Mathematics III

LAPLACE TRANSFORMS

denote - L

Laplace is defined as

$$L(f(x)) = \int_0^\infty e^{-px} \cdot f(x)\,dx$$

p, any parameter (positive Real No.)

$p > 0$

- $\int f \cdot g\,dx = f\int g\,dx - \int \frac{df}{dx} \cdot \int g\,dx$

①

→ If $f(x)=1$

$$L(1) = \int_0^\infty e^{-px} \cdot 1\,dx$$

- $\int e^{2x}dx = \frac{e^{2x}}{2}$

- $e^{-\infty} = 0$

$$= \left(\frac{e^{-px}}{-p}\right)_0^\infty$$

$$= \frac{-1}{p}\left(e^{-\infty} - e^{0}\right)$$

$$= \frac{1}{p}$$

p, any +ve Real No.

②

→ If $f(x) = x$

$$L(f(x)) = \int_0^\infty e^{-px} \cdot f(x)\,dx$$

$$= \int_0^\infty e^{-px} \cdot x\,dx$$

$$= \int_0^\infty x \cdot e^{-px}\,dx$$

$$= x \cdot \int_0^\infty e^{-px}dx - \int_0^\infty 1 \cdot \int_0^\infty e^{-px}dx$$

$$= x \left(\frac{e^{-px}}{-p}\right)_0^{\infty} - \int_0^{\infty} \frac{e^{-px}}{-p} dx$$

$$= \left(\frac{-1}{p}\left(x.e^{-px}\right) - \frac{1}{p^2} e^{-px}\right)_0^{\infty}$$

$\cdot e^{-px} = e^{-p\times\infty}$
$= e^{-\infty}$
$= \underline{\underline{0}}$

taking $x = \infty$

$$= \left(\frac{-1}{p} \cdot 0\right) - \left(0 - 0 + \frac{1}{p^2}\right)$$

$$\lim_{x\to\infty} x.e^{-px} = \cdot \lim_{x\to\infty} \frac{x}{e^{px}} = \frac{1}{pe^{px}} = \frac{1}{\infty}$$
$$= \underline{\underline{0}}$$

$$\therefore \quad L(x) = \underline{\underline{\frac{1}{p^2}}}$$

→ Gamma Function

$$\int_0^{\infty} e^{-x} . x^{n-1} dx = \Gamma n$$

$$\boxed{\cdot \Gamma n+1 = \cancel{\lfloor n} = n!}$$

$$L(x) = \int_0^{\infty} e^{-px} . x \, dx$$

put $px = t \Rightarrow p dx = dt \Rightarrow dx = dt/p$

$$= \int_0^{\infty} e^{-t} . \frac{t}{p} . \frac{dt}{p}$$

$$= \frac{1}{p^2} \int_0^{\infty} e^{-t} . t . dt$$

$$= \frac{1}{p^2} \int_0^{\infty} e^{-t} . t^{2-1} . dt \Rightarrow \textit{Similar to } \int_0^{\infty} e^{-x} . x^{n-1} . dx$$

here, $\Gamma n+1 = \Gamma 2-1+1 = \Gamma 2 = \underline{\lfloor 1}$
$= \Gamma 1+1$

$= \Gamma 2$

$= 1!$

$= \underline{\underline{1}}$

③ $L(x^n) = \int_0^\infty e^{-px} . x^n dx$

put, $px = t$
$p dx = dt$

$\cancel{put} = \int_0^\infty e^{-t} . \left(\frac{t}{p}\right)^n . \frac{dt}{p}$

$= \frac{1}{p^{n+1}} \underbrace{\int_0^\infty e^{-t} . t^{n+1-1} . dt}_{\Gamma n+1}$

$\Gamma n = \int_0^\infty e^{-x} . x^{n-1} dx$

$L(x^n) = \frac{\Gamma n+1}{p^{n+1}}$ for n is not integers.

$L(x^n) = \frac{\lfloor n}{p^{n+1}}$ for +ve integer

→ Differentiation of two fn. (differentiation operation satisfies addition and subtraction)

$D(x + \sin x) = D(x) + D(\sin x)$

Laplace satisfy linear property (i.e)

(i) $L(f+g) = L(f) + L(g)$

$L(1+x+x^2) = \frac{1}{p} + \frac{1}{p^2} + \frac{2}{p^3}$

(ii) $L(\lambda g) = \lambda L(g)$

$L(2+3x) = 2L(1) + 3L(x)$

$= \frac{2}{p} + \frac{3}{p^2}$

④

Laplace of any exponential fn,

$$L(e^{ax}) = \int_0^\infty e^{-px} \cdot e^{ax}\, dx$$

$$= \int_0^\infty e^{-px+ax}\, dx$$ (add the powers, since both are exponential)

$$= \int_0^\infty e^{-x(p-a)}\, dx$$

$$= \left(\frac{e^{-(p-a)x}}{-(p-a)} \right)_0^\infty$$

$(e.g.)$ $\int e^{-2x}\, dx = \frac{e^{-2x}}{-2}$

$$= \left(0 + \frac{1}{p-a}\right)$$

$\therefore L(e^{ax}) = \frac{1}{p-a}$ and $L(e^{-ax}) = \frac{1}{p-a}$

if $a = -3$

$$L(e^{-3x}) = \frac{1}{p+3}$$

* In every Laplace equation, we get in terms of p

- Laplace Transforms for $\sin ax$

(a) $L(\sin ax) = \int_0^\infty e^{-px} \cdot \sin ax\, dx$

$$\boxed{\int e^{ax} \cdot \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)}$$

$$= \left(\frac{e^{-px}}{a^2+p^2}(-p \sin ax - a \cos ax)\right)_0^\infty$$

$$\boxed{\begin{array}{l} \sin 0 = 0 \\ \cos 0 = 1 \end{array}}$$

$$= 0 - \frac{1}{a^2+p^2} \cdot -a$$

$\therefore L(\sin ax) = \dfrac{a}{a^2+p^2}$

(b) $L(\cos ax) = \int_0^\infty e^{-px} \cdot \cos ax\, dx$

$$\boxed{\int e^{ax} \cdot \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)}$$

$\therefore L(\cos ax) = \dfrac{p}{a^2+p^2}$

- Inverse

1. $L(1) = 1/p$ $\qquad L^{-1}(1/p) = 1$

2. $L(x^2) = 2/p^3$ $\qquad L^{-1}(1/p^3) = x^2/2$

3. $L(\sin ax) = \dfrac{a}{a^2+p^2}$ $\qquad L^{-1}\left(\dfrac{1}{1+p^2}\right) = \sin x$

$a=1$

· Section 48

- $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$
- $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

1. Find $L(\cos^2 ax)$ and $L(\sin^2 ax)$

- $L(\cos^2 ax) = L\left(\dfrac{1+\cos 2ax}{2}\right)$

$= L\left(\dfrac{1}{2}+\dfrac{1}{2}\cos 2ax\right)$

$= L\left(\dfrac{1}{2}\right) + L\left(\dfrac{1}{2}\cos 2ax\right)$

$= \dfrac{1}{2}L(1) + \dfrac{1}{2}L(\cos 2ax)$

$L(\cos ax) = P/a^2+p^2$

$\because L(1) = 1/P$

$L(\cos 2ax) = \dfrac{P}{P^2+4a^2}$

$= \dfrac{1}{2}\cdot\dfrac{1}{P} + \dfrac{1}{2}\cdot\dfrac{P}{P^2+4a^2}$

2. Without integrating find $L(\cosh ax)$

- $\cosh x = \dfrac{e^x+e^{-x}}{2}$
- $\sinh x = \dfrac{e^x-e^{-x}}{2}$

$L(\cosh ax) = L\left(\dfrac{e^{ax}+e^{-ax}}{2}\right)$

$= \dfrac{1}{2}\left(L(e^{ax}) + L(e^{-ax})\right)$

$\because L(e^{ax}) = \dfrac{1}{P-a}$

$= \dfrac{1}{2}\left(\dfrac{1}{P-a} + \dfrac{1}{P+a}\right)$ → take LCM

$= \dfrac{P}{P^2-a^2}$

Similarly,

$L(\sinh ax) = \dfrac{a}{P^2-a^2}$

4.

(d) Find $L(4\sin x\cos x + 2e^{-x})$

$\to \frac{a}{a^2+p^2}$ $\to \frac{1}{p+a}$

$$L(2\sin 2x + 2.e^{-x}) = 2L(\sin 2x) + 2L(e^{-x})$$
$$= 2.\frac{2}{p^2+4} + 2.\frac{1}{p+1}$$
$$= \frac{4}{p^2+4} + \frac{2}{p+1}$$

(e) $L(x^6\sin^2 3x + x^6\cos^2 3x)$

$$L(x^6) = \frac{6!}{p^7}$$

$\cdot\ L(x^n) = \frac{\lfloor n}{p^{n+1}}$

5. Find a fn. $f(x)$ whose transform is : $\to$ i.e: to find inverse $f(x)$ from

(e) $\frac{1}{p^4+p^2}$

$$L^{-1}\left(\frac{1}{p^4+p^2}\right) = L^{-1}\left(\frac{1}{p^2(p^2+1)}\right)$$ take p^2 common

$$= L^{-1}\left(\frac{1}{p^2} - \frac{1}{p^2+1}\right)$$ · when you take LCM.

· since in subtraction, you can use linear property.

$$= L^{-1}\left(\frac{1}{p^2}\right) - L^{-1}\left(\frac{1}{p^2+1}\right)$$

· since 'constant' in numerator = "$\sin x$"

$\therefore\ f(x) = x - \sin x$

(c) $L^{-1}\left(\frac{4}{p^3}+\frac{6}{p^2+4}\right) = L^{-1}\left(\frac{4}{p^3}\right) + L^{-1}\left(\frac{6}{p^2+4}\right)$

$$= 2 \cdot L^{-1}\left(\frac{2}{p^3}\right) + 3\,L^{-1}\left(\frac{2}{p^2+2^2}\right)$$

$$= \underline{\underline{2x^2 + 3\sin 2x}}$$

• <u>Section 49</u>

2. Unit step fn. $u(x) = \begin{cases} 0\,; & \text{if } x<0 \\ 1\,; & \text{if } x \geq 0 \end{cases}$, find $L(u(x-a))$

$\rightarrow$ $L(u(x-a) = \begin{cases} 0; & \text{if } \; x-a<0 \rightarrow x<a \\ 1; & \text{if } \; x-a \geq 0 \rightarrow x \geq a \end{cases}$

$$L(f(x)) = \int_0^\infty e^{-px} f(x)\, dx$$

$$L(u(x-a)) = \cancel{\int_0^a e^{-px} . 0\, dx}^{\,0} + \int_a^\infty e^{-px} . 1\, dx$$

$$= \int_a^\infty e^{-px}\, dx$$

$$= \left(\frac{e^{-px}}{-p}\right)_a^\infty$$

$$= -\frac{1}{p}\left(\cancel{e^{-\infty}}^{\,0} - e^{-pa}\right)$$

$$= \underline{\underline{\frac{e^{-pa}}{p}}}$$

5. $f_\varepsilon x = \begin{cases} 1/\varepsilon, & \text{if } 0 \le x \le \varepsilon \\ 0, & \text{if } x > \varepsilon \end{cases}$

Also,

$\delta(x) = \lim_{\varepsilon \to 0} f_\varepsilon(x)$. Find $L(\delta(x))$.

→ Taking both side Laplace Transformation:

$$L(\delta(x)) = \lim_{\varepsilon \to 0} L(f_\varepsilon(x))$$

$$\Rightarrow L(f_\varepsilon(x)) = \int_0^\varepsilon e^{-px} \cdot f(x)\, dx + \int_\varepsilon^\infty e^{-px} \cdot f(x)\, dx$$

$$= \int_0^\varepsilon e^{-px} \cdot \frac{1}{\varepsilon}\, dx + \cancel{\int_\varepsilon^\infty e^{-px} \cdot 0 \cdot dx}^{\,0}$$

$$= \frac{1}{\varepsilon} \int_0^\varepsilon e^{-px}\, dx$$ ∵ integration with x, we can take $1/\varepsilon$ outside.

$$= \frac{1}{\varepsilon} \left(\frac{e^{-px}}{-p} \right)_0^\varepsilon$$

$$= \frac{-1}{\varepsilon p} \left(e^{-p\varepsilon} - e^{-p \cdot 0} \right)$$

$$\underline{L(f_\varepsilon(x)) = \frac{1 - e^{-p\varepsilon}}{p\varepsilon}}$$

Now taking limit,

$$\lim_{\varepsilon \to 0} L(f_\varepsilon(x)) = \lim_{\varepsilon \to 0} \frac{1 - e^{-p\varepsilon}}{p\varepsilon}$$ If we put $\varepsilon = 0$, we get 0/0 form

$$= \lim_{\varepsilon \to 0} \frac{\cancel{p} \cdot e^{-p\varepsilon}}{\cancel{p}}$$ so apply

"Dirac Delta Function"

$$\therefore L(\delta(x)) = \underline{\underline{1}}$$

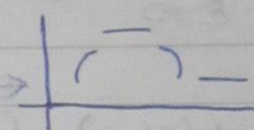
• Existence of Laplace Transform

$$L(f(x)) = \int_0^\infty e^{-px}. f(x). dx$$

For integration the fn should be:- continuous, and bounded

$\int_0^\infty x\,dx$ - continous but not bounded.

For $L(f(x)) = \int_0^\infty e^{-px}. f(x)\ dx$, Laplace fn. exist if:

1. f(x) is piece wise continous. 
2. f(x) should have exponential order.

   (i.e) take modulus,

   $$|f(x)| \le M.e^{cx}$$

   where,

   c, constant

   M, big +ve no:

REMARK:

These two above conditions are sufficient for the existence of Laplace Transforms but not necessary.

which means if both ~~conditions~~ conditions satisfied, Laplace will exist, if one or both are not satisfied, then Laplace may or maynot exist.

(e.g) Find Laplace of $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

Not piecewise condition because it wont satisfy in the interval zero (not defined at zero)

and,

$$|f(x)| \le M.e^{cx}$$

$$\left|\frac{1}{\sqrt{x}}\right| \le M.e^{cx}$$

$$\frac{1}{|\sqrt{x}|\cdot e^{cx}} \le M \qquad \lim_{x\to\infty} \frac{1}{|\sqrt{x}|e^{cx}} = \frac{1}{\infty} = \underline{0}$$

$0 \le M$ is satisfied.

$$L(f(x)) = \int_0^\infty e^{-px} \cdot f(x)\,dx$$

$$L(1/\sqrt{x}) = \int_0^\infty e^{-px} \cdot x^{-1/2}\,dx$$

put,

$px = t$, $p\,dx = dt$

$$\therefore L(1/\sqrt{x}) = \int_0^\infty e^{-t} \cdot \left(\frac{t}{p}\right)^{-1/2} \frac{dt}{p}$$

$$= \frac{1}{\sqrt{p}} \int_0^\infty e^{-t} \cdot t^{-1/2}\,dt \qquad \because p^{-1/2} \cdot p^{1} = p^{-1/2+1} = p^{1/2} = \underline{\underline{\sqrt{p}}}$$

$$-1/2 = 1/2 - 1$$

$$= \frac{1}{\sqrt{p}} \int_0^\infty e^{-t} \cdot t^{1/2-1}\,dt$$

$$= \frac{1}{\sqrt{p}} \cdot \Gamma(1/2)$$

$$= \frac{1}{\sqrt{p}} \cdot \sqrt{\pi} \qquad \because \Gamma(1/2) = \sqrt{\pi}$$

$$= \underline{\underline{\sqrt{\pi/p}}}$$

Laplace of

- Does $f(x) = \frac{1}{x}$ exist.

$$L(1/x) = \int_0^\infty e^{-px} \cdot 1/x \cdot dx$$

$$= \int_0^\infty e^{-px} \cdot x^{-1} \, dx$$

can be done without integration,

$$L(x^n) = \frac{\lceil n+1}{p^{n+1}} = \frac{n!}{p^{n+1}}$$

$$\therefore L(x^{-1}) = \frac{\lfloor -1}{p^0}$$

$\boxed{\lfloor 0 \rightarrow \text{not defined}}$

$\because \lfloor -1$ doesn't exist

$f(x) = 1/x$ doesn't exist.

- Shifting Property

If $L(f(x)) = F(P)$

then,

$$L(e^{ax} \cdot f(x)) = F(P-a)$$

(e.g) $L(\sin x) = \frac{1}{1+p^2} = F(P)$

$$L(e^{2x} \cdot \sin x) = \frac{1}{1+(P-2)^2}$$

wherever p is there put p-a.
here e^{2x}, a = +2, so p = p-2
if
e^{-2x}, then p = p+2

Section 50

1.

(a) $L(x^5 \cdot e^{-2x})$

$$L(x^5) = \frac{\lfloor 5}{p^6} = F(p)$$

$$L(e^{-2x} \cdot x^5) = \frac{\lfloor 5}{(p+2)^6} = F(p-(-2))$$

(b) $L\{(1-x^2)e^{-x}\}$

$$L(1-x^2) = L(1) - L(x^2)$$
$$= \frac{1}{p} - \frac{\lfloor 2}{p^3}$$

$$L(e^{-x}(1-x^2)) = \frac{1}{p+1} - \frac{2}{(p+1)^3}$$

2.

(a) $L^{-1}\left(\frac{6}{(p+2)^2+9}\right)$

$$= e^{-2x} \cdot L^{-1}\left(\frac{2 \cdot 3}{p^2+3^2}\right)$$

$$= e^{-2x} \cdot 2 \cdot L^{-1}\left(\frac{3}{p^2+3^2}\right)$$

$$= 2e^{-2x} \sin 3x$$ → since constant in the numerator

(b) $L^{-1}\left(\frac{12}{(P+3)^4}\right)$

Remove 3 by e^{-3x}

$\therefore$

$$L^{-1}\left(\frac{12}{(P+3)^4}\right) = 12.e^{-3x}.L^{-1}\left(\frac{1}{P^4}\right)$$

$$= 12.e^{-3x}\ \frac{x^3}{3.2.1}$$

$$= \underline{\underline{2x^3e^{-3x}}}$$

(c) $L^{-1}\left(\frac{P+3}{P^2+2P+5}\right)$

Factor,

P^2+2P+5 cannot be applied.

so,

$$L^{-1}\left(\frac{P+3}{P^2+2P+5}\right) = L^{-1}\left(\frac{(P+1)+2}{(P+1)^2+4}\right) \quad p-(-1) = e^{-x}$$

$$= e^{-x}.L^{-1}\left(\frac{P+2}{P^2+4}\right)$$

$$= e^{-x}\ L^{-1}\left(\frac{P}{P^2+4}\right) + L^{-1}\left(\frac{2}{P^2+2^2}\right)$$

$$= \underline{\underline{e^{-x}(\cos 2x + \sin 2x)}}$$

• Laplace of first derivative : $L(f'(x))$ or $L(y')$

$$L(f'(x)) = \int_0^\infty e^{-px} \cdot f'(x)\, dx$$

(or)

$$L(y') = \int_0^\infty e^{-px} \cdot y'\, dx$$

If product of fn. is given,

$$\int f \cdot g\, dx = f \int g\, dx - \int \frac{df}{dx} \cdot \int g\, dx$$

$$\therefore L(y') = \int_0^\infty e^{-px} \cdot y'\, dx$$

$$= \left(e^{-px} \cdot y + p \int e^{-px} \cdot y\, dx \right)_0^\infty$$

$x = \infty$ $x = 0$

$$= 0 - y(0) + p \cdot L(y)$$

(*) $\boxed{L(y') = p\, L(y) - y(0)}$

$$L(y'') = p\, L(y') - y'(0)$$

(*) $\boxed{L(y'') = p^2 L(y) - p\, y(0) - y'(0)}$

• Solve $\frac{dy}{dx} + y = 3e^{2x}$ by Laplace equation

first order

$y = f(x) = x^2 \sin x$

y is dependent, so dependent variable in the nameralor → $\frac{dy}{dx}$

independent variable

$y' + y = 3e^{2x}$ → first Order differential equ.

with $y(0) = 0$

Apply Laplace both sides,

$L(y') + L(y) = L(3e^{2x})$

formula

$$pL(y) - y(0) + L(y) = \frac{3}{p-2}$$

Taking L(y) as common,

$$L(y)(p+1) - \underset{y(0)}{0} = \frac{3}{p-2}$$

$$L(y) = \frac{3}{(p-2)(p+1)}$$

Partial fraction

$$\frac{3}{(p+1)(p-2)} = \frac{A}{(p+1)} + \frac{B}{(p-2)}$$

$$= \frac{A(p-2) + B(p+1)}{(p+1)(p-2)}$$

$$3 = AP - 2A + Bp + B$$

$$3 = P(A+B) - 2A + B$$

$$A + \cancel{B} = 0 \longrightarrow (1)$$

$$-2A \cancel{+ B} = 3 \longrightarrow (2)$$

$$3A = -3$$

$$A = -1$$

$$B = +1$$

$$y = L^{-1}\left(\frac{3}{(p+1)(p-2)}\right)$$

$$y = L^{-1}\left(\frac{-1}{p+1} + \frac{1}{p-2}\right)$$

$$= L^{-1}\left(\frac{-1}{p+1}\right) + L^{-1}\left(\frac{1}{p-2}\right)$$

$$\underline{y = -e^{-x} + e^{2x}}$$

$$\underline{y(x) = -e^{-x} + e^{2x}} \rightarrow \text{Solution.}$$

- Solve: $y'' + 2y' + 5y = 3e^{-x}\sin x$ with $y(0) = 0$ and $y'(0) = 3$

Applying laplace both sides,

$$L(y'') + 2L(y') + 5L(y) = 3L(e^{-x}\sin x)$$

$$p^2 L(y) - py(0) - y'(0) + 2\big(pL(y) - y(0)\big) + 5L(y) = \frac{3}{1+(1+p)^2}$$

$$L(y)(p^2 + 2p + 5) - 3 = \frac{3}{1+(1+p)^2}$$

$$\boxed{\begin{aligned} L(\sin x) &= \frac{1}{1+p^2} \\ L(e^{-x}\sin x) &= \frac{1}{1+(1+p)^2} \end{aligned}}$$

$$L(y) = \frac{1}{p^2 + 2p + 5}\left(\frac{3}{1+(1+p)^2} + 3\right)$$

$$= \frac{3(2 + (1+p^2))}{(p^2 + 2p + 5)(1 + (1+p)^2)}$$

$$y = 3L^{-1}\left(\frac{2 + (1+p)^2}{((1+p)^2 + 4)(1 + (1+p)^2)}\right)$$

$$y = 3 \cdot e^{-x} L^{-1}\left(\frac{2 + p^2}{(p^2+4)(1+p^2)}\right)$$

* $p+1$ becomes p after taking out e^{-x} out as common

Applying Partial Fraction,

$$\frac{2+p^2}{(p^2+4)(1+p^2)} = \frac{A}{(p^2+4)} + \frac{B}{(1+p^2)}$$

$$2+p^2 = A(1+p^2) + B(p^2+4)$$

$$2+p^2 = A + Ap^2 + Bp^2 + 4B$$

$$2+p^2 = p^2(A+B) + A + 4B$$

$$\cancel{A} + B = 1 \longrightarrow (1)$$

$$\overset{(-)}{\cancel{A}} \overset{(-)}{+} 4B = \overset{(-)}{2} \longrightarrow (2)$$

$$-3B = -1$$

$$\underline{\underline{B = 1/3}}$$

$$A + \frac{1}{3} = 1$$

$$\underline{\underline{A = \frac{2}{3}}}$$

$$\therefore y = 3e^{-x}\left[L^{-1}\left(\frac{2}{3} \times \frac{1}{p^2+4}\right) + L^{-1}\left(\frac{1}{3} \times \frac{1}{1+p^2}\right)\right]$$

$$= 3e^{-x}\left[L^{-1}\left(\frac{1}{3} \times \frac{2}{p^2+4}\right) + L^{-1}\left(\frac{1}{3} \times \frac{1}{1+p^2}\right)\right]$$

$$= 3e^{-x}\left(\frac{1}{3}\sin 2x + \frac{1}{3}\sin x\right)$$

$$= e^{x}(\sin 2x + \sin x)$$

$$\therefore \underline{\underline{y(x) = e^{-x}(\sin 2x + \sin x)}}$$

5. If $L(f(x)) = F(p)$, prove that $L\left(\int_0^x f(x)dx\right) = \frac{F(p)}{p}$, then verify this by finding $L^{-1}\left(\frac{1}{p(p+1)}\right)$

$$L^{-1}\left(\frac{1}{p(p+1)}\right) = L^{-1}\left(\frac{1}{p} - \frac{1}{p+1}\right)$$
$$= L^{-1}\left(\frac{1}{p}\right) - L^{-1}\left(\frac{1}{p+1}\right)$$
$$= \underline{\underline{1-e^{-x}}}$$

Proof:

Let $y(x) = \int_0^x f(x)dx \qquad y'(x) = F(x)$

Taking Laplace both sides,

$$L(y'(x)) = L(F(x))$$
$$pL(y) - y(0) = L(F(x))$$

Apply zero to x
→ We can find $y(0)$,
$y(0) = \int_0^0 f(0)\,do = \underline{\underline{0}}$

$$L(y) = \frac{1}{p} L(f(x)) \quad (\because y(0) \text{ is zero})$$
$$L(f(x)) = \frac{1}{p} \times F(p)$$
$$L(f(x)) = \frac{F(p)}{p}$$

Hence Proved

Now we need to get the expression as $1-e^{-x}$.

$$\int_0^x f(x)dx = L^{-1}\left(\frac{F(p)}{p}\right)$$

We have to find $L^{-1}\left(\frac{1}{p(1+p)}\right)$

$$L^{-1}\left(\frac{1/1+p}{p}\right)$$

$$F(p) = \frac{1}{1+p}$$

$$L(f(x)) = \frac{1}{1+p}$$

$$f(x) = L^{-1}\left(\frac{1}{1+p}\right)$$

$$\boxed{f(x) = e^{-x}}$$

so, $$\int_0^x f(x)\,dx = \int_0^x e^{-x}\,dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^x$$

$$= 1.e^{-x}$$

Result

$$\boxed{L(x^n . f(x)) = (-1)^x \frac{d^n}{dp^n}(L(f(x)))}$$

(ex) $$L(x^1 . \sin x) = (-1)^1 \frac{d}{dp}(L(\sin x))$$

$$= -1 \frac{d}{dp}\left(\frac{1}{1+p^2}\right)$$

$$= (1+p^2)^{-2} 2p$$

Section 5.1

1. Show that $L(x.\cos ax) = \dfrac{p^2-a^2}{(p^2+a^2)^2}$, use above result to find

$L^{-1}\left(\dfrac{1}{(p^2+a^2)^2}\right)$

$$L(f(x)) = L(\cos ax)$$
$$= \frac{p}{a^2+p^2}$$
$$L(x.\cos ax) = (-1)^1 \frac{d}{dp}\left(\frac{p}{p^2+a^2}\right)$$
$$= -\frac{(p^2+a^2).1 - 2p.p}{(p^2+a^2)^2}$$
$$= \frac{p^2-a^2}{(p^2+a^2)^2}$$

$$L^{-1}\left(\frac{1}{(p^2+a^2)^2}\right) = \frac{-1}{2a^2} L^{-1}\left(\frac{p^2-a^2}{(p^2+a^2)^2} - \frac{1}{p^2+a^2}\right)$$
$$= \frac{-1}{2a^2} L^{-1}\left(\frac{p^2-a^2}{(p^2+a^2)^2}\right) + \frac{1}{2a^2} L^{-1}\left(\frac{1}{p^2+a^2}\right)$$
$$= \frac{-1}{2a^2} \times x\cos ax + \frac{1}{2a^3} L^{-1}\left(\frac{a}{p^2+a^2}\right)$$
$$= \frac{-1}{2a^2} x\cos ax + \frac{1}{2a^3} \sin ax$$

$$L(x^{-1/2}) = \sqrt{\pi/p}$$

- Find $L(x^{3/2})$

We know $L(x^{-1/2}) = \sqrt{\pi/p}$

$$L(x^{3/2}) = L(x^{2-1/2})$$
$$= L(x^2 \cdot x^{-1/2})$$
$$L(f(x)) = L(x^{-1/2}) = \sqrt{\pi/p}$$

$$\therefore L(x^{3/2}) = (-1)^2 \cdot \frac{d^2}{dp^2} L(f(x))$$
$$= \frac{d^2}{dp^2}\left(\sqrt{\frac{\pi}{p}}\right)$$
$$= \sqrt{\pi}\,\frac{d^2}{dp^2}\left(p^{-1/2}\right)$$
$$= \sqrt{\pi} \times \frac{-1}{2} \times \frac{-3}{2}\, p^{-5/2}$$ → Second derivative of $p^{-1/2}$
$$= \frac{3}{4}\sqrt{\pi}\, p^{-5/2}$$

<u>Result</u>

$$\boxed{\int_0^\infty \frac{f(x)}{x}\,dx = \int_0^\infty L(f(x))\cdot dp}$$

<u>eg</u> $\int_0^\infty \frac{\sin x}{x}\,dx$

here, $f(x) = \sin x$

$$= \int_0^\infty L(\sin x)\,dp = \int_0^\infty \frac{1}{p^2+1}\,dp$$
$$= \left(\tan^{-1} p\right)_0^\infty$$
$$= \frac{\pi}{2} - 0$$
$$= \pi/2$$

Section 53

- Convolution of two fns. f and g.

Convolution of $f(x)$ and $g(x)$ is defined as,

$$\boxed{(f*g)(t) = \int_0^\infty f(t-\tau)\, g(\tau)\, d\tau}$$

Convolution Theorem

$$\boxed{L(f*g) = L(f) \cdot L(g)}$$

2.(a) Find convolution of $f(t)=1$, $g(t)=\sin at$ and verify convolution Theorem.

$f(\text{any no;})=1$

RHS

$$L(f(t)) \cdot L(g(t)) = L(1) \cdot L(\sin at)$$

$$= \frac{1}{p} \cdot \frac{a}{a^2+p^2}$$

$$= \underline{\underline{\frac{a}{p(a^2+p^2)}}}$$

LHS

$$(f*g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

$$= \int_0^t 1 \cdot \sin a\tau\, d\tau$$

$$= \int_0^t \sin a\tau\, d\tau = -\left(\frac{\cos a\tau}{a}\right)_0^t$$

$$= \frac{-1}{a}(\cos a\tau)_0^t = \frac{-1}{a}(\cos at - 1) = \underline{\frac{1-\cos at}{a}}$$

Now, $L(f*g) = L\left(\frac{1-\cos at}{a}\right)$

$$= \frac{1}{a}\left(L(1) - L(\cos at)\right)$$

$$= \frac{1}{a}\left(\frac{1}{p} - \frac{p}{a^2+p^2}\right)$$

$$= \frac{1}{\cancel{a}}\left(\frac{a^2+\cancel{p^2}-\cancel{p^2}}{p(a^2+p^2)}\right)$$

$$= \frac{a}{p(a^2+p^2)}$$

LHS = RHS

Hence Verified

2.(c) $f(t) = t$, $g(t) = e^{at}$

RHS $L(t) \cdot L(e^{at}) = \frac{1}{p^2} \cdot \frac{1}{p-a}$

LHS

$$L(f*g)(t) = (f*g)(t)$$

$$= \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

$$= \int_0^t \underset{u}{(t-\tau)} \cdot \underset{v}{e^{a\tau}}\, d\tau$$

$$= \left\{(t-\tau)\int e^{a\tau} d\tau - \int \frac{d}{d\tau}(t-\tau) \cdot \int e^{a\tau} d\tau\right\}_0^t$$

$$= \left((t-\tau)\frac{e^{a\tau}}{a} + 1\int \frac{e^{a\tau}}{a} d\tau\right)_0^t$$

$$= \left((t-\tau) \cdot \frac{1}{a} \cdot e^{a\tau} + \frac{1}{a^2} e^{a\tau}\right)_0^t$$

$$= 0 + \frac{1}{a^2} e^{at} - \left(t \times \frac{1}{a} + \frac{1}{a^2} \right)$$

$$= \frac{1}{a^2} e^{at} - \frac{t}{a} - \frac{1}{a^2}$$

$$L(f * g) = L\left(\frac{1}{a^2} e^{at} - \frac{t}{a} - \frac{1}{a^2} \right)$$

$$= \frac{1}{a^2} \times \frac{1}{p-a} - \frac{1}{a} \times \frac{1}{p^2} - \frac{1}{a^2} \times \frac{1}{p}$$

$$= \frac{1}{a^2(p-a)} - \frac{1}{ap^2} - \frac{1}{a^2 p}$$

$$= \frac{p^2 - a(p-a) - p(p-a)}{a^2(p-a)p^2}$$

$$= \frac{\cancel{p^2} - \cancel{ap} + a^2 - \cancel{p^2} + \cancel{ap}}{a^2 p^2 (p-a)}$$

$$= \frac{\cancel{a^2}}{\cancel{a^2} p^2 (p-a)}$$

$$= \frac{1}{p^2(p-a)}$$

LHS = RHS

Hence Verified.

* Use convolution theorem to solve the differential eq:-

$y'' + 5y' + 6y = 5e^{3t}$ with 2nd order diff. equ.

$y'(0) = y(0) = 0$

Applying laplace on both sides,

$$L(y'') + 5L(y') + 6L(y) = L(5e^{3t})$$

$$p^2 L(y) - py(0) - y'(0) + 5pL(y) - 5y(0) + 6L(y) = 5L(e^{3t})$$

$$(p^2 + 5p + 6) L(y) = 5 \times \frac{1}{p-3}$$

$$L(f*g) = L(f).L(g)$$

$$(f*g) = L^{-1}(L(f).L(g))$$

$$L(y) = \frac{5}{(P-3)(P^2+5P+6)}$$

$$y = L^{-1}\left(\frac{5}{(P-3)(P^2+5P+6)}\right) \rightarrow \text{factorize.}$$

$$P^2+5P+6$$

$$\underline{\underline{3, 2}} \qquad S=5 \quad P=6$$

$$= L^{-1}\left(\frac{5}{(P-3)(P+3)(P+2)}\right)$$

$$y(t) = L^{-1}\left(\frac{5}{(P^2-3^2)(P+2)}\right)$$

From convolution theorem,

$$f*g = L^{-1}\left(L(f(t)).L(g(t))\right)$$

$$y(t) = 5L^{-1}\left(\frac{1}{P+2} \cdot \frac{1}{P^2-3^2}\right)$$

Multiply and Divide by 3

result for sinh at

$$= \frac{5}{3}L^{-1}\left(\frac{1}{P+2} \cdot \frac{3}{P^2-3^2}\right)$$

$$= \frac{5}{3}L^{-1}\Big(L(\underset{f(t)}{e^{-2t}}).L(\underset{g(t)}{\sinh 3t})\Big)$$

$$= \frac{5}{3}(\underbrace{e^{-2t} * \sinh 3t}_{f*g})$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad f*g = L^{-1}\left(L(f(t)).L(g(t))\right)$$

$$= \frac{5}{3}e^{-2t} * \sinh 3t$$

$$= \frac{5}{3}\int_0^t e^{-2(t-\tau)}.\sinh 3\tau \, d\tau$$

$$= \frac{5}{3}e^{-2t}\int_0^t e^{2\tau}\left(\frac{e^{3\tau}-e^{-3\tau}}{2}\right)d\tau$$

$$= \frac{5}{3}e^{-2t}\int_0^t (e^{5\tau}-e^{-\tau})d\tau$$

$$= \frac{5}{6}e^{-2t}\left[\frac{e^{5\tau}}{5} + e^{-\tau}\right]_0^t$$

$$= \frac{5}{6} e^{-2t} \left[\frac{e^{5t}}{5} + e^{-t} - \left(\frac{1}{5} + 1 \right) \right]$$

$$= \frac{5}{6} e^{-2t} \left[\frac{e^{5t}}{5} + e^{-t} - \frac{1}{5} - 1 \right]$$

$$= \frac{e^{3t}}{6} + \frac{5e^{-3t}}{6} - e^{-2t}$$

• 4 (c). Solve $y'' - y' = t^2$ with $y'(0) = y(0) = 0$

$$L(y'') - L(y') = L(t^2)$$

$$p^2 L(y) - p\,y(0) - y'(0) - pL(y) + y(0) = \frac{2!}{p^3}$$

$$Lt^2 = \frac{2!}{p^{2+1}}$$

$$(p^2 - p)\,L(y) = \frac{2}{p^3}$$

$2! = 2$

$$L(y) = \frac{2}{p^3(p^2 - p)}$$

$$= \frac{2}{p^4(p-1)}$$

$$y = 2\,L^{-1}\left(\frac{1}{p^4(p-1)} \right)$$

$$y = 2\,L^{-1}\left(\frac{1}{p^4} \cdot \frac{1}{p-1} \right)$$

$$L(x^3) = \frac{3!}{p^4} = \frac{6}{p^4}$$

Multi. & Divide by 3!,

$$y = \frac{2}{6} L^{-1}\left(\frac{6}{p^4} \cdot \frac{1}{p-1} \right)$$

$$\int f.g = f \int g\,dx - \int \frac{df}{dx} \int g\,dx$$

$$= \frac{2}{6} L^{-1}\left(\underset{f(t)}{L(t^3)} \cdot \underset{g(t)}{L(e^t)} \right)$$

From Convolution Theorem,

$$y = \frac{1}{3}(t^3 * e^t)$$

$$f * g = L^{-1}\left(L(f(t)) \cdot L(g(t)) \right)$$

$$= \frac{1}{3} \int_0^t (t - \tau)^3 . e^{\tau}\, d\tau$$

$$= \frac{1}{3}\left[(t-\tau)^3 \times e^\tau + \int 3(t-\tau)^2 e^\tau \, d\tau\right]$$

$$= \frac{1}{3}\left[(t-\tau)^3 e^\tau + 3\left((t-\tau)^2 e^\tau + 2\int (t-\tau) e^\tau \, d\tau\right)\right]$$

$$= \frac{1}{3}\left[(t-\tau)^3 e^\tau + 3(t-\tau)^2 e^\tau + 6(t-\tau)e^\tau + 6e^\tau\right]_0^t$$

$$= \frac{1}{3}\left(6e^t - (t^3 + 3t^2 + 6t + 6)\right)$$

$$= 2e^t - \frac{t^3}{3} - t^2 - 2t - 2$$

* $L(|x|) = L(x) = \frac{1}{p^2}$

$$\int_0^\infty e^{-px} |x| \cdot dx$$

$$L([x]) = \int_0^\infty e^{-px}[x]\,dx$$

$$= \int_0^1 e^{-px} x \, 0 dx + \int_1^2 e^{-px} \cdot 1 dx + \int_2^3 e^{-px} 2 dx + \ldots$$

* $[x] \rightarrow$ greatest Integers, $[1.2] = 1$

$[-1.2] = 2$

6-FOURIER SERIES - series only when f(x) is periodic

Periodic function.

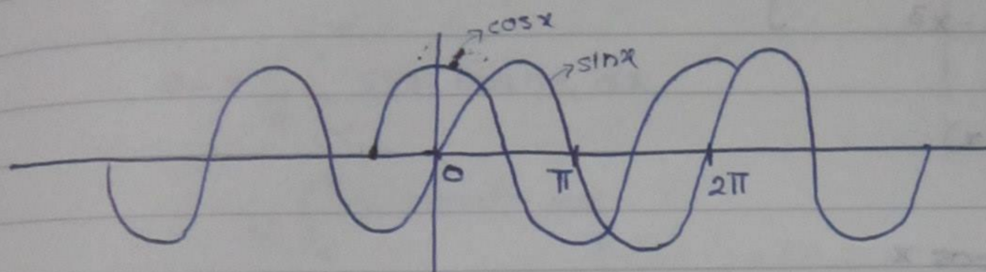
If $f(x+T) = f(x)$, then, fn. is period with T period

$f(x) = \sin x$

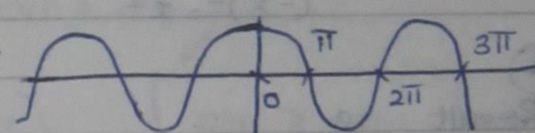
$f(x+2\pi) = \sin(2\pi + x)$ sin x has 2π period

$= \sin x$

$= f(x)$

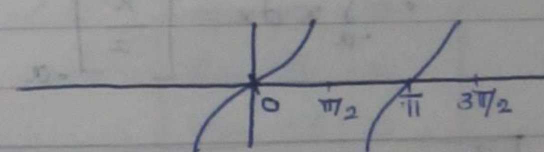


$f(x) = \cos x$ period $\to 2\pi$



$f(x) = \tan x$ period $\to \pi$

$= \tan(\pi + x)$



- If f(x) is periodic with period 2π in $(-\pi, \pi)$, then f(x) can be expressed in terms of Fourier series

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

where,

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx \, dx$$

<u>Odd fn</u>: $f(-x) = -f(x)$

(e.g) $\sin(-x) = -\sin x$
$(-x)^3 = -x^3$

$\Big\}$ $x, x^3, \sin x, \tan x$ are <u>ODD</u>

<u>Even fn</u>: $f(-x) = f(x)$

(e.g) $\cos(-x) = \cos x$
$(-x)^2 = x^2$

(*)
<u>Result</u>

- $\int_{-a}^{a} x \, dx = \left[\frac{x^2}{2}\right]_{-a}^{a} = \underline{\underline{0}}$

$$\boxed{\int_{-a}^{a} f(x) \, dx = 0 \quad \text{if } f(x) \text{ is } \underline{\underline{\text{ODD}}}}$$

(e.g) $\frac{1}{\pi} \int_{-\pi}^{\pi} \underset{\text{odd}}{\underset{\downarrow}{x^2}} \; \underset{\text{even}}{\underset{\downarrow}{\sin x}} \cdot dx$

odd

· Find Fourier Series of $f(x)=x$ for $-\pi \le x \le \pi$

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = 1/\pi \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi}$$

$$= \underline{\underline{0}}$$

$$\boxed{\begin{array}{l} \cos\pi = -1 \\ \cos 2\pi = 1 \\ \cos n\pi = (-1)^n \end{array}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left(x\int \cos nx\,dx - \int \frac{d}{dx}x \int \cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(x \cdot \frac{\sin nx}{n} - \frac{1}{n}\int \sin x\,dx\right)$$

$$= \frac{1}{\pi}\left(x \cdot \frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(0 + \frac{\cos n\pi}{n^2} - \frac{\cos(-n\pi)}{n^2}\right)$$

$$= \underline{\underline{0}}$$

$$\boxed{\begin{array}{l} \sin\pi = 0 \\ \sin 2\pi = 0 \\ \sin 3\pi = 0 \\ \sin n\pi = 0 \end{array}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{x}_{\text{odd}}\ \underbrace{\cos nx}_{\text{even}}}_{\text{odd}}\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underset{\text{odd}}{x}\cdot\underset{\text{even}}{\sin nx}}_{\text{even}}\,dx$$

$$= \frac{1}{\pi}\left(x\left(\frac{-\cos nx}{n}\right) + \frac{1}{n}\int \cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{-x\cos nx}{n} + \frac{1}{n^2}\sin x\right)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{-\pi\cos n\pi}{n} - \frac{\pi\cos n\pi}{n}\right)$$

$$= \frac{-2}{n}\cos n\pi \longrightarrow b_n = \frac{-2}{n} \text{ where } n \text{ is even}$$

$$= \frac{-2}{n}(-1)^n \qquad b_n = \frac{2}{n} \text{ where } n \text{ is odd}$$

$$= \frac{2(-1)^{n+1}}{n} \quad (\because \cos n\pi = (-1)^n)$$

Fourier series is:

$$F(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n \sin nx$$

$$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx$$

(or)

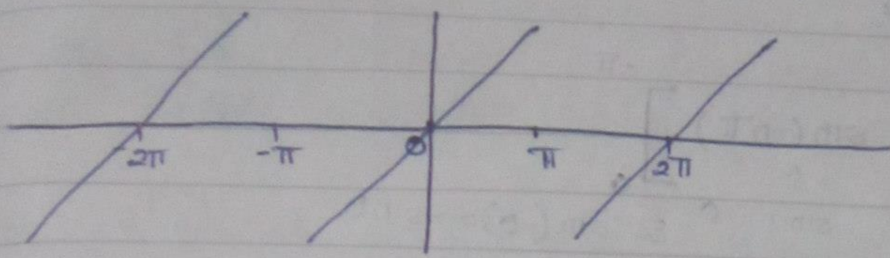
$$x = 2\sum_{n=1,3,5\ldots} \frac{\sin nx}{n} - 2\sum_{n=2,4,\ldots} \frac{\sin nx}{n}$$

$$x = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x \ldots$$

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 5x}{5} \ldots\right)$$

• $f(x) = x$

1) above problem → we assumed it is periodic



• Find the Fourier series for the periodic function with period 2π, i.e, $f(x+2\pi) = f(x)$

$$f(x) = \begin{cases} \pi & ; \ -\pi \le x \le \pi/2 \\ 0 & , \ \pi/2 < x \le \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi/2} \pi\,dx + \int_{\pi/2}^{\pi} 0 . dx\right]$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi/2} \pi\,dx$$

$$= [x]_{-\pi}^{\pi/2}$$

$$= \pi/2 - (-\pi)$$

$$= 3\pi/2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi/2} \pi\cos nx\,dx + \int_{-\pi/2}^{\pi} 0 \ \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi/2} \pi\cos nx\,dx$$

$$= \int_{-\pi}^{\pi/2} \cos nx \, dx$$

$$= \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi/2}$$

$$= \frac{1}{n}\left[\sin \frac{n\pi}{2} - \underset{\substack{\uparrow \\ \sin n\pi = 0}}{\sin(-n\pi)}\right]_0^{-\pi}$$

& $\sin(-\theta) = -\sin\theta$

$$\therefore a_n = \frac{\sin n\pi/2}{n} \quad ; \; n = 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi/2} \pi \sin nx \, dx + 0$$

$$= \left[\frac{-\cos nx}{n}\right]_{\pi}^{+\pi/2}$$

$$= \left(\frac{\cos n\pi - \cos n\pi/2}{n}\right)$$

$$\therefore b_n = \left(\frac{1 - \cos n\pi/2}{n}\right) \quad ; \; n = 2, 4, 6, \ldots$$

&

$$b_n = \frac{\cos n\pi}{n} \quad \text{for odd } n = 1, 3, 5, \ldots$$

Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{3\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{\sin n\pi/2}{n}\right)\cos nx + \sum_{n=2,4,6\ldots}^{\infty}\left(\frac{1-\cos n\pi/2}{2}\right)\sin nx$$

$$+ \sum_{n=1,3,5\ldots}^{\infty}\left(\frac{\cos n\pi}{n}\right)\sin nx$$

. Find Fourier Series for periodic function with 2π period
$f(x+2\pi) = f(x)$

$$f(x) = \begin{cases} 0 & ; -\pi \le x < 0 \\ \sin x & ; 0 \le x \le \pi \end{cases}$$

$$\boxed{2\sin A \cos B = \sin(A+B) + \sin(A-B)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.dx + \int_{0}^{\pi} \sin x\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$\boxed{\cos(-\theta) = \cos\theta}$$

$$= \frac{-1}{\pi}\left[\cos\pi - \cos 0\right]$$

$$= \frac{-1}{\pi}\left[-1-1\right]$$

$$= \underline{\underline{2/\pi}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[0 + \int_0^{\pi} \sin x \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\int_0 \sin x \cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin x \cos nx\,dx \quad \leftarrow \text{applying formula.}$$

$$= \frac{1}{2\pi}\int_0^{\pi} \left(\sin(1+n)x + \sin(1-n)x\right)dx$$

$$\boxed{\begin{array}{l} \sin n\pi = 0 \\ \cos n\pi = (-1)^n \end{array}}$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$\cos(n\pi+\pi) = \cos(n+1)\pi = (-1)^{n+1}$$

$$\cos(n-1)\pi = (-1)^{n-1}$$

$$= \frac{-1}{2\pi}\left[\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{-1}{2\pi}\left[\frac{\overset{(-1)^{n+1}}{\cos(n+1)\pi}}{n+1} - \frac{\cos(n-1)\pi}{n-1} - \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{-1}{2\pi}\left[\frac{((-1)^{n+1}-1)}{n+1} + \frac{(1-(-1)^{n-1})}{n-1}\right]$$

= If n is ODD, ~~substitute n = odd~~

~~so~~

$$\underline{\underline{a_n = 0}} \quad ; \quad n = 3, 5, 7 \ldots$$

but, n+1 is not defined

= If n is even.

$$a_n = \frac{1}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right)$$

$$= \frac{1}{\pi}\left(\frac{n-1-n-1}{n^2-1}\right)$$

$$= \underline{\underline{\frac{-2}{\pi(n^2+1)}}}$$

We need to find a_n for $n = 1$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$a_1 = \frac{1}{\pi}\left(0 + \int_0^{\pi} \sin \cos x\, dx\right.$$

$$a_1 = \frac{1}{2\pi}\int_0^{\pi} \sin 2x\, dx$$

$$= \underline{\underline{0}}$$

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left[0 + \int_0^{\pi} \sin x \sin nx \, dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left(\cos(1-n)x - \cos(1+n)\right) dx$$

$$\sin(n+1)\pi = 0$$
$$\sin(n-1)\pi = 0$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left(\cos(n-1)x - \cos(n+1)\right) dx$$

$$= \frac{1}{2\pi}\left(\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right)_0^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{\sin(n-1)\pi}{n-1} - \frac{\sin(n+1)\pi}{n+1} - 0 + 0\right)$$

$$= \frac{1}{2\pi}\left(\frac{\sin(n-1)\pi}{n-1} - \frac{\sin(n+1)\pi}{n+1}\right)$$

$b_n = 0$ except for $n=1$

We need to find b_n for $n=1$,

$$b_n = \frac{1}{\pi}\left[0 + \int_0^{\pi} \sin x \sin nx \, dx\right]$$

for $n=1$,

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \sin x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{1-\cos 2x}{2} \, dx$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} dx - \int_0^{\pi} \cos 2x \, dx\right] = \frac{1}{2\pi}\left[(x)_0^{\pi} - \left(\frac{\sin 2x}{2}\right)_0^{\pi}\right] = \frac{1}{2\pi}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi}$$

$$b_1 = \frac{1}{2\pi}(\pi) = 1/2$$

$a_1 = 0$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{1}{\pi} + \underbrace{0}_{a_1}.\cos nx_{\,n=1} + \sum_{n=3,5,\ldots}^{\infty} 0.\cos nx - \frac{2}{\pi}\sum_{n=2,4..}^{\infty} \frac{\cos nx}{n^2-1} +$$

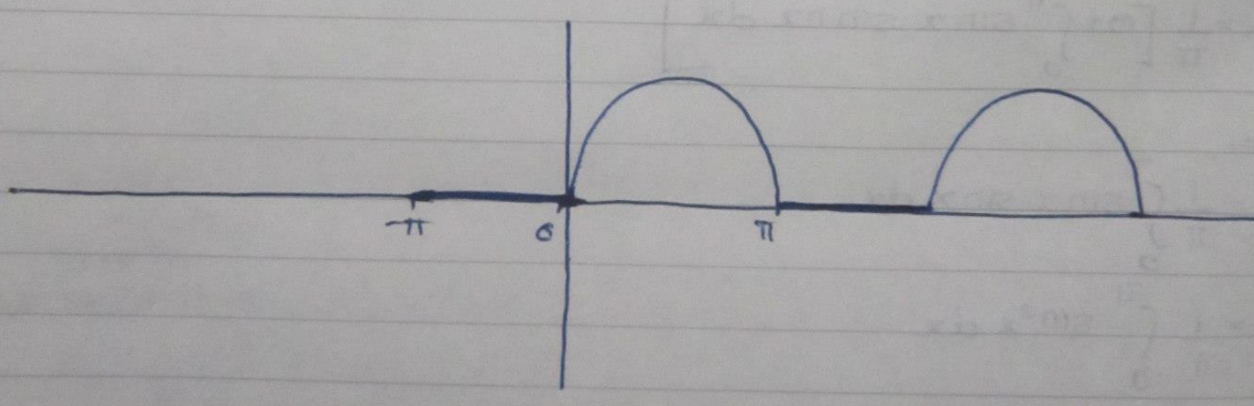
$$\underbrace{1/2}_{b_1} \cdot \sin x_{\,n=1} + \sum_{n=2}^{\infty} 0.\sin nx$$

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2,4,6,\ldots}^{\infty} \frac{\cos nx}{n^2-1} + \frac{1}{2}\sin x$$

(or)

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1} + \frac{\sin x}{2}$$

Graph



$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$$

Sec 34

Prob 4(a) Show that Fourier Series for the periodic fun.

$$f(x) = \begin{cases} 0 & , -\pi \le x < 0 \\ x^2 & , n \le x < \pi \end{cases}$$

is given by

$$f(x) = \frac{\pi^2}{6} + 2\sum_{\pi=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} + \pi \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)^n}{(2n-1)^3}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left(\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} x^2\, dx \right)$$

$$a_0 = \frac{1}{\pi} \left(\frac{x^3}{3} \right)_0^{\pi} \rightarrow \boxed{a_0 = \frac{\pi^2}{3}}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} x^2 \cos nx\, dx .$$

$$a_n = \frac{1}{\pi} \left(x^2 . \frac{\sin nx}{n} - \frac{2}{n} \int x \sin nx\, dx \right)$$

$$a_n = \frac{1}{\pi} \left(x^2 \frac{\sin nx}{n} - \frac{2}{n} \left(x . \left(-\frac{\cos nx}{n} \right) + \frac{1}{n^2} \sin nx \right) \right)$$

$$a_n = \frac{1}{\pi} \left(x^2 \frac{\sin nx}{n} + \frac{2x}{n^2} \cos nx - \frac{2}{n^3} \sin nx \right)_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left(\frac{2\pi}{n^2} \cdot \cos n\pi\right)$$

$$a_n = \frac{2\cos n\pi}{n^2} \Rightarrow \boxed{a_n = \frac{2(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{0}^{\pi} x^2 \sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left(x^2 \cdot \frac{-\cos nx}{n} + \frac{2}{n}\int x \cdot \cos nx\,dx\right)$$

$$b_n = \frac{1}{\pi}\left(\frac{-x^2\cos nx}{n} + \frac{2}{n}\left(\frac{x \cdot \sin nx}{n} + \frac{\cos nx}{n^2}\right)\right)$$

$$= \frac{1}{\pi}\left(\frac{-x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2}{n^3}\cos nx\right)_0^{\pi}$$

$$\boxed{b_n = \frac{1}{\pi}\left(\frac{-\pi^2\cos n\pi}{n} + \frac{2}{n^3}\cos n\pi - \frac{2}{n^3}\right)}$$

$$b_n = \frac{1}{\pi}\left(\frac{-\pi^2\cos n\pi}{n} + \frac{2}{n^3}(\cos n\pi - 1)\right)$$

$$\boxed{b_n = \frac{1}{\pi}\left(\frac{-\pi^2(-1)^n}{n} + \frac{2}{n^3}\left((-1)^n - 1\right)\right)}$$

$b_n = \dfrac{-\pi(-1)^n}{n}$; if n is even.

$$b_n = \frac{-\pi(-1)^n}{n} - \frac{4}{\pi n^3} \quad ; \text{ if } n \text{ is odd.}$$

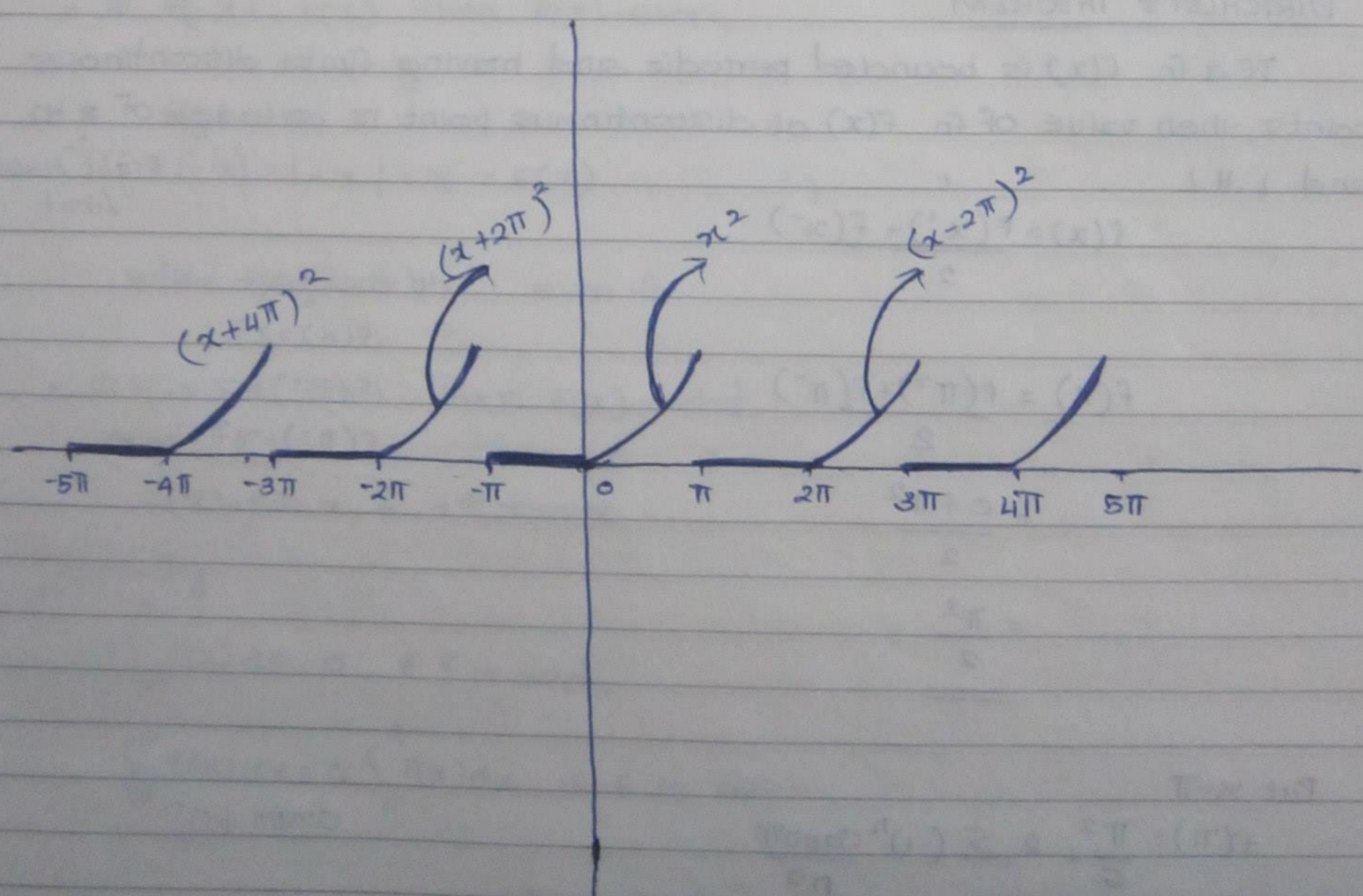
Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1} a_n \cos nx + \sum_{n=1} b_n \sin nx$$

$$f(x) = \frac{\pi^2}{6} + 2\sum_{n=1} \frac{(-1)^n}{n^2} \cos nx + \pi \sum \frac{(-1)^{n+1}}{n} \sin nx - \frac{4}{\pi} \sum \frac{1}{(2n-1)^3} \sin(2n-1)x$$

4 (b) Sketch the graph between $-5\pi \le x \le 5\pi$

$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ x^2 & 0 \le x < \pi \end{cases}$$



(c) Put $x=0$, in Fourier Series and show

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\cdots=\frac{\pi^2}{12}$$

$$f(x)=\frac{\pi^2}{6}+2\sum\frac{(-1)^n}{n^2}\times 1+0+0$$

$$f(0)=0=\frac{\pi^2}{6}+2\left(\frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\cdots\right)$$

$$-\frac{\pi^2}{6}=-2\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\right)$$

DIRICHLET'S THEOREM

If a fn. $f(x)$ is bounded periodic and having finite discontinous points, then value of fn. $f(x)$ at discontinous point is average of R.H.L (Right Hand Limit) and L.H.L

$$f(x)=\frac{f(\overset{R}{x^+})+f(\overset{L}{x^-})}{2}$$

In the graph above,

$f(x)=x^2$

$f(\pi^+)=0$

$f(\pi^-)=\pi^2$

$$f(\pi)=\frac{f(\pi^+)+f(\pi^-)}{2}$$

$$=\frac{0+\pi^2}{2}$$

$$=\frac{\pi^2}{2}$$

Put $x=\pi$

$$f(\pi)=\frac{\pi^2}{6}+2\sum(-1)^n\frac{\cos n\pi}{n^2}$$

$\cos n\pi=(-1)^n$

$$= \frac{\pi^2}{6} + 2 \sum \frac{(-1)^{2n}}{n^2}$$

$$= \frac{\pi^2}{6} + 2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + 2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

$$\boxed{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}}$$

Section 35

Even and Odd Functions

- If $f(-x) = f(x)$ then $f(x) \Rightarrow$ even.

$f(x) = |x|$

$f(-x) = |-x| = x = f(x)$

$|x|, \cos x, x^2, x^4,$ even fn.

- If $f(-x) = -f(x)$, then $f(x) \to$ odd

(e.g)

$f(x) = \sin x, x, x^3, \tan x$

Property:

$\int_{-a}^{a} f(x)\,dx = 0$ if f is odd

$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if f is even.

• Fourier series of $f(x)$ in $(-\pi, \pi)$ with period 2π

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$\boxed{a_0 = 0 \text{ , if } f \text{ is odd}}$$

$$\boxed{a_0 = \frac{2}{\pi} \int_{0}^{\pi} f(x) dx \text{ , if } f \text{ is even}}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{odd}} \underbrace{\cos nx}_{\text{even}}\, dx$$

$-$ $+$

$-$ (odd) * $\int$ odd is zero

$$\boxed{a_n = 0 \text{ , if } f \text{ is odd}}$$

$$\boxed{a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx\, dx \text{ ; } f \text{ is even}}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx\, dx$$

odd odd
$-$ $-$
$+$ (even)

even odd
$+$ $-$
$-$ (odd).

$$\boxed{b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx\, dx \text{ ; if } f \text{ is odd}}$$

$$\boxed{b_n = 0 \quad ; \text{ if } f \text{ is even}}$$

• Fourier series of $f(x)$ → even with period 2π in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad ; \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

If $f(x)$ is odd in $(-\pi, \pi)$
fourier series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

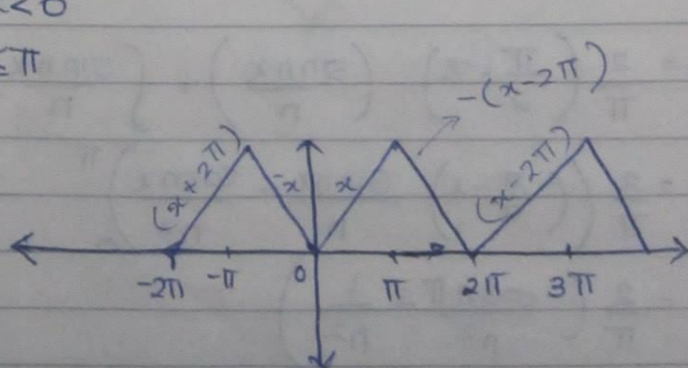
• Find Fourier series for:

1. $f(x) = \begin{cases} -x & -\pi \le x < 0 \\ x & 0 \le x \le \pi \end{cases}$

2. $f(x) = \begin{cases} x + \pi/2 & -\pi \le x < 0 \\ -x + \pi/2 & 0 \le x \le \pi \end{cases}$

An:

1. $f(x) = |x|$; $-\pi \le x \le \pi$
   ↓
   even



2. $f(x) = \frac{\pi}{2} - |x|$; $-\pi \leq x \leq \pi$

neither odd nor even.

$f(-x) = \frac{\pi}{2} - |-x|$

$= \frac{\pi}{2} - x$

$= f(x)$

$\therefore f$ is even.

$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$

since taking only positive value

$a_0 = \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi}{2} - x\right) dx$

$= \frac{2}{\pi} \left(\frac{\pi}{2} x - \frac{x^2}{2} \right)_0^{\pi}$

$\boxed{a_0 = 0}$

$a_n = \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi}{2} - x\right) \cos nx\,dx$

$a_n = \frac{2}{\pi} \left(\left(\frac{\pi}{2} - x\right) \cdot \left(\frac{\sin nx}{n}\right) + \int \frac{\sin nx}{n}\,dx \right)$

$= \frac{2}{\pi} \left(\left(\frac{\pi}{2} - x\right) \frac{\sin nx}{n} - \frac{\cos nx}{n^2} \right)_0^{\pi}$

$= \frac{2}{\pi} \left(-\frac{\cos n\pi}{n^2} + \frac{1}{n^2} \right)$

$$a_n = \frac{2}{\pi}\left(\frac{1-(-1)^n}{n^2}\right)$$

$a_n = 0$ for even

$a_n = \frac{4}{\pi n^2}$ for odd

Fourier Series is

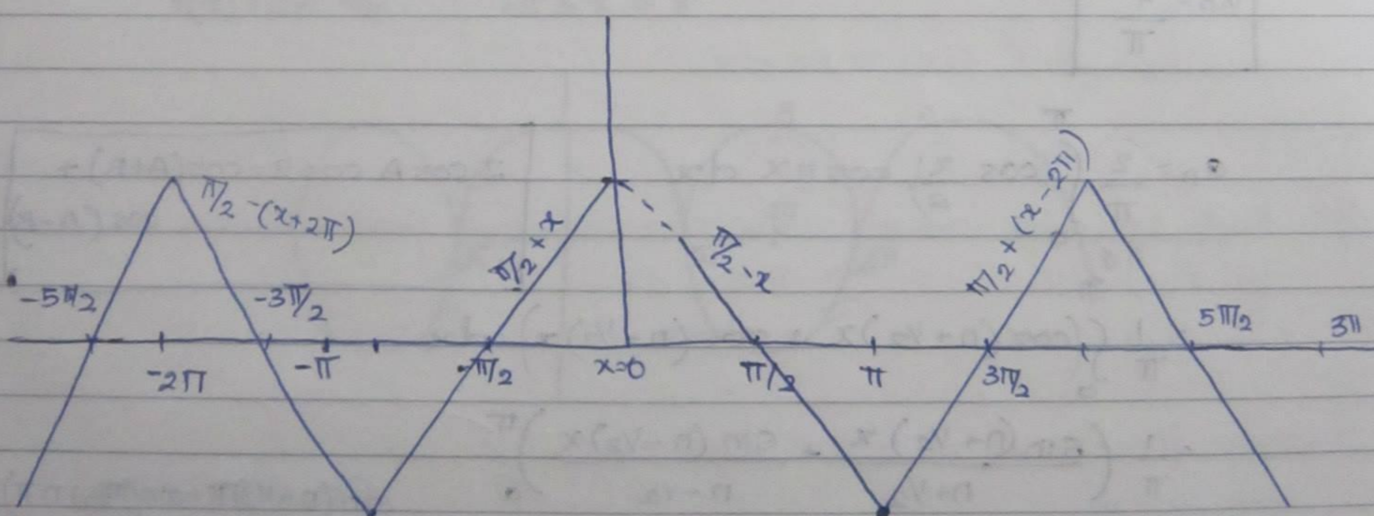
$$f(x) = 0 + \frac{4}{\pi} \sum_{n=1,3,5,\dots} \frac{\cos nx}{n^2}$$

$$\boxed{f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}}$$

Graph: $f(x) = \begin{cases} \pi/2 + x & -\pi \le x < 0 \\ \pi/2 - x & 0 \le x \le \pi \end{cases}$

(or)

$f(x) = \pi/2 - |x|$



• Find Fourier Series in $(-\pi, \pi)$ with 2π period for

$f(x) = \cos x/2 \qquad -\pi \le x \le \pi$

Sketch between $-5\pi \le x \le 5\pi$

$$f(-x) = \cos\left(\frac{-x}{2}\right) = \cos\frac{x}{2} = f(x)$$

$\Rightarrow$ even function.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos\frac{x}{2}\,dx$$

$$= \frac{2}{\pi}\left(2\sin\frac{x}{2}\right)_0^{\pi}$$

$$\boxed{a_0 = \frac{4}{\pi}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos\overset{A}{\frac{x}{2}}\,\cos\overset{B}{nx}\,dx$$

$$\boxed{2\cos A\cos B = \cos(A+B) + \cos(A-B)}$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\cos(n+1/2)x + \cos(n-1/2)x\right)dx$$

$$= \frac{1}{\pi}\left(\frac{\sin(n+1/2)x}{n+1/2} + \frac{\sin(n-1/2)x}{n-1/2}\right)_0^{\pi}$$

$\sin(n+1/2)\pi = \sin(\pi/2 + n\pi)$

$= \cos n\pi$

$\sin(n-1/2)\pi = \sin(n\pi - \pi/2)$

$= -\sin(\pi/2 - n\pi)$

$= -\cos n\pi$

$$= \frac{1}{\pi}\left(\frac{\cos n\pi}{n+1/2} - \frac{\cos n\pi}{n-1/2}\right)_0^{\pi}$$

$$= \frac{\cos n\pi}{\pi}\left(\frac{2}{2n+1} - \frac{2}{2n-1}\right)_0^{\pi}$$

$$= \frac{2\cos n\pi}{\pi}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right)_0^{\pi}$$

$$= \frac{2\cos n\pi}{\pi}\left(\frac{2n-1-2n-1}{4n^2-1}\right)$$

$$= \frac{-4}{\pi}\left(\frac{(-1)^n}{4n^2-1}\right)$$

Fourier Series is

$$f(x) = \cos x/2 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos nx$$

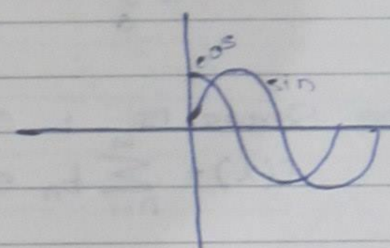
Sketch:

$\sin(\pi/2 + \theta)$ (or) $\sin(\pi - (\pi/2 - \theta))$

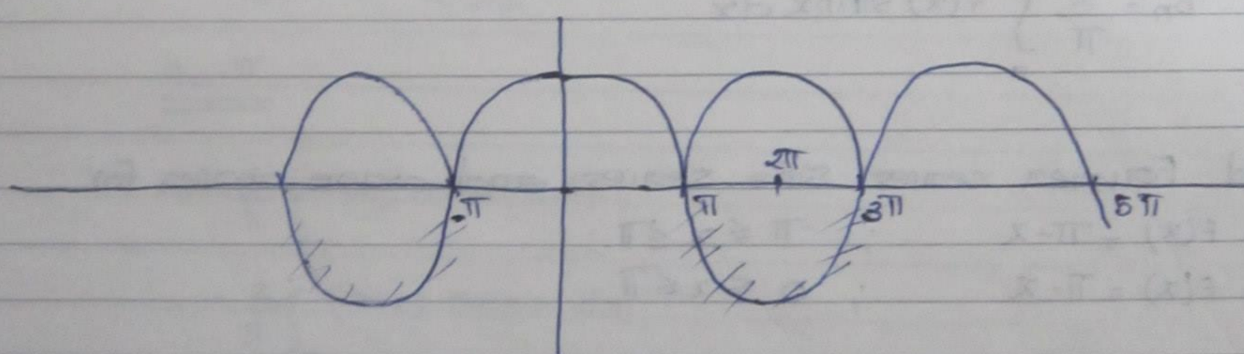
$\sin(\pi - x)$

$\sin(\pi/2 - \theta)$

$= \cos\theta$



$f(x) = \cos x/2 \qquad -\pi \le x \le \pi$



$$f(x) = f(x+2\pi)$$

$$= \cos\left(\frac{x+2\pi}{2}\right)$$

$$= \cos(x/2 + \pi)$$

$$= -\cos\frac{x}{2}$$

COSINE AND SINE SERIES

For given $f(x)$ is defined in interval $(0,\pi)$

Cosine ~~Fourier~~ series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Sine Series is : sine series in $(0,\pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Find Fourier series, Sine series and cosine series for

(i) $f(x) = \pi - x$; $-\pi \leq x \leq \pi$

(ii) $f(x) = \pi - x$; $0 \leq x \leq \pi$

- For $f(x) = \pi - x$; $0 \leq x \leq \pi$

sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left((\pi-x)\left(\frac{-\cos nx}{n}\right) - \frac{\sin nx}{n^2}\right)_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{\pi}{n}\right)$$

$$b_n = \frac{2}{n}$$

Sine series $f(x) = \pi - x = \sum_{n=1}^{\infty} b_n \sin nx$

$$\pi - x = 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

- $F(x) = \pi - x \quad ; -\pi \le x \le \pi$

Cosine series

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\,dx$$

$$= \frac{2}{\pi}\left(\pi x - \frac{x^2}{2}\right)_0^{\pi}$$

$$\underline{\underline{a_0 = \pi}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left((\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right)_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\cos n\pi}{n^2} + \frac{1}{n^2}\right)$$

$$= \frac{2}{\pi n^2}(1 - \cos n\pi)$$

$$= \frac{2}{\pi n^2}(1 - (-1)^n)$$

$a_n = 0$ for even. $\quad a_n = \frac{4}{\pi n^2}$ for odd $n = 1, 3, 5, \ldots$

cosine series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

- Fourier Series for any interval $(-L, L)$

If $f(x)$ is periodic (i.e) $f(x+2L) = f(x)$ with period $2L$ defined in $(-L, L)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{L} + \sum_{n=1}^{\infty} b_n \frac{\sin n\pi x}{L}$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx$$

- Find Fourier Series for $f(x+4) = f(x)$

$f(x) = |x|$ in $-2 \leq x \leq 2$

Here $L = 2$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{2} \int_{-2}^{2} |x|\,dx$$

$$= 1/2$$

$$= \frac{2}{\cancel{2}} \int_{0}^{2} |x|\,dx \quad = \frac{2}{2} \int_{0}^{2} |x|\,dx$$

$$\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x\,dx \quad \text{if } f \text{ is even}}$$

$$= \int_{0}^{2} x\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^2$$

$$= 2$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cdot \frac{\cos n\pi x}{L}\, dx$$

even x even = even

$$= \frac{1}{2}\int_{-2}^{2} |x| \frac{\cos n\pi x}{2} \cdot dx$$

$$= \frac{1}{2} \times 2 \int_{0}^{2} x \frac{\cos n\pi x}{2} dx$$

$$= \int_{0}^{2} x \frac{\cos n\pi x}{2} dx$$

$$= \left(x \cdot \left(\frac{\sin n\pi/2\, x}{n\pi/2} \right) + \frac{4}{n^2\pi^2} \frac{\cos \pi n x}{2} \right)_0^2$$

$$= \frac{4}{n^2\pi^2} \cos n\pi - \frac{4}{n^2\pi^2}$$

$$= \frac{4}{n^2\pi^2} \left((-1)^n - 1 \right)$$

$a_n = 0$; n = even

$a_n = \frac{-8}{n^2\pi^2}$; n = odd

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \frac{\sin n\pi x}{L} dx$$

$$= \frac{1}{2}\int_{-2}^{2} |x| \frac{\sin n\pi x}{2} dx$$

even x odd = odd

$$= \int_{-a}^{a} f(x) dx = 0 \Rightarrow b_n = 0$$

odd

Fourier Series is

$$f(x) = \frac{2}{2} - \frac{8}{\pi^2} \sum \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{2}$$

cosine and sine series in $(0,L)$

Fn. $f(x)$ in $(0,L)$ can be expressed in cosine series

$$f(x) = \frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{2}$$

where $a_0 = \frac{2}{L} \int_0^L f(x)\,dx$

$$a_n = \frac{2}{L} \int_0^L f(x) \frac{\cos n\pi x}{L} dx$$

sine series can be expressed as

$$f(x) = \sum_{n=1}^{\infty} b_n \frac{\sin n\pi x}{L}$$

where $b_n = \frac{2}{L} \int_0^L f(x) \frac{\sin n\pi x}{L} dx$

- Find cosine series.

$$f(x) = \begin{cases} 2, & 0 \le x \le 1 \\ 0, & 1 < x \le 2 \end{cases}$$

for $(0,L) \xrightarrow{L} (0,2)$

$$a_0 = \frac{2}{L} \int_0^L f(x)\,dx$$

$$= \frac{2}{2} \int_0^2 f(x)\,dx$$

$$= \int_0^1 2\,dx + \int_1^2 0\,dx$$

$$= \int_0^1 2\,dx$$

$$= 2(x)_0^1 \qquad \underline{a_0 = 2}$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\, dx$$

$$= \frac{2}{2}\int_0^2 f(x) \cos\frac{n\pi x}{2}\, dx$$

$$= \int_0^1 2\cos\frac{n\pi x}{2}\, dx + \int_1^2 0 . \cos\frac{n\pi x}{2}\, dx$$

$$= 2\left(\frac{\sin n\pi x/2}{n\pi/2}\right)_0^1$$

$$= \frac{4}{n\pi}\left(\sin\frac{n\pi}{2}\right)$$

$$= \frac{4}{n\pi}\sin\frac{n\pi}{2}$$

n is even, $a_n = 0$

$a_n = \frac{4}{n\pi}(\pm 1)$ for odd

cosine series

$$f(x) = \frac{2}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1} . \frac{\cos(2n-1)\pi x}{2}$$

- Find the sine series for $f(x) = 1$, $0 \le x \le 2$

Sine series: $f(x) = \sum b_n \sin\frac{n\pi x}{L}\, dx.$

where,

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{2\pi x}{L}\, dx$$

Here

$(0, L) = (0, 2)$

$\underline{\underline{L = 2}}$

$$b_n = \frac{2}{2}\int_0^2 1.\frac{\sin n\pi x}{2}\,dx = \int_0^2 \frac{\sin n\pi x}{2}\,dx$$

$$= \left(\frac{-\cos n\pi x/2}{n\pi/2}\right)_0^2$$

$$= \frac{-2}{n\pi}\left(\cos\frac{n\pi x}{2}\right)_0^2$$

$$= \frac{-2}{n\pi}(\cos n\pi - 1)$$

$$= \frac{2}{\pi}\left(\frac{1-\cos n\pi}{n}\right)$$

$2n-1$ odd
$2n$ even

$b_n = 0$, for even

$b_n = \frac{4}{\pi n}$, for odd

Sine series $\rightarrow f(x) = \sum_{n=1}^{\infty} b_n \frac{\sin n\pi x}{L}\,dx$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)} \frac{\sin (2n-1)\pi x}{2}$$

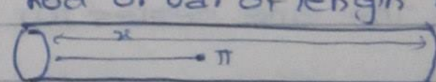
Application

1. One Dimensional Heat Equation.

Heat equation is given by,

$$\boxed{\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}}$$

Rod or bar of length $L=\pi$



$u(x,t) \rightarrow$ temp fn

Assume end point of bar is having 0 temp

$u(0,t) = 0$
$u(L,t) = 0$ — ①

Initial temperature, $u(x,0) = f(x)$ — ②

By using ①, ②, solu. of heat equ. $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$

(i.e) $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2a^2t} . \sin nx$

where,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x). \sin nx \, dx$$

$a^2 = \frac{K}{\rho c}$ k, thermal conductivity

ρ, density c, Specific heat.

- The two end of the metal bar of length π are maintained at $0°$ at $t=0$. And the initial temperature distribution in the body of the bar is given by $kx(\pi-x)$. Find the temp. distribution in the metal bar at any time t.

Here initial temp, $u(x,0) = f(x)$
$= kx(\pi-x)$

Temperature distribution:

$$u(x,t) = \sum_{n=1} b_n e^{-n^2a^2t} . \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} kx(\pi-x) \sin nx \, dx$$

$$= \frac{2k}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$

$$= \frac{2k}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) + \frac{1}{n}\int (\pi - 2x)(\cos nx \, dx)\right]$$

$$= \frac{2k}{\pi}\left((x^2-\pi x)\frac{\cos nx}{n} + \frac{1}{n}\left((\pi-2x)\frac{\sin nx}{n} - \frac{2\cos nx}{n^2}\right)\right)_0^{\pi}$$

$$= \frac{2K}{\pi}\left((x^2-\pi x)\frac{\cos nx}{n} + \frac{(\pi-2x)\sin nx}{n^2} - \frac{2}{n^3}\cos nx \right)_0^{\pi}$$

$$= \frac{2K}{\pi}\left(\frac{-2\cos n\pi}{n^3} + \frac{2}{n^3} \right)$$

$$= \frac{4K}{\pi}\left(\frac{1-\cos n\pi}{n^3} \right)$$

$\cos n\pi \to$ odd $= 1$

even $= 0$

For n even,

$b_n = 0$

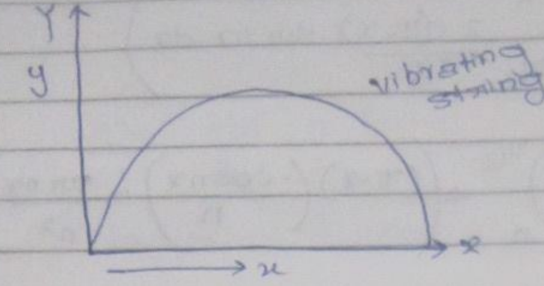
n odd,

$$b_n = \frac{4K}{\pi}\left(\frac{2}{n^3}\right) = \frac{8K}{\pi}\cdot\frac{1}{n^3}$$

Temperature distribution $u(x,t) = \sum_{n=1}^{\infty} b_n \cdot e^{-n^2a^2t} \cdot \sin nx$

$$u(x,t) = \frac{8K}{\pi}\sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2a^2t}}{(2n-1)^3} \cdot \sin(2n-1)x$$

One dimensional wave equ.



One dim. wave equ

$$\boxed{\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}}$$

$a^2 = T/m$

$a^2 = \frac{\text{Tension}}{\text{mass}}$

y - vertical displacement

x - position

t - time

y - is a fn. of x and t

Assumption

1. $y(0,t) = 0$
   $y(\pi,t) = 0$ > Boundary condition.

2. Initial motion is zero

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$$

Initial shape of string

$$y(x,0) = f(x)$$

> Initial Condition.

Solution of wave equ. by using boundary and initial condition is:

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} b_n \sin x \cos n a t}$$

where,

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) . \sin nx \, dx$$

$f(x) = y(x,0)$ = Initial shape

Section 40

1. Solve the vibrating string prob. if the initial shape is given by the fn. $f(x) = \begin{cases} 2cx/\pi, & 0 \le x \le \pi/2 \\ \frac{2c(\pi - x)}{\pi} & \pi/2 \le x \le \pi \end{cases}$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left(\int_0^{\pi/2} \frac{2cx}{\pi}\sin nx\,dx + \int_{\pi/2}^{\pi} \frac{2c(\pi-x)}{\pi}\sin nx\,dx\right)$$

$$= \frac{4c}{\pi^2}\left[\left(x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right)_0^{\pi/2} + \left((\pi-x)\left(\frac{-\cos nx}{n}\right) - \frac{\sin nx}{n^2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{4c}{\pi^2}\left(-\frac{\pi}{2}\frac{\cos n\pi/2}{n} + \frac{\sin n\pi/2}{n^2} - \left(\frac{\pi}{2}\left(\frac{-\cos n\pi/2}{n}\right) - \frac{\sin n\pi/2}{n^2}\right)\right)$$

$$= \frac{4c}{\pi^2}\left(\frac{2\sin n\pi/2}{n^2}\right)$$

$$= \frac{8c}{\pi^2}\left(\frac{\sin n\pi/2}{n^2}\right)$$

$\sin n\pi/2 = \pm 1$ odd

$\cos n\pi/2 = \pm 1$ even

$b_n = 0$, for even n

Solution is,

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \cos nat$$

$$= \frac{8c}{\pi^2}\sum_{n=1} \frac{\sin n\pi/2}{n^2}\cos nat \sin nx$$

$$= \frac{8c}{\pi^2}\sum_{n=1} \frac{(-1)^{n+1}}{(2n-1)^2}\sin(2n-1)x\cos(2n-1)at$$

Laplace Equation

Two dimensional Heat equ. is given by

$$\frac{\partial \omega}{\partial t} = a^2\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

ω - Temperature

t - time

x, y - dimensions

For steady state condition

$$\frac{\partial \omega}{\partial t} = 0$$

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} = 0$$

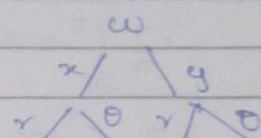
known as Laplace equ

put $x = r\cos\theta$ (polar)
$y = r\sin\theta$

Equ ① will be

$$\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \omega}{\partial \theta^2} = 0$$

Laplace in polar form



Solution of ② is given by,

$$\omega(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{r} r^n (a_n \cos\theta + b_n \sin\theta)$$

a_0, a_n, b_n same as Fourier coeff

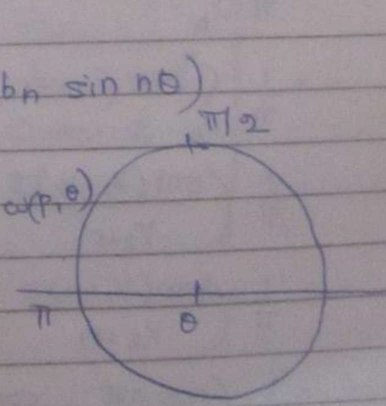
Dirichlet's Problem

In a disc of radius ρ, the temperature distribution region $R: \{0 \le r < \rho,\ 0 < \theta \le 2\pi\}$

Solution is

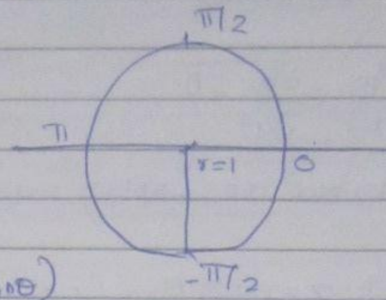
$$\omega(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r_n (a_n \cos n\theta + b_n \sin n\theta)$$

under the boundary condition
$\omega(\rho,\theta) = f(\theta)$
for $0 \le \theta \le 2\pi$



Problem

Solve Dirichlet's problem for the unit circle, if the boundary condition is $f(\theta)=\cos\theta/2$, $-\pi\le\theta\le\pi$



$$u(1,\theta)=f(\theta)=\cos\theta/2$$

Solution is given by

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^n\left(a_n\cos n\theta+b_n\sin n\theta\right)$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\,d\theta$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos\theta/2\,d\theta$$

$$=\frac{2}{\pi}\int_{0}^{\pi}\cos\theta/2\,d\theta \quad (\because \cos\theta/2 \text{ is even})$$

$$=\frac{2}{\pi}\left(\frac{\sin\theta/2}{1/2}\right)_0^{\pi}$$

$$=\frac{4}{\pi}(1-0)$$

$$=\underline{\underline{4/\pi}}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\cos n\theta\,d\theta=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underbrace{\cos\theta/2}_{\text{even}}\cdot\underbrace{\cos n\theta}_{\text{even}}}_{\text{even}}\,d\theta$$

$$a_n=\frac{2}{\pi}\int_{0}^{\pi}\cos\theta/2\cdot\cos n\theta\,d\theta$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\cos\left(1/2+n\right)\theta+\cos\left(1/2-n\right)\theta\,d\theta$$

$$=\frac{1}{\pi}\left(\frac{\sin(1/2+n)\theta}{1/2+n}+\frac{\sin(1/2-n)}{1/2-n}\right)_0^{\pi}$$

$$=\frac{1}{\pi}\left(\frac{\sin(1/2+n)\pi}{1/2+n}+\frac{\sin(1/2-n)\pi}{1/2-n}\right)$$

$$=\frac{1}{\pi}\left(\frac{\cos n\pi}{1/2+n}+\frac{\cos n\pi}{1/2-n}\right)$$

$$= \frac{\cos n\pi}{\pi}\left(\frac{1}{1+2n/2} + \frac{1}{1-2n/2}\right)$$

$$= \frac{(-1)^n}{\pi}\left(\frac{2(1-2n)+2(1+2n)}{(1+2n)(1-2n)}\right)$$

$$\underline{\underline{a_n = \frac{(-1)^n \cdot 4}{(1-4n^2)\pi}}} \qquad n = 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cdot \sin n\theta \, d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\cos \theta/2}_{\text{even}} \cdot \underbrace{\sin \theta}_{\text{odd}} \, d\theta = \underline{\underline{0}}$$

odd

Solu. (Temperature distribution)

$$w(r,\theta) = \frac{a_0}{2} + \sum_{n=1} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

$$\underline{= \frac{2}{\pi} + \frac{4}{\pi}\sum r^n\left(\frac{(-1)^n}{1-4n^2} \cdot \cos n\theta\right)}$$

for $r=1$

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin x$	$\rightarrow$ Linear	Order: 2, degree: 1
$\frac{dy}{dx} + y^2 = 1$	$\rightarrow$ Non-Linear	Order = 1, degree = 2
$\frac{d^2y}{dx} + e^x = y$	$\longrightarrow$ Linear	Order: 2, degree: 1

1 dependent 'x'

↳ Next chapter

Linear Differential Equation

1. If Degree is dependent variable $y=1$.
2. Degree of derivatives $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\right)$ should be 1
3. No product of dependent variable (y) with derivatives should occur.

$$(e.g)\left(x\frac{dy}{dx}\right)+\left(\frac{d^2y}{dx}\right)+\left(\frac{d^3y}{dx^3}\right)=\sin x \quad \text{(linear)}$$

$$\left(y\frac{dy}{dx}\right)+\left(\frac{d^2y}{dx^2}\right)^2=\left(\frac{d^3y}{dx^3}\right)^{1/2}=\sin y \quad \text{(non-linear)}$$

$$\frac{dy}{dx}+\frac{y}{\sin x}=c^x.\sin 3x \quad \text{(linear)}$$

Section 8

First Order

Let a fn. of 2 variable $f(x,y)=C$ given

$f(x,y)=C$

Using chain rule $\frac{\partial f}{\partial x}.dx+\frac{\partial f}{\partial y}.dy=0$

$$\boxed{M\,dx+N\,dy=0 \quad \text{is exact if} \quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$$

$$\frac{dy}{dx}=-\frac{M}{N}$$

- Differential equ :- An equ. containing various order derivatives is called differential equ.

$$\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\sin 2x+x^2 \qquad \text{order}=2$$

- Ordinary diff. equ :- If diff equ. has only 1 independent variable
- Partial diff. equ :- If there exists more than 1 independent variable

$$(e.g)\ \frac{d^2y}{dx^2}+\frac{d^2y}{dz^2}=xz$$

• Order of a diff. equ :- It is an order of highest order derivative occuring in diff. equ.

(eg) $\left(x\frac{dy}{dx}\right)^3 + \left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{d^3y}{dx^3}\right)^1 = \sin x$ order = 3

• Degree of diff. equ :- It is the degree or power of highest order derivative in differential equation.

$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{d^3y}{dx^3}\right) + y^5 = 0$ degree = 1 ; Order = 3

2. Check if following is exact or not : $\underbrace{(\sin x \tan y + 1)}_{M}dx + \underbrace{(\cos x \sec^2 y)}_{N}dy = 0$

$\frac{\partial M}{\partial y} = \sin x(\sec^2 y)$ $\frac{\partial N}{\partial x} = -\sin x \sec^2 y$

Not exact.

$x+y^3$

* First order diff equ. $Mdx + Ndy = 0$ is said to be exact if

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

then solution is given $\int Mdx + \int Ndy$ = constant C

Take only those terms who are free from x.

Section 8

(3) Solve : $(y - x^3)dx + (x + y^3)dy = 0$

$M = y - x^3$

$N = x + y^3$

Partially differentiate with respect to y : $\frac{\partial M}{\partial y} = 1$

" " " to x - $\frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ⇒ exact diff. equ.

choose only y term.

Solution is : $\int Mdx + \int Ndy = C$

$\int(y - x^3)dx + \int y^3 dy = C$

$yx - \frac{x^4}{4} + \frac{y^4}{4} = C$

(7) $(\sin x \sin y - xe^y)dy = (e^y + \cos x \cos y)dx$. Solve.

$\downarrow N \qquad \downarrow M$

Non-linear, Order: 1

$M = e^y + \cos x \cos y$

$\frac{\partial M}{\partial y} = e^y + \cos x \cdot (-\sin y)$

$= e^y - \cos x \sin y$

$N = -(\sin x \sin y - xe^y)$

$\frac{\partial N}{\partial x} = -\cos x \sin y + e^y$

$= e^y - \cos x \sin y$.

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ exact diff. equ.

$\therefore$ The solution is: $\int M dx + \int N dy = C$

nothing is free of x, hence zero.

$\int e^y + \cos x \cos y \, dx + \int 0 \cdot dy = C$

$e^y \cdot x + \sin x \cos y + c_1 = c_2$

$\underline{e^y \cdot x + \sin x \cos y = C}$

(12) $(2xy^4 + \sin y)dx + (4x^2y^3 + x \cos y)dy = 0$

$M dx + N dy = 0$

$M = 2xy^4 + \sin y$

$\frac{\partial M}{\partial y} = 8xy^3 + \cos y$

$N = 4x^2y^3 + x \cos y$

$\frac{\partial N}{\partial x} = 8xy^3 + \cos y$.

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ exact diff. equ.

$\therefore$ The solution is: $\int M dx + \int N dy = C$

$\int 2xy^4 + \sin y \, dx + \int 4x^2y^3 + x \cos y \, dy = C$

$\underline{x^2y^4 + \sin y \cdot x = C}$

(16) $(e^{y^2} - \mathrm{cosec}^2x \cdot \mathrm{cosec}\, y)dx + (2xye^{y^2} - \mathrm{cosec}\, y \cot y \cot x)dy = 0$

$M = e^{y^2} - \mathrm{cosec}^2x \cdot \mathrm{cosec}\, y$

$\frac{\partial M}{\partial y} = e^{y^2} \cdot 2y - \mathrm{cosec}^2x\,(-\mathrm{cosec}\, y \cot y)$

$$\boxed{\begin{array}{l} \frac{d}{dx}\cot x = -\mathrm{cosec}^2x \\ \frac{d}{dx}\mathrm{cosec}\, x = -\mathrm{cosec}\, x \cdot \cot x \end{array}}$$

$\frac{\partial N}{\partial x} = 2ye^{y^2} - \mathrm{cosec}\, y \cdot \cot y\,(-\mathrm{cosec}^2x)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y} \Rightarrow$ exact diff. equ.

∴ The solution is: $\int M dx + \int N dy = C$

$\cancel{\int e^{y^2} - \mathrm{cosec}^2x \cdot \mathrm{cosec}\, y\, dx + \int 2xye^{y^2} \mathrm{cosec}\, y \cot y \cot x\, dy = C}$

$\int (e^{y^2} - \mathrm{cosec}^2x \cdot \mathrm{cosec}\, y)dx + \int 0 \cdot dy = C$

$e^{y^2} \cdot x + \cot x \cdot \mathrm{cosec}\, y = C$

• Solve: $(xy-1)dx + (x^2 - xy)dy = 0$

$M = xy - 1$ $\quad$ $N = x^2 - xy$

$\frac{\partial M}{\partial y} = x$ $\quad$ $\frac{\partial N}{\partial x} = 2x - y$

It is not exact

If any first order diff. equ is not exact, then we have to multiply with some function μ to make it exact.

μ - known as integrating factors.

When $\partial M/\partial y$ and $\partial N/\partial x$ are not equal:

case 1: Take difference and divide by N

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \quad \text{then,} \quad I.F = \mu = e^{\int f(x)dx}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{x-(2x-y)}{x^2-xy} = \frac{y-x}{x(x-y)} = \frac{-1}{x} = f(x)$$

$$\text{I. factor} = \mu = e^{-\int 1/x\,dx}$$
$$= e^{-\log x}$$
$$= e^{\log 1/x}$$
$$= \frac{1}{x}$$

$$\mu\left[(xy-1)\,dx + (x^2-xy)\,dy\right] = 0$$

$M' = \frac{xy-1}{x}$ $\qquad$ $N' = \frac{x^2-xy}{x}$

$= y - \frac{1}{x}$ $\qquad$ $= x-y$

$\frac{\partial M'}{\partial y} = 1$ $\qquad$ $\frac{\partial N'}{\partial x} = 1$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \Rightarrow \text{exact}$$

$\therefore$ The solution is : $\int (y - 1/x)\,dx + \int -y\,dy = C$

$$yx - \log x - \frac{y^2}{2}$$

$$2xy - \log x^2 - y^2 = 2C = C_1$$

Case II

If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = f(y)$ then integrating factor (I.F)

$$\mu = e^{\int f(y)\,dy}$$

Section 9

Problem 2(d)

Solve: $e^x dx + [e^x \cot y + 2y \operatorname{cosec} y]\, dy = 0$

Here, $M = e^x \quad \partial M/\partial y = 0$

$N = e^x \cot y + 2y \operatorname{cosec} y \quad \partial N/\partial x = e^x \cot y$

Here,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{not exact}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0 - e^x \cot y$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{-e^x \cot y}{-e^x} = \cot(y) = f(y)$$

$$\mu = I.F = e^{\int \cot y\, dy}$$
$$= e^{\log \sin y}$$
$$= \sin y$$

$$\boxed{\begin{aligned}\int \cot x\, dx &= \log \sin x \\ \int \tan x\, dx &= \log \sec x\end{aligned}}$$

Multiplicating with μ

$$\sin y \left[e^x dx + (e^x \cot y + 2y \operatorname{cosec} y)\, dy\right] = 0$$

$M' = \sin y\, e^x \qquad N' = \sin y\, e^x \cot y + 2y \sin y \operatorname{cosec} y$

$\frac{\partial M'}{\partial y} = \cos y . e^x \qquad N' = e^x \cos y + 2y$

$\frac{\partial N'}{\partial x} = e^x \cos y$

Now it is an exact diff. equ.

Solu. is

$$\int M' dx + \int N' dy = C$$
$$\int \sin y . e^x\, dx + \int 2y\, dy = C$$

$$\sin y . e^x + y^2 = C$$

2(e). Solve : $(x+2)\sin y\,dx + x\cos y\,dy = 0$

$M = (x+2)\sin y$ $\qquad$ $N = x\cos y$

$\frac{\partial M}{\partial y} = (x+2)\cos y$ $\qquad$ $\frac{\partial N}{\partial x} = \cos y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Not exact}$$

Take the difference of the partial difference.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = (x+2)\cos y - \cos y$$

$$= \cos y\,(x+2-1)$$

$$= \underline{\cos y\,(x+1)}$$

- if we divide this by M we get a fn. of x and y bt with N we get fn. of x alone

So divide in such a way that fn. of x or y alon.

Use first case.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(x+1)\cos y}{x\cos y} = \frac{x+1}{x} = 1 + \frac{1}{x} = \underline{f(x)}$$

$$\text{I.F} = \mu = e^{\int (1+1/x)dx}$$

$$= e^{x+\log x}$$

$$= e^x \cdot e^{\log x}$$

$$= \underline{e^x \cdot x}$$

$\because e^x \cdot x$ is IF

$$xe^x(x+2)\sin y\,dx + xe^x \cdot x\cos y\,dy = 0$$

$M' = xe^x(x+2)\sin y$ $\qquad$ $N' = xe^x \cdot x\cos y$

$\frac{\partial M'}{\partial y} = xe^x(x+2)\cos y$ $\qquad$ $\frac{\partial N'}{\partial y} = (e^x \cdot 2x + x^2 e^x)\cos y$

$$= xe^x(2+x)\cos y$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial y} \Rightarrow \text{exact.}$$

Solution is,

$$\int M'dx + \int N'dy = C$$

$$\int xe^x(x+2)\sin y\,dx + \int 0\,dy = C$$

product rule

$$\sin y \int (x^2 e^x + 2x e^x)\,dx = c$$

$$\sin y \left(x^2 e^x - \cancel{\int 2x e^x\,dx}\right) + \cancel{\sin y \int 2x e^x\,dx} = c$$

$$\underline{\underline{\sin y \; x^2 e^x = c}}$$

2(c) Solve : $(y \log y - 2xy)\,dx + (x+y)\,dy = 0$

$M = y\log y - 2xy$ $\qquad\qquad$ $N = (x+y)$

$\frac{\partial M}{\partial y} = y \cdot \frac{1}{y} + 1 \cdot \log y - 2x$ $\qquad\qquad$ $\frac{\partial N}{\partial x} = 1$

$= \underline{1 + \log y - 2x}$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{\log y - 2x}{-y\log y + 2xy} = \frac{-1}{y}$$

$$\text{I.F.} = \mu = e^{\int f(y)\,dy} = e^{-\int 1/y\,dy}$$

$$\mu = e^{-\log y} = e^{\log 1/y} = \underline{\underline{1/y}}$$

$$\frac{1}{y}(y\log y - 2xy)\,dx + (x+y)\,dy = 0$$

$$(\log y - 2x)\,dx + \left(\frac{x}{y} + 1\right)dy = 0$$

$$\frac{\partial M'}{\partial y} = \frac{1}{y}$$

$$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \Rightarrow \text{exact}$$

$$\frac{\partial N'}{\partial x} = \frac{1}{y}$$

Solution is,

$$\int M'dx + \int N'dy = C \Rightarrow x\log y - x^2 + \int 1\,dy = C$$

$$\underline{\underline{x\log y - x^2 + y = C}}$$

$y^2 = x$

$2y\,dy = 1.dx$

$dx - 2y\,dy = 0$

$M = 1 \quad N = -2y$

$\frac{\partial M}{\partial y} = 0 \quad \frac{\partial N}{\partial x} = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ exact

$\int M\,dx + \int N\,dy = C$

$\int 1\,dx + \int -2y\,dy = C$

$\boxed{x - y^2 = C}$

→ so, you can directly integrate

$\int 2y\,dy = \int dx$

$x\,dy + y\,dx = 0$

$d(xy) = 0$

$xy = C$

1. $d(xy) = x\,dy + y\,dx$

2. $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

3. $d(\log x/y) = \frac{y}{x} \cdot \frac{y\,dx - x\,dy}{y^2} = \frac{y\,dx - x\,dy}{xy}$

4. $d(\tan^{-1} x/y) = \frac{1}{1 + x^2/y^2} \times \frac{y\,dx - x\,dy}{y^2}$

$= \frac{y\,dx - x\,dy}{x^2 + y^2}$

Section 9

4(b). solve: $y\,dx - x\,dy = xy^3\,dy$

$\frac{y\,dx - x\,dy}{xy} = \int y^2\,dy$

$\log \frac{x}{y} = \frac{y^3}{3} + C$

a. $x\,dy - y\,dx = (1+y^2)\,dy$

don't use this method if 'integration method' is asked

$$\frac{x\,dy - y\,dx}{y^2} = \frac{1+y^2}{y^2}\,dy$$

$$-d(x/y) = (y^{-2}+1)\,dy$$

$$-\int d(x/y) = \int (y^{-2}+1)\,dy$$

$$-x/y = \frac{y^{-1}}{-1} + y + C$$

$$-x = -1 + y^2 + cy$$

c. $x\,dy = (x^5 + x^3y^2 + y)\,dx$

$$x\,dy - y\,dx = x^3(x^2+y^2)\,dx$$

$$\frac{x\,dy - y\,dx}{x^2+y^2} = x^3\,dx$$

$$-\int d\left(\tan^{-1}\frac{x}{y}\right) = \int x^3\,dx$$

$$-\tan^{-1}\frac{x}{y} = \frac{x^4}{4} + C$$

f. $(y^2 - y)\,dx + x\,dy = 0$

$$x\,dy - y\,dx = -y^2\,dx$$

$$\frac{y\,dx - x\,dy}{y^2} = dx$$

$$d\left(\frac{x}{y}\right) = dx$$

$$\frac{x}{y} = x + C$$

h. $x\,dy + y\,dx = \sqrt{xy}\,dy$

$$d(xy) = \sqrt{xy}\,dy$$

$$\frac{d(xy)}{\sqrt{xy}} = dy$$

$$\int \frac{d(xy)}{\sqrt{xy}} = \int dy$$

$$\frac{(xy)^{-1/2+1}}{-1/2+1} = y + C$$

$$2\sqrt{xy} = y + C$$

e. $x\,dy = (y + x^2 + 9y^2)\,dx$

$$x\,dy - y\,dx = (x^2 + 9y^2)\,dx$$

$$d\left(\tan^{-1}\left(\frac{x}{3y}\right)\right) = \frac{1}{\cancel{1+x^2/9y^2}} \times$$

$$\frac{1}{3}(3x\,dy - 3y\,dx) = (x^2 + 9y^2)\,dx$$

$$= \frac{\cancel{9y^2}}{x^2+9y^2} \times \frac{3y\,dx - 3x\,dy}{\cancel{9y^2}}$$

$$\frac{1}{3}\,\frac{3x\,dy - 3y\,dx}{x^2 + 9y^2} = dx$$

$$\cancel{\frac{3y\,dx - 3x\,dy}{x^2 + (3y)^2} = -3\,dx}$$

$$\frac{3x\,dy - 3y\,dx}{x^2 + 9y^2} = 3\,dx$$

$$-\tan^{-1}\frac{x}{3y} = 3x + C$$

$$\cancel{d\left(\tan^{-1}\frac{x}{3y}\right) = -3\,dx}$$

- $\dfrac{d^2y}{dx^2} + x^2y = \sin x$ → linear → Second Order

$y'\dfrac{dy}{dx} + x = 5$ → non-linear

$\dfrac{d^2y}{dx^2} + y^2 = \sin x$

Section 10

First order linear diff. equ.

$$\frac{dy}{dx} + x^3 y = \tan x$$

A diff. equ. of first order and linear

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

can be solved by,

$$y \cdot e^{\int P(x)\,dx} = \int Q \cdot e^{\int P(x)\,dx} \cdot dx$$

1. Solve.

$$x \frac{dy}{dx} - 3y = x^4$$

divide by x to get first derivative as 1

$$\frac{dy}{dx} - \frac{3y}{x} = x^3$$

Here,

$$P(x) = \frac{-3}{x} \qquad Q(x) = x^3$$

$$I.F = e^{\int P\,dx} = e^{-\int \frac{3}{x}\,dx} = e^{-3\log x} = e^{\log x^{-3}} = x^{-3}$$

Solution is,

$$y \cdot e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx} \cdot dx$$

$$y \cdot x^{-3} = \int x^3 \cdot x^{-3}\,dx$$

$$y \cdot x^{-3} = \int dx$$

$$y \cdot x^{-3} = x + C$$

$$y = x^4 + Cx^3$$

• Solve $y' + y = \frac{1}{1+e^{2x}}$

$\frac{dy}{dx} + P(x).y = Q(x)$

It is linear 1st order diff. equ.

$IF = e^{\int P dx}$
$= e^{\int 1 dx} = e^{x}$

Solution is
$y.e^{\int P dx} = \int Q.e^{\int P dx} dx$

$ye^{x} = \int \frac{1}{1+e^{2x}} . e^{x} dx$

put $e^{x} = t$
$e^{x} dx = dt$
$ye^{x} = \int \frac{dt}{1+t^{2}}$

$y.e^{x} = \tan^{-1} t + C$
$y.e^{x} = \tan^{-1} e^{x} + C$
$y = e^{-x} \tan^{-1}(e^{x}) + Ce^{-x}$

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h. $\frac{dy}{dx} - 2xy = 6xe^{2}$

$\frac{dy}{dx} + P.y = Q(x)$

$P = -2x$
$e^{\int P dx} = e^{\int -2x dx} = e^{-x^2}$

Solution is:
$y.e^{\int P dx} = \int Q e^{\int P dx} dx$

$y.e^{-x^2} = \int 6xe^{x^2}. e^{-x^2} dx$
$= \int 6x dx = \frac{6x^2}{2} = 3x^{2} + C$

$\therefore y = 3x^{2}e^{x^2} + Ce^{x^2}$

2e. $y' + y\cot x = 2x\,\mathrm{cosec}\,x$

$\int \cot x\,dx = \log\sin x$

$\int \tan x\,dx = \log\sec x$

$P(x) = \cot x$

$e^{\int P dx} = e^{\int \cot x\,dx}$

$= e^{\log\sin x}$

$\underline{\underline{e^{\int P dx} = \sin x}}$

The solution is

$y.e^{\int P dx} = \int Q . e^{\int P dx}$

$y . \sin x = \int 2x\,\mathrm{cosec}\,x . \sin x\,dx$

$\underline{\underline{y = \sin x = x^2 + c}}$

4. Type II

Solve: $(e^x - 2xy)y' = y^2$

$\frac{dy}{dx} + Py = Q$

$\frac{dy}{dx} = \frac{y^2}{(e^x - 2xy)}$

Not linear

cannot integrate

$\frac{dx}{dy} = \frac{e^{x} - 2xy}{y^2}$

x dependent $= \frac{e^x}{y^2} - \frac{2x}{y}$

$\frac{dx}{dy} + P(y).x = Q(y)$

$\frac{dx}{dy} + \frac{2x}{y} = \frac{e^y}{y^2}$ → y fn.

order 1

$P = \frac{2}{y}$

$I.F = e^{\int P dy} = e^{\int 2/y\,dy} = e^{2\int 1/y\,dy} = \underline{\underline{e^{2\log y}}} = e^{\log y^2}$

$= \underline{\underline{y^2}}$

The solution is: $xe^{\int P dy} = \int Q \cdot e^{\int P dy} dy$

$$x \cdot y^2 = \int \frac{e^y}{y^2} x y^2 dy$$

$$x \cdot y^2 = e^y + c$$

$$x = \frac{e^y}{y^2} + \frac{c}{y^2}$$

4b. $y - xy' = y' \cdot y^2 e^y$

$$y'(y^2 e^y + x) = y$$

$$y' = \frac{dy}{dx} = \frac{y}{y^2 e^y dx} \quad \frac{y}{y^2 e^y + x}$$

$$\frac{dx}{dy} = \frac{y^2 e^y + x}{y}$$

$$\frac{dx}{dy} - \frac{1}{y} x = y \cdot e^y$$

$$P(y) = \frac{-1}{y}$$

$$IF = e^{\int P(y) dy} = \frac{1}{y}$$

solution is

$$x \cdot e^{\int P dy} = \int Q \cdot e^{\int P dy} dy$$

$$x \cdot \frac{1}{y} = \int y \cdot e^y \frac{1}{y} dy$$

$$\frac{x}{y} = e^y + c$$

$$x = ye^y + cy$$

Bernoulli's Equ

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

divide by y^n

$$y^{-n} \frac{dy}{dx} + P(x)\, y^{-n+1} = Q(x)$$

put $y^{-n+1} = m$ and solve

~~Solution~~

3.a solve: $xy' + y = x^4 y^3$

$$x \frac{dy}{dx} + y = x^4 \cdot y^3$$

$$\frac{dy}{dx} + y \cdot \frac{1}{x} = x^3 y^3$$

Divide through out by y^3

$$\frac{1}{y^3} \frac{dy}{dx} + y^{-2} \cdot x = x^3$$

Put, $y^{-2} = m$

$$-2y^{-2} \frac{dy}{dx} = \frac{dm}{dx}$$

$$\frac{1}{y^3} \frac{dy}{dx} = -\frac{1}{2} \frac{dm}{dx}$$

$$-\frac{1}{2} \frac{dm}{dx} + m \cdot \frac{1}{x} = x^3$$

$$\frac{dm}{dx} - \frac{2}{x} \cdot m = -2x^3$$

$$\cancel{\frac{dm}{dx} - \frac{2}{x} \cdot m = -2x}$$

$P = \frac{-2}{x}$, $Q = -2x^3$

$$e^{\int P dx} = e^{\int -2/3\, dx}$$
$$= e^{-2\log x}$$
$$= e^{\log x^{-2}}$$
$$= \frac{1}{x^2}$$

Solution is

$$m \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx}\, dx$$

$$m \cdot \frac{1}{x^2} = \int -2x^3 \cdot \frac{1}{x^2}\, dx$$

$$m \cdot \frac{1}{x^2} = -x^2 + C$$

$$m = -x^4 + Cx^2$$

$$\frac{+1}{y^2} = -x^4 + Cx^2$$

Section 10

Solve

3(c) $x\,dy + y\,dx = xy^2\,dx$

$$x\frac{dy}{dx} + y = xy^2$$

$$\frac{dy}{dx} + \frac{1}{x}\cdot y = y^2 \text{ (Bernoullis)}$$

Divide by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = 1 \longrightarrow (1)$$

put $\frac{1}{y} = m$

diff. both side w.r.t x

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{dm}{dx}$$

Now eq ① becomes

$$-\frac{dm}{dx} + \frac{1}{x}\cdot m = 1$$

$$\frac{dm}{dx} - \frac{1}{x} m = -1$$

Here $P = \frac{-1}{x}$, $Q = -1$

$$I.F = e^{\int P dx} = e^{\int -1/x dx} = e^{-\log x} = \frac{1}{x}$$

The Solution is

$$y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} dx$$

$$\Rightarrow m \cdot \frac{1}{x} = \int -1 \cdot \frac{1}{x} dx$$

$$\frac{m}{x} = -\log x + C$$

$$m = -x \log x + Cx$$

$$\frac{1}{y} = -x \log x + Cx$$

$$\Rightarrow 1 + xy \log x = cxy$$

Section II

Reduction of Order

(e.g) : $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \frac{3}{x} \cdot y = x$

dependent variable $y \rightarrow$ degree 1

$\rightarrow$ order 2

$\rightarrow$ It is linear

Case 1 : when dependent variable y is missing

Put $\frac{dy}{dx} = p$ $\quad \frac{d^2y}{dx^2} = \frac{dp}{dx}$

Solve

1(g) $xy''+y'=4x$

Second Order so cannot apply linear

Only x is present

$\therefore$

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=4x$$

Now put $\frac{d^2y}{dx^2}=\frac{dp}{dx}$, $\frac{dy}{dx}=p$

$$\therefore x.\frac{dp}{dx}+p=4x$$

$$\frac{dp}{dx}+\frac{1}{x}\cdot p=4$$

$P(x)=1/x \quad Q(x)=4$

$I.F=e^{\int Pdx}=e^{\int 1/x\, dx}=e^{\log x}=\underline{\underline{x}}$

$\therefore$ The solution is :

$$p.e^{\int Pdx}=\int Q.e^{\int Pdx}dx$$

$$p.x=\int 4.x\,dx$$

$$px=2x^2+c$$

$$p=2x+\frac{c}{x}$$

$\frac{dy}{dx}=2x+\frac{c}{x}$ $\rightarrow$ first order diff. equ.

Since all y and x is in one side you can integrate.

$$\frac{dy}{\cancel{dx}}=\left(2x+\frac{c}{x}\right)dx$$

$$\underline{\underline{y=x^2+c\log x+d}}$$

Case 2: When independent variable x is missing.
Then,

Put $\frac{dy}{dx} = p$ $\quad \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx}$

$$\therefore \frac{d^2y}{dx^2} = p \cdot \frac{dp}{dy}$$

(1F) Solve: $y y'' - (y')^2 = 0$

(or)

$$y \cdot \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$$

Second-order derivative

x is no where.

→ Non-linear

degree = 1

$$y \cdot p \cdot \frac{dp}{dy} - p^2 = 0$$

$$p\left(y \frac{dp}{dy} - p\right) = 0$$

if $p = 0 \Rightarrow \frac{dy}{dx} = 0$

$$y = c$$

$$y \cdot \frac{dp}{dy} - p = 0$$

$$y \frac{dp}{dy} = p$$

$$\frac{dp}{p} = \frac{dy}{y}$$

$\log_a x = b$

$x = a^b$

$y = c_1 e^{c_2 x}$

Integrate both side.

$\log p = \log y + \log c$

$p = yc$

$\frac{dy}{dx} = yc \Rightarrow \frac{dy}{y} = c\,dx \Rightarrow \log y = cx + d$

$y = e^{cx+d} = e^{cx} \cdot e^{d}$

$= c_1 e^{cx}$

2a. Solve: $(x^2+2y')y'' + 2xy' = 0$ with initial condition $y(0)=1$

$y'(0)=0$

For Laplace we need initial condition.

Here, x is there and y is missing so <u>Case 1</u>

$\frac{dy}{dx} = p \quad \frac{d^2y}{dx^2} = \frac{dp}{dx}$

$(x^2+2p)\frac{dp}{dx} + 2xp = 0$

$\frac{2xp}{x^2+2p}$

$\underbrace{(x^2+2p)}_{M} dp + \underbrace{2xp}_{N} dx = 0$

$M\,dx + N\,dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\frac{\partial M}{\partial x} = 2x \quad \frac{\partial N}{\partial p} = 2x$

The solution is,

$\int M dx + \int N dy = C$

$\int (x^2+2p)\,dp + \int 0 . dx = C$

$x^2p + p^2 = C$

at $x=0$;

$0 \times 0 + 0 = C$

$\underline{\underline{C=0}}$

$x^2p + p^2 = 0$

$p(x^2+p) = 0$

if

$p = 0$ y is constant

$\frac{dy}{dx} = 0 \Rightarrow y = c_1$

$\because y(0) = 1$

$y(0) = c_1 \Rightarrow c_1 = 1$

$\underline{\underline{y=1}}$

if $x^2 + p = 0$

$p = -x^2$

$$\frac{dy}{dx} = -x^2$$

$$y = \frac{-x^3}{3} + c_2$$

$\because y(0) = 1$

$1 = 0 + c_2 \Rightarrow \underline{\underline{c_2 = 1}}$

$$\Rightarrow \underline{\underline{y = \frac{-x^2}{3} + 1}}$$

• Solve

$x^2 y'' = 2xy' + (y')^2$

y is missing. Case I

Put

$\frac{dy}{dx} = p \; ; \; \frac{d^2y}{dx^2} = \frac{dp}{dx}$

$x^2 \cdot \frac{dp}{dx} - 2xp = p^2$

$$\boxed{\begin{array}{l} \frac{dy}{dx} + P.y = Q \\ \frac{dy}{dx} + P.y = Qy^n \end{array}}$$

Divide by $p^2 \cdot x^2$

$\frac{1}{p^2} \frac{dp}{dx} - \frac{2}{px} = \frac{1}{x^2}$

Put $\frac{1}{p} = m$

$\frac{-1}{p^2} \cdot \frac{dp}{dx} = \frac{dm}{dx}$

$-\frac{dm}{dx} - \frac{2}{x} \cdot m = \frac{1}{x^2} \Rightarrow \frac{dm}{dx} + \frac{2}{x} \cdot m = \frac{-1}{x^2}$

$$\frac{dy}{dx} + Py = Q \qquad y.e^{\int Px} = \int Q$$

Solution is,

$$m.e^{\int 2/x\,dx} = \int \frac{-1}{x^2} . e^{\int 2/x\,dx}\,dx$$

$$m.x^2 = -x + c$$

$$m = \frac{-1}{x} + \frac{c}{x^2}$$

$$\frac{1}{p} = \frac{-1}{x} + \frac{c}{x^2}$$

$$\frac{dx}{dy} = \frac{-1}{x} + \frac{c}{x^2} = \frac{-x+c}{x^2}$$

$$\frac{x^2\,dx}{c-x} = dy$$

$$\int \frac{x^2}{c-x}\,dx = \int dy$$

Put $c-x = t \quad x = c-t \quad dx = -dt$

$$\therefore -\int \frac{(c-t)^2}{t}\,dt = y + c_1$$

$$-\int \frac{c^2+t^2-2ct}{t}\,dt = y + c_1$$

$$-c\log t - \frac{t^2}{2} + 2ct + c_1 = y$$

$$-c\log(c-x) - \frac{(c-x)^2}{2} + 2c(c-x) + c_1 = y$$

Chapter-3

Second Order Linear diff. equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y' = 5\sin x$$

linear because the dependent variable y is with a one y^{1}

General form for second order linear diff. equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x).y = R(x)$$

Result:

Above diff. equ. have solution in $[a,b]$ if fn. $P(x)$, $Q(x)$ and $R(x)$ is continous in $[a,b]$

1. $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + x^2y = x^3+5$ ✓ $Q(x) = x^2$, $P(x) = \frac{1}{x}$ → is not continous at 0

   x^2 is continous everywhere

   x^3+5 ..

2. Right hand side should always be a fn. of x. If it is zero then it is known as homogeneous differential equation

   (i.e) $\frac{d^2y}{dx^2} + \cancel{x}^{P(x)}\frac{dy}{dx} + Q(x).y = 0$

   always for homogeneous

   Order: 2 ; Linear

   General Solution is given by :

   linear combination of two solution

   $$y_g(x) = c_1y_1(x) + c_2y_2(x)$$

   where,

   y_1 and y_2 ~~are~~ two independent solution of above homogeneous ~~equ~~ diff. equ.

If $R(x) \neq 0$, then diff. equ. is non-homogeneous and we have to find y_p (particular solu.)

Then complete solution

$$y_c(x) = y_g(x) + y_p(x)$$

general sol. + particular sol.

WRONSKIAN

If $y_1(x)$ and $y_2(x)$ are two solution then wronskian $(y_1(x), y_2(x))$

$$W(y_1(x), y_2(x)) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

1. If $W(y_1(x), y_2(x)) = 0$ y_1 & y_2 are dependent.
2. If $W(y_1(x), y_2(x)) \neq 0$ then linearly independent

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2. Show that $y = C_1 x + C_2 x^2$ is general solu. of $x^2 y'' - 2xy' + 2y = 0$ also check both dependent or Independent

$$W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

$$= 2x^2 - x^2$$

$$= x^2$$

Independent except $x = 0$

$$x^2 y'' - 2xy' + 2y = 0$$

$$y'' - \frac{2}{x} y' + \frac{2}{x^2} y = 0$$

It is continous at everywhere except at $x = 0$

Here, $y_1 = x$ $y_1' = 1$ $y_1'' = 0$

$x^2 . 0 - 2x . 1 + 2x$

$0 = 0$

y_1 is solution

$y_2 = x^2$ $y = x^2$ $y' = 2x$ $y'' = 2$

$x^2 . 2 - 2x . 2x + 2x^2 = 2x^2 - 4x^2 + 2x^2 = 0$

x^2 is also solution

6 (c) $y''+y'-2y=0$

$y_1=e^{x}$; $y(0)=8$

$y_2=e^{-2x}$; $y'(0)=2$

Homogeneous → General solu.
Non-Homogeneous → General & Particular

Show. 1. y_1 and y_2 are independent
2. y_1 and y_2 are solu. of above diff. equ.
3. Find particular solu.

→ Wronskian $W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-2x} \\ e^{x} & -2e^{-2x} \end{vmatrix} = -2e^{-x}-e^{-x}$

$= -3e^{-x} \neq 0$ in $[0,2]$ $\Rightarrow$ y_1 and y_2 are independent. A(1)

For; $y_1=e^{x}$, $y_1'=e^{x}$, $y_1''=e^{x}$

Now, $y''+y'-2y = e^{x}+e^{x}-2e^{x} = \underline{\underline{0}}$ $\Rightarrow$ y_1' is solution.

A(2)

$y_2=e^{-2x}$, $y_2'=-2e^{-2x}$, $y_2''=4e^{-2x}$

$y_2''+y_2'-2y_2 = 4e^{-2x}-2e^{-2x}-2e^{-2x}=0 \Rightarrow y_2$ is also solution

For given homoge. equ; General solu. is, $c_1y_1+c_2y_2$

$y_g = c_1e^{x}+c_2e^{-2x}$

Since, $y(x)=c_1e^{x}+c_2e^{-2x}$

$y(0) = c_1+c_2$

$8 = c_1+c_2 \longrightarrow (1)$

$y'(x) = c_1e^{x}-2c_2e^{-2x}$

$y'(0) = c_1-2c_2 \Rightarrow 2=c_1-2c_2 \longrightarrow (2)$

$c_1=6$, $c_2=2$ (On solving)

$y(x) = c_1e^{x}+c_2e^{-2x}$

A(3) - Particular solu.

$\underline{\underline{y_p = 6e^{x}}}+2e^{-2x}$

6d. $y''+y=0$; $y_1=1$; $y(2)=0$; $y_2=e^{-x}$; $y'(2)=e^{-2}$

Show. 1. y_1 and y_2 are independent
2. " " " are solus. of above diff. equ.
3. Find particular solu.

→ $W(1,e^{-x}) = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} = -e^{-x}-0 = -e^{-x} \neq 0$ in $[0,2]$

→ Linearly Independent A(1)

General solu, $y = c_1 y_1 + c_2 y_2$

$$y(x) = c_1 \cdot 1 + c_2 e^{-x} \qquad y'(x) = -c_2 e^{-x}$$
$$y(2) = c_1 + c_2 e^{-2} \qquad y'(2) = -c_2 e^{-2}$$
$$0 = c_1 + c_2 \cdot e^{-x} \rightarrow (1) \qquad e^{-2} = -c_2 e^{-2} \longrightarrow (2)$$

$c_1 = e^{-2}$, $c_2 = -1 \Rightarrow y_p = e^{-2} - e^{-x}$

Section 16

For a homogeneous diff. equ. of second order,

$$\frac{d^2y}{dx^2} + P(x) \cdot \frac{dy}{dx} + Q(x) \cdot y = 0$$

If one solu. is given, y_1, then, the second solu. is

$y_2 = v y_1$ where v is,

$$v = \int \frac{1}{y_1^2} \cdot e^{-\int P dx} dx$$

Quiz (1)
(Exact & Not exact)

General solu. is, $y_g(x) = c_1 y_1 + c_2 y_2$

2(a) $y'' + y = 0$

First solu, $y_1 = \sin x$

Find another one

$y'' + Py' + Q \cdot y = 0$ Here, $P = 0$

$$v = \int \frac{1}{y^2} \cdot e^{-\int P dx} dx = \int \frac{1}{\sin^2 x} \cdot e^0 dx$$

$$v = \int \operatorname{cosec}^2 x \cdot 1 \, dx = -\cot x$$

$y_2 = v y_1 = -\cot x \cdot \sin x \Rightarrow y_2 = -\cos x$

General solu. is, $y = c_1 y_1 + c_2 y_2$

$$y_g = c_1 \sin x + c_2 \cos x$$

7(c) $x^2 y'' - x(x+2) y' + (x+2) y = 0$

Divide by x^2

$$y'' - \frac{(x+2)}{x} y' + \frac{(x+2) y}{x^2} = 0 \Rightarrow \text{Here}, \; P = \frac{-x+2}{x}$$

$$v = \int \frac{1}{y_1^2} \cdot e^{-\int P dx} dx = \int \frac{1}{x^2} \cdot e^{-\int -\frac{(x+2)}{x} dx} dx = \int \frac{1}{x^2} \cdot e^{\int 1 + \frac{2}{x} dx} dx$$

$$= \int \frac{1}{x^2} e^{x + \log x^2} dx = \int \frac{1}{x^2} \cdot e^x \cdot e^{\log x^2} dx = \int \frac{1}{x^2} e^x \cdot x^2 dx = \int e^x dx = e^x$$

$\therefore y_2 = v \cdot y_1$, $y_2 = e^x \cdot x \Rightarrow y_g = c_1 x + c_2 x e^x$

(4) Verify $y_1 = x^2$ is one solu. of $x^2y'' + xy' - 4y = 0$. Find y_2 and general solu.

$$y_1 = x^2$$
$$y_1' = 2x$$
$$y_1'' = 2$$

Substituting in diff. equ.

$$x^2(2) + x(2x) - 4x^2 = 4x^2 - 4x^2$$
$$= 0$$

$\Rightarrow y_1$ is solu. of diff. equ.

Second solution, y_2.

$$y_2 = V \cdot y_1$$

$$V = \int \frac{1}{y^2} e^{-\int P dx} dx$$

$$P = \frac{1}{x}$$

$$e^{-\int P dx} = e^{-\int \frac{1}{x} dx} = \frac{1}{x}$$

$$V = \int \frac{1}{y_1^2} \times \frac{1}{x} dx$$

$$= \int \frac{1}{x^4} \times \frac{1}{x} dx$$

$$V = \frac{x^{-4}}{-4}$$

$$y_2 = V \cdot y_1$$
$$y_2 = \frac{-x^{-2}}{4}$$

$$y_g = c_1 y_1 + c_2 y_2 = \frac{c_1 x^2 + c_2 x^{-2}}{4}$$

7a. For $y_1 = x$ one solu.

$$y'' - \frac{x}{x-1} y' + \frac{1}{x-1} y = 0$$

Find y_2 and general solution.

$$P = \frac{-x}{x-1}$$

$$e^{-\int P dx} = e^{\int \frac{x}{x-1} dx} = e^{\int \frac{x-1+1}{x-1} dx} = e^{\int \left(1+\frac{1}{x-1}\right) dx}$$

$$= e^{x + \log(x-1)}$$

$$= e^{x} \cdot (x-1)$$

$$v = \int \frac{1}{y_1^2} \cdot e^{-\int P dx} dx = \int \frac{1}{x^2} \cdot e^{x} (x-1) dx$$

$$= \int e^{x} \left(\frac{1}{x} - \frac{1}{x^2} \right) dx \qquad \boxed{\int e^{x} (f(x) + f'(x)) dx = e^{x} f(x)}$$

$$= e^{x} \cdot \frac{1}{x}$$

$y_2 = v \cdot y_1 = e^{x}$

General Solution:

$$y_g = c_1 y_1 + c_2 y_2$$

$$= c_1 x + c_2 \cdot e^{x}$$

$$\left\{ \begin{array}{l} x^2 y'' + (x+1) y' + y = 0 \\ y'' + y' + y = \sin x \quad \text{non homogenous} \end{array} \right\}$$

· Homogeneous differential equation of 2^{nd} order with constant coefficient:

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Q \cdot y = 0$$ this was homogeneous.

here,

$$\frac{d^2y}{dx^2} + p \frac{dy}{dx} + q y = 0$$ where p and q are constant

Let the trial solu. is $y = e^{mx}$.

$$\frac{dy}{dx} = m.e^{mx}$$

$$\frac{d^2y}{dx} = m^2 e^{mx}$$

Now diff. equ.

$$m^2 e^{mx} + pme^{mx} + qe^{mx} = 0$$

$$e^{mx}(m^2 + pm + q) = 0$$ exponential fn. can never be 0

$\therefore$ $\boxed{m^2 + mp + q = 0}$ Auxiliary equ.

Case 1: When two real and different root.

$m = m_1 . m_2$

General Solution: $\boxed{y_g = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$

Page 97.

1a. $y'' + y' - 6y = 0$

Auxiliary equ: $m^2 + mp + q = 0$

$m^2 + m - 6 = 0$

$m^2 + 3m - 2m - 6 = 0$

$m(m+3) - 2(m+3) = 0$

$(m+3)(m-2) = 0$

$m = -3, 2$

$$y_g = c_1 e^{-3x} + c_2 e^{2x}$$

Case 2: When both are real and same

$m = m \ m$

General solution: $\boxed{y_g = c_1 e^{mx} + c_2 . x . e^{mx}}$

16. $y'' + 2y' + y = 0$

$m^2 + 2m + 1 = 0$

$(m+1)^2 = 0 \Rightarrow (m+1)(m+1) = 0$

$m = -1, -1$

$y_g = c_1 e^{-x} + c_2 x e^{-x}$

Case 3 : When roots are complex

$m = a \pm ib$

Then general solu. is

$y_g = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

Section

1d. Find general solu. of $2y'' - 4y' + 8y = 0$

Auxillary equ. is $2m^2 - 4m + 8 = 0$

$ax^2 + bx + c = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$m = \frac{4 \pm \sqrt{16 - 4 \cdot 2 \cdot 8}}{2 \cdot 2}$

$= \frac{4 \pm \sqrt{-48}}{4}$

$= \frac{4 \pm 4\sqrt{-3}}{4}$

$m = 1 \pm i\sqrt{3}$

General Solution is,

$y_g = e^{x}(c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x)$

2a. Find the solus of following d.e. with initial conditions

$$y'' - 5y' + 6y = 0$$

with $y(1) = e^2$, $y'(1) = 3e^2$

Auxillary equ. is

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

Case 1: $m = m_1 . m_2$

$$y_g = c_1 e^{m_1 x} + c_2$$

$$y_g = c_1 e^{2x} + c_2 e^{3x} \qquad y(1) = c_1 e^2 + c_2 e^3 = e^2 \longrightarrow 1$$

$$y(x) = c_1 e^{2x} + c_2 e^{3x} \cancel{e^2} \cancel{\longrightarrow 1}$$

$$y'(x) = 2c_1 e^{2x} + 3c_2 e^{3x}$$

$$y'(1) = 2c_1 e^2 + 3c_2 e^3$$
$$= 3e^2 \longrightarrow 2$$

$$c_1 e^2 + c_2 e^3 = e^2 \longrightarrow 1$$
$$2c_1 e^2 + 3c_2 e^3 = 3e^2 \longrightarrow 2$$

$$c_2 e^3 = e^2 \quad (2 - 2.1)$$

Solving 1, 2

$$c_2 = e^{-1}$$
$$c_1 = 0$$

Particular Solution is, $y(x) = 0.e^{2x} + e^{-1}.e^{3x}$

$$y(x) = e^{3x-1}$$

2d. $y'' + 4y' + 5y = 0$ with $y(0) = 1$, $y'(0) = 0$

Auxiliary equ, $m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 4.1.5}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$m = \underset{a}{-2} \pm \underset{b}{1i}$$

$$y_g = e^{-2x}(c_1 \cos x + c_2 \sin x)$$

$y(0) = 1$

$y(0) = 1.(c_1 + c_2 \sin 0)$

$\underline{\underline{1 = c_1}}$

$$y'(x) = e^{-2x}(-c_1 \sin x + c_2 \cos x) - 2e^{-2x}(c_1 \cos x + c_2 \sin x)$$

$y'(0) = 0$

$y'(0) = 1(0 + c_2) - 2.1(c_1 + 0)$

$0 = c_2 - 2c_1$

$\underline{\underline{c_2 = 2}}$

Particular sol. is : $\underline{\underline{y_p = e^{-2x}(\cos x + 2\sin x)}}$

EULER'S DIFF. EQU. OF IInd ORDER

A diff. equ.

$$x^{2} \frac{d^2y}{dx^2} + p.x.\frac{dy}{dx} + qy = 0$$

Put $x = e^z$

Take log both side

$\therefore z = \log x$

differentiate

$$\frac{dz}{dx} = \frac{1}{x}$$

Now,

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$$

$$\boxed{x\frac{dy}{dx} = \frac{dy}{dz}} \rightarrow (B)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x} \frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x} \frac{d}{dz} \cdot \frac{dy}{dx}$$

$$= \frac{-1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x} \cdot \frac{d}{dz}\left(\frac{dy}{dz} \cdot \frac{dz}{dx}\right)$$

$$= \frac{-1}{x^2} \frac{dy}{dz} + \frac{1}{x^2} \frac{d^2y}{dz^2}$$

$$\boxed{x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}}$$

Solve

5c. $x^2y'' + 2xy' - 12y = 0$

It's Euler's diff. equation

Put $x = e^z$

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + 2\frac{dy}{dz} - 12y = 0$$

Auxiliary equ,

$m^2 + m - 12 = 0$

$m = +3, -4$

The General Solution is

$y_g = c_1e^{+3z} + c_2e^{-4z}$

$= c_1(e^z)^3 + c_2(e^z)^{-4}$

$= c_1x^3 + c_2x^{-4}$

Section 17

Find general solution of following Euler's diff. equ.

5.a. $x^2y'' + 3xy' + 10y = 0$

b. $2x^2y'' + 10xy' + 8y = 0$

An:-

5. a. $x^2y'' + 3xy' + 10y = 0$

Put $x = e^z \longrightarrow ①$

$x\frac{dy}{dx} = \frac{dy}{dz} \longrightarrow ②$

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \longrightarrow ③$

$\therefore \frac{d^2y}{dz^2} - \frac{dy}{dz} + \frac{3dy}{dz} + 10y = 0$

$\frac{d^2y}{dz^2} + \frac{2dy}{dz} + 10y = 0$

Auxillary equ.

$m^2 + 2m + 10 = 0$

$m = \frac{-2 \pm \sqrt{4-40}}{2}$

$m = -1 \pm 3i$

$a \pm ib$

General Solution.

$x = e^z$

$\frac{1}{x} = e^{-z}$

$y_g = e^{az}(c_1 \cos \overset{b}{z} + c_2 \sin \overset{b}{z})$

$= e^{-z}(c_1 \cos 3z + c_2 \sin 3z)$

$= \frac{1}{x}(c_1 \cos 3\log x + c_2 \sin 3\log x)$

$= \frac{1}{x}(c_1 \cos(\log x^3) + c_2 \sin(\log x^3))$

b. $2x^2y'' + 10xy' + 8y = 0$

Put $x = e^z \longrightarrow ①$

$x\frac{dy}{dx} = \frac{dy}{dz} \longrightarrow ②$

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \longrightarrow ③$

$\therefore 2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + 10\left(\frac{dy}{dz}\right) + 8y = 0$

$2\frac{d^2y}{dz^2} + 8\frac{dy}{dz} + 8y = 0$

Auxillary equ.

$2m^2 + 8m + 8 = 0$

$m^2 + 4m + 4 = 0$

$2(m+2)^2 = 0$

$m = -2, -2$

General Solution.

$$y_g = c_1 e^{mx} + c_2 x . e^{mx}$$
$$= c_1 e^{-2z} + c_2 . z . e^{-2z}$$
$$= c_1 x^{-2} + c_2 (\log x) . x^{-2}$$

Section 18

Method of undetermined co-efficient

To find particular solution of II^{nd} order diff. equ.

Let $\frac{d^2y}{dx^2} + p . \frac{dy}{dx} + Q . y = R(x)$

non.homogenous diff. equ.

Case 1

If $R(x) = e^{ax}$ also $m = m_1, m_2$ (two auxilliary roots)

Then trial solution

$$y = Ae^{ax}$$

1.a. Find complete solution of $y'' + 3y' - 10y = 6 . e^{4x}$

Auxilliary equation

$m^2 + 3m - 10 = 0$

$m^2 + 5m - 2m - 10 = 0$

$(m+5)(m-2) = 0$

$m = 2, -5$

The General Solution,

$y_g = c_1 e^{2x} + c_2 e^{-5x}$

Let $y = Ae^{4x}$ is solution.

$y' = 4 . Ae^{4x}$

$y'' = 16 . Ae^{4x}$

$\therefore y'' + 3y' - 10y = 6 . e^{4x}$

$(16 . Ae^{4x}) + 3(4 . Ae^{4x}) - 10(Ae^{4x}) = 6e^{4x}$

$18 Ae^{4x} = 6e^{4x}$

$A = 1/3$

Particular Solution , $y = Ae^{4x}$

$$y_p = \frac{1}{3} e^{4x}$$

Complete Solution , $y = y_g + y_p$

$$= c_1 e^{2x} + c_2 e^{-5x} + \frac{1}{3} e^{4x}$$

Case 2

$R(x) = e^{ax}$

$m = m_1\, a$

then trial solution $= Axe^{ax}$

Case 3

If $R(x) = e^{ax}$, $m = a, a$

then $y_p = A.x^2.e^{ax}$

Section 18

1.e. Find complete solution of. $y'' - y' - 6y = 20e^{-2x}$

Auxiliary equation is,

$m^2 - m - 6 = 0$

$(m-3)(m+2) = 0$

$m = 3, -2$

General Solution is,

$y_g = c_1 e^{3x} + c_2 e^{-2x}$

Let, $y = Axe^{-2x}$ is solution of different equation.

$y' = Ae^{-2x} - 2Axe^{-2x}$

$y'' = -2Ae^{-2x} - 2Ae^{-x2} + 4Axe^{-2x}$

$= -4Ae^{-2x} + 4Axe^{-2x}$

From differei. equ.

$$-4Ae^{-2x}+4A\cancel{xe^{-2x}}-Ae^{-2x}+2A\cancel{xe^{-2x}}-6\cancel{Ax}e^{-2x}=20e^{-2x}$$

$$A=-4$$

$$y_p=-4xe^{-2x}$$

complete Solution,

$$y_c=y_g+y_p$$

$$=c_1e^{3x}+c_2e^{-2x}-4xe^{-2x}$$

• Find complete solution of $y''-2y'+y=6e^{x}$

Auxillary equation is,

$$m^2-2m+1=0$$

$$(m-1)^2=0$$

$$m=1,1$$

General Solution is,

$$y_g=c_1e^{x}+c_2xe^{x}$$

Let $y=A.x^2e^{x}$ is solution of diff. equ.

$$y'=2Axe^{x}+Ax^2e^{x}$$

$$y''=2Ae^{x}+2Axe^{x}+2Axe^{x}+Ax^2e^{x}$$

From diff. equation,

$$2Ae^{x}+4\cancel{Axe^{x}}+A\cancel{x^2e^{x}}-4A\cancel{xe^{x}}-2A\cancel{x^2e^{x}}+\cancel{A.x^2e^{x}}=6e^{x}$$

$$2Ae^{x}=6e^{x}$$

$$A=3$$

$$y_p=3x^2e^{x}$$

Complete Solution,

$$y_c=y_g+y_p$$

$$=c_1e^{x}+c_2xe^{x}+3x^2e^{x}$$

Case 4

IF $R(x) = \sin ax$ (or) $\cos ax$ (or) both

$m = m_1, m_2$

then trial solution,

$y = A(\sin ax + \cos ax)$

- Solve $y'' - 3y' + 2y = 14 \sin 2x - 18 \cos 2x$

Auxillary equation,

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$m = 2, 1$

$y_g = c_1 e^{2x} + c_2 e^{x}$

Trial solution is

$y = A(\sin 3x + \cos 3x)$

$y' = A(3\cos 3x - 3\sin 3x)$

$y'' = A(-9\sin 3x - 9\cos 3x)$

Substituting above in diff. equ.

~~Particular solution when R(x) = sin ax or cos ax~~

~~Then Trial solution y =~~

Particular Solution when $R(x) = \sin ax$ or $\cos ax$
Then trial solution $y = A\sin ax + B\cos ax$.

Type I

- Find complete solution : $y'' + 4y = 3\sin x$

An: Auxillary equation

$m^2 + 4 = 0$

$m = \sqrt{-4}$

$m = \pm 2i$

$m = a \pm ib$

$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

The General Solution

$y_g = c_1 \cos 2x + c_2 \sin 2x$

Here,

$R(x) = 3\sin x$

which is not equal to $\cos 2x$ or $\sin 2x$.

$\Rightarrow$ Trial solution will be,

$y = A\sin x + B\cos x$

$y' = A\cos x - B\sin x$

$y'' = -A\sin x - B\cos x$

Diff. equ $y'' + 4y = 3\sin x$

Sub y', y'' and y

$\Rightarrow (-A\sin x - B\cos x) + 4(A\sin x + B\cos x) = 3\sin x$

$-A\sin x - B\cos x + 4A\sin x + 4B\cos x = 3\sin x$

$3A\sin x + 3B\cos x = 3\sin x$

$3A = 3 \qquad 3B = 0$

$\underline{\underline{A = 1}} \qquad \underline{\underline{B = 0}}$

Particular Solution,

$y = A\sin x + B\cos x$

$y_p = \sin x$

$y_g = c_1 \cos 2x + c_2 \sin 2x$

$y_{complete} = y_g + y_p$

• $y'' - 3y' + 2y = 14 \sin 2x - 18 \cos 2x$

Auxillary Equation is

$$m^2 + 3m + 2 = 0$$

$$\underline{\underline{m = 2, 1}}$$

The General Solution,

$$y_g = c_1 \underline{e^{2x}} + c_1 e^{x}$$

this should not match with R.H.S

Trial solution is,

$$y = A \sin 2x + B \cos 2x$$

$$y' = 2A \cos 2x - 2B \sin 2x$$

$$y'' = -4A \sin 2x - 4B \cos 2x$$

Sub. in diff. equ,

$$y'' - 3y' + 2y = 14 \sin 2x - 18 \cos 2x$$

$$-4A \sin 2x - 4B \cos 2x - 6A \cos 2x + 6B \sin 2x + 2A \sin 2x + 2B \cos 2x = 14 \sin 2x - 18 \cos 2x$$

$$-2A \sin 2x - 2B \cos 2x - 6A \cos 2x + 6B \sin 2x = 14 \sin 2x - 18 \cos 2x$$

$$(-2A + 6B) \sin 2x + (-2B - 6A) \cos 2x = 14 \sin 2x - 18 \cos 2x$$

$\therefore -2A + 6B = 14 \rightarrow (1)$

$-2B - 6A = -18 \rightarrow (2)$

$-6A + 18B = 42 \rightarrow 3 \times (1)$

$-6A - 2B = -14$

$20B = 28$

$B = \frac{28}{20}$

$-A + 3B = 7 \rightarrow (1)$

$-3A - B = -9 \Rightarrow 3A + B = 9 \rightarrow (2)$

$-A + 3B = 7$

$-9A - 3B = -27$

$-10A = -20$

$\underline{\underline{A = 2}}$

$-2 + 3B = 7$

$3B = 9$

$\underline{B = 3}$

$\therefore \underline{\underline{A = 2, B = 3}}$

$\therefore$ Particular Solution is,

$$\underline{\underline{y_p = 2 \sin 2x + 3 \cos 2x}}$$

Type II

• Find y_p : $y''+y=2\cos x$.

$\therefore$ Trial solution is,

$y=A\sin x+B\cos x$

$y=x(A\sin x+B\cos x)$

Since in product form,

$y'=A\sin x+B\cos x+x(A\cos x-B\sin x)$

$y''=A\cos x-B\sin x+A\cos x-B\sin x+x(-A\sin x-B\cos x)$

Auxillary Equation is

~~$m^2+m=0$~~ $m^2+1=0$

$m^2=-1$

$m=\sqrt{-1}$

$m=\pm i$

$m=0\pm i$

The General Solution is,

$y_g=c_1\cos x+c_2\sin x$

one of the solution is matching with $R(x)$

Differential equ is,

$\therefore y''+y=2\cos x$

$A\cos x-B\sin x+A\cos x-B\sin x-\cancel{xA\sin x}-\cancel{xB\cos x}+\cancel{xA\sin x}+\cancel{xB\cos x}=2\cos x$

$2A\cos x-2B\sin x=2\cos x$

$2A=2 \qquad -2B=0$

$A=1 \qquad B=0$

Particular Solution is,

$y_p=x\sin x$

$y_{complete}=y_p+y_g$

(*)

• Solve: $y'' - 2y' + 2y = e^{x}\sin x$

Auxillary equation is,

$$m^2 - 2m + 2 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 4.2}}{2}$$

$$\underline{\underline{m = 1 \pm i}}$$

General solution is,

$$y_g = e^{x}(c_1 \cos x + c_2 \sin x)$$

$$y_g = c_1 e^{x}\cos x + c_2 e^{x}\sin x$$

Trial solution is,

$$y = e^{x}(A\sin x + B\cos x)$$

Multiply by x in RHS. because your general solu. is matching with R.H.S

$$y = x.e^{x}(A\sin x + B\cos x)$$

$y' =$

$y'' =$

Diff. equ is;

$A = -1/2 \quad B = 0$

Particular Solution

$$\underline{\underline{y_p = \frac{-1}{2} x e^{x}\cos x}}$$

Case IV

If $R(x)$ is polynomial and y is missing in $y'' + py' + y = R(x)$

Then trial solution: $y = x\,(Ax^n + Bx^{n-1} + \cdots)$

1-K. Find Particular solution of $y'' + y' = 10x^4 + 2$

Since y is missing

Trial solution is

$$y = x\,(Ax^4 + Bx^3 + Cx^2 + Dx + E)$$
$$y = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex$$
$$y' = 5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E$$
$$y'' = 20Ax^3 + 12Bx^2 + 6Cx + 2D$$

Substituting in diff. equ. and comparing the co.eff. both side

$5A = 10 \Rightarrow \boxed{A = 2}$

$4B + 20A = 0 \Rightarrow \boxed{B = -10}$

$3C + 12B = 0 \Rightarrow \boxed{C = 40}$

$2D + 6C = 0 \Rightarrow \boxed{D = -120}$

$E + 2D = 2 \Rightarrow \boxed{E = 242}$

Substitute in y.

Particular solution is: $y_p = 2x^5 + 10x^4 + 40x^3 - 120x^2 + 242x$

• $y'' - 7y + 10 = \sin x + e^x$

$y_p = y_{p_1}$ (for $\sin x$) $+ \, y_{p_2}$ (e^x)

Section 19

Variation of Parameter method

To find Particular Solution

For any IInd order diff. equ,

$$y'' + py' + qy = R(x)$$

Let general solution: $y_g = c_1 y_1 + c_2 y_2$

Then particular " : y_p with variation of parameters

$$y_p = v_1 y_1 + v_2 y_2$$

where

$$v_1 = -\int \frac{y_2 \cdot R(x)}{W(y_1, y_2)}\, dx \qquad v_2 = \int \frac{y_1 \cdot R(x)}{W(y_1, y_2)}\, dx$$

4a. Find Particular solu. of $y''+y=\sec x$

Auxillary equ. is : $m^2+1=0$

$$m=\sqrt{-1}$$
$$=0\pm i$$

$a\pm ib$

$e^{ax}(c_1\cos bx+c_2\sin bx)$

General solution $y_g=c_1\cos x+c_2\sin x$

$$y_1=\cos x$$
$$y_2=\sin x$$

$$W(y_1,y_2)=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}$$
$$=\cos^2 x+\sin^2 x$$
$$=1$$

$$R(x)=\sec x$$
$$W(y_1,y_2)=1$$

$$\therefore v_1=-\int\frac{\sin x.\sec x}{1}dx$$
$$=-\int\tan dx$$
$$=-\log\sec x$$
$$=\log\cos x$$

$$v_2=\int\frac{y_1 R(x)}{W(y_1,y_2)}dx$$
$$-\int\frac{\cos x\ \sec x}{1}dx$$
$$=1$$

$$\therefore y_p=v_1y_1+v_2y_2$$
$$=(\log\cos x)\cos x+x.\sin x$$

Section 19

3a. Find particular solution of $y''+4y=\tan 2x$ by using variation of parameters.

Auxillary equation is

$$m^2+4=0$$
$$m=\pm 2i$$

General solution

$$y_g=c_1\cos 2x+c_2\sin 2x$$

$$c_1y_1+c_2y_2$$

Here,

$y_1 = \cos 2x$

$y_2 = \sin 2x$

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$w(\cos 2x, \sin 2x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= \underline{\underline{2}}$$

$$w(y_1, y_2) = 2$$

$y_p = v_1 y_1 + v_2 y_2$

where,

$\int \tan x = \log \sec x$

$\int \cot x \, dx = \log \sin x$

$\int \sec x \, dx = \log(\sec x + \tan x)$

$\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x + \cot x)$

$$v_1 = -\int \frac{y_2 R(x)}{w} dx$$

$$= -\int \frac{\sin 2x \cdot \tan 2x}{2} dx$$

$$= \frac{-1}{2} \int \frac{\sin^2 2x}{\cos 2x} dx$$

$$= \frac{-1}{2} \int \frac{1 - \cos^2 2x \, dx}{\cos 2x}$$

$$= \frac{-1}{2} \left[\int \sec 2x \, dx - \int \cos 2x \, dx \right]$$

$$= \frac{-1}{2} \left[\frac{1}{2} \log(\sec^2 2x + \tan 2x) - \frac{1}{2} \sin 2x \right]$$

$$= \frac{-1}{4} \log(\sec 2x + \tan 2x) + \frac{1}{4} \sin 2x$$

$$v_2 = \int \frac{y_1 R(x)}{w(y_1, y_2)} dx$$

$$= \int \frac{\operatorname{cosec} x \cdot \tan x}{2} dx$$

$$= \frac{1}{2} \int \sin 2x \, dx$$

$$= \underline{\underline{-\frac{1}{4} \cos 2x}}$$

Particular soluton is

$$y_p = v_1 y_1 + v_2 y_2$$
$$= \left(-\frac{x^2}{2}\log x + \frac{x^2}{4}\right)e^{-x} + x(\log x - 1)\, x e^{-x}$$
$$= \frac{x^2 e^{-x}}{2}\log x - \frac{3}{4}x^2 e^{-x}$$

4b. $y'' + y = \cot^2 x$

Auxillary equ is,

$$m^2 + 1 = 0$$
$$m = \pm i$$

General solu. is,

$$y_g = c_1 \cos x + c_2 \sin x$$
$$y_1 = \cos x$$
$$y_2 = \sin x$$

$$w(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$
$$= \cos^2 x + \sin^2 x$$
$$= 1$$

$$v_1 = -\int \frac{(y_2)\, R(x)}{w(y_1, y_2)}\, dx = -\int \frac{\sin x \cdot \cos^2 x}{1}\, dx$$
$$= -\int \frac{\cos^2 x}{\sin x}\, dx$$
$$= -\int \frac{1 - \sin^2 x}{\sin x}\, dx$$
$$= \int \sin x - \operatorname{cosec} x\, dx$$
$$= -\cos x + \log(\operatorname{cosec} x + \cot x)$$

$$v_2 = \int \frac{y_1\, R(x)}{w(y_1, y_2)}\, dx = \int \frac{\cos x \cdot \cot^2 x}{1}\, dx$$
$$= \int \frac{\cos^2 x}{\sin^2 x}\, dx$$

$$= \int \frac{\cos^2 x}{\sin^2 x} \cdot \cos x \, dx$$

Put $\sin x = t$

$\cos x \, dx = dt$

$$v_2 = \int \frac{(1-t^2)}{t^2} \, dt = \int (t^{-2} - 1) \, dt$$

$$= \frac{t^{-1}}{-1} - t$$

$$= -\operatorname{cosec} x - \sin x$$

$y_p = v_1 y_1 + v_2 y_2$

$= -\cos^2 x + \cos x \log(\operatorname{cosec} x + \cot x) + (\operatorname{cosec} x + \sin x) \sin x$

$= -2 + \cos x \log(\operatorname{cosec} x + \cot x)$

Section 22

High Order Linear diff. equation

1. Solve $\frac{d^4 y}{dx^4} - y = 0$

1. $y''' - 3y'' + 2y' = 0$

Auxillary equ is.

$m^3 - 3m^2 + 2m = 0$

$m(m^2 - 3m + 2) = 0$

$m = 0, m = 1, m = 2$

General solu. is

$y_g = c_1 e^0 + c_2 e^x + c_3 e^{2x}$

2. $y''' - 3y'' + 4y' - 2y = 0$

Auxillary equ is,

$m^3 - 3m^2 + 4m - 2 = 0$

$m = 1$ is one solution

$(m-1)(m^2 - 2m + 2) = 0$

$m^2 - 2m + 2 = 0$

$$\begin{array}{r|l} & m^2 - 2m + 2 \\ \hline m - 1 & m^3 - 3m^2 + 4m - 2 \\ & m^3 - m^2 \\ \hline & -2m^2 + 4m - 2 \\ & -2m^2 + 2m \\ \hline & 2m - 2 \end{array}$$

$$m = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\underline{\underline{m = 1 \pm i}}, 1$$

$$y_g = \underline{\underline{c_1 e^{x} + e^{x}}}(c_2 \cos x + c_3 \sin x)$$

8. $y^{IV} + 5y'' + 4y = 0$

Auxillary equ is

$$m^4 + 5m^2 + 4 = 0$$

$$m^4 + 4m^2 + m^2 + 4 = 0$$

$$m^2(m^2+4) + 1(m^2+4) = 0$$

$$(m^2+4)(m^2+1) = 0$$

$$m^2 + 4 = 0$$

$$m = \pm 2i, \quad m = \pm i$$

$$\underline{\underline{y_g = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos x + c_4 \sin x}}$$

11. $y^{IV} + 2y''' + 2y'' + 2y' + y = 0$

Auxillary equ. is

$$m^4 + 2m^3 + 2m^2 + 2m + 1 = 0$$

$m = -1$ is one root

$$\begin{array}{r|l}
 & m^3 + m^2 + m + 1 \\
\hline
m+1 & m^4 + 2m^3 + 2m^2 + 2m + 1 \\
 & \underline{-m^4 \mp m^3} \\
 & m^3 + 2m^2 + 2m + 1 \\
 & \underline{-m^3 \mp m^2} \\
 & m^2 + 2m + 1 \\
 & \underline{-m^2 \mp m} \\
 & m + 1 \\
 & \underline{-m \mp 1} \\
 & \underline{\underline{0}}
\end{array}$$

(or)

$$(m^2 + 2m + 1)(m^2 + 1) = 0$$

$$(m+1)^2 (m^2+1) = 0$$

$$m = -1, -1$$

$$\underline{\underline{m = \pm i}}$$

$$\therefore y_g = c_1 e^{-x} + c_2 x e^{-x} + c_3 \cos x + c_4 \sin x$$

(eg), If roots are 1 1 1 1

$$y = c_1 e^{x} + x c_2 e^{x} + x^2 c_1 e^{x} + x^3 c_2 e^{x}$$

Section 23

Operator methods to find particular solu. (y_P)

$$\frac{dy}{dx} = Dy \Rightarrow D = \frac{d}{dx}$$

$$D^2y = \frac{d^2y}{dx^2}$$

Method I : Successive Integration

(e.g)

$$\frac{dy}{dx} = F(x)$$

$$Dy = F(x)$$

$$y = \boxed{\frac{1}{D}} F(x)$$

$$y = \int F(x)\,dx$$

$$\boxed{\frac{1}{D-r} f(x) = e^{rx} \int e^{-rx} F(x)\,dx}$$

Section 23

4. Find y_P : $y'' - 2y' + y = e^x$

$D^2 - 2D + 1$

$a^2 - 2ab + b^2 = (a-b)^2$

$$D^2y - 2Dy + y = e^x$$

$$(D^2 - 2D + 1)\,y = e^x$$

$$y = \frac{1}{(D-1)(D-1)} e^x$$

$$= \frac{1}{(D-1)} \cdot e^x \int e^{-x} \cdot e^x\,dx$$

$$= \frac{1}{(D-1)} e^x \cdot x$$

$$= e^x \int e^{-x} \cdot e^x x\,dx$$

$$y_P = \frac{e^x \cdot x^2}{2}$$

3. $y'' + 4y' + 4y = 10x^3 e^{-2x}$

$$(D^2+4D+4)y = 10x^3 e^{-2x}$$

$$y = \frac{1}{(D^2+4D+4)} \cdot 10x^3 e^{-2x}$$

$$= \frac{1}{(D+2)^2} \cdot 10x^3 \cdot e^{-2x}$$

$$= \frac{1}{(D+2)(D+2)} \cdot 10x^3 e^{-2x}$$

$$y_P = \frac{1}{(D+2)} \left(e^{-2x} \int e^{2x} \cdot 10x^3 e^{-2x}\, dx\right)$$

$$= \frac{1}{D+2} \underbrace{e^{-2x} \cdot 10x^4/4}_{f(x)}$$

$$= e^{-2x} \int e^{2x} e^{-2x} \cdot \frac{5}{2} x^4\, dx$$

$$y_P = e^{-2x} \cdot \frac{1}{2} x^5$$

6. $y'' - 2y' - 3y = 6e^{5x}$

$$(D^2-2D-3)y = 6e^{5x}$$

$$y = \frac{1}{(D^2-2D-3)} \cdot 6e^{5x}$$

$$= \frac{1}{(D+1)(D-3)} \cdot 6e^{5x}$$

$$= \frac{1}{D+1} \left(e^{3x} \int e^{-3x} \cdot 6e^{5x}\, dx\right)$$

$$= \frac{1}{D+1} \cdot e^{3x} \cdot 3e^{2x}$$

$$= \frac{1}{(D+1)} 3e^{5x}$$

$$= e^{-x} \int e^{x} \cdot 3e^{5x}\, dx$$

$$= e^{-x} \cdot 3 \cdot \frac{e^{6x}}{6}$$

$$= \frac{1}{2} e^{5x}$$

Method 2: Series Expansion method

$(1+x)^{-1} = 1-x+x^2-x^3+\dots$

$(1-x)^{-1} = 1+x+x^2+x^3+\dots$

$(1+x)^{-2} = 1-2x+3x^2-4x^3+5x^4-\dots$

$(1-x)^{-2} = 1+2x+3x^2+\dots$

13. $y^{(iv)}+y = x^4$

$D^4y+y = x^4$

$(D^4+1)y = x^4$

$y = \frac{1}{D^4+1} \cdot x^4$

$y = (1+D^4)^{-1}(x^4)$ → infinite terms, so we need to find which terms we'll take.

$= (1+D^4+D^8\dots)x^4$ Diff. x^4, we can max. diff x^4 twice

$= x^4 - D^4(x^4)$ the third term, it will be zero. so D^8 term and all would be zero.

$= x^4 - 24$

$4x^3$
$12x^2$
$24x$
24

9. $4y''+y = x^4$

$4D^2y+3y = x^4$

$y = \frac{1}{(4D^2+3)} \cdot x^4$

$= \frac{1}{3(1+\frac{4}{3}D^2)} \quad x^4$

$= \frac{1}{3}\left(1+\frac{4}{3}D^2\right)^{-1}(x^4)$

$= \frac{1}{3}\left(1-\frac{4}{3}D^2+\frac{16}{9}D^4-\left(\frac{4}{3}D^2\right)^{8},\dots\right)x^4$

$= \frac{1}{3}\left(x^4-\frac{4}{3}\cdot 12x^2+\frac{16}{9}\cdot 24\right)$

$= \frac{x^4}{3} - \frac{16}{3}x^2 + \frac{128}{3}$

14. Find particular solution of

$$y''' - y'' = 12x - 2$$

$$D^3y - D^2y = 12x - 2$$

$$(D^3 - D^2)y = 12x - 2$$

$$y''' = \frac{d^3y}{dx^3} = D^3$$

$$y = \frac{1}{D^3 - D^2} 12x - 2$$

$$(1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$$

$$(1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$$

$$= \frac{1}{-D^2(1-D)}(12x - 2)$$

$$= \frac{-1}{D^2}(1-D)^{-1}(12x-2)$$

diff. will be zero.

$$= \frac{-1}{D^2}(1 + D + D^2 + D^3 + D^4 + \dots)(12x - 2)$$

First divide by D^2

$$\left(\frac{-1}{D^2} - \frac{1}{D} - 1 - D - D^2 + \dots\right)(12x - 2)$$

We will take only one D because diff. of 12x will be.

$1^{st} \to 12$

$2^{nd} \to 0$

so till '$-D$'

double diff. of $(12x - 2)$

$$\int 12x - 2 = \int 6x^2 - 2x$$

$$= 2x^3 - x^2$$

$$= -2x^3 + x^2 - (6x^2 - 2x) - (12x - 2) - 12$$

$$= -2x^3 - 5x^2 - 10x - 10$$

• Find particular solution of

$$y''' + y'' = 9x^2 - 2x + 1$$

$$(D^3 + D^2)y = 9x^2 - 2x + 1$$

$$y = \frac{1}{D^3 + D^2}(9x^2 - 2x + 1)$$

$$= \frac{1}{D^2(D+1)}(9x^2 - 2x + 1)$$

$$= \frac{1}{D^2}(1+D)^{-1}(9x^2 - 2x + 1)$$

$$= \frac{1}{D^2}(1 - D + D^2 - D^3 + D^4 \dots)(9x^2 - 2x + 1)$$

$$= \left(\frac{1}{D^2} - \frac{1}{D} + 1 - D + D^2 \dots\right)(9x^2 - 2x + 1)$$

$= \frac{3}{4}x^4 - \frac{x^3}{3} + \frac{x^2}{2} - 3x^3 + x^2 - x + 9x^2 - 2x + 1$

$\left[\int (9x^2 - 2x + 1)\, dx = \int 3x^3 - x^2 + x = \frac{3}{4}x^4 - \frac{x^3}{3} + \frac{x^2}{2} \right.$

$-18x + 2 + 18$

$= \frac{3}{4}x^4 - \frac{10}{3}x^3 + \frac{21}{2}x^2 - 21x + 21$

Auxillary equ,

$m^3 + m^2 = 0$

$m = 0, 0, -1$

④ $= y_P$

General solution,

$y_g = c_1 + c_2 x + c_3 e^{-x}$

7. Find particular solution of : $y'' - y' + y = x^3 - 3x^2 + 1$

$D^2 y - Dy + y = x^3 - 3x^2 + 1$

$(D^2 - D + 1)y = x^3 - 3x^2 + 1$

$y = \frac{1}{(D^2 - D + 1)} \cdot x^3 - 3x^2 + 1$

$= [1 + (D^2 - D)]^{-1} (x^3 - 3x^2 + 1)$

$(1 + x)^{-1} = 1 - x + x^2 - x^3$

this term is necessary bcoz $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$ we need upto D^3

$= \left[1 - (D^2 - D) + (D^2 - D)^2 - (D^2 - D)^3\right] (x^3 - 3x^2 + 1)$

$= 1 - \cancel{D^2} + D + D^4 + \cancel{D^2} - 2D^3 - (D^6 - D^3 - 3D^3 (D^2 - D))$

D^4, D^6 of $(x^3 - 3x^2 + 1)$ is zero

$= (1 + D + \cancel{D^4} - 2D^3 + D^3)(x^3 - 3x^2 + 1)$

$= x^3 - \cancel{3x^2} + 1 + \cancel{3x^2} - 6x - 6$

$= x^3 - 6x - 5$

Method 3

Shifting Property

If $\frac{1}{f(D)} \cdot e^{ax} \cdot v = e^{ax} \cdot \frac{1}{f(D+a)} \cdot v$

(e.g)

$$y = \frac{1}{D^2-D} \cdot e^{3x} \cdot (x^2+5)$$

$$= e^{3x} \cdot \frac{1}{(D+3)^2-(D+3)} \cdot x^2+5$$

16. Find y_p

$$y'' - 4y' + 3y = x^2 e^{2x}$$

$$(D^2-4D+3)y = e^{2x} \cdot x^2$$

$$y = \frac{1}{(D^2-4D+3)} \, e^{2x} x^2$$

$$= e^{2x}\left(\frac{1 \text{ add } +2}{(D+2)^2-4(D+2)+3}\right) \cdot x^2$$

$$y_p = e^{2x} \cdot \frac{1}{D^2+\cancel{4D}+4-\cancel{4D}-8+3} \cdot x^2$$

$$= e^{2x} \cdot \frac{1}{D^2-1} \cdot x^2$$

$$= e^{2x} \cdot \frac{1}{-(1-D^2)} \cdot x^2$$

$$= -e^{2x}(1-D^2)^{-1} x^2$$

$$= -e^{2x}(1+D^2+D^4+\dots)x^2$$

$$= -e^{2x}(x^2+2)$$

***Shifting Method**

$$\frac{1}{\phi(D)} e^{ax} g(x) = e^{ax} \cdot \frac{1}{\phi(D+a)} \cdot g(x)$$

Find particular solu. of : $y''-3y'+2y = xe^x$

$$D^2y - 3Dy + 2y = xe^x$$

$$(D^2-3D+2)y = xe^x$$

$$y = \frac{1}{D^2-3D+2} . x.e^x$$

$$= e^x . \frac{1}{(D+1)^2 - 3(D+1)+2} . x$$

$$= e^x . \frac{1}{D^2-D} . x$$

$$= e^x . \frac{1}{-D(1-D)} . x$$

$$= -e^x . \frac{1}{D}(1-D)^{-1} . x$$

Always check fn. of x here diff. of x once is zero so we need $(1+D$

$$= -e^x\left(\frac{1}{D}.(1+D+D^2+D^3)\right)x$$

$$= -e^x\left(\frac{1}{D}+1+D+D^2\right)x$$

$$= -e^x\left(\frac{x^2}{2}+x+1+0+\ldots\right)$$

Particular Solution $= -e^x\left(\frac{x^2}{2}+x+1\right)$

25.a. Find complete solu. of $(y'-2)^3 y = e^{2x}$

$$(D-2)^3 y = e^{2x}$$

$$y = \frac{1}{(D-2)^3} . e^{2x} . 1$$

$$= e^{2x} . \frac{1}{(D+2-2)^3} . 1$$

$$= e^{2x} . \frac{1}{D^3} . 1$$

→ 3 times diff. of 1

$$\int 1 = \int x = \int \frac{x^2}{2} = \frac{x^3}{6}$$

$$= e^{2x} . \frac{x^3}{6}$$

General solution,

Auxillary equ : $(m-2)^3 = 0$

$$m = 2, 2, 2$$

$y_g = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}$

Complete solution $= y_g + y_p$

$= e^{2x}\left(c_1 + x c_2 + x^2 c_3 + \frac{x^3}{6}\right)$

(~~[illegible]~~)

$y'' - 3y' + 5y = e^{3x}$

$(D^2 - 3D + 5)y = e^{3x}$

$y = \frac{1}{D^2 - 3D + 5} \cdot e^{3x}$

$= \frac{e^{3x}}{3^2 - 3.3 + 5}$

$= \frac{e^{3x}}{5}$ (This method is not there in book)

$y'' - 4y' + 4y = e^{2x}$ solve by short method

$y = \frac{1}{D^2 - 4D + 4} \cdot e^{2x}$

$= \frac{e^{2x}}{4 - 8 + 4} \rightarrow 0$

$(D-2)^3 y = e^{2x}$

$y = \frac{1}{(D-2)^3} e^{2x}$

$\rightarrow 0$

cannot define as we get zero in the denominator

So in this case we differentiate the denominator

- multiply x

$\therefore y'' - 4y' + 4y = e^{2x}$

$y = \frac{1}{D^2 - 4D + 4} \cdot e^{2x}$

$= x \cdot \frac{1}{2D - 4} \cdot e^{2x}$ (differ. deno. again)

$= x^2 \cdot \frac{1}{2} e^{2x}$

and

$$y = \frac{1}{(D-2)^3} e^{2x}$$

$$= x \cdot \frac{1}{3(D-2)^2} \cdot e^{2x}$$

$$= x^2 \cdot \frac{1}{6(D-2)} \cdot e^{2x}$$

$$= x^2 \cdot x \cdot \frac{1}{6} e^{2x}$$

25.b $(D-2)^2 y = e^{2x} \cdot \sin x$. Find the complete solution.

Auxillary equation: $(m-2)^2 = 0$

$m = 2, 2$

General solution: y_g

$$y_g = c_1 e^{2x} + c_1 \cdot x \cdot e^{2x}$$

Particular solution $y_p = \frac{1}{(D-2)^2} \cdot e^{2x} \cdot \sin x$

By shifting rule,

$$y_p = e^{2x} \cdot \frac{1}{(D+2-2)^2} \cdot \sin x$$

$$= e^{2x} \cdot \frac{1}{D^2} \cdot \sin x$$

$$= e^{2x}(-\sin x)$$

Method 4

* Partial Fraction Method

can be done using 3rd method (Always check first three method, then go for this)

• $y'' - 3y' + 2y = x \cdot e^x$

$$D^2y - 3Dy + 2y = x \cdot e^x$$

$$y = \frac{1}{D^2 - 3D + 2} \cdot x \cdot e^x$$

factor this.

$$y = \frac{1}{(D-2)(D-1)} \cdot x.e^{x}$$

$$\frac{1}{(D-2)(D-1)} = \frac{A}{(D-2)} + \frac{B}{(D-1)}$$

$$= \frac{(D-1)A + B(D-2)}{(D-2)(D-1)}$$

$$= \frac{AD - A + 3D - 2B}{(D-2)(D-1)}$$

$$= \frac{(A+B)D - A - 2B}{(D-2)(D-1)}$$

$$A + B = 0$$

$$-A - 2B = 1$$

$$B = -1$$

$$\underline{\underline{A = 1}}$$

$$\therefore \; y_p = \left(\frac{1}{D-2} - \frac{1}{D-1}\right) \cdot x.e^{x}$$

$$= \frac{1}{D-2} \cdot xe^{x} - \frac{1}{D-1} xe^{x}$$ $\longrightarrow$ if it was $\frac{1}{D} \cdot xe^{x}$ it's differentiation.

$$= e^{2x}\int e^{-2x} \cdot xe^{x}dx - e^{x}\int e^{-x} \cdot xe^{x}dx$$

Now. in Method 1

$$\frac{1}{D-r} \cdot F(x) = e^{rx} \cdot \int e^{-rx} \cdot F(x).dx$$

$$= e^{2x}\int x.e^{-x}dx - e^{x}\int x\,dx$$

$$= e^{2x}\left(x \cdot -e^{-x} - e^{-x}\right) - e^{x} \cdot \frac{x^2}{2}$$

$$= -xe^{x} + e^{x} - e^{x} \cdot \frac{x^2}{2}$$

Chapter-5 :

Power Series solutions to diff. Equ.

An infinite series $\sum_{n=0}^{\infty} a_n . x^n = a_0 + a_1 x + a_2 x^2 + \dots$ is known as power series expanded about $x=0$. Convergent if $|x|<R$.
R-radius of convergence.

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

A power series $\sum_{n=1}^{\infty} a_n (x-x_0)^n$ is power series about $x = x_0$

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \dots$$

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \dots$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots\right)$$

Section. 27

1a. Solve $y' = 2xy$ by using power series.

Direct method: $\frac{dy}{dx} = 2xy$

$$\frac{dy}{y} = 2x\,dx$$

Integrate both side.

$$\log y = x^2 + c$$
$$y = e^{x^2+c}$$
$$y = e^{x^2} . e^c$$
$$y = \underline{\underline{a . e^{x^2}}}$$

Let $y = \sum_{n=0}^{\infty} a_n x^n$ is solution of $y' = 2xy \longrightarrow (3)$

$\hookrightarrow (1)$

$$y = \sum_{n=0}^{\infty} a_n . x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$$y' = a_1 + 2a_2 x + \dots + n . a_n x^{n-1}$$

$$y' = \sum_{n=1}^{\infty} n \, a_n x^{n-1} \longrightarrow (2) \quad (\text{integration of } (1))$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 2x \sum_{n=0}^{\infty} a_n . x^n$$

$$\sum n a_n x^{n-1} = 2 \sum_{n=0}^{\infty} a_n . x^{n+1}$$

$n \to n+1 \qquad n \to n-1$

$1 = n+1 \qquad 0 = n-1$

$n = 0$

$$\sum_{n=0} (n+1) a_{n+1} x^{n+1-1} = 2 \sum_{n=1} a_{n-1} x^n$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} \boxed{x^n} = 2 \sum_{n=1}^{\infty} a_{n-1} \boxed{x^n}$$

$$\cancel{a_0 x^0} + \cancel{\sum_{n=1}^{\infty} (n+1) a_{n+1} x^n} =$$

$$1 . a_1 . x^0 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n = 2 \sum_{n=1} a_{n-1} x^n$$

On comparing co-eff.

$a_1 = 0$

$$(n+1) a_{n+1} = 2a_{n-1}$$

$$\underline{\underline{a_1 = 0}}$$

$$\boxed{a_{n+1} = \frac{2a_{n-1}}{n+1} \; ; \; n \geq 1 \quad \text{Recurrence relation.}}$$

when $n=1$

$$a_2 = \frac{2a_0}{2} = a_0$$

$n=2$

$$a_3 = \frac{2}{3} a_1 = 0$$

$n=3$

$$a_4 = \frac{2a_2}{4} = \frac{a_0}{2} \qquad (\because a_2 = a_0)$$

$n=4$

$$a_5 = \frac{2}{5} a_3$$

$$a_5 = 0$$

$n=5$

$$a_6 = \frac{2 \cdot a_4}{6} = \frac{a_4}{3}$$

$$a_6 = \frac{a_0}{3 \cdot 2}$$

Solution is

$$y = \sum_{n=0}^{\infty} a_n \cdot x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$y = a_0 + 0 \cdot x + a_0 \cdot x^2 + 0 \cdot x^3 + \frac{a_0}{2} \cdot x^4 + 0 \cdot x^5 + \frac{a_0}{3 \cdot 2} x^6 + \ldots$$

$$y = a_0 \left(1 + x^2 + \frac{x^4}{2 \cdot 1} + \frac{x^6}{3 \cdot 2 \cdot 1} + \ldots\right)$$

$$\underline{\underline{y = a_0 e^{x^2}}}$$

1. $y'+y=1$

let $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1} n.a_n x^{n-1}$$

Now

$$y'+y=1$$

$$\sum_{n=1}^{\infty} n.a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 1$$

$n \to n+1$

$1 = n+1$

$0 = n$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 1$$

$$1.a_1 x^0 + a_0 . x^0 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} a_n x^n = 1$$

On comparing

$a_1 + a_0 = 1 \Rightarrow a_1 = 1 - a_0$

$(n+1) a_{n+1} + a_n = 0$

$$a_{n+1} = \frac{-a_n}{n+1} \quad ; n \geq 1$$

2. $y'+y=1$

Recurrence relation

$$a_{n+1} = \frac{-a_n}{n+1} \quad ; n \geq 1$$

also $a_1 = (1-a_0)$

For $n=1$

$$a_2 = \frac{-a_1}{2} = \frac{-(1-a_0)}{2}$$

$n=2$

$$a_3 = \frac{-a_2}{3} = \frac{1-a_0}{3.2}$$

$n=3$

$$a_4 = \frac{-a_3}{4} = \frac{-(1-a_0)}{4.3.2}$$

Solution is

$$y = \sum_{n=0} a_n x^n$$

$$y = a_0 + a_1 x + a_2 x^2 + \dots$$

$$y = a_0 + (1-a_0)x - \frac{(1-a_0)}{2} x^2 + \frac{(1-a_0)}{3.2} x^3 - \frac{(1-a_0)}{4.3.2} x^4 + \dots$$

$$y = 1 + -1 + a_0 + (1-a_0)x - (1-a_0)\frac{x^2}{\lfloor 2} + (1-a_0)\frac{x^3}{\lfloor 3} - (1-a_0)\frac{x^4}{\lfloor 4} \dots$$

$$y = 1 + (a_0 - 1)\left[1 - x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} \dots\right]$$

$$\underline{\underline{y = 1 + (a_0 - 1)e^{-x}}}$$

<u>Direct Method</u> : $\frac{dy}{dx} + y = 1$

$$\int \frac{dy}{1-y} = \int dx$$

$$x = -\log(1-y) + \log C$$

$$x = \log \frac{C}{1-y}$$

$$e^x = \frac{C}{1-y} \Rightarrow 1-y = C.e^{-x}$$

$$\underline{\underline{y = 1 - Ce^{-x}}}$$

- Find series solu. of : $y' = x^2 + y$

Let,

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = x^2 + y$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' = x^2 + y$$

$$\sum_{n=1}^{\infty} n.a_n x^{n-1} = x^2 + \sum_{n=0}^{\infty} a_n x^n$$

$n \to n+1$

$1 = n+1$

$n = 0$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = x^2 + \sum_{n=0} a_n x^n$$

$$1.a_1 . x^0 + 2a_2 x + 3a_3 x^2 + \sum_{n=3}^{\infty} (n+1) a_{n+1} . x^n = x^2 + a_0 + a_1 x + a_2 x^2 + \sum_{n=3} a_n x^n$$

Now comparing both side co.eff of x

$a_1 = a_0$

$2a_2 = a_1 \Rightarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2}$

$3a_3 = 1 + a_2 \Rightarrow a_3 = \frac{1+a_2}{3}$

$$\underline{a_3 = \frac{2+a_0}{6}}$$

<u>Reccurrence relation</u>

$(n+1) a_{n+1} = a_n$

$$\boxed{a_{n+1} = \frac{a_n}{n+1} \quad \text{valid for } n \geq 3}$$

$$a_3 = \frac{1}{3} + \frac{a_0}{3.2}$$

<u>For $n=3$</u>

$$a_4 = \frac{a_3}{4} = \frac{1}{3.4} + \frac{a_0}{4.3.2}$$

<u>$n=4$</u>

$$a_5 = \frac{a_4}{5} = \frac{1}{3.4.5} + \frac{a_0}{5.4.3.2}$$

<u>$n=5$</u>

$$a_6 = \frac{a_5}{6} = \frac{1}{3.4.5.6} + \frac{a_0}{6.5.4.3.2}$$

Solution is

$$y = \sum_{n=0} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots$$

$$y = a_0 + a_0 x + \frac{a_0}{2} x^2 + \left(\frac{1}{3} + \frac{a_0}{3.2}\right) x^3 + \left(\frac{1}{3.4} + \frac{a_0}{4.3.2}\right) x^4 + \ldots$$

$$y = a_0 \left(1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots\right) + \frac{x^3}{3} + \frac{x^4}{4.3} + \frac{x^5}{5.4.3} + \ldots\ldots$$

$$y = a_0 e^x + 2\left(1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \cdots\right)$$

$$-2 - 2x - x^2$$

$$y = a_0 e^x + 2(e^x) - 2 - 2x - x^2$$

$$\underline{\underline{y = (a_0 + 2) e^x - 2 - 2x - x^2}}$$

<u>Section 28</u>

3. ~~Solve by~~ Find two solution y_1 & y_2 with the help of series solution : $y'' + y' - xy = 0$

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1} n a_n x^{n-1}$$

$$y'' = \sum_{n=2} n(n-1) a_n x^{n-2}$$

$$y'' + y' - xy = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n . x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0} a_n x^{n+1} = 0$$

$n \to n+2$ $\qquad$ $n \to n+1$ $\qquad$ $n \to n-1$

$2 = n+2$ $\qquad$ $1 = n+1$ $\qquad$ $0 = n-1$

$0 = n$ $\qquad$ $n = 0$ $\qquad$ $n = 1$

$$\sum_{n=0} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0} (n+1) a_{n+1} x^n - \sum_{n=1} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + a_1 + \sum_{n=1} (n+1) a_{n+1} \cancel{x^n} = 0$$

On comparing both side

$$2a_2 + a_1 = 0 \implies \underline{\underline{a_2 = -\frac{a_1}{2}}}$$

$$(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} - a_{n-1} = 0$$

$$a_{n+2} = \frac{a_{n-1} - (n+1)a_{n+1}}{(n+2)(n+1)} \quad \text{valid for } n \geq 1$$

$$\bullet\ a_2 = \frac{-a_1}{2}$$

$$a_{n+2} = \frac{a_{n-1} - (n+1)a_{n+1}}{(n+2)\ (n+1)} \quad ; n \geq 1$$

For $n=1$

$$a_3 = \frac{a_0 - 2a_2}{3.2}$$

$$a_3 = \frac{a_0 + a_1}{3.2}$$

For $n=2$

$$a_4 = \frac{a_1 - 3a_3}{4.3}$$

$$= \frac{a_1 - \frac{a_0 + a_1}{2}}{4.3}$$

$$= \frac{a_1 - a_0}{4.3}$$

For $n=3$

$$a_5 = \frac{a_2 - 4a_4}{5.4}$$

$$= \frac{a_2 - \frac{a_1 - a_0}{3}}{5.4}$$

$$= \frac{\frac{-a_1}{2} - \frac{a_1 - a_0}{3}}{5.4}$$

$$= \cancel{\frac{-5a_1 + 2a_0}{6.5.4.3.2}} \qquad = \frac{-3a_1 - a_1 + a_0}{5.4.3.2}$$

Solu. is

$$y = \sum_{n=0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0 + a_1 x - a_1 \cdot \frac{x^2}{2} + \left(\frac{a_0 + a_1}{3.2}\right) x^3 + \left(\frac{a_1 - a_0}{4.3.2}\right) x^4 + \left(\frac{-5a_1 + 2a_0}{6.5.4.3.2}\right) x^5 + \cdots$$

$$= a_0\left(1 + \frac{x^3}{3.2} - \frac{x^4}{4.3.2} + \frac{2}{6.5.4.3.2} x^5 + \cdots\right) + a_1\left(x - \frac{x^2}{2} + \frac{x^3}{3.2} + \frac{x^4}{4.3.2} + \cdots\right)$$

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Section 28

1. Solve

$(1+x^2) y'' + 2xy' - 2y = 0$ by series method

Let $y = \sum_{n=0}^{\infty} a_n x^n$ is solu. of above diff. equ.

$$y = \sum_{n=0} a_n x^n$$

$$y' = \sum_{n=1} n . a_n . x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(1+x^2) \sum_{n=2} n(n-1) a_n x^{n-2} + 2x \sum_{n=1} n a_n x^{n-1} - 2 \sum_{n=0} a_n x^n = 0$$

$$\sum_{n=2} n(n-1) a_n x^{n-2} + \sum_{n=2} n(n-1) a_n x^n + 2 \sum_{n=1} n a_n x^n - 2 \sum_{n=0} a_n x^n = 0$$

Replace n by $n+2$ to get x^n

$n \Rightarrow n+2$

$2 = n+2 \Rightarrow n = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2} n(n-1) a_n x^n + 2 \sum_{n=1} n a_n x^n - 2 \sum_{n=0} a_n x^n = 0$$

$$\Rightarrow 2a_2 x^0 + 3.2 a_3 x^1 + \sum_{n=2} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2} n(n-1) a_n x^n$$

$$+ 2a_1 x^1 + 2 \sum_{n=2} n a_n x^n - 2a_0 - 2a_1 x^1 - 2 \sum_{n=2} a_n x^n = 0$$

comparing the co-eff of x^n both side.

$2a_2 - 2a_0 = 0 \Rightarrow a_2 = a_0$

$6a_3 + 2a_1 - 2a_1 = 0 \Rightarrow a_3 = 0$

$(n+2)(n+1)a_{n+2} + n(n-1)a_n + 2na_n - 2a_n = 0$

$\Rightarrow a_{n+2} = \left(\frac{2-2n-n(n-1)}{(n+2)(n+1)}\right)a_n$; valid for $n \geq 2$

$a_{n+2} = \left(\frac{1-n}{n+1}\right)a_n$; $n \geq 2$ Recurrence relation

For $n=2$

$a_4 = \frac{-1}{3} a_2$

$a_4 = \frac{-1}{3} a_0$

For $n=3$

$a_5 = \frac{-2}{4} a_3$

$a_5 = 0$

For $n=4$

$a_6 = \frac{-3}{5} \quad a_4 = \frac{1}{5.3} a_0$

For $n=5$

$a_7 = 0$

Solution 15.

$y = \sum_{n=0}^{\infty} a_1 . x^n = a_0 + a_1 x + a_2 x^2 + \dots$

$y = a_0 + a_1 x + a_0 . x^2 + 0 . x^3 - \frac{1}{3} a_0 x^4 + 0 . x^5 + \frac{1}{5} a_0 x^6 + \dots$

$$y = a_0\left(1+x^2-\frac{x^4}{3}+\frac{x^6}{5}+\dots\right)+a_1 x$$

$$= a_0(1+x\tan^{-1}x)+a_1 x$$

Example

Lengendre's diff. equ.

$$(1+x^2)y''-2xy'+p(p+1)y=0$$

p is any constant.

$y=\sum_{n=0} a_n x^n$ is solution of above diff. equ.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}+\sum_{n=2}^{\infty} n(n-1)a_n x^n-2\sum_{n=1}^{\infty} n a_n x^n+p(p+1)\sum_{n=0} a_n x^n=0$$

$$\sum_{n=0} (n+2)(n+1)a_{n+2}x^n+\sum_{n=2} n(n-1)a_n x^n-2\sum n a_n x+p(p+1)\sum_{n=0} a_n x^n=0$$

$$\Rightarrow 2a_2x^0+3.2.a_3x^1+\sum_{n=2}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2} n(n+1)a_n x^n-2a_1x-2\sum_{n=2} n a_n x^n+p(p+1)a_0x^0+$$

$$p(p+1)a_1x+p(p+1)\sum a_n x^n=0$$

Comparing the co-efficients

$$2a_2+p(p+1)a_0=0$$

$$a_2=\frac{-p(p+1)a_0}{2}$$

$$6a_3-a_1+p(p+1)a_1=0$$

$$a_3=\frac{1-p(p+1)}{6}a_1$$

Recurrence relation.

$$a_{n+2}=\frac{2n-n(n+1)-p(p+1)}{(n+2)(n+1)}a_n$$

$$a_{n+2}=\frac{n-n^2-p^2-p}{(n+2)(n+1)} \quad -\frac{(p+n+1)(n-p)}{(n+1)(n+2)}$$

· **Singular Point**

A point $x = x_0$ is singular point at which fn. is not analytic

$$f(x) = \frac{x^2+3}{(x-1)(x+4)}$$

$x = 1, -4$ singular Points.

$f(x) = \frac{\sin x}{(x+1)^2}$

$x=0$ is ordinary pt.

$x=-1$ singular pt.

Two kind of Singular pts. (*)

① **Regular Singular points**

$y'' + P(x).y' + Q(x).y = 0$ is said to have regular singular point at $x = x_0$ if

$(x-x_0)P(x)$ is analytic

$(x-x_0)^2 Q(x)$ " " Otherwise $x = x_0$ is irregular singular pt.

· $x^2y'' + (\sin x)y = 0$

$y'' + \frac{\sin x}{x^2} y = 0$

$y'' + Py' + Qy = 0$

$P(x) = 0$ ($\because$ no y')

$Q(x) = \frac{1}{\left(\frac{\sin x}{x^2}\right)}$

$x_0 = 0$ is singular point.

1. $(x-x_0)(P(x)) = (x-0).0$ analytic at $x=0$

11. $(x-x_0)^2 Q(x) = (x-0)^2 \frac{\sin x}{x^2}$

$= \sin x$ is analytic at $x=0$

$x=0$ is regular singular pt.

2d • $x^3 y'' + (\sin x) y = 0$

$P(x) = 0 \qquad Q(x) = \frac{\sin x}{x^3}$

$(x-0) \cdot P(x) = x \cdot 0 = 0$ analytic at 0

$(x-0)^2 \cdot Q(x) = x^2 \cdot \frac{\sin x}{x^3} = \frac{\sin x}{x}$

not analytic at 0

$\therefore x=0$ is irregular singular pt.

1.b $x^2(x^2-1)^2 y'' - x(1-x) y' + 2y = 0$

$$y'' - \frac{x(1-x)}{x^2(x^2-1)^2} y' + \frac{2}{x^2(x^2-1)^2} y = 0$$

$$P(x) = \frac{-x(1-x)}{x^2(x^2-1)^2}$$

$$Q(x) = \frac{2}{x^2(x^2-1)^2}$$

Singular pts. are

$x = 0, 1, -1$

For $x=0$

$$(x-0) \cdot P(x) = \frac{x \cdot (-x(1-x))}{(x^2(x^2-1)^2)}$$

$$= \frac{1}{(x+1)^2(x-1)}$$

It is analytic at $\overset{x\to}{0}$ and not at 1 and -1

$$(x-0)^2 Q(x) = x^2 \cdot \frac{2}{x^2(x^2-1)^2}$$

$$= \frac{2}{(x^2-1)^2} \text{ is analytic at } x=0$$

$\Rightarrow x=0$ is regular singular point.

$x=1$ is also " " " (H.W)

At $x=-1$

$x - x_0$

$$(x+1)\,p(x) = (x+1)\left(\frac{-x(1-x)}{x^2(x^2+1)^2}\right)$$

$$= \frac{x^2-1}{x(x^2-1)^2}$$

$$= \frac{1}{x(x^2-1)}$$

at $x=-1$ it is not analytic

$$(x+1)^2 Q(x) = (x+1)^2 \cdot \frac{2}{x^2(x^2-1)^2}$$

$$= \frac{2}{x^2(x-1)}$$

~~is~~ analytic at $x=-1$

$\Rightarrow x=-1$ is not regular or irregular singular pt.

- Frobenius series method

$xy'' + 3y' + 5y = 0$

Here

$x=0$ is regular singular pt. so power series $y = \sum a_n x^n$ can't be applied

Then Frobenius series

$y = x^m(a_0 + a_1 x + a_2 x^2 + \dots)$ will be used as trial solution

$$y = \sum_{n=0}^{\infty} a_n x^{m+n} \qquad a_0 \neq 0$$

$$y' = \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

$$y'' = \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2}$$

Indicial Equation

coefficient of lowest power of diff. equ $= 0$

- Find indicial equ. for $4x^2y'' + (2x^4 - 5x)y' + (3x^2+2)y = 0$

Let $y = \sum a_n x^{m+n}$ be solution

$$4x^2 \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2} + (2x^4 - 5x) \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

$$(3x^2+2) \sum a_n x^{m+n} = 0$$

Frobenius series method

$xy'' + 3y' + 5y = 0$

Here

$x=0$ is regular singular pt. so power series $y = \sum a_n x^n$ can't be applied

Then Frobenius series

$y = x^m (a_0 + a_1 x + a_2 x^2 + \dots)$ will be used as trial solution

$$y = \sum_{n=0}^{\infty} a_n x^{m+n} \qquad a_0 \neq 0$$

$$y' = \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

$$y'' = \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2}$$

※

Indicial Equation

coefficient of lowest power of diff. equ = 0

- Find indicial equ. for $4x^2 y'' + (2x^4 - 5x) y' + (3x^2 + 2) y = 0$

  Let $y = \sum a_n x^{m+n}$ be solution

$$4x^2 \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2} + (2x^4 - 5x) \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

$$(3x^2+2) \sum a_n x^{m+n} = 0$$

Indicial Equation

coefficient of lowest power of diff. equ = 0

- Find indicial equ. for $4x^2y'' + (2x^4 - 5x)y' + (3x^2 + 2)y = 0$

Let $y = \sum a_n x^{m+n}$ be solution

$$4x^2 \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2} + (2x^4 - 5x) \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

$$(3x^2+2) \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$\underline{x^{m+n}}$, x^{m+n+3}, $\underline{x^{m+n}}$, x^{m+n+2}, $\underline{x^{m+n}}$ → lowest power

$4m(m-1)a_0 - 5ma_0 + 2a_0 = 0$

$a_0(4m(m-1) - 5m + 2) = 0$

$a_0 \neq 0$

Indicial eq : $4m(m-1) - 5m + 2 = 0$

$4m^2 - 4m - 5m + 2 = 0$

$4m^2 - 9m + 2 = 0$

$4m^2 - 8m - m + 2 = 0$

$4m(m-2) - 1(m-2) = 0$

$m = 2, 1/4$ (indicial root)

3a. $x^3y''+(\cos 2x-1)y'+2xy=0$

$x=0$ is reg. singular pt.

Let $y=\sum a_n x^{m+n}$ be solution

$$y'=\sum_{n=0}(m+n)a_n x^{m+n-1}$$

$$y''=\sum_{n=0}(m+n)(m+n-1)a_n x^{m+n-2}$$

$$x^3\sum(m+n)(m+n-1)a_n x^{m+n-2}+(\cos 2x-1)$$

expand and take lowest power

$$\left(1-\frac{(2x)^2}{2!}\right)+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}\ldots$$

$\Rightarrow x^3\sum(m+n)(m+n-1)a_n x^{m+n-2}+\left(-\frac{(2x)^2}{2!}\right)\sum(m+n)a_n x^{m+n-1}+2x\sum a_n x^{m+n}=0$

x^{m+n+1}, x^{m+n+1}, x^{m+n+1}

$\Rightarrow x^3 \sum (m+n)(m+n-1) a_n x^{m+n-2} + \left(-\frac{(2x)^2}{2!}\right) \sum (m+n) a_n x^{m+n-1} + 2x \sum a_n x^{m+n} = 0$

$x^{m+n+1}, x^{m+n+1}, x^{m+n+1}$

$a_0 m(m-1) - 2a_0(m) + 2a_0 = 0$

$a_0(m^2 - m - 2m + 2) = 0 \qquad a_0 \neq 0$

$m^2 - 3m + 2 = 0 \longrightarrow$ Indicial eq.

$m^2 - 2m - 1m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$m = 2, 1 \longrightarrow$ Indicial root

4a. Find indicial equ. and solve: $4xy'' + 2y' + y = 0$

$$4x \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2} + 2 \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1} + \sum_{n=0} a_n x^{m+n} = 0$$

$x^{m+n-1}, x^{m+n-1}, x^{m+n}$

$4m(m-1)a_0 + 2m a_0 = 0$

$a_0(4m^2 - 4m + 2m) = 0 \quad a_0 \neq 0 \Rightarrow 2m^2 - m = 0 \Rightarrow m(2m-1) = 0$ indicial equ

$m = 0, 1/2 \rightarrow$ Indicial root

4. (a) Verify that $x=0$ is regular singular point for $4xy''+2y'+y=0$ also solve by Frobenius series method.

$y''+P.y'+Q.y=0$

$P(x)=\frac{2}{4x}=\frac{1}{2x}$

$Q(x)=\frac{1}{4x}$

$x=0$ is regular singular

if $\left.\begin{array}{l}(x-0).P(x)\\(x-0)^2.Q(x)\end{array}\right\}$ analytic at $x=0$

$(x-0).\frac{1}{2x}=\frac{1}{2}$ analytic at $x=0$

$(x-0)^2.\frac{1}{4x}=\frac{x}{4}$ analytic at $x=0$

$x=0$ is regular singular point.

$4xy''+2y'+y=0$

$(x-0)^2 \cdot \frac{1}{4x} = \frac{x}{4}$ analytic at $x=0$

$x=0$ is regular singular point.

$4xy'' + 2y' + y = 0$

let $y = \sum_{n=0}^{\infty} a_n . x^{m+n}$

Frobenius series is solution of above diff. eqn.

$$y' = \sum_{n=0} (m+n)\, a_n\, x^{m+n-1}$$

$$y'' = \sum_{n=0} (m+n)(m+n-1)\, a_n\, x^{m+n-2}$$

$$4x . \sum (m+n)(m+n-1)\, a_n\, x^{m+n-2}$$

$$4 . \sum_{n=0} (m+n)(m+n-1)\, a_n x^{m+n-1} + 2 \sum_{m=0} (m+n) a_n x^{m+n-1} +$$

$$\sum_{n=0} a_n x^{m+n} = 0$$

$n \to n-1$

$0 = n-1$

$n = 1$

$$4\sum_{n=0}(m+n)(m+n-1)a_n x^{m+n-1} + 2\sum_{n=0}(m+n)a_n x^{m+n-1} + \sum_{n=1} a_{n-1} x^{m+n-1} = 0$$

Take 1^{st} term out because all the powers of x is same but starting value of n is different

$$\Rightarrow 4(m)(m-1)a_0 x^{m-1} + 4\sum_{n=1}(m+n)(m+n-1)a_n x^{m+n-1} + 2ma_0 x^{m-1} + 2\sum_{n=1}(m+n)a_n$$

$$x^{m+n-1} + \sum_{n=1} a_{n-1} x^{m+n-1} = 0$$

Sum of coeff. of
Lowest power of x.

$$4m(m-1)a_0 + 2ma_0 = 0 \quad \text{(Indicial Equ.)}$$

$$a_0(4m^2 - 4m + 2m) = 0$$

Since $a_0 \neq 0$

$$4m^2 - 4m + 2m = 0$$

$$4m^2 - 2m = 0$$

$$2m(2m-1) = 0 \qquad m = 0$$

$$m = 1/2$$

comparing the coeff. of x^{m+n-1}

$2m(2m-1)=0$

$m = 1/2.$

comparing the coeff. of x^{m+n-1}

$$4(m+n)(m+n-1)a_n + 2(m+n)a_n + a_{n-1} = 0$$

$$a_n = \frac{-a_{n-1}}{4(m+n)(m+n-1)+2(m+n)} \quad ; n \geq 1$$

$$a_n = \frac{-a_{n-1}}{(m+n)(4m+4n-2)} \quad ; n \geq 1 \qquad \text{Recurrence Relation}$$

Case I when $m=0$

$$a_n = \frac{-a_{n-1}}{n(4n-2)} \quad ; n \geq 1$$

$$a_1 = \frac{-a_0}{1.2} \qquad a_2 = \frac{-a_1}{2.6} = \frac{a_0}{6.4} = \frac{a_0}{4.3.2} \qquad a_3 = \frac{-a_2}{3.10} = \frac{-a_0}{10.6.4.3} = \frac{-a_0}{6.5.4.3.2}$$

Solution is,

$$y = \sum_{n=0} a_n . x^{m+n}$$

$$y = x^m(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots .)$$

$$= x^0\left(a_0 - \frac{a_0}{2} x + \frac{a_0}{4.3.2} x^2 - \frac{a_0}{6.5.4.3.2} x^3 + \ldots\right)$$

$$= a_0\left(1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} + \frac{(\sqrt{x})^6}{6!} + \ldots\right) = a_0 \cos\sqrt{x}$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} \ldots$$

Case II

when $m = +1/2$

$$a_n = \frac{-a_{n-1}}{(2n+1)(2n)} \quad ; n \geq 1$$

$\underline{n=1}$

$$a_1 = \frac{-a_0}{3.2} = \frac{-a_0}{6}$$

$\underline{\underline{n=1}}$

$$a_1 = \frac{-a_0}{3.2} = \frac{-a_0}{\underline{|3}}$$

$$a_2 = \frac{-a_1}{5.4} = \frac{a_0}{\underline{|5}}$$

$$a_3 = \frac{-a_2}{7.6} = \frac{-a_0}{\underline{|7}}$$

Solution is.

$$y = x^m(a_0 + a_1 x) + a_2 x^2 + \ldots)$$

$$y = x^{1/2}\left(a_0 - \frac{a_0}{3!}x + \frac{a_0}{5!}x^2 - \frac{a_0}{7!}x^3 + \ldots\right)$$

$$y = a_0\left(x^{1/2} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \frac{x^{7/2}}{7!} = \ldots\right)$$

$$\underline{\underline{y = a_0 \sin\sqrt{x}}}$$

Case I

When indicial roots $m_1 = m_2 = m$

Then there will be only one solution.

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1. $x^2y'' - 3xy' + (4x+4)y = 0$

Here $x=0$ regular singular pt.

Let $y = \sum_{n=0} a_n x^{m+n}$ is solution

$$\sum_{n=0} (m+n)(m+n-1) a_n x^{m+n} - 3\sum_{n=0} (m+n)a_n x^{m+n} + 4\sum_{n=0} a_n x^{m+n+1} + 4\sum_{n=0} a_n x^{m+n} = 0$$

replace this

$n \to n-1$

$0 = n-1$

$n = 1$

$$\sum_{n=0} (m+n)(m+n-1) a_n x^{m+n} - 3\sum_{n=0} (m+n)a_n x^{m+n} + 4\sum_{n=1} a_{n-1} x^{m+n} + 4\sum_{n=0} a_n x^{m+n} = 0$$

replace this

$n \rightarrow n-1$

$0 = n-1$

$n = 1$

$$\sum_{n=0} (m+n)(m+n-1)a_n x^{m+n} - 3\sum_{n=0}(m+n)a_n x^{m+n} + 4\sum_{n=1} a_{n-1}x^{m+n} + 4\sum_{n=0} a_n x^{m+n} = 0$$

$$(m(m-1)a_0 - 3ma_0 + 4a_0)x^m + \sum_{n=1}(m+n)(m+n-1)a_n x^{m+n} - 3\sum_{n=1}(m+n)a_n x^{m+n} +$$

$$4\sum_{n=1} a_{n-1} x^{m+n} + 4\sum_{n=1} a_n x^{m+n} = 0$$

Indicial Equ

co.eff. of lowest power $(x^m) = 0$

$(m(m-1)3m+4)a_0 = 0$

$(m^2 - m - 3m + 4) = 0 \qquad (\because a_0 \neq 0)$

$(m-2)^2 = 0$

$m = 2, 2$

replace this.

$n \to n-1$

$0 = n-1$

$n = 1$

$$\sum_{n=0}(m+n)(m+n-1)a_n x^{m+n} - 3\sum_{n=0}(m+n)a_n x^{m+n} + 4\sum_{n=1} a_{n-1}x^{m+n} + 4\sum_{n=0} a_n x^{m+n} = 0$$

$$(m(m-1)a_0 - 3ma_0 + 4a_0)x^m + \sum_{n=1}(m+n)(m+n-1)a_n x^{m+n} - 3\sum_{n=1}(m+n)a_n x^{m+n} +$$

$$4\sum_{n=1} a_{n-1}x^{m+n} + 4\sum_{n=1} a_n x^{m+n} = 0$$

Indicial Equ

co.eff. of lowest power $(x^m) = 0$

$(m(m-1)3m+4)a_0 = 0$

$(m^2 - m - 3m + 4) = 0 \quad (\because a_0 \neq 0)$

$(m-2)^2 = 0$

$m = 2, 2$

$$a_n = \frac{-4\, a_{n-1}}{(2+n)(1+n) - 3(2+n) + 4}$$

$$= -\frac{4 a_{n-1}}{n^2}$$

$$= \frac{-4}{n^2}\, a_{n-1} \; ; \; n \geq 1$$

$a_1 = -4a_0$

$a_2 = -4/4\, a_1$

$= 4a_0$

$a_3 = -4/9\, a_2$

$= -16/9\, a_0$

Solution is,

$y = x^m (a_0 + a_1 x + a_2 x^2 + \dots)$

$= x^2 (a_0 - 4a_0 x + 4a_0 x^2 - \frac{16}{9} a_0 x^3 + \dots)$

$y = a_0 x^2 (1 - 4x + 4x^2 - \frac{16}{9} x^3 + \dots)$

Case II

When $m_1 - m_2$ = fraction.

Then there will be two solu.

Case III:

When $m_1 - m_2$ = integer

Then there may be one or two solutrons.

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3. b. $x^2 y'' - x^2 y' + (x^2 - 2) y = 0$

Here $x=0$ is regular singular point

$\Rightarrow$ use only Frobenius senes

Let Frobenius Series $y = \sum_{n=0}^{\infty} a_n x^{m+n}$ is trial solution

$$x^2 \sum_{n=0} (m+n)(m+n-1) a_n x^{m+n-2} - x^2 \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1} + (x^2-2) \sum_{n=0} a_n x^{m+n} = 0$$

$$\sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n} - \sum_{n=0} (m+n) a_n x^{m+n+1} + \sum_{n=0} a_n x^{m+n+2} - 2 \sum_{n=0} a_n x^{m+n} = 0$$

$n \to n-1$ $n \to n-2$

$0 = n-1$ $0 = n-2$

$n = 1$ $n = 2$

$$\sum_{n=0} (m+n)(m+n-1) a_n x^{m+n} - \sum_{n=1} (m+n-1) a_{n-1} x^{m+n} + \sum_{n=2} a_{n-2} x^{m+n} - 2 \sum_{n=0} a_n x^{m+n} = 0$$

$$\Rightarrow m(m-1) a_0 x^m + (m+1)(m) a_1 x^{m+1} + \sum_{n=2} (m+n)(m+n-1) a_n x^{m+n} -$$

$$m a_0 x^m - \sum_{n=2} (m+n-1) a_{n-1} x^{m+n} + \sum_{n=2} a_{n-2} x^{m+n} - 2a_0 x^m - 2a_1 x^{m+1} - 2 \sum_{n=2} a_n x^{m+n} = 0$$

Indicial equ. is co.eff. of lowest power of $x = 0$

$(m^2 - m - 2) a_0 = 0$

$m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$$ma_0x^m - \sum_{n=2}(m+n-1)a_{n-1}x^{m+n} + \sum_{n=2}a_{n-2}x^{m+n} - 2a_0x^m - 2a_1x^{m+1} - 2\sum_{n=2}a_nx^{m+n} = 0$$

Indicial equ. is co.eff. of lowest power of $x = 0$

$$(m^2 - m - 2)a_0 = 0$$

$$m^2 - m - 2 = 0$$

$$(m-2)(m+1) = 0$$

$$m = 2, -1$$

$$m(m-1)a_0 - 2a_0 = 0$$

Comparing co-eff. of x^{m+1} and x^{m+n} both side,

$$(m+1)(m)a_1 - ma_0 - 2a_1 = 0$$

$$a_1 = \frac{ma_0}{m^2+m-2}$$

$$a_n = \frac{(m+n-1)a_{n-1} - a_{n-2}}{(m+n)(m+n-1) - 2} \; ; \; n \geq 2$$

$$a_1 = \frac{ma_0}{m^2+m-2}$$

$$m = 2, -1$$

$m_1 \wedge m_2$ = Integer (one or two soln.)

For $m=2$ $\quad a_1 = \frac{a_0}{2}$

$$a_n = \frac{(n+1)a_{n-1} - a_{n-2}}{(n+2)(n+1) - 2}$$

$$a_n = \frac{(n+1)\,a_{n-1} - a_{n-2}}{n^2+3n} \; ; \; n \geq 2$$

$n=2$;

$$a_2 = \frac{3a_1 - a_0}{10} = \frac{a_0}{20} \qquad a_3 = \frac{4a_2 - a_1}{15}$$

Solution is

$$y = x^m(a_0 + a_1 x + a_2 x^2 + \ldots\ldots)$$

$$y = x^2\left(a_0 + \frac{a_0}{2}x + \frac{a_0}{20}x^2 + \ldots\right)$$

For $m=-1$

$$y = x^2\left(a_0 + \frac{a_0}{2}x + \frac{a_0}{20}x^2 + \cdots\right)$$

For $m = -1$

$$a_1 = \frac{-a_0}{-2} = \frac{a_0}{2}$$

$$\therefore a_n = \frac{(n-2)a_{n-1} - a_{n-2}}{(n-1)(n-2)-2}$$

$$a_n = \frac{(n-2)a_{n-1} - a_{n-2}}{n^2 - 3n} \quad ; n \geqslant 2$$

For $n = 2$

$$a_2 = \frac{-a_0}{-2} = \frac{a_0}{2}$$

$n = 3$

$$a_3 = \frac{a_2 - a_1}{3^2 - 3.3}$$

$$0.a_3 = a_2 \, a_1$$

$$= \frac{a_0}{2} - \frac{a_0}{2}$$

$$0.a_3 = 0$$

a_3 is arbitrary constant

$$\underline{\underline{a_3 = 0}}$$

$\underline{n=4}$

$$a_4 = \frac{2a_3 - a_2}{4}$$

$$= \frac{0 - a_0/2}{4}$$

$$= \frac{-a_0}{8}$$

Solution is

$$y = x^m \left(a_0 + a_1 x + a_2 x^2 + \dots\right)$$

$$= x^{-1}\left(a_0 + \frac{a_0}{2} x + \frac{a_0}{2} x^2 + 0 - \frac{a_0}{8} x^4 + \dots\right)$$

Section 31

Hyper Geometric diff. equ.

A diff. equ. of second order $x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$ is known as Hypergeometric diff. equ.

$x=0$, $x=1$ is regular singular pts.

$m=0$ and $m=1-c$ are the indicial roots

solution at $m=0$ is

$$y = x^0\left(1 + \frac{ab}{1.c}x + \frac{a(a+1)\,b(b+1)}{1.2.\,c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}x^3 + \ldots\right.$$

(or)

$$y = x^0 F(a,b,c,x)$$

Similarly at $m=1-c$

$$y = x^{1-c}F(a+1-c,\, b+1-c,\, 2-c,\, x)$$

General solution

$$\boxed{y = c_1 x^0 F(a,b,c,x) + c_2 x^{1-c} F(a+1-c,\, b+1-c,\, 2-c,\, x)}$$

Pg 202

General solution

$$\boxed{y = c_1 x^0 F(a, b, c, x) + c_2 x^{1-c} F(a+1-c, b+1-c, 2-c, x)}$$

Pg 202.

2(a). Find general sol. of $x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0$

$$x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0$$

on comparing

$$\left.\begin{aligned} -ab &= 2 \\ a+b+1 &= 2 \\ a+b &= 1 \end{aligned}\right]$$

$a = 2 \quad b = -1$

$$a - b = \sqrt{(a+b)^2 - 4ab}$$

$$c = \frac{3}{2}$$

Solu. is

$$y = c_1 x^0 F(2, -1, 3/2, x) + c_2 x^{1-3/2} F(3/2, -3/2, 1/2, x)$$

1.b. Show that

$$x\,F(1,1,2,-x) = \log(1+x)$$

LHS

$$F(a,b,c,x) = 1 + \frac{ab}{1\cdot c}x + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}x^2 + \frac{a(a+1)(a+2)\,b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}x^3 + \dots$$

$$x\,F(1,1,2,-x) = x\left(1 + \frac{1\cdot 1}{1\cdot 2}(-x) + \frac{\cancel{1}\cdot\cancel{2}\cdot 1\cdot\cancel{2}}{\cancel{1}\cdot\cancel{2}\cdot\cancel{2}\cdot 3}(-x)^2 + \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot 1\cdot\cancel{2}\cdot\cancel{3}}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{3}}(-x)^3 + \dots\right.$$

$$= x\left(1 - x/2 + x^2/3 - x^3/4 + \dots\right)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$= \log(1+x)$$

– Show that $e^x = \lim_{b\to\infty} F(a,b,a,x/b)$

RHS

$$F(a,b,c,x) = 1 + \frac{ab}{1\cdot c}x + \frac{a(a+1)(b+1)b}{1\cdot 2\cdot c\cdot(c+1)}x^2 + \frac{a(a+1)(a+2)\,b(b+1)(b+2)}{1\cdot 2\cdot 3\,c(c+1)(c+2)}x^3 + \dots$$

RHS

$$F(a,b,c,x) = 1 + \frac{ab}{1\cdot c}x + \frac{a(a+1)(b+1)b}{1.2.c.(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3c(c+1)(c+2)}x^3 + \dots$$

$$F(a,b,a,x/b) = 1 + \frac{\cancel{a}b}{1\cdot \cancel{a}}\left(\frac{x}{\cancel{b}}\right) + \frac{\cancel{a(a+1)}b(b+1)}{1.2.\cancel{a(a+1)}}\left(\frac{x^2}{b}\right) + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.a(a+1)(a+b)}\left(\frac{x^3}{b}\right) + \dots$$

$$= 1 + x + \frac{b(b+1)}{2b^2}x^2 + \frac{b(b+1)(b+2)}{2\cdot 3.\, b^3}x^3 + \dots$$

$$= \lim_{b\to\infty}\left(1 + x + \frac{1}{2}\left(\frac{b^2+b}{b^2}\right)x^2 + \frac{1}{6}\left(\frac{b(b^2+3b+6)}{b^3}\right)x^3 + \dots\right.$$

$$= \lim_{b\to\infty}\left(1 + x + \frac{1}{2}\left(1+\frac{1}{b}\right)x^2 + \frac{1}{6}\left(1+\frac{3}{b}+\frac{6}{b^2}\right)x^3 + \dots\right.$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

$$= e^x$$

c. Find general soln. of $(x^2-1)y'' + (5x+4)y' + 4y = 0$

near singular point $x=-1$

Generalized Hyper geometric diff. equ.

$(x-A)(x-B)y'' + (C+Dx)y' + Ey = 0$

Put

$t = \frac{x-A}{B-A}$ for singular pt. near $x=A$.

$(x-1)(x-1)y'' + (5x+4y)y' + 4y = 0$

$t = \frac{x-A}{B-A}$ $\quad A = -1 \quad , B = -1$

$t = \frac{x+1}{1-(-1)}$

$= \frac{x+1}{2}$

$\frac{dt}{dx} = \frac{1}{2}$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$

$$t = \frac{x+1}{1-(-1)} = \frac{x+1}{2}$$

$$\frac{dt}{dx} = \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{2}\frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{2}\frac{dy}{dt}\right) = \frac{1}{2}\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$y'' = \frac{1}{2}\frac{d}{dt}\left(\frac{1}{2}\frac{dy}{dt}\right) = \frac{1}{4}\frac{d^2y}{dt^2}$$

Substituting in diff. equ.

$$(2t)(2t-2)\frac{1}{4}y'' + \left(5\cdot(2t\cdot 1) + 4\right)\frac{1}{2}y' + 4y = 0$$

$$t(t-1)y'' + \left(\frac{10t-1}{2}\right)y' + 4y = 0$$

$$t(1-t)y'' + \left(\frac{1}{2} - 5t\right)y' - 4y = 0$$

$$x(1-x)y'' + \left(c - (a+b+1)x\right)y' - aby = 0$$

$$c = \frac{1}{2}$$

$$\left.\begin{aligned} a+b+1 &= 5 \\ a+b &= 4 \\ ab &= 4 \end{aligned}\right\} \begin{aligned} a &= 2 \\ b &= 2 \\ c &= 1/2 \end{aligned}$$

$$t = \frac{x+1}{2}$$

Solu. is

$$y = c_1 t^0 F(2, 2, 1/2, t) + c_2 t^{1+1/2}(2+1-1/2, 2+1-1/2, 2-1/2, t)$$

$$= c_1 F(2, 2, 1/2, x+1/2) + c_2 (x+1/2)^{1/2} * F(5/2, 5/2, 3/2, 1+x/2)$$

Section 44 : Legendre's Diff. Equation

A second order differential eq^n

$(1-x^2)y''-2xy'+n(n+1)y=0$ is known as Legendre's diff. equ.

$x=\pm 1$ are 2 regular singular pts

Near singular pt. $x=1$; put $t=\frac{1-x}{2}$

$t(1-t)y''+(1-2t)y'+n(n+1)y=0$ is hypergeometric diff. equ

Solution of Legendre's diff. equ

$$y=F\left(-n,\, n+1,\, 1,\, \frac{1-x}{2}\right)$$

$$= P_n(x) = \text{Legendre's Polynomial}$$

Generating Relation

$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$ generating fn.

Pg. 341 : Section 44

1.a. Prove $P_n(1)=1$

$$\quad (1-2xt+t^2)^{-1/2} = \sum^{\infty} P_n(x)\, t^n$$

Pg. 341 : Section 14

1.a. Prove $P_n(1) = 1$

since $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$

Put $x = 1$

$(1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1) t^n$

$(1-t)^{-1} = \sum_{n=0} P_n(1) t^n$

$1+t+t^2+\dots t^n = P_0(1) + P_1(1)t + P_2(1)t^2 + \dots P_n(1) t^n$.

comparing coeff. of t^n $P_n(1) = 1$

1.b. Prove : $P_n(-1) = (-1)^n$

Put $x = -1$

$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$

$(1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(-1) t^n$

$$(1+t)^{-1} = \sum_{n=0} P_n(-1)\, t^n$$

$$1-t+t^2-t^3 \cdots t^n = P_0(-1) + P_1(-1)\,t + \cdots P_n(-1)\, t^n$$

comparing coeff. $\underline{\underline{(-1)^n = P_n(-1)}}$

1.c. <u>Prove</u> $P_{2n}(0) = \dfrac{(-1)^n\, 1.3.5 \ldots (2n-1)}{2^n (n!)}$

$P_{2n+1}(0) = 0$

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Put $x = 0$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$(1+t^2)^{-1/2} = 1 + (-1/2)\,t^2 + \frac{-1/2 \times -3/2}{2!}(t^4) + \frac{(-1/2)(-3/2)(-5/2)\,t^6}{3!} + \cdots + \sum_{n=0}^{\infty} P_n(0)\, t^n$$

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots$$

$$(1+t^2)^{-1/2} = 1+(-1/2)t^2+\frac{-1/2 \times -3/2}{2!}(t^4)+\frac{(-1/2)(-1/2)(-5/2)t^6}{3!}+$$

$$\dots + \sum_{n=0}^{\infty} P_n(0)t^n$$

$$\Rightarrow 1+\left(\frac{-1}{2}\right)t^2+\frac{1.3}{2^2\, 2!}(t^2)^2-\frac{1.3.5}{2^3\, 3!}(t^2)^5+\dots+\frac{(-1)^n 1.3.5\dots(2n-1)(t^2)^n}{2^n (n)!}\dots$$

$$= P_0(0)+P_1(0)t+P_2(0)t^2+\dots P_n(0)t^n\dots$$

On comparing

$$P_{2n}(0) = \frac{(-1)^n\, 1.3.5\dots(2n-1)}{2^n(n!)}$$

$P_{2n+1}(0) =$ left hand side has no odd powers of !

$= 0$

2.a. Prove that

$$(x-t)\sum_{n=0} P_n(x)t^n = (1-2xt+t^2)\sum_{n=1}^{\infty} n.P_n(x)t^{n-1}$$

Generating fn $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$

Diff. w.r.t t both sides

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x-2t) = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

Multiply both sides with $(1-2xt+t^2)$ in order to get $(-1/2)$ in LHS

$$(1-2xt+t^2)^{-1/2}(x-t) = (1-2xt+t^2)\sum_{n=1} P_n(x)nt^{n-1}$$

$$\left(\sum_{n=0} P_n(x)t^n\right)(x-t) = (1-2xt+t^2)\sum_{n=1} P_n(x)nt^{n-1}$$

Hence Proved

2.b. Show that $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

Recursion Formula

2.a. Prove that

$$(x-t)\sum_{n=0} P_n(x)t^n = (1-2xt+t^2)\sum_{n=1}^{\infty} n.P_n(x)t^{n-1}$$

Generating fn $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$

Diff. w.r.t t both sides

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x-2t) = \sum_{n=0}^{\infty} n P_n(x)t^{n-1}$$

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$$(1-2xt+t^2)^{-1/2}(x-t) = (1-2xt+t^2)\sum_{n=1} P_n(x)nt^{n-1}$$

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Hence Proved

2.b. Show that $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

Recursion Formula

2.b. Show that $(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$

Recursion Formula

We know

$$(x-t) \sum_{n=0} P_n(x) t^n = (1-2xt+t^2) \sum_{n=1}^{\infty} n P_n(x) t^{n-1}$$

$$= x \sum_{n=0} P_n(x) t^n - \sum_{n=0} P_n(x) t^{n+1} \quad n \to n-1$$

$$= \sum_{n=1} n P_n(x) t^{n-1} - 2x \sum_{n=1} P_n(x) t^{n+1}$$

$n \to n+1$ $\qquad n \to n-1$

$$\Rightarrow x \sum_{n=0} P_n(x) t^n - \sum_{n=1} P_{n-1}(x) t^n = \sum_{n=0}^{\infty} (n+1) P_{n+1}(x) t^n - 2x \sum_{n=1}^{\infty} n P_n(x) t^n + \sum_{n=2}^{\infty} (n-1) P_{(n-1)}(x) t^n$$

Comparing coefficient of t^2

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xn\,P_n(x) + (n-1)P_{n-1}(x)$$

$$(n+1)P_{n+1}(x) = (2n+1)\,x\,P_n(x) - n\,P_{n-1}(x)$$

2.C. If $P_0(x) = 1$, $P_1(x) = x$ Find $P_2(x)$, $P_3(x)$

At $n=1$

$$2P_2(x) = 3x\,P_1(x) - P_0(x)$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

At $n=2$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Rodrigue's Formula

From this formula $P_0, P_1, P_2, \ldots$ Legendre Polynomials can be determined

$$P_n(x) = \frac{1}{2^n . n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n \qquad \text{for } n = 0, 1, 2, \ldots$$

$\underline{n=0}$

$$P_0(x) = 1$$

$\underline{n=1}$

$$P_1(x) = \frac{1}{2(1)!} \frac{d}{dx} (x^2-1)^1$$

$$= \frac{1}{2} 2x$$

$$P_1(x) = x$$

$\underline{n=2}$

$$P_2(x) = \frac{1}{2^2 2!} \cdot \frac{d^2}{dx^2} (x^2-1)^2$$

$\underline{n=2}$

$$P_2(x) = \frac{1}{2^2 \cdot 2!} \cdot \frac{d^2}{dx^2} (x^2-1)^2$$

$$= \frac{1}{8} \cdot \frac{d^2}{dx^2} (x^4+1-2x^2)$$

$$= \frac{1}{8} (12x^2-4)$$

$$= \frac{3x^2-1}{2}$$

Section. 45

Legendre's Series

Any function $f(x)$ can be expressed in Legendre's series as

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot P_n(x) = a_0 P_0(x) + a_1 P_1(x) + \dots$$

where

$$a_n = \left(n+\frac{1}{2}\right) \int_{-1}^{1} f(x) \cdot P_n(x)\, dx$$

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4.a. Find first three terms of Legendre series for

$$f(x) = \begin{cases} 0 & ; -1 \le x < 0 \\ x & ; 0 \le x < 1 \end{cases}$$

b. $f(x) = e^{x}$

(a) $\sum_{n=0}^{2} a_n P_n(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$

$$a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)\, P_n(x)\, dx$$

$$= \left(n+\frac{1}{2}\right)\left(\int_{-1}^{0} 0 . P_n(x)\, dx + \int_{0}^{1} x . P_n(x)\, dx\right)$$

$$= \left(n+\frac{1}{2}\right)\int_{0}^{1} x . P_n(x)\, dx$$

Put $n=0$

$$a_0 = \frac{1}{2}\int_{0}^{1} x . P_0(x)\, dx$$

$$= \frac{1}{2}\int_{0}^{1} x . 1 . dx$$

Put $n=0$

$$a_0 = \frac{1}{2}\int_0^1 x \cdot P_0(x)\,dx$$

$$= \frac{1}{2}\int_0^1 x \cdot 1 \cdot dx$$

$$= \frac{1}{2} \cdot \frac{1}{2}$$

$$a_0 = \frac{1}{4}$$

$n=1$

$$a_1 = \frac{3}{2}\int_0^1 x \cdot P_1(x)\,dx$$

$$= \frac{3}{2}\int_0^1 x \cdot x \cdot dx$$

$$= \frac{3}{2} \cdot \frac{1}{3}$$

$$= \frac{1}{2}$$

$a_1 = \frac{1}{2}$

$\underline{n=2}$

$a_2 = \frac{5}{2}\int_0^1 x \cdot P_2(x)\, dx$

$= \frac{5}{2}\int_0^1 x \cdot \left(\frac{3x^2-1}{2}\right) dx$

$= \frac{5}{4}\int_0^1 (3x^3 - x)\, dx$

$= \frac{5}{4}\left(\frac{3}{4}x^4 - \frac{x^2}{2}\right)_0^1$

$= \frac{5}{16}$

First three terms of Legendre's series

$= a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$

$= \frac{1}{4} P_0(x) + \frac{1}{2} P_1(x) + \frac{5}{16} P_2(x)$

$= \frac{1}{4} P_0(x) + \frac{1}{2} P_1(x) + \frac{5}{16} P_2(x)$

Orthogonal property of Legendre's Poly. ($P_n(x)$)

If $P_n(x)$ and $P_m(x)$ are two Legendre's polynomial then

$$\int_{-1}^{1} P_n(x) . P_m(x) dx = 0 \quad \text{if } m \neq n$$

$$= \frac{2}{2n+1} \quad \text{if } m = n$$

✓ $\int_{-1}^{1} P_0 . P_1 dx =$

$\int_{-1}^{1} 1 . x \, dx = \left(\frac{x^2}{2}\right)_{-1}^{1} = 0$

✓ $\int_{-1}^{1} P_1 \times P_1 \, dx$ - P_1-Polynomial

$\int_{-1}^{1} x . x \, dx = \left(\frac{x^3}{3}\right)_{-1}^{1} = \frac{2}{3} = \frac{2}{2.1+1}$

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3. Prove $\int_{-1}^{1} (P_n(x))^2 dx = \frac{2}{2n+1}$

– Generating relation

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) . t^n \longrightarrow (1)$$

For m

$$(1-2xt+t^2)^{-1/2} = \sum_{m=0}^{\infty} P_m(x)\, t^m \rightarrow (2)$$

Multiplying (1) and (2)

$$(1-2xt+t^2)^{-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_n(x) \cdot P_m(x)\, t^{m+n}$$

Put $m=n$

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} (P_n(x))^2 t^{2n}$$

Integrate both side from -1 to 1

$$\int_{-1}^{1} \frac{1}{1-2xt+t^2}\, dx = \int_{-1}^{1} \sum\sum (P_n(x))^2 \cdot t^{2n}$$

Put $1-2xt+t^2 = z$

$-2t\,dx = dz$

put $1-2xt+t^2 = z$

$-2t\,dx = dz$

$\underline{x=-1}$

$1+2t+t^2 = z$

$z = (1+t)^2$

$\underline{x=1}$

$z = (1-t)^2$

$$\frac{-1}{2t}\int_{(1+t)^2}^{(1-t)^2} \frac{1}{z}\cdot dz = \frac{-1}{2t}\left(\log z\right)_{(1+t)^2}^{(1-t)^2}$$

$$= \sum_{n=0}^{\infty} \int_{-1}^{1} (P_n(x))^2 t^{2n}\, dx$$

$$= \frac{-1}{2t}\left(2\log(1-t) - 2\log(1+t)\right)$$

$$\boxed{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots}$$

$$\boxed{\log(1-x) = -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)}$$

$$\frac{-1}{2t}\left[-2\left(t+\frac{t^3}{3}+\frac{t^5}{5}+\cdots\right)\right] = 1+\frac{t^2}{3}+\frac{t^4}{5}+\cdots+\frac{t^{2n}}{2n+1}+\cdots$$

$$= \sum \int_{-1}^{1} (P_n(x))^2 dx . t^{2n}$$

comparing coeff of t^{2n}

$$\int_{-1}^{1} (P_n(x))^2 dx = \frac{2}{2n+1}$$

Chebyshev's Diff. Equ.

$(1-x^2)y'' - xy' + n^2 y = 0$

$x = 1, -1$ are regular singular pt.

Solution is

$$y = F(n, -n, 1/2, 1-x/2)$$

$F(n, -n, 1, \frac{1-x}{2})$ is also called $T_n(x)$ - Chebyshev Polynomial

Solution is

$$y = F(n, -n, 1/2, 1-x/2)$$

$F(n, -n, 1, \frac{1-x}{2})$ is also called $T_n(x)$ - Chebyshev Polynomial

$T_n(x) = \cos(n \cos^{-1} x)$ for $-1 \le x \le 1$

$T_0(x) = \cos 0 = 1$

$T_1(x) = \cos(\cos^{-1} x) = x$

Put $x = \cos\theta$

$\boxed{T_n(\cos\theta) = \cos n}$ $\quad 0 \le \theta \le \pi$

- Prove

$$\frac{1}{2}\left((x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right) = T_n(x)$$

LHS

Put $x = \cos\theta$

$$\frac{1}{2}\left[(\cos\theta + \sqrt{-\sin^2\theta})^n + (\cos\theta - \sqrt{-\sin^2\theta})^n\right]$$

$$= \frac{1}{2}\left[(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\right]$$

$$= \frac{1}{2}\left[\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta\right]$$

$= \cos n\theta$

$= T_n(\cos\theta)$

$= T_n(x)$

*

- **Prove** Recurrence relation

$$2x\, T_{n-1}(x) = T_n(x) + T_{n-2}(x)$$

LHS =

$2x\, T_{n-1}(x)$

Put

$x = \cos\theta$

$2\cos\theta \cdot T_{n-1}(\cos\theta)$ $\qquad$ $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\cos\theta \cdot \cos(n-1)\theta$

$= \cos n\theta + \cos(n-2)\theta$

$= T_n(\cos\theta) + T_{n-2}(\cos\theta)$

$2x\, T_{n-1}(x)$

Put

$x = \cos\theta$

$2\cos\theta \cdot T_{n-1}(\cos\theta)$

$2\cos\theta \cdot \cos(n-1)\theta$ $\qquad\qquad 2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$= \cos n\theta + \cos(n-2)\theta$

$= T_n(\cos\theta) + T_{n-2}(\cos\theta)$

$= T_n(x) + T_{n-2}(x) = \underline{RHS}$

$T_0(x) = 1$

$T_1(x) = x$

Put $n=2$ in recurrence relation

$\cancel{2x \cdot x = T_2(x) + 1}$ $\qquad 2x\, T_1(x) = T_2(x) + T_0(x)$

$\cancel{T_2(x) = 2}$ $\qquad 2x \cdot x = T_2(x) + 1$

$T_2(x) = 2x^2 - 1$

at $n=3$

$2x\, T_2(x) = T_3(x) + T_1(x)$

$2x(2x^2-1) = T_3(x) + x$

$T_3(x) = 4x^3 - 3x$

at $n=4$

$$T_4(x)=8x^4-8x^2+1$$

• Prove

$$x^4=\frac{1}{8}\left(T_4(x)+4T_2(x)+3T_0(x)\right)$$

since

$$T_4(x)=8x^4-8x^2+1$$

$$x^4=\frac{1}{8}\left(T_4(x)+8x^2-1\right)$$

$$=\frac{1}{8}\left(T_4(x)+8\left(\frac{T_2(x)+1}{2}\right)-1\right)$$

$$=\frac{1}{8}\left(T_4(x)+4T_2(x)+3.T_0\right)$$

• Prove

$$2(T_n(x))^2=1+T_{2n}(x)$$

• Prove

$$2(T_n(x))^2 = 1 + T_{2n}(x)$$

R.H.S

$1 + T_{2n}(x)$

Put $x = \cos\theta$

$$1 + T_{2n}(\cos\theta) = 1 + \cos 2n\theta = 1 + 2\cos^2 n\theta - 1$$
$$= 2(\cos^2 n\theta)$$
$$= 2(\cos n\theta)^2$$
$$= 2(T_n(\cos\theta))^2$$
$$= 2T_n^2(x)$$

$T_{m+n}(x) + T_{m-n}x = 2T_m(x) \cdot T_n(x)$. Prove

$2\cos m\theta \cdot 2\cos n\theta$

Put $x = \cos\theta$

$$T_{m+n}(\cos\theta) = \cos(m+n)\theta$$
$$T_{m-n}(\cos\theta) = \cos(m+n)\theta$$
$$\cos(m+n)\theta + \cos(m-n)\theta = 2\cos m\theta \cdot \cos n\theta$$
$$= 2T_m(\cos\theta) \cdot T_n(\cos\theta)$$
$$= 2T_m(x) \cdot T_n(x)$$

Section 55-56

System of Linear Differential Equation

$$\frac{dx}{dt} + ax + by = 0$$

$$\frac{dy}{dt} + ax + by = 0$$ are system of Homogeneous Linear diff. equation

(question)

• Solve

1.(a) $\frac{dx}{dt} = -3x + 4y$ (1) $\qquad \frac{dy}{dt} = -2x + 3y$ (2)

differentiating (1) w.r.t t

$$\frac{d^2x}{dt^2} = -3\frac{dx}{dt} + 4\frac{dy}{dt}$$

substitude $\frac{dy}{dt}$ value from (2)

$$\frac{d^2x}{dt^2} = -3\frac{dx}{dt} + 4(-2x+3y)$$

differentiating (1) w.r.t t

$$\frac{d^2x}{dt^2} = -3\frac{dx}{dt} + 4\frac{dy}{dt}$$

substitude $\frac{dy}{dt}$ value from (2)

$$\frac{d^2x}{dt^2} = -3\frac{dx}{dt} + 4(-2x+3y)$$

$$\frac{d^2x}{dt^2} = -3\frac{dx}{dt} - 8x + 3.4y$$

Substitude value of $4y$ from (1)

$$\frac{d^2x}{dt^2} = -3\frac{dx}{dt} - 8x + 3.\left(\frac{dx}{dt} + 3x\right)$$

$$\frac{d^2x}{dt^2} = x \qquad \frac{d^2y}{dx^2} - y = 0 \qquad (m^2-1)y = 0$$

A.E is $m^2 - 1 = 0$

$m = \pm 1$

$$x = c_1e^{t} + c_2e^{-t}$$

$$x(t) = c_1 e^{t} + c_2 e^{-t}$$

Differentiate w.r.t t

$$\frac{dx}{dt} = c_1 e^{t} - c_2 e^{-t}$$

substitude 3 , 4 in 1

$$c_1 e^{t} - c_2 e^{-t} = -3(c_1 e^{t} + c_2 e^{-t}) + 4y$$

$$y(t) = c_1 e^{t} + \frac{c_2}{2} e^{-t}$$

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1.b. Solve the following system of linear diff. equ.

$$\frac{dx}{dt} = 4x - 2y \longrightarrow (1)$$

$$\frac{dy}{dt} = 5x + 2y \longrightarrow (2)$$

Diff. equ (1) wrt t

$$\frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 2\frac{dy}{dt}$$

From equ. (2)

$$\frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 2(5x + 2y) \longrightarrow (3)$$

Also from (1)

$$y = 4x - \frac{dx}{dt}$$

Substituting $2y$ in (3)

$$\frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 10x - 8x + 2\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 18x = 0$$

Auxillary equ. is

$$m^2 - 6m + 18 = 0$$

$$m = \frac{6 \pm \sqrt{36-72}}{2}$$

$$= \frac{6 \pm \sqrt{-36}}{2}$$

$$= 3 \pm 3i$$

Solution is

$$x = e^{3t}(c_1 \cos 3t + c_2 \sin 3t)$$

$$\frac{dx}{dt} = e^{3t}(-3c_1 \sin 3t + 3c_2 \cos 3t) + 3e^{3t}(c_1 \cos 3t + c_2 \sin 3t)$$

Substituting $x, \frac{dx}{dt}$ in (1)

$$e^{3t}(-3c_1 \sin 3t + 3c_2 \cos 3t) + 3e^{t}(c_1 \cos 3t + c_2 \sin 3t) = 4e^{3t}(c_1 \cos 3t) + (c_2 \sin 3t) - 2y$$

$$y = \frac{1}{2}\left(e^{3t}(c_1 \cos 3t + c_2 \sin 3t - e^{3t}(-3c_1 \sin 3t + 3c_2 \cos 3t)\right)$$

(*) **Bessel's diff. Equ.**

A diff. equ. $x^2y'' + xy' + (x^2 - p^2)y = 0$ known as Bessel's diff. equ.

Solution is

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{(n)!\,(n+p)!}$$

Bessel's Function

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\,(n+p)!}$$

- Prove $\frac{d}{dx}(x^p J_p(x)) = x^p J_{p-1}(x)$

also

$$\frac{d}{dx}(x J_1(x)) = x J_0(x)$$

$\Rightarrow$ L.H.S $\frac{d}{dx} x^p (J_p(x))$

$$\frac{d}{dx} x^p \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\ (n+p)!} = \frac{d}{dx} x^p \sum \frac{(-1)^n x^{2n+p}}{2^{2n+p} . n! (n+p)!}$$

$$= \frac{d}{dx} \sum_{n=0} \frac{(-1)^n x^{2n+2p}}{2^{2n+p} (n!) (n+p)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n+2p) x^{2n+2p-1}}{2^{2n+p} (n!) (n+p)!}$$

$$\cancel{x^p} x \sum_{n=0} \frac{(-1)^n x^{2n+2p-1}}{2^{2n+p-1} (n)! \ (n+p-1)!}$$

$$= x^p \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p-1}}{2^{2n+p-1} (n)! \ (n+p-1)!}$$

$$= x^p \sum_{n=0} \frac{(-1)^n (x/2)^{2n+p-1}}{n! \ (n+p-1)!}$$

$$\boxed{\frac{d}{dx} (x^p J_p(x)) = x^p . J_{p-1}(x)}$$

For $p=1$,

$$\frac{d}{dx} (x \ J_1(x)) = x J_0(x)$$

• Prove: $\frac{d}{dx}\left(x^{-p} J_p(x)\right) = -x^{-p} J_{p+1}(x)$ also prove

$\frac{d}{dx}\left(J_0(x)\right) = -J_1(x)$

LHS $\frac{d}{dx}\left(x^{-p} \cdot J_p(x)\right) = \frac{d}{dx} x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{(n)!\,(n+p)!}$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+p}\,(n)!\,(n+p)!}$$

$$= \frac{(-1)^n\, 2n\, x^{2n-1}}{2^{2n+p}\,(n)!\,(n+p)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n+p-1}\,(n-1)!\,(n+p)!}$$

$$= x^{-p} \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+p-1}}{2^{2n+p-1}\,(n-1)!\,(n+p)!}$$

$n \rightarrow n+1$

$1 \rightarrow n+1$

$n=0$

$$= x^{-p} \sum_{n=0} \frac{(-1)^{n+1} (x/2)^{2(n+1)+p-1}}{n!\,(n+p+1)!}$$

$$= -x^{-p} \sum_{n=0} \frac{(-1)^n (x/2)^{2n+p+1}}{(n)!\,(n+p+1)!}$$

$$= \underline{-x^{-p} J_{p+1}}$$

Prove that: $J'_p(x) = \frac{1}{2}\left(J_{p+1}(x) - J_{p-1}(x)\right)$

$$\frac{2p}{x} J_p(x) = J_{p+1} + J_{p-1}$$

also express $J_2(x), J_3(x), J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$.

We know that,

$$\frac{d}{dx}\left(x^p J_p(x)\right) = x^p J_{p-1}(x)$$

$$x^p \cdot J'_p(x) + p \cdot x^{p-1} J_p(x) = x^p \cdot J_{p-1}(x)$$

$$\Rightarrow J'_p(x) + p x^{-1} J_p(x) = J_{p-1}(x) \longrightarrow (1)$$

Also we know that

$$\frac{d}{dx}\left(x^{-p} J_p(x)\right) = -x^{-p} J_{p+1}(x)$$

$$J'_p(x) - p \cdot x^{-1} J_p(x) = -J_{p+1}(x) \longrightarrow (2)$$

Adding (1) and (2)

$$J'_p(x) = \frac{J_{p-1}(x) - J_{p+1}(x)}{2}$$

Substituting (2) from (1)

$$2 p x^{-1} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$$

$$\boxed{\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x) \quad \text{Recurrance Relation}}$$

for Bessel's

Put $p = 1$

$$\frac{2}{x} J_1(x) = J_0(x) + J_2(x)$$

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

put $p=2$,

$$\frac{4}{x}J_2(x) = J_1(x) + J_3(x)$$

$$\cancel{J_3(x) = \frac{4}{x}J_1(x) - J_0(x)}$$

$$J_3(x) = \frac{4}{x}\left(\frac{2}{x}J_1(x) - J_0(x)\right) - J_1(x)$$

$$J_3(x) = \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0$$

Similarly find $J_4(x)$

- Prove

(i) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x$

(ii) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x$

We know that $J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\,(n+p)!}$

$$J_p(x) = \frac{x^p}{2^p} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\,(n+p)!}$$

$$J_p(x) = \frac{x^p}{2^p} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{\Gamma(n+1)\,\Gamma(n+p+1)}$$

When $n=0$ $n=1$ $n=2$

$$= \frac{x^p}{2^p}\left(\frac{1}{\Gamma(p+1)} - \frac{(x/2)^2}{\Gamma 2\,\Gamma(p+2)} + \frac{(x/2)^4}{\Gamma 3 \cdot \Gamma(p+3)} - \frac{(x/2)^6}{\Gamma 4\,\Gamma(p+4)} \dots\right)$$

$(p+3)(p+2)(p+1)$

$\Gamma(p+2) = p+1\,\Gamma(p+1)$

$(p+2)(p+1)\,\Gamma(p+1)$

$\int_0^{\infty} e^{-x} x^{n-1}\,dx = \Gamma n$ Gamma fn

$\Gamma 1/2 = \sqrt{\pi}$

$\Gamma(n+1) = n\,\Gamma n$

$\Gamma 3/2 = 1/2\,\Gamma 1/2 = 1/2\sqrt{\pi}$

$\Gamma 5 = 4\,\Gamma 4 = 4\,3\,\Gamma 3$

$= 4 \cdot 3 \cdot 2 \cdot \Gamma 2$

$\Gamma 7/2 = 5/2 \cdot 3/2 \cdot 1/2\,\Gamma 1/2$

$= \frac{5 \cdot 3 \cdot 1}{2 \cdot 2 \cdot 2}\sqrt{\pi}$

$\boxed{n! = \Gamma(n+1)}$ $\frac{1}{2}! = \Gamma\frac{3}{2} = \frac{1}{2}\Gamma\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$

$$= \frac{x^p}{2^p \Gamma(p+1)} \left(1 - \frac{x^2}{2^2 . 1 (p+1)} + \frac{x^4}{2^4 . 2 (p+2)(p+1)} - \frac{x^6}{2^6 . 6 (p+3)(p+2)(p+1)} \right) \to \text{(A)}$$

Substitute $p = \frac{1}{2}$ in (A)

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2} . \Gamma\frac{3}{2}} \left(1 - \frac{x^2}{4 . \frac{3}{2}} + \frac{x^4}{16 . 2 . \frac{5}{2} . \frac{3}{2}} - \frac{x^6}{\cancel{64}8 . 6 . \frac{7}{2} . \frac{5}{2} . \frac{3}{2}} + \cdots \right)$$

$$= \frac{x^{1/2}}{2^{1/2} \frac{1}{2} \Gamma 1/2} \left(1 - \frac{x^2}{3.2} + \frac{x^4}{5.4.3.2} - \frac{x^6}{7.6.5.4.3.2} + \cdots \right)$$

$$= \frac{\sqrt{x} . \sqrt{2}}{\sqrt{\pi}} . \frac{1}{x} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)$$

$$* \quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} (\sin x)$$

Substitute $p = -1/2$ in equ (A)

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2} \Gamma 1/2} \left(1 - \frac{x^2}{2} + \frac{x^4}{2^5 . \frac{3}{2} . \frac{1}{2}} - \frac{x^6}{\cancel{64}8 . 6 . \frac{5}{2} . \frac{3}{2} . \frac{1}{2}} + \cdots \right)$$

$$= \sqrt{\frac{2}{\pi x}} \left(1 - \frac{x^2}{2} + \frac{x^4}{4.3.2} - \frac{x^6}{6.5.4.3.2} + \cdots \right)$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} . \cos x$$

• Find $J_{3/2}(x)$, $J_{5/2}(x)$, $J_{-3/2}(x)$ and $J_{-5/2}(x)$

Recurrence relation is given by

$$\frac{2p}{x} J_p(x) = J_{p+1}(x) + J_{p-1}(x)$$

Put $p = 1/2$

$$1/x \, J_{1/2}(x) = J_{3/2}(x) + J_{-1/2}(x)$$

$$J_{3/2}(x) = 1/x \, J_{1/2}(x) - J_{-1/2}(x)$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$$

Put $p = -1/2$

$$-\frac{1}{x} J_{-1/2}(x) = J_{1/2}(x) + J_{-3/2}(x)$$

$$J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x)$$

$$J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x \right)$$

Put $p = 3/2$

$$\frac{3}{x} J_{3/2}(x) = J_{5/2}(x) + J_{1/2}(x)$$

$$J_{5/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}(x)$$

$$= \frac{3}{x} \left(\sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right) \right) - \sqrt{\frac{2}{\pi x}} \cdot \sin x$$

$$= \sqrt{\frac{2}{\pi x}} \left(\frac{3 \sin x}{x^2} - \frac{3 \cos x}{x} - \sin x \right)$$

For $J_{-5/2}(x)$ put $p = \frac{-3}{2}$ in recurrence relation (H.W)

- Sturm-Liouville's Diff Equation

A diff $^{=n}$

$$\frac{d}{dx}\left(P(x)\cdot\frac{dy}{dx}\right)+\lambda q(x)+r(x)y=0$$

which is continuous in $[a,b]$ with boundary condition

$y(a)+y'(a)=0$

$y(b)+y'(b)=0$

is called sturm-Liouville's diff. equ.

Special case

if $P(x)=q(x)=1$ and $r(x)=0$; $\frac{d^2y}{dx^2}+\lambda y=0$ where $\lambda \to$ any +ve no

Auxillary equation is $m^2+\lambda=0$

$m=\pm i\sqrt{\lambda}$

Solution is,

$$\boxed{y=c_1\cos\sqrt{\lambda}\,x+c_2\sin\sqrt{\lambda}\,x}$$

$\lambda_2 \to$ eigen values

$y_n \to$ " functions

(8) • Find the eigen values and eigen fns for $y''+\lambda y=0$ under boundary conditions $y(0), y(\pi/2)=0$

$\Rightarrow$ Auxillary equ is $m^2+\lambda=0$

$m=\pm i\sqrt{\lambda}$

For $\lambda>0$, sturm Liouville Diff. equ $y''+\lambda y=0$

$\therefore$ Solution is,

$y(x)=c_1\cos\sqrt{\lambda}\,x+c_2\sin\sqrt{\lambda}\,x$

Given that $y(0)=0$

$0=c_1\cos 0+c_2\sin 0$

$0=c_1+c_2\cdot 0$

$c_1=0$

Also

Now solution is

$y(x) = c_2 \sin\sqrt{\lambda}.x$

We have

$y(\pi/2) = 0$

$0 = c_2 \sin\sqrt{\lambda}.\frac{\pi}{2}$

For non-zero solution, $c_2 \neq 0$

$\sin\sqrt{\lambda}.\frac{\pi}{2} = 0$

$\sin\sqrt{\lambda}.\frac{\pi}{2} = \sin n\pi$

$n \leftarrow$ Integer

$\sqrt{\lambda}.\frac{\pi}{2} = n\lambda$

$\frac{\lambda\pi^2}{4} = n^2\pi^2$

$\lambda = (2n)^2 \Rightarrow$ Eigen value

when $n = \pm 1, \pm 2$

$\therefore$ Eigen value $= 4n^2$

(or) $\lambda_n = 4, 16,$ ...

Eigen function,

$y_n(x) = c_2 \sin 2n x$

For $c_2 = 1$, $\rightarrow$ bcz c_2 is constant, so let us take it as 1

$y_n = \sin 2n x$

- Find Eigen values and Eigen fn. for $y'' + \lambda y = 0$ under boundary conditions.

$y(0) = 0, \quad y(L) = 0, \quad L > 0$

Solution of $y'' + \lambda y = 0$ is,

$y(x) = c_1 \cos\sqrt{\lambda}.x + c_2 \sin\sqrt{\lambda}.x$

since $y(0) = 0$

$0 = c_1 \cos 0 + c_2 \sin 0$

$c_1 \cdot 1 = 0$

$c_1 = 0$

$y(l) = 0 = c_1 \cos\sqrt{\lambda}\, l + c_2 \sin\sqrt{\lambda}\, l$

For non-zero solu.

$c_2 \neq 0$

$\therefore \sin\sqrt{\lambda}\, l = 0$

$\sin\sqrt{\lambda}\, l = \sin n\pi$

$\sqrt{\lambda} = \frac{n\pi}{l}$

$\lambda = \left(\frac{n\pi}{l}\right)^2$

Eigen values $\lambda_n = \frac{\pi^2}{l^2}, \frac{4\pi^2}{l^2}, \frac{9\pi^2}{l^2}, \ldots$

Eigen fn, $y_n(x) = c_2 \sin \frac{n\pi}{l} x$

for $c_2 = 1$

$y_n(x) = \sin \frac{n\pi}{l} x$