SONAL SONARUHARE

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MATHS ASSIGNMENT-2

1. Find the Fourier series for $f(x) = \sqrt{1-\cos x}$ in $(0, 2\pi)$
Hence, Deduce that $\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x)\,dx \qquad a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Given: $0, 2\pi$ $\therefore c = 0$ $2l = 2\pi$ $\therefore l = \pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{— (1)}$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx \qquad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\,dx$$

Solving:

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} \sqrt{1-\cos x}\,dx$$

$$a_0 = \frac{1}{\pi} \times \sqrt{2} \int_0^{2\pi} \left[\sin\frac{x}{2}\right] dx \quad \ldots \quad 1-\cos x = 2\sin^2 x/2$$

$$a_0 = \frac{\sqrt{2}}{\pi} \left[-\cos\frac{x}{2}(2)\right]_0^{2\pi}$$

$$a_0 = -\frac{2\sqrt{2}}{\pi} (\cos\pi - \cos 0)$$

$$a_0 = -\frac{2\sqrt{2}}{\pi} (-1-1) \quad \therefore \boxed{a_0 = \frac{4\sqrt{2}}{\pi}}$$

Solving:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{1-\cos x}\ \cos nx\, dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\cos nx\, dx$$

$$\sin A\cos B = \frac{1}{2}\left(\sin(A+B)+\sin(A-B)\right)$$

$$a_n = \frac{\sqrt{2}}{\pi}\frac{1}{2}\int_0^{2\pi}\left(\sin\left(n+\frac{1}{2}\right)x + \sin\left(\frac{1}{2}-n\right)x\right)dx$$

$$a_n = \frac{\sqrt{2}}{2\pi}\left[\frac{-\cos\left(\frac{2n+1}{2}\right)x}{\frac{2n+1}{2}} - \frac{\cos\left(\frac{1-2n}{2}\right)x}{\frac{1-2n}{2}}\right]_0^{2\pi}$$

$$a_n = -\frac{2\sqrt{2}}{2\pi}\left[\frac{\cos\left(\frac{2n+1}{2}\right)(2\pi)}{2n+1} + \frac{\cos\left(\frac{1-2n}{2}\right)(2\pi)}{1-2n} - \frac{\cos(0)}{2n+1} - \frac{\cos(0)}{1-2n}\right]$$

$$a_n = -\frac{2\sqrt{2}}{2\pi}\left[\frac{\cos(2n\pi+\pi)}{2n+1} + \frac{\cos(\pi-2n\pi)}{1-2n} - \frac{1}{2n+1} - \frac{1}{1-2n}\right]$$

$$a_n = -\frac{2\sqrt{2}}{2\pi}\left(-\frac{\cos 2n\pi}{2n+1} - \frac{\cos 2n\pi}{1-2n} - \frac{1}{2n+1} - \frac{1}{1-2n}\right)$$

$$a_n = -\frac{2\sqrt{2}}{2\pi}\left(\frac{-1}{2n+1} - \frac{1}{1-2n} - \frac{1}{2n+1} - \frac{1}{1-2n}\right) \quad \because \cos 2n\pi = 1.$$

$$a_n = -\frac{2\sqrt{2}}{2\pi}\left(\frac{-2}{2n+1} - \frac{2}{1-2n}\right)$$

$$a_n = \frac{2\sqrt{2}}{\pi}\left(\frac{2}{1-4n^2}\right) \quad \therefore \boxed{a_n = -\frac{4\sqrt{2}}{\pi(4n^2-1)}}$$

Solving:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{1-\cos x}\,\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\sqrt{2}\int_0^{2\pi} \sin\frac{x}{2}\,\sin nx\,dx$$

$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

$$\therefore b_n = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left(\cos\left(\frac{1}{2}-n\right)x - \cos\left(\frac{1}{2}+n\right)x\right)dx$$

$$\therefore b_n = \frac{\sqrt{2}}{2\pi}\left[\frac{2\sin\left(\frac{1-2n}{2}\right)x}{1-2n} - \frac{2\sin\left(\frac{1+2n}{2}\right)x}{1+2n}\right]_0^{2\pi}$$

$\because \sin 2\pi = 0$.

$$\boxed{b_n = 0}$$

Hence putting the values in eq (1) we get.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\boxed{f(x) = \frac{4\sqrt{2}}{2\pi} + \sum_{n=1}^{\infty}\left(\frac{-4\sqrt{2}\cos nx}{\pi(4n^2-1)}\right) + 0.}$$

$$\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty}\left(-\frac{4\sqrt{2}\cos nx}{\pi(4n^2-1)}\right) \quad \text{— (2)}$$

Put $x=0$ in eq (2).

$$\therefore 1 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{2\sqrt{2}}{\pi} = \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$

$$\boxed{\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}}$$

Hence Proved:

2. Find Fourier Series expansion for $f(x) = 4-x^2$ in $(0,2)$
Deduce $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx \qquad a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Given: $0, 2$ $\quad c=0 \quad 2l = 2 \quad \therefore l = 1.$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \text{— (1)}$$

$$a_0 = \int_0^2 f(x)\,dx \qquad a_n = \int_0^2 f(x)\cos n\pi x\,dx \qquad b_n = \int_0^2 f(x)\sin n\pi x\,dx$$

Solving:

$$a_0 = \int_0^2 (4-x^2)\,dx$$

$$a_0 = \left[4x - \frac{x^3}{3}\right]_0^2$$

$$a_0 = 8 - \frac{8}{3} - 0 \qquad \boxed{\therefore a_0 = \frac{16}{3}}$$

Solving:

$$a_n = \int_0^2 (4-x^2)\cos n\pi x \, dx$$

$$\int uv = uv_1 - u'v_2 + u''v_3$$

$$a_n = \left[(4-x^2)\frac{\sin n\pi x}{n\pi} - (-2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + (-2)\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_0^2$$

$$a_n = \left[(4-x^2)\frac{\sin n\pi x}{n\pi} - \frac{2x\cos n\pi x}{n^2\pi^2} + \frac{2\sin n\pi x}{n^3\pi^3}\right]_0^2$$

$$a_n = \left[0 - \frac{4\cos 2n\pi}{n^2\pi^2} + \frac{2\sin 2n\pi}{n^3\pi^3} - 4\sin(0) + 0 - 0\right]$$

$$\boxed{a_n = \frac{-4}{n^2\pi^2}} \quad \cdots \quad \cos 2n\pi = 1, \ \sin 2n\pi = 0.$$

$$b_n = \int_0^2 (4-x^2)\sin n\pi x \, dx$$

$$b_n = \left[(4-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) - \left(-2x\left(-\frac{\sin n\pi x}{(n\pi)^2}\right)\right) + \left(-2\left(+\frac{\cos n\pi x}{(n\pi)^3}\right)\right)\right]_0^2$$

$$b_n = 0 - \frac{2\cdot 2\sin 2n\pi}{(n\pi)^2} - \frac{2\cos 2n\pi}{(n\pi)^3} + \frac{4\cos(0)}{n\pi} + 0 + \frac{2\cos 0}{(n\pi)^3}$$

$$b_n = -\frac{2}{n^3\pi^3} + \frac{4}{n\pi} + \frac{2}{n^3\pi^3}$$

$$\boxed{b_n = \frac{4}{n\pi}}$$

Hence putting all values in eq 1.

$$f(x) = \frac{8}{3} + \sum_{n=1}^{\infty}\left(-\frac{4}{n^2\pi^2}\right)\cos n\pi x + \sum_{n=1}^{\infty}\frac{4}{n\pi}\sin n\pi x$$

$$4-x^2 = \frac{8}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}$$

$$4-x^2 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^3}\cos 3\pi x \cdots\right]$$

$$+ \frac{4}{\pi}\left[\frac{\sin\pi x}{1} + \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} \cdots\right]$$

Substitute $x=0$.

$$\therefore 4 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]+0.$$

$$\therefore \frac{1}{3} = -\frac{1}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] \quad \text{— I}$$

Substitute $x=2$.

$$4-4 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\frac{2}{3} = \frac{1}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] \quad \text{— II}.$$

Adding (I) and (II)

$$-\frac{1}{3}+\frac{2}{3} = \frac{2}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\frac{1}{3} = \frac{2}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\boxed{\therefore \frac{\pi^2}{6} = \left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]}$$

Hence Proved.

3. Find the Fourier Series for $f(x) = \begin{cases} x+\frac{\pi}{2}, & -\pi<x<0 \\ \frac{\pi}{2}-x & 0<x<\pi \end{cases}$

Deduce that $\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$

Solution: $f(-x) = \begin{cases} -x+\frac{\pi}{2} & -\pi<-x<0 \\ \frac{\pi}{2}+x & 0<-x<\pi \end{cases}$

$$f(-x) = \begin{cases} -x + \frac{\pi}{2} & +\pi > x > 0 \\ \frac{\pi}{2} + x & 0 > x > -\pi \end{cases}$$

$$= \begin{cases} \frac{\pi}{2} + x & -\pi < x < 0 \\ \frac{\pi}{2} - x & 0 < x < \pi \end{cases}$$

$$= f(x)$$

$\therefore$ $f(x)$ is even function.

$b_n = 0$ $\quad l = \pi$

Hence: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ — (1)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi}{2} - x\right) dx$$

$$a_0 = \frac{2}{\pi}\left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2} - 0\right]$$

$$\boxed{\therefore a_0 = 0.}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi}{2} - x\right)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left[\left(\frac{\pi}{2} - x\right)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\left(\frac{\pi}{2}-\pi\right)\frac{\sin \pi n}{n} - \frac{\cos n\pi}{n^2} - \frac{\pi}{2}\sin 0 + \frac{\cos 0}{n^2}\right]$$

$$a_n = \frac{2}{\pi}\left(0 - \frac{(-1)^n}{n^2} + \frac{1}{n^2}\right)$$

$$\boxed{a_n = \frac{2}{\pi}\left(\frac{1-(-1)^n}{n^2}\right)}$$

Putting all values in eq 1.

$$f(x) = 0 + \frac{2}{\pi}\sum_{n=1}^{\infty} \cancel{(1-(-1)}\left(\frac{1-(-1)^n}{n^2}\right)\cos nx$$

$$f(x) = \frac{2}{\pi}\left(\frac{2}{1^2}\cos x + \frac{2}{3^2}\cos 3x + \frac{2}{5^2}\cos 5x + \dots\right)$$

$$f(x) = \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2}\right) \quad -(2)$$

Put $x=0$. in eq (2).

$$f(0) = \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right]$$

$$= \frac{1}{2}\left[\lim_{x\to 0^-}\left(\frac{\pi}{2}+x\right) + \lim_{x\to 0^+}\left(\frac{\pi}{2}-x\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = \frac{\pi}{2}$$

$$\therefore \frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$$

$$\therefore \boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots}$$

Hence proved

4. Expand $f(x) = lx - x^2,\ 0 < x < l$ in half range sine series.

Deduce $\frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots$

Solution:

Sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{—①}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi x}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi x}{l}\right)^2}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{2}{l}\left[0 - 0 - \frac{2l^3}{n^3\pi^3}(\cos n\pi) - 0 - 0 + \frac{2\cos 0\, l^3}{n^3\pi^3}\right)$$

$$= \frac{2}{l}\left[\frac{-2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{2}{l}\left(\frac{2l^3}{n^3\pi^3}\right)\left[-1(-1)^n + 1\right]$$

$$\boxed{b_n = \frac{4l^2}{n^3\pi^3}\left[1 - (-1)^n\right]}$$

Substituting value in eq ①.

$$f(x) = \sum_{n=1}^{\infty} \frac{4l^2}{n^3\pi^3}(1-(-1)^n)\sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^3}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{4l^2}{\pi^3}\left(\frac{2}{1^3}\sin\frac{\pi x}{l} + \frac{2}{3^3}\sin\left(\frac{3\pi x}{l}\right)+\cdots\right)$$

$$\underset{lx-x^2}{f(x)} = \frac{8l^2}{\pi^3}\left(\frac{\sin\pi x}{1^3 l} + \frac{1}{3^3}\sin\left(\frac{3\pi x}{l}\right)+\cdots\right)$$

~~put $x = l/2$, we get.~~

$$\cancel{\frac{l^2}{2} - \frac{l^2}{4} = \frac{8l^2}{\pi^3}\left(\frac{1}{1^3}(1) + \frac{1}{3^3}\sin(-1)+\cdots\right)}$$

$$\cancel{\frac{2l^2}{8} = \frac{8l^2}{\pi^3}\left(\frac{1}{1^3}(1) + \frac{1}{3^3}\sin(-1)+\cdots\right.}$$

$$\cancel{\frac{\pi^3}{32} = \frac{1}{1^3} + \frac{1}{3^3}(-1)+\cdots}$$

Hence proved.

By Parseval's Identity for sine series

$$\frac{2}{l}\int_0^l [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2$$

$$\frac{2}{l}\int_0^l (lx-x^2)^2\,dx = \left[\sum_{n=1}^{\infty}\left[\frac{4l^2}{n^3\pi^3}(1-(-1)^n)\right]^2\right]$$

$$\frac{2}{l}\int_0^l (l^2x^2-2lx^3+x^4)\,dx = \sum_{n=1}^{\infty}\frac{16l^4}{n^6\pi^6}(1-(-1)^n)^2$$

$$\frac{2}{l}\left[\frac{l^2x^3}{3} - \frac{2lx^4}{4} + \frac{x^5}{5}\right]_0^l = \frac{16l^4}{\pi^6}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)^2}{n^6}$$

$$\frac{2}{l}\left[\frac{l^5}{3} - \frac{2l^5}{4_2} + \frac{l^5}{5} - 0\right] = \frac{16l^4}{\pi^6}\left[\frac{2^2}{1^6} + 0 + \frac{2^2}{3^6} + 0\cdots\right]$$

$$\frac{2l^5-3l^5}{6}$$

$$\frac{2}{l}\left[\frac{-1l^5}{6} + \frac{l^5}{5}\right] = \frac{16l^4}{\pi^6}2^2\left[\frac{1}{1^6} + \frac{1}{3^6}+\cdots\right]$$

$$\frac{-l^5}{6}+\frac{l^5}{5}$$

$$\frac{-5l^5+6l^5}{30} \quad \frac{l^5}{30}$$

$$\frac{2}{l}\left[\frac{l^5}{30}\right]\frac{\pi^6}{16l^4} = 4\left[\frac{1}{1^6}+\frac{1}{3^6}+\cdots\right]$$

$$\frac{\pi^6}{30\times8\times4\times} = \frac{1}{1^6}+\frac{1}{3^6}+\cdots$$

$$\frac{\pi^6}{960} = \frac{1}{1^6}+\frac{1}{3^6}+\cdots$$

5. Find Fourier Series of $f(x)=x^2, -\pi<x<\pi$

Deduce $\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$

Solution:

$f(-x) = (-x)^2 = x^2 = f(x)$

Hence $f(x)$ is even

$\therefore b_n = 0$.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{— (1)}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \qquad a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$a_0 = \frac{2}{\pi}\left[\frac{\pi^3}{3}\right]$$

$$a_0 = \frac{2\pi^2}{3} \qquad a_n = \frac{2}{\pi}\left[0 + 2\pi\frac{\cos n\pi}{n^2} + 0 - 0 - 0 + 0\right]$$

$$a_n = \frac{4(-1)^n}{n^2}.$$

$$\boxed{\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx.}$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

Put $x = \pi$

$$\therefore \pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos n\pi}{n^2}$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n^2}$$

$$\boxed{\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots}$$

Hence proved.

6. Half range cosine series $f(x) = e^x$, $0 < x < 1$

Solution:

Cosine series: $b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \text{—} (1)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$\therefore a_0 = \frac{2}{l}\int_0^l e^x\,dx \qquad a_n = \frac{2}{l}\int_0^l e^x \cos\left(\frac{n\pi x}{l}\right)dx$$

$$a_0 = \frac{2}{l}\left[e^x\right]_0^l \qquad a_n = 2\int_0^1 e^x \cos(n\pi x)\,dx$$

$$a_0 = \frac{2}{l}\times(e^l - e^0) \qquad a_n = 2\left[\frac{e^x}{(1^2+n^2\pi^2)}(\cos n\pi x + n\pi \sin n\pi x)\right]_0^1 \ldots$$

$$a_0 = \frac{2}{l}(e^l - 1) \qquad \int e^{ax}\cos bx\,dx = e^{ax}\frac{(a\cos bx + b\sin bx)}{a^2+b^2}$$

$l = 1$

$$a_n = 2\left[\frac{e}{1+n^2\pi^2}(\cos n\pi + n\pi \sin n\pi) - \frac{1}{1^2+n^2\pi^2}(\cos 0 + n\pi \times 0)\right]$$

$$\therefore a_0 = \frac{2(e^1 - 1)}{1}$$

$$a_n = 2\left[\frac{e((-1)^n)}{(1+n^2\pi^2)} - \frac{1}{1+n^2\pi^2}\right]$$

$\therefore\ a_n = \frac{2}{1+n^2\pi^2}\left(e(-1)^n - 1\right)$.

$$\boxed{\therefore\ f(x) = e - 1 + 2\sum_{n=1}^{\infty}\left(\frac{e(-1)^n - 1}{1+n^2\pi^2}\right)\cos n\pi x}$$

7. Find k $\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{kx}{y}\right)$ is analytic.

Solution.

Let $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{kx}{y}\right)$

$= u + iv$.

Take $u = \frac{1}{2}\log(x^2+y^2)$ $v = \tan^{-1}\left(\frac{kx}{y}\right)$

If the function is an analytic then CR function is satisfied

CR equation:

$u_x = v_y$ $u_y = -v_x$

$\therefore\ u_x = \frac{1}{2}\left(\frac{1}{x^2+y^2}(2x)\right)$ $v_y = \frac{1}{1+\frac{k^2x^2}{y^2}}\, kx(-1y^{-2})$

$\therefore\ u_x = \frac{x}{x^2+y^2}$ $v_y = \frac{-kx \cdot y^2}{y^2(y^2+k^2x^2)}$

$v_y = \frac{-kx}{k^2x^2+y^2}$

$\therefore\ \frac{x}{x^2+y^2} = \frac{-kx}{k^2x^2+y^2}$

$\therefore\ \boxed{k=-1}$

$u_y = \frac{1}{2}\frac{1}{(x^2+y^2)}(2y)$ $v_x = \frac{1}{1+\frac{k^2x^2}{y^2}}\,\frac{k}{y}(1)$

$u_y = \frac{y}{x^2+y^2}$ $v_x = \frac{ky}{kx^2+y^2}$

$\therefore\ \frac{y}{x^2+y^2} = -\frac{ky}{kx^2+y^2}$ $\therefore\ \boxed{k=-1}$

$\therefore$ By $U_x = V_y$ and $U_y = -V_x$

$\underline{\underline{K = -1}}$

Hence when $f(x)$ is analytic function when $\underline{K = -1}$

10. Imaginary part $V = x^2 - y^2 + \frac{x}{x^2+y^2}$ find real part.

Solution:

Let $f(z) = U + iV$ be analytic function $V = x^2 - y^2 + \frac{x}{x^2+y^2}$

$\therefore V_x = 2x + \left(\frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}\right)$ $\quad V_y = -2y - \frac{2xy}{(x^2+y^2)^2}$

$V_x = 2x + \left(\frac{-x^2+y^2}{(x^2+y^2)^2}\right)$

$\therefore$ $U_x = V_y$ and $U_y = -V_x$ $\because$ $f(z)$ is analytic

As $f'(z) = U_x + iV_x = V_y + iV_x$

$\therefore f'(z) = -2y - \frac{2xy}{(x^2+y^2)^2} + i\left(2x - \left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)\right)$

By Milne Thompson's Method. $x = z$ $\quad y = 0$.

$f'(z) = 0 + i\left(2z - \frac{z^2}{z^4}\right) = i\left(2z - \frac{1}{z^2}\right)$

On Integrating $f'(z)$ we get

$\int f'(z)\,dz = \int i\left(2z - \frac{1}{z^2}\right)dz$

$= i\left(\frac{2z^2}{2} + \frac{1}{z}\right) + c$

$f(z) = i\left(z^2 + \frac{1}{z}\right) + c$

Put $z = x + iy$

$f(z) = i\left((x+iy)^2 + \frac{1}{x+iy}\right) + c = i\left(x^2 - y^2 + 2xyi + \frac{(x-iy)}{(x-iy)(x+iy)}\right)$

$= i\left((x^2 - y^2 + 2xyi) + \frac{x-iy}{x^2+y^2}\right)$

$\therefore f(z) = \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + c$.

$\therefore$ **Real part**: $-2xy + \frac{y}{x^2+y^2}$.

11. Find an analytic function $f(z) = u + iv$, where $u + v = e^x(\cos y + \sin y)$

Solution:

let $f(z) = u + iv \Rightarrow if(z) = i(u+iv) = iu - v = -v + iu$

$\therefore f(z) + if(z) = (u+iv) + -v + iu)$

$= (u-v) + i(u+v)$

$f(z)(1+i) = (u-v) + i(u+v)$.

$f(z)(1+i) = U + iV$ where $U = u - v$, $V = u + v$

as $f(z)$ is analytic $U_x = V_y$ and $U_y = -V_x$ and $(1+i)f(z)$ is also analytic

Also $f'(z) = U_x + iV_x$

$\therefore V = u + v = e^x(\cos y + \sin y)$

$\therefore V_x = e^x(\cos y + \sin y)$

$\therefore V_y = e^x(-\sin y + \cos y)$ $F(z) = (1+i)f(z)$.

$f'(z) = U_x + iV_x = V_y + iV_x$

By Milne Thompson's Method: $x = z$, $y = 0$

$\therefore f'(z) = e^x(-\sin y + \cos y) + i(e^x(\cos y + \sin y))$

$= e^z(0+1) + i(e^z(1)$

$f'(z) = e^z + ie^z$

$(1+i)(f'(z)) = \cancel{(1+i)}(e^z + ie^z)$

$= (1+i)e^z$

On Integrating.

$\int (1+i)(f'(z)dz$.

$(1+i)f(z) = \int (1+i)e^z dz$.

$(1+i)F(z) = (1+i)e^z + c$

$\therefore \boxed{f(z) = e^z + c}$

12. construct an analytic function whose real part is $\frac{\sin 2x}{\cosh 2y + \cos 2x}$

Solution:

Let $f(z) = u + iv$ is an analytic function : $u_x = v_y \quad u_y = -v_x$

$$\therefore u_x = \frac{(\cosh 2y + \cos 2x)(2\cos 2x) - \sin 2x(\sinh 2y(0) - 2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$\therefore u_x = \frac{2\cos 2x(\cosh 2y + \cos 2x) + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$$

$$\therefore u_y = \frac{(\cosh 2y + \cos 2x)(0) - \sin 2x(\sinh 2y(2)}{(\cosh 2y + \cos 2x)^2}$$

$$\therefore u_y = -\frac{2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$$

By Milne Thompson's Method

$x = z \qquad y = 0$

$$\therefore u_x = \frac{2\cos 2z(1 + \cos 2z) + 2\sin^2 2z}{(1 + \cos 2z)^2}$$

$$\therefore u_x = \frac{2(\cos 2z + \cos^2 2z + \cancel{2}\sin^2 2z)}{4\cos^4 z}$$

$$u_x = \frac{2(1 + \cos 2z)}{4\cos^4 z}$$

$$u_x = \frac{2 \cdot 2\cos^2 z}{4\cos^4 z}$$

$$\therefore \underline{u_x = \sec^2 z}$$

$\therefore \underline{u_y = 0}$

$\therefore f'(z) = \sec^2 z$

$$\int f'(z)\,dz = \int \sec^2 z\,dz$$

$$f(z) = \tan x + c$$

FOR EDUCATIONAL USE

Sundaram

13. Determine the constants a, b, c, d, e if

$f(z) = ax^3 + bxy^2 + 3x^2 + cy^2 + x + i(dx^2y - 2y^3 + exy + y)$

is analytic

Solution:

$f(z) = u + iv$

$\therefore u = ax^3 + bxy^2 + 3x^2 + cy^2 + x$

$v = dx^2y - 2y^3 + exy + y$

Given $f(z)$ is analytic $\Rightarrow$ $f(z)$ satisfies CR equation

$\therefore u_x = v_y$ and $u_y = -v_x$

$\therefore u_x = 3ax^2 + by^2 + 6x + 1$

$u_y = 2bxy + 2cy$

$\therefore v_x = 2dxy + ey$

$v_y = dx^2 - 6y^2 + ex + 1$

$\therefore u_x = v_y$

$3ax^2 + by^2 + 6x + 1 = dx^2 - 6y^2 + ex + 1$

$\therefore \boxed{3a = d}$ $\boxed{b = -6}$ $\boxed{e = 6}$

$u_y = -v_x$

$2bxy + 2cy = -2dxy - ey$

$\therefore 2b = -2d$ $\qquad$ $2c = -e$

$\therefore 2(-6) = -2d$ $\qquad$ $2c = -6$

$\therefore d = 6$ $\qquad$ $\boxed{c = -3}$

$\therefore 3a = d \quad \therefore a = \frac{6}{3} \quad \therefore \boxed{a = 2}$

$\boxed{a = 2, \quad b = -6, \quad c = -3 \quad d = 6, \quad e = 6}$

14. Find probability density function of a random variable X is

X =	0	1	2	3	4	5	6
P(X=x)	K	3K	5K	7K	9K	11K	13K

Find $P(X<4)$, $P(3<X\le 6)$

Solution:

$\sum_x Px = 1$

$K+3K+5K+7K+9K+11K+13K = 1$

$49K = 1$

$\boxed{\therefore K = \frac{1}{49}}$

$\therefore$ The probability density function of x is given by

x =	0	1	2	3	4	5	6
P(X=x)	0.02	0.06	0.1	0.14	0.18	0.22	0.26

To find $P(X<4)$

$= P(X=3) + P(X=2) + P(X=1) + P(X=0)$

$= \cancel{0.22 + 0.26} = 0.14 + 0.1 + 0.06 + 0.02$

$= \cancel{0.48} = \boxed{0.32}$

$P(3<x\le 6)$

$= P(X=4) + P(X=5) + P(X=6)$

$= 0.18 + 0.22 + 0.26$

$= \boxed{0.66.}$

15. Probability function of a discrete random variable x

X	0	1	2	3	4	5	6	7
$P(X=x)$	0	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2+c$

Solution:

i) $\sum_x Px = 1.$

$0+c+2c+2c+3c+c^2+2c^2+7c^2+c = 1$

$10c^2+9c = 1$

$10c^2+9c-1 = 0$

$\therefore c = 0.1$ and $c=-1$.

$c=-1$ is not possible

$\therefore \boxed{c=0.1}$

$\therefore$ The probability function of a discrete random variable x

X	0	1	2	3	4	5	6	7
$P(X=x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

iii) To find $P(X \leq 6)$

$= P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$
$+ P(X=5) + \cancel{P(X=6)}$

$= 0+0.1+0.2+0.2+0.3+0.01+\cancel{0.02}$

$= \underline{0.81}$

ii) $P(X \geq 6) = P(X=6)+P(X=7)$

$= \cancel{2c}\ 0.02 + 0.17$

$= \underline{0.19}$

iv) $P(1.5 < x < 4.5 / x > 2)$

$P(A/B) = \frac{P(A \cap B)}{P(B)}$

$1.5 < x < 4.5 / x > 2)$

$$= \frac{P(1.5 < x < 4.5 \cap x > 2)}{P(x > 2)}$$

$$= \frac{P(X = 2, 3, 4) \cap P(X = 3, 4, 5, 6, 7)}{P(X = 3, 4, 5, 6, 7)}$$

$$= \frac{(0.2 + 0.3 + 0.2) \cap (0.2 + 0.3 + 0.01 + 0.02 + 0.17)}{0.2 + 0.3 + 0.01 + 0.02 + 0.17}$$

$$= P(X = 3, 4) = \frac{0.2 + 0.3}{0.2 + 0.3 + 0.01 + 0.02 + 0.17}$$

$$= \frac{0.5}{0.7}$$

$$= \boxed{\frac{5}{7}}$$

16. Continuous random variable x with probability density

$$f(x) = \begin{cases} \frac{3}{4}(2x - x^2), & 0 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

i) not more than 1 inch

ii) greater than 1.5

iii) between 1 and 1.5 inch

Solution:

i) P(rainfall is not more than 1 inch)

$$= P(X \leq 1) = \int_{-\infty}^{1} f(x)\,dx$$

$$= \int_{-\infty}^{1} \frac{3}{4}(2x - x^2)\,dx$$

$$= \frac{3}{4}\left[\left(\frac{2x^2}{2} - \frac{x^3}{3}\right)\right]_0^1$$

$$= \frac{3}{4}\left(1 - \frac{1}{3} - 0\right)$$

$$= \frac{3}{4} \times \frac{2}{3} = \boxed{\frac{1}{2}}$$

ii) P(rainfall greater than 1.5)

$$P(X > 1.5) = \int_{1.5}^{\infty} f(x)\,dx.$$

$$= \int_{1.5}^{2} \frac{3}{4}(2x - x^2)\,dx$$

$$= \frac{3}{4}\left(x^2 - \frac{x^3}{3}\right)\Big|_{1.5}^{2}$$

$$= \frac{3}{4}\left(4 - \frac{8}{3} - \frac{9}{4} + \frac{9}{8}\right)$$

$$= \frac{3}{4} \times \frac{5}{24}$$

$$= \frac{5}{32} = \underline{0.15625}$$

iii) P(ranifall between 1 inch and 1.5 inch) = $P(1 < X < 1.5)$

$$= \int_{1}^{1.5} f(x)\,dx$$

$$= \int_{1}^{1.5} \frac{3}{4}(2x - x^2)\,dx$$

$$= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_{1}^{1.5}$$

$$= \frac{3}{4}\left(\frac{9}{4} - \frac{9}{8} - 1 + \frac{1}{3}\right)$$

$$= \frac{3}{4} \times \frac{11}{24} = \underline{0.343}$$

17. Find k and Expectation & Variance if x has the p.d.f

$$f(x) = \begin{cases} kx(2-x), & 0 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Solution:

f(x) is called p.d.f if

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\therefore \int_{0}^{2} f(x)\,dx = 1$$

$$\int_{0}^{2} kx(2-x)\,dx = 1$$

$$k \int_{0}^{2} (2x - x^2)\,dx = 1$$

$$k \left[\frac{2x^2}{2} - \frac{x^3}{3} \right]_0^2 = 1$$

$$k \left(4 - \frac{8}{3} \right) = 1$$

$$k \left(\frac{12-8}{3} \right) = 1$$

$$k \left(\frac{4}{3} \right) = 1$$

$$\boxed{k = \frac{3}{4}.}$$

Expectation:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{0}^{2} x \cdot kx(2-x)\,dx$$

$$E(X) = K\int_0^2 (2x^2 - x^3)\,dx$$

$$= K\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= K\left(\frac{2\times 8}{3} - \frac{16}{4} - 0\right)$$

$$= K\left(\frac{16}{3} - \frac{16}{4}\right)$$

$$= \frac{3}{4}\left(\frac{4}{3}\right)$$

$$\therefore \boxed{E(X) = 1}$$

Varience: $Var(X) = E(X)^2 - (E(X))^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$E(X^2) = \int_0^2 x^2 (Kx(2-x))\,dx$$

$$= K\int_0^2 (2x^3 - x^4)\,dx$$

$$= K\left(\frac{2x^4}{4} - \frac{x^5}{5}\right)_0^2$$

$$= K\left(\frac{2\times 16}{4} - \frac{2^5}{5}\right)$$

$$= \frac{3}{4}\left(8 - \frac{2^5}{5}\right)$$

$$\boxed{E(X^2) = \frac{6}{5}}$$

$$\therefore \text{Varinu} = \sigma^2 = E(X^2) - (E(X))^2$$

$$= \frac{6}{5} - 1 = \boxed{\frac{1}{5}}$$

18 Mean of distribution: 16

Find m and n.

x	8	12	16	20	24
$P(X=x)$	$\frac{1}{8}$	m	n	$\frac{1}{4}$	$\frac{1}{12}$

Solution:

$$\sum_x px = 1.$$

$$\frac{1}{8} + m + n + \frac{1}{4} + \frac{1}{12} = 1$$

$$m + n + \frac{11}{24} = 1$$

$$\therefore m + n = \frac{13}{24}. \quad — ①$$

$$\text{Mean} = E(X) = 16$$

$$\therefore \sum_x x\,px = 16$$

$$8 \times \frac{1}{8} + 12m + 16n + 20 \times \frac{1}{4} + \frac{24}{12} = 16$$

$$1 + 12m + 16n + 5 + 2 = 16$$

$$12m + 16n = 8 \quad — ②$$

Solving 1 and 2 simultaneously

$$\boxed{m = \frac{1}{6}} \qquad \boxed{n = \frac{3}{8}}$$