following, data for varying degrees of advertising :*

(i) No advertising, medium advertising, and large advertising for both firms will result in equal market shares.

(ii) Firm A with no advertising: 40% of the market with medium advertising by firm B and 28% of the market with large advertising by firm B.

(iii) Firm A using medium advertising: 70% of the market with no advertising by firm B and 45% of the market with large advertising by firm B.

(iv) Firm A using large advertising 75% of the market with no advertising by firm B and 47.5% of the market with medium advertising by firm B.

(a) Based upon the foregoing information, answer the marketing executive's questions.

(b) What advertising policy should firm A persue when consideration is given to the above factors : selling price Rs. 4 *per unit; variable cost of product* Rs. 2.50 *per unit, annual volume of* 30,000 *units for firm A; cost of annual medium advertising* Rs. 5,000 *and cost of annual large advertisitng* Rs. 15,000? *What contribution, before other fixed costs, is available to the firm?*

[Delhi (MBA) 2000]

Solution (*a*) The payoff matrix between the firms *A* and *B* is as follows :

		Firm B		
		No Advertising	Med. Adv.	Large Adv.
Firm A	No Adv.	50	40	28
	Med. Adv.	70	50	45
	Large Adv.	75	47.5	50

Obviously, the *first row* is dominated by *second row* and *first column* is dominated by *second column*. Therefore, eleminating first row and first column, we get the reduced payoff matrix;

	Medium Advestising	Large Advestising
Medium Advestising	50	45
Large Advestising	47.5	50

Since the reduced 2×2 matrix has no saddle point, the strategies are of mixed type.

By solving the 2×2 game, the optimum strategies for the two players and the values of the game are obtained as (0, 1/3, 2/3), (0, 2/3, 1/3), and $v = 145/3$.

Economically, the optimum strategy for firm *A* has to apply medium strategy with probability 1/3 and large strategy with prob. 2/3 on any one play of the game.

Adopting this policy, the firm is expected to gain 145/3% of the market share.

(*b*) Since the annual value of the firm *A* is of 30,000 units, its market share will be as follows :

		Firm B		
		No Adv.	Medium Adv.	Large Adv.
Firm A	No Adver.	$0.50 \times 30{,}000 = 15{,}000$	$0.40 \times 30{,}000 = 12{,}000$	$0.28 \times 30{,}000 = 8{,}400$
	Med. Adv.	$0.70 \times 30{,}000 = 21{,}000$	$0.50 \times 30{,}000 = 15{,}000$	$0.45 \times 30{,}000 = 13{,}500$
	Large Adv.	$0.75 \times 30{,}000 = 22{,}500$	$0.475 \times 30{,}000 = 14{,}250$	$0.50 \times 30{,}000 = 15{,}000$

Given the expenditure on *medium* and *large* advertisements as Rs. 5,000 respectively, the net profit of firm *A* can be computed as follows :

Net Profit = (Sales price – cost price) × Sales volume – Adver Expenditure

= (42.50) × Sales Volume – Adver Expenditions

Net profit of firm *A* is given in the following table :

		Firm B		
		No. Adver.	Medium Adv.	Large Adv.
Firm A	No. Adv.	$1.5 \times 15{,}000 - 0 = 22{,}500$	$1.5 \times 12{,}000 - 0 = 18{,}000$	$1.5 \times 8{,}400 - 0 = 12{,}600$
	Med.Adv.	$1.5 \times 21{,}000 - 5{,}000 = 26{,}000$	$1.5 \times 15{,}000 - 5{,}000 = 17{,}500$	$1.5 \times 13{,}500 - 5{,}000 = 15{,}250$
	Large Adv.	$1.5 \times 22{,}500 - 15{,}000 = 18{,}750$	$1.5 \times 14{,}250 - 15{,}000 = 6{,}375$	$1.5 \times 15{,}000 - 15{,}000 = 7{,}500$

From above table, we observe the following

(*i*) If firm *A* selects '*No Adv.*', then minimum profit will be of Rs. 12,600, because *firm B* can use the strategy '*Large Adver.*'.

(*ii*) If firm *A* selects '*Medium Adver.*', then minimum profit will be of Rs. 15,250 because firm *B* can again use the strategy '*Large Adv.*'.

(*iii*) If firm *A* selects '*Large Adver.*', then *minimum* profit will be of Rs. 6,375.

Thus, on the basis of above observations, the firm *A* must use the policy of 'Medium Advertising' to gain maximum profit of Rs. 15,250 among these there choices and must spend Rs. 5,000 for advertising.

NOTE If we apply principle of dominance to the pay-off matrix having a saddle point, then we get a reduced matrix of single element of only. So the students are advised to use the principle of dominance for solving the games without saddle point (s) only, until unless otherwise stated.