$$16A = \begin{bmatrix} \ldots & \ldots & \ldots \\ -32 & -48 & -64 \\ 48 & 80 & 112 \end{bmatrix}$$

$$12I = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

It can be easily seen that $A^3 - 7A^2 + 16A - 12I = O$. .... (1)

Thus, the Cayley-Hamilton theorem is verified.

Now, by actual division, we see that

$$A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I = (A^3 - 7A^2 + 16A - 12I)(A^3 + A^2) + 2A - I$$

$$= O + 2A - I \qquad [\because A^3 - 7A^2 + 16A - 12I = O \text{ by (1)}]$$

$$= 2\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 20 & 10 \\ -4 & -7 & -8 \\ 6 & 10 & 13 \end{bmatrix}$$

Example 6 : Find the characteristic equation of the matrix A given below and hence, find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$ **(M.U. 2000, 03, 19)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\therefore \quad (2-\lambda)[(1-\lambda)(2-\lambda) - 0] - 1[0 - 0] + 1[0 - 1(1-\lambda)] = 0$

$\therefore \quad (4 - 4\lambda + \lambda^2)(1-\lambda) - (1-\lambda) = 0$

$\therefore \quad 4 - 4\lambda + \lambda^2 - 4\lambda + 4\lambda^2 - \lambda^3 - 1 + \lambda = 0$

$\therefore \quad \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$

This equation is satisfied by A.

Now dividing $\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1$ by $\lambda^3 - 5\lambda^2 + 7\lambda - 3$, we get the quotient $\lambda^5 + \lambda$ and the remainder $\lambda^2 + \lambda + 1$.

In terms of the matrix A this means

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$
$$= (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I)$$

But $\quad (A^3 - 5A^2 + 7A - 3I) = 0$

$\therefore \quad$ L.H.S. $= A^2 + A + I$

Now, $\quad A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

$$\therefore \quad A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

$$\therefore \quad \text{The given expression} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 7 : Use Cayley-Hamilton theorem to find $2A^4 - 5A^3 - 7A + 6I$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$.

(M.U. 2004, 10, 16, 19)

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (1-\lambda)(2-\lambda) - 4 = 0$$

$$\therefore \quad 2 - 3\lambda + \lambda^2 - 4 = 0$$

$$\therefore \quad \lambda^2 - 3\lambda - 2 = 0$$

Number	Variables	x_1	x_2	s_1	s_2	s_3	R.H.S. Solution
		Coefficients of					
1	z	1/3	0	0	0	4/3	20/3
	s_1	5/3	0	1	0	−1/3	4/3
	s_2	4/3	0	0	1	−5/3	2/3
	x_2	1/3	1	0	0	1/3	5/3

Since, all the coefficients in the objective equation (above the dotted line) are positive, the optimum solution is obtained. The required results are given by the last column. Thus, since x_1 does not appear in the second column, $x_1 = 0$, $x_2 = 5/3$, $z_{Max} = 20/3$.

Example 2 : Using Simplex Method solve the following L.P.P.

Maximise $\quad z = 10x_1 + x_2 + x_3$

subject to $\quad x_1 + x_2 - 3x_3 \leq 10$

$$4x_1 + x_2 + x_3 \leq 20$$

$$x_1, x_2, x_3 \geq 0.$$

Sol. : We first express the given problem in standard form

$$z - 10x_1 - x_2 - x_3 + 0s_1 + 0s_2 = 0$$

$$x_1 + x_2 - 3x_3 + s_1 + 0s_2 = 10$$

$$4x_1 + x_2 + x_3 + 0s_1 + s_2 = 20$$

We now put this information in tabular form. Since there are two constraints, there will be two stack variables.

Simplex Table

Iteration Number	Basic Variables	Coefficients of x_1	x_2	x_3	s_1	s_2	R.H.S. Solution	Ratio
0	z	-10	-1	-1	0	0	0	
s_2 leaves	s_1	1	1	-3	1	0	10	$10/1 = 10$
x_1 enters	s_2	4^*	1	3	0	1	20	$20/4 = 5 \leftarrow$
		$\uparrow$						

The **most minimum value** in the row of z is -10. The column headed by -10 is the **key-column** or **pivot column**. We put an arrow-head below this column and put this column in a rectangular box. We now divide the elements in the column of R.H.S. by the corresponding elements in the key column and obtain the column of ratios. Neglecting the negative values, we find the **minimum positive ratio**. It is 5. The row having the element 5 is the **key row** or **pivot row**. We put an arrow head on the right and also put this row in a rectangular box. The element common to the key-column and the key-row is the **key-element** or **pivot element**. It is 4. The element on the left of key row is the **outgoing element**. Thus, s_2 leaves. The element, at the top of the key column is the **incoming element**. Thus, x_1 enters.

Using the **key element** and by using **elementary row operations**, we bring **zeros** at all other **places in the key column**. For this we divide the elements of the key row by 4 and write the resulting row in another table. Thus, we get the following row. (See the third row of the next table)

x_1	1	1 / 4	3 / 4	0	1 / 4	5

Using this row, we bring zeros at all other places in the key-column. So multiply these elements of this row by 10 and add them to the corresponding elements of the first row. Thus, we get new row above the dotted line. (See the first row of the next table)

z	0	3 / 2	13 / 2	0	5 / 2	50

Now multiply the elements of the **same** row by 1 and subtract them from the corresponding elements of the second row. Thus, we get the new row. (See the second row of the next table)

s_1	0	3 / 4	$-15/4$	1	$-1/4$	5

Table (For Example 2)

Iteration Number	Basic Variables	Coefficients of x_1	x_2	x_3	s_1	s_2	R.H.S. Solution
1	z	0	3 / 2	13 / 2	0	5 / 2	50
	s_1	0	3 / 4	$-15/4$	1	$-1/4$	5
	x_1	1	1 / 4	3 / 4	0	1 / 4	5

Since all the coefficients in the objective equation in the row of z are positive this is a optimal ...tion. The values of the variables and of z are given by the last column. Since x_2, x_3 do not ...pear in the second column, they are zero.

$$\therefore \quad x_1 = 5, x_2 = 0, x_3 = 0, z_{Max} = 50.$$

Example 3 : Solve the following L.P.P. by Simplex Method.

Maximise $\quad z = 5x_1 + 4x_2$

subject to $\quad 6x_1 + 4x_2 \le 24 ; \quad x_1 + 2x_2 \le 6$

$-x_1 + x_2 \le 1 ; \quad x_2 \le 2 ; \quad x_1, x_2 \ge 0.$

Sol. : We first express the given problem in standard form

$$z - 5x_1 - 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 = 0$$

$$6x_1 + 4x_2 + s_1 + 0s_2 + 0s_3 + 0s_4 = 24$$

$$x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 + 0s_4 = 6$$

$$-x_1 + x_2 + 0s_1 + 0s_2 + s_3 + 0s_4 = 1$$

$$0x_1 + \ldots + 0s_2 + 0s_3 + s_4 = 2$$

Since, Δ is negative, X_0 is a minima.

Hence, $x_1 = \frac{7}{31}, x_2 = \frac{42}{31}, z_{Min} = \frac{294}{31}$.

Example 2 : Using the method of Lagrange's multipliers solve the following N.L.P.P.

Optimise $z = 4x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $x_1 + x_2 = 4,$

$x_1, x_2 \geq 0.$

Sol. : We have the Lagrangian function

$$L(x_1, x_2, \lambda) = (4x_1 + 8x_2 - x_1^2 - x_2^2) - \lambda(x_1 + x_2 - 4)$$

We, now, obtain the following partial derivatives

$$\frac{\partial L}{\partial x_1} = 4 - 2x_1 - \lambda, \quad \frac{\partial L}{\partial x_2} = 8 - 2x_2 - \lambda, \quad \frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 4)$$

Solving the equations $\frac{\partial L}{\partial x_1} = 0, \frac{\partial L}{\partial x_2} = 0, \frac{\partial L}{\partial \lambda} = 0$, we get

$$\therefore \ 4 - 2x_1 - \lambda = 0, \ 8 - 2x_2 - \lambda = 0, \ x_1 + x_2 = 4$$

Adding the first two, we get

$$12 - 2(x_1 + x_2) - 2\lambda = 0 \qquad \therefore \ 12 - 8 = 2\lambda \qquad \therefore \ \lambda = 2$$

Hence, from the first equation, we get

$$4 - 2x_1 - 2 = 0 \quad \therefore\ 2x_1 = 2 \quad \therefore\ x_1 = 1$$

And from the second equation, we get

$$8 - 2x_2 - 2 = 0 \quad \therefore\ 2x_2 = 6 \quad \therefore\ x_2 = 3.$$

Hence, X_0 is (1, 3).

Now, $h(x_1, x_2) = x_1 + x_2 - 4 = 0$

$\therefore\ \dfrac{\partial h}{\partial x_1} = 1,\ \dfrac{\partial h}{\partial x_2} = 1$ and all other partial derivatives are zero.

And $f(x_1, x_2) = 4x_1 + 8x_2 - x_1^2 - x_2^2$

$$\therefore\ \frac{\partial f}{\partial x_1} = 4 - 2x_1,\ \frac{\partial^2 f}{\partial x_1 x_2} = 0,\ \frac{\partial^2 f}{\partial x_1^2} = -2,$$

$$\frac{\partial f}{\partial x_2} = 8 - 2x_2,\ \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0,\ \frac{\partial^2 f}{\partial x_2^2} = -2.$$

$$\therefore\ \Delta = \begin{vmatrix} 0 & \dfrac{\partial h}{\partial x_1} & \dfrac{\partial h}{\partial x_2} \\ \dfrac{\partial h}{\partial x_1} & \dfrac{\partial^2 f}{\partial x_1^2} - \lambda \dfrac{\partial^2 h}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \dfrac{\partial^2 h}{\partial x_1 \partial x_2} \\ \dfrac{\partial h}{\partial x_2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \dfrac{\partial^2 h}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} - \lambda \dfrac{\partial^2 h}{\partial x_2^2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = -1 \begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} = 2 + 2 = 4$$

Since, Δ is positive, X_0 is a maxima.

Hence, $x_1 = 1,\ x_2 = 3,\ z_{Max} = 18.$

EXERCISE - II

Using Lagrange's multipliers, solve the following N.L.P.P.

1. Optimise $z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

subject to ...

Example 1 : Verify Cayley-Hamilton Theorem for the matrix A and hence, find A^{-1}, A^{-2} and A^4 where

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

(M.U. 2005, 06, 10, 15, 16, 17)

(M.U. 2017)

Prove that $A^{-1} = A^2 - 5A + 9I$.

Sol. : The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (1-\lambda)[(3-\lambda)(1-\lambda)-0] - 2[-1(1-\lambda)-0] - 2[2-0] = 0$$

$$\therefore \quad (1-\lambda)[3-4\lambda+\lambda^2] + 2(1-\lambda) - 4 = 0$$

$$\therefore \quad 3-4\lambda+\lambda^2-3\lambda+4\lambda^2-\lambda^3+2-2\lambda-4 = 0 \qquad \therefore \quad \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0.$$

Cayley-Hamilton theorem states that this equation is satisfied by the matrix A,

i.e. $\quad A^3 - 5A^2 + 9A - I = 0$ .... (1)

Now, $\quad A^2 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$

and $\quad A^3 = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$

It can now be easily seen that

$$A^3 - 5A^2 + 9A - I = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Now, multiply the above equation by A^{-1}.

$$\therefore \quad A^2 - 5A + 9I - A^{-1} = 0$$

$$\therefore \quad A^{-1} = A^2 - 5A + 9I \qquad \text{.....................(2)}$$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

(b) To find A^{-2} multiply (2) by A^{-1}.

$$\therefore \quad A^{-1} \cdot A^{-1} = A^{-1} \cdot A^2 - 5 \cdot A^{-1} \cdot A + 9A^{-1} I$$

$$\therefore \quad A^{-2} = A - 5I + 9A^{-1}$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 9\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 23 & 20 & 52 \\ 8 & 7 & 18 \\ 18 & -16 & 41 \end{bmatrix}$$

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To find A^4 multiply (1) by

$$\therefore \quad A^4 - 5A^3 + 9A^2 - A = 0$$

$$\therefore \quad A^4 = 5A^3 - 9A^2 + A.$$

$$\therefore \quad A^4 = 5\begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 9\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & 13 \end{bmatrix}$$

(You should verify that $AA^{-1} = 1$)

Example 2 : Find the characteristic equation of the matrix A where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

...tic equation and hence, find (a) A^{-1}, (b)

(M.U. 1998, 2003

xample 8 : Evaluate $\int_0^{1+i}(x^2 + iy)\,dz$, along the path (i) $y = x$, (ii) $y = x^2$.

Is the line integral independent of the path ? **(M.U. 1996, 2009, 14)**

) Along the path $y = x$: $\because\ y = x,\ dy = dx$

$\therefore\ dz = dx + i\,dy = dx + i\,dx = (1 + i)\,dx$. And x varies from 0 to 1.

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$$\int_0^{1+i}(x^2+iy)\,dz=\int_0^1(x^2+ix)(1+i)\,dx=(1+i)\left[\frac{x^3}{3}+\frac{ix^2}{2}\right]_0^1$$

$$=(1+i)\left(\frac{1}{3}+\frac{i}{2}\right)=(1+i)\frac{(2+3i)}{6}$$

$$=\frac{(2+2i+3i-3)}{6}=\frac{-1+5i}{6}$$

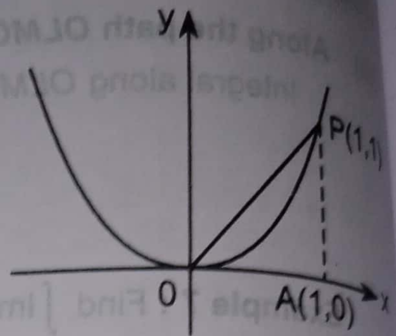

Fig. 2.12

(ii) Along the path $y = x^2$: $y=x^2,\ dy=2x\,dx$

$$\int_0^1(x^2+ix^2)(dx+2ix\,dx)=(1+i)\int_0^1 x^2(1+2ix)\,dx$$

$$=(1+i)\int_0^1(x^2+2ix^3)\,dx$$

$$=(1+i)\left[\frac{x^3}{3}+i\cdot\frac{x^4}{2}\right]_0^1=(1+i)\left(\frac{1}{3}+\frac{i}{2}\right)=\frac{-1+5i}{6}$$

The two line integrals are equal.

(iii) Now, consider the integral along a third path, say, along OA and then along AP. Along OA, x varies from 0 to 1 and $y = 0$

$\therefore\ dy=0 \qquad \therefore\ dz=dx.$

$$\therefore\ \int_{OA}(x^2+iy)\,dz=\int_0^1 x^2\,dx=\left[\frac{x^3}{3}\right]_0^1=\frac{1}{3}$$

Along AP, $x=1$ $\therefore\ dx=0$ and y varies from 0 to 1. $\therefore\ dz=i\,dy.$

$$\therefore\ \int_{AP}(x^2+iy)\,dz=\int_0^1(1+iy)i\,dy=i\left[y+\frac{iy^2}{2}\right]_0^1=i\left(1+\frac{i}{2}\right)=i-\frac{1}{2}$$

$$\therefore\ \int_0^{1+i}(x^2+iy)\,dz=\frac{1}{3}+i-\frac{1}{2}=-\frac{1}{6}+i$$

Thus, the third integral is not equal to the first two. Hence, the integral is not independent of the path.

(iv) Again let $f(z)=x^2+iy=u+iv$. $\therefore\ u=x^2,\ v=y.$

$\therefore\ u_x=2x,\ u_y=0,\ v_x=0,\ v_y=1.$

Hence, Cauchy-Rieman equations $u_x=v_y$ and $u_y=-v_x$ are not satisfied. $f(z)$ is not analytic and hence, integral is not independent of the path.

(See the corollary given on page 2-18)

Example 9 : Evaluate $\int f(z)\,dz$ ...

$$= \frac{az^2 - a^2z + z - az^2}{(1-az)(z-a)} = \frac{z(1-a^2)}{(1-az)(z-a)}.$$

$$\boxed{Z\{a^{|k|}\} = \frac{z(1-a^2)}{(1-az)(z-a)}}$$

The series in G.P. are convergent if $1 > |az|$ and $|z| > a$ i.e. $\frac{1}{a} > |z|$ and $|z| > a$.

$\therefore$ The ROC is $(1/a) > |z| > a$.

Example 2 : Find the Z-transform of $\left\{\left(\frac{1}{3}\right)^{|k|}\right\}$. **(M.U. 2014)**

Sol. : We have

$$Z\left\{\left(\frac{1}{3}\right)^{|k|}\right\} = \Sigma\left(\frac{1}{3}\right)^{|k|} \cdot z^{-k} = \sum_{k=-\infty}^{-1}\left(\frac{1}{3}\right)^{-k} z^{-k} + \sum_{k=\infty}^{\infty}\left(\frac{1}{3}\right)^{k} z^{-k}$$

$$= \left[\ldots + \left(\frac{1}{3}\right)^3 z^3 + \left(\frac{1}{3}\right)^2 z^2 + \left(\frac{1}{3}\right) z\right] + \left[1 + \frac{1}{3}\cdot z^{-1} + \left(\frac{1}{3}\right)^2 z^{-2} + \left(\frac{1}{3}\right)^3 z^{-3} + \ldots\right]$$

$$= \left[\frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \ldots\right] + \left[1 + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \ldots\right]$$

$$= \frac{z}{3}\left[1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \ldots\right]\left[1 + \left(\frac{1}{3z}\right) + \left(\frac{1}{3z}\right)^2 + \ldots\right]$$

$$= \frac{z}{3}\cdot\frac{1}{1-(z/3)} + 1\cdot\frac{1}{1-[1/(3z)]}, \quad \left|\frac{z}{3}\right| < 1, \left|\frac{1}{3z}\right| < 1$$

$$= \frac{z}{3}\cdot\frac{3}{3-z} + \frac{3z}{3z-1} = \frac{z}{3-z} + \frac{3z}{3z-1}, \quad |z| < 3, \frac{1}{3} < |z|$$

$$= \frac{3z^2 - z + 9z - 3z^2}{(3-z)(3z-1)} = \frac{8z}{(3-z)(3z-1)}, \quad \frac{1}{3} < |z| < 3.$$

Remark ....

The above Ex. 2 is a particular case of Ex. 1 where $a = 1/3$.

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$$-4=0$$

$\therefore\ (m^2+4)(m^2-1)=0 \quad \therefore\ m^2=-4$ or $m^2=1$

$\therefore$ The mean is 1 since $m > 0$.

$\therefore$ The statement is correct.

Example 5 : A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as Poisson variate with mean 1·5. Calculate the proportion of days on which (i) neither car is used, (ii) some demand is refused. **(M.U. 1996, 98)**

Sol. : We have $P(x) = e^{-m}\dfrac{m^x}{x!} = \dfrac{e^{-1\cdot5}\cdot(1\cdot5)^x}{x!}, \quad x = 0, 1, 2, \ldots..$

(i) Probability that there is no demand is

$$P(X=0) = e^{-1\cdot5}\,\frac{(1\cdot5)^0}{0!} = 0\cdot2231$$

(ii) Probability that some demand is refused means there was demand for more than two cars.

$$\therefore\ P(X>2) = P(X=3) + P(X=4) + \ldots..$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left[e^{-1\cdot5}\,\frac{(1\cdot5)^0}{0!} + e^{-1\cdot5}\,\frac{(1\cdot5)^1}{1!} + e^{-1\cdot5}\,\frac{(1\cdot5)^2}{2!}\right]$$

$$= 1 - [0\cdot2231 + 0\cdot3347 + 0\cdot2510] = 0\cdot1912.$$

$\therefore$ Proportion of days on which (i) neither car is used is 0·2231.

(ii) some demand is refused is 0·1912.

Example 6 : If a random variable X follows Poisson distribution such that $P(X=1) = 2P(X=2)$, find the mean and the variance of the distribution. Also find $P(X=3)$. **(M.U. 2002, 05, 16)**

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Remark ....

Since there is no 1 in the first column Cramer's rule is more convenient than elementary operations.

Type II : A is a non-symmetric matrix and eigenvalues are repeated

Example 1 : Find the eigenvalues and eigenvectors of the following matrix.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

(M.U. 2001, 09

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

After simplification, we get,

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0 \quad \therefore \quad (\lambda-1)(\lambda-1)(\lambda+5) = 0 \quad \therefore \quad \lambda = 1, 1, 5$$

Hence, 1, 1, 5 are the eigenvalues.

(I) For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \; x_1 + 2x_2 + x_3 = 0.$

We see that the rank of the matrix is 1 and number of variables is 3. Hence, there are $3 - 1 = 2$ linearly independent solutions *i.e.*, there are two parameters. We shall denote these parameters by s and t.

Putting $x_2 = -s$, $x_3 = -t$, we get $x_1 = -2x_2 - x_3 = 2s + t$

$$\therefore \quad X = \begin{bmatrix} 2s+t \\ -s+0 \\ 0-t \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \therefore \; X_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The vectors $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are linearly independent.

[Let us verify that the vectors X_1 and X_2 are independent.

Consider $k_1X_1 + k_2X_2 = 0$.

$\therefore \quad k_1(2, -1, 0) + k_2(1, 0, -1) = 0$

$\therefore \quad 2K_1 + K_2 = 0, \; -K_1 = 0, \; -K_2 = 0$

Hence, the vectors are linearly independent.]

Hence, corresponding to $\lambda = 1$, the eigenvectors are $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(ii) For $\lambda = 5$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{By } R_{13} \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_2 - R_1 \\ R_3 + 3R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 8 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{By } R_3 + 2R_2 \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 + 2x_2 - 3x_3 = 0$ and $-4x_2 + 4x_3 = 0$.

Putting $x_3 = t$, we get $x_2 = t$ and $x_1 = -2x_2 + 3x_3 = -2t + 3t = t$.

$$\therefore \quad X = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 5$, the eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Note ....

When the matrix A is non-symmetric and the eigenvalues are repeated, the eigenvectors corresponding to **repeated root** may or may not be linearly independent. Verify that the vectors X_1 and X_2 are independent. See Examples 1, 2, 3. In Ex. 1, page 1-25, we see that the eigenvectors corresponding to the repeated root $\lambda = 2$ are **not** independent.

Example 2 : Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \end{bmatrix}$$ **(M.U. 2016)**

$$= \lim_{z \to 0} \frac{\sin z - z\cos z}{\sin^2 z} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z \to 0} \frac{\cos z - \cos z + z\sin z}{2\sin z\cos z} \qquad [\text{ By Hospital's Rule }]$$

$$= \lim_{z \to 0} \frac{z}{2\cos z} = 0$$

$$\therefore \int_C \frac{dz}{z\sin z} = 0.$$

Example 4 : Evaluate $\int_C \frac{z^2}{(z-1)^2(z-2)}\,dz$ where C is the circle $|z| = 2{\cdot}5$.

(M.U. 1993, 2001, 03, 05)

Sol. : $f(z)$ has a simple pole at $z = 2$ and a pole of order 2 at $z = 1$.

Both poles lie inside C.

$$\therefore \text{ Residue (at } z = 2) = \lim_{z \to 2} (z-2) \cdot \frac{z^2}{(z-1)^2(z-2)} = \lim_{z \to 2} \frac{z^2}{(z-1)^2} = 4.$$

$$\text{Residue (at } z = 1) = \lim_{z \to 1} \frac{d}{dz}(z-1)^2 \cdot \frac{z^2}{(z-1)^2(z-2)} = \lim_{z \to 1} \frac{d}{dz} \cdot \frac{z^2}{z-2}$$

$$= \lim_{z \to 1} \frac{(z-2)\,2z - z^2}{(z-2)^2} = -3$$

$$\therefore \int_C f(z)\,dz = 2\pi i\,(\text{Sum of residues}) = 2\pi i\,(4-3) = 2\pi i$$

Example 1 : Find inverse Z-transform of $F(z) = \dfrac{z}{(z-1)(z-2)}$, $|z| > 2$. **(M.U. 2016)**

Sol. : We have $F(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{2}{z-2} - \dfrac{1}{z-1}$

Since $|z| > 2$ clearly $|z| > 1$.

$\therefore \quad |z/2| > 1$ and $|z| > 1$.

$\therefore \quad |2/z| < 1$ and $|1/z| < 1$. $\therefore$ We take z common.

$$\therefore \quad F(z) = \frac{2}{z[1-(2/z)]} - \frac{1}{z[1-(1/z)]} = \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{2}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\ldots+\frac{2^{k-1}}{z^{k-1}}+\ldots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots+\frac{1}{z^{k-1}}+\ldots\right)$$

$$= \left(\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\ldots+\frac{2^k}{z^k}+\ldots\right) - \left(\frac{1}{z}+\frac{1}{z^2}+\ldots+\frac{1}{z^k}+\ldots\right)$$

Coefficient of $z^{-k} = 2^k - 1, \ k \geq 1$

$\therefore \quad Z^{-1}[F(z)] = \{2^k - 1], \ k \geq 1.$

5. ... of gr... ...are required when the constraints are ...implex method artificial variables are not required.

6. The dual simplex method is more convenient than the regular simplex method.

Example 1 : Use the dual simplex method to solve the following L.P.P.

Minimise $z = 2x_1 + 2x_2 + 4x_3$

subject to $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_2, x_3 \geq 0.$

(M.U. 2001, 03, 06, 09, 15)

Sol. : We first express the given problem using $\leq$ in the first constraint.

Minimise $z = 2x_1 + 2x_2 + 4x_3$

subject to $-2x_1 - 3x_2 - 5x_3 \leq -2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

Introducing the slack variables s_1, s_2, s_3, we have

Minimise $z = 2x_1 + 2x_2 + 4x_3 - 0s_1 - 0s_2 - 0s_3$

i.e. $z - 2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3 = 0$

subject to $-2x_1 - 3x_2 - 5x_3 + s_1 + 0s_2 + 0s_3 = -2$

$3x_1 + x_2 + 7x_3 + 0s_1 + s_2 + 0s_3 = 3$

$x_1 + 4x_2 + 6x_3 + 0s_1 + 0s_2 + s_3 = 5$

Simplex Table

Iteration Number	Basic Variables	Coefficients of x_1	x_2	x_3	s_1	s_2	s_3	R.H.S. Solution
0	z	-2	-2	-4	0	0	0	0
s_1 leaves	s_1	-2	-3^*	-5	1	0	0	-2
x_2 enters	s_2	3	1	7	0	1	0	3
	s_3	1	4	6	0	0	1	5
Ratio		1	2/3 ↑	4/5				
1	z	$-2/3$	0	$-2/3$	$-2/3$	0	0	4/3
	x_2	2/3	1	5/3	$-1/3$	0	0	2/3
	s_2	7/3	0	16/3	1/3	1	0	7/3
	s_3	$-5/3$	0	$-2/3$	4/3	0	1	7/3

$\therefore \; x_1 = 0, \; x_2 = \dfrac{2}{3}, \; x_3 = 0, \; z_{Min} = \dfrac{4}{3}.$

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Example 5 : In an examination marks obtained by students in Mathematics, Physics ... Chemistry are normally distributed with means 51, 53 and 46 with standard deviation 15, 12 ... respectively. Find the probability of securing total marks (i) 180 or above, (ii) 80 or below.

(M.U. 2005, ...

Sol. : Let X_1, X_2, X_3 denote the marks obtained in the three subjects. Then X_1, X_2, X_3 are nor... variates with mean 51, 53, 46 and variance $15^2, 12^2, 16^2$.

Assuming the variates to be independent, $Y = X_1 + X_2 + X_3$ is distributed normally with m... $m = 51 + 53 + 46 = 150$ and $\sigma^2 = 15^2 + 12^2 + 16^2 = 625 = 25^2$.

$$\therefore \quad \text{S.N.V. } Z = \frac{Y - m}{\sigma} = \frac{Y - 150}{25}$$

When $Y = 180$, $\quad Z = \dfrac{180 - 150}{25} = \dfrac{30}{25} = 1{\cdot}2$.

$$\therefore \quad P(Y \geq 150) = P(Z \geq 1{\cdot}2)$$

$= $ Area to the right of $Z = 1{\cdot}2$

$= 0{\cdot}5 -$ (area between $Z = 0$ and $Z = 1{\cdot}2$)

$= 0{\cdot}5 - 0{\cdot}3849 = 0{\cdot}1151$.

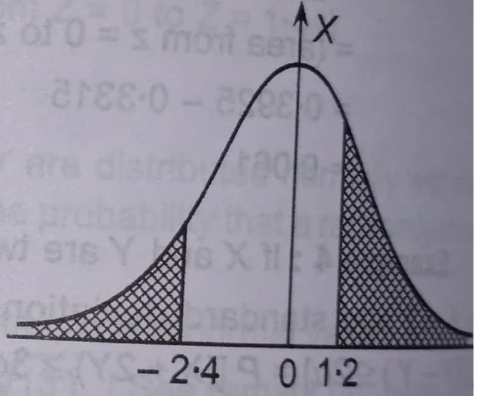


Fig. 4.18

When $Y = 80$, $\quad Z = \dfrac{80 - 150}{25} = -\dfrac{70}{25} = -2{\cdot}4$

$$\therefore \quad P(Y \leq 80) = P(Z \leq -2{\cdot}4)$$

$= $ Area to the left of $Z = -2{\cdot}4$

$= 0{\cdot}5 -$ area from $Z = 0$ to $Z = 2{\cdot}4$

$= 0{\cdot}5 - 0{\cdot}4918 = 0{\cdot}0082$.

Type V

From the table we find that the area from $Z = 0$ to $Z = 1{\cdot}96$ is $0{\cdot}475$.

$\therefore$ The required value of $Z = 1{\cdot}96$

When $Z = 1{\cdot}96$, $\quad 1{\cdot}96 = \dfrac{X - 750}{50}$

$\therefore X - 750 = 1{\cdot}96 \times 50 \qquad \therefore X = 848$

$\therefore$ The lowest income of richest 250 persons = ₹ 848.

Example 3 : In a competitive examination the top 15% of the students appeared will get grade A, while the bottom 20% will be declared fail. If the grades are normally distributed with mean % of marks 75 and S.D. 10, determine the lowest % of marks to receive grade A and the lowest % of marks that passes.

(M.U. 2014)

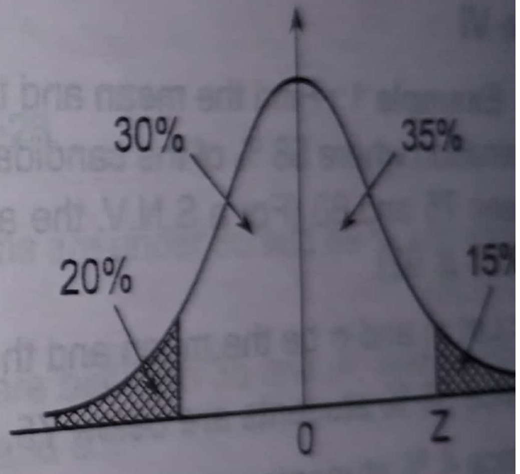


Fig. 4.20

Sol. : This is a reverse problem as above.

We have $Z = \dfrac{X - m}{\sigma} = \dfrac{X - 75}{10}$.

(i) Grade A is given for 15%. We have to find the value of Z to the right of which the area is 0·15.
But the area to the right of $Z = 0$ is 0·5.

$\therefore$ Area from ($Z = 0$ to Z = this value) $= 0 \cdot 5 - 0 \cdot 15 = 0 \cdot 35$.

From the table, we find that the area between $Z = 0$ to $Z = 1 \cdot 04$ is 0·35.

$\therefore$ The required value of $Z = 1 \cdot 04$.

But $Z = \dfrac{X - 75}{10}$ $\quad\therefore\quad 1 \cdot 04 = \dfrac{X - 75}{10}$ $\quad\therefore\quad X = 75 + 10 \cdot 4 = 85 \cdot 4$

(ii) Lowest 20% students are declared fail. We have to find the value of Z to the left of which the area is 0·20. But the area to the left of $Z = 0$ is 0·5.

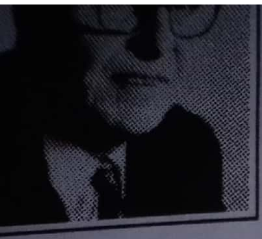
$\therefore$ Area from ($Z = 0$ to Z = this value) $= 0 \cdot 5 - 0 \cdot 2 = 0 \cdot 3$.

From the table we find that the area between $Z = 0$ and $Z = 0 \cdot 84$ is 0·3.

But this ordinate is on the left and hence negative.

$\therefore$ The required value of $Z = -0 \cdot 84$.

But $Z = \dfrac{X - 75}{10}$ $\quad\therefore\quad -0 \cdot 84 = \dfrac{X - 75}{10}$ $\quad\therefore\quad X = 75 - 8 \cdot 4 = 66 \cdot 6$.



University and head of the departm... years.
On a number of papers and models he worked with Harold W. Kuhn. One of his Ph.D. students John Forbes Nash got Nobel prize in Economics in 1994. He is well known for Karush-Kuhn-Tucker conditions in non-linear programming (Historically W. Karush was the st to develop these KKT conditions as part of M.S. thesis at the university of Chicago in 1939. The me conditions were developed independently in 1951 by Kuhn and Tucker.)

Example 1 : Solve the following N.L.P.P.

Maximise $z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

subject to $2x_1 + x_2 \le 5$

$x_1, x_2 \ge 0.$ **(M.U. 2010, 15, 16)**

ol. : We rewrite the given problem as

$$f(x_1, x_2) = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$$

and $h(x_1, x_2) = 2x_1 + x_2 - 5.$

Now, Kuhn-Tucker conditions are

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0,$$

$$\lambda h(x_1, x_2) = 0, \quad h(x_1, x_2) \le 0, \quad \lambda \ge 0$$

$\therefore$ We get

$$10 - 4x_1 - 2\lambda = 0 \quad \ldots\ldots\ldots (1) \qquad 4 - 2x_2 - \lambda = 0 \quad \ldots\ldots\ldots (2)$$

$$\lambda(2x_1 + x_2 - 5) = 0 \quad \ldots\ldots\ldots (3) \qquad 2x_1 + x_2 - 5 \le 0 \quad \ldots\ldots\ldots (4)$$

$$x_1, x_2, \lambda \ge 0 \quad \ldots\ldots\ldots (5)$$

From (3), we get either $\lambda = 0$ or $2x_1 + x_2 - 5 = 0$.

Case 1 : If $\lambda = 0$, from (1) and (2), we get

$$4x_1 = 10 \text{ i.e. } x_1 = 5/2 \text{ and } 2x_2 = 4 \text{ i.e. } x_2 = 2.$$

Putting these values of x_1 and x_2 in l.h.s. of (4), we get

$$\text{l.h.s.} = 5 + 2 - 5 = 2 \not\le 0$$

Thus, these values do not satisfy (4). Hence, $\lambda = 0$ does not yield a feasible solution. We eject these values.

Case 2 : If $\lambda \ne 0$, $2x_1 + x_2 - 5 = 0$ .... (6)

We now solve the equations (1), (2) and (6). Subtracting twice the second equation from the rst.

$$10 - 4x_1 - 8 + 4x_2 = 0 \quad \therefore \; 2x_1 - 2x_2 = 1$$

Multiply (6) by 2, $\therefore \; 4x_1 + 2x_2 = 10$

$\therefore$ By addition, we get $6x_1 = 11$ $\quad\therefore\quad x_1 = 11/6$

$\therefore\quad x_2 = 5 - 2x_1 = 5 - (11/3) = 4/3$

$\therefore$ From (2), $\quad \lambda = 4 - 2x_2 = 4 - (8/3) = 4/3.$

These values satisfy all the necessary conditions.

$\therefore$ The optimal solution is $x_1 = 11/6,\ x_2 = 4/3.$

$$\therefore\quad z_{Max} = 10 \times \frac{11}{6} + 4 \times \frac{4}{3} - 2\left(\frac{11}{6}\right)^2 - \left(\frac{4}{3}\right)^2 = \frac{91}{6}.$$

Example 2 : Use the Kuhn-Tucker conditions to solve the following N.L.P.P.

(M.U. 2004, 07, 11

Maximise $\quad z = 2x_1^2 - 7x_2^2 + 12x_1x_2$

subject to $\quad 2x_1 + 5x_2 \le 98$

$x_1, x_2 \ge 0.$

Sol. : We rewrite the given problem as

$$\ldots 2 - 7x_2^2 + 12x_1x_2$$