Matrix Basic concepts

Matrix: A matrix is a rectangular array of numbers (or functions) enclosed in brackets.
These numbers (or functions) are called entries or elements of the matrix.

Ex.

$$\begin{array}{ll} & \downarrow \text{columns} \\ \text{Rows} \rightarrow & \begin{bmatrix} 2 & 0.4 & 8 \\ 5 & -32 & 0 \end{bmatrix}, \end{array} \quad \begin{bmatrix} 6 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \begin{bmatrix} e^x & 3x \\ e^{2x} & x^2 \end{bmatrix}$$

Coefficient matrix of a linear system of equations

In a system of equations such as

$$5x - 2y + z = 0$$

$$3x + 4z = 0$$

the coefficient of the unknowns x, y, z are the entries of the coefficient matrix.

$$A = \begin{bmatrix} 5 & -2 & 1 \\ 3 & 0 & 4 \end{bmatrix}$$

Ex. Suppose there are three products I, II, III in a store, Sales amounts are given for a day of week so this data can be arranged in a matrix as follow

$$A = \begin{array}{c} \begin{array}{cccccc} M & T & W & TH & F & S \end{array} \\ \begin{bmatrix} 40 & 33 & 81 & 0 & 21 & 47 \\ 0 & 12 & 78 & 50 & 50 & 96 \\ 10 & 0 & 0 & 27 & 43 & 78 \end{bmatrix} \end{array} \begin{array}{c} \\ I \\ II \\ III \end{array}$$

If company has ten stores, we can set up ten such matrices one for each store.
Then by adding corresponding entries of these matrics we get a matrix of total sales of each product on each day.

Ex. transportation
Storage
in recording phone calls
listing distances in a n/w of roads

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

→ second is column

first subscription alway denotes the row

if $m = n$ then A is square matrix

$a_{11}, a_{22}, a_{33} \cdots a_{nn}$ is called the main diagonal or principal diagonal.

A matrix that is not square is called a rectangular matrix.

Vectors :-

A vector is a matrix thas has only one row than that matrix is called a row vector.
Only one column then called it a column vector.

Row vector

$a = [a_1 \quad a_2 \; --- \; a_n]$

for ex. $a = [5 \quad 3 \quad 8]$

Column vector

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad \text{for ex. } \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}$$

Row vector can be converted to column vector and vice versa. by operation that is called transposition and it is denoted by ^{T}.

a^T is a column vector

b^T is a row vector

$$a^T = \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix} \qquad b^T = \begin{bmatrix} 4 & 0 & -7 \end{bmatrix}$$

Transposition of a matrix

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}$$

Then

$$A^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$$

Equality of matrices

$A = [a_{jk}]$ and $B = [b_{jk}]$ are equal

$A = B$ if and only if they have the same size and the corresponding entries are equal i.e. $a_{11} = b_{11}$, $a_{12} = b_{12}$ and so on.

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}$$

Matrix Addition

~~Ad~~ we can add two or more matrices only if the size of matrix are same.

$A = [a_{jk}]$ and $B = [b_{jk}]$ of same size

Then $A + B$

$$A = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$

Scalar multiplication
(Multiplication by a number)

If $A = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}$ then

$$-A = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}$$

$$10A = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}$$

$$0A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A + B = B + A$$

$$(U+V) + W = U + (V+W) = U+V+W$$

$$A + 0 = A$$

$$A + (-A) = 0$$

for Scalar multiplication

$$c(A+B) = cA + cB$$

$$(c+k)A = cA + kA$$

$$c(kA) = (ck)A$$

$$1(A) = A$$

$$(A+B)^T = A^T + B^T$$

$$(cA)^T = cA^T$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 5 \\ 0 & 8 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 0 & 3 \\ 1 & 0 & -5 \end{bmatrix} \qquad D = \begin{bmatrix} 4 & 0 & -4 \\ -3 & 4 & 9 \end{bmatrix}$$

calculate 1) $5D - 3C$

2) $A - A^T$

3) $3C^T + 2D^T$

Addition & scalar multiplication of vectors

$$a = [3 \quad 0 \quad 4] \qquad b = [-1 \quad 8 \quad 2]$$

$$c = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix} \qquad d = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}$$

1. $7a^T - 5b^T$

2. $c - a^T$

3. $3c - 12d$

Matrix Multiplication

① $A = \begin{bmatrix} 4 & 7 \\ 7 & 2 \\ 9 & 0 \end{bmatrix}_{3\times 2}$ $\quad B = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}_{2\times 2}$

$$AB = \begin{bmatrix} 4\cdot 2 + 3\cdot 1 & 4\cdot 5 + 3\cdot 6 \\ 7\cdot 2 + 2\cdot 1 & 7\cdot 5 + 2\cdot 6 \\ 9\cdot 2 + 0\cdot 1 & 9\cdot 5 + 0\cdot 6 \end{bmatrix} = \begin{bmatrix} 11 & 38 \\ 16 & 47 \\ 18 & 45 \end{bmatrix}$$

② Multiplication of a matrix and a vector

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{pmatrix} 12 + 10 \\ 3 + 40 \end{pmatrix} = \begin{pmatrix} 22 \\ 43 \end{pmatrix}$$

③ Multiplication of row and column vector

$$\begin{bmatrix} 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}$$

Triangular Matrices

Upper triangular Matrices

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 4 & 2 & 2 & 0 \\ 0 & -3 & 5 & 1 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Lower Triangular Matrices

$$\begin{bmatrix} 5 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix}$$

Diagonal Matrices

Unit Matrix I

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transpose of a Product

$(AB)^T = B^T A^T$

$$(AB)^T = \left(\begin{bmatrix} 4 & 9 \\ 0 & 2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 2 & 8 \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} 30 & 100 \\ 4 & 16 \\ 15 & 55 \end{bmatrix}^T$$

$$= \begin{bmatrix} 30 & 4 & 15 \\ 100 & 16 & 55 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 3 & 2 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ 9 & 2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 4 & 15 \\ 100 & 16 & 55 \end{bmatrix}$$

System of linear equations

$$x_1 + x_2 + x_3 = 3 \quad (1)$$
$$x_1 - x_2 + 2x_3 = 2 \quad (2)$$
$$x_2 + x_3 = 2 \quad (3)$$

from (1) & (3) equation

$$\boxed{x_1 = 1.}$$

from eq^n (1) & (2)

(1) + (2) we get

$$2x_1 + 3x_3 = 5$$

put value of $\boxed{x_1 = 1}$

$$\therefore \boxed{x_3 = 1}$$

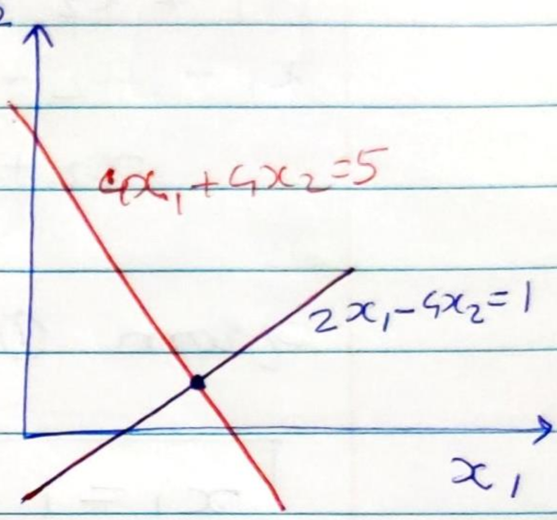
from eq^n (3) we get

$$\boxed{x_2 = 1}$$

$\therefore$ (1, 1, 1) is the unique solution

Every linear equation represents a line.

$$4x_1 + 4x_2 = 5 \quad \text{— (1)}$$
$$2x_1 - 4x_2 = 1 \quad \text{— (2)}$$



add eq^n (1) & (2)

$$6x_1 = 6$$
$$\boxed{x_1 = 1}$$

Put value of x_1 in eq^n (1)

$$4 \times 1 + 4x_2 = 5$$
$$4x_2 = 5 - 4$$
$$\boxed{x = \frac{1}{4}}$$

Solution is $(x_1, x_2) = (1, 1/4)$

For a systematic approach to solving system of linear equations there is a compact notation.

We collect the coefficients a_{ij} into vectors and collect the vectors into matrices.

Given system of linear equations as follow

$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$

$\vdots$

$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$

where $a_{ij} \in R$ and $b_i \in R$

From above equation we can write the linear equation in the following form

$$\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \cdots \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Ex. Compact representation of system of linear equations

$$2x_1 + 3x_2 + 5x_3 = 1$$
$$4x_1 - 2x_2 - 7x_3 = 8$$
$$9x_1 + 5x_2 - 3x_3 = 2$$

we can write the linear equations to the matrix form

$$\begin{bmatrix} 2 & 3 & 5 \\ 4 & -2 & -7 \\ 9 & 5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}$$

Geometric interpretation - Existence of Solutions

Two equation are given

$a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

If we intercept x_1, x_2 as coordinates in the x_1x_2 - plane, then each of the two equations represents a straight line and (x_1, x_2) is a solution if and only if the point P with coordinates x_1, x_2 lies on both lines.

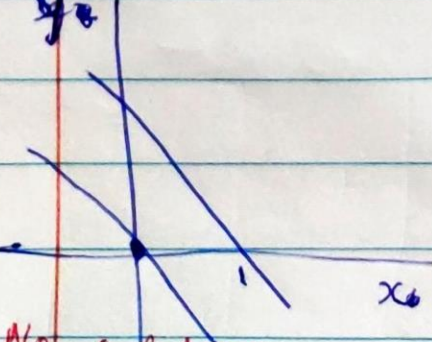
Hence there are three possible cases:

a) No solution if the lines are parallel

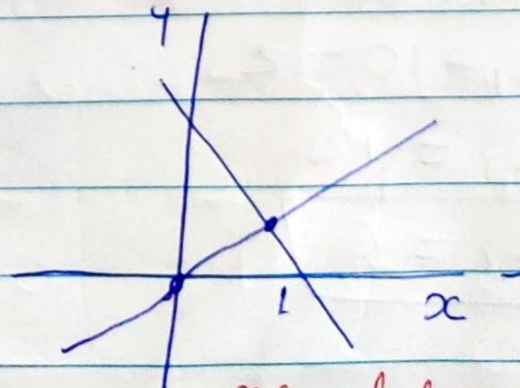
b) Precisely one solution if they intersect

c) Infinity many solution if they coincide

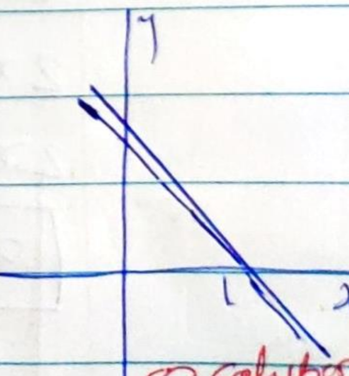
$x + y = 1$	$x + y = 1$	$x + y = 1$
$x + y = 0$	$x - y = 0$	$2x + 2y = 2$



No solution



one solution



∞ solution

How to solve linear equations

$2x_1 + 5x_2 = 2$ — ①

$4x_1 + 3x_2 = 18$ — ②

multiply the ① eq^n by 2

it gives

$4x_1 + 10x_2 = 4$ —— ③

Subtract it from ② equation

it gives

$-7x_2 = 14$

$x_2 = \frac{14}{-7} = -2$

$\boxed{x_2 = -2}$

put value of x_2 in eq^n ①

$2x_1 + 5 \times (-2) = 2$

$2x_1 - 10 = 2$

$2x_1 = 12$

$\boxed{x_1 = 6}$

Solve the linear system

$-x_1 + x_2 + 2x_3 = 2$ — ①

$3x_1 - x_2 + x_3 = 6$ — ②

$-x_1 + 3x_2 + 4x_3 = 4$ — ③

Subtract eq^n ① from ③ we get

$2x_2 + 2x_3 = 2$ — ④

add eq^n ② and $3 \times eq^n$ ①

$-3x_1 + 3x_2 + 6x_3 = 6$

$3x_1 - x_2 + x_3 = 6$

$2x_2 + 7x_3 = 12$ — ⑤

Subtract eq^n ④ from eq^n ⑤ we get

$5x_3 = 10$

$\boxed{x_3 = 2}$

Back substitution

put value of x_3 in eq^n ④ we get

$2x_2 + 2 \times 2 = 2$

$2x_2 = \frac{-2}{2}$

$\boxed{x_2 = -1}$

put value of x_3 and x_2 in eq^n 1
we get

$$-x_1 + (-1) + 2 \times 2 = 2$$

$$-x_1 + 3 = 2$$

$$\boxed{x_1 = 1}$$

Same solution with
Gauss elimination

Augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ -1 & 3 & 4 & 4 \end{array}\right] \begin{array}{l} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array}$$

first step:- Elimination of x_1 from 3 equation

Row 3 − Row 1 we get

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ 0 & 2 & 2 & 2 \end{array}\right]$$

elemination of x_1 from 2nd equation

Row 2 + 3 Row 1

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 2 & 2 & 2 \end{array}\right]$$

Step 2: Elimination of x_2 from third equation

$$-x_1 + x_2 + 2x_3 = 2$$
$$2x_2 + 7x_3 = 12$$
$$2x_2 + 2x_3 = 2$$

Row 3 − Row 2 it gives

$$-5x_3 = -10 \quad \therefore \boxed{x_3 = 2}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 2 & 2 & 2 \end{array}\right]$$

Back substitution we get

$x_3 = 2$

$x_2 = -1$

$x_1 = 1$

Inner Product / Dot Product / Scalar product

The inner product or Dot product of Two $n \times 1$ matrices, $\vec{u}$ and $\vec{v}$ Denoted by $\vec{u} \cdot \vec{v}$ is computed by $\vec{u}^T \vec{v}$ and result is a 1×1 matrix which is a scalar.

Ex. $\vec{u} = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 0 \\ +1 \\ 1 \end{bmatrix}$

Comput $\vec{u} \cdot \vec{v}$.

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$$

$$= \begin{bmatrix} 4 & 2 & -1 & 0 \end{bmatrix}_{1\times4} \begin{bmatrix} 3 \\ 0 \\ +1 \\ 1 \end{bmatrix}_{4\times1}$$

$$= \left[4(3) + 2(0) + (-1)(+1) + 0(1) \right]$$

$$= 12 - 1$$

$$= 11$$

Compute
$\vec{u}\cdot\vec{v}$ and $\vec{v}\cdot\vec{u}$ for

$\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix}$

$$\vec{u}\cdot\vec{v} = \vec{u}^T \vec{v} = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix}$$

$$= [0\times(-3) + 1(3) + 2(2)]$$
$$= [3+4]$$
$$= [7]$$

$$\vec{v}\cdot\vec{u} = v^T\cdot\vec{u} = \begin{bmatrix} -3 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= [(-3)\times 0 + 3\times 1 + 2\times 2]$$

$$= [3+4]$$
$$= [7]$$

Properties

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in R^n and c be a scalar

① $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

② $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

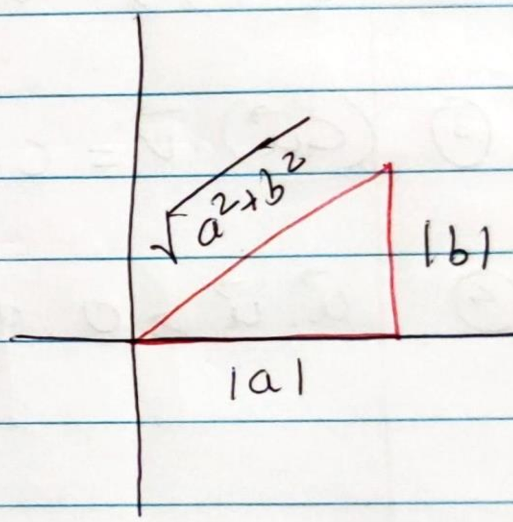
③ $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$

④ $\vec{u} \cdot \vec{u} \geq 0$ and $\vec{u} \cdot \vec{u} = 0$ iff $\vec{u} = 0$

The length or Norm of a vector is non-negative scalar $\|V\|$ defined by

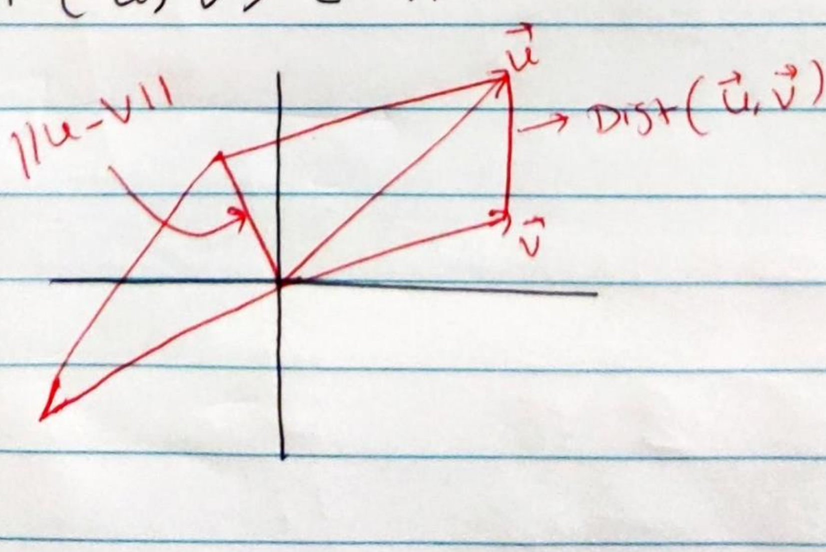
$$\|V\| = \sqrt{V \cdot V} = \sqrt{V_1^2 + V_2^2 + \cdots + V_n^2}$$

Suppose $\vec{V} = \begin{bmatrix} a \\ b \end{bmatrix}$



The ~~s~~ distance bet^n $\vec{u}$ and $\vec{v}$ can be found by

$$\text{Dist}(\vec{u}, \vec{v}) = \|u - v\|$$



Compute dist $(\vec{u}, \vec{v})$ for $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

$$\vec{u} - \vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$\|\vec{u} - \vec{v}\| = \sqrt{4^2 + (-2)^2} = \sqrt{16 + 4} = \sqrt{20}$$

Find the distance betⁿ $\vec{u} = \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$

$$\vec{u} - \vec{v} = \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 5 \end{bmatrix}$$

$$\text{Distance}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{9^2 + 3^2 + 5^2}$$

$$= \sqrt{81 + 9 + 25} = \sqrt{115}$$

Orthogonal Vectors

Two vectors are parallel or orthogonal or neighter

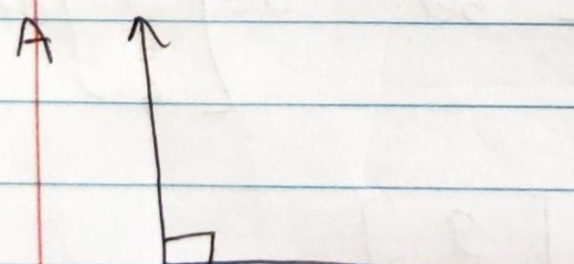
Parallel

$A \longrightarrow$ $\qquad$ $A \longrightarrow$

$B \longrightarrow$ $\qquad$ $\longleftarrow B$

$\theta = 0^\circ$ $\qquad$ $\theta = 180^\circ$

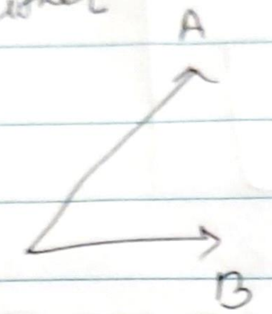
Orthogonal



$\theta = 90^\circ$

$= \pi/2$

Neither



$A = \langle 3, 5 \rangle$ $\qquad$ $B = \langle 5, -3 \rangle$

slope →

$m_A = \frac{A_y}{A_x} = \frac{5}{3}$ $\qquad$ $m_B = \frac{B_y}{B_x} = \frac{-3}{5}$

$\boxed{A \cdot B = 0}$

Dot product

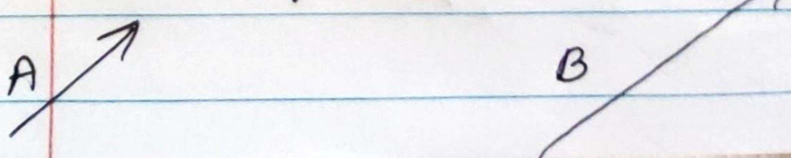
$A \cdot B = 3 \times 5 + 5 \times (-3) = 15 - 15 = 0$

If dot product of two vector is 0 then that two vectors are perpendicular

$A = \langle 4, 3 \rangle$ $\qquad$ $B = \langle 12, 9 \rangle$

$m_A = \frac{3}{4}$ $\qquad$ $m_B = \frac{9}{12} = \frac{3}{4}$

Slopes are same



$B = 3A$

$$\theta = \cos^{-1}\left[\frac{A \cdot B}{||A|| \cdot ||B||}\right]$$

$A \cdot B = 4(12) + 3(9) = 48 + 27 = 75$

$||A|| = \sqrt{(x^2) + (y^2)} = \sqrt{4^2 + 3^2}$

$= \sqrt{16 + 9} = 5$

$||B|| = \sqrt{12^2 + 9^2} = \sqrt{144 + 81} = \sqrt{225}$

$||B|| = 15$

$$\theta = \cos^{-1}\left[\frac{75}{5 \times 15}\right]$$

$$\theta = \cos^{-1}\left[\frac{75}{75}\right] = \cos^{-1}(1)$$

$\theta = 0^\circ$

$A = \langle 5, -6 \rangle$ $\quad B = \langle 3, -4 \rangle$

$m_A = \frac{-6}{5}$ $\quad m_B = \frac{-4}{3}$

$$A \cdot B = 5(3) + (-6)(-4)$$
$$= 15 + 24$$
$$= 39$$

$$\|A\| = \sqrt{25 + 36} = \sqrt{61}$$

$$\|B\| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\theta = \cos^{-1}\left[\frac{A \cdot B}{\|A\| \cdot \|B\|}\right]$$

$$= \cos^{-1}\left[\frac{39}{(\sqrt{61} \cdot 5)}\right]$$

$$\theta = \cos^{-1}(\quad)$$

$$\theta = 2.936^{\circ}$$

Unit vector

$U = \frac{V}{\|V\|}$

$\|U\| = 1$ is called unit vector

every unit vector has magnitude 1

$\boxed{U = \frac{V}{\|V\|}}$

$V = \|V\| \cdot U$

$V = \langle 4, -3 \rangle$

$U = \frac{\langle 4, -3 \rangle}{\sqrt{4^2 + (-3)^2}} = \frac{\langle 4, -3 \rangle}{5}$

$U = \langle \frac{4}{5}, \frac{-3}{5} \rangle$

$\|U\| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{-3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}}$

$= \sqrt{\frac{25}{25}} = \sqrt{1}$

$\|U\| = 1$

Finding orthogonal vector vectors in $\mathbb{R}^3$

Let $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ find $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Such that $x \perp v$

$\Rightarrow \quad v \cdot x = 0$

$\Rightarrow x_1 \cdot v_1 + x_2 v_2 + x_3 v_3 = 0$

Let $x_1 = \alpha_1, x_2 = \alpha_2$

Then $x_3 v_3 = -\alpha_1 v_1 - \alpha_2 v_2$

$$x_3 = \frac{-\alpha_1 v_1}{v_3} - \frac{\alpha_2 v_2}{v_3} \quad ; \; v_3 \neq 0$$

$\cancel{v_2 \neq 0}$

So, $x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \frac{-\alpha_1}{v_3} - \frac{\alpha_2}{v_3} \end{bmatrix}$ for any α_1, α_2

Check for orthogonal vector

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad x = \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}$$

$$v \cdot x = (1)(3) + (2)(6) + (3)(-5)$$
$$= 3 + 12 - 15$$
$$= 0$$

Find vectors that are orthogonal to [1, 2, 3]
Explain why we can have infinite number of such vectors.

$\bar{u} = [1, 2, 3]$ Let $\vec{v}$ be perpendicular to $\vec{u}$

$\vec{v} = [x, y, z]$

$\bar{u} \cdot \bar{v} = 0$, $\bar{u} \neq 0$

$\bar{v} \neq 0$

$$\bar{u} \cdot \bar{v} = |\bar{u}| \cdot |\bar{v}| \cos\theta$$
$$0 = |\bar{u}| \cdot |\bar{v}| \cos\theta$$
$$\cos\theta = 0$$
$$\theta = 90^\circ$$

$\bar{a} \cdot \bar{v} = [1 \quad 2 \quad 3][x \quad y \quad z]$

$\Rightarrow x + 2y + 3z = 0$

x	y	z
1	0	...
-5	1	1

$1 + (2)0 + 3(z) = 0$

$z = -1/3$

$\bar{v} = [1, 0, -1/3]$

All these ~~vectors~~ values ~~corresponding~~ correspond to vectors which is orthogonal to given vectors

Symmetric Positive Definite Matrix

Symmetric Matrix

A square matrix $A = [a_{ij}]$ is called symmetric matrix, if $a_{ij} = a_{ji}$ for all i, j.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}_{3\times 3}$$

if $\boxed{A = A^T}$

Such matrices are called as symmetric matrix.

$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A^T = [1 \ 2]$

A is not symmetric

Ex.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$A = A^T$ symmetric

$$D = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 1 & 4 \\ 2 & 1 & 0 \end{bmatrix}$$

$$D^T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 1 & 4 & 0 \end{bmatrix}$$

$D \neq D^T$ Not symmetric

$$E = \begin{bmatrix} \sqrt{2} & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & -1 \end{bmatrix} \qquad E^T = \begin{bmatrix} \sqrt{2} & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & -1 \end{bmatrix}$$

$E = E^T$

E is symmetric

Symmetric Positive Definite Matrix

A symmetric matrix A is positive definite if $x^T A x > 0$ for any $x \neq 0$

Ex. $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$x^T A x = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3x_1 + x_2 \\ x_1 + 3x_2 - x_3 \\ -x_2 + 3x_3 \end{bmatrix}$$

$$= x_1(3x_1 + x_2) + x_2(x_1 + 3x_2 - x_3) + x_3(-x_2 + 3x_3)$$

$$= 3x_1^2 + x_1x_2 + x_1x_2 + 3x_2^2 - x_2x_3 - x_2x_3 + 3x_3^2$$

$$= 3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_2x_3$$

$$= (x_1 + x_2)^2 + 2x_1^2 + 2x_2^2 + (x_2 - x_3)^2 - x_2^2 + 2x_3^2$$

$$= (x_1 + x_2)^2 + (x_2 - x_3)^2 + 2x_1^2 + x_2^2 + 2x_3^2 > 0$$

6. $\bar{A}$ is an SPD matrix.

Theorem

1. A is a non-singular matrix is $|A| \neq 0$

2. Diagonal elements are positive ie. $a_{ii} > 0$

3. $\max_{\substack{1 \le k \le n \\ 1 \le j \le n}} |a_{kj}| \le \max_{1 \le k \le n} |a_{ii}|$

other than diagonal

diagonal

4) $(a_{ij})^2 < a_{ii} \cdot a_{jj}$ $\quad i \neq j$

checking for each element other than diagonal

$|A| = 21 \neq 0$

$a_{11} = 3 > 0$, $a_{22} = 3 > 0$, $a_{33} = 3 > 0$

$1 \leq 3 \rightarrow \quad 1 < 3$

$(a_{12})^2 = (a_{21})^2 \overset{?}{<} a_{11}\, a_{12} \rightarrow \quad 1^2 < 3(3)$ ✓

$(a_{13})^2 = (a_{31})^2 \overset{?}{<} a_{33}\, a_{11} \rightarrow \quad 0^2 < 3(3)$ ✓

$(a_{23})^2 = (a_{32})^2 \overset{?}{<} a_{22} a_{33} \rightarrow (-1)^2 < 3(3)$ ✓

Since all conditions are satisfied

$\bar{A}$ is a symmetric positive Definite matrix.

Calculate symmetric matrix & skew symmetric matrix

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

$\downarrow$ Symmetric matrix

$\searrow$ Skew symmetric

$A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$ Express this matrix as a sum of symmetric and a skew symmetric matrix

$$A^T = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$$

$$\frac{A + A^T}{2} = \frac{1}{2}\left(\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix}$$ symmetric matrix

$$\frac{A - A^T}{2} = \frac{1}{2}\left(\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightarrow$$ skew symmetric matrix

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

$$= \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$$

Determinant

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = ad - cb$$

① $$A = \begin{bmatrix} 3 & 5 \\ -4 & 7 \end{bmatrix}$$

$$|A| = 3(7) - 5(-4)$$
$$= 21 + 20$$
$$= 41$$

② $$A = \begin{bmatrix} -7 & 8 \\ 4 & -3 \end{bmatrix}$$

$$|A| = (-7)(-3) - 4(8)$$
$$= 21 - 32$$
$$= -11$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$|A| = a_1 \begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} - b_1 \begin{bmatrix} a_2 & c_2 \\ a_3 & c_3 \end{bmatrix} +$$

$$c_1 \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

Ex. $A = \begin{bmatrix} 2 & 4 & -3 \\ 5 & 7 & 6 \\ -8 & 1 & 9 \end{bmatrix}$

$$= 2 \begin{bmatrix} 7 & 6 \\ 1 & 9 \end{bmatrix} - 4 \begin{bmatrix} 5 & 6 \\ -8 & 9 \end{bmatrix} + (-3) \begin{bmatrix} 5 & 7 \\ -8 & 1 \end{bmatrix}$$

$$= 2[7(9) - 1(6)] - 4[5(9) - (6)(-8)] -$$

$$3[5-(-56)]$$

$$= 2[63-6] - 4[45+48] - 3[61]$$

$$= 2(57) - 4(93) - 183$$

$$= 114 - 372 - 183$$

$$= -441$$

Ex.

$$A = \begin{bmatrix} 5 & 7 & -8 \\ 4 & -3 & 6 \\ 1 & 7 & -9 \end{bmatrix}$$

$|A| = -29$

Trace :-

The trace of a square matrix is the sum of its diagonal entries.

let A be a $k \times k$ matrix then

trace is defined as

$$\text{trace}(A) = \text{tr}(A) = \sum_{k=1}^{K} A_{kk}$$

Ex. $A = \begin{bmatrix} 2 & 1 & 5 \\ 2 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix}$

$$\text{tr}(A) = A_{11} + A_{22} + A_{33}$$
$$= 2 + 3 + 0$$
$$= 5$$

$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 2 & 1 & 5 & 1 \\ 4 & 0 & 7 & -1 \end{bmatrix}$

$$\text{tr}(A) = A_{11} + A_{22} + A_{33} + A_{44}$$
$$= 1 + 2 + 5 - 1$$
$$= 7$$

Eigen values :-

How to find the eigen values of a matrix

1) check whether the given matrix is a square matrix or not

if "yes" follow 2^{nd} step

2) Determine identity matrix(I)

3) Estimate the matrix $A - \lambda I$

4) find the determinant of $A - \lambda I$

5) Equate the determinant of $A - \lambda I$ to zero. $\{|A - \lambda I| = 0\}$

6) Calculate the possible values of λ.

Ex. find the eigen values of matrix

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

Sol^n $A - \lambda I = \begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$(1-\lambda)(2-\lambda) - 12 = 0$$

$$2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$\lambda^2 - 5\lambda + 2\lambda - 10 = 0$$

$$(\lambda + 2)(\lambda - 5) = 0$$

$$\lambda = -2, 5$$

$\therefore$ eigen values will be $(-2, 5)$

(2) find the eigen values of the given matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Solⁿ. $A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 0-\lambda \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$(1-\lambda)[(1-\lambda)(0-\lambda) - 2] = 0$$

$$(1-\lambda)(\lambda^2 - \lambda - 2) = 0$$

$$-\lambda^3 + 2\lambda + \lambda - 2 = 0$$

$$\lambda = 1, 0$$

(3) Find eigen values of a given matrix

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 3 & 8 \\ 0 & 0 & 2 \end{bmatrix}$$

Solⁿ- eigen values will be 1, 3, 2

Eigen vectors

Eigen vector equation is

$$AX = \lambda X$$

X is a eigen vector

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ -4 & -7 \end{bmatrix}$$

Soln-

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 4 \\ -4 & -7-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ -4 & -7-\lambda \end{vmatrix}$$

$$(1-\lambda)(-7-\lambda) - 4(-4) = 0$$

$$(\lambda+3)^2 = 0$$

$$\lambda = -3, 3$$

$Ax = \lambda x \Rightarrow |A - \lambda I| = 0$

Using the eigen vector equation

$Ax = -3x$ for $\lambda = -3$

$(A + 3I)x = 0$ $\therefore |A + 3I| = 0$

$$\left(\begin{bmatrix} 1 & 4 \\ -4 & -7 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives

$4x_1 + 4x_2 = 0$

$x_1 + x_2 = 0$

Let us set $x_1 = k$ then $x_2 = -k$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

eigen vector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

for $\lambda = 3$

$Ax = 3x$

$$\left(\begin{bmatrix} 1 & 4 \\ -4 & -7 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x_1 + 4x_2 = 0$

$-4x_1 - 10x_2 = 0$

Eigen vector of 2x2 matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \quad \text{——— (1)}$$

$$(5-\lambda)(2-\lambda) - (4)(1) = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 6)(\lambda - 1) = 0$$

$$\lambda = 6 \ , \ \lambda = 1$$

Find the eigen vector for eigen values 1 and 6

when $\lambda = 1$

$$AX = \lambda X$$

$$(A - \lambda I) X = 0$$

Put $\lambda = 1$ in eqn ①

$$\begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \to 4R_2 - R_1$

$$\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x + 4y = 0$$

assume $y = t$ then

$$4x + 4t = 0 \Rightarrow \quad 4x = -4t \Rightarrow x = -t$$

So,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

when $\lambda = 6$

Put $\lambda = 6$ in eq^n 1

$$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \to R_2 + R_1$

$$\begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x+4y=0$

$y=t$ then $-x+4t=0 \Rightarrow x=4t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vector is $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Eigen vector for 3x3 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$ λ is eigenvalue

$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigen vector of A

$\rightarrow$ $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

Calculate determinant we get

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)(1-\lambda) - 1\right] - \left[(1-\lambda)-1\right] + \left[1-(1-\lambda)\right]$$

$$= (1-\lambda)\left[1-2\lambda+\lambda^2-1\right] + \lambda + \lambda$$

$$= (1-\lambda)(\lambda^2-2\lambda) + 2\lambda$$

$$= \lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2 + 2\lambda$$

$$|A-\lambda I| = -\lambda^3 + 3\lambda^2$$

$$|A-\lambda I| = 0$$

$$\therefore -\lambda^3 + 3\lambda^2 = 0$$

$$\lambda^2(-\lambda+3) = 0$$

$$\lambda = 0, \lambda = 3$$

$\therefore$ The eigen values are 0, 0 and 3.

Let us find the corresponding eigen vectors.

when $\lambda = 0$:

$(A - \lambda I) V = 0$

$$\begin{bmatrix} 1-0 & 1 & 1 \\ 1 & 1-0 & 1 \\ 1 & 1 & 1-0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Expanding the above matrix, we get

$x + y + z = 0$

we have only one equation with three unknowns. So let us assume two of the variables to be $y = t_1$ and $z = t_2$.

Then the above eq^n becomes

$$x + t_1 + t_2 = 0$$

$$\Rightarrow \quad x = -t_1 - t_2$$

Thus the eigen vector ~~is~~ at $\lambda = 0$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

where $\lambda =$

where $\lambda = 3$

$(A - \lambda I) v = 0$

$$\begin{bmatrix} 1-3 & 1 & 1 \\ 1 & 1-3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \rightarrow 2R_2 + R_1$ and $R_3 \rightarrow 2R_3 + R_1$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now apply $R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Expanding the matrix we get

$-2x+y+z=0$

$-3y+3z=0$

Let $z=t$ then

$-3y+3z=0 \Rightarrow z=y=t$

$-2x+y+z=0 \Rightarrow -2x+t+t=0$

$x=t$

Thus the eigen vector is :

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus the eigen vector that correspond to $\lambda = 3$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ Thus the eigen vectors of the given 3×3 matrix are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Finding Eigenspace

The eigenspace of a matrix (linear transformation) is the set of all of its eigen vectors i.e. to find the eigen space:

- Find eigen values first
- Then find the corresponding eigen vectors
- Just enclose all the eigenvectors in a set (order doesn't matter).

From previous ex. the eigenvectors of given matrix A are

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So, eigenspace of A is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Note: The vectors in the eigenspace are linearly independent.

Diagonalize Matrix Using Eigenvalues and Eigenvectors.

Diagonalizing a matrix A is the process of writing it as the product of three matrices such that the middle one is a diagonal matrix i.e. $\boxed{A = XDX^{-1}}$

where D is the matrix of eigenvalues (to find D, take the identity matrix of the same order as A, replace 1s in it by eigen values) and X is the matrix of eigen vectors that are written in the same order as eigenvalues in D.

Ex. Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Sol^n- Finding eigen values
The characteristic equation is
$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

calculate the determinant we get

$$(1-\lambda)\left[(1-\lambda)(1-\lambda)-1\right] - 1\left[(1-\lambda)-1\right] + 1\left[1-1(1-\lambda)\right]$$

calculate

$$= -\lambda^3 + 3\lambda^2$$

$$|A - \lambda I| = 0$$

$$\therefore \quad -\lambda^3 + 3\lambda^2 = 0$$

$$\lambda^2(-\lambda+3) = 0$$

eigen values $\boxed{\lambda = 0, \ \lambda = 3}$

Calculate corresponding eigen vectors for eigen values 0, 0, 3

$\therefore$ eigenvectors are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Calculate D

D → take identity matrix of same order
Replace 1s in it by eigenvalues

$$\therefore D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

X is the matrix of eigenvectors that are written in the same order as eigen values in D

$$\therefore X = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The inverse of the matrix X is X^{-1}

$$X^{-1} = \begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$\therefore$ The diagonalization of A is

$$A = X D X^{-1}$$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Note:- We can diagonalize a matrix A only when determinant of the corresponding matrix X is NOT ZERO because if $\det(X) = 0$, we cannot find X^{-1}.

Inverse of Matrix

$$A^{-1} = \frac{adj(A)}{det(A)}$$

2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = (ad) - (bc)$$

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Adjoint of a 3x3 matrix

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

Calculator minors M_{11}, M_{12}, M_{13}
M_{21}, M_{22} M_{23}

Cofactor matrix is

$$\begin{bmatrix} +m_{11} & -m_{12} & +m_{13} \\ -m_{21} & +m_{22} & -m_{23} \\ +m_{31} & -m_{32} & +m_{33} \end{bmatrix}$$

$$\begin{bmatrix} -3 & \cancel{+}4 & 5 \\ \cancel{+}4 & 0 & \cancel{+}4 \\ 5 & \cancel{+}4 & -3 \end{bmatrix} \qquad \begin{bmatrix} -3 & -4 & 5 \\ -4 & 0 & -4 \\ 5 & -4 & -3 \end{bmatrix}$$

Transposing the cofactor matrix we get the adjoint matrix

$$\text{adj } A = \begin{bmatrix} -3 & -4 & 5 \\ -4 & 0 & -4 \\ 5 & -4 & -3 \end{bmatrix}$$

$|A| = -16$

$$A^{-1} = \frac{\text{Adj}(A)}{\det(A)}$$

$\boxed{\text{Note } AA^{-1} = A^{-1}A = I}$

Trick to calculate determinant

$$\begin{matrix} a & b & c & a & b & c \\ p & q & r & p & q & r \\ x & y & z & x & y & z \end{matrix}$$

$|A| = aqz + brx + cpy$
$- ary - bpz - cqx$

Singular Value Decomposition (SVD)

Singular value decomposition (SVD) of a matrix is a factorization of that matrix into three matrices.
It has some important applications in data science.

SVD of $m \times n$ matrix A is given by

$$A = UWV^T$$

Where,
$U = m \times n$ matrix of the orthogonal eigen vector

$V^T =$ Transpose of a $n \times n$ matrix containing the orthogonal vector of $A^T A$

$W =$ a $n \times n$ diagonal matrix of the singular values which are the square root of the eigenvalues

Ex.

Calculate SVD of given matrix

$$A = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}$$

$t^4 6 - t^2 9$

$\begin{matrix} 64 & -32 \\ -32 & 16 \end{matrix}$

$\begin{matrix} 4 & -2 \\ -2 & 9 \end{matrix}$

$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0.5 \end{bmatrix}$

Sol^n.

$$A^T = \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix}$$

Now

$$AA^T = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 65 & -32 \\ -32 & 17 \end{bmatrix}$$

Calculate eigenvector for AA^T

$$[A \cdot A^T - \lambda I] = 0$$

$$\begin{bmatrix} (65-\lambda) & -32 \\ -32 & (17-\lambda) \end{bmatrix} = 0$$

$$(65-\lambda)(17-\lambda) - (-3) \times (-32) = 0$$

$$(1105 - 82\lambda + \lambda^2) - 1024 = 0$$

$$(\lambda^2 - 82\lambda + 81) = 0$$

$A - \lambda I = 0 \qquad A - 1I = 0$

$$(\lambda - 1)(\lambda - 81) = 0$$

$$(\lambda - 1) = 0 \quad \text{or} \quad \lambda - 81 = 0$$

$$\boxed{\lambda = 1} \qquad \boxed{\lambda = 81}$$

eigen values of the matrix AA^T are given by $\lambda = 1, 81$

Eigenvectors for $\lambda = 81$ are:

$$v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Eigenvectors for $\lambda = 1$ are

$$v_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

Calculate Eigen vector for orthogonal matrix

as calculated matrix is

$$\begin{bmatrix} 65-\lambda & -32 \\ -32 & +17-\lambda \end{bmatrix} = 0$$

for $\lambda = 81$ eigen vector will be

$$\begin{bmatrix} 65-81 & -32 \\ -32 & 17-81 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -16 & -32 \\ -32 & -64 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 0.5 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

from above matrix it gives

$0.5x + y = 0$ — ①

$x + 2y = 0$ — ②

As our matrix is orthogonal we can calculate eigen vector as follows

From eq^n ①

Put $x = -2$

x	y
-2	1

$0.5 \times (-2) + y = 0$

$-1 + y = 0$

$\boxed{y = 1}$

Hence eigen vector is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

For eigen value $\lambda = 1$ eigen vector is calculated as follow

$$\begin{bmatrix} 65-1 & -32 \\ -32 & 16 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0.5 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 0.5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

x	y
0.5	1

$2x - y = 0 \Rightarrow y = +2x$

$\cancel{0.5x - y = 0} \Rightarrow \cancel{y = 0.5x}$

$x = \dfrac{y}{2}$

put $\boxed{x = 0.5}$

$\boxed{y = 1}$

for eigen vector 1 $(-2, 1)$

length $l = \sqrt{(-2)^2 + (1)^2} = 2.236$

normalizing give

$u_1 = \left(\frac{-2}{2.236}, \frac{1}{2.236}\right) = (-0.894, 0.447)$

for eigen vector 2 $(0.5, 1)$

Length $l = \sqrt{0.5^2 + 1^2} = 1.118$

so normalizing gives $u_2 = \left(\frac{0.5}{1.118}, \frac{1}{1.118}\right)$

$= (0.447, 0.894)$

Eigen vectors for ~~z~~ ~~square~~

$v_1 = \begin{bmatrix} \cancel{0.84} \\ 1 \end{bmatrix}$

$U = [u_1, u_2] = \begin{bmatrix} -0.894 & 0.447 \\ 0.447 & 0.894 \end{bmatrix}$

$$V_i = \frac{1}{\sigma_i} A^T \cdot u_i$$

$$V = \begin{bmatrix} 0.447 & +0.894 \\ -0.894 & 0.447 \end{bmatrix}$$

$$W = \begin{bmatrix} \sqrt{81} & 0 \\ 0 & \sqrt{1} \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 0.447 & -0.894 \\ 0.894 & 0.447 \end{bmatrix}$$

$$A = U W V^T$$

$$= \begin{bmatrix} -0.894 & 0.447 \\ 0.447 & 0.894 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.447 & -0.894 \\ 0.894 & 0.447 \end{bmatrix}$$

For calculating $V \rightarrow$

Calculate eigen vectors of $A^T A$ as eigenvector

for $\lambda = 81$ is

$$V_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

for $\lambda = 1$ is

$$V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

for eigen vector $(0.5, 1)$

length $L = \sqrt{(0.5)^2 + 1^2}$

$= 1.118$

normalizing gives $V_1 = \left(\frac{0.5}{1.118}, \frac{1}{1.118} \right)$

$V_1 = (0.447, 0.894)$

For eigen vector $(-2,1)$

length $L = \sqrt{(-2)^2+1^2} = 2.236$

normalizing gives $v_2 = \left(\frac{-2}{2.236}, \frac{1}{2.236}\right)$

$$v_2 = (-0.894, 0.447)$$

$$V = \begin{bmatrix} 0.447 & +0.894 \\ -0.894 & 0.447 \end{bmatrix}$$

$\therefore$

$$A = UWV^T$$

$$A = \begin{bmatrix} -0.894 & 0.447 \\ 0.447 & 0.894 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.447 & -0.894 \\ 0.894 & 0.447 \end{bmatrix}$$

Ex. SVD for the matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

Soln →

$$A^T = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -3 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

characteristics eq^n for above matrix is

$$AA^T - \lambda I = 0$$

$$\begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{bmatrix} = 0$$

$(17-\lambda)(17-\lambda)-64=0$

$\lambda^2-34\lambda+225=0$

$(\lambda-25)(\lambda-9)=0$

$\therefore$ Eigen values are

$\lambda_1 = 25 \qquad \lambda_2 = 9$

~~Since~~

Calculate for eigen vectors

$AX = \lambda X$

$|A-\lambda I| X = 0$

for $\lambda = 25$

$$\begin{vmatrix} 17-25 & 8 \\ 8 & 17-25 \end{vmatrix} = \begin{vmatrix} -8 & 8 \\ 8 & -8 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}$$

∴ eigenvectors for $\lambda = 25$ ~~are~~

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$-x + y = 0$

$x - y = 0$

x	y
1	1

$x = y$

∴ eigen vector for $\lambda = 25$ is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

eigen vector for $\lambda = 9$ is

$$\begin{bmatrix} 17-9 & 8 \\ 8 & 17-9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x+y=0$

$x+y=0$

$x=-y$

$\therefore$ eigen vector for $\lambda = 9$ is

x	y
1	-1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

→ Vector normalization for eigen vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Length, $L = \sqrt{1^2+1^2} = \sqrt{1+1} = \sqrt{2}$

Normalization gives

$u_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

for eigen vector $(1, -1)$

$$L = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

Normalization gives

$$u_2 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right)$$

$$\therefore \quad U = [u_1, u_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

→ For calculating W

$$W = \begin{bmatrix} \sqrt{25} & 0 \\ 0 & \sqrt{9} \end{bmatrix}$$

$$W = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

For calculating V

$$A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}_{3\times 2} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 3\times3+2\times2 & 3\times2+2\times3 & 3\times2+2\times(-2) \\ 2\times3+3\times2 & 2\times2+3\times3 & 2\times2+3\times(-2) \\ 2\times3+3\times(-2) & 2\times2+(-2)\times3 & 2\times2+(-2)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 9+4 & 6+6 & 6-4 \\ 6+6 & 4+9 & 4-6 \\ 6-6 & 4-6 & 4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 0 & -2 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} \quad \end{bmatrix}$$