Semester: VIII   Subject: AIFB   Academic Year: 2024-25

BETA DISTRIBUTION:

The betā distribution plays a significant role in finance, particularly in risk modelling portfolio management, and probability estimation. Since it is defined on the interval (0,1), it is useful for modelling probabilities, proportion and uncertain financial variables.

Defination of the Betā Distribution:

The Betā distribution is parameterized by two shape parameters, α and β:

$$\boxed{f(x;\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad 0 < x < 1}$$

where:

α, β > 0 are shape parameters

→ B(α, β) is the Betā function, defined as:

$$\boxed{B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt} \quad (or) \quad \boxed{B(\alpha,\beta) = \frac{\Gamma\alpha . \Gamma\beta}{\Gamma\alpha+\beta}}$$

→ The mean of the Betā distribution:

$$\boxed{E[X] = \frac{\alpha}{\alpha+\beta}}$$

→ The variance

$$\boxed{Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

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Shape of the Beta Distribution:-

* The Beta Distribution is very flexible and takes different shapes on α and β:
* Uniform Distribution: $\alpha = 1, \beta = 1 \rightarrow$ Flat.
* Left skewed: $\alpha < \beta \rightarrow$ More probability mass near 0.
* Right skewed: $\alpha > \beta \rightarrow$ More probability mass near 1.
* Bell shaped: $\alpha = \beta > 1$

~~Applications of the Beta distribution in Finance.~~

Example:-

Suppose a stock analyst estimates the probability of a bull market using a Beta distribution with parameters $\alpha = 3$, and $\beta = 2$.

(a) Find the mean of the distribution.
(b) Compute the variance.
(c) Find the probability density for $x = 0.5$.

Solution:

(a) Mean of the Beta Distribution:-

The mean of a beta distribution is:

$$E(X) = \frac{\alpha}{\alpha+\beta} = \frac{3}{3+2} = \frac{3}{5} \boxed{= 0.6}$$

(b) Variance of the Beta Distribution:-

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{(3)(2)}{(3+2)^2(3+2+1)}$$

$$= \frac{6}{(5)^2(6)} = \frac{6}{150} \boxed{= 0.04}$$

(c) Probability Density at $x = 0.5$

The probability density function is:

$$\boxed{f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}}$$

First compute beta function:

$$\boxed{B(2,3) = \frac{\Gamma(3)\Gamma(2)}{\Gamma(3+2)}}$$

$$\Gamma(3) = 2! = 2, \ \Gamma(2) = 1! = 1, \ \Gamma(5) = 4! = 24$$

$$B(2,3) = \frac{(2)(1)}{24} = \frac{2}{24} \boxed{= \frac{1}{12}}$$

Now compute $f(0.5)$:-

$$f(0.5) = \frac{(0.5)^{3-1}(1-0.5)^{2-1}}{1/12} = \frac{(0.5)^2(0.5)}{1/12}$$

$$= \frac{(0.25)(0.5)}{1/12} = \frac{0.125}{1/12}$$

$$\boxed{= 1.5}$$

Mean = 0.6, Variance = 0.04, $f(0.5) = 1.5$

This means that the probability density at $x = 0.5$ is 1.5, which indicates the likelihood of observing a probability of 0.5 for bull market is relatively high.

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Applications of Betā Distribution in Finance:

(A) Modeling Stock Return Probabilities:

* The Betā distribution is used to model the probability distribution of stock returns.
* It helps in estimating expected future returns when historical data is limited.

(B) Bayesian Portfolio Optimization:

* The Betā Distribution is a conjugate prior for the Bernoulli and Binomial distributions in Bayesian statistics.
* In Bayesion portfolio theory, investors can update their beleifs about expected returns based on new market data.

Example:-

If an investor initially beleives that a stock has a 30% chance of outperformance but receives new information suggesting better performance, the Betā distribution can update the probability dynamically.

(C) Risk Management & Value-at-Risk

* The Betā Distribution is useful for estimating probabilities of extreme loses or gains.
* It can be applied to stress testing and scenario analysis to model bounded risks.