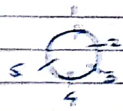
Discrete → means countable



Discrete
(Regulator of fan)

Steering Wheel
(Continuous)

SET THEORY

Set → It is a collection of unordered object.

Definition of Set

→ Explicit listing

Eg:- $A = \{a, b, c, d\}$ $a \in A$ ← belongs

$B = \{1, 2, 3, 4\}$ $4 \in B$

$C = \{2, 4, 6\}$

→ Set Builder

Eg:-

$A = \{x \mid x \text{ is an positive integer} < 10\}$

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$V = \{x \mid x \text{ is a vowel}\}$

$V = \{a, e, i, o, u\}$

Special sets

N → Natural nos.

N = {1, 2, 3, 4, 5, ...}

Z → Set of Integers

Z = {.... -2, -1, 0, 1, 2, ...}

Z^+ → Set of +ve integers

Z = {1, 2, 3, 4, ...}

R → Set of Real Numbers

R = {... -2.4, -1.6, 0, 1.4, 2.7, ...}

Q → Rational numbers

Q = {x | x = p/q ; where p, q ∈ Z}

C → Complex numbers

C = {a+ib | ~~a+ib~~ a, b ∈ R, $i^2 = -1$}

Null Set / Empty set

Notation :- ϕ or { }

Singletone set

Eg:- C = {5}

Subset

If A, B are 2 sets

~~A⊂B~~ A ⊂ B (A is a subset of B)
if any element of A is in B

If A ≠ B
A is a proper subset of B
A ⊊ B

Eg:- A = {1, 2, 3, 4}
B = {1, 2, 3, 4, 5, 6, 7}

A ⊆ A
ϕ ⊆ A .. (Null set is subset of every set)

* $\boxed{N \subset Z \subseteq Q \subseteq R \subseteq C}$

Eg A = {1, 2, 3}

No of subsets = 2^n

Subset of ϕ = 2^0 = 1 ie ϕ ⊆ ϕ

A = {ϕ}

Subset of {ϕ} = 2^1 = 2 ie {ϕ}, ϕ

Cardinality of a set

$|A|$ $n(A)$

It is the no. of elements in the set

Eg :- $A = \{1,2,3,4\}$
$n(A) = 4$

Power set → set of subsets of a set

$A = \{1,2,3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

☆ $p(\phi) = \{\phi\}$

☆ $p(\{\phi\}) = \{\{\phi\}, \phi\}$

$p(\{\phi\}) = p(p(\phi))$

Find the power set of ϕ, $\{\phi\}$

⇒ Operation on a Set

- Union $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Eg :- $A = \{1,2,3\}$ $B = \{2,3,5\}$

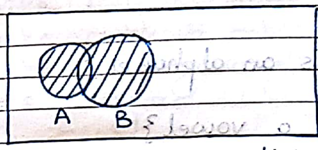
$A \cup B = \{1,2,3,5\}$

- Intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Eg :- $A = \{1,2,3\}$ $B = \{2,3,5\}$

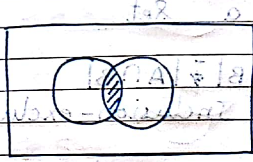
$A \cap B = \{2,3\}$

Venn Diagram



$A \cup B$

Universal Set.

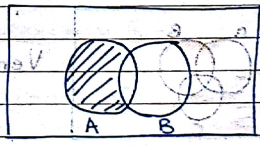


$A \cap B$

- Difference $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

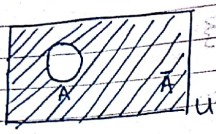
Eg :- $A = \{1,2,3,4,5\}$, $B = \{1,3,5\}$

$\therefore A - B = \{2,4\}$



→ Complement :-

$\bar{A} = U - A$



Also,
A = {x | x is an alphabet}

B = {x | x is a vowel}

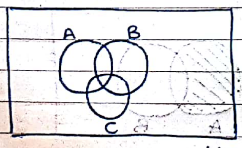
A - B = {x | x is a consonent}

⇒ Cardinality of a Set.

* $|A \cup B| = |A| + |B| - |A \cap B|$
This is called Inclusion-exclusion principle

When $A \cap B = \phi$, then it is called disjoint set.

* $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$



Venn diagram.

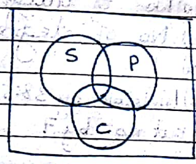
Probs :-

1) In a survey about hobbies, it is found that 35 people like ~~only~~ singing, 20 people like playing, 25 like cooking, 10 people like singing and playing, 15 people like playing and cooking, 8 people like cooking and singing and 5 people like all the 3.

Find :- 1) No. of people
2) Who like singing only
3) Who like cooking only
4) Who like playing only
5) ~~People ...~~

Soln :-

$|S| = 35$
$|P| = 20$
$|C| = 25$
$|S \cap P| = 10$
$|P \cap C| = 15$
$|S \cap C| = 8$
$|S \cap P \cap C| = 5$



$|S \cup P \cup C| = |S| + |P| + |C| - |S \cap P| - |S \cap C| - |P \cap C| + |S \cap P \cap C|$
$= 35 + 20 + 25 - 10 - 15 - 8 + 5$
$= 55 - 8 + 5$
$= 52$

No. of people $= 52$

Only singing $= |S| - |S \cap P| - |S \cap C| + |S \cap P \cap C|$
$= 35 - 10 - 8 + 5$
$= 22$

Cooking only $= |C| - |C\cap P| - |C\cap S| + |C\cap P\cap S|$

$= 25 - 15 - 8 + 5$

$= 7$

Playing only $= |P| - |P\cap C| - |P\cap S| + |C\cap P\cap S|$

$= 20 - 15 - 10 + 5$

$= 0$

2) From 1 to 500, Find out

i) No. of integers divisible by 3

ii) No. of integers divisible by 5

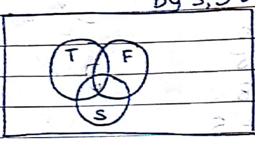
iii) No. of integers divisible by 7.

iv) No. of integers divisible by 3 and 5 both

v) No. of integers divisible by 5 and 7 both

vi) " " " by 3,5 & 7

vi) Divisible by 3&5 but not by 7



i) $|T| = \left\lfloor \frac{500}{3} \right\rfloor = 166$

ii) $|F| = \left\lfloor \frac{500}{5} \right\rfloor = 100$

iii) $|S| = \left\lfloor \frac{500}{7} \right\rfloor = 71$

iv) $|T\cap F| = \left\lfloor \frac{500}{3\times 5} \right\rfloor = 33$ $\quad |T\cap S| = \left\lfloor \frac{500}{3\times 7} \right\rfloor = 23$

v) $|F\cap S| = \left\lfloor \frac{500}{5\times 7} \right\rfloor = 14$

vi) $|F\cap S\cap T| = \left\lfloor \frac{500}{3\times 5\times 7} \right\rfloor = 4$

vii) Divisible by 3&5 but not by 7

$= |T\cap F| - |T\cap S\cap F|$

$= 33 - 4 = 29$

viii) Neither by 3, nor by 5, nor by 7

$|T\cup S\cup F| = |T| + |S| + |F| - |T\cap S| - |S\cap F| - |T\cap F| + |T\cap S\cap F|$

$= 166 + 71 + 100 - 23 - 33 - 14 + 4$

$= 271$

Ans $= 500 - 271 = 229$

ix) Atleast by 2 nos

$= |T\cap F| + |T\cap S| + |F\cap S| - 2|T\cap F\cap S|$

$= 33 + 23 + 14 - 2\times 4$

$= 70 - 8$

$= 62$

x) Only by 1 no.

$= |T\cup F\cup S| - |T\cap F| - |T\cap S| - |F\cap S| + 2|T\cap F\cap S|$

$= 271 - 62$

$= 209$

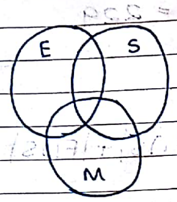
~~Out of 200 students~~

It was found that in a class, 80 students are passed in English, 60 in Science, 50 in Mathematics. It was also found that 30 students passed both in English & science, 15 students passed in both English & mathematics and 20 students passed in both Maths and Science. 10 students passed in all the three.

a) How many students passed in atleast one subject?

b) How many passed in English only

c) How many failed in all subjects.



$|E| = 80$, $|S| = 60$ & $|M| = 50$

$|E \cap S| = 30$

$|E \cap M| = 15$

$|M \cap S| = 20$

$|E \cap S \cap M| = 10$

$|E \cup S \cup M| = |E| + |S| + |M| - |E \cap S| - |E \cap M| - |M \cap S| + |E \cap S \cap M|$

$= 80 + 60 + \cancel{50} - \cancel{30} - 15 - \cancel{20} + 10$

$= 135$

No. of students passed $\}$ = 135 students

English only $= |E| - |E \cap S| - |E \cap M| + |E \cap S \cap M|$

$= 80 - 30 - 15 + 10$

$= 45$ students

Failed in all $= |U| - |E \cup S \cup M|$

$= 200 - 135$

$= 65$ students

★ Sometimes to twist the question, the word 'only' is used.

Eg:- It was found in a class, 30 students passed only in English, 60 in science, ...

*make lot of difference

Using the formula we can find out no of students passing in English.

⇒ Set Identities / Set Theoretic Laws

1) $A \cup A = A$ $\quad$ $A \cap A = A$ $\quad$ Idempotent

2) $A \cup B = B \cup A$ $\quad$ $A \cap B = B \cap A$ $\}$ Commutative

3) $A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$ $\}$ Associative

4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $\}$ Distributive

5) $\overline{A \cup B} = \bar{A} \cap \bar{B}$
$\overline{A \cap B} = \bar{A} \cup \bar{B}$ $\}$ De Morgan's Law

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6) $A \cup \phi = A$ & $A \cap \phi = \phi$
$A \cup U = U$ & $A \cap U = A$ } Complement
$\bar{\bar{A}} = A$

☆ $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$

☆ $A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$

⇒ De Morgan's Law for n sets

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$$

⇛ Propositional Logic

It is a declarative sentence having value True or False.

p → true
$\neg p$ → false

Operations / connections

$\wedge$	conjunction	AND
$\vee$	Disjunction	OR
$\neg$	Negation	NOT
$\rightarrow$	Implication / Conditional	
$\leftrightarrow$	Double implication / Bidirectional.	

Truth Table

p	q	$p \vee q$	$p \wedge q$	$p \rightarrow q$	$p \leftrightarrow q$	$\neg p$	$\neg q$
T	T	T	T	T	T	F	F
T	F	T	F	F	F	F	T
F	T	T	F	T	F	T	F
F	F	F	F	T	T	T	T

$p \rightarrow q \equiv \neg p \vee q$
$\neg(p \rightarrow q) = p \wedge \neg q$ } Elimination of implication

$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$
$= (\neg p \vee q) \wedge (\neg q \vee p)$ } Elimination of double implication

⊛ If $p \rightarrow q$

Converse :- $q \rightarrow p$
Inverse :- $\sim p \rightarrow \sim q$ Logically equivalent
Contrapositive :- $\neg q \rightarrow \neg p$

6) $\{A \cup \phi\} = A$ & $A \cap \phi = \phi$ } Complement
$A \cup U = U$ & $A \cap U = A$
$\bar{\bar{A}} = A$

☆ $A_1 \cup A_2 \cup A_3 \cup \ldots \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$

☆ $A_1 \cap A_2 \cap A_3 \cap \ldots \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$

⇒ De Morgan's Law for n sets

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$$

⇛ Propositional Logic

It is a declarative sentence having value True or False

$p \rightarrow$ true
$\neg p \rightarrow$ false

Operations / connections

$\wedge$	Conjunction	AND
$\vee$	Disjunction	OR
$\neg$	Negation	NOT
$\rightarrow$	Implication / Conditional	
$\leftrightarrow$	Double implication / Bidirectional	

Truth Table

p	q	$p \vee q$	$p \wedge q$	$p \rightarrow q$	$p \leftrightarrow q$	$\neg p$	$\neg q$
T	T	T	T	T	T	F	F
T	F	T	F	F	F	F	T
F	T	T	F	T	F	T	F
F	F	F	F	T	T	T	T

$p \rightarrow q \equiv \neg p \vee q$ } Elimination of implication
$\neg(p \rightarrow q) = p \wedge \neg q$

$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$ } Elimination of double implication
$= (\neg p \vee q) \wedge (\neg q \vee p)$

⊛ If $p \rightarrow q$

Converse :- $q \rightarrow p$
Inverse :- $\sim p \rightarrow \sim q$ } Logically equivalent
Contrapositive :- $\neg q \rightarrow \neg p$

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Laws of logic

1) $p \vee p \equiv p$, $p \wedge p \equiv p$ Idempotent
2) $p \vee q = q \vee p$, $p \wedge q = q \wedge p$ Commutative
3) $p \vee (q \vee r) = (p \vee q) \vee r$, $p \wedge (q \wedge r) = (p \wedge q) \wedge r$ } Associative
4) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$, $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ } Distributive
5) $\neg(p \vee q) = \neg p \wedge \neg q$, $\neg(p \wedge q) = \neg p \vee \neg q$ } Demorgan
6) $\overline{\overline{p}} = p$
7) $p \wedge T = p$, $p \wedge F = F$, $p \vee T = T$, $p \vee F = p$

* Tautology $\rightarrow$ Always True
Contradiction $\rightarrow$ Always False
Contingency $\rightarrow$ Sometimes true & sometimes false

Show that it is a tautology.

$\neg p \wedge (p \vee q) \rightarrow q$

$$\equiv ((\neg p \wedge p) \vee (\neg p \wedge q)) \rightarrow q$$

$$\equiv (F \vee (\neg p \wedge q)) \rightarrow q$$

$$\equiv (\neg p \wedge q) \rightarrow q$$

$$\equiv \neg(\neg p \wedge q) \vee q \quad \text{(Elimination of implication)}$$

$$\equiv p \vee (\neg q \vee q)$$

$$\equiv p \vee T$$

$$\equiv T$$

Mathematical Induction

1) $1+2+3+4+\ldots+n = \frac{n(n+1)}{2}$

$p(1)$:
LHS = 1
RHS = $\frac{1(1+1)}{2} = \frac{2}{2} = 1$

$\therefore$ LHS = RHS
$\therefore p(1)$ is true

$p(k)$: $1+2+3+4+\ldots+k = \frac{k(k+1)}{2}$

$p(k+1)$: $1+2+3+4+\ldots+k+(k+1) = \frac{(k+1)(k+2)}{2}$

LHS $= 1+2+3+\ldots+k+(k+1)$

$= \frac{k(k+1)}{2} + (k+1)$

$= (k+1)\left[\frac{k}{2}+1\right]$

$= \frac{(k+1)(k+2)}{2}$

= RHS

2) $1+a+a^2+a^3+\ldots+a^{n-1} = \frac{a^n-1}{a-1}$

$p(1)$: LHS = 1
RHS = $\frac{a^1-1}{a-1} = 1$

$\therefore$ LHS = RHS

$p(k)$: $1+a+a^2+a^3+\ldots+a^{k-1} = \frac{a^k-1}{a-1}$

$p(k+1)$: $1+a+a^2+a^3+\ldots+a^{k-1}+a^k = \frac{a^{k+1}-1}{a-1}$

$p(k+1)$ is true holds

LHS $= 1+a+a^2+a^3+\ldots+a^{k-1}+a^k$

$= \frac{a^k-1}{a-1} + a^k$

$= \frac{a^k-1+a^k(a-1)}{a-1}$

$= \frac{a^k-1+a^{k+1}-a^k}{a-1}$

$= \frac{a^{k+1}-1}{a-1}$

= RHS

3) $1+2^1+2^2+2^3+\dots+2^{n-1} = 2^n-1$

$p(1)$: LHS = 1
RHS = $2^1-1 = 1$
LHS = RHS

$p(k)$: $1+2^1+2^2+\dots+2^{k-1} = 2^k-1$

$p(k+1)$: $1+2^1+2^2+\dots+2^{k-1}+2^k = 2^{k+1}-1$

$p(k+1)$ is true holds

LHS $= 1+2^1+2^2+2^3+\dots+2^{k-1}+2^k$

$= 2^k-1+2^k$

$= 2\times2^k-1$

$= 2^{k+1}-1$

= RHS

4) $1\times2+2\times3+3\times4+\dots+n(n+1) = \frac{n(n+1)(n+2)}{3}$

$\sum_{a=1}^{n} a(a+1)$

$p(1)$: LHS $= 1\times2 = 2$
RHS $= \frac{1(1+1)(1+2)}{3} = \frac{1\times2\times3}{3} = 2$
$\therefore$ LHS = RHS

Suppose $p(k)$ is true

$p(k)$: $1\times2+2\times3+3\times4+\dots+k(k+1) = \frac{k(k+1)(k+2)}{3}$

$\therefore p(k+1)$: $1\times2+2\times3+3\times4+\dots+k(k+1)+(k+1)(k+2)$
$= \frac{(k+1)(k+2)(k+3)}{3}$

LHS $= 1\times2+2\times3+3\times4+\dots+k(k+1)+(k+1)(k+2)$

$= \frac{k(k+1)(k+2)}{3}+(k+1)(k+2)$

$= (k+1)(k+2)\left[\frac{k}{3}+1\right]$

$= \frac{(k+1)(k+2)(k+3)}{3}$

= RHS

$\therefore$ the values of are true for all n

5) $1+2\times2^1+3\times2^2+4\times2^3+5\times2^4+\dots+n2^{n-1} = 2^n(n-1)+1$

$p(1)$: LHS = 1
RHS $= 2^1(1-1)+1 = 1$
$\therefore$ LHS = RHS

$p(k)$:
$1+2\times2^1+3\times2^2+4\times2^3+\dots+k2^{k-1} = 2^k(k-1)+1$

$p(k+1)$
$1+2\times2^1+3\times2^2+4\times2^3+\dots+k2^{k-1}+(k+1)2^k = 2^{k+1}(k)+1$

$LHS = 1 + 2\times2^1 + 3\times2^2 + 4\times2^3 + \dots + k.2^{k-1} + (k+1)2^k$

$= 2^k(k-1) + 1 + (k+1)2^k$

$= 2^k[k+1+k-1] + 1$

$= 2^k(2k) + 1$

$= k.2^{k+1} + 1$

$= RHS$

$\therefore$ the values are true for all n

6) $\sum_{a=1}^{n} a^2 = \frac{1}{6} n(n+1)(2n+1)$

$1 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6} n(n+1)(2n+1)$

p(1) : $LHS = 1$

$RHS = \frac{1}{6}(1)(1+1)(2\times1+1)$

$= \frac{1}{6}(1)(2)(3) = 1$

$\therefore LHS = RHS$.

p(k):

$1 + 2^2 + 3^2 + \dots + k^2 = \frac{1}{6} k(k+1)(2k+1)$

p(k+1):

$1 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{1}{6}(k+1)(k+2)(2(k+1)+1)$

$= \frac{1}{6}(k+1)(k+2)(2k+3)$

$LHS = 1 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2$

$= \frac{1}{6} k(k+1)(2k+1) + (k+1)^2$

$= k+1 \left(\frac{k(2k+1)}{6} + k+1 \right)$

$= \frac{(k+1)}{6} \left[k(2k+1) + 6(k+1) \right]$

$= \frac{(k+1)}{6} \left[2k^2 + k + 6k + 6 \right]$

$= \frac{(k+1)}{6} \left[2k^2 + 7k + 6 \right]$

$= \frac{(k+1)}{6} \left[2k^2 + 3k + 4k + 6 \right]$

$= \frac{(k+1)}{6} \left[k(2k+3) + 2(2k+3) \right]$

$= \frac{(k+1)(k+2)(2k+3)}{6}$

$= RHS$

$\therefore$ the values are true for all n.

7) $3+33+333+\ldots+\underbrace{33\ldots333}_{n\text{-repetitions}} = \frac{10^{n+1}-9n-10}{27}$

$p(1)$: LHS $= 3$

RHS $= \frac{10^{1+1}-9\times1-10}{27} = \frac{100-9-10}{27} = \frac{81}{27} = 3$

It holds for $n=1$.

Let us assume that $p(k)$ holds.

$p(k)$:

$3+33+333+\ldots+\underbrace{333\ldots333}_{k\text{-repetitions}} = \frac{10^{k+1}-9k-10}{27}$

$\therefore$ for $(k+1)$

$p(k+1)$

LHS

$3+33+333+\ldots+\underbrace{333\ldots333}_{k\text{-repetions}} + \underbrace{333\ldots333}_{k+1 \text{ repetition}}$

$= \frac{10^{k+2}-9(k+1)-10}{27}$

LHS $= 3+33+333+\ldots+\underbrace{33\ldots33}_{k\text{-reps}} + \underbrace{333\ldots333}_{(k+1)\text{reps}}$

$= \frac{10^{k+1}-9k-10}{27} + \frac{10^{k+1}-1}{3}$

$= \frac{10^{k+1}-9k-10+9(10^{k+1}-1)}{27}$

$= \frac{10^{k+1}-9k-10+9\times10^{k+1}-9}{27}$

$= \frac{(1+9)10^{k+1}-9(k+1)-10}{27}$

$= \frac{10^{k+2}-9(k+1)-10}{27}$

$=$ RHS.

$p(k+1)$ holds true

$\therefore$ this true for all n.

8) P.T. :- $7^n - 2^n$ is divisible by 5, $n \geq 0$

$p(1)$: LHS $= 7-2 = 5$

Suppose $p(k)$ is true, let m be an integer

$p(k)$: $7^k - 2^k = 5m$ $\quad m \geq 0$

$p(k+1)$:

$7^{k+1} - 2^{k+1}$

$= (7)7^k - (2)2^k$

$= (5+2)7^k - (2).2^k$

$= 5.7^k + 2.7^k - 2.2^k$

$= (5)7^k + 2(7^k - 2^k)$

$= (5)7^k + 2(5m)$

$= 5(7^k + 2m)$

Also (7^k+2m) is integer value.

$\therefore p(k+1)$ holds true.

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9) PT: $4^{n+1} + 5^{2n-1}$ is divisible by 21 $n \geq 0$.

p(1): $4^{1+1} + 5^{2\times1-1}$

$= 4^2 + 5^1$

$= 16 + 5$

$= 21$ which is divisible by 21.

p(k): $4^{k+1} + 5^{2k-1} = 21m$

p(k+1): $4^{k+2} + 5^{2k+1}$

$= (4)(4)^{k+1} + 5^2 . 5^{2k-1}$

$= (25-21)4^{k+1} + (25)5^{2k-1}$

$= (25)4^{k+1} - (21)4^{k+1} + (25)5^{2k-1}$

$= (25)\left(4^{k+1} + 5^{2k-1}\right) - (21)4^{k+1}$

$= (25)(21m) - (21)4^{k+1}$

$= 21\left[25m - 4^{k+1}\right]$

∴ it holds true

10) $11^{n+1} + 12^{2n-1}$ is divisible by 133 $n \geq 0$.

p(1): $11^{1+1} + 12^{2(1)-1}$

$= 121 + 12$

$= 133$

p(k): $11^{k+1} + 12^{2(k)-1} = 133m$

p(k+1): $11^{k+2} + 12^{2(k+1)-1}$

$= (11)(11^{k+1}) + 12^2 . (12^{2k-1})$

$= (144-133)(11^{k+1}) + 144(12^{2k-1})$

$\cancel{= 144(11^{k+1} + 12^{2k-1}) - 133(11^{k+1})}$

$= (144)(11^{k+1}) - (133)(11^{k+1}) + 144(12^{2k-1})$

$= 144(11^{k+1} + 12^{2k-1}) - 133(11^{k+1})$

$= 144(133m) - 133(11^{k+1})$

$= 133\left[144m - 11^{k+1}\right]$

∴ it holds true

11) $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$

$P(1)$: $\overline{A_1} = \overline{A_1}$

$P(2)$: $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$

$P(k)$: $\overline{\bigcup_{i=1}^{k} A_i} = \bigcap_{i=1}^{k} \overline{A_i}$

$P(k+1)$: $\overline{\bigcup_{i=1}^{k+1} A_i} = \bigcap_{i=1}^{k+1} \overline{A_i}$

$LHS = \overline{\bigcup_{i=1}^{k} A_i \cup A_{k+1}}$

$= \cancel{\overline{\bigcap_{i=1}^{k} A_i \cup A_{k+1}}}$

$= \overline{\bigcup_{i=1}^{k} A_i} \cap \overline{A_{k+1}} = \bigcap_{i=1}^{k} \overline{A_i} \cap \overline{A_{k+1}}$

$= \bigcap_{i=1}^{k+1} \overline{A_i} = RHS$

∴ it holds True.

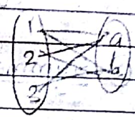
Quantifier

- Universal $[\forall x \; p(x)]$ → all value of x
- Existential $[\exists x \; p(x)]$ → There exist some value of x

Relation :-

$A = \{1, 2, 3\}$ $\quad B = \{a, b\}$

Cartesian product,



$A \times B = \{(x, y) \mid \forall x \in A \wedge \forall y \in B\}$

$= \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

R is a relation from A to B if $R \subset A \times B$

If R is a relation from A to A ⇓ replaced by "on A"

if ~~R×R~~ $R \subset A \times A$

$(a, b) \in R = aRb$

$a|b$ (a divides b) means b is divisible by a

R is relation on Z^+

aRb iff (if and only if) $b = ka$ $(k \in Z)$

$R = \{(1,1),(2,2),(3,3)\}$

Properties

① **Reflexive** :-
aRa, $\forall a \in A$

Eg: $(1,1), (2,2), (3,3)$

Irreflexive
$a \not R a$, $\forall a \in A$

② **Symmetric** :-
whenever $aRb \Rightarrow bRa$ or aRa

$R = \{(1,2), (1,3), (2,1), (3,1)\}$ is symmetric

$R = \{(1,1), (2,2)\}$ is also symmetric

Asymmetric :-
whenever $aRb \Rightarrow b \not R a$

$R = \{(1,2), (1,3), (2,3), (3,3)\}$ is ~~anti~~ asymmetric

Antisymmetric

$aRb \wedge bRa \Rightarrow a = b$
aRb iff $a \leq b$
aRb $a \leq b$
bRa $b \leq a$ $a=b$

In divisibility,
$aRb \Rightarrow a|b$

$b = k_1 a$
$a = k_2 b$
$a = k_2 k_1 a$
$k_1 k_2 = 1$
k_1 & k_2 are integers,
$\therefore k_1 = k_2 = 1$

③ **Transitive**

aRb & $bRc \Rightarrow aRc$

Eg:- i) $a|b$ & $b|c \Rightarrow a|c$
Suppose $b = k_1 a$
$c = k_2 b$
$c = k_2 k_1 a$
$\therefore c = ka$
Hence proved.

ii) aRb $a \leq b$
bRc $b \leq c$

$\therefore aRc$ ie $a \leq c$

* A relation having Reflexive, symmetric & transitive properties is called <u>Equivalence Relation</u>.

- A relation having reflexive, antisymmetric & transitive properties is called <u>Partial Order</u>.

Eg of Equivalence Relation

1) R is a relation on Z^+
aRb iff $a \equiv b \pmod 4$

$a = q_1 4 + r$
$b = q_2 4 + r$

$\therefore (a-b) = (q_1 - q_2)\,4$

$\therefore (a-b)$ is divisible by 4

a) for reflexive,
aRa
$\therefore a-b = a-a = 0$
$\frac{0}{4} = 0$

$\therefore$ it is reflexive

b) for symmetric
aRb $\quad a-b = 4k$
$\quad b-a = -4k = 4(-k)$
$\therefore$ bRa
$(aRb) \Rightarrow (bRa)$
$\therefore$ it is symmetric

c) for transitive
Suppose
aRb & bRc
$\therefore a-b = k_1 4 \quad \text{(i)}$
$b-c = k_2 4 \quad \text{(ii)}$
Adding both eqns.

$a-c = (k_1+k_2)4$
$\therefore (a-c)$ is also divisible by 4

$\therefore$ This is an equivalence relation

2) R is a relation on Z^+
xRy iff $2x+5y$ is divisible by 7.
Show that R is an equivalence relation

a) for reflexive,
suppose xRx

$\therefore 2x+5x = 7x$
$\therefore$ it is divisible by 7.

$\therefore$ it is reflexive.

b) ~~for symmetric~~
~~xRy~~
~~$2x+5y = 7k$~~
~~$5y+2x = 7k$~~
~~$\therefore$ yRx~~
~~$\therefore$ it is symmetric~~

~~$2y+5x$~~
~~$= 2y + 5\left[\frac{7k-5y}{2}\right]$~~
~~$= 2y + \frac{35}{2}k - \frac{15}{2}y$~~
~~$= \left(\frac{4-25}{2}\right)y + \frac{35}{2}k$~~

$$= \cancel{-\frac{21}{2}y + \frac{35}{2}k}$$
$$= \cancel{\frac{7}{2}\left[5k - 3y\right]}$$

ii) for symmetric

xRy

$\therefore 2x + 5y = 7k_1$ ...(i)

Suppose $2y + 5x = k_2$ ...(ii)

(i) + (ii)

$7x + 7y = 7k_1 + k_2$

$7(x+y) = 7k_1 + k_2$

$k_2 = 7(x + y - k_1)$

$\therefore$ k_2 is divisible by 7.

$\therefore$ yRx

$\therefore$ $xRy \Rightarrow yRx$

$\therefore$ it is symmetric

iii) for transitive

xRy & yRz

$2x + 5y = 7k_1$

$2y + 5z = 7k_2$

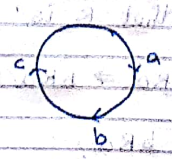
$2x + 7y + 5z = 7(k_1 + k_2)$ (Adding)

$\therefore 2x + 5z = 7(k_1 + k_2 - y)$

$\therefore$ xRz

$\therefore$ it is transitive

R is a relation on A. R is circular if $aRb \wedge bRc \Rightarrow cRa$. Show that R is reflexive & circular iff it is equivalence.



$\rightarrow$) Assume R is reflexive & circular

i) aRb & bRb

bRa ...(circular)

$\therefore$ It is symmetric

ii) aRb & $bRc \Rightarrow cRa$ (circular prop)

But $cRa = aRc$ (symmetric)

$\therefore$ aRb & $bRc \Rightarrow aRc$

$\therefore$ It is transitive

$\therefore$ it is equivalence

$\leftarrow$ Assume R is ~~ref~~ equivalent

$\therefore$ it is reflective

aRb & $bRc \Rightarrow aRc$

But $aRc \Rightarrow cRa$ (By symmetry)

$\therefore$ aRb & $bRc \Rightarrow cRa$

$\therefore$ circular

→ If aRb & $aRc \Rightarrow bRc$ and R is reflexive Show that R is an equivalence Relation.

It is given that R is reflexive.

aRb and $aRa \Rightarrow bRa$.

$\therefore aRb = bRa$.

$\therefore$ it is symmetric.

bRa & $bRc \Rightarrow aRc$.

→ Suppose R is a relation on $Z \times Z$
$(a,b) R (c,d)$ iff $a+b = c+d$

① $(a,b) R (a,b)$ $\therefore a+b = a+b$
$\therefore$ the given relation is reflexive

② $(a,b) R (c,d)$
~~By reflexive~~
$\therefore a+b = c+d$
$\therefore c+d = a+b$
$\therefore (c,d) R (a,b)$
$\therefore$ the relation is symmetric.

③ $(a,b) R (c,d)$ & $(c,d) R (e,f)$

$a+b = c+d$ & $c+d = e+f$.
$\therefore a+b = e+f$
$\therefore (a,b) R (e,f)$.

→ R is a relation on $Z \times Z$
$(a,b) R (c,d)$ iff $a+d = b+c$.

i) $(a,b) R (a,b)$
$\therefore a+b = a+b$
$\therefore$ it is reflexive.

ii) $(a,b) R (c,d)$

$a+d = b+c$
$c+b = d+a$
$\therefore (c,d) R (a,b)$

$\therefore$ it is ~~re~~ symmetric

iii) $(a,b) R (c,d)$ & $(c,d) R (e,f)$.

$a-b = c-d$.
$c-d = e-f$.
$\therefore a-b = e-f$.
$\therefore (a,b) R (e,f)$.

$\therefore (a,b) R (c,d)$ & $(c,d) R (e,f) = (a,b) R (e,f)$

→ R is a relation on $Z \times Z$
$(a,b) R (c,d)$ iff $ad = bc$
PT it is equivalence

① $(a,b) R (a,b)$
$\therefore ab = ba$
$\therefore$ it is reflexive

② $(a,b) R (c,d) \Rightarrow (c,d) R (a,b)$.

We know $\frac{a}{b} = \frac{c}{d}$

Also $\frac{c}{d} = \frac{a}{b}$

$\therefore \cancel{b}cb = ad$

$\therefore (c,d) R (a,b)$

$\therefore$ it is symmetric.

③ $(a,b) R (c,d)$ & $(c,d) R (e,f)$

$\therefore \frac{a}{b} = \frac{c}{d}$

Also $\frac{c}{d} = \frac{e}{f}$

$\therefore \frac{a}{b} = \frac{e}{f}$

$\therefore af = be$

$\therefore (a,b) R (e,f)$

$\therefore$ the given relation is transitive.

$\therefore$ it is an equivalence relation.

→ R is a relation on $Z \times Z$

$(a,b) R (c,d)$ iff $ad(b+c) = bc(a+d)$.

PT it is equivalence.

① $(a,b) R (a,b)$.

$ab(b+a) = ba(a+b)$

$\therefore$ it is reflexive.

② $(a,b) R (c,d)$

$ad(b+c) = bc(a+d)$

$cb(d+a) = da(c+b)$

$\therefore (c,d) R (a,b)$

$\therefore$ it is symmetric.

③ $(a,b) R (c,d)$ & $(c,d) R (e,f)$

$\therefore ad(b+c) = bc(a+d)$ .. ①

$cf(d+e) = de(c+f)$ .. ②

Divide eqn ① by abcd

$\therefore \frac{b+c}{bc} = \frac{a+d}{ad}$

$\therefore \frac{1}{c} + \frac{1}{b} = \frac{1}{d} + \frac{1}{a}$ .... ③

Divide eq ② by cdef

$\therefore \frac{d+e}{de} = \frac{c+f}{cf}$

$\therefore \frac{1}{e} + \frac{1}{d} = \frac{1}{f} + \frac{1}{c}$ .... ④

Adding ③ & ④

$$\cancel{\frac{1}{c}}+\frac{1}{b}+\frac{1}{e}+\cancel{\frac{1}{d}} = \cancel{\frac{1}{d}}+\frac{1}{a}+\frac{1}{f}+\cancel{\frac{1}{c}}$$

$$\frac{1}{b}+\frac{1}{e} = \frac{1}{a}+\frac{1}{f}$$

$$\therefore \frac{e+b}{eb} = \frac{f+a}{af}$$

$$af(e+b) = eb(f+a)$$

$$\therefore af(b+e) = eb(a+f)$$

$\therefore (a,b)R(e,f)$

$\therefore$ it is transitive

Equivalence classes

R is equivalence relation on A

$[a] = \{b \mid aRb\}$

Eg -1) $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$

$[1] = \{1,2\}$ $[3] = \{3,4\}$

$[2] = \{1,2\}$ $[4] = \{3,4\}$

2) R is equivalence relation on A

$A = \{1,2,3,4,5\}$

$R = \{(1,1),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4)\}$

$[1] = \{1\}$

$[2] = \{2,3,4\}$

$[3] = \{2,3,4\}$ × (No need to write

$[4] = \{2,3,4\}$ × these two as they are sam

$A/R = \{[1],[2]\}$

$= \{\{1\},\{2,3,4\}\}$

2) R is a relation on Z

aRb iff $a \equiv b \pmod 5$

$a \div 5$ $r=1$
$b \div 5$ $r=1$
$\therefore a \equiv b \pmod 5$

$a = 5k_1 + r$

$b = 5k_2 + r$

$\therefore a-b = 5(k_1-k_2)$

$[0] = \{\ldots,-10,-5,0,5,10,15,\ldots\}$

$[1] = \{\ldots,-14,-9,-4,1,6,11,\ldots\}$

$[2] = \{\ldots,-13,-8,-3,2,7,12,\ldots\}$

$[3] = \{\ldots,-12,-7,-2,3,8,13,\ldots\}$

$[4] = \{ \ldots -11, -6, -1, 4, 9, 14, \ldots \}$

$Z/R = \{[0], [1], [2], [3], [4]\}$

3) $A = \{1, 2, 3, 4\}$
R is a relation on $A \times A$
$(a,b) R (c,d)$ if $a+b = c+d$.
Find $(A \times A)/R$

$(a,b) R (c,d)$

$\therefore (a,b) R (a,b)$

$\because a+b = a+b$
$\therefore$ it is reflexive

Also $(a,b) R (c,d)$

$\therefore a+b = c+d$
$\therefore c+d = a+b$

$\therefore (c,d) R (a,b)$
$\therefore$ it is symmetric

Also T.P. $(a,b) R (c,d)$ & $(c,d) R (e,f)$

$a+b = c+d$ ... ①
$c+d = e+f$ ... ②
Adding ① & ②

$a+b+c+d = c+d+e+f$
$a+b = e+f$
$\therefore (a,b) R (e,f)$
$\therefore$ it is transitive

$\therefore$ it is an equivalence relation

$[(1,1)] = \{(1,1)\}$

$[(1,2)] = \{(1,2), (2,1)\}$

$[(1,3)] = \{(1,3), (2,2), (3,1)\}$

$[(1,4)] = \{(1,4), (2,3), (3,2), (4,1)\}$

$[(2,4)] = \{(2,4), (3,3), (4,2)\}$

$[(3,4)] = \{(3,4), (4,3)\}$

$[(4,4)] = \{(4,4)\}$

$(A \times A)/R = \{[(1,1)], [(1,2)], [(1,3)], [(1,4)], [(2,4)], [(3,4)], [(4,4)]\}$

4) $S = \{1, 2, 3, 4, \ldots, 14, 15\}$
Let $A = S \times S$ and define a relation R on A: $(a,b) R (c,d)$ iff $ad = bc$
Find equivalence class of $(2,3)$. $\frac{a}{b} = \frac{c}{d}$

It is given that, it is equivalence relation

~~$[(2,3)] = \{(2,6), (2,3), (3,2), (6,3)\}$~~

$[(2,3)] = \{(2,3), (4,6), (6,9), (8,12), (10,15)\}$

Operation on a relation.

- Inverse
- Complement
- Composition

1) Inverse:-

Whenever aRb , $bR^{-1}a$

$A = \{1,2,3\}$

$R = \{(1,1),(2,1),(2,2),(3,2)\}$

$R^{-1} = \{(1,1),(1,2),(2,2),(2,3)\}$

2) Complement:-

$\bar{R} = A \times A - R$

$= \{(1,2),(1,3),(2,3),(3,1),(3,3)\}$

$\therefore (a,b) \notin R \Rightarrow (a,b) \in \bar{R}$

3) Composition:-

$A = \{1,2,3,4\}$

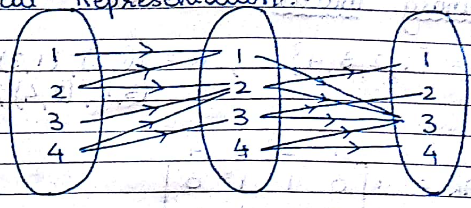
$R = \{(1,1),(2,1),(2,2),(3,2),(4,2),(4,3)\}$

$S = \{(1,3),(2,1),(2,3),(3,2),(3,3),(4,3),(4,4)\}$

$R \circ S = \{(a,c) \mid aRb \ \& \ bSc\}$

$= \{(1,3),(2,3),(2,1),(3,1),(3,3),(4,1),(4,3),(4,2)\}$

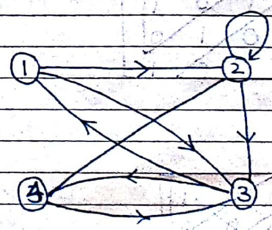
Graphical Representation.



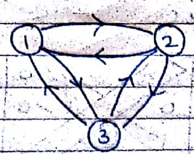
Representation of a relation:

$A = \{1,2,3,4\}$

$R = \{(1,2),(1,3),(2,2),(2,3),(3,1),(3,4),(4,2),(4,3),(4,4)\}$



Symmetric representation



→ Binary Matrix

$A = \{1, 2, 3, 4\}$; $R = \{(1,2), (1,3), (2,2), (2,3), (3,1), (3,4), (4,2), (4,3), (4,4)\}$

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 0 & 1 & 1 & 1 \end{array}$$

Reflexive :-

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Symmetric

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$0 \to 0$ and $1 \to 1$

Assym ~~A~~nti-symmetric

$$\begin{bmatrix} & 1 & 0 & 0 \\ 0 & & 1 & 0 \\ 1 & 0 & & 0 \\ 1 & 1 & 1 & \end{bmatrix}$$

No restriction on diagonal elements. upper and lower triangular elements are anti-symmetric ($0 \to 1$ & $1 \to 0$)

$M_{R \cup S} = M_R \vee M_S$

$M_{R \cap S} = M_R \wedge M_S$

$M_{R \circ S} = M_R \odot M_S \Rightarrow$ (matrix multiplication)

↓

composition

Closure of a Relation

$A = \{1, 2, 3\}$

R lacks some elements

$R = \{(1,1), (1,2), (2,1), (2,3), (3,2), (3,3)\}$.

$(2,2)$ is not there, ∴ no reflexivity

Δ_A = Identity Relation

$= \{a \mid a, a \in A\}$

$R^* = R \cup \Delta_A$

↓

reflexive closure, it is also symmetric closure.

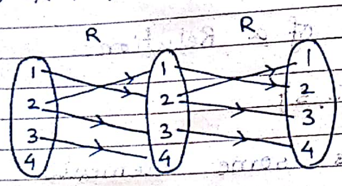
Whenever $aRb \Rightarrow bR^{-1}a$

$R^* = R \cup R^{-1}$

$R^* = R \cup R^2 \cup R^3 \cup R^4 \ldots \cup R^n$

$R^2 = R \circ R$
$R^3 = R^2 \circ R$
$R^4 = R^3 \circ R$

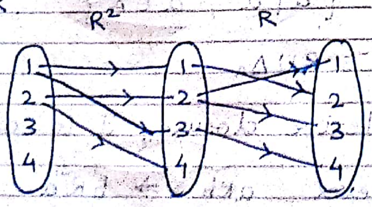
$R = \{(1,2),(2,1),(2,3),(3,4)\}$



$R^2 = R \circ R$

$R^2 = \{(1,1),(1,3),(2,2),(2,4)\}$

$R^3 = R^2 \circ R$



$R^3 = \{(1,2),(2,1),(2,3),(1,4)\}$

Warshall's Algorithm

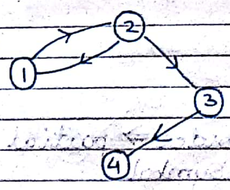
$M_{R^2} = M_{R \circ R} = M_R \odot M_R$
$M_{R^3} = M_{R^2} \odot M_R$
$M_{R^4} = M_{R^3} \odot M_R$

$M_{R^*} = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4}$

⊛ According to warshall's Algorithm,

$M_R = W_0$

Eg:-



$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = W_0$

$W_1 = \begin{array}{cccc} a_{11} & a_{12} & a_{13} & \\ \end{array}$
$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$a_{21} = 1$
$a_{12} = 1$
$\cancel{a_{23}}$
$\therefore b_{22} = 1$

$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$a_{12} = 1$
$a_{21} = 1$
$a_{22} = 1$
$a_{23} = 1$

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$a_{13} = 1$
$a_{23} = 1$
$a_{34} = 1$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$a_{14} = 1$
$a_{24} = 1$
$a_{34} = 1$

Partial Order Relation

Reflexive
Anti-symmetric
Transitive

Set & partial order $\Rightarrow$ partial order set
$\langle A, R \rangle$ $\leftarrow$ Symbol

eg ① R is a relation on Z
R is defined
① ∴ aRb iff $a \leq b$
$a \leq a$ ∴ aRa
∴ it is reflexive
② aRb & bRa
$a \leq b$ & $b \leq a$ $\Rightarrow$ $a = b$
∴ it is anti-symmetric
③ aRb & bRc

$a \leq b$ & $b \leq c$
∴ $a \leq c$
∴ it is transitive.

② $A = \{2, 3, 4, 6\}$
∴ relation is $a|b$
① $a|a$ aRa
it is reflexive
② $a|b$ & $b|a$ $\Rightarrow$ $a = b$

$b = k_1 a$
$a = k_2 b$

$a = k_2 k_1 a$
∴ $k_1 k_2 = 1$
∴ $k_1 = k_2 = 1$
∴ $a = b$

whenever ARB $A \subseteq B$
① $A \subseteq A$
② $A \subseteq B$ & $B \subseteq A$ $\Rightarrow$ $A = B$
③ $A \subseteq B$ & $B \subseteq C$
∴ $A \subseteq C$

eg R is a relation on Z, aRb iff $a = b^r$, $r \in Z$
∴ $a = a^1$ (∵ $r = 1$)
aRa
∴ it is reflexive
aRb & bRa
$a = b^{r_1}$ $b = a^{r_2}$
∴ $a = (a^{r_2})^{r_1}$
∴ $a = a^{r_1 . r_2}$

Case 1	Case 2
$r_1 r_2 = 1$	$a = 1$

∴ it is antisymmetric

$\therefore aRb \;\&\; bRc$

$\therefore a = b^{r_1} \quad \& \quad b = c^{r_2}$

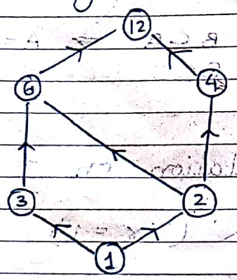
$\therefore a = c^{r_2 \cdot r_1}$

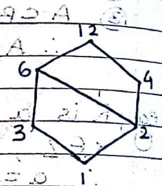
$\therefore aRc$

$\therefore$ it is transitive

eg $A = \{1, 2, 3, 4, 6, 12\}$

(Divisibility diagram)



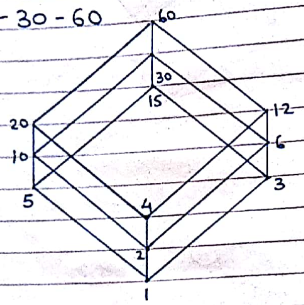


Hasse diagram

Eg:- $D_{60} = \{1, 2, 3, 4\}$

1 - 2 - 4
3 - 6 - 12
5 - 10 - 20
15 - 30 - 60

1	2	4
3	6	12
5	10	20
15	30	60

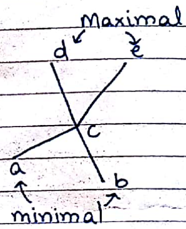


Special Elements

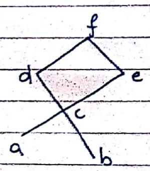
- Extreme elements:-

① Minimal
② Maximal
③ Greatest:

$x \leq g \quad \forall x \in A$



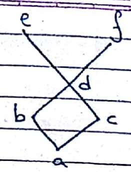
Eg of greatest



$a \leq f$
$b \leq f$
$c \leq f$
$d \leq f$
$e \leq f$
$f \leq f$

$\therefore$ f is greatest element.

④ Least element

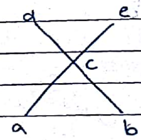


$l \leq x \quad \forall x \in A$

Eg:- $a \leq a$
$a \leq b$
$a \leq c$
$a \leq d$
$a \leq e$
$a \leq f$

$\therefore$ a is the least element.

Eg:-



there there is no least or greatest element as a & b cannot be compared & d & e cannot be compared.

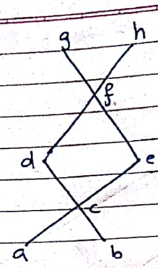
- There can be only 1 greatest or least element.

Proof of above statement:-
Let us assume g_1 & g_2 both are greatest elements.
When g_1 is greater
$g_2 \leq g_1$
When g_2 is greater
$g_1 \leq g_2$

$\therefore$ $g_1 = g_2$ using anti-symmetric prop.

$\therefore$ there can be only one greatest element

Similarly, there can be only one least element.



$A = \{a, b, c, d, e, f, g, h\}$, $B = \{c, d, e\}$

$b \in B$, lower bound of b

$x \leq b$, $x \in A$

Contition for
Greatest lower bound

$= x \leq g$, x is a lower bound

For upper bound ,
$u \in A$

if $x \leq u$ $\forall x \in B$

For least upper bound,

$l \leq x$, x is an upper bound

Lattice

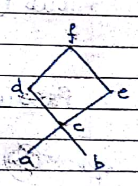
$\langle L, \leq \rangle$ that satisfies that every pair of elements of L is having least upper bound & greatest lower bound.

$\langle L, \wedge, \vee \rangle$

$glb = a \wedge b$ $\wedge \rightarrow$ meet

$lub = a \vee b$ $\vee \rightarrow$ join

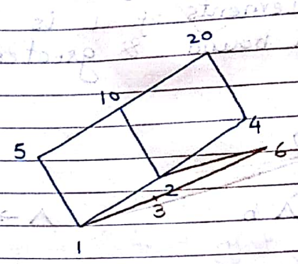
$\wedge$	a	b	c	d	e	f
a	a	–	a	a	a	a
b	–	b	b	b	b	b
c	a	b	c	c	c	c
d	a	b	c	d	–	d
e	a	b	c	–	e	e
f	a	b	c	d	e	f



$\vee$	a	b	c	d	e	f
a	a	c	c	d	e	f
b	c	b	c	d	e	f
c	c	c	c	d	e	f
d	d	d	d	d	f	f
e	e	e	e	f	e	f
f	f	f	f	f	f	f

$D_{20} = \{1, 2, 4, 5, 10, 20\}$

$A = \{1, 2, 3, 4, 5, 6, 10, 20\}$



Lub $(6, 10)$ = LCM $(6, 10)$ = 30
Not there

Product partial order

$\langle A, \leq_1 \rangle$ $\langle B, \leq_2 \rangle$

$\langle A \times B, \leq \rangle$: $(a_1, b_1) \leq (a_2, b_2)$

iff $a_1 \leq a_2$ & $b_1 \leq_2 b_2$

① $(a, b) \leq (a, b)$

$a \leq_1 a$ & $b \leq_2 b$

∴ Reflexive

② $(a_1, b_1) \leq (a_2, b_2)$

$(a_2, b_2) \leq (a_1, b_1)$

∴ ~~antisymmetric~~

③ $(a, b) \leq (c, d)$ & $(c, d) \leq (e, f)$.

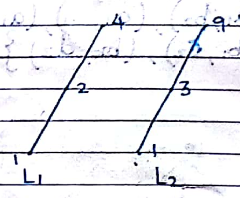
$a \leq_1 c$ & $b \leq_2 d$ & $c \leq_1 e$ & $d \leq_2 f$

∴ $a \leq_1 e$ & $b \leq_2 f$.

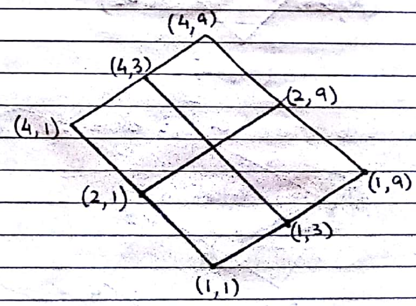
∴ $(a, b) \leq (e, f)$.

∴ transitive

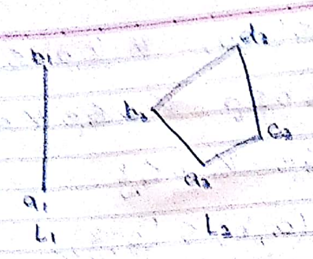
⇒



$L = L_1 \times L_2 = \{(1,1), (1,3), (1,9), (2,1), (2,3), (2,9), (4,1), (4,3), (4,9)\}$

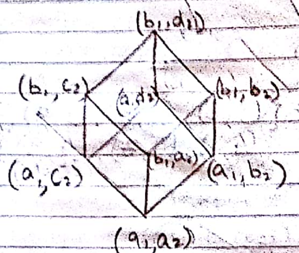


$L_1 \times L_2$

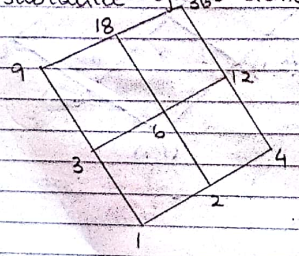


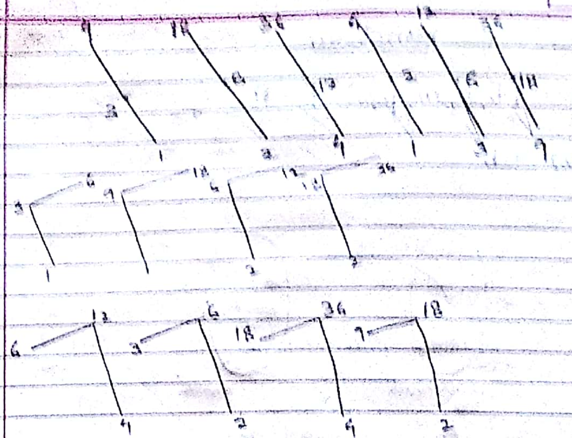
$L = L_1 \times L_2$

$= \{(a_1, a_2), (a_1, b_2), (a_1, c_2), (a_1, d_2), (b_1, a_2), (b_1, b_2), (b_1, c_2), (b_1, d_2)\}$



Construct sublattice of 36 3 elements





Types of Lattice
- Bounded

Properties of lattice

$a \vee b = b \vee a$ $\quad a \wedge b = b \wedge a$

$a \vee (a \wedge b) = a$ $\quad a \wedge (a \vee b) = a$ Absorption

$a \vee (b \vee c) = (a \vee b) \vee c$ $\quad a \wedge (b \wedge c) = (a \wedge b) \wedge c$ Associative

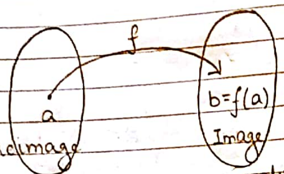
$a \vee a = a$ $\quad a \wedge a = a$ Idempotent

$\left.\begin{array}{l} a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \end{array}\right\}$ Distributive

Function

$[a] = \{b, c, m, p, \dots\}$

$f(a) = b$



preimage — argument — A — domain

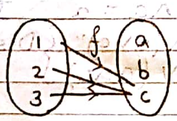
value — B — codomain (f)

Types of functions

- One-to-one / Injective function

$a \neq b \qquad f(a) \neq f(b)$

$f(a) = f(b) \Rightarrow a = b$



f is a constant function

range $(f) = \{c\}$

Onto / Surjective.

one-to-one & onto

$f: A \longrightarrow B$

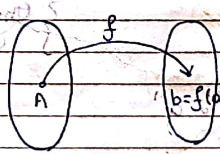
$1_A(x) = x$ [Identity Relation]

A

$f(x) = x$

$f \equiv 1_A$

$\Rightarrow$ If the image of 2 functions are equal then their premages are also equal.



$f: A \rightarrow B$

$I_A(a) = a$

$I_B(b) = b$

$f \circ f^{-1} = I_B \qquad f^{-1} \circ f = I_A$

* $(f\circ f^{-1})(b)$

$= f(f^{-1}(b))$

$= f(a)$

$= b$

$\therefore (f\circ f^{-1})(b) = b = I_B(b)$

$\therefore f\circ f^{-1} = I_B$

* $(f^{-1}\circ f)(a)$

$= f^{-1}(f(a))$

$= f^{-1}(b)$

$= a$

$= I_A(a)$

$\therefore f^{-1}\circ f = I_A\cancel{(a)}$

* $f : A \to B$
  $g : B \to C$ } One-to-one correspondence.

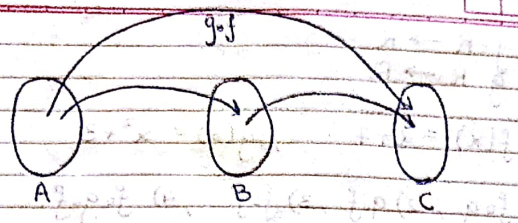
To Show: $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$

$(g\circ f)(x_1) = (g\circ f)(x_2)$

$g(f(x_1)) = g(f(x_2))$

$\therefore f(x_1) = f(x_2)$

$\therefore x_1 = x_2$



$(g\circ f)(a)$

$= g(f(a))$

$= g(b)$

$= c$

ST: $(g\circ f)^{-1}(c) = a$

$\therefore (f^{-1}\circ g^{-1})(c)$

$= f^{-1}(g^{-1}(c))$

$= f^{-1}(b)$

$= a$

$\therefore (g\circ f)^{-1}(c) = (f^{-1}\circ g^{-1})(c)$

$\therefore \boxed{(g\circ f)^{-1} = f^{-1}\circ g^{-1}}$

$f: R \to R$
$g: R \to R$

$f(x) = x+7$, $g(x) = x^2+3$

1) $f \circ g$, 2) $g \circ f$, 3) $f \circ f$, 4) $f \circ g \circ f$

1) $(f \circ g)(x) = f(g(x))$
$= f(x^2+3)$
$= x^2+3+7$
$= x^2+10$

2) $(g \circ f)(x) = g(f(x))$
$= g(x+7)$
$= (x+7)^2+3$
$= x^2+14x+52$

3) $(f \circ f)(x) = f(f(x))$
$= f(x+7)$
$= x+7+7$
$= x+14$

4) $(f \circ g \circ f)(x) = f(g(f(x)))$
$= f(g(x+7))$
$= f((x+7)^2+3)$
$= (x+7)^2+3+7$
$= x^2+14x+49+10$
$= x^2+14x+59$

⇒ Recurrence relation :-

$a_n = 5 \times 2^n = 5 \times 2 \times 2^{n-1} = 2 \times (5 \times 2^{n-1})$
$= 2a_{n-1}$

$\boxed{\therefore a_n = 2a_{n-1}}$

$\therefore a_n - 2a_{n-1} = 0$

$a_n = 3 \times 2^n = 3 \times 2 \times 2^{n-1} = 2 \times (3 \times 2^{n-1}) = 2a_{n-1}$

$\therefore \boxed{a_n = 2a_{n-1}}$

To prove :- $a_n = 2a_{n-1}$

Let $a_n = b r^n$
$= b \times r \times r^{n-1}$
$= (b \times r^{n-1}) \times r$
$a_n = r a_{n-1}$
But $a_n = 2a_{n-1}$
$\therefore r = 2$

→ Solve

$a_r = 5a_{r-1} - 6a_{r-2}$ $\quad a_0 = 0, \; a_1 = 1$

$a_r = bx^r$

$\therefore bx^r = 5bx^{r-1} - 6bx^{r-2}$

Dividing by bx^{r-2}

$\therefore x^2 = 5x - 6$

$\therefore x^2 - 5x + 6 = 0$

$x^2 - 2x - 3x + 6 = 0$

$x(x-2) - 3(x-2) = 0$

$x = 2, 3.$

$\therefore a_r = b_1 2^r + b_2 3^r$

$\therefore a_0 = 0$

$\therefore 0 = b_1(1) + b_2(1)$

$b_1 + b_2 = 0 \quad \text{①}$

$a_1 = b_1(2) + b_2(3)$

$\therefore 1 = 2b_1 + 3b_2 \quad \text{②}$

① × 2

$\therefore 2b_1 + 2b_2 = 0 \quad \text{③}$

② − ③

$\therefore 2b_1 + 3b_2 = 1$

$- \; 2b_1 + 2b_2 = 0$

$b_2 = 1$

$\therefore \boxed{b_1 = -1}$

$\therefore a_r = -2^r + 3^r = 3^r - 2^r$

→ Solve

$a_r = 10a_{r-1} - 25a_{r-2}$; $a_0 = 1$, $a_1 = 4$

$a_r = bx^r$

$\therefore a_r = 10a_{r-1} - 25a_{r-2}$

$bx^r = 10bx^{r-1} - 25bx^{r-2}$

Divide by bx^{r-2}

$\therefore x^2 = 10x - 25$

$\therefore x^2 - 10x + 25 = 0$

$x^2 - 5x - 5x + 25 = 0$

$x(x-5) - 5(x-5) = 0$

$x = 5, 5.$

$a_r = b_1 5^r + b_2 5^r$

$= (b_1 + b_2 r) 5^r$

$\cancel{a_r = b 5^r}$

$\therefore a_r = (b_1 + b_2 r) 5^r$

$\therefore$ Put $r = 0$

$\therefore a_0 = (b_1 + 0) 5^0 = b_1 5^0 = b_1$

$\boxed{\therefore b_1 = 1}$

Put $r = 1$

$\therefore a_1 = (b_1 + b_2) 5$

$4 = (1 + b_2) 5$

$\therefore b_2 = \frac{4}{5} - 1 = -\frac{1}{5}$

$\therefore a_r = \left(1 - \frac{r}{5}\right) 5^r$

$= 5^r - r\,5^{r-1}$

$\boxed{a_r = 5^{r-1}(5 - r)}$

Solve

$a_r = 6a_{r-1} - 11a_{r-2} + 6a_{r-3}$, $a_0 = 2$, $a_1 = 5$, $a_2 = 15$

$x^3 = 6x^2 - 11x + 6$

$\therefore x^3 - 6x^2 + 11x - 6 = 0$

$x = 1, 2, 3$

$\therefore a_r = b_1 1^r + b_2 2^r + b_3 3^r$

$a_0 = b_1(1) + b_2(1) + b_3 = 2$

$b_1 + b_2 + b_3 = 2$ ... ①

$a_1 = b_1 + 2b_2 + 3b_3 = 5$

$b_1 + 2b_2 + 3b_3 = 5$ ... ②

$a_2 = b_1 + 4b_2 + 9b_3 = 15$ ... ③

$b_1 = 1$, $b_2 = -1$, $b_3 = 2$.

$\therefore a_r = 1(1)^r - 1(2)^r + 2(3)^r$

$c_n = 3c_{n-1} - 2c_{n-2}$, $c_1 = 5$, $c_2 = 3$

$x^2 = 3x - 2$

$\therefore x^2 - 3x + 2 = 0$

$x^2 - 2x - x + 2 = 0$

$x(x-2) - 1(x-2) = 0$

$(x-2)(x-1) = 0$

$x = 2$, $x = 1$

$\therefore c_n = b_1(1^n) + b_2 2^n$

$= b_1 + b_2 2^n$

$c_1 = b_1 + b_2 2^1$

$5 = b_1 + 2b_2$

$\therefore b_1 + 2b_2 = 5$ ... ①

$c_2 = b_1 + b_2 4$ ... ②

$b_1 + 4b_2 = 3$ ... ②

① − ②

$-2b_2 = 2$

$\boxed{b_2 = -1}$

$\therefore -1 + 2b_2 = 5$

$2b_2 = 6$

$\boxed{b_2 = 3}$

$\therefore c_n = -1 + 3(2)^n = 3(2)^n - 1$

→ Solve $a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$, $a_0 = 5$, $a_1 = -9$, $a_2 = 15$

$x^3 + 3x^2 + 3x + 1 = 0$

$x = -1, -1, -1$

∴ $a_n = (b_1 + b_2 n + b_3 n^2)(-1)^n$

∴ $a_0 = (b_1 + b_2 n + b_3 n^2)(-1)^0$

$5 = b_1 + b_2 n + b_3 n^2$

$5 = b_1 + 0 + 0$

$\boxed{b_1 = 5}$ ...... (1)

Put $n = 1$

∴ $a_1 = (b_1 + b_2 + b_3)(-1)^1 = -9$

$\boxed{b_1 + b_2 + b_3 = 9}$ ...... (2)

Put $n = 2$

∴ $a_2 = [b_1 + b_2(2) + b_3(4)](-1)^2$

∴ $15 = b_1 + 2b_2 + 4b_3$ ...... (3)

Solving (1), (2), (3)

$b_1 = 5$, $b_2 = 3$, $b_3 = 1$

∴ $a_n = (5 + 3n + n^2)(-1)^n$

→ $a_n - 4a_{n-1} + 4a_{n-2} = 0$, $a_0 = a_1 = 1$.

$r^2 - 4r + 4 = 0$

∴ $(r-2)^2 = 0$

$r = 2, 2$

∴ $a_n = (b_1 + b_2 n)2^n$

$a_0 = b_1 = 1$

$a_1 = (b_1 + b_2)2 = 1$

∴ $b_1 + b_2 = \frac{1}{2}$

∴ $1 + b_2 = \frac{1}{2}$

∴ $b_2 = \frac{1}{2} - 1 = -\frac{1}{2}$

∴ $a_n = \left(1 - \frac{n}{2}\right)2^n$

→ $a_n = 7a_{n-2} + 6a_{n-3}$, with $a_0 = 9$, $a_1 = 10$, $a_2 = 32$.

∴ $a_n - 7a_{n-2} - 6a_{n-3} = 0$

$r^3 - 7r - 6 = 0$

$(r+1)(r^2 - r - 6) = 0$

$(r+1)(r+2)(r-3) = 0$

$r = -1, -2, 3$

$a_n = b_1(-1)^n + b_2(-2)^n + b_3(3)^n$

Put $n=0$

$\therefore a_0 = b_1 + b_2 + b_3 = 9$ ...... ①

Put $n=1$

$a_1 = -b_1 - 2b_2 + 3b_3 = 10$ ②

Put $n=2$

$a_2 = b_1 + 4b_2 + 9b_3$ ...... ③

$\therefore b_1 =$
$b_2 =$
$b_3 =$

→ If $a_n + ca_{n-1} + \ldots = f(n)$

When $f(n) = 0$, it is called homogeneous recurrence relation.

$f(n) \neq 0$ ...... non-homogeneous

$a_n + a_{n-1} - 2a_{n-2} = 6$; $a_0 = 1$, $a_1 = 6$

Homogeneous part

$a_n + a_{n-1} - 2a_{n-2} = 0$

$r^2 + r - 2 = 0$
$r^2 + 2r - r - 2 = 0$
$r(r+2) - 1(r+2) = 0$
$(r+2)(r-1) = 0$
$r = -2, 1$

$\therefore a_n^{(h)} = b_1(1)^n + b_2(-2)^n$

$a_n^{(p)} = cn(1)^n = cn$

$cn + c(n-1) - 2c(n-2) = 6$
$cn + cn - c - 2cn + 4c = 6$
$3c = 6$
$c = 2$

$\therefore a_n^{p} = 2n$

$\therefore a_n = a_n^{(h)} + a_n^{(p)}$
$= b_1 + b_2(-2)^n + 2n$

Put $n=1$

$\therefore a_1 = b_1 - 2b_2 + 2 = 6$
$\therefore b_1 - 2b_2 = 4$ ... ①

Put $n=0$

$a_0 = b_1 + b_2 = 1$
$\therefore b_1 + b_2 = 1$ ②

Solving ① & ②
$b_1 = 2$, $b_2 = -1$

$a_n = 2 - (-2)^n + 2n$

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$a_n + 5a_{n-1} + 6a_{n-2} = 21 \times 4^n$, $a_0 = 11$, $a_1 = 24$

$r^2 + 5r + 6 = 0$ (Homogeneous part)

$r^2 + 2r + 3r + 6 = 0$
$r(r+2) + 3(r+2) = 0$
$r = -2, -3$

$\therefore a_n^{(h)} = b_1(-2)^n + b_2(-3)^n$

$a_n^{(p)} = c.4^n$

$\because f(n) = 21 \times 4^n$

$\therefore c4^n + 5c\,4^{n-1} + 6c4^{n-2} = 4^n \times 21$

$\div$ by 4^{n-2}

$\therefore c(4)^2 + 5c(4) + 6c = (4)^2 \times 21$

$\therefore 16c + 20c + 6c = 16 \times 21$
$\therefore 42c = 16 \times 21$

$\therefore c = \frac{16 \times \cancel{21}}{\cancel{42}_2} = 8$

$\therefore \boxed{c = 8}$

$a_n = a_n^{(h)} + a_n^{(p)}$

$= b_1(-2)^n + b_2(-3)^n + 8(4)^n$

Put $n = 0$

$a_0 = b_1 + b_2 + 8 = 11$
$b_1 + b_2 = 3$ ... (1)

Put $n = 1$

$a_1 = -2b_1 - 3b_2 + 8(4)^1 = 24$
$-2b_1 - 3b_2 = 24 - 32 = -8$
$2b_1 + 3b_2 = 8$ (2)

(1) $\times 2$

$2b_1 + 2b_2 = 6$ (3)

(2) - (3)

$2b_1 + 3b_2 = 8$
$-\ 2b_1 + 2b_2 = 6$

$\boxed{b_2 = 2}$

$\boxed{\therefore b_1 = 1}$

$\boxed{\therefore a_n = (1)(-2)^n + 2(-3)^n + 8(4)^n}$

$\rightarrow a_n = 2a_{n-1} + 3.2^n$

$a_n - 2a_{n-1} = 3.2^n$

$\therefore x - 2 = 0$ (Homogeneous)

$\therefore x = 2$

$a_n^{(h)} = b_1(2)^n$

$a_n^{(p)} = cn(2)^n$

$\therefore f(n) = 3 \times (2)^n$

$\therefore cn(2)^n - 2c(n-1)(2)^{n-1} = 3.(2)^n$

$\therefore 2cn = 2cn - 2c + 6$

$2c = 6$

$c = 3.$

$\therefore a_n = a_n^{(h)} + a_n^{(p)}$

$= b(2)^n + 3n2^n$

$\therefore a_n = (b+3n)2^n$

$\Rightarrow a_n - 7a_{n-1} + 10a_{n-2} = 8n+6, a_0 = 1, a_1 = 5$

$r^2 - 7r + 10 = 0$ ... (Homogeneous eqn)

$\therefore r^2 - 5r - 2r + 10 = 0$

$r(r-5) - 2(r-5) = 0$

$(r-5)(r-2) = 0$

$r = 5, 2.$

$a_n^{(h)} = b_1(5)^n + b_2(2)^n$

$a_n^{(p)} = c_0 + c_1 n$

$\therefore (c_0 + c_1 n) - 7(c_0 + c_1(n-1)) + 10(c_0 + c_1(n-2)) = 8n+6$

$4c_0 + 4c_1 n - 13c_1 = 8n + 6$

$\therefore (4c_0 - 13c_1) + 4c_1 n = 8n + 6$

$\therefore 4c_0 - 13c_1 = 6$ ... 1

$4c_1 = 8$

$c_1 = 2$

$\therefore 4c_0 - 13(2) = 6$

$4c_0 = 6 + 26 = 32$

$c_0 = \frac{32}{4} = 8.$

$\boxed{c_0 = 8}$

$\therefore a_n^{(p)} = 8 + 2n$

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = b_1(5)^n + b_2(2)^n + 8 + 2n$

Put $n = 0$.

$\therefore a_0 = b_1(1) + b_2(1) + 8 + 2(0)$

$1 = b_1 + b_2 + 8$

$\therefore b_1 + b_2 = -7$ ... 1

Put $n = 1$

$\therefore a_1 = b_1(5) + 2(b_2) + 8 + 2$

$5 = 5b_1 + 2b_2 + 10$

$5b_1 + 2b_2 = -5$ ... 2

1 × 2

$2b_1 + 2b_2 = -14$ ... 3

2 − 3

$5b_1 + 2b_2 = -5$

$-\ 2b_1 + 2b_2 = -14$

$-\quad -\quad +$

$3b_1 = 9$

$\boxed{b_1 = 3}$

$\therefore b_1 + b_2 = -7$

$\therefore 3 + b_2 = -7$

$\boxed{b_2 = -10}$

$\therefore a_n = 3(5)^n - 10(2)^n + 8 + 2n$

$\Rightarrow a_n - 5a_{n-1} + 6a_{n-2} = 2^n + 3n \; ; \; a_0 = 10, \; a_1 = 5$

$r^2 - 5r + 6 = 0$ .. (Homogeneous eqn)

$r^2 - 2r - 3r + 6 = 0$ $\quad a_0^{(p)} = (c_1 n 2^n + c_2 n + c_3$

$r(r-2) - 3(r-2) = 0$

$(r-2)(r-3) = 0$

$r = 2, 3.$

$a_n^{(h)} = b_1(2)^n + b_2(3)^n$

$a_n^{(p)} = c_0 + c_1 n + c_2 n 2^n$

$\therefore (c_0 + c_1 n + c_2 n 2^n) - 5(c_0 + c_1(n-1) + c_2(n-1)2^{n-1}) + 6(c_0 + c_1(n-2) + c_2(n-2)2^{n-2}) = 2^n + 3n$

$\therefore c_0 + c_1 n + c_2 2^n n - 5c_0 - 5c_1 n + 5c_1 - 5c_2 (2)^n n + 5c_2 2^n + 6c_0 + 6c_1 n - 12c_1 + 6c_2 n 2^n - 12 c_2 2^n = 2^n + 3n.$

$2c_0 + 2c_1 n + 2c_2 n 2^n - 7c_1 - 7c_2 2^n = 2^n + 3n.$

$(2c_0 - 7c_1)$

$(2c_0 + 2c_1 n) + (c_2 n 2^n - 5c_2 n 2^{n-1} + 6c_2 n 2^{n-2})$

$+ (5c_2 2^{n-1} - 12 c_2 2^{n-2}) - 7c_1 = 2^n + 3n.$

$2c_1 n + (2c_0 - 7c_1) - 2 \cdot 2^{n-2} c_2 = 2^n + 3n$

$\therefore 2c_1 n + (2c_0 - 7c_1) - 2^n \left(\frac{c_2}{2}\right) = 2^n + 3n$

$\frac{c_2}{2} = -1$

$\boxed{\therefore c_2 = -2}$

$2c_1 = 3$

$\boxed{c_1 = \frac{3}{2}}$

$2c_0 - 7c_1 = 0.$

$\therefore 2c_0 = 7c_1$

$c_0 = \frac{7}{2} \times \frac{3}{2} = \frac{21}{4}.$

$\boxed{c_0 = \frac{21}{4}}$

$\therefore a_n^{(p)} = \frac{21}{4} + \frac{3}{2}n - 2n 2^n$

$a_n = a_n^{(h)} + a_n^{(p)}$

$= b_1(2)^n + b_2(3)^n + \frac{21}{4} + \frac{3}{2}n - 2n(2)^n.$

$= b_1(2)^n + b_2(3)^n + \frac{21}{4} + \frac{3}{2}n - n(2)^{n+1}$

Put $n = 0$

$a_0 = b_1 + b_2 + \frac{21}{4} + 0 + 0$

$\therefore 10 = b_1 + b_2 + \frac{21}{4}$

$b_1 + b_2 = 10 - \frac{21}{4} = +\frac{19}{4}$ $\quad$ (1)

Put $n = 1$

$\therefore a_1 = b_1(2) + b_2(3) + \frac{21}{4} + \frac{3}{2}(1) - (1)(2)^2$

$$5 = 2b_1 + 3b_2 + \frac{21}{4} + \frac{6}{4} - 4$$

$$= 2b_1 + 3b_2 + \frac{27}{4} - 4$$

$$9 - \frac{27}{4} = 2b_1 + 3b_2$$

$$2b_1 + 3b_2 = \frac{9}{4} \quad \ldots (2)$$

$(1) \times 2$

$$2b_1 + 2b_2 = \frac{19}{2} \quad \ldots (3)$$

$$-\quad 2b_1 + 3b_2 = \frac{9}{4}$$

$$-b_2 = \frac{19}{2} - \frac{9}{4}$$

$$\therefore b_2 = \frac{9}{4} - \frac{19}{2} = \frac{9-38}{4} = \frac{-29}{4}$$

$$\therefore b_1 + b_2 = \frac{19}{4}$$

$$b_1 - \frac{29}{4} = \frac{19}{4}$$

$$b_1 = \frac{19+29}{4} = \frac{48}{4} = 12$$

$$\boxed{b_1 = 12}$$

$$\therefore a_n = 12(2)^n - \frac{29}{4}(3)^n + \frac{21}{4} + \frac{3}{2}n - 2n(2)^n.$$

$$G(x) = a_0 + a_1(x) + a_2(x)^2 + a_3(x)^3 + a_4(x)^4 + \ldots$$

$$= \sum_{r=0}^{\infty} a_r x^r$$

$$G(x) = 1 - x + x^2 - x^3 + x^4 - x^5 \ldots$$

$$= 1 + (-x)^1 + (-x)^2 + (-x)^3 + (-x)^4 + (-x)^5 + \ldots$$

$$= \frac{1}{1-(-x)} = \frac{1}{1+x} \quad (1, -1, 1, -1, 1, -1, \ldots)$$

$$G(x) = 0 + 0x + x^2 + x^3 + x^4 + x^5 + \ldots$$

$$= x^2\left[1 + x + x^2 + x^3 + \ldots\right]$$

$$= x^2\left(\frac{1}{1-x}\right)$$

$$= \frac{x^2}{1-x} \quad (0, 0, 1, 1, 1, 1, 1, 1, 1, 1)$$

For $(1, 0, 1, 1, 1, 1, 1, 1, 1, 1 \ldots)$

$$G(x) = 1 + 0x + x^2 + x^3 + x^4 + x^5 + \ldots$$

$$= 1 + x^2(1 + x + x^2 + x^3 + \ldots)$$

$$= 1 + \frac{x^2}{1-x}$$

$$= \frac{1 - x + x^2}{1-x} = \frac{x^2 - x + 1}{1-x}$$

$$G(x) = 1 + x + 0x^2 + x^3 + x^4 + x^5 + x^6 + \ldots$$

$$= (1+x) + x^3(1 + x + x^2 + x^3 + \ldots)$$

$$= 1 + x + \frac{x^3}{1-x} = \frac{(1+x)(1-x) + x^3}{(1-x)}$$

$$= \frac{1 - x^2 + x^3}{(1-x)}$$

1, 2, 3, 4, 5, 6, 7, ...

$G(x) = 1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+\dots$

$xG(x) = x+2x^2+3x^3+4x^4+5x^5+6x^6+\dots$

$G(x)-xG(x) = 1+x+x^2+x^3+x^4+x^5+\dots$

$\therefore (1-x)G(x) = \frac{1}{1-x}$

$\therefore G(x) = \frac{1}{(1-x)^2}$

Using derivatives,

$G(x) = 1+2x+3x^2+4x^3+5x^4+\dots$

$H(x) = 1+x+x^2+x^3+x^4+x^5+\dots = \frac{1}{1-x}$

Diff both sides wrt x

$\therefore 0+1+2x+3x^2+4x^3+5x^4+\dots = \frac{-1}{(1-x)^2}(-1)$

$= \frac{1}{(1-x)^2}$

1, 0, 1, 0, 1, 0

$G(x) = 1+0x+x^2+x^4+x^6+x^8+\dots$

$G(x) = 1+x^2(1+x^2+x^4+x^6+\dots)$

$G(x) = 1+x^2G(x)$

$\therefore G(x)(1-x^2) = 1$

$\therefore G(x) = \frac{1}{1-x^2}$

Alternative method:-

$A(x) = 1+x+x^2+x^3+x^4+x^5+x^6+x^7 = \frac{1}{1-x}$ ①

$B(x) = 1-x+x^2-x^3+x^4-x^5+x^6-x^7 = \frac{1}{1+x}$ ②

① + ②

$2+2x^2+2x^4+2x^6+\dots = \frac{1}{1-x}+\frac{1}{1+x} = \frac{2}{1-x^2}$

$\therefore 1+x^2+x^4+x^6+\dots = \frac{1}{1-x^2}$

Generating function:-

→ 1, 3, 5, 7, 9, 11, ...

$G(x) = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + \dots$

$xG(x) = x + 3x^2 + 5x^3 + 7x^4 + \dots$

$G(x) - xG(x) = 1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + \dots$

$(1-x)G(x) = 1 + 2x(1 + x + x^2 + x^3 + x^4 + \dots)$

$= 1 + \frac{2x}{1-x} = \frac{1-x+2x}{1-x} = \frac{1+x}{1-x}$

$G(x) = \frac{1+x}{(1-x)^2}$

→ 0, 2, 4, 6, 8, 10, ...

$G(x) = 0 + 2x + 4x^2 + 6x^3 + 8x^4 + 10x^5 \dots$

$xG(x) = 0 + 2x^2 + 4x^3 + 6x^4 + 8x^5 \dots$

$G(x) - xG(x) = 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 \dots$

$(1-x)G(x) = 2x[1 + x + x^2 + x^3 + x^4 + \dots]$

$(1-x)G(x) = \frac{2x}{1-x}$

$G(x) = \frac{2x}{(1-x)^2}$

$\Rightarrow$ 1, 4, 9, 16, ...

$G(x) = 1+4x+9x^2+16x^3+25x^4+\dots$

$xG(x) = x+4x^2+9x^3+16x^4+\dots$

$(1-x)G(x) = 1+3x+5x^2+7x^3+9x^4+\dots$

$x(1-x)G(x) = x+3x^2+5x^3+7x^4+\dots$

$(1-x)^2 G(x) = 1+2x+2x^2+2x^3+2x^4+\dots$

$(1-x)^2 G(x) = 1+2x\left[1+x+x^2+x^3+\dots\right]$

$(1-x)^2 G(x) = 1+\frac{2x}{1-x}$

$G(x) = \frac{1-x+2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}$

$\Rightarrow$ 1, 3, 7, 15, 31, 63, ...

$G(x) = 1+3x+7x^2+15x^3+31x^4+63x^5+\dots$

$xG(x) = x+3x^2+7x^3+15x^4+31x^5+\dots$

$(1-x)G(x) = 1+2x+4x^2+8x^3+16x^4+32x^5$

$= 1+2x\left[1+2x+4x^2+8x^3+\dots\right]$

$\cancel{= \frac{1+}{1-2x}}$

$(1-x)G(x) = 1+2x\left[(1-x)G(x)\right]$

$(1-x)G(x) - 2x(1-x)G(x) = 1$

$\therefore (1-x)G(x)(1-2x) = 1$

$G(x) = \frac{1}{(1-x)(1-2x)}$

Generic power for $3^n = \frac{1}{1-3x}$,

$4^n = \frac{1}{1-4x}$

$a_n = 2\cdot 3^n - 3\cdot 4^n$

$G(x) = \frac{2}{1-3x} - \frac{3}{1-4x}$

$= \frac{2(1-4x)-3(1-3x)}{(1-3x)(1-4x)}$

$G(x) = \frac{x-1}{1-7x+12x^2}$

Recurrence relation using Generating function

1) $a_n = 3a_{n-1} - 2a_{n-2}$, $a_0 = 1$, $a_1 = 3$

$$\sum_{n=2}^{\infty} a_n x^n = 3\sum_{n=2}^{\infty} a_{n-1} x^n - 2\sum_{n=2}^{\infty} a_{n-2} x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

$$= G(x) - a_0 - a_1 x$$

$$= G(x) - 1 - 3x$$

$$\sum_{n=2}^{\infty} a_{n-1} x^n = a_1 x^2 + a_2 x^3 + a_3 x^4 + \ldots$$

$$= x\left[a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots\right]$$

$$= x\left[G(x) - a_0\right]$$

$$= x\left[G(x) - 1\right]$$

$$\sum_{n=2}^{\infty} a_{n-2} x^n = a_0 x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 + \ldots$$

$$= x^2\left[a_0 + a_1 x + a_2 x^2 + \ldots\right]$$

$$= x^2 G(x).$$

$$\therefore G(x) - 1 - 3x = 3\left[x G(x) - x\right] - 2\left[x^2 G(x)\right]$$

$$G(x) - 1 - 3x = 3x G(x) - 3x - 2x^2 G(x)$$

$$G(x)\left[1 - 3x + 2x^2\right] = 1$$

$$G(x) = \frac{1}{1-3x+2x^2} = \frac{1}{1-2x-x+2x^2}$$

$$= \frac{1}{(1-2x)-x(1-2x)}$$

$$G(x) = \frac{1}{(1-x)(1-2x)}$$

$$\frac{1}{(1-x)(1-2x)} = \frac{A}{(1-x)} + \frac{B}{(1-2x)}$$

$$1 = A(1-2x) + B(1-x).$$

Put $x = 1$

$\therefore 1 = A(1-2) + 0$

$1 = -A$

$\boxed{A = -1}$

~~Put~~ Put $x = \frac{1}{2}$

$\therefore 1 = 0 + B\left(1 - \frac{1}{2}\right)$

$\boxed{B = 2}$

$$G(x) = \frac{-1}{(1-x)} + \frac{2}{(1-2x)}$$

$$a_n = -1^n + 2 \cdot 2^n$$

$$= 2^{n+1} - 1.$$

$a_n = 2a_{n-1} + 3 \cdot 2^n$ $a_0 = 5$

$a_n - 2a_{n-1} = 3 \cdot 2^n$

$\sum_{n=1}^{\infty} a_n x^n - 2\sum_{n=1}^{\infty} a_{n-1} x^n = 3\sum_{n=1}^{\infty} 2^n x^n$

$G(x) - 5 - 2x\,G(x) = \frac{3}{1-2x}$

$(1-2x)\,G(x) = \frac{3}{1-2x} + 5 = \frac{3+5(1-2x)}{1-2x} = \frac{8-10x}{1-2x}$

$G(x) = \frac{8-10x}{(1-2x)^2}$

$G(x) = \frac{3}{(1-2x)^2} + \frac{5}{(1-2x)}$

$= 3\cdot(2)^{2n} + 5\,(2)^n$

29-08-18

ALGEBRAIC STRUCTURES.

→ Closure property:-
$a, b \in A$

If $a * b \in A$ $*$ is closed

$(a, b) \in N$
$\therefore a+b \in N$
$a \cdot b \in N$
But $a-b \notin N$ - is not closed.

→ Associative

$a, b, c \in A$

$(a*b)*c = a*(b*c)$

→ Identity (e)

$a*e = e*a = a$
i) $(Z, +)$
$0 \in Z$
$0+a = a+0 = a$
ii) $(Z, \times)$
$1 \in Z$
$1 \cdot a = a \cdot 1 = a$

→ Inverse
$a, b \in A$
$a*b = b*a = e$

1) $a*b = ab/2$

$e = ?$

$a^{-1} = ?$

$a*e = e*a = \frac{ae}{2}$

$a*a^{-1} = 2$

$\therefore \frac{aa^{-1}}{2} = 2$

$\therefore a^{-1} = \frac{4}{a}$

For verifying;

$(a*b)*c = a*(b*c)$

$\left(\frac{ab}{2}\right)*c = a*\left(\frac{bc}{2}\right)$

$\frac{abc}{4} = \frac{abc}{4}$

$\therefore$ LHS = RHS

Hence verified.

2) $(R, *)$

$a*b = a+b-ab.$

$a*e = a+e-ae = a$

$\therefore e-ae = 0$

$e = 0.$

$a^{-1} = ?$

$a*a^{-1} = e$

$\therefore a+a^{-1}-a.a^{-1} = 0$

$\therefore a+a^{-1}(1-a) = 0$

$\therefore a = a^{-1}(a-1)$

$\therefore a^{-1} = \frac{a}{a-1}$

⇒ $(S, *)$

If $*$ is closed

$*$ is assosiative,

then it is called a semigroup.

⇒ If $*$ is closed

$*$ is associative

& having identity.

then it is called Monoid.

⇒ If $*$ is closed, $*$ is associative, & having identity & every element having its inverse.

Then it is called a Group.

If Group is commutative → Abelion Group

$(G, *)$

$a * b = \frac{ab}{2}$.

$(a, b) \in G$

$a * b = \frac{ab}{2} \in G$.

LHS:-

$(a * b) * c$

$= \left(\frac{ab}{2}\right) * c = \frac{ab}{2} \times \frac{c}{2} = \frac{abc}{4}$

RHS :- $a * (b * c)$

$= a * \left(\frac{bc}{2}\right) = \frac{a \times bc}{2 \times 2} = \frac{abc}{4}$

LHS = RHS.

$\therefore$ it is associative

$a * e = a$

$\therefore \frac{ae}{2} = a$

$e = 2$

$\therefore$ identity

$a * a^{-1} = e$

$\therefore \frac{aa^{-1}}{2} = 2$

$a^{-1} = \frac{4}{a}$

$\therefore$ inverse

$\therefore$ it is a group.

$\Rightarrow$ Isomorphism.

$f : (G_1, *) \longrightarrow (G_2, *')$

$f(a * b) = f(a) *' f(b)$ If only this is there it is called Homomorphism

Eg:-

$f : D_{30} \rightarrow D_{42}$

$f(1) = 1$

$f(2) = 2$

$f(3) = 3$

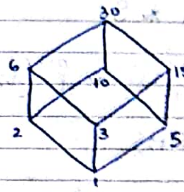
$f(5) = 7$

$f(6) = 6$

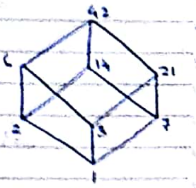
$f(10) = 14$

$f(15) = 21$

$f(30) = 42$



D_{30}



D_{42}

1) If $(S, *)$ and $(T, *')$ are two semigroup and if $f : S \rightarrow T$ is an isomorphism from $(S, *)$ to $(T, *')$ then $f^{-1} : T \rightarrow S$ is also an isomorphism from $(T, *')$ to $(S, *)$.

Let $(s_1, s_2) \in S$

f onto $t_1, t_2 \in T$

$s_1, s_2 \in S$

$f(s_1) = t_1 \Rightarrow s_1 = f^{-1}(t_1)$

$f(s_2) = t_2 \Rightarrow s_2 = f^{-1}(t_2)$

T.P. $\therefore f^{-1}(t_1 *' t_2) = f^{-1}(t_1) * f^{-1}(t_2)$

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LHS :- $f^{-1}(t_1 *' t_2)$

$= f^{-1}(f(s_1) *' f(s_2))$

$= f^{-1}(f(s_1 * s_2))$

$= f_o^{-1} f(s_1 * s_2)$

$= s_1 * s_2$

$= f^{-1}(t_1) * f^{-1}(t_2) = RHS$

$\therefore$ LHS = RHS.

Hence it is an isomorphism

2) Let S be the set of all even integers. Show that $(z, +)$ and $(s, +)$ are isomorphic

$f : Z \rightarrow S$

i) For one to one correspondence

$f(a) = 2a$ ①

$\therefore f(a_1) = f(a_2)$

$\therefore 2a_1 = 2a_2$

$\therefore a_1 = a_2$

ii) ~~For~~ onto

$b \in S$

$f(a) = b$

$\therefore 2a = b$

$\therefore a = \frac{b}{2} \in Z$

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T.S.T :- $f(a + b) = f(a) + f(b)$.

LHS =

$f(a+b) = 2(a+b)$ from (i)

$= 2a + 2b$

$= f(a) + f(b) = RHS$

$\therefore$ LHS = RHS

$\therefore$ Isomorphic.

3) Let R^+ is set of positive real nos. Show that the function $f : (R^+, +) \rightarrow (R, +)$ defined by $f(x) = \log x$. is an isomorphism.

i) One to one correspondence

$f(x_1) = f(x_2)$

$\log x_1 = \log x_2$

$\therefore x_1 = x_2$

$\therefore f$ is one to one.

$b = \log(x) \in R$

$\therefore x = e^b \in R^+$

$e^b \rightarrow b.$

$f(ab) = \log(ab)$

$= \log a + \log b$

$= f(a) + f(b)$

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4) $(G, *)$ & $(G', *')$ e and e'
Let $f: G \to G'$ is an isomorphism
prove that $f(e) = e'$
$a \in G$ $b \in G'$

$f(a) = b$
$a = a * e = e * a$

$b = f(a) = f(a * e)$
$= f(a) *' f(e)$
$= b *' f(e)$

$b = f(a) = f(e * a)$
$= f(e) *' f(a)$
$b = f(e) *' b$

$\therefore b = b *' f(e) = f(e) *' b$

$\therefore f(e)$ must be identity
$f(e) = e'$

5) If f is a homomorphism from a commutative semigroup $(S, *)$ onto a semigroup $(T, *')$, then show that $(T, *')$ is also commutative

$f: (S, *) \to (T, *')$

$s_1 * s_2 = s_2 * s_1$

$f(s_1 * s_2) = f(s_1) *' f(s_2)$
$t_1, t_2 \in T$, $s_1, s_2 \in S$

$f(s_1) = t_1$ $f(s_2) = t_2$

$t_1 *' t_2 = f(s_1) *' f(s_2)$

$= f(s_1 * s_2)$
$= f(s_2 * s_1)$

$= f(s_2) *' f(s_1)$
$= t_2 *' t_1$

$\therefore t_1 *' t_2 = t_2 *' t_1$
Hence it is also commutative.

6) Prove that the set of cube roots of unity is a group under multiplication of complex number

$C = \{1, \omega, \omega^2\}$

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$(1 \times \omega) \times \omega^2$
$= 1 \times (\omega \times \omega^2)$

$*$ is associative.

$(a * b) * c = a * (b * c)$

- Identity $= 1$

- Inverse.
$(1)^{-1} = 1$
$(\omega)^{-1} = \omega^2$
$(\omega^2)^{-1} = \omega$

→ Let G be the set of rational numbers other than 1.
Define $a * b = a + b - ab$ for all $a, b \in G$
Prove that $(G, *)$ is a group.

① closure

$a, b \in G$

$a * b = a + b - ab \in G$

② $(a*b)*c = a*(b*c)$

LHS :- $(a*b)*c$
$= (a+b-ab)*c$
$= (a+b-ab)+c - (a+b-ab)c$
$= a+b+c-ab-ac-bc+abc$

RHS :- $a*(b*c)$
$= a*(b+c-bc)$
$= a+b+c-bc-a(b+c-bc)$
$= a+b+c-bc-ab-ac+abc$

$\therefore$ LHS = RHS
$\therefore$ associative.

$a*e = a$
$\therefore a+e-ae = a$
$e - ae = 0$
$e(1-a) = 0$
$e = 0$ when $a \neq 1$

$a \in G$
$\therefore a * a^{-1} = e = 0.$
$a + a^{-1} - aa^{-1} = 0.$

$\therefore a - a^{-1}(a-1) = 0$

$\boxed{a^{-1} = \frac{a}{a-1}} \in G$

⇒ Let $\mathbb{Q}$ be the set of all positive rational numbers which can be expressed as $2^a . 3^b$, $a, b \in Z$
Prove that $(\mathbb{Q}, *)$ is a group where $*$ is usual multiplication.

1) $a, b \in \mathbb{Q}$
$\therefore a \times b \in \mathbb{Q}$
$\therefore *$ closure

2) Since it is multiplication of 2 nos. it is associative.
ie $2^a 3^b = 3^b . 2^a$

3) $a * e = a$
$\therefore 2^a . 3^e = a$
$3^e = \frac{a}{2^a}$

$e \log 3 = \log\left(\frac{a}{2a}\right)$

$e = \frac{1}{\log 3} \log\left(\frac{a}{2a}\right)$

4) $a * a^{-1} = e$
$\therefore 2^a . 3^{a^{-1}} = e$
$3^{a^{-1}} = \frac{e}{2^a}$

$\therefore a^{-1} \log 3 = \log\left(\frac{e}{2^a}\right)$

$a^{-1} = \frac{1}{\log 3}\left\{\log\left[\frac{1}{\log 3} \log\left(\frac{a}{2a}\right)\right] - \log 2^a\right\}$

$\therefore a^{-1}$ exist
$\therefore$ it is a group

Prove that the set $Z_6 = \{0, 1, 2, 3, 4, 5\}$ is an Abelian group under modulo 6.

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4
~~6~~	~~6~~					

$\therefore$ it is closed.

Prove that the set $A = \{1, 2, 3, 4, 5, 6\}$ is an Abelian group under modulo 7

$\times_7$	1	2	3	4	5	6	~~7~~
1	1	2	3	4	5	6	~~7~~
2	2	4	6	1	3	5	
3	3	6	2	5	1	4	
4	4	1	5	2	6	3	
5	5	3	1	6	4	2	
6	6	5	4	3	2	1	
~~7~~							

1) $2, 5 \in A$
$2 \times_7 5 \in A$
$\therefore$ it is closed.

2) $2^{-1} = 4$
Also $4^{-1} = 2$.
$\therefore$ inverse

3) $a * e = a$ (Identity).
$\therefore a \times e = a$
$\therefore e = \frac{a}{a} = 1$
$\boxed{e = 1}$

4) ~~$a \times_7 b = b \times_7 a$~~
~~$\therefore$ It is associative.~~
LHS $(a \times_7 b) \times c$
$= (2 \times_7 5) \times_7 3$
$= \cancel{2 \times_7 (8 \times_7 3)}$
$= 3 \times_7 3$
$= 2$

RHS:- $a \times_7 (b \times_7 c)$
$= 2 \times_7 (5 \times_7 3)$
$= 2 \times_7 1$
$= 2$
LHS = RHS
$\therefore$ associative.

⇒ Determine whether the set A of all ordered pair (a,b) of real number (a ≠ 0) under * defined by $(a,b)*(c,d) = (ac, bc+d)$ is an Abelian group.

T.P. * is closed.

i) $(a,b), (c,d) \in A$.

$\therefore (a,b)*(c,d) = (ac, bc+d) \in A$.

* is closed.

ii) LHS =

$(a,b)*((c,d)*(e,f))$

$= (a,b)*(ce, de+f)$

$= (ace, bce+de+f)$

RHS =

$((a,b)*(c,d))*(e,f)$

$= (ac, bc+d)*(e,f)$

$= (ace, bce+de+f)$

$\therefore$ LHS = RHS.

$\therefore$ it is associative.

* is associative.

$(a,b)*(x,y) = (a,b)$

Also $(a,b)*(x,y) = (ax, bx+y) = (a,b)$.

$\therefore ax = a$

$\therefore x = 1$

$bx+y = b$.

$\therefore b+y = b$. $[\because x = 1]$

$y = 0$.

$\therefore (a,b)*(1,0) = (a,b)$ (Identity)

$e = (1,0)$.

$(a,b)*(x,y) = (1,0)$.

$\therefore (ax, bx+y) = (1,0)$

$ax = 1$

$x = \frac{1}{a}$

$bx+y = 0$

$\frac{b}{a}+y = 0$

$y = -\frac{b}{a}$

$\therefore$ Inverse $= \left(\frac{1}{a}, \frac{-b}{a}\right)$

LHS :-

$(a,b)*(x,y)$

$= (ax, bx+y)$

RHS :-

$(x,y)*(a,b)$

$= (xa, ay+b)$

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Identity element is unique.

e, e'

$e = e \times e' = e'$

Inverse of an element is unique.

$a \quad a' \quad a''$

$(a'a)a'' = e a'' = a''$

$\therefore (a'a)a'' = a'(aa'')$

$a'(aa'') = a'e = a'$

$\therefore a'' = a'$

⇒ Let $(G,*)$ be a group. Prove that G is an Abelian group if and only if $(a*b)^2 = a^2 * b^2$, where $a^2 = a*a$

Let G be an Abelian group

i) $(a*b)^2 = (a*b)*(a*b)$

$= a*(b*(a*b))$

$= a*((b*a)*b)$

$= a*((a*b)*b)$

$= a*(a*b^2)$

$= (a*a)*b^2$

$= a^2*b^2$

ii) Conversely, Let $(a*b)^2 = a^2*b^2$.

$(a*b)^2 = a^2*b^2$

$(a*b)*(a*b) = (a*a)*(b*b)$

$a^{-1}*(a*b)*(a*b)*b^{-1} = a^{-1}*(a*a)*(b*b)*b^{-1}$

$(a^{-1}*a)*b*a*(b*b^{-1}) = (a^{-1}*a)*a*b*(b*b^{-1})$

$(e*b)*(a*e) = (e*a)*(b*e)$

$b*a = a*b.$

⇒ If $(G,*)$ is an Abelian group then prove that $(a*b)^n = a^n * b^n$ by mathematical induction.

⇒ ~~$(a*b)^n$~~

For $n=1$

$(a*b)^1 = (a*b)$

$\therefore (a*b)^1 = a^1 * b^1$

For $n=2$

$(a*b)^2 = (a*b)*(a*b)$

$= a*(b*(a*b))$

$= a*((b*a)*b)$

$= a*((a*b)*b)$

$= a*(a*(b*b))$

$= a*(a*b^2)$

$= a^2*b^2.$

~~Put n=k~~

Suppose it is true for $n=k$

$(a*b)^k = a^k * b^k$

T.P. :- $(a*b)^{k+1} = a^{k+1} * b^{k+1}$

LHS :- $(a*b)^{k+1}$

$= (a*b)^k * (a*b)$

$= (a^k * b^k) * (a*b)$

$= a^k * (b^k * a) * b$

$= a^k * (a * b^k) * b = a^k * a * (b^k * b)$

$= (a^k * a) * (b^k * b)$

$= a^{k+1} * b^{k+1} = RHS$

Hence

$(a*b)^n = a^n * b^n$

$\Rightarrow$ $S = \{x \mid x \in R$ and $x \neq 0, x \neq 1\}$

Consider the following functions

$f_i : S \longrightarrow S$, $i = 1, 2, 3, 4, 5, 6$

$f_1(x) = x$, $f_2(x) = 1 - x$, $f_3(x) = \frac{1}{x}$,

$f_4(x) = \frac{1}{1-x}$, $f_5(x) = 1 - \frac{1}{x}$, $f_6(x) = \frac{x}{x-1}$

Show that $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ is a group under the operation of function composition.

o	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_5	f_6	f_3	f_4
f_3	f_3	f_4	f_1	f_2	f_6	f_5
f_4	f_4	f_3	f_6	f_5	f_2	f_1
f_5	f_5					
f_6	f_6					

$f_2 \circ f_2(x) = f_2(f_2(x))$

$= f_2(1-x)$

$= 1 - (1-x)$

$= x$

$= f_1(x)$

$f_2 \circ f_3(x) = f_2(f_3(x))$

$= f_2\left(\frac{1}{x}\right)$

$= 1 - \frac{1}{x} = f_5$

$f_2 \circ f_4(x) = f_2(f_4(x))$

$= f_2\left(\frac{1}{1-x}\right) = 1 - \frac{1}{1-x} = \frac{1-x-1}{1-x}$

$= \frac{-x}{1-x} = \frac{x}{x-1} = f_6$

$f_2 \circ f_5(x) = f_2(f_5(x))$

$= f_2\left(1 - \frac{1}{x}\right) = 1 - \left(1 - \frac{1}{x}\right) = 1 - 1 + \frac{1}{x} = f_3$

$f_2 \circ f_6(x) = f_2(f_6(x)) = f_2\left(\frac{x}{x-1}\right) = 1 - \frac{x}{x-1} = \frac{x-1-x}{x-1}$

$= \frac{-1}{x-1}$

$= \frac{1}{1-x} = f_4$

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$$f_3 \circ f_2(x) = f_3(f_2(x)) = f_3(1-x) = \frac{1}{1-x} = f_4$$

$$f_3 \circ f_3(x) = f_3(f_3(x)) = f_3\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} = x = f_1$$

$$f_3 \circ f_4(x) = f_3(f_4(x)) = f_3\left(\frac{1}{1-x}\right) = 1-x = f_2$$

$$f_3 \circ f_5(x) = f_3(f_5(x)) = f_3\left(1-\frac{1}{x}\right) = \frac{x}{x-1} = f_6$$

$$f_3 \circ f_6(x) = f_3(f_6(x)) = f_3\left(\frac{x}{x-1}\right) = \frac{x-1}{x} = 1-\frac{1}{x} = f_5$$

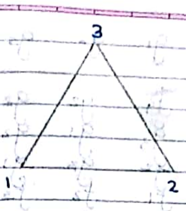
$$f_4 \circ f_4(x) = f_4(f_4(x)) = f_4\left(\frac{1}{1-x}\right) = \frac{1}{1-\frac{1}{1-x}}$$

$$= \frac{1-x}{1-x-1}$$

$$= \frac{1-x}{-x} = \frac{x-1}{x}$$

$$= 1-\frac{1}{x} = f_5$$

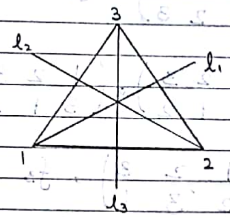
$\therefore$ it is a group $\because$ it is close, associative, inverse



$$f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$ //when rotated by 120°

$$f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$ //when rotated by 240°

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$ //when rotated by 360°



$$g_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$g_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$g_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$G = \{f_1, f_2, f_3, g_1, g_2, g_3\}$

$\circ$	f_1	f_2	f_3	g_1	g_2	g_3
f_1	f_1	f_2	f_3	g_1	g_2	g_2
f_2	f_2	f_3	f_1	g_3	g_1	g_2
f_3	f_3	f_1	f_2	g_2	g_3	g_1
g_1	g_1	g_2	g_3	f_1	f_2	f_3
g_2	g_2	g_3	g_1	f_3	f_1	f_2
g_3	g_3	g_1	g_2	f_2	f_3	f_1

$$f_1 \circ f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = f_1$$

$$f_2 . f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = f_3$$

$$f_2 . f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = f_1$$

$$f_2 . g_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = g_3$$

$$f_2 . g_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = g_1$$

$$f_2 . g_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = g_2$$

$$f_3 . f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = f_3$$

$$f_3 . f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = f_1$$

$$f_3 . f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = f_2$$

$$f_3 \circ g_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = g_2$$

$$f_3 \circ g_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = g_3$$

$$f_3 \circ g_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = g_1$$

$$g_1 \circ f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = g_1$$

$$g_1 \circ f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = g_2$$

$$g_1 \circ f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = g_3$$

$$g_1 \circ g_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = f_1$$

$$g_1 \circ g_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = f_2$$

$$g_1 \circ g_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = f_3$$

$$g_2 \circ f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = g_3$$

$$g_2 \circ f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = g_1$$

$$g_2 \circ g_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = f_3$$

$$g_2 \circ g_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = f_1$$

$$g_2 \circ g_3 = f_2$$

$$g_3 \circ f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = g_1$$

⇒ Group Code

1) Show that the (2,5) encoding function $e : B^2 \to B^5$ defined by

$e(00) = 00000$
$e(01) = 01110$
$e(10) = 10101$
$e(11) = 11011$ is a group code.

$+_2$	00000	01110	10101	11011
00000	00000	01110	10101	11011
01110	01110	00000	11011	10101
10101	10101	11011	00000	01110
11011	11011	10101	01110	00000

∴ it is closed.

For associative

$00000 + (01110 + 10101) = 00000 + 11011 = 11011$
$(00000 + 01110) + 10101 = 01110 + 10101 = 11011$
∴ Associative

Identity is 00000.

Inverse
- Every element is an inverse of itself.

⇒ Consider the parity check matrix H given by

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determine the group code.

$e_H : B^2 \to B^5$

$e(00) = 0\,0\,x_1 x_2 x_3$
$x_1 = 0.1 + 0.0 = 0$
$x_2 = 0.1 + 0.1 = 0$
$x_3 = 0.0 + 0.1 = 0$
$e(00) = 00000$

$e(01) = 0\,1\,x_1 x_2 x_3$

$x_1 = 0.1 + 1.0 = 0$
$x_2 = 0.1 + 1.1 = 1$
$x_3 = 0.0 + 1.1 = 1$
$e(01) = 01011$

$e(10) = 1\,0\,x_1 x_2 x_3$
$x_1 = 1.1 + 0.0 = 1$
$x_2 = 1.1 + 0.1 = 1$
$x_3 = 1.0 + 0.1 = 0$
$e(10) = 10110$

$e(11) = 1\,1\,x_1 x_2 x_3$
$x_1 = 1.1 + 1.0 = 1$
$x_2 = 1.1 + 1.1 = 0$
$x_3 = 1.0 + 1.1 = 1$
$e(11) = 11101$

⇒ Consider the parity check matrix H given by

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determine the group code

$e_H : B^3 \to B^5$

$e(000) = 000\,x_1 x_2 x_3$

$x_1 = 0.1 + 0.0 + 0.1 = 0$

$x_2 = 0.0 + 0.1 + 0.1 = 0$

$x_3 = 0.0 + 0.1 + 0.1 = 0$

$e(000) = 000\,000$

$e(101) = 101\,x_1 x_2 x_3$

$x_1 = 1.1 + 0.0 + 1.1 = 0$

$x_2 = 1.0 + 0.1 + 1.1$

$x_3 = 1.0 + 0.1 + 1.1 = 1$

$e(101) = 101011$

Consider (2,6) group encoding function $e : B^2 \to B^6$ defined as

$e(00) = 000\,000$

$e(01) = 011110$

$e(10) = 101101$

$e(11) = 110011$

Decode the following codes relative to maximum likelihood decoding function.

i) 001110

ii) 111101

iii) 110010

Decoding table:-

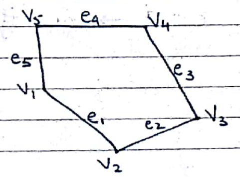
000000	011110	101101	110011
000001	011111	101100	110010
000010	011100	101111	110001
000100	011010	101001	110111
001000	010110	100101	111011
010000	$\boxed{001110}$	111101	100011
100000	111110	001101	010011

Graph Theory:-

$G' = \langle V, E \rangle$

V : set of vertices

E : set of edges



$V = \{v_1, v_2, v_3, v_5, v_4\}$

$E = \{e_1, e_2, e_3, e_4, e_5\}$.

t_2

undirected graph

$$\sum_{i=1}^{n} d(v_i) = 2e$$

ie total no. of degree of vertices = 2 x no. of edges

Q) Is it possible to draw a graph of 2 vertices of degree 4 and 4 vertices of degree 2. Draw such graph.

$\therefore 2 \times 4 + 4 \times 2 = 2e$

$\therefore \frac{8+8}{2} = e$

$e = 8$

$\therefore$ Possible

If $3 \times 3 + 4 \times 2 = 2e$

$9 + 8 = 2e$

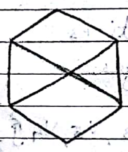
$e = \frac{17}{2} = 8.5$

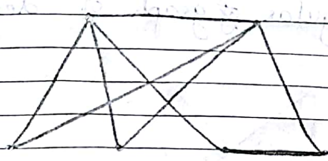
$\therefore$ Not possible

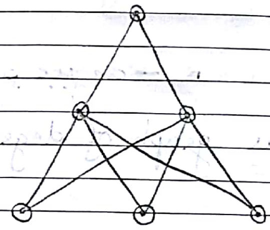
Q) If $n = 6$, $e = 8$, make the graph

$n = 6 \rightarrow$ no. of vertices

$e = 8 \rightarrow$ no. of edges



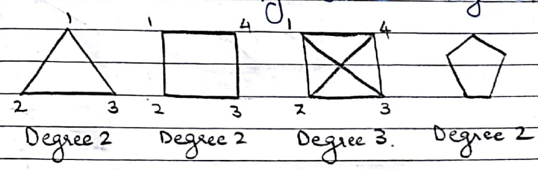




Regular Graph :-

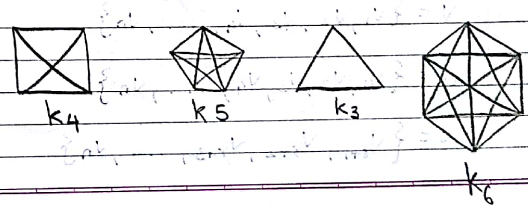
Each vertex is having same degree.

Eg :-

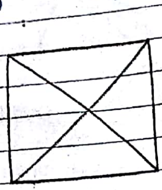


Complete Graph

Every vertex is connected to all other vertex.

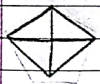


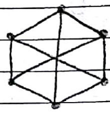
Draw a regular graph of degree 3

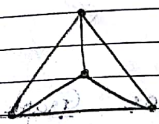


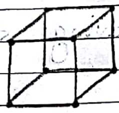
$K_4 \Rightarrow$ degree 3

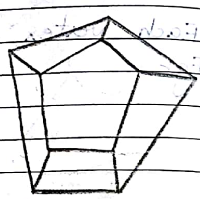
Draw regular graph of degree 3 other than K_4











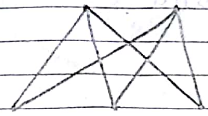
Peterson's graph

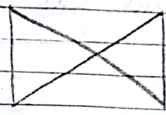
Bipartite graph

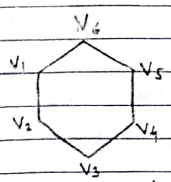
$V = \{v_1, v_2, v_3, v_4, \dots, v_n\}$

$U = \{v_1, v_2, v_3, v_4, \dots, v_n\}$

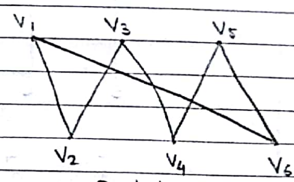
$W = \{v_{r+1}, v_{r+2}, v_{r+3}, \dots, v_n\}$





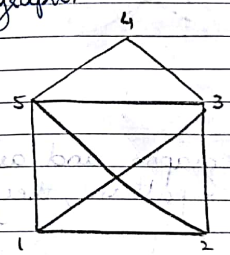


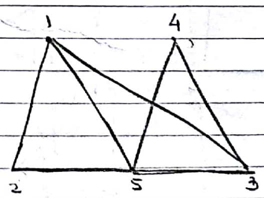
Non-BiPartite



Bi-Partite

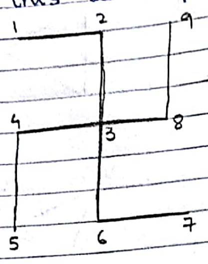
Represent this graph as a Bi-Partite graph.

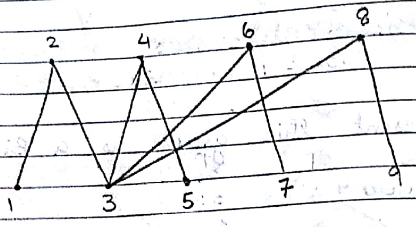




$\therefore$ it is not BiPartite

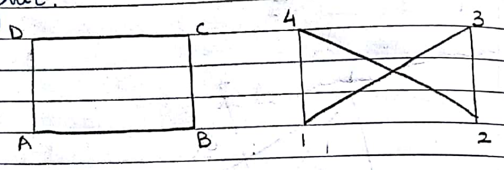
Represent this as bipertite.





Graph Isomorphism

If G1 & G2 are two graphs and are represented in similar pattern, they are isomorphic.



$f : G1 \rightarrow G2$

$f(A) = 1$
$f(B) = 3$
$f(C) = 2$
$f(D) = 4$

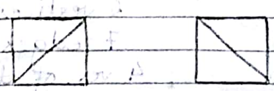
This is called preservation of adjacency.

$u, v \in G1$

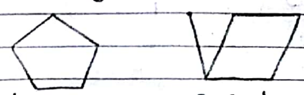
then $f(u), f(v) \in G2$

⇒ For two graphs to be isomorphic,
- No. of vertices must be same,
- No. of edges must be same,
- The vertices & degree of two Graphs must be same.

Eg:-



They are isomorphic.



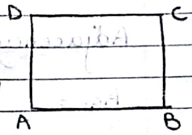
5 of degree 2

3 of degree 2
1 of degree 3
1 of degree 1

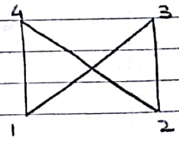
They are not isomorphic.

- Adjacency
  → Adjacency matrix.

$$A_{G1} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



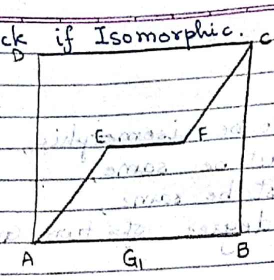
$$A_{G2} = \begin{array}{c} \\ 1 \\ 3 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 2 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

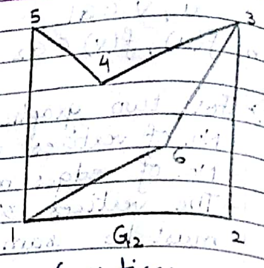


∴ Adjacency is preserverd

Q) Check if Isomorphic.

1)





~~4~~ 6 vertices
7 edges
4 ver. of deg. 2
2 ver. of deg 3

6 vertices
7 edges
4 ver. of deg. 2
2 ver. of degree 3

∴ Isomorphic.

$f : G_1 \to G_2$

$f(A) = 1$
$f(B) = 2$
$f(C) = 3$
$f(D) = 6$
$f(E) = 5$
$f(F) = 4$

Deg of A = 3 = Deg. of 1
Deg of B = 2 = Degree of 2.

Here we didn't choose 4 because it is not directly connected ~~to 1 but D is connected to A~~

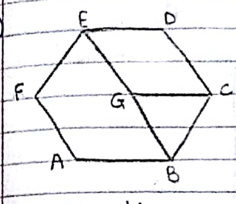
Adjacency Matrix

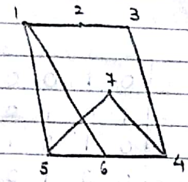
$$A_{G_1} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$A_{G_2} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 6 & 5 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$\therefore A_{G_1} = A_{G_2}$.

2)





7 vertices
9 edges
4 nodes of deg 3
3 nodes of deg 2

7 vertices
9 edges.
4 nodes of deg. 3.
3 nodes of deg 2.

$f : G_1 \to G_2$

$f(A) = 2$
$f(B) = 1$
$f(C) = 5$
$f(D) = 7$
$f(E) = 4$
$f(F) = 3$
$f(G) = 6$

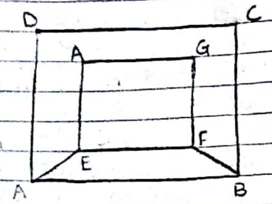
Adjacency matrix

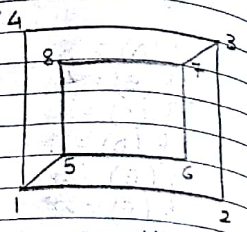
$$A_{G_1} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} \begin{array}{c} \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$A_{G_2} = \begin{array}{c} \\ 2 \\ 1 \\ 5 \\ 7 \\ 4 \\ 3 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 2 & 1 & 5 & 7 & 4 & 3 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$\therefore A_{G_1} = A_{G_2}$

3)





8 vertices
10 edges
4 nodes of deg 3
4 nodes of deg 2

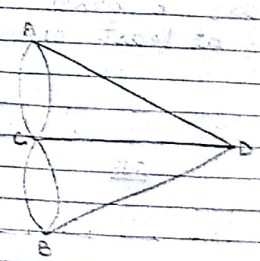
8 vertices
10 edges
4 nodes of deg 3
4 nodes of deg 2

$f : G_1 \rightarrow G_2$

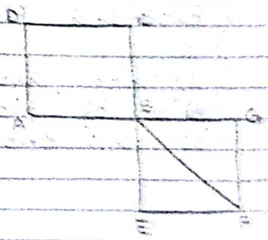
$f(A) = 1, 3, 5, 7$
but adjacency is not preserved
$\therefore f(A)$ is not defined

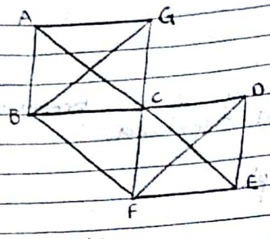
Eulerian Graph

- All the nodes are of even degree
- The graph must be of exactly zero or 2 nodes of odd degree



π : B → E → F → G → B → A → D → C → B → F





Path not possible

⇒ ~~Travelling salesman Problem (TSP).~~

⇒ Pigeon Hole Principle

- If there are k boxes and n objects, & $n > k$, then at least one box contains 2 objects

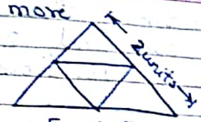
4 boxes
5 object.

OR

⇒ Extended Pigeon Hole Principle

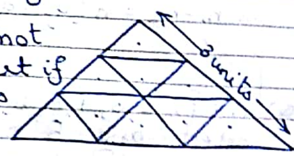
- If there are k no. of pigeon holes & N no. of pigeons where $k < N$, then at least one pigeon hole contains $\left\lceil \frac{N}{k} \right\rceil$ pigeons.

At least 2 points are not more than 1 unit apart if we have 5 points to be ploted



If 3 units is the length

At least 2 points are not more than 1 unit apart if we have 10 points to be plotted.



Q) Show that among 4 numbers one can find 2 nos. so that their difference is divisible by 3.

Atleast 2 nos will have same remainder

$\therefore n_1 = 3q_1 + r$ options for remainder $\boxed{0, 1, 2}$

$n_2 = 3q_2 + r$

$\therefore n_1 - n_2 = 3(q_1 - q_2)$

Eg: $12 = 3(5-1) = 12$ ✓

$\therefore (n_1 - n_2)$ is divisible by 3.

Q) Show that among n+1 numbers one can find 2 nos. so that their diff is divisible by n.

→ $n_1 = nq_1 + r$
$n_2 = nq_2 + r$

$\therefore n_1 - n_2 = n(q_1 - q_2)$.

Q) $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
select any 5 numbers.
Show that at least 2 nos add up to nine.

→ The sets will be
$\{1,8\}, \{2,7\}, \{3,6\}, \{4,5\}$.
We have to select 5 nos, so atleast 1 set will become true.

Eg:- $\{1, 2, 4, 6, 7\}$ This no will be our solution

Q) In a set from 1 to 20, show that there are atleast 2 nos which is a multiple of other

Let us take some nos, suppose

1, 2, 3, 5, 7, 11, 13, 17, 19

$n = 2^k \cdot q$ [q is odd part]

$\therefore n_1 = 2^{k_1} \cdot q$
$n_2 = 2^{k_2} \cdot q$

If $k_1 < k_2$, n_2 is multiple of n_1.
If $k_1 > k_2$, n_1 is multiple of n_2.

Q) Given 12 different 2-dit nos. show that one can choose two of them so that their difference is a two digit no with identical first & second digits.

→ If there are 12 digits atleast two digits will have same remainder when divided by 11.

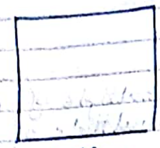
$\therefore n_1 = 11 \cdot q_1 + r$
$n_2 = 11 \cdot q_2 + r$

$\therefore (n_1 - n_2) = 11(q_1 - q_2)$.

For two digits to be identical it must be ~~div~~ multiple of 11.

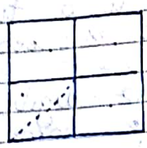
$\therefore$ The diff of atleast any two 2-digit nos is equal to multiple of 11 ie identical first & second digit.

Q)



2 units.

Show that if there are 5 points, no two points are $\sqrt{2}$ units apart.



If we divide square into 4 parts.

Let us put 1 point in each square.
$\therefore$ the fifth point will be in any of the squares.

$\sqrt{1^2 + 1^2} = \sqrt{2}$

$\therefore$ No two points are $\sqrt{2}$ units apart.

Q) What is the minimum no. of students in a class so that at least 6 students will receive same grade if there are 5 possible grades A, B, C, D, E.

$5 \times 5 = 25$

$25 + 1 = 26$.

26 students have same grade.

26-9

COUNTING

→ Product rule

(Element)
E_1 can occupy n_1 ways
E_2 can occupy n_2 ways.

∴ Total ways $= n_1 \times n_2$

Q) 5 chairs
5 students.

How can students occupy the chairs.

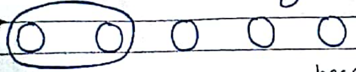
→ $5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$.

→ Sum Rule

- If two diff. sets A & B are present then if an event is to be performed then no of ways it can be done $= n(A) + n(B)$

Q) If there are 5 chairs and 5 students at 2 students must always sit side by side

consider 2 students & 2 chairs as 1 →



∴ No of ways $= 4! \times 2$ ← because 2 students can exchange their places
$= 24 \times 2$
$= 48$.

Q) ~~What~~ How is th many ways are there such that the 2 students never sit together.

P(of 5 students) $= 5! = 120$

P(of 2 students sitting together) $= 4! \times 2 = 48$.

∴ P(of 2 students not siting together) $= 120 - 48$
$= 72$.

Q) A registeration of 4 digits has to be made using 0, 1, 2, 3, 4, 5.

→ 1st digit can be selected in 5 ways.
2nd digit can be selected in 6 ways.
3rd " " " " " 6 ways.
4th " " " " " 6 ways.

∴ Total no. of ways if there is repetition $\}$ $= 5 \times 6 \times 6 \times 6 = 5 \times 216 = 1080$.

∴ Total no. of ways if there is no repetition $\}$ $= 5 \times 5 \times 4 \times 3 = 15 \times 20$
$= 300$.

Q) We have to create a 6 character log like SS05BC. How many diff ways can it be represented with repetition of characters and without repetition of character.

Always same.

55/05 BC

→ With repetition:-

No. of ways = $1 \times 10 \times 10 \times 26 \times 26 = 67600$

Without repetition

No. of ways = $1 \times 10 \times 9 \times 26 \times 25 = 90 \times 25 \times 25 = 58500$

Q) There are 4 boys and 5 girls in a group of students. Find the no. of ways a committee of 5 students can be formed with the following condition

a) There are two boys in the committee.

b) There are at least 2 girls in the committee

c) There are at most 2 girls in the committee

d) No restriction.

→ a) If two boys in committee $= {}^4C_2 \cdot {}^5C_3 = \frac{4!}{2! \times 2!} \times \frac{5!}{3! \times 2!}$

$= \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} \times \frac{5!}{3!\,2!}$

$= 6 \times 10$

$= 60$

b) If atleast 2 girls:

i) If 2 girls & 3 boys $= {}^5C_2 \times {}^4C_3$

$= \frac{5!}{3!\,2!} \times \frac{4!}{3! \times 1!}$

$= 10 \times 4 = 40.$

ii) If 3 girls & 2 boys $= {}^5C_3 \times {}^4C_2$

$= \frac{5!}{3!\,2!} \times \frac{4!}{2! \times 2!}$

$= 10 \times 6$

$= 60$

iii) If 4 girls & 1 boy $= {}^5C_4 \times {}^4C_1$

$= \frac{5!}{4! \times 1!} \times \frac{4!}{3! \times 1!}$

$= 5 \times 4 = 20$

iv) If 5 girls & 0 boys $= {}^5C_5 \times {}^4C_0$

$= \frac{5!}{5!} \times \frac{4!}{4!}$

$= 1 \times 1 = 1$

∴ Total ways $= 40 + 60 + 20 + 1 = 121$

c) Atmost 2 girls in committee.

i) If 1 girls & 4 boys $= {}^5C_1 \times {}^4C_4$

$= \frac{5!}{4! \times 1!} \times \frac{4!}{4!}$

$= 5 \times 1 = 5.$

ii) If 2 girls & 3 boys $= {}^5C_2 \times {}^4C_3$

$= \frac{5!}{3!\,2!} \times \frac{4!}{3! \times 1!}$

$= \frac{5 \times 4}{2 \times 1} \times \frac{4 \times 3!}{3!}$

$= 10 \times 4 = 40.$

∴ Total ways $= 5 + 40 = 45.$

→ Permutation :-

$^nP_n = n!$

$^nP_r = n(n-1)(n-2)\dots(n-r+1)$

$= \frac{n(n-1)(n-2)\dots(n-r+1)(n-r)(n-r-1)\dots3.2.1}{(n-r)(n-r-1)\dots3.2.1}$

$\boxed{^nP_r = \frac{n!}{(n-r)!}}$

→ In word 'SUCCESS'

Permutation of S = $^3P_3 = 3! = 6$
Permutation of C = $^2P_2 = 2! = 2$

No. of distinct string that can be made $= \frac{7!}{2!\,3!}$

$= \frac{7\times6\times5\times\cancel{4}^2}{\cancel{2}\times1}$

$= 42\times10$

$= 420$

→ In word 'BENZENE' how many strings can be generated.

Permutation of E = $^3P_3 = 3! = 6$.
Permutation of N = $^2P_2 = 2! = 2$

No. of strings generated $= \frac{7!}{3!\,2!}$

$= 420$.

→ How many bit strings of length 10 contain.

① exactly four 1's
② at most four 1s
③ atleast four 1s
④ Equal no. of 0's & 1s

① No. of bit string having exactly ④ 1s $= \frac{10!}{6!\,4!}$

$= \frac{10\times\cancel{9}^3\times\cancel{8}\times7}{\cancel{4}\times\cancel{3}\times\cancel{2}\times1}$

$= 210$

② No. of bits having at mos four 1s $= {}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4$

$= \frac{10!}{10!} + \frac{10!}{9!\,1!} + \frac{10!}{8!\,2!} + \frac{10!}{7!\,3!} + \frac{10!}{6!\,4!}$

$= 1 + 10 + \frac{10\times9}{2} + \frac{10\times\cancel{9}\times\cancel{8}}{\cancel{3}\times\cancel{2}} + \frac{10\times\cancel{9}\times\cancel{8}\times7}{\cancel{4}\times\cancel{3}\times\cancel{2}\times1}$

$= 1+10+45+120+210$

$= 386$

Scanned by CamScanner

③ No. of bits having atleast 4 1s $= {}^{10}C_4 + {}^{10}C_5 + {}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}$

$= \frac{10!}{6!4!} + \frac{10!}{5!5!} + \frac{10!}{6!\times4!} + \frac{10!}{7!3!} + \frac{10!}{8!2!} + \frac{10!}{9!1!} + \frac{10!}{10!0!}$

$= 210 + 252 + 210 + 120 + 45 + 10 + 1$

$= 848$

④ No. of bits having exact 5 1's $= \frac{10!}{5!5!}$

$= 252$

Q) How many permutations of the letters ABCDEFGH contains ⓐ the string ED

ⓑ the string CDE

ⓒ the string BA & FGH

ⓓ the string AB, DE, & GH

ⓔ the string CAB and BED

ⓕ the string BCA and ABF

→ ⓐ String should contain ED, ∴ consider ED to be 1

∴ No. of permutation $= {}^7P_7 = 7! = 5040$

ⓑ String should contain CDE, ∴ consider CDE to be 1

∴ No. of permutation $= {}^6P_6 = 6! = 720$

ⓒ String should contain BA & FGH.

∴ BA, C, D, E, FGH

No. of permutations $= {}^5P_5 = 5! = 120$.

ⓓ String should contain AB, DE, GH.

AB, C, DE, F, GH.

No. of permutations $= {}^5P_5 = 5! = 120$.

ⓔ String should contain CAB & BED

CABED FGH.

No. of permutation $= {}^4P_4 = 4! = 24$.

ⓕ String should BCA and ABF

No. of permutations $= 0$.

Q) One hundred tickets numbered 1, 2, 3, ..., 100. are sold to 100 different people for drawing. Four different prizes are awarded, including a grand prize. How many ways are there to award the prizes if

ⓐ There is no restriction.

ⓑ The person holding 47 wins the grand prize.

ⓒ the person holding 47 wines 1 of the prizes.

(d) the person holding 47 doesnot win a prize.
(e) The person holding ticket 19 & 47 both win prizes.
(f) the people holding tickets 19, 47 & 73 all win prizes.
(g) the people holding 19, 47, 73 & 97 win a prize.
(h) None of the people 19, 47, 73 & 97 win a prize.
(i) the grand prize winner is a person holding the ticket 19, 47, 73 & 97.
(j) 19 & 47 win prize but not 73 & 97.

a) No restriction = ${}^{100}P_4 = 100 \times 99 \times 98 \times 97$
$= 94109400$.

b) ${}^{99}P_3 = 1 \times 99 \times 98 \times 97 = 941094$.
47 can get only 1 prize.

c) $4 \times {}^{99}P_3 = 4 \times 941094 = 3764376$

d) ${}^{99}P_4 = 90345024$

e) $4 \times 3 \times {}^{98}P_2 = 114072$

f) $4 \times 3 \times 2 \times {}^{97}P_1 = 24 \times 97 = 2328$.

g) $4 \times 3 \times 2 \times 1 = 24$

h) ${}^{96}P_4 = 79727040$.

i) $4 \times {}^{96}P_3 = 3429120$.

j) $4 \times 3 \times {}^{96}P_2 = 12 \times 96 \times 95 = 109440$.

PROBABILITY

$0 \le P(E) \le 1$

$P(E) = \frac{|E|}{|S|} = \frac{\text{Probable outcomes}}{\text{sample spaces}}$

Sample spaces :-

1) 2 flips of a coin.

HH, HT, TH, TT.

2) 3 flips of a coin.

HHH, HHT, HTT, THH, TTH, TTT, HTH, THT.

$\Rightarrow |E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$

$\frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|}$

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Q) What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?

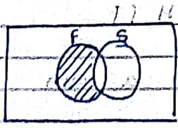
$$|F \cup S| = |F| + |S| - |F \cap S|$$

$$p(F \cup S) = p(F) + p(S) - p(F \cap S)$$

$$= \frac{20}{100} + \frac{14}{100} - \frac{2}{100}$$

$$= \frac{32}{100} = 0.32$$

Q) Find the probability of a randomly selected integer which is divisible by 5 but not by 7 from 1 to 100.



$$P(\text{only divisible by 5}) = P(F) - P(S \cap F)$$

$$= \frac{20}{100} - \frac{2}{100} = \frac{18}{100} = 0.18$$

Q) Suppose that 100 people enter a contest and that different winners are selected at random for first, second & third prize. What is the probability that John wins one of these prizes if he is one of the contestent.

$$\text{John} = \frac{3}{3 \times 99 \times 98} \qquad \frac{3}{3 \times 99 \times 98} = \frac{1}{99 \times 98}$$

Solve

1) $a_r + 4a_{r-1} + 3a_{r-2} = 4$ $a_0 = 5.5$, $a_1 = -8.5$

Take $a_r = bx^r$

$\therefore bx^r + 4bx^{r-1} + 3bx^{r-2} = 4$

Taking homogeneous eqn.

$bx^r + 4bx^{r-1} + 3bx^{r-2} = 0$

$\div$ by bx^{r-2}

$\therefore x^2 + 4x + 3 = 0$

$x^2 + 3x + x + 3 = 0$

$x(x+3) + 1(x+3) = 0$

$(x+3)(x+1) = 0$

$x = -3, -1$

$\therefore a_r^{(h)} = b_1(-3)^r + b_2(-1)^r$

$a_r^{(p)} = \cancel{an(4)}\ c$

$\therefore c + 4c + 3c = \cancel{8}4$

$8c = 4$

$c = \frac{1}{2} = 0.5$

$\therefore a_r^{p} = 0.5$

$\therefore a_r = a_r^{(h)} + a_r^{(p)}$

$= b_1(-3)^r + b_2(-1)^r + 0.5$

$a_r = 2(-3)^r + 3(-1)^r + 0.5$

$a_0 = 2 + 3 + 0.5 = 5.5$

$a_1 = -6 - 3 + 0.5 = -8.5$

2) $a_r \cancel{+} 6a_{r-1} - 9a_{r-2} + 5.2^r$

$\therefore a_r - 6a_{r-1} + 9a_{r-2} = 0$ (Homogeneous).

$a_r = bx^r$

$bx^r - 6bx^{r-1} + 9bx^{r-2} = 0$

$x^2 - 6x + 9 = 0$

$x^2 - 3x - 3x + 9 = 0$

$x(x-3) - 3(x-3) = 0$

$x = 3, 3$

$\therefore a_r^{(h)} = (b_1 + b_2 r)3^r$

$a_r^{(p)} = c.2^r$

$\therefore c.2^r - 6c2^{r-1} + 9c.2^{r-2} = 5.2^r$

$\therefore \div$ by 2^{r-2}

$\therefore c.2^2 - 6c.2^1 + 9c = 5(2)^2$

$\therefore 4c - 12c + 9c = 5\times4$

$-8c + 9c = 20$

$c = 20$

$\therefore a_r^{(p)} = 2^r(20)$.

$a_r = a_r^{(h)} + a_r^{(p)}$

$a_r = (b_1 + b_2 r)3^r + (20)2^r$

Q) An urn contains 4 blue balls & 5 red balls. What is the probability that a ball choosen from the urn is blue.

A) $|S| = 9$

$|E| = 4$

$p(E) = \frac{|E|}{|S|} = \frac{4}{9} = 0.444$

Q) What is the probability that when two dice are rolled, the sum of numbers on two dice is 7?

A) $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$|S| = 36$

$|E| = 6$

$p(E) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6}$

Q) What probabilities should be assign to the outcomes H(heads) and T (tail) when a fair coin is flipped? What will be the probability of the outcomes when a biased coin is flipped so that head comes up twice as often as tails.

→ For fair coin

$p(H) = 0.5$

$p(T) = 0.5$

For biased coin

Let no. of tail $= x$.

∴ No. of head $= 2x$.

∴ $|S| = 2x + x = 3x$.

∴ $p(T) = \frac{x}{3x} = \frac{1}{3}$

$p(H) = \frac{2x}{3x} = \frac{2}{3}$

$p(E|F) = \frac{P(E \cap F)}{P(F)}$

Q) A bit string of length four is generated random so that each of the 16 bit strings of length four is equally likely. What is the probability that

What probabilities can be assigned to each outcome when a die (biased) is rolled, if a 3 is twice as likely as each of the five numbers on the dice.

$|1| = x = |2| = |4| = |5| = |6|$

$|3| = 2x$

$p(1) = \frac{x}{7x} = \frac{1}{7}$

$p(3) = \frac{2x}{7x} = \frac{2}{7}$

Q) What is the probability of these events when we randomly select a permutation of $\{1,2,3\}$?

a) 1 preceeds 3.
b) 3 preceeds 1.
c) 3 preceeds 1 & 3 preceeds 2.

$S = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$

$E_1 = \{(1,2,3), (1,3,2), (2,1,3)\}$

$p(E_1) = \frac{3}{6} = \frac{1}{2} = 0.5$

$E_2 = \{(2,3,1), (3,1,2), (3,2,1)\}$

$p(E_2) = \frac{3}{6} = \frac{1}{2} = 0.5$

$E_3 = \{(3,1,2), (3,2,1)\}$

$p(E_3) = \frac{2}{6} = \frac{1}{3}$

Q) What is the probability of these events we randomly select a permutation of 26 lowercase letters of the English alphabet.

${}^{26}P_{26} = 26!$

z is first letter

$p(b) = \frac{25!}{26!} = \frac{1}{26}$

$p(c) = \frac{25!}{2.26!} = \frac{1}{2 \times 26} = \frac{1}{52}$

TUTORIAL

1) b) i) 1, 2, 3, 4, 5, ...

$$G(x) = 1+2x+3x^2+4x^3+5x^4+\dots$$

Take $H(x) = 1+x+x^2+x^3+x^4+\dots = \frac{1}{1-x}$

Diff. both sides

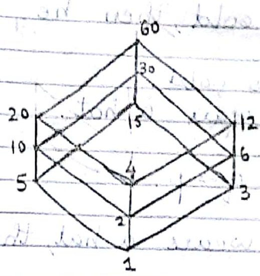
$$1+2x+3x^2+4x^3+5x^4+\dots = \frac{0-1(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$\therefore G(x) = \frac{1}{(1-x)^2}$$

ii) 0, 0, 0, 3, 3, 3, 3, 3, ...

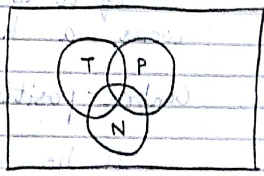
$$G(x) = 0+0x+0x^2+3x^3+3x^4+3x^5+3x^6+\dots$$
$$= 3x^3\left[1+x+x^2+x^3+x^4+\dots\right]$$
$$= 3x^3\left[\frac{1}{1-x}\right] = \frac{3x^3}{(1-x)}.$$

c) D60 :-



It is a lattice.

Q2) a)



$|T| = 88$
$|P| = 73$
$|N| = 46$
$|T\cap P| = 34$
$|T\cap N| = 16$
$|P\cap N| = 12$
$|T\cap P\cap N| = 5$

Total no. of people surveyed = 166.

$|T\cup P\cup N| = |T|+|P|+|N| - |T\cap P| - |T\cap N| - |P\cap N| + |T\cap N\cap P|$
$= 88+73+46-34-16-12+5$
$= 150$.

No. of people who got news = 150

i) No. of people got news other than TV or paper or news $= |U| - |T\cup P\cup N|$
$= 166 - 150$
$= 16$ people.

ii) Only through news magazine $= |N| - |N\cap T| - |N\cap P| + |N\cap P\cap T|$
$= 46 - 16 - 12 + 5$
$= 23$ people.

b) i) If it is cold, then he wears a hat.

p : It is cold.
q : he wears a hat.

Converse :- $q \to p$

It he wears a hat, then it is cold

Inverse :- $\sim p \to \sim q$

If it is ~~m~~not cold, then he does not wear a hat

Contra-positive :- $\sim q \to \sim p$.

If he doesnot wear a hat, then it is not cold.

ii) If an integer is a multiple of 2, then it is an even

p : An integer is a multiple of 2,
q : it is an even

Converse :- $q \to p$

If it is even, then an integer is a multiple of 2.

Inverse :- ~~q~~ $\sim p \to \sim q$

If an integer is not a multiple of 2 then it is not even.

Contrapositive :- $\sim q \to \sim p$.

If it is not even, then an integer ~~it~~ is not a multiple of 2.

c) $f_1(x) = x$

$f_2(x) = 1 - x$

$f_3(x) = \dfrac{x}{x-1}$

$f_4(x) = \dfrac{1}{x}$

$f_5(x) = \dfrac{1}{1-x}$

$f_6(x) = 1 - \dfrac{1}{x} = \dfrac{x-1}{x}$

o	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_5	f_6	f_3	f_4
f_3	f_3	f_6	f_1	f_5	f_4	f_2
f_4	f_4	f_5	f_6	f_1	f_2	f_3
f_5	f_5	f_4	f_2	f_3	f_6	f_1
f_6	f_6	f_3				

$f_1 \circ f_1 = f_1(f_1(x)) = f_1(x) = x$.

$f_1 \circ f_2 = f_1(f_2(x)) = f_1(1-x) = 1 - x = f_2$.

$f_2 \circ f_1 = f_2(f_1(x)) = f_2(x) = f_2$

$f_2 \circ f_2 = f_2(f_2(x)) = f_2(1-x) = 1-(1-x) = x = f_1$

$f_2 \circ f_3 = f_2(f_3(x)) = f_2\left(\dfrac{x}{x-1}\right) = 1 - \dfrac{x}{x-1} = \dfrac{x-1-x}{x-1}$

$= \dfrac{-1}{x-1} = \dfrac{1}{1-x}$

$= f_5$

$$f_2 \circ f_4 = f_2(f_4(x)) = f_2\left(\frac{1}{x}\right) = 1 - \frac{1}{x} = \frac{x-1}{x}$$

$$f_2 \circ f_5 = f_2(f_5(x)) = f_2\left(\frac{1}{1-x}\right) = 1 - \frac{1}{1-x} = \frac{1-x-1}{1-x} = \frac{-x}{1-x} = \frac{x}{x-1}$$

$$f_2 \circ f_6 = f_2(f_6(x)) = f_2\left(\frac{x-1}{x}\right) = 1 - \frac{x-1}{x} = \frac{x-x+1}{x} = \frac{1}{x} = f_4$$

$$f_3(f_1(x)) = f_3(x) = f_3$$

$$f_3 \circ f_2 = f_3(f_2(x)) = f_3(1-x) = \frac{1-x}{1-x-1} = \frac{1-x}{-x} = \frac{x-1}{x} = f_6$$

$$f_3 . f_3 = f_3(f_3(x)) = f_3\left(\frac{x}{x-1}\right) = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{\frac{x}{(x-1)}}{\frac{x-x+1}{(x-1)}}$$

$$= \frac{x}{1} = f_1$$

$$f_3(f_4(x)) = f_3\left(\frac{1}{x}\right) = \frac{1/x}{\frac{1}{x} - 1} = \frac{1/x}{(1-x)/x} = \frac{1}{1-x} = f_5.$$

$$f_3(f_5(x)) = f_3\left(\frac{1}{1-x}\right) = \frac{\frac{1}{1-x}}{\frac{1}{1-x} - 1} = \frac{\frac{1}{1-x}}{\frac{1-1+x}{1-x}} = \frac{1}{x} = f_4.$$

$$f_4(f_2(x)) = f_4(1-x) = \frac{1}{1-x} = f_5.$$

$$f_4(f_3(x)) = f_4\left(\frac{x}{x-1}\right) = \frac{x-1}{x} = 1 - \frac{1}{x} = f_6$$

$$f_4(f_4(x)) = f_4\left(\frac{1}{x}\right) = x = f_1$$

$$f_4(f_5(x)) = f_4\left(\frac{1}{1-x}\right) = 1-x = f_2$$

Tutorial

1) Let $H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be parity check matrix

Determine the group code $e_H : B^2 \to B^5$

$e(00) = 00x_1x_2x_3$

$x_1 = 0.1 +_2 0.1 = 0$

$x_2 = 0.0 +_2 0.1 = 0$

$x_3 = 0.0 +_2 0.0 = 0$

$e(00) = 00000$

$e(01) = 01x_1x_2x_3$

$x_1 = 0.1 +_2 1.1 = 1$

$x_2 = 0.0 +_2 1.1 = 1$

$x_3 = 0.0 +_2 1.0 = 0$

$e(01) = 01110$

$e(10) = 10x_1x_2x_3$

$x_1 = 1.1 +_2 0.1 = 1$

$x_2 = 1.0 +_2 0.1 = 0$

$x_3 = 1.0 +_2 0.0 = 0$

$e(10) = 10100$

$e(11) = 11x_1x_2x_3$

$x_1 = 1.1 +_2 1.1 = 0$

$x_2 = 1.0 +_2 1.1 = 1$

$x_3 = 1.0 +_2 1.0 = 0$

2) Let G be the set of real numbers and let ~~G be the~~ $a * b = ab/2$. Show that $(G, *)$ is an Aabelian group

① $a*b = ab/2 \in G$.

$\therefore$ Closure

② T.P:- $(a*b)*c = a*(b*c)$.

LHS:- $(a*b)*c$

$= \frac{ab}{2} * c$

$= \frac{abc}{2.2} = \frac{abc}{4}$

RHS $= a*(b*c)$

$= a*\left(\frac{bc}{2}\right) = \frac{abc}{4}$

LHS = RHS

$\therefore$ associative.

③ $a*e = a$

$a*e = \frac{ae}{2} = a$

$\therefore e = 2$

Identity.

4) $a * a^{-1} = e$

$$\frac{a\,a^{-1}}{2} = 2$$

$$a^{-1} = \frac{4}{a}$$

Ans : It is a group.

For group to be ~~same~~ Abelion, it has to be commutative.

T.P :- $a*b = b*a$

LHS :- $a*b$
$= \frac{ab}{2}$

RHS :- $b*a$
$= \frac{ba}{2} = \frac{ab}{2}$

LHS = RHS.

∴ Abelion Group.

3) $G = \{1,2,3,4,5,6\}$ modulo 7.

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

4) $(ab)^2 = a^2b^2$

$a, b * G$

Let G be an Abelian group.

i) $(ab)^2 = (ab)(ab)$
$= a(b(ab))$
$= a((ba)b)$
$= a((ab)b)$
$= a(ab^2)$
$= (aa)b^2$
$= a^2b^2$