MODULE 6 :- PROBABILITY

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Basics of Probability :-

(1) If 'A' is an event and 'S' is the $\underbrace{\text{sample space}}_{\text{Total cases}}$

then $\boxed{P(A) = \dfrac{n(A)}{n(S)}}$

(2) $\boxed{0 \le P(A) \le 1}$

(3) If 'A' is an event and '$\bar{A}$' is complementry event (opposite event)

then $\boxed{P(A) + P(\bar{A}) = 1} \Rightarrow \boxed{P(\bar{A}) = 1 - P(A)}$

(4) Conditional Probability :-

$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$

Intersection (common)

{Probability of A when B has occured}

OR

$$\boxed{P(B/A) = \frac{P(A \cap B)}{P(A)}}$$

Probability of B when A has occured.

PROBABILITY DISTRIBUTION

DISCRETE

FORMULAE :-

(1) Sum of all Probabilities is ONE.

$$\sum P(X=x) = 1$$

p.d.f. — Probability density fn

(2) Expectation (Mean) :-

$$E(X) = \sum x \cdot P(X=x)$$

(3) Variance {Var(x) or V(x) or σ^2} :-

$$\boxed{Var(x) = E(x^2) - [E(x)]^2}$$

where,

$$E(x^2) = \sum x^2 P(X=x)$$

(4) Standard deviation (S.D.) or σ :-

$$\sigma = +\sqrt{Var(x)}$$

(5) Median (M) :-

$$P(X<M) = P(X \geq M) = \frac{1}{2}$$

CONTINUOUS

FORMULAE :-

(1) Sum of all Probabilities is ONE.

$$\int_a^b f(x)\, dx = 1$$

p.d.f.

$$P(m<X<n) = \int_m^n f(x)\, dx$$

(2) Expectation (Mean) :-

$$E(X) = \int_a^b x \cdot f(x)\, dx$$

(3) Variance {Var(x) or V(x) or σ^2 :-

$$\boxed{Var(x) = E(x^2) - [E(x)]^2}$$

where,

$$E(x^2) = \int_a^b x^2 f(x)\, dx$$

(4) Standard deviation (S.D.) or σ :-

$$\sigma = +\sqrt{Var(x)}$$

(5) Median (M) :-

$$P(X<M) = P(X \geq M) = \frac{1}{2}$$

(6) Harmonic Mean {H.M.}

$$\frac{1}{H} = \int_a^b \frac{1}{x} f(x)\, dx$$

PROBLEMS ON DISCRETE DISTRIBUTION

(1) The probability density function of a random variable X is

X	0	1	2	3	4	5	6
P(X = x)	k	3k	5k	7k	9k	11k	13k

Find : (i) k, (ii) $P(X \le 4)$, (iii) $P(3 < X \le 6)$ (iv) $E(X)$ (v) $V(X)$ (vi) σ (vii) $P(X \ge 4 / 2 < X \le 5)$ (viii) c.d.f.

A, B

Soln:-

(i) As, sum of all probabilities is ONE.

$$\therefore \sum P(X=x) = 1$$

$$\therefore k + 3k + 5k + 7k + 9k + 11k + 13k = 1$$

$$\therefore 49k = 1 \qquad \therefore \boxed{k = \frac{1}{49}}$$

$\therefore$ p.d.f. is,

X	0	1	2	3	4	5	6
P(X=x)	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$

(ii) $P(X \le 4) = 1 - P(X > 4)$

$$= 1 - [P(X = 5, 6)]$$

$$= 1 - \left[\frac{11+13}{49}\right] = 1 - \frac{24}{49}$$

$$= \frac{25}{49}$$

$$\therefore \boxed{P(X \le 4) = 0.5102}$$

(iii) $P(3 < X \le 6) = P(X = 4, 5, 6)$

$$= \frac{9 + 11 + 13}{49} = \frac{33}{49}$$

$$\therefore \boxed{P(3 < X \le 6) = 0.6734}$$

(vii) $\because P(X \ge 4 / 2 < X \le 5) = \dfrac{P\{X \ge 4 \cap 2 < X \le 5\}}{P(2 < X \le 5)}$

conditional probability

$$= \frac{P\{X = 4,5,6 \cap X = 3,4,5\}}{P(2 < X \le 5)}$$

$$= \frac{P\{X = 4, 5\}}{P\{X = 3, 4, 5\}}$$

$$= \frac{\left(\frac{9+11}{49}\right)}{\left(\frac{7+9+11}{49}\right)} = \frac{20}{27}$$

$$= 0.7407$$

(viii) C.d.f. {Cumulative distribution function} :-

X	0	1	2	3	4	5	6
F(X=x)	$\frac{1}{49}$	$\frac{4}{49}$	$\frac{9}{49}$	$\frac{16}{49}$	$\frac{25}{49}$	$\frac{36}{49}$	1

$$\left(\frac{1}{49} + \frac{3}{49}\right)$$

$$\left(\frac{1}{4} + \frac{3}{49} + \frac{5}{49}\right)$$

(iv) Expectation:-

$$\because E(X) = \sum x \cdot P(X = x)$$

$$\therefore E(X) = \frac{0 \times 1 + 1 \times 3 + 2 \times 5 + 3 \times 7 + 4 \times 9 + 5 \times 11 + 6 \times 13}{49}$$

$$\therefore \boxed{E(X) = 4.1428}$$

(v) Variance:-

$$\because \boxed{V(X) = E(X^2) - [E(X)]^2} \longrightarrow (1)$$

where,

$$E(X^2) = \sum x^2 P(X = x)$$

$$= \frac{0^2 \times 1 + 1^2 \times 3 + 2^2 \times 5 + 3^2 \times 7 + 4^2 \times 9 + 5^2 \times 11 + 6^2 \times 13}{49}$$

$$\therefore \boxed{E(X^2) = 19.8571}$$

$\therefore$ from (1),

$$V(X) = 19.8571 - (4.1428)^2$$

$$\therefore \boxed{V(X) = 2.6943}$$

(vi) S.D.

$$\sigma = +\sqrt{Var(X)}$$

$$\therefore \sigma = +\sqrt{2.6943}$$

$$\therefore \boxed{\sigma = 1.6410}$$

(2) A random variable X has the following probability function

X:	1	2	3	4	5	6	7
P (X):	k	2k	3k	k^2	$k^2 + k$	$2k^2$	$4k^2$

find : (i) k (ii) P(X < 5) (iii) P(X > 5) (iv) P (0 ≤ X ≤ 5)

(v) $\bar{x}$ (vi) σ_x^2 (vii) σ_x

E(x)/Mean/Expectation

soln:

(i) Sum of all Probab's is ONE.

$\therefore \sum P(X=x) = 1$

$\therefore k + 2k + 3k + k^2 + k^2 + k + 2k^2 + 4k^2 = 1$

$\therefore 8k^2 + 7k - 1 = 0$

$\therefore k = \frac{1}{8}, -1$

But $0 \le P(X=x) \le 1$

$\therefore$ we discard $k = -1$

$\therefore k = \frac{1}{8}$

$\therefore$ P.d.f. is,

X	1	2	3	4	5	6	7
P(X=x)	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{2}{64}$	$\frac{4}{64}$

(ii) $P(X<5) = P(X=1,2,3,4)$

$= \frac{1}{8} + \frac{2}{8} + \frac{3}{8} + \frac{1}{64}$

$= 0.7656$

(iii) $P(X>5) = P(X=6,7)$

$= \frac{2}{64} + \frac{4}{64} = \frac{6}{64}$

$= 0.0937$

(iv) $P(0 \le X \le 5) = P(X=0,1,2,3,4,5)$

$= 1 - P(X>5)$

$= 1 - [P(X=6,7)]$

$= 1 - 0.0937$

$= 0.9064$

(v) $E(X) / \bar{x}$:-

$\because \bar{x} = \sum x \cdot P(X=x)$

$\therefore \bar{x} = 1 \times \frac{1}{8} + 2 \times \frac{2}{8} + 3 \times \frac{3}{8} + 4 \times \frac{1}{64} + 5 \times \frac{9}{64} + 6 \times \frac{2}{64} + 7 \times \frac{4}{64}$

$\therefore \bar{x} = 3.1406$

(vi) $Var(x)\ (\sigma_x^2)$:- $\because \boxed{Var(x) = E(x^2) - [E(x)]^2}$ — (1)

where,

$E(x^2) = \sum x^2 P(X=x)$

$\therefore E(x^2) = 1^2 \times \frac{1}{8} + 2^2 \times \frac{2}{8} + 3^2 \times \frac{3}{8} + 4^2 \times \frac{1}{64} + 5^2 \times \frac{9}{64} + 6^2 \times \frac{2}{64} + 7^2 \times \frac{4}{64}$

$\therefore \boxed{E(x^2) = 12.4531}$

$\therefore$ from (1),

$Var(x) = 12.4531 - (3.1406)^2$

$\therefore \boxed{Var(x) = 2.5897}$

(vii) σ_x :- (S.D.)

$\because \sigma_x = +\sqrt{Var(x)}$

$\therefore \sigma_x = +\sqrt{2.5897}$

$\therefore \boxed{\sigma_x = 1.6092}$

(3) Find the value of K for the following data : (P(X) is the probability density function)

X :	0	10	15
P(X) :	$\frac{K-6}{5}$	$\frac{2}{K}$	$\frac{14}{5K}$

Also find the cumulative probability distribution function and mean of X.

Soln:

As we know,

$$\sum P(X=x) = 1$$

$$\therefore \frac{(K-6)^{\times K}}{5_{\times K}} + \frac{2^{\times 5}}{K_{\times 5}} + \frac{14}{5K} = 1$$

$$\therefore \frac{K^2-6K+10+14}{5K} = 1$$

$$\therefore K^2-6K+24 = 5K$$

$$\therefore K^2-11K+24 = 0$$

$$\therefore K = 3, 8$$

But for K=3, P(X=0) becomes negative

$\therefore$ we discard K=3.

$$\therefore \boxed{K=8}$$

$\therefore$ p.d.f. is,

X	0	10	15
P(X=x)	$\frac{2}{5}$	$\frac{2}{8}$	$\frac{7}{20}$

Now c.d.f. is given by.

X	0	10	15
F(X=x)	$\frac{2}{5}$	$\frac{13}{20}$	1

$\frac{2}{5}+\frac{2}{8}$ $\qquad$ $\frac{2}{5}+\frac{2}{8}+\frac{7}{20}$

$\rightarrow$ E(X) = FOR YOU !!
(Mean)

(4) If the following distribution of a discrete random variable X has mean = 16 then find m, n and the variance of X. **[D-10; CIVIL : D-13; MECH/AUTO : M-17]**

X	8	12	16	20	24
P(X = x)	1/8	m	n	¼	1/12

[**Ans.:** m = 1/6, n = 3/8]

Soln:-

As we know,

$$\sum P(X=x) = 1$$

$$\therefore \frac{1}{8} + m + n + \frac{1}{4} + \frac{1}{12} = 1$$

$$\therefore m+n+\frac{11}{24} = 1$$

$$\therefore \boxed{m+n = \frac{13}{24}} \rightarrow ①$$

Now we have,

mean = 16 .... (given)

$$\therefore E(X) = 16$$

$$\therefore \sum x \cdot P(X=x) = 16$$

$$\therefore \cancel{8}\times\frac{1}{\cancel{8}} + 12m + 16n + \cancel{20}^{5}\times\frac{1}{\cancel{4}} + \cancel{24}^{2}\times\frac{1}{\cancel{12}} = 16$$

$$\therefore 12m+16n = 8$$

$$\therefore \boxed{3m+4n=2} \longrightarrow ②$$

Solving ① & ② in calc,

$$m = \frac{1}{6} \quad , \quad n = \frac{3}{8}$$

$\therefore$ p.d.f. is,

X	8	12	16	20	24
P(X=x)	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{12}$

FOR YOU !!

(5) The probability density function of a random variable X is zero expect at X = 0, 1, 2, and P (0) = $3C^3$, P(1) = $4C - 10C^2$, P(2) = 5C – 1. Find (i) C and (ii) P ($0 < X \le 2$).
[**Ans.:** (i) C = 1/3, (ii) 8/9] **[INFT : D-18; EXTC : M-18, D-19]**

Soln:- ∵ p.d.f. is,

X	0	1	2
P(X=x)	$3c^3$	$4c-10c^2$	$5c-1$

As we know,

$$\sum P(X=x) = 1$$

$$\therefore 3c^3 + 4c - 10c^2 + 5c - 1 = 1$$

$$\therefore 3c^3 - 10c^2 + 9c - 2 = 0$$

$$\therefore c = 2, 1, \frac{1}{3}$$

But for c = 2, 1, P(X=0) > 1

∴ we discard c = 2, 1

$$\therefore c = \frac{1}{3}$$

↓

FOR YOU!!

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(6) If the random variable X takes the values 1, 2, 3, 4 such that 2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4), find the probability distribution and cumulative distribution of X.

Soln:- Here we have,

$$2 \cdot P(X=1) = 3P(X=2) = P(X=3) = 5 \cdot P(X=4)$$

Let $P(X=3) = K$

→ ∴ $2P(X=1) = P(X=3)$

$\therefore P(X=1) = \frac{K}{2}$

→ $3P(X=2) = P(X=3)$

$\therefore P(X=2) = \frac{K}{3}$

→ $5 \cdot P(X=4) = P(X=3)$

$\therefore P(X=4) = \frac{K}{5}$

∴ p.d.f. is,

X	1	2	3	4
P(X=x)	$\frac{K}{2}$	$\frac{K}{3}$	K	$\frac{K}{5}$

↓

For You !!

HOMEWORK PROBLEMS :-

(7) The probability density function of a random variable x is **[CMPN/INFT : D-15; CMPN : M-18, D-18; MECH/AUTO/PROD/CIVIL : D-18]**

x :	–2	–1	0	1	2	3
P(x)	0.1	k	0.2	2k	0.3	k

Find : (i) k, (ii) mean, (iii) variance.
[**Ans.:** (i) 0.1, (ii) 0.8, (iii) 2.16]

(8) A random discrete variable x has the probability density function given **[INFT : M-18]**

x :	–2	–1	0	1	2	3
P(x) :	0.2	k	0.1	2k	0.1	2k

Find (i) k (ii) E(X) (iii) V(X)
[**Ans.:** (i) 0.2, (ii) 0.64, (iii) 3.3104]

(9) A discrete random variable has the probability distribution given below: **[CMPN/INFT : M-17]**

x	–2	–1	0	1	2	3
p(x)	0.2	k	0.1	2k	0.1	2k

Find k, the mean and variance
[**Ans.:** k = 0.12, Mean = 0.64, Variance = 3.3104]

(10) A discrete random variable has the probability density function given below **[MECH/AUTO : M-13; CMPN : D-19]**

X :	–2	–1	0	1	2	3
P(X = x) :	0.2	k	0.1	2k	0.1	2k

Find k, the mean and variance.
[**Ans.:** 0.12, 0.64, 3.3104]

(11) Given the following probability function of a discrete random variable X.

x :	0,	1,	2,	3,	4,	5,	6,	7,
P(x = x) :	0,	C,	2C,	2C,	3C,	C^2,	$2C^2$,	$7C^2 + C$

Find c, P(x ≥ 6) and P(X < 6). **[PROD : D-13]**

$$\left[\textbf{Ans.}: C = \frac{1}{10}, P(x \ge 6) = \frac{19}{100}, P(X < 6) = \frac{81}{100}\right]$$

(12) The number of messages sent per hour over a computer network has the following probability distribution **[ETRX : D-18]**

x	10	11	12	13	14	15
P(X = x)	0.08	3k	6k	4k	4k	0.07

[**Ans.:** k = 0.05, mean = 12.5]

Find the mean and variance of number of messages sent per hour.

PROBLEMS ON CONTINUOUS DISTRIBUTION

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13) The PDF of random variable X is given by

$f(x) = kx^2(2-x) \quad 0 \le x \le 2$
$= 0 \quad$ otherwise

Find : (i) k (ii) Mean (iii) Variance (iv) $P(1.5 \le X \le 2)$ (v) $P(X > 1)$ (vi) $P(X \le 1.5)$ **[D-12]**

equality doesn't matter, i.e. $P(X>2) = P(X \ge 2)$

[**Ans. :** $K = \frac{3}{4}$, Mean $= \frac{6}{5}$, Var. $= \frac{4}{25}$]

Soln:- $\because f(x) = kx^2(2-x) \quad \ldots\ldots \quad 0 \le x \le 2$
$= 0 \quad \ldots\ldots \quad$ otherwise

(i) As we know,

$$\int_0^2 f(x)\,dx = 1$$

$$\therefore k\int_0^2 (2x^2 - x^3)\,dx = 1$$

$$\therefore k\left[2\cdot\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = 1$$

$$\therefore k\left[\frac{16}{3} - 4\right] = 1$$

$$\therefore k\left(\frac{4}{3}\right) = 1$$

$$\therefore k = \frac{3}{4}$$

$\therefore$ p.d.f. is,

$$\boxed{\begin{aligned} f(x) &= \frac{3}{4}(2x^2 - x^3) \quad \ldots\ldots \quad 0 \le x \le 2 \\ &= 0 \quad \ldots\ldots \quad \text{otherwise} \end{aligned}}$$

(ii) Mean:-

$$\because E(x) = \int_0^2 x \cdot f(x)\,dx$$

$$= \int_0^2 \frac{3}{4}(2x^3 - x^4)\,dx$$

$$= \frac{3}{4}\left[2\frac{x^4}{4} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{3}{4}\left[8 - \frac{32}{5}\right]$$

$$= \frac{3}{\cancel{4}}\left[\frac{\cancel{8}^{\,2}}{5}\right] = \frac{6}{5}$$

$$\therefore \boxed{\text{Mean} = \frac{6}{5}}$$

(iii) Variance:-

$$\because \boxed{Var(x) = E(x^2) - [E(x)]^2} \rightarrow (1)$$

where,

$$E(x^2) = \int_0^2 x^2 f(x)\,dx$$

$$\therefore E(x^2) = \int_0^2 \frac{3}{4}(2x^4 - x^5)\,dx$$

$$\therefore E(x^2) = \frac{3}{4}\left[2\cdot\frac{x^5}{5} - \frac{x^6}{6}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{64}{5} - \frac{64}{6}\right]$$

$$= \frac{\cancel{3}}{\cancel{4}} \times \cancel{64}^{\,16} \times \frac{1}{\cancel{30}_{\,10}} = \frac{8}{5}$$

$$\therefore \boxed{E(x^2) = \frac{8}{5}}$$

$\therefore$ from (1),

$$Var(x) = \frac{8}{5} - \left(\frac{6}{5}\right)^2 = \frac{8}{5} - \frac{36}{25}$$

$$\therefore \boxed{Var(x) = \frac{4}{25}}$$

(iv) $P(1.5 \le X \le 2)$:-

$$= \int_{1.5}^2 f(x)\,dx$$

$$= \int_{1.5}^2 \frac{3}{4}(2x^2 - x^3)\,dx$$

$\downarrow$

(v) $P(X > 1)$:-

$$= \int_1^2 f(x)\,dx$$

$$= \int_1^2 \frac{3}{4}(2x^2 - x^3)\,dx$$

$\downarrow$

(vi) $P(X \le 1.5)$:-

$$= \int_0^{1.5} f(x)\,dx$$

$$= \int_0^{1.5} \frac{3}{4}(2x^2 - x^3)\,dx$$

$\downarrow$

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14) Let X be continuous random variable with probability distribution

$P(X = x) = \begin{cases} \frac{x}{6} + k, & \text{if } 0 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$ Evaluate k and find $P(1 \le x \le 2)$, $\bar{x}$, σ_x **[EXTC : D-18; ETRX : D-19]**

[Ans.: $k = 1/12$, $P(1 \le x \le 2) = 0.33$]

Soln: $\because$ P.d.f. is,

$$f(x) = \frac{x}{6} + K \quad \cdots\cdots \quad 0 \le x \le 3$$
$$= 0 \quad \cdots\cdots \quad \text{elsewhere.}$$

As we know,

$$\int_0^3 f(x)dx = 1$$

$$\therefore \int_0^3 \left(\frac{x}{6} + K\right) dx = 1$$

$$\therefore \left(\frac{x^2}{12} + Kx\right)_0^3 = 1$$

$$\therefore \frac{9}{12} + 3K = 1$$

$$\therefore \cancel{3}K = \frac{\cancel{3}}{12} \qquad \therefore K = \frac{1}{12}$$

Now,

$$P(1 \le X \le 2) = \int_1^2 f(x)dx$$

$$= \int_1^2 \left(\frac{x}{6} + \frac{1}{12}\right) dx$$

$$\downarrow$$

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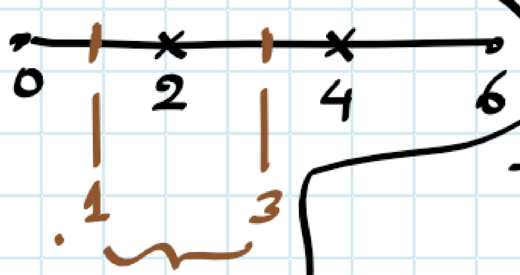
15) If X is a Random Variable with probability density function : **[CIVIL : M-13; MECH/AUTO/PROD/CIVIL : M-19]**

$f(x) = kx \qquad 0 \le x \le 2$
$= 2k \qquad 2 \le x \le 4$
$= 6k - kx \qquad 4 \le x \le 6$

Find k, expectation and $P(1 \le x \le 3)$ **[Ans.:** $k = 1/8$, Mean = 3, $P(1 \le x \le 3) = 0.4375$]

Soln: $\because f(x) = Kx \quad \cdots\cdots \quad 0 \le x \le 2$
$= 2K \quad \cdots\cdots \quad 2 \le x \le 4$
$= 6K - Kx \quad \cdots\cdots \quad 4 \le x \le 6$



As we know,

$$\int_0^6 f(x)dx = 1$$

$$\therefore \int_0^2 Kx\,dx + \int_2^4 2K\,dx + \int_4^6 (6K - Kx)dx = 1$$

$$\therefore K\left(\frac{x^2}{2}\right)_0^2 + 2K(x)_2^4 + \left(6Kx - K\frac{x^2}{2}\right)_4^6 = 1$$

$$\therefore 2K + 2K(2) + (36K - 18K) - (24K - 8K) = 1$$

$$\therefore 2K + 4K + 18K - 16K = 1$$

$$\therefore 8K = 1 \qquad \therefore K = \frac{1}{8}$$

$\therefore$ P.d.f. is,

$$f(x) = \frac{1}{8}x \quad \cdots\cdots \quad 0 \le x \le 2$$
$$= \frac{1}{4} \quad \cdots\cdots \quad 2 \le x \le 4$$
$$= \left(\frac{6}{8} - \frac{x}{8}\right) \quad \cdots \quad 4 \le x \le 6$$

Now,

$$E(x) = \int_0^6 x f(x) dx$$

$$\therefore E(x) = \int_0^2 \frac{1}{8}x^2 dx + \int_2^4 \frac{1}{4} \cdot x\,dx + \int_4^6 \left(\frac{6}{8}x - \frac{x^2}{8}\right) \cdot dx$$

$$\downarrow$$

Now,

$$P(1 \le X \le 3) = \int_1^3 f(x)dx$$

$$= \int_1^2 \frac{1}{8}x\,dx + \int_2^3 \frac{1}{4} \cdot dx$$

$$= \frac{1}{8}\left(\frac{x^2}{2}\right)_1^2 + \frac{1}{4}(x)_2^3$$

$$= \frac{1}{8}\left(2 - \frac{1}{2}\right) + \frac{1}{4}(1)$$

$$= \frac{1}{8}\left(\frac{3}{2}\right) + \frac{1}{4}$$

$$= \frac{3}{16} + \frac{1 \times 4}{4 \times 4}$$

$$= \frac{7}{16}$$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16) A continuous random variable X has the following probability law $f(x) = kx^2e^{-x}$, $x \geq 0$. Find k, mean and variance. also find $P(X \leq 2)$ & $P(X > 3)$ **[CMPN/INFT : D-14]**
[**Ans.:** k = 1/2, m = 3, Var(x) = 3]

$$\int_0^\infty e^{-x} x^n dx = \sqrt{n+1}$$

$\rightarrow \sqrt{n+1} = n! \quad \cdots n \in I$

$= n\sqrt{n} \quad \cdots n \notin I$

$\rightarrow \sqrt{\tfrac{1}{2}} = \sqrt{\pi}$

Soln:-

$\because$ P.d.f. is,

$$f(x) = kx^2 e^{-x} \quad \cdots\cdots \quad x \geq 0$$

As we know,

$$\int_0^\infty f(x)\,dx = 1$$

$$\therefore k\int_0^\infty e^{-x} x^2 dx = 1$$

$$\therefore k\left[\sqrt{3}\right] = 1 \quad \cdots \left\{ \text{By Gamma function} \right\}$$

$$\therefore k(2!) = 1$$

$$\therefore k = \frac{1}{2}$$

$\therefore$ P.d.f. is,

$$\boxed{f(x) = \frac{1}{2}x^2 e^{-x} \quad \cdots \quad x \geq 0}$$

Now,

$$E(X) = \int x \cdot f(x)\,dx$$

$$\therefore E(X) = \int_0^\infty \frac{1}{2} x^3 e^{-x} dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x} x^3 dx$$

$$= \frac{1}{2}\left[\sqrt{4}\right]$$

$$= \frac{1}{2}(3!) = \frac{1}{2}(6)$$

$$\therefore \boxed{E(X) = 3}$$

also,

$$\boxed{Var(X) = E(X^2) - [E(X)]^2} \longrightarrow ①$$

where,

$$E(X^2) = \int x^2 f(x)\,dx$$

$$\therefore E(X^2) = \int_0^\infty \frac{1}{2} x^4 e^{-x} dx$$

$$= \frac{1}{2}\left[\sqrt{5}\right] = \frac{1}{2}(4!)$$

$$= \frac{24}{2}$$

$$\therefore \boxed{E(X^2) = 12}$$

$\therefore$ from ①,

$$Var(X) = 12 - (3)^2$$

$$\therefore \boxed{Var(X) = 3}$$

Now,

$$P(X \leq 2) = \int_0^2 f(x)\,dx$$

$$= \int_0^2 \frac{1}{2} \underset{u}{x^2} \cdot \underset{v}{e^{-x}} dx$$

$$\therefore P(X \leq 2) = \frac{1}{2}\left[x^2\left(\frac{e^{-x}}{-1}\right) - (2x)\left(\frac{e^{-x}}{1}\right) + (2)\left(\frac{e^{-x}}{-1}\right)\right]_0^2 \quad ②$$

$$= \frac{1}{2}\left[\left\{-4e^{-2} - 4e^{-2} - 2e^{-2}\right\} - \left\{-2\right\}\right]$$

$$= \frac{1}{2}\left[-10e^{-2} + 2\right]$$

$$\therefore \boxed{P(X \leq 2) = 0.3233}$$

Similarly,

$$P(X > 3) = \int_3^\infty f(x)\,dx$$

$$= \int_3^\infty \frac{1}{2} \underset{u}{x^2} \cdot \underset{v}{e^{-x}} dx$$

$$= \frac{1}{2}\left[\quad - \quad \cdots \right]_3^\infty$$

$$e^{-\infty} \rightarrow 0$$

17) Define Random variable with an example. Find k if the following is a p.d.f.

$f(x) = k\,x\,e^{-4x^2}, \quad 0 \le x < \infty$ $\left[\textbf{Ans. :} \ k=8, \text{ Mean} = \frac{\sqrt{\pi}}{4}\right]$

Also find mean.

Soln:

$\because f(x) = k x e^{-4x^2} \ldots\ldots 0 \le x < \infty \ (x \ge 0)$

As we know,

$$\int_0^\infty f(x)\,dx = 1$$

$$\therefore K\int_0^\infty x\cdot e^{-4x^2}\cdot dx = 1$$

put $\boxed{4x^2 = t}$

$\therefore x^2 = \frac{t}{4}$

$\therefore x = \frac{t^{1/2}}{2}$

$\therefore dx = \frac{1}{2}\times\frac{1}{2}t^{-1/2}\cdot dt$

$$\therefore K\int_0^\infty \frac{\cancel{t^{1/2}}}{2}\cdot e^{-t}\cdot\frac{1}{4}\cancel{t^{-1/2}}\cdot dt = 1$$

$$\therefore \frac{K}{8}\int_0^\infty e^{-t}\cdot t^0\cdot dt = 1$$

$$\therefore \frac{K}{8}\left[\Gamma 1\right] = 1$$

$$\therefore \frac{K}{8} = 1$$

$$\therefore K = 8$$

$\therefore$ P.d.f is,

$$f(x) = 8xe^{-4x^2} \ldots\ldots x \ge 0$$

also,

$$E(x) = \int_0^\infty x\cdot f(x)\,dx$$

$$\therefore E(x) = 8\int_0^\infty x^2 e^{-4x^2}\cdot dx$$

using same substitution,

$$E(x) = 8\int_0^\infty \frac{t}{4}\cdot e^{-t}\cdot\frac{1}{4}t^{-1/2}dt$$

$$= \frac{\cancel{8}}{\cancel{16}_{2}}\int_0^\infty e^{-t}\cdot t^{1/2}\,dt$$

$$\therefore E(x) = \frac{1}{2}\left[\Gamma\tfrac{3}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\times\sqrt{\pi}\right]$$

$$\therefore \boxed{E(x) = \frac{\sqrt{\pi}}{4}} \rightarrow 0.4431$$

---

18) Let 'X' be a continuous random variable with p.d.f. $f(x) = kx\,(1-x),\ 0 \le x \le 1$. Find 'k' and determine a number 'b' such that $P(x \le b) = P(X \ge b)$. **[Ans.:** k = 6, b = 1/2]

[M-09, MECH/AUTO : M-15; BIOM : M-18,19; INST : M-19]

Soln:

$\because$ P.d.f is, (0 —— b —— 1)

$$\boxed{f(x) = K(x - x^2) \ldots\ldots 0 \le x \le 1}$$

$\downarrow$ Find 'K'

$$\boxed{K = 6}$$

$\therefore$ P.d.f is,

$$\boxed{f(x) = 6(x - x^2) \ldots\ldots 0 \le x \le 1}$$

Now we have to determine 'b' such that

$$P(x \le b) = P(x \ge b)$$

i.e. Number 'b' is nothing but "Median".

$$\therefore P(X \le b) = \frac{1}{2}$$

$$\therefore \int_0^b f(x)\,dx = \frac{1}{2}$$

$$\therefore \int_0^b 6(x - x^2)\,dx = \frac{1}{2}$$

$$\therefore 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b = \frac{1}{2}$$

$$\therefore 6\left[\frac{b^2}{2} - \frac{b^3}{3}\right] = \frac{1}{2}$$

$$\therefore 3b^2 - 2b^3 = \frac{1}{2}$$

$$\therefore 2b^3 - 3b^2 + \frac{1}{2} = 0$$

Solving in calc,

$\therefore b = -0.3660,\ 1.3660,\ \frac{1}{2}$

But Range of x is $0 \le x \le 1$

$\therefore$ we discard $b = -0.3660,\ 1.3660$

$$\therefore b = \frac{1}{2}$$

19) The length of time (in minutes), a lady speaks on telephone is found to be a random variable with probability density function

$f(x) = A\,e^{-x/5}$ for $x \ge 0$
$= 0$ elsewhere

find 'A' and the probability that she will speak for,
(i) more than 10 minutes, (ii) less than 5 minutes. [**Ans.:** A = 1/5, (i) $1 - e^{-2}$, (ii) e^{-1}]

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Soln:- $\because$ P.d.f. is,

$$\boxed{\begin{aligned} f(x) &= Ae^{-x/5} \;\dots\dots\; x \ge 0 \\ &= 0 \;\dots\dots\; \text{elsewhere} \end{aligned}}$$

As we know,

$$\int_0^\infty f(x)\,dx = 1$$

$$\therefore A\int_0^\infty e^{-x/5}\,dx = 1$$

$$\therefore A\left[\frac{e^{-x/5}}{-1/5}\right]_0^\infty = 1$$

$$\therefore A\left[\{0\} - \left\{\frac{1}{-1/5}\right\}\right] = 1$$

$$\therefore A(5) = 1$$

$$\therefore \boxed{A = \frac{1}{5}}$$

$\therefore$ p.d.f. is,

$$\boxed{\begin{aligned} f(x) &= \tfrac{1}{5}e^{-x/5} \;\dots\dots\; x \ge 0 \\ &= 0 \;\dots\dots\; \text{elsewhere} \end{aligned}}$$

(i) P(More than 10 minutes) :-

$$\therefore P(X > 10) = \int_{10}^\infty f(x)\,dx$$

$$= \int_{10}^\infty \frac{1}{5}e^{-x/5}\,dx$$

$$= \frac{1}{\cancel{5}}\left\{\frac{e^{-x/5}}{-1/\cancel{5}}\right\}_{10}^\infty$$

$$= \{0\} - \{-e^{-2}\}$$

$$= e^{-2}$$

$$= 0.1353 \rightarrow 13.53\%$$

(ii) P(less than 5 minutes) :-

$$\therefore P(X < 5) = \int_0^5 f(x)\,dx$$

---

20) The mileage (in thousands of miles) which car owners get with a certain kind of tyres is a random variable having p.d.f.

$\rightarrow$ 10 x 1000 miles

$f(x) = \frac{1}{20}e^{-x/20}$, for $x > 0$
$= 0$, for $x \le 0$

[**Ans.:** (i) $1 - e^{-0.5}$, (ii) $e^{-0.8} - e^{-1.2}$]

Find the probability that one of these tyres will last
(i) at most 10000 miles (ii) anywhere from 16000 to 24000 miles.

$P(X \le 10)$ $\qquad$ $P(16 \le X \le 24)$ $\leftarrow$ IMP*

At most $\rightarrow$ $\le$
At least $\rightarrow$ $\ge$
Exactly $\rightarrow$ $=$

---

21) The daily consumption of electric power (in millions of kWh) is a r.v. X with p.d.f.

$$f(x) = \begin{cases} kxe^{-x/3} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

[CMPN/INFT : M-14]

Find the value of k and the probability that on a given day the electric consumption is more than the expected electric consumption. [**Ans.:** k = 1/9, prob = $3\,e^{-2} = 0.4060$] $\rightarrow P[X > E(X)]$

(Expectation)

Now we have to calculate probability of electric consumption more than the expected electric consumption i.e. E(X).

Soln:- $\because$ P.d.f. is,

$$f(x) = \begin{cases} Kxe^{-x/3} \;\dots\dots\; x > 0 \\ 0 \;\dots\dots\; \text{elsewhere,} \end{cases}$$

As we know,

$$\int_0^\infty f(x)\,dx = 1$$

$$\therefore K\int_0^\infty \underset{u}{x}\cdot \underset{v}{e^{-x/3}}\,dx = 1$$

$$\therefore K\left[x\left(\frac{e^{-x/3}}{-1/3}\right) - (1)\left(\frac{e^{-x/3}}{1/9}\right)\right]_0^\infty = 1$$

$$\therefore K\left[\{0\} - \{-9\}\right] = 1$$

$$\therefore K = \frac{1}{9}$$

$\therefore$ P.d.f. is,

$$\boxed{f(x) = \begin{cases} \tfrac{1}{9}xe^{-x/3} \;\dots\; x > 0 \\ 0 \;\dots\; \text{elsewhere} \end{cases}}$$

Now we calculate

$$E(X) = \int_0^\infty x\cdot f(x)\,dx$$

$$\therefore E(X) = \frac{1}{9}\int_0^\infty x^2 e^{-x/3}\,dx$$

$$\therefore E(X) = \frac{1}{9}\left[x^2\left(\frac{e^{-x/3}}{-1/3}\right) - (2x)\left(\frac{e^{-x/3}}{1/9}\right) + (2)\left(\frac{e^{-x/3}}{-1/27}\right)\right]_0^\infty$$

$$= \frac{1}{9}\left[\{0\} - \{-54\}\right] = \frac{\cancel{54}}{\cancel{9}}\,6$$

$$\therefore \boxed{E(X) = 6}$$

$$\therefore P(X > E(X)) = P(X > 6)$$

$$= \int_6^\infty f(x)\,dx$$

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22) The diameter say X of an electric cable is assumed to be a continuous random variable with p.d.f. ; $f(x) = 6x(1 - x)$; $0 \le x \le 1$. **[M-11]**
(i) Is it probability distribution function? → check $\int f(x) \leftrightarrow$ it should be one.
(ii) Obtain cumulative distribution function. → c.d.f. $F_X(X = x) = \int_0^x f(x)\,dx$
(iii) Compute $P\left\{X \le \frac{1}{2} \Big/ \frac{1}{3} \le X \le \frac{2}{3}\right\}$ (Rare)
(iv) Determine K, so that $P(X < K) = P(X > K)$. Median
(v) H.M.

[**Ans.:** (i) Yes, (ii) $F(x) = 3x^2 - 2x^3$, (iii) 0.499, (iv) $K = 1/2$]

Soln:- $\boxed{f(x) = 6(x - x^2) \;\cdots\; 0 \le x \le 1}$

$$F_X(X = x) = \int_0^x f(x)\,dx = \int_0^x 6(x - x^2)\,dx = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^x = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right) = (3x^2 - 2x^3)$$

(iii) $$P\Big\{\underbrace{X \le \tfrac{1}{2}}_{A} \Big/ \underbrace{\tfrac{1}{3} \le X \le \tfrac{2}{3}}_{B}\Big\} = \frac{P\left\{X \le \frac{1}{2} \cap \frac{1}{3} \le X \le \frac{2}{3}\right\}}{P\left\{\frac{1}{3} \le X \le \frac{2}{3}\right\}}$$

$$= \frac{P\left\{\frac{1}{3} \le X \le \frac{1}{2}\right\}}{P\left\{\frac{1}{3} \le X \le \frac{2}{3}\right\}}$$

$$= \frac{\int_{1/3}^{1/2} f(x)\,dx}{\int_{1/3}^{2/3} f(x)\,dx} \longrightarrow$$

HOMEWORK PROBLEMS:-

23) For a continuous random variable 'x' its probability density function given by
$f(x) = k(2 - x)$ $\quad 0 < x < 2$
$= kx(x - 2)$ $\quad 2 \le x \le 3$
$= 0$ $\quad$ otherwise
Find : (i) k, (ii) mean and (iii) median for the distribution. [**Ans.:** (i) 0.3, (ii) 1.475, (iii) 1.18]

24) The probability density function of a random variable X is $f(x) = kx^2(1 - x^3)$, $0 \le x \le 1$.
Find k, expectation and variance of x. **[MECH/AUTO/PROD/CIVIL : M-18]**
[**Ans.:** k = 6, mean = 9/14, Variance = 9/245]

25) If x is a continuous random variable with the probability density function given by
$$f(x) = \begin{cases} k(x - x^3) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$ **[CMPN/INFT : M-16]**
Find : (i) k, (ii) the mean of the distribution. [**Ans.:** (i) k = 4, (ii) mean = 8/15]

26) Find k and then E(x) for the p.d.f. **[CMPN/INFT : D-16]**
$$f(x) = \begin{cases} k(x - x^2), & 0 \le x \le 1,\ k > 0 \\ 0 & \text{, otherwise} \end{cases}$$ [**Ans.:** k = 6, E(x) = 1/2]

27) A continuous random variable 'x' has probability density function $f(x) = kx^3$, $0 \le x \le 1$, hence find k, mean and $P(0.3 < x < 0.6)$. **[BIOM : D-19]**
[**Ans.:** k = 4, mean = 4/5, P(0.3 < x < 0.6) = 0.1215]

28) A continuous random variable has p.d.f.
$f(x) = 1 - x$, $\quad 0 < x < 1$
$= x - 1$, $\quad 1 < x < 2$
$= 0$, $\quad$ otherwise
Find mean and variance. [**Ans.:** Mean = 1, Variance = 0.5]

29) (i) If f(x) is probability density function of a continuous random variate k, mean and variance
$f(x) = kx^2$ $\quad 0 \le x \le 1$
$= (2 - x)^2$ $\quad 1 \le x \le 2$ [**Ans.:** k = 2, mean = 11/12, s.d. = 0.3051]
(ii) A continuous random variable x has p.d.f. $f(x) = kx^2$, $0 \le x \le 2$. Determine k, $P(0.2 \le x \le 0.5)$ and $P(x \ge 0.75/x \ge 0.5)$ [**Ans.:** k = 3/8, 0.015, 0.9623]

30) A random variable X has the following density function **[EXTC : M-18]**
$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0 \;, & x \le 0 \end{cases}$$ $\left[\textbf{Ans.:}\ \text{m.g.f.} = \frac{2}{z - t},\ \text{mean} = \frac{1}{2},\ \text{variance} = \frac{1}{4}\right]$
Find the m.g.f. and hence, its mean and variance.

31) Find the value of k such that $f(x) = k(1 - x^2)$ $\quad 0 < x < 1$ **[ETRX : M-19]**
$= 0$ $\quad$ otherwise
is a probability function and hence find $P(0.1 < x < 0.2)$ and $P(x > 0.5)$
[**Ans.:** k = 3/2, P(0.1 < x < 0.2) = 0.1465 and P(x > 0.5) = 0.3125]

32) Find the value of k and mean if the function : **[ELEC : M-19]**
$f(x) = kx^2(1 - x^3)$ $\quad$ if $0 \le x \le 1$
$= 0$ $\quad$ otherwise [**Ans.:** k = 6, Mean = 9/14]

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33) If three coins are tossed find the expectation and variance of the number of heads.

[Ans.: $\frac{3}{2}, \frac{3}{4}$]

Soln.- $\because$ Three coins are tossed.

$\therefore S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Let 'X' be the no. of Heads (r.v.)

$\therefore$ P.d.f. is,

X	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$\rightarrow E(X) = ?$

$\rightarrow V(X) = ?$

आप !! ☺

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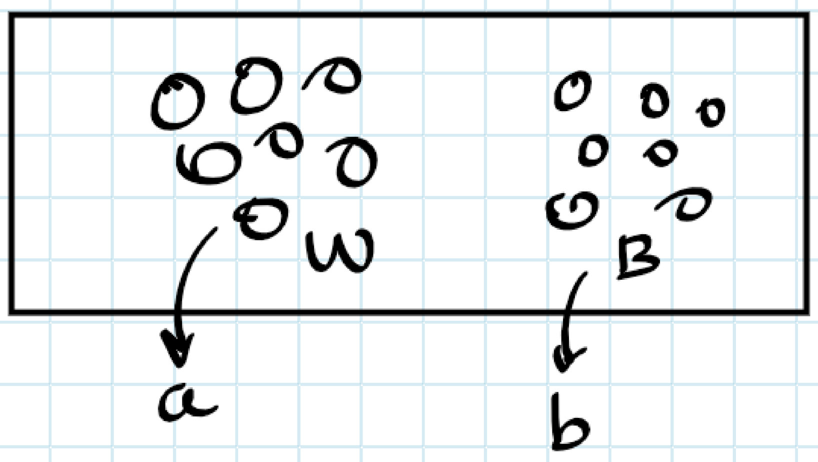
34) A box contains 'a' white balls and 'b' black balls. Now 'c' balls are drawn at random from the box. Find the expected value of the number of white balls.

[Ans.: ac/(a + b)]

Soln:-



$\because P\{\text{The ball is white}\} = \frac{a}{(a+b)}$

Now expected number of white balls is same as E(X) for 'c' balls.

$\because E(X) = \sum x \, P(X=x)$ ($x \to c$)

$\therefore E(X) = c \cdot P(X = \text{white ball})$

$= c \cdot \frac{a}{(a+b)} = \frac{ac}{a+b}$ //.

---

35) A coin is tossed until a head appears. What is the expectation of the number of tosses required? **[M-10]**

Soln:-

	T	TH	TTH	TTTH	......
$P(X=x)$:	$\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$	......
x :	1	2	3	4	......

Now we know,

$E(X) = \sum x \cdot P(X=x)$

$\therefore E(X) = \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^4 + \cdots \infty$

$\therefore E(X) = \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots \infty\right]$

$\therefore E(X) = \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-2}$ .... $\{(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \infty\}$

$= \frac{1}{2}\left(\frac{1}{2}\right)^{-2} = \frac{1}{\cancel{2}}(\cancel{4}^{2})$ $\therefore \boxed{E(X) = 2}$

---

36) A and B toss a fair coin alternately one who gets a head first wins Rs. 12. A starts. Find their expectation. [Ans.: A = 8, B = 4]

Soln:- $\because$ Probabilities of 'A' wining,

	H	TTH	TTTTH	.....
$P(X=x)$:	$\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$	....
	$\left(\frac{1}{2}\right)$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^5$	......
x :	1	1	1	......

$\because E(X) = \sum x \cdot P(X=x)$

$= \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \cdots \infty$

$= \frac{1}{2}\left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \cdots \infty\right]$

$= \frac{1}{2}\left[1 - \left(\frac{1}{2}\right)^2\right]^{-1}$ .... $\{(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \infty\}$

$= \frac{1}{2}\left[\frac{3}{4}\right]^{-1} = \frac{1}{\cancel{2}}\left(\frac{\cancel{4}^{2}}{3}\right)$

$\therefore \{E(X)\}_{A\ \text{wining}} = \frac{2}{3}$

i.e. $\{E(X)\}_{B\ \text{wining}} = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore$ wining amount.

$A \rightarrow \frac{2}{\cancel{3}} \times \cancel{12}^{4}₹ = 8₹$

$B \rightarrow \frac{1}{\cancel{3}} \times \cancel{12}^{4}₹ = 4₹$

PROPERTIES OF EXPECTATION :-

(1) $E(\underset{\text{const.}}{a}) = a$

(2) $E(ax) = a\,E(x)$

(3) $E(ax \pm b) = a\,E(x) \pm b$

(4) $E(ax \pm bY) = a\,E(x) + b\,E(Y)$

PROPERTIES OF VARIANCE :-

(1) $V(\underset{\text{const.}}{a}) = 0$

(2) $V(ax) = a^2 \cdot V(x)$

(3) $V(ax \pm b) = a^2 \cdot V(x)$

(4) $V(ax \pm bY) = a^2 V(x) \pm b^2 \cdot V(Y)$

---

37) If X_1 has mean 5 and variance 5, X_2 has mean −2 and variance 3 and if X_1, X_2 are independent, find :

(i) $E(X_1 + X_2)$, $V(X_1 + X_2)$ (ii) $E(X_1 - X_2)$, $V(X_1 - X_2)$

(iii) $E(2X_1 + 3X_2 - 5)$, $V(2X_1 + 3X_2 - 5)$

[Ans.: (i) 3, 8, (ii) 8, 8, (iii) −1, 47]

Soln:-

$$\left.\begin{array}{ll} E(X_1) = 5, & E(X_2) = -2 \\ V(X_1) = 5, & V(X_2) = 3 \end{array}\right\}\text{(given)}$$

(iii) $E(2X_1 + 3X_2 - 5) = 2\,E(X_1) + 3\,E(X_2) - 5$

$= 2(5) + 3(-2) - 5$

$= -1$ //.

also,

$V(2X_1 + 3X_2 - 5) = V(2X_1) + V(3X_2) - V(5)$

$= 4 \cdot V(X_1) + 9\,V(X_2) - 0$

$= 4(5) + 9(3)$

$= 20 + 27 = 47$ //.

---

38) If X_1 has mean 4 and variance 9 and X_2 has mean −2 and variance 4 where X_1 & X_2 are independent, find $E(2X_1 + X_2 - 3)$ and $V(2X_1 + X_2 - 3)$. **[EXTC : M-19]**

[Ans.: 3, 40]

↓

FOR you !!

MOMENT GENERATING FUNCTION (M.G.F)

Formulae :-

(1) M.G.F $\{M_x(t)\}$

DISCRETE

$$M_x(t) = E(e^{tx})$$

i.e. $M_x(t) = \sum e^{tx}.P(X=x)$

CONTINUOUS

$$M_x(t) = E(e^{tx})$$

i.e. $M_x(t) = \int_a^b e^{tx}.f(x)\,dx$

(2) **MOMENTS :-**

(A) Moment about Origin (Raw Moment) $\rightarrow \mu_r'$ $\rightarrow$ r^{th} Raw moment :-

$$\boxed{\mu_r' = E(x^r)} \quad \text{OR} \quad \boxed{\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}} \quad \text{OR} \quad \boxed{\mu_r' = \text{co-efficient of } \frac{t^r}{r!}}$$

→ First four Raw Moments will be,

$\mu_1' = E(x)$ → Mean $(\bar{x})$ OR $\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$ OR $\mu_1' =$ co-efficient of t

$\mu_2' = E(x^2)$ $\quad$ $\mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}$ $\quad$ $\mu_2' =$ co-efficient of $\frac{t^2}{2!}$

$\mu_3' = E(x^3)$ $\quad$ $\mu_3' = \left[\frac{d^3}{dt^3} M_x(t)\right]_{t=0}$ $\quad$ $\mu_3' =$ co-efficient of $\frac{t^3}{3!}$

$\mu_4' = E(x^4)$ $\quad$ $\mu_4' = \left[\frac{d^4}{dt^4} M_x(t)\right]_{t=0}$ $\quad$ $\mu_4' =$ co-efficient of $\frac{t^4}{4!}$

(B) Moment about Mean (Central Moment) $\rightarrow \mu_r$:-

$$\boxed{\mu_r = E[x - E(x)]^r}$$

⊛ First Raw Moment = Mean

⊛ Second central moment = Vari

→ First four Central moments will be,

→ $\mu_1 = 0$

→ $\mu_2 = \mu_2' - (\mu_1')^2 = E(x^2) - [E(x)]^2 = Var(x) = \sigma_x^2$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$ and $\mu_4 = \mu_4' - 4\mu_3'\mu_1 + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

(C) Moment about any Arbitrary point 'a' $\rightarrow \mu_{ar}$:-

$$\boxed{\mu_{ar} = E(x-a)^r}$$

39) The probability distribution of random variable 'X' is [INFT : M-10; CMPN/INFT : M-15]

X	−2	3	1
P(X)	1/3	1/2	1/6

Find : (i) MGF, (ii) First four moments about the origin, (iii) First four moments about the mean.

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Soln: $\because$ P.d.f. is, Discrete

X	−2	3	1
P(X)	1/3	1/2	1/6

(i) M.G.F. :- $M_x(t) = E(e^{tx})$

$$\therefore M_x(t) = \sum e^{tx} \cdot P(X=x)$$

$$\therefore \boxed{M_x(t) = e^{-2t} \cdot \frac{1}{3} + e^{3t} \cdot \frac{1}{2} + e^{t} \cdot \frac{1}{6}}$$

(ii) Raw Moment :-

$\because$ We know r^{th} Raw Moment is,

$$\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}$$

$\therefore$ First four Raw moments are,

$$\rightarrow \therefore \mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = \left[\frac{d}{dt}\left\{e^{-2t} \cdot \frac{1}{3} + e^{3t} \cdot \frac{1}{2} + e^{t} \cdot \frac{1}{6}\right\}\right]_{t=0}$$

$$= \left[\frac{1}{3}(-2)e^{-2t} + \frac{1}{2}(3)e^{3t} + \frac{1}{6}(e^t)\right]_{t=0}$$

$$= \frac{-2}{3} + \frac{3}{2} + \frac{1}{6} = 1 \longrightarrow \text{Mean}$$

$$\rightarrow \mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = \left[\frac{-2}{3}(-2)e^{-2t} + \frac{3}{2}(3)e^{3t} + \frac{1}{6}(e^t)\right]_{t=0}$$

$$= \frac{4}{3} + \frac{9}{2} + \frac{1}{6} = 6$$

$$\rightarrow \mu_3' = \left[\frac{d^3}{dt^3} M_x(t)\right]_{t=0} = \left[\frac{4}{3}(-2)e^{-2t} + \frac{9}{2}(3)e^{3t} + \frac{1}{6}(e^t)\right]_{t=0}$$

$$= \left[\frac{-8}{3} + \frac{27}{2} + \frac{1}{6}\right] = 11$$

$$\rightarrow \mu_4' = \left[\frac{d^4}{dt^4} M_x(t)\right] = \left[\frac{-8}{3}(-2)e^{-2t} + \frac{27}{2}(3)e^{3t} + \frac{1}{6}(e^t)\right]_{t=0}$$

$$= \frac{16}{3} + \frac{81}{2} + \frac{1}{6} = 46$$

(iii) Central Moments :-

$$\rightarrow \mu_1 = 0$$

$$\rightarrow \mu_2 = \mu_2' - (\mu_1')^2 = 6 - (1)^2 = 5 \longrightarrow \text{Variance}$$

$$\rightarrow \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 = 11 - 3(6)(1) + 2(1)^3 = -5$$

$$\rightarrow \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 = (46) - 4(11)(1) + 6(6)(1)^2 - 3(1)^4 = 35$$

40) **Suppose a fair die is thrown such that a chance of occurring each face is equally likely. Find MGF and using MGF obtain first four moments.**

Soln:- outcome

X	1	2	3	4	5	6
P(X=x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\rightarrow M_X(t) = E(e^{tx}) = \sum e^{tx} P(X=x)$$

$$\therefore M_X(t) = e^{t}\cdot\frac{1}{6} + e^{2t}\frac{1}{6} + e^{3t}\frac{1}{6} + e^{4t}\cdot\frac{1}{6} + e^{5t}\cdot\frac{1}{6} + e^{6t}\cdot\frac{1}{6} = \frac{1}{6}\left(e^{t}+e^{2t}+e^{3t}+e^{4t}+e^{5t}+e^{6t}\right)$$

---

41) A random variable X has the following probability distribution

x	0	1	2	3
p(x)	1/6	1/3	1/3	1/6

Find M.G.F. about the origin and hence first four raw moments.

42) A random variable x has the probability distribution $P(X = x) = \frac{1}{8}\,{}^3C_x$, X = 0, 1, 2, 3.

Find the moment generating function of x. **[CMPN : D-19]**

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

Soln:-

x	0	1	2	3
P(X=x)	$\frac{1}{8}{}^3C_0$	$\frac{1}{8}{}^3C_1$	$\frac{1}{8}{}^3C_2$	$\frac{1}{8}{}^3C_3$
	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

HW

---

43) A continuous r.v. X has the p.d.f $f(x) = \frac{4x(9-x^2)}{81}; 0 \le x \le 3$; 0 otherwise. Find the first four moments about the origin and about the mean. **[D-09]**

Soln:- $\because f(x) = \frac{4x(9-x^2)}{81} \;\ldots\; 0 \le x \le 3$

$$\boxed{\begin{aligned} \therefore f(x) &= \frac{4}{81}(9x - x^3) \;\ldots\; 0 \le x \le 3 \\ &= 0 \;\ldots\ldots\; \text{otherwise} \end{aligned}}$$

Now let's calculate M.G.F.,

$$\therefore \text{M.G.F. } \{M_X(t)\} = \int_a^b e^{tx} f(x)\,dx$$

$$= \int_0^3 \underbrace{e^{tx}}_{v} \cdot \frac{4}{81} \underbrace{(9x - x^3)}_{u} \cdot dx$$

$$\therefore M_X(t) = \frac{4}{81}\left[(9x-x^3)\left(\frac{e^{tx}}{t}\right) - (9-3x^2)\left(\frac{e^{tx}}{t^2}\right) + (-6x)\left(\frac{e^{tx}}{t^3}\right) - (-6)\left(\frac{e^{tx}}{t^4}\right)\right]_0^3$$

$\Downarrow$ put limits

→ First Four Raw Moments are :-

$$\mu_1' = E(x) = \int_0^3 x\cdot f(x)\,dx = \int_0^3 \frac{4}{81}(9x^2 - x^4)\,dx = \frac{4}{81}\left[9\cdot\frac{x^3}{3} - \frac{x^5}{5}\right]_0^3 = \frac{4}{81}\left[\frac{243}{3} - \frac{243}{5}\right] = \frac{4}{\cancel{81}} \times \cancel{243}\left[\frac{2}{15}\right] = \frac{8}{5}$$

$$\mu_2' = E(x^2) = \int_0^3 x^2 f(x)\,dx = \int_0^3 \frac{4}{81}(9x^3 - x^5)\,dx = \frac{4}{81}\left[9\cdot\frac{x^4}{4} - \frac{x^6}{6}\right]_0^3 = 3$$

$\mu_3' = E(x^3) = \int_0^3 x^3 f(x)\,dx =$ —

$\mu_4' =$ —

} FOR you !!

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44) **First four moments about the value 5 are 2, 20, 40, 50. Calculate the value of mean, variance, μ_3 and μ_4.**

Soln:- $\because$ Moment about any point 'a' is,

$$\mu_{ar} = E(x-a)^r$$

put $\boxed{a=5}$,

$$\therefore \boxed{\mu_{5r} = E(x-5)^r}$$

Now we have,

$1 \quad 2 \quad 1 \quad \text{---} (\ )^2$
$1 \quad 3 \quad 3 \quad 1 \quad \text{---} (\ )^3$
$1 \quad 4 \quad 6 \quad 4 \quad 1 \quad \text{---} (\ )^4$

$\rightarrow \mu_{51} = 2$

$\therefore E(x-5) = 2$

$\therefore E(x) - E(5) = 2$

$\therefore E(x) - 5 = 2$

$\therefore \boxed{E(x) = 7} \rightarrow \mu_1'$ (mean)

$\rightarrow \mu_{52} = 20$

$\therefore E(x-5)^2 = 20$

$\therefore E(x^2 - 10x + 25) = 20$

$\therefore E(x^2) - 10E(x) + 25 = 20$

$\therefore E(x^2) - 10(7) + 25 = 20$

$\therefore \boxed{E(x^2) = 65} \rightarrow \mu_2'$

$\rightarrow \mu_{53} = 40$

$\therefore E(x-5)^3 = 40$

$\therefore E(x^3 - 15x^2 + 75x - 125) = 40$

$\therefore E(x^3) - 15E(x^2) + 75E(x) - 125 = 40$

$\therefore E(x^3) - 15(65) + 75(7) - 125 = 40$

$\therefore \boxed{E(x^3) = 615} \rightarrow \mu_3'$

$\rightarrow \mu_{54} = 50$

$\therefore E(x-5)^4 = 50$

$\therefore E\left(x^4 - 4x^3(5) + 6x^2(5)^2 - 4x(5)^3 + (5)^4\right) = 50$

$\therefore E(x^4) - 20E(x^3) + 150E(x^2) - 500E(x) + 625 = 50$

$\therefore E(x^4) - 20(615)$

$\therefore \boxed{E(x^4) = \quad} \leftarrow \mu_4'$

45) If a random variable has the moment generating function $M_t = \frac{3}{3-t}$, obtain the mean $\rightarrow \mu_1'$ and the standard deviation. $\rightarrow \sqrt{\mu_2}$ [Ans.: 1/3, 1/3]

Soln:- $\because M_t = \frac{3}{(3-t)} = \frac{\cancel{3}}{\cancel{3}\left(1-\frac{t}{3}\right)}$

$\therefore M_t = \left(1-\frac{t}{3}\right)^{-1}$

$\therefore M_t = \left[1 + \frac{t}{3} + \left(\frac{t}{3}\right)^2 + \left(\frac{t}{3}\right)^3 + \cdots \infty\right]$

$\cdots\left\{(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \infty\right\}$

$\therefore M_t = 1 + \frac{1}{3}t + \frac{1}{9}t^2 + \frac{1}{27}t^3 + \cdots \infty$

$\therefore \boxed{M_t = 1 + \frac{1}{3}t + \frac{2}{9}\cdot\frac{t^2}{2} + \frac{1}{27}t^3 + \cdots \infty}$

$\therefore$ First two Raw Moments are,

$\therefore \mu_1' =$ co-efficient of $t = \frac{1}{3}$ (Mean)

$\mu_2' =$ co-efficient of $\frac{t^2}{2} = \frac{2}{9}$

$\therefore$ central moments are,

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2$

$Var(x) = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

$\therefore \sigma = +\sqrt{Var(x)} = \sqrt{\frac{1}{9}} = \frac{1}{3}$ //.

46) If X is a r.v. whose moment generating function is given by $M_X(t) = e^{t^2/2}$. ~~Prove that~~ then find mean and variance.

Soln:- $\because M_x(t) = e^{(t^2/2)}$

$\therefore M_x(t) = 1 + \frac{t^2}{2} + \frac{(t^2/2)^2}{2!} + \frac{(t^2/2)^3}{3!} + \cdots \infty$

$\cdots\left\{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty\right\}$

$\therefore M_x(t) = 1 + \frac{t^2}{2} + \frac{t^4}{8} + \frac{t^6}{48} + \cdots \infty$

$\therefore$ First two Raw moments are,

$\mu_1' =$ co-efficient of $t = 0 \rightarrow$ Mean

$\mu_2' =$ coefficient of $\frac{t^2}{2} = 1$

& First two central moments are,

$\mu_1 = 0$

$\mu_2 = $ Variance $= \mu_2' - (\mu_1')^2 = 1$ //.

47) The first three moments of a distribution about the value 2 of the variable are 1, 16 and –40. Show that the mean = 3, the variance = 15 and $\mu_3 = -86$.

$\rightarrow$ Same as Q(44) $\longrightarrow$ HW!

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BAYE'S THEOREM

$$\rightarrow \begin{Bmatrix} \text{Required} \\ \text{Probability} \\ (P_i) \end{Bmatrix} = \frac{P_i P_i'}{P_1P_1' + P_2P_2' + P_3P_3' + \cdots + P_nP_n'} \rightarrow \begin{matrix} P = \text{Result Probability} \\ P' = \text{Cause Probability} \end{matrix}$$

PROBLEMS :-

48) In a factory, machines A, B & C produce 30%, 50% & 20% of the total production of an item. Out of their production 80%, 50% & 10% are defective respectively. An item is chosen at random and found to be defective. What is the probability that it was produced by machine A. [**Ans.:** 0.47] **[EXTC : M-19]**

Soln:-

	A	B	C	
{production probability}	$\frac{30}{100} = 0.3 = P_1$	$\frac{50}{100} = 0.5 = P_2$	$\frac{20}{100} = 0.2 = P_3$	Result
{defective probability}	$\frac{80}{100} = 0.8 = P_1'$	$\frac{50}{100} = 0.5 = P_2'$	$\frac{10}{100} = 0.1 = P_3'$	Cause

$$\therefore \begin{Bmatrix} \text{Probability that} \\ \text{defective item} \\ \text{is from A} \end{Bmatrix} = \frac{P_1P_1'}{\underbrace{P_1P_1' + P_2P_2' + P_3P_3'}_{\text{Total probability}}} = \frac{(0.3)(0.8)}{(0.3)(0.8) + (0.5)(0.5) + (0.2)(0.1)}$$

$$= 0.4705 \rightarrow (47.05\%)$$

49) The Probabilities of Shivam, Surojit, Sakshi becoming managers are 4/9, 2/9 and 1/3 respectively. The probabilities that the Bonus scheme will be introduced if Shivam, Surojit and Sakshi become managers are 3/10, 1/2 and 4/5 respectively.

(i) What is the probability that the Bonus scheme will be introduced.
(ii) If the Bonus scheme has been introduced, What is the probability that the manager appointed was Shivam.

Soln:-

	Shivam	Surojit	Sakshi
{probab. of becoming manager}	$P_1' = \frac{4}{9}$	$P_2' = \frac{2}{9}$	$P_3' = \frac{1}{3}$
{probab. of introducing Bonus scheme}	$P_1 = \frac{3}{10}$	$P_2 = \frac{1}{2}$	$P_3 = \frac{4}{5}$

(i) $P(\text{Bonus scheme will be introduced}) = \text{Total Probability} = P_1P_1' + P_2P_2' + P_3P_3' = \frac{4}{9}\times\frac{3}{10} + \frac{2}{9}\times\frac{1}{2} + \frac{1}{3}\times\frac{4}{5}$

$= 0.5111$

(ii) $P\begin{Bmatrix} \text{manager appointed} \\ \text{was shivam} \end{Bmatrix} = \frac{P_1P_1'}{P_1P_1' + P_2P_2' + P_3P_3'} = \frac{(4/9)(3/10)}{0.5111} = 0.2608$

50) The number of COVID patients in three cities at one point was found to be – DELHI – 3000, CHENNAI – 2500 and Mumbai – 4500
The recovery rate of these cities were 72%, 81% and 86% respectively. A patient is selected at random and found to have been recovered from the Coronavirus. What is the probability that the patient belongs to –
(i) DELHI
(ii) CHENNAI } HW
(iii) MUMBAI } HW

$\rightarrow$ Total patients $= 10,000$

Soln:-

	DELHI	CHENNAI	MUMBAI
{Probability Covid infected people}	$P_1' = \frac{3000}{10,000} = 0.3$	$P_2' = \frac{2500}{10,000} = 0.25$	$P_3' = \frac{4500}{10,000} = 0.45$
{Probab. of recovery Rate}	$P_1 = \frac{72}{100} = 0.72$	$P_2 = \frac{81}{100} = 0.81$	$P_3 = \frac{86}{100} = 0.86$

(i) $\begin{Bmatrix} \text{Patient} \\ \text{belongs to delhi} \end{Bmatrix} = \frac{P_1P_1'}{P_1P_1' + P_2P_2' + P_3P_3'}$

$$= \frac{(0.3)(0.72)}{(0.3)(0.72) + (0.25)(0.81) + (0.45)(0.86)}$$

$= 0.2681$

51) The chances that Dr. Subodh will diagnose COVID correctly is 60%. The Chances that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of Dr. Subodh, who had COVID, died. What is the chance that his disease was diagnosed correctly?

Soln:

{Probability of diagnosis} $P_1' =$ correct $= 0.6 \rightarrow 60\%$ $P_2' =$ Incorrect $= 1 - 0.6 = 0.4 \rightarrow 40\%$

{probab. of death} $P_1 = 0.4$ $P_2 = 0.7$

$\therefore$ {probability of death by correct diagnosis} $= \dfrac{P_1 P_1'}{P_1 P_1' + P_2 P_2'} = \dfrac{(0.6)(0.4)}{(0.6)(0.4) + (0.4)(0.7)} = 0.4615$

& {Probability of death by incorrect diagnosis} $= \dfrac{P_2 P_2'}{P_1 P_1' + P_2 P_2'} = \dfrac{(0.4)(0.7)}{(0.6)(0.4) + (0.4)(0.7)} = 0.5384$

HOMEWORK PROBLEMS :-

52) Ina factory, Machines A, B and C produce 30%,50% and 20% of the total production of an item. Out of their production 80%, 50% and 10% are defective respectively. An item is chosen at random and found to be defective. What is the probability that it was produced by machine A?

53) In 2002, there will be three candidates for the position of principal – Mr. Subodh, Mr. Vaibhav and Mr. Ghanshyam – whose chances of getting the appointment are in proportion 4:2:3 respectively. The probability that Mr. Subodh selected would introduce co-education in the college is 0.3. The probabilities of Mr. Vaibhav and Mr. Ghanshyam doing the same are 0.5 and 0.8 respectively.

(i) What is the Probability that there will be co-education in the college in 2003?
(ii) If there is co-education in the college in 2003. What is the Probability that Mr. Subodh is the principal.

	Subodh	Vaibhav	Ghanshyam
{probab. of principal}	$P_1' = \frac{4}{9}$	$P_2' = \frac{2}{9}$	$P_3' = \frac{3}{9}$
{introduction of co-ed}	$P_1 = 0.3$	$P_2 = 0.5$	$P_3 = 0.8$

54) The chances that doctor A will diagnose a disease X correctly is 60%. The chances that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of doctor A, who had disease X, died. What is the chance that his disease was diagnosed correctly? [**Ans.:** 6/13]