Applied Mathematics-IV

Chapter 1 : Complex Integration

Q. 1 **Evaluate $\int_c (\bar{z} + 2z)\, dz$ along the circle c : $x^2 + y^2 = 1$.** May 2014

Ans. :

$$x^2 + y^2 = 1 \text{ is a circle with centre } (0, 0)$$

$$\text{and radius} = 1$$

$$\text{put } z = re^{i\theta} = 1\, e^{i\theta}$$

$$dz = e^{i\theta} \cdot i\, d\theta$$

$$\text{and } \bar{z} = e^{-i\theta}$$

$$\therefore \int_c (\bar{z} + 2z)\, dz = \int_0^{2\pi} [e^{-i\theta} + 2\, e^{i\theta}] \cdot i\, e^{i\theta}\, d\theta$$

$$= i \int_0^{2\pi} [e^0 + 2\, e^{2i\theta}]\, d\theta = i \left[1\,\theta + \frac{2\, e^{2i\theta}}{2\, i} \right]_0^{2\pi}$$

$$= i \left\{ \left[2\pi + \frac{e^{4i\pi}}{i} \right] - \left[0 + \frac{e^0}{1} \right] \right\}$$

$$\text{Now } e^{i4\pi} = \cos 4\pi + I \sin 4\pi$$

$$= 1 + i\,(0) = 1$$

$$\therefore \int_c (\bar{z} + 2z)\, dz = i \left\{ 2\pi + \frac{1}{i} - \frac{1}{i} \right\}$$

$$\therefore \int_c (\bar{z} + 2z)\, dz = 2\, i\, \pi$$

Q. 2 **Using Cauchy's integral formula, evaluate $\int_c \frac{(12\, z - 7)\, dz}{(z - 1)^2\, (2z + 3)}$ where C : | z + I | = $\sqrt{3}$** May 2014

Ans. :

$$\text{Let } I = \int_c \frac{12\, z - 7}{(z - 1)^2\, (2z + 3)}\, dz$$

Circle | z + i | = $\sqrt{3}$ has centre (0, – 1) and radius $\sqrt{3}$

Here z_0 = – 3/2 lies outside while $z_0 = 1$ lies inside the circle

z_0 = 1 is a pole of order 2

$$\therefore I = \int_c \frac{(12\, z - 7) / (2\, z + 3)}{(z - 1)^2}\, dz$$

$$\text{Let } f(z) = \frac{12\, z - 7}{2\, z + 3} \text{ and } z_0 = 1$$

$$\therefore f'(z) = \frac{(2z + 3) \cdot 12 - (12\, z - 7)\, 2}{(2\, z + 3)^2}$$

$$\therefore f'(z_0) = f'(1) \frac{(2 \times 1 + 3) - (12 \times 1 - 7) \cdot 2}{(2 \times 1 + 3)^2} = \frac{50}{25} = 2$$

By Cauchy's integral formula

$$\int_c \frac{f(z)}{(z - Z_0)^n}\, dz = \frac{2\, \pi i}{(n - 1)\, i}\, f^{n-1}(Z_0)$$

$$\therefore \int_c \frac{(12\, z - 7) / (2\, z + 3)}{(z - 1)^2}\, dz = \frac{2\, \pi i}{(2 - 1)\, i}\, f'(Z_0)$$

$$\therefore \int_c \frac{12\, z - 7}{(z - 1)^2 - (2\, z - 3)}\, dz = \frac{2\, \pi i}{1 i} \times 2 = 4\pi$$

Q. 3 **Obtain all Taylor's and Laurent's series expansions of function $\frac{(z - 2)\,(z + 2)}{(z + 1)\,(z + 4)}$ about z = 0.** May 2014

Ans. :

$$\text{Consider } \frac{(z - 2)\,(z + 2)}{(z + 1)\,(z + 4)} = \frac{z^2 - 4}{z^2 + 5z + 4}$$

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2 + 5z + 4 & z^2 + 0z - 4 \\ & z^2 + 5z + 4 \\ \hline & \quad -5z - 8 \end{array}$$

$$\therefore \frac{(z - 2)\,(z + 2)}{(z + 1)\,(z + 4)} = 1 + \frac{(-5z - 8)}{z^2 + 5z + 4}$$

$$= 1 - \frac{1}{z + 1} - \frac{4}{z + 4} \text{ (by partial fractions)}$$

Case 1 : **For | z | < 1 obviously | z | < 4 :**

$\therefore \; |z| < 1$ and $\left|\frac{z}{4}\right| < 1$

$$\therefore \; f(z) = 1 - \frac{1}{1+z} - \frac{4}{4(z/4+1)}$$

$$= 1 - (1+z)^{-1} - \left(1 + \frac{z}{4}\right)^{-1}$$

$$= 1 - (1 - z + z^2 - z^3 + \ldots) - \left(1 - \frac{z}{4} + \frac{z^2}{4^2} - \frac{z^3}{4^3} + \ldots\right)$$

$$= (z - z^2 + z^3 - z^4 + \ldots) - \left(1 - \frac{z}{4} + \frac{z^2}{4^2} - \frac{z^3}{4^3} + \ldots\right)$$

Case 2 : For $1 < |z| < 4$:

$\therefore \; 1 < |z|$ and $|z| < 4$

$\therefore \; \left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{4}\right| < 1$

$$f(z) = 1 - \frac{1}{z(1+1/z)} - \frac{4}{4(z/4+1)}$$

$$= 1 - \frac{1}{z}\left(1 + \frac{1}{z}\right)^{-1} - \left(1 + \frac{z}{4}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\right) - \left(1 - \frac{z}{4} + \frac{z^2}{4^2} - \frac{z^3}{4^3} + \ldots\right)$$

$$= -\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \ldots\right) + \left(\frac{z}{4} + \frac{z^2}{4^2} - \frac{z^3}{4^3} - \frac{z^4}{z^4} + \ldots\right)$$

Case 3 : Fro $|z| > 4$ obviously $|z| > 1$:

$\therefore \; \left|\frac{1}{z}\right| < 1$ and $\left|\frac{4}{z}\right| < 1$

$$f(z) = 1 - \frac{1}{z(1+1/z)} - \frac{4}{z(1+4/z)}$$

$$= 1 - \frac{1}{z}\left(1 + \frac{1}{z}\right)^{-1} - \frac{4}{z}\left(1 + \frac{4}{z}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\right) - \frac{4}{z}\left(1 - \frac{4}{z} + \frac{4^2}{z^2} - \frac{4^3}{z^3} + \ldots\right)$$

$$= 1 - \left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} \ldots\right) - \left(\frac{4}{z} - \frac{4^2}{z^2} + \frac{4^3}{z^3} - \frac{4^4}{z^4} \ldots\right)$$

Q. 4 Using Cauchy's residue theorem, show the $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \frac{\pi}{6}$.

May 2014

Ans. :

Consider a circle $|z| = 1$ which has centre (0, 0) and radius 1.

Put $z = re^{i\theta} = 1\,e^{i\theta} = e^{i\theta}$

$\therefore \; dz = e^{i\theta} \cdot i d\theta = iz\, d\theta$

$\therefore \; d\theta = \frac{dz}{iz}$

Also, $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{z^2+1}{2z}$

And $\cos 2\theta = \text{R.P.}(e^{i2\theta}) = \text{R.P.}(e^{i\theta})^2 = \text{R.P.}(z^2)$

On substituting,

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \int_c \frac{\text{R.P.}(z)^2}{5 + 4[(z^2+1)/2z]} \cdot \frac{dz}{iz}$$

$$= \text{R.P.} \int_c \frac{z \cdot z^2}{5z + 2z^2 + 2} \cdot \frac{dz}{iz}$$

$$= \text{R.P.} \int_c \frac{z^2\, dz}{i(2z^2 + 4z + z + 2)}$$

$$= \text{R.P.} \int_c \frac{z^2\, dz}{i(2z+1)(z+2)}$$

$\therefore$ Here, $z_o = \frac{-1}{2}$ and $z_o = -2$ are simple poles.

$Z_o = -2$ lies outside while $z_o = \frac{-1}{2}$ lies inside the circle $|z| = 1$.

R_1 = Residue of f (z) at $z_o = \frac{-1}{2}$

$$= \lim_{z \to z_o} (z - z_o) \times f(z)$$

$$= \lim_{z \to -1/2} \left(z + \frac{1}{2}\right) \times \frac{z^2}{i \times 2(z+1/2)(z+2)}$$

$$= \lim_{z \to -1/2} \frac{z^2}{2i(z+2)} = \frac{(-1/2)^2}{2i(-1/2+2)}$$

$$= \frac{1}{12i}$$

By Cauchy's Residue theorem,

$$\int_c f(z)\, dz = 2\pi i(R_1 + R_2 + \ldots)$$

$$\therefore \; \text{R.P.} \int_c \frac{z^2\, dz}{i(2z+1)(z+2)} = \text{R.P.}\left[2\pi i \times \frac{1}{12i}\right]$$

$$\therefore \; \int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \text{R.P.}\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$$

$$\therefore \; \int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \frac{\pi}{6}$$

Q. 5 A continuous random variable x has the following probability law

Dec. 2014

$f(x) = kx^2 e^{-x}$, $x \geq 0$. Find k, mean and variance.

Ans. :

1. Here x is a continuous function.

$$\int_{-\infty}^{\infty} f(x) = 1$$

$$\int_{0}^{\infty} k x^2 e^{-x} = 1$$

$$K\int_{0}^{\infty} x^2 e^{-x} = 1K\left[x^2\frac{e^{-x}}{-1} + 2x\frac{e^{-x}}{-1} + 2\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$K[(0+0+0)-(0+0-2e^0)] = 1$$

$$K \times 2 = 1$$

$$K = \frac{1}{2}$$

$$\therefore \quad f(x) = \frac{1}{2}x^2 e^{-x}$$

2. Mean,

$$\mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$= \int_{0}^{\infty} x \cdot \frac{1}{2}x^2 e^{-x}\,dx = \frac{1}{2}\int_{0}^{\infty} x^3 e^{-x}\,dx$$

$$= \frac{1}{2}\left[x^3\frac{e^{-x}}{-1} + 3x^2\frac{e^{-x}}{-1} + 6x\frac{e^{-x}}{-1} + 6\frac{e^{-x}}{-1}\right]_0^{\infty}$$

$$= \frac{1}{2}[(0+0+0+0)-(0+0+0-6)] = \frac{6}{2}$$

$$\text{Mean} = 3$$

3. $$E[x^2] = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_{0}^{\infty} x^2 \cdot \frac{1}{2}x^2 e^{-x}\,dx$$

$$= \frac{1}{2}\int_{0}^{\infty} x^4 e^{-x}\,dx$$

$$= \frac{1}{2}\left[x^4\frac{e^{-x}}{-1} + 4x^3\frac{e^{-x}}{-1} + 12x^2\frac{e^{-x}}{-1} + 24x\frac{e^{-x}}{-1} + 24\frac{e^{-x}}{-1}\right]_0^{\infty}$$

$$= \frac{1}{2}[(0+0+0+0+0)-(0+0+0+0-24)]$$

$$= \frac{24}{2} = 12$$

$$\text{Variance} \quad \sigma^2 = E[x^2] - (E[x])^2$$

$$= 12 - 9 = 3$$

Q. 6 Show that $\int_C \log z\,dz = 2\pi i$, where C is the unit circle in the z-plane. May 2015

Ans. :

$|z| = 1$ is unit circle with centre (0, 0) and radius = 1

$$\text{Let } I = \int_c \log z\,dz$$

$$\text{Put } z = re^{i\theta} = 1e^{i\theta} = e^{i\theta}$$

$$\therefore dz = e^{i\theta} \cdot i\,d\theta$$

$$\therefore I = \int_0^{2\pi} \log(e^{i\theta}) \cdot e^{i\theta} \cdot i\,d\theta$$

$$= \int_0^{2\pi} i\theta \log e \cdot e^{i\theta}\, i\,d\theta = i^2 \int_0^{2\pi} \theta e^{i\theta}\,d\theta$$

$$= i^2\left[\theta \cdot \frac{e^{i\theta}}{i} - 1 \cdot \frac{e^{i\theta}}{i^2}\right]_0^{2\pi}$$

$$= [i\theta e^{i\theta} - e^{i\theta}]_0^{2\pi} = [e^{i\theta}(i\theta - 1)]_0^{2\pi} = e^{i2\pi}(i \cdot 2\pi - 1) - e^0(i \cdot 0 - 1)$$

$$= (\cos 2\pi + i\sin 2\pi)(2\pi i - 1) + 1 = (1+0)(2\pi i - 1) + 1$$

$$= 2\pi i - 1 + 1 = 2\pi i$$

$$\text{Hence, } \int_c \log z\,dz = 2\pi i.$$

Q. 7 Evaluate $\int_c \frac{z+2}{z^3 - 2z^2}dz$, where C is the circle $|z-2-i| = 2$ May 2015

Ans. :

Circle, $|z-2-i| = 2$ has centre (2, 1) and radius = 2

$$\text{Let } I = \frac{z+2}{z^3-2z^2}dz$$

$$\text{For singularity, } z^3 - 2z^2 = 0$$

$$\therefore z^2(z-2) = 0$$

$$\therefore z = 0 \text{ or } z = 2$$

$z_0 = 0$ lies outside while $z_0 = 2$ lies inside the given circle

$\therefore$ The given integrand is not analytic at $z = 2$.

Now, $I = \int_c \frac{(z+2)/z^2}{z-2} dz$

Let $f(z) = \frac{z+2}{z^2}$

$\therefore I = 2\pi i \, f(z_0)$

... (Cauchy's integrated formula)

$$= 2\pi i \times \frac{z_0+2}{z_0^2} = 2\pi i \times \frac{2+2}{2^2} = 2\pi i$$

Q. 8 Find all possible Laurent's expansions of the function $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ about $z = -1$.

May 2015

Ans. : By partial fractions,

$$f(z) = \frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2}$$

Put $u = z + 1$

$\therefore z = u - 1$

$$\therefore f(z) = \frac{1}{u-1} - \frac{3}{(u-1)+1} + \frac{2}{(u-1)-2} = \frac{1}{u-1} - \frac{3}{u} + \frac{2}{u-3}$$

Case 1 : For $|u| < 1$

Obviously, $|u| < 3$

$\therefore |u| < 1$ and $\left|\frac{u}{3}\right| < 1$

$$\therefore f(z) = \frac{1}{-(1-u)} - \frac{3}{u} + \frac{2}{3(u/3-1)} = -(1-u)^{-1} - \frac{3}{u} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$= -(1+u+u^2+\dots) - \frac{3}{u} - \frac{2}{3}\left(1+\frac{u}{3}+\frac{u^2}{3^2}+\dots\right)$$

$$= -[1+(z+1)+(z+1)^2+\dots] - \frac{3}{(z+1)} - \frac{2}{3}$$
$$\left[1+\frac{(z+1)}{3}+\frac{(z+1)^2}{3^2}+\dots\right]$$

Case 2 : $1 < |u| < 2$

$\therefore 1 < |u|$ and $|u| < 3$

$\therefore \left|\frac{1}{u}\right| < 1$ and $\left|\frac{u}{3}\right| < 1$

$$\therefore f(z) = \frac{1}{u(1-1/u)} - \frac{3}{u} + \frac{2}{3(u/3-1)}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{3}{u} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$= \frac{1}{u}\left(1+\frac{1}{u}+\frac{1}{u^2}+\dots\right) - \frac{3}{u} - \frac{2}{3}\left(1+\frac{u}{3}+\frac{u^2}{3^2}+\dots\right)$$

$$= \left(\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\dots\right) - \frac{3}{u} - \frac{2}{3}\left(1+\frac{u}{3}+\frac{u^2}{3^2}+\dots\right)$$

$$= \frac{1}{(z+1)} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3}$$
$$+ \dots - \frac{3}{(z+1)} - \frac{2}{3}\left[1+\frac{(z+1)}{3}+\frac{(z+1)^2}{3^2}+\dots\right]$$

Case 3 : For $|u| > 3$

Obviously, $|u| > 1$

$\therefore 1 < |u|$ and $3 < |u|$

$\therefore \left|\frac{1}{u}\right| < 1$ and $\left|\frac{3}{u}\right| < 1$

$$\therefore f(z) = \frac{1}{u(1-1/u)} - \frac{3}{u} + \frac{2}{u(1-3/u)}$$

$$= \frac{1}{u}\cdot\left(1-\frac{1}{u}\right)^{-1} - \frac{3}{u} - \frac{2}{u}\left(1-\frac{3}{u}\right)^{-1}$$

$$= \frac{1}{u}\left(1+\frac{1}{u}+\frac{1}{u^2}+\dots\right) - \frac{3}{u} + \frac{2}{u}\left(1+\frac{3}{u}+\frac{3^2}{u^2}+\dots\right)$$

$$= \left(\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\dots\right) - \frac{3}{u} + 2\left(\frac{1}{u}+\frac{3}{u^2}+\frac{3^3}{u^3}+\dots\right)$$

$$= \left(\frac{1}{u^2}+\frac{1}{u^3}+\frac{1}{u^4}+\dots\right) + 2\left(\frac{3}{u^2}+\frac{3^2}{u^3}+\frac{3^3}{u^4}+\dots\right)$$

$$= \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \frac{1}{(z+1)^4} + \dots$$
$$+ 2\left[\frac{1}{(z+1)^2} + \frac{3}{(z+1)^3} + \frac{3^2}{(z+1)^4} + \dots\right]$$

Q. 9 Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^3}$, $a > 0$ using contour integration.

May 2015

Ans. :

S1 : Consider the contour of a large semicircle with diameter on real axis; centre at origin and above the real axis.

S2 : Let $f(z) = \frac{1}{(z^2+a^2)^3}$

As $z \to \infty$, $z\,f(z) \to 0$

S3 : For singularity

$\therefore (z^2+a^2)^3 = 0$

$\therefore z^2 = -a^2 = i^2 a^2$

$\therefore z = \pm ai, \pm ai$

Here $z_0 = -ai$ lies outside while $z_0 = ai$ lies inside the contour.

$z_0 = ai$ is a pole of order 3

S4 : R_1 = Residue of f (z) at "$z = ai$"

$$= \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n \times f(z)$$

$$= \frac{1}{(3-1)!} \lim_{z \to ai} \frac{d^2}{dz^2} (z-ai)^3 \times \frac{1}{(z-ai)^3 (z+ai)^3}$$

$$= \frac{1}{2!} \lim_{z \to ai} \frac{d^2}{dz^2} (z+ai)^{-3}$$

$$= \frac{1}{2} \lim_{z \to ai} \frac{d}{dz} [-3 \cdot (z+a)^{-4} \cdot 1]$$

$$= \frac{1}{2} \times -3 \lim_{z \to ai} [-4 \cdot (z+ai)^{-5} \cdot 1]$$

$$= \frac{-3}{2} \times -4 \times \frac{1}{(ai+ai)^5} = \frac{6}{2^5 a^5 i^5} = \frac{3}{16a^5 i}$$

By Cauchy's Residue theorem $\int_c f(z)\, dz = 2\pi i\, (R_1 + R_2 + \ldots)$

$$\therefore \int_c \frac{1}{(z^2+a^2)^3} dz = 2\pi i \left(\frac{3}{16a^5 i}\right)$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} dx = \frac{3\pi}{8a^5}$$

$$\therefore 2\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^3} dx = \frac{3\pi}{8a^5} \quad \ldots \text{(Since Even Function)}$$

$$\therefore \int_{0}^{\infty} \frac{dx}{(x^2+a^2)^3} dx = \frac{3\pi}{16a^5}$$

Q. 10 Evaluate the line integral $\int_0^{1+i} (x^2 - iy)\, dz$ along the path $y = x$. Dec. 2015

Ans. : Let

$$I = \int_0^{1+i} (x^2 - iy)\, dz$$

$$dz = dx + idy$$

For $y = x$

$dx = dy \qquad x = 0 \text{ to } 1$

$dz = dx + idx = (1+i)\, dx$, $f(z) = x^2 - ix$

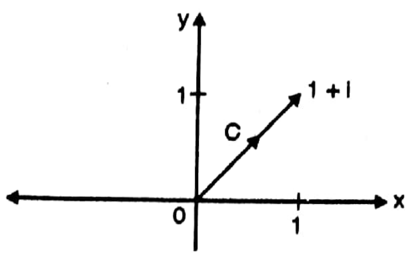

Fig. 1.1

$$I = \int_0^1 (x^2 - ix)(1+i)\, dx$$

$$= (1+i) \int_0^1 (x^2 - ix)\, dx$$

$$= (1+i) \left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$$= (i+1)\left(\frac{1}{3} - \frac{i}{2}\right) = \frac{(1+i)(2-3i)}{6}$$

$$I = \frac{5-i}{6}$$

Q. 11 Evaluate $\int_c \frac{dz}{z^3 (z+4)}$ where C is the circle $|Z| = 2$. Dec. 2015

Ans. : Let,

$$I = \oint_c f(z)\, dz$$

Where C is the circle $|Z| = 2$ and $f(Z) = \frac{1}{z^3 (z+4)}$ is analytic within and on the simple closed curve $|Z| = 2$.

$\therefore$ f(Z) has a pole at $Z = 0$ of order 3 inside the curve C : $|Z| = 0$

Consider residue at $Z = 0$

$$\therefore \text{Residue (at } Z = 0) = \frac{1}{(3-1)!} \lim_{Z \to 0} \frac{d^2}{dZ^2}\left(Z^3 \cdot \frac{1}{Z^3 (Z+4)}\right)$$

$$= \frac{1}{2} \lim_{Z \to 0} \frac{d^2}{dZ^2} \left[\frac{1}{(Z+4)}\right]$$

$$= \frac{1}{2} \lim_{Z \to 0} \frac{d}{dZ} \left[\frac{-1}{(Z+4)^2}\right]$$

$$= \frac{1}{2} \lim_{Z \to 0} \frac{d}{dz} \left[\frac{2}{(Z+4)^3}\right] = \frac{1}{64}$$

$$\text{Residue (at } Z = 0) = \frac{1}{64}$$

$\therefore$ By Cauchy's Residue Theorem,

$$I = 2\pi\, [\text{Residue}] = 2\pi \left[\frac{1}{64}\right] = \frac{\pi i}{32}$$

$$\therefore \int_c \frac{dz}{z^3 (z+4)} = \frac{\pi i}{32}$$

Q. 12 Expand $f(Z) = \frac{2}{(Z-2)(Z-1)}$ is the regions (i) $|Z| < 1$ (ii) $1 < |Z| < 2$ (iii) $|Z| > 2$. Dec. 2015

Ans. : Let, $f(Z) = \dfrac{2}{(Z-2)(Z-1)} = \dfrac{-2}{(Z-1)} + \dfrac{2}{Z-2}$

For region of convergence $|Z| < 1$

$$f(Z) = \frac{2}{1-Z} + \frac{2}{-2\left(1-\frac{Z}{2}\right)}$$

$$= 2[1-Z]^{-1} - \left[1-\frac{Z}{2}\right]^{-1}$$

$$= 2[1+Z+Z^2+Z^3+\ldots] - \left[1-\left(\frac{Z}{2}\right)+\left(\frac{Z}{2}\right)^2+\left(\frac{Z}{2}\right)^3+\ldots\right]$$

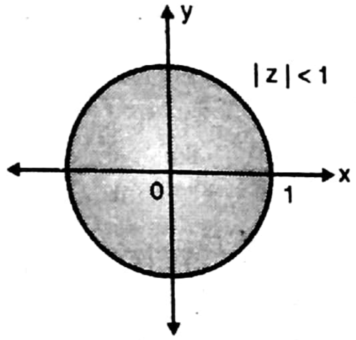

Fig. 1.2

For region of convergence $1 < |Z| < 2$

$$f(Z) = \frac{-2}{z\left(1-\frac{Z}{2}\right)} + \frac{2}{-2\left(1-\frac{Z}{2}\right)}$$

$$f(Z) = \frac{-2}{Z}\left[1-\frac{1}{Z}\right]^{-1} - \left[1-\frac{Z}{2}\right]^{-1}$$

$$= \frac{-2}{Z}\left[1+\frac{1}{Z}+\frac{1}{Z^2}+\frac{1}{Z^3}+\ldots\right]^{1}$$

$$- \left[1+\left(\frac{2}{Z}\right)+\left(\frac{2}{Z}\right)^2+\left(\frac{2}{Z}\right)^3+\ldots\right]$$

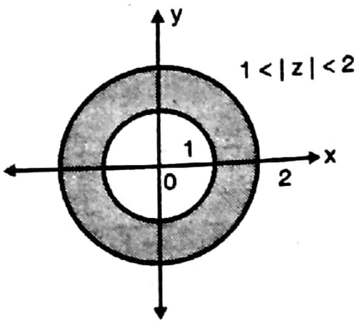

Fig. 1.3

For region of convergence $|Z| > 2$

$$f(Z) = \frac{-2}{z\left(1-\frac{1}{Z}\right)} + \frac{2}{z\left(1-\frac{2}{Z}\right)}$$

$$= \frac{-2}{z}\left[1-\frac{1}{z}\right]^{-1} + \frac{2}{z}\left[1-\frac{2}{z}\right]^{-1}$$

$$f(Z) = -\frac{2}{z}\left[1+\frac{1}{Z}+\frac{1}{Z^2}+\frac{1}{Z^3}+\ldots\right]$$

$$+ \frac{2}{z}\left[1+\left(\frac{2}{Z}\right)+\left(\frac{2}{Z}\right)^2+\left(\frac{2}{Z}\right)^3+\ldots\right]$$

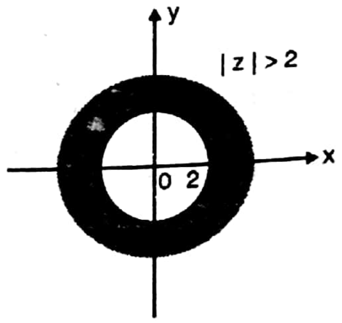

Fig. 1.4

Q. 13 Evaluaste using Cauchy's Residue Theorem

$$\int_C \frac{1-2Z}{Z(Z-1)(Z-2)}\,dt \text{ where C is } |Z| = 1.5.$$

Dec. 2015

Ans. : Let, $Z = \oint_c f(Z)\,dt$

where C is the circle $|Z| = 1.5$ and $f(Z) = \dfrac{1-2Z}{Z(Z-1)(Z-2)}$ is

Analytic within and on the closed curve C except at $Z = 0, 1, 2$.

The points $Z = 0$ and $Z = 1$ are inside the curve $|Z| = 1.5$.

Consider residues at these simple poles.

$$\text{Residue (at } Z=0) = \lim_{Z\to 0} Z \cdot f(x)$$

$$= \lim_{Z\to 0} \frac{Z(1-2Z)}{Z(Z-1)(Z-2)} = \frac{1}{2}$$

$$\text{Residue (at } Z=1) = \lim_{Z\to 1} (Z-1) \text{ for}$$

$$= \lim_{Z\to 1} \frac{(Z-1)(1-2Z)}{Z(Z-1)(Z-2)} = 1$$

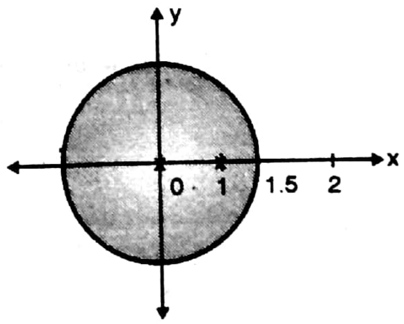

Fig. 1.5

By Cauchy's Residue Theorem,

$$I = 2\pi i \text{ [sum of Residues]}$$

$$I = 2\pi i\left[\frac{1}{2}+1\right] = 2\pi i\left[\frac{3}{2}\right] = 3\pi i$$

Q. 14 **Evaluate the line integral $\int_0^{1+i}(x^2+iy)\,dz$ along the path $y = x$** **May 2016**

Ans. :

As $I = \int_0^{1+i}(x^2+iy)\,dz$ and the path is $y = x$, $\therefore\ dy = dx$

$$\therefore\ dz = dx + idy = dx + idx = (1+i)\,dx$$

$$f(z) = x^2 + iy = x^2 + ix$$

$\therefore$ x : 0 to 1

Consider, $I = \int_0^1 (x^2+ix)(1+i)\,dx$

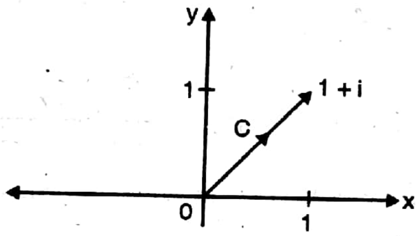

Fig. 1.6

$$= (1+i)\int_0^1 (x^2+ix)\,dx = (1+i)\left[\frac{x^3}{3}+i\frac{x^2}{2}\right]_0^1$$

$$= (1+i)\left[\frac{2+3i}{6}\right] = \frac{-1+5i}{6}$$

$$\therefore\ I = \frac{-1+5i}{6}$$

Q. 15 **Evaluate $\oint_C \frac{e^{2z}}{(z-1)^4}\,dz$ where C is the circle $|z| = 2$.** **May 2016**

Ans. :

Let, $I = \oint_C \frac{e^{2z}}{(z-1)^4}\,dz$ where C is $|z| = 2$

Consider, $f(z) = \frac{e^{2z}}{(z-1)^4}$ is analytic function within and on the simple closed curve C : $|z| = 2$ except at $z = 1$. The pole $z = 1$ is of order 4.

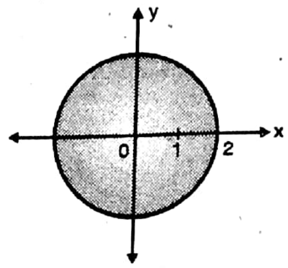

Fig. 1.7

Consider,

$$\text{Residue (at } z = 1) = \frac{1}{(4-1)!}\lim_{z\to 1}\frac{d^3}{dz^3}\left[(z-1)^4 - f(z)\right]$$

$$= \frac{1}{6}\lim_{z\to 1}\frac{d^3}{dz^3}\left[(z-1)^4\cdot\frac{e^{2z}}{(z-1)^4}\right]$$

$$= \frac{1}{6}\lim_{z\to 1}\frac{d^3}{dz^3}(e^{2z}) = \frac{1}{6}\lim_{z\to 1} 8e^{2z}$$

$$= \frac{1}{6}\cdot 8e^2 = \frac{4}{3}e^2$$

By Cauchy Residue theorem,

$$I = 2\pi i\,[\text{Residue}] = 2\pi i\cdot\frac{4}{3}e^2$$

$$I = \frac{8\pi i\, e^2}{3}$$

Q. 16 **Expand $f(z) = \frac{1}{z(z-2)(z+1)}$ in the regions.**

(i) $|z| < 1$ (ii) $1 < |z| < 2$

(iii) $|z| > 2$ **May 2016**

Ans. :

Let, $f(x) = \frac{1}{z(z-2)(z+1)}$

Consider $\frac{1}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-2}$

$$\therefore\ 1 = A(z+1)(z-2) + Bz(z-2) + Cz(z+1)$$

$$\therefore\ f(z) = \frac{-1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)}$$

Case I : Let, $\theta < |z| < 1$,

$$\therefore\ f(z) = \frac{-1}{2z} + \frac{1}{3(1+z)} + \frac{1}{(-12)\left(1-\frac{z}{2}\right)}$$

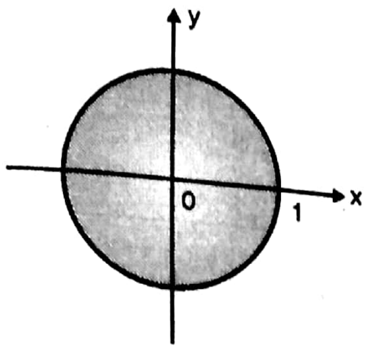

Fig. 1.8

$$\therefore \quad f(z) = \frac{-1}{2z} + \frac{1}{3}(1+z)^{-1} - \frac{1}{12}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{-1}{2z} + \frac{1}{3}\left[1 - z + z^2 - z^3 + \ldots\ldots\right]$$

$$- \frac{1}{12}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots\ldots\ldots\right]$$

Case II : Let, $1 < |z| < 2$,

$$f(z) = \frac{-1}{2z} + \frac{1}{3z\left(1+\frac{1}{z}\right)} - \frac{1}{12\left(1-\frac{z}{2}\right)}$$

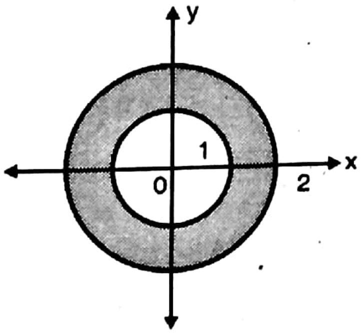

Fig. 1.9

$$= \frac{-1}{2z} + \frac{1}{3z}\left[1+\frac{1}{z}\right]^{-1} - \frac{1}{12}\left[1-\frac{z}{2}\right]^{-1}$$

$$= \frac{-1}{2z} + \frac{1}{3z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\ldots\right]$$

$$\frac{-1}{12}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots\ldots\ldots\right]$$

Case III : Let, $|z| > 2$

$$\therefore \quad f(z) = \frac{-1}{2z} + \frac{1}{3z\left(1+\frac{1}{z}\right)} - \frac{1}{6z\left(1-\frac{2}{z}\right)}$$

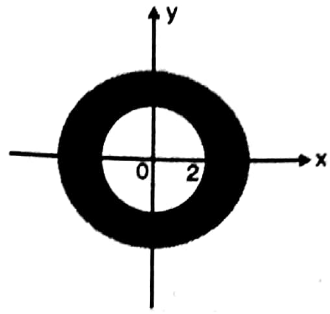

Fig. 1.10

$$= \frac{-1}{2z} + \frac{1}{3z}\left[1+\frac{1}{z}\right]^{-1} + \frac{1}{6z}\left[1-\frac{2}{z}\right]^{-1}$$

$$= \frac{-1}{2z} + \frac{1}{3z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\ldots\right]$$

$$+ \frac{1}{6z}\left[1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \ldots\ldots\ldots\right]$$

Q. 17 Evaluate using Cauchy's Residue Theorem.

$$\oint_C \frac{2z-1}{z(2z+1)(z+2)}\, dz \quad \text{where } C \text{ is } |z| = 1$$

May 2016

Ans. :

Let, $I = \oint_C f(z)\, dz$ where C is $|z| = 1$ and

$f(z) = \dfrac{2z-1}{z(2z+1)(z+2)}$ is analytic within and on the simple closed where except at $z = 0, -1/2, -2$. The function f(z) has two poles $z = 0$ and $z = -1/2$ lie inside the circle consider

$$\text{Residue (at } z = 0) = \lim_{z \to 0} z \cdot f(z)$$

$$= \lim_{z \to 0} \frac{z \cdot (2z-1)}{z(2z+1)(z+2)}$$

$$= \frac{-1}{2} \qquad \therefore \ \textbf{Residue (at } z = 0) = \frac{-1}{2}$$

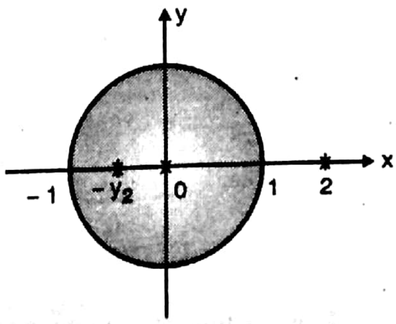

Fig. 1.11

Now

$$\text{Residue}\left(\text{at } z=-\frac{1}{2}\right) = \lim_{z\to\frac{-1}{2}} \left(z+\frac{1}{2}\right) f(x)$$

$$= \lim_{z\to\frac{-1}{2}} \frac{\left(z+\frac{1}{z}\right)\cdot(2z-1)}{z\,(2z+1)\,(z+2)}$$

$$= \lim_{z\to\frac{-1}{2}} \frac{\left(z+\frac{1}{z}\right)\cdot(2z-1)}{2z\left(z+\frac{1}{z}\right)(z+2)} = \frac{4}{3}$$

$$\textbf{Residue}\left(\text{at } z=-\frac{1}{2}\right) = \frac{4}{3}$$

By Cauchy Residue Theorem

$$I = 2\pi i\,[\text{ Sum of Residues}]$$

$$= 2\pi i\left[-\frac{1}{2}+\frac{4}{3}\right] = 2\,\pi i\left[\frac{5}{6}\right]$$

$$I = \frac{5\pi i}{3}$$

Q. 18 **Evaluate** $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$ **using contour integration.** May 2016

Ans. :

Let, $\displaystyle I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$

Consider, $f(z) = \dfrac{x^2}{(z^2+1)(z^2+4)}$ be the function with simple poles at $z = \pm i \pm 2i$

The poles $z = i$ and $z = 2i$ die in upper half of the plane. Consider residences at these poles

$$\text{Residue (at } z = i) = \lim_{z\to i} (z-i)\,f(z)$$

$$= \lim_{z\to i} \frac{(z-i)\,z^2}{(z+i)\,(z-i)(z^2+4)}$$

$$= \lim_{z\to i} \frac{z^2}{(z+i)(z^2+4)} = \frac{-1}{6i}$$

$$\text{Residence(at } z = 2i) = \lim_{z\to 2i} (z-2i)\,f(z)$$

$$= \lim_{z\to 2i} \frac{(z-2i)z^2}{(z+2i)\,(z-2i)(z^2+1)} = \frac{1}{3i}$$

By Cauchy Residue Theorem,

$$I = 2\pi i\ [\text{sum of residues}]$$

$$= 2\pi i\left[\frac{1}{3i}-\frac{1}{6i}\right]\ 2\pi i\left[\frac{1}{6i}\right]$$

$$I = \frac{\pi}{3}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{3}$$

Q. 19 **Evaluate the line integral** $\displaystyle\int_0^{1+i} (x^2+iy)\,dz$ **along the path y = x.** Dec 2016

Ans. :

Let, $I = \displaystyle\int_c f(z)\,dz$ where C is the path along $y = x$ and $f(z) = x^2 + iy$

$$\therefore\ dx = dy$$

$$dz = dx + idy = dx + idx = (1+i)\,dx$$

$$x\ :\ 0 \text{ to } 1$$

$$\therefore\ f(z) = x^2 + ix$$

As

$$I = \int_0^1 (x^2+ix)\,(1+i)\,dx$$

$$= (1+i)\left[\frac{x^3}{3}+i\frac{x^2}{2}\right]_0^1 = (1+i)\left(\frac{2+3i}{6}\right)$$

$$I = \frac{-1+5i}{6}$$

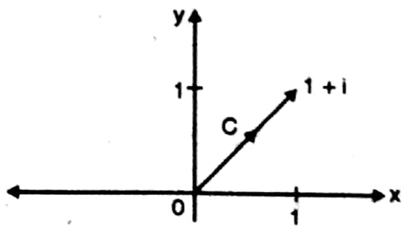

Fig. 1.12

Q. 20 **Find k and then E(x) for the pdf**

$$f(x) = \begin{cases} k(x-x^2) & 0 \le x \le 1,\ k > 0 \\ 0 & \text{otherwise} \end{cases}$$

Dec. 2016

Ans. :

$$\text{Let } f(x) = \begin{cases} k(x-x^2) & 0 \le x \le 1,\ k > 0 \\ 0 & \text{otherwise} \end{cases}$$

As $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\therefore k\int_0^1 (x - x^2)dx = 1$$

$$k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1 \quad \therefore k\left[\frac{1}{2} - \frac{1}{3}\right] = 1$$

$$\frac{k}{6} = 1 \qquad \therefore \mathbf{k = 6}$$

Now consider

$$E(x) = \int_0^1 x\, f(x)\, dx = \int_0^1 x \cdot 6\,(x - x^2)\, dx$$

$$= 6\int_0^1 (x^2 - x^3)\, dx$$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left[\frac{1}{3} - \frac{1}{4}\right] = 6 \cdot \frac{1}{12}$$

$$\therefore E(x) = \frac{1}{2}$$

Q. 21 **Evaluate $\oint_c \frac{e^{2z}}{(z+1)^4}\, dz$ where C is the circle $|z - 1| = 3$.** **Dec. 2016**

Ans. :

$$\text{Let } I = \oint_c \frac{e^{2z}}{(z+1)^4}\, dz$$

Where C is the circle $|z - 1| = 3$.

Consider $f(z) = e^{2z}$ be the analytic function within bound on the simple closed curve

$C : |z - 1| = 3$ and the point $z = -1$

Lies inside the circle.

$\therefore$ By Cauchy Integral Formula

$$\oint_c \frac{f(z)}{(z - z_0)^n}\, dz = \frac{2\pi i}{(n-1)!}\, f^{(n-1)}(z_0)$$

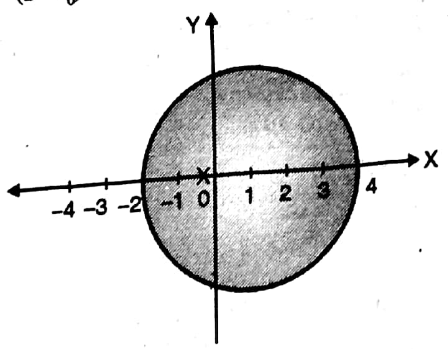

Fig. 1.13

Consider $f(t) = e^{2z}$, $n = 4$

$$f'(t) = 2e^{2z}$$

$$f''(t) = 4e^{2z}$$

$$f'''(t) = 8e^{2z}$$

As

$$I = \frac{2\pi i}{(4-1)!}\, f'''(-1) = \frac{2\pi i}{3!} \cdot 8 \cdot (e^{-2})$$

$$= \frac{2\pi i \cdot 8}{6 \cdot e^2} = \frac{8\pi i}{3e^2}$$

$$\therefore I = \frac{8\pi i}{3e^2}$$

Q. 22 **Find the Laurent's series which represents $f(z) = \frac{2}{(z-1)(z-2)}$ when (I) $|z| < 1$ (II) $1 < |z| < 2$ (III) $|z| > 2$.** **Dec. 2016**

Ans. :

$$\text{Let, } f(z) = \frac{2}{(z-1)(z-2)}$$

$$\text{As } \frac{2}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\therefore 2 = A(z-2) + B(z-1)$$

$$2 = A(z-2) + B(z-1)$$

Put $z = 1$ Put $z = 2$

$2 = -A$ $2 = B$

$\therefore$ **A = –2**   **B = 2**

As

$$f(z) = \frac{-2}{z-1} + \frac{2}{z-2}$$

for $|z| < 1$,

$$f(z) = \frac{2}{1-z} - \frac{2}{2\left(1 - \frac{z}{2}\right)} = 2\,[1 - z]^{-1} - \left[1 - \frac{z}{2}\right]^{-1}$$

$$= 2\,[1 + z + z^2 + z^3 + \ldots] - \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots\right]$$

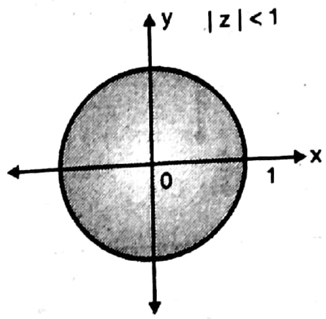

Fig. 1.14

For $1 < |z| < 2$

$$f(z) = \frac{-2}{z\left(1-\frac{1}{z}\right)} - \frac{2}{2\left(1-\frac{z}{2}\right)}$$

$$= \frac{-2}{z}\left[1-\frac{1}{z}\right]^{-1} - \left[1-\frac{z}{2}\right]^{-1}$$

$$= \frac{-2}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right]$$

$$-\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\ldots\right]$$

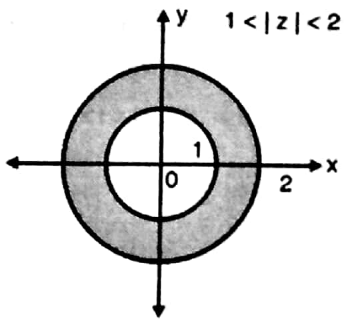

Fig. 1.15

For $|z| > 2$

$$f(z) = \frac{-2}{z\left(1-\frac{1}{z}\right)} + \frac{2}{z\left(1-\frac{2}{z}\right)}$$

$$= \frac{-2}{z}\left[1-\frac{1}{z}\right]^{-1} + \frac{2}{z}\left[1-\frac{2}{z}\right]^{-1}$$

$$= \frac{-2}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right]$$

$$+\frac{2}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\ldots\right]$$

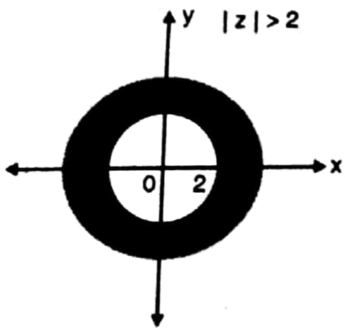

Fig. 1.16

Q. 23 Evaluate $\oint_c \frac{z^2}{(z-1)^2(z+1)}\,dz$ where C is $|z| = 2$ using residue theorem. Dec 2016

Ans. :

Let, $I = \oint_c f(z)\,dz$ where C is the circle $|z| = 2$ and

$$f(z) = \frac{z^2}{(z-1)^2(z+1)}$$

The function f(z) has poles at $z = 1$ and $z = -1$ inside the closed curve $C : |z| = 2$.

The pole $z = 1$ is of order 2 and the pole $z = -1$ has order 1.

Consider the residues at these poles

$$\text{Residue (at } z = -1) = \lim_{z \to -1} (z+1) \cdot f(z)$$

$$\lim_{z \to -1} \frac{(z+1)z^2}{(z-1)^2(z+1)} = \lim_{z \to -1} \frac{z^2}{(z-1)^2}$$

$$= \frac{(-1)^2}{(-1-1)^2} = \frac{1}{4}$$

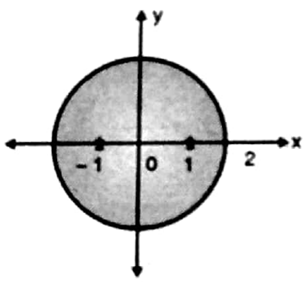

Fig. 1.17

$$\text{Residue (at } z = -1) = \frac{1}{4}$$

As

$$\text{Residue (at } z = 1) = \frac{1}{(2-1)!}\lim_{z \to 1}\frac{d}{dz}[(z-1)^2 f(z)]$$

$$= \lim_{z \to 1}\frac{d}{dz}\left[\frac{(z-1)^2 \cdot z^2}{(z-1)^2(z+1)}\right] = \lim_{z \to 1}\frac{d}{dz}\left[\frac{z^2}{z+1}\right]$$

$$= \lim_{z \to 1}\frac{(z+1)(2z)-z^2}{(z+1)^2} = \frac{3}{4} = \frac{3}{4}$$

$$\therefore \text{Residue (at } z = 1) = \frac{3}{4}$$

By Cauchy Residue Theorem

$$I = 2\pi i\,[\text{sum of residues}] = 2\pi i\left[\frac{1}{4}+\frac{3}{4}\right] = 2\pi i\,(2) = 4\pi i$$

$$I = 4\pi i$$

Q. 24 **Evaluate $\int_{-\infty}^{\infty} \frac{x^2 + x + 2}{x^4 + 10x^2 + 9} dx$ using contour integration.**

Dec. 2016

Ans. :

Let, $I = \int_{-\infty}^{\infty} \frac{x^2 + x + 2}{x^4 + 10x^2 + 9} dx$

Now, consider C be the semi-circle with centre at origin and a large radius in upper half of the plane and its diameter to be real axis.

Consider $f(z) = \frac{z^2 + z + 2}{z^4 + 10z^2 + 9}$ is analytic function within and on the simple closed curve C, except at its poles $z = \pm i, \pm 3i$.

The poles $z = i$ and $z = 3i$ are inside the curve C.

Consider residues at these simple pole.

$$\text{Residue (at } z = i) = \lim_{z \to i} (z - i) f(z)$$

$$= \lim_{z \to i} \frac{(z - i)(z^2 + z + 2)}{(z - i)(z + i)(z^2 + 9)}$$

$$= \lim_{z \to i} \frac{z^2 + z + 2}{(z + i)(z^2 + 9)} = \frac{1 + i}{16i}$$

$$\text{Residue (at } z = 3i) = \lim_{z \to 3i} \frac{(z - 3i)(z^2 + z + 2)}{(z^2 + 1)(z + 3i)(z - 3i)}$$

$$= \lim_{z \to 3i} \frac{z^2 + z + 2}{(z^2 + 1)(z + 3i)} = \frac{7 - 3i}{48i}$$

By Cauchy Residue Theorem,

$$I = 2\pi i \left[\frac{1 + i}{16i} + \frac{7 - 3i}{48i} \right] = \frac{5\pi}{12}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 + x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$$

Chapter 2 : Matrices

Q. 1 **If $A = \begin{bmatrix} x & 4x \\ 2 & y \end{bmatrix}$ has eigen values 5 and –1 then find values of x and y.**

May 2014

Ans. :

Given eigen values = 5, – 1

As sum of the eigen values = sum of the diagonal elements

$$\therefore \; 5 + (-1) = x + y$$

$$\therefore \; y = 4 - x$$

And product of the eigen values = | A |

$$\therefore \; 5(-1) = xy - 8x$$

$$-5 = x(4 - x) - 8x$$

$$-5 = 4x - x^2 - 8x$$

$$x^2 + 4x - 5 = 0$$

$$x = -5 \text{ or } x = 1$$

where $x = -5, \; y = 4 - (-5)$

$y = 9$

when $x = 1 \; y = 4 - 1$

$y = 3$

$x = -5$ or 1, $y = 9$ or 3

Q. 2 **Determine whether matrix A is derogatory $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$**

May 2014

Ans. :

Case 1 :

Let $f(x) = x - 2$

$$\therefore \; f(A) = A - 2I$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$$

$\therefore$ $f(x) = x - 2$ does not anihilates A so, f (x) is not minimal polynomial

Case 2 :

Let $g(x) = (x - 2)(x - 2) = x^2 - 4x + 4$

$$\text{Now } A^2 = A \times A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\therefore \; A^2 - 4A + 4I = \begin{bmatrix} 4 & 4 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} - 4\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$$

$\therefore$ $g(x) = (x - 2)(x - 2)$ does not anihilates A so g(x) is not minimal polynomial.

Hence matrix A is not derogatory

Q. 3 Show that the matrix A is diagonalizable, find its diagonal form and transforming matrix, if $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ **May 2014**

Ans. :

Let λ be eigen value and X be corresponding eigen vector of matrix A.

Characteristic equation is $|A - \lambda I| = 0$.

$$\therefore \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

On solving,

λ^2 – (sum of diagonal elements) λ^2 + (sum of the minors of diagonal elements) $\lambda - |A| = 0$.

$$\therefore \lambda^3 - (-9+3+7)\lambda^2 + (-11+1+5)\lambda - (3) = 0$$

$$\therefore \lambda^3 - \lambda^2 - 5\lambda - 3 = 0$$

$\therefore$ Eigen values (λ) are 3, – 1, – 1.

Case 1 : $\lambda = -1$

$$\therefore [A - \lambda I]X = 0$$

$$\therefore \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$; $R_3 - 2R_1$; $R_1/4$;

$$\therefore \begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...(1)$$

Number of unknowns (n) = 3.

Rank (r) = number of non-zero rows = 1.

Here, algebraic multiplicity (A.M.) = No. of times "$\lambda = -1$" is repeated = 2.

Geometric multiplicity (G.M.) = n – r = 3 – 1 = 2.

$\therefore$ A.M. = G.M. for "$\lambda = -1$".

From Equation (1),

$$-2x_1 + x_2 + x_3 = 0 \quad ...(2)$$

Put $x_1 = 0$; $x_2 = 1$

$$\therefore 0 + 1 + x_3 = 0$$

$$\therefore x_3 = -1$$

$\therefore$ Eigen vector $x_1 = [0 \quad 1 \quad -1]$.

If k is non-zero scalar then k x_1 is also an eigen vector.

Put $x_1 = 1$; $x_2 = 0$ in Equation (2)

$$\therefore -2 + 0 + x_3 = 0$$

$$\therefore x_3 = 2$$

$\therefore$ Eigen vector $x_2 = [1 \quad 0 \quad 2]$.

If k is non-zero scalar then k X_2 is also an eigen vector.

Case 2 : $\lambda = 3$

$$\therefore [A - \lambda I]X = 0$$

$$\therefore \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 - R_2$; $R_3 - R_2$; $R_2/4$;

$$\therefore \begin{bmatrix} -4 & 4 & 0 \\ -2 & 0 & 1 \\ -8 & 8 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 2R_1$; $R_1/4$;

$$\therefore \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...(3)$$

Here n = 3 and r = 1.

A.M. = No. of times "$\lambda = 3$" is repeated = 1.

G.M. = n – r = 3 – 2 = 1

$\therefore$ A.M. = G.M. for "$\lambda = 3$".

Expanding Equation (3),

$$-2x_1 + x_3 = 0 \quad ...(4)$$

And

$$-x_1 + x_2 = 0 \quad ...(5)$$

Put $x_1 = 1$

From Equation (4), $-2(1) + x_3 = 0$

$$\therefore x_3 = 2$$

From Equation (5), $-1 + x_2 = 0$

$$\therefore x_2 = 1$$

$\therefore$ Eigen vector $x_3 = [1 \quad 1 \quad 2]$.

If k is non-zero scalar then k x_3 is also an eigen vector.

Since, A.M. = G.M. for all eigen values, matrix A is diagonalizable.

$$\therefore M^{-1}AM = D$$

So the given matrix A is diagonalized to diagonal matrix D by the transforming matrix M, where

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

Q. 4 **If $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ then find the eigen values of $4A^{-1} + 3A + 2I$.** May 2015

Ans. :

Since A is a lower triangular matrix, eigen values (λ) = diagonal elements

$\therefore \lambda = 1, 4$

As, $f(\lambda)$ is eigen value of $f(A)$.

Let $f(A) = 4A^{-1} + 3A + 2I$

$\therefore f(\lambda) = 4\lambda^{-1} + 3\lambda + 2 = \frac{4}{\lambda} + 3\lambda + 2$

When $\lambda = 1$

$f(1) = \frac{4}{1} + 3(1) + 2 = 4 + 3 + 2 = 9$

When $\lambda = 4$

$f(4) = \frac{4}{4} + 3(4) + 2$

$= 1 + 12 + 2 = 15$

$\therefore$ Eigen values of $4A^{-1} + 3A + 2I$ are 9, 15.

Q. 5 **Show that $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$ is derogatory.** May 2015

Ans. :

Let λ be eigen value of matrix A.

Characteristic equation is $|A - \lambda I| = 0$

$$\therefore \begin{bmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & -4 & 4-\lambda \end{bmatrix} = 0$$

On solving

λ^3 – (sum of diagonal elements) λ^2 + (sum of the minors of diagonal elements) $\lambda - |A| = 0$

$\therefore \lambda^3 - (7 + 7 + 4)\lambda^2 + (24 + 24 + 33)\lambda - 108 = 0$

$\therefore \lambda^3 - 18\lambda^2 + 81\lambda - 108 = 0$

$\therefore$ Eigen values (λ) are 3, 3, 12

Let, $f(x) = (x-3)(x-12) = x^2 - 15x + 36$

$$\text{Now, } A^2 = A \times A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} \times \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix}$$

$$\therefore A^2 - 15A + 36I = \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix} - 15\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} + 36\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore f(x) = x^2 - 15x + 36$ annihilates A

$\therefore f(x)$ is a minimal polynomial.

Degree of $f(x) = 2$ and order of $A = 3$.

$\therefore$ Degree of $f(x) <$ order of A

Hence matrix A is derogatory.

Q. 6 **Show that the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalisable. Find the transforming matrix and the diagonal matrix.** May 2015

Ans. :

Let λ be eigen value and X be corresponding eigen vector of matrix A.

Characteristic equation is $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

On solving,

λ^3 – (sum of diagonal elements) λ^2 + (sum of the minors of diagonal elements) $\lambda - |A| = 0$

$\therefore \lambda^3 - (8 + 7 + 3)\lambda^2 + (5 + 20 + 20)\lambda - 0 = 0$

$\therefore \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\therefore$ Eigen values (λ) are 0, 3, 15

Case 1 : $\lambda = 0$

$\therefore [A - \lambda I]X = 0$

$$\therefore \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

Using crammer's rule

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\therefore \frac{x_1}{10} = \frac{-x_2}{-20} = \frac{x_3}{20}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$\therefore$ Eigen vector $X_1 = [1 \ 2 \ 2]$

If k is non-zero scalar then k X_1 is also an eigen vector.

Case 2 : $\lambda = 3$

$$\therefore [A - \lambda I]\, X = 0$$

$$\therefore \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore 5x_1 - 6x_2 + 2x_3 = 0$ and $-6x_1 + 4x_2 - 4x_3 = 0$

Using crammer's rule

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\therefore \frac{x_1}{16} = \frac{-x_2}{-8} = \frac{x_3}{-16}$$

$$\therefore \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$\therefore$ Eigen vector $X_2 = [2 \ 1 \ -2]$

If k is non-zero scalar then k X_2 is also eigen vector.

Case 3 : $\lambda = 15$

$$\therefore [A - \lambda I]\, X = 0$$

$$\therefore \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -7x_1 - 6x_2 + 2x_3 = 0$ and $-6x_1 + 8x_2 - 4x_3 = 0$

Using crammer's rule

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\therefore \frac{x_1}{40} = \frac{-x_2}{40} = \frac{x_3}{20}$$

$$\therefore \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$\therefore$ Eigen vector $X_3 = [2 \ -2 \ 1]$

If k is non-zero scalar then k X_3 is also an vector.

Since all eigen values are distinct, matrix A is diagonalizable.

$$\therefore M^{-1} AM = D$$

So the given matrix A is diagonalized to diagonal matrix D by the transforming matrix M,

Where $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$ and $M = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Q. 7 State Cayley Hamilton Theorem and verify the same for $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$.

Dec. 2015

Ans. : Cayley Hamilton theorem

Every square matrix satisfies its own characteristics equation.

Let, $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$, as $|A - \lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(2-\lambda) = 0$$

$$\lambda^2 - 3\lambda + 2 - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0 \quad \ldots(1)$$

Consider, $A^2 = AA = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 6 & 10 \end{bmatrix}$

As,

$$A^2 - 3A - 4I = \begin{bmatrix} 7 & 9 \\ 6 & 10 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

i.e. $A^2 - 3A - 4I = 0$

$\therefore$ Cayley Hamilton Theorem is satisfies.

Q. 8 Find the eigen values and the eigen vectors of the matrix.

Dec. 2015

$$\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$$

Ans. : Let $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

As $|A| = \begin{vmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{vmatrix} = 4$

Consider the characteristic equation of A

$$|A - \lambda I| = 0,$$

$$\begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{bmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

The eigen values are 1, 2, 2

For $\lambda = 1$

$$[A - \lambda I]\, X = 0$$

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

By Crammer's rule,

$$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ -5 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ -1 & -5 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{x_2}{1} = \frac{x_3}{-3}$$

$$X = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$$

For $\lambda = 2$

$$[A - \lambda I]\, X = 0$$

$$\begin{bmatrix} 1 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Crammer's Rule,

$$\frac{x_1}{\begin{vmatrix} 1 & 2 \\ -5 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 1 \\ -1 & -5 \end{vmatrix}}$$

$$\frac{x_1}{3} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$\therefore X = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

Q. 9 **Show that the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable. Find the transforming matrix and diagonal matrix.** Dec. 2015

Ans. : Let, $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

Consider characteristic equation of A

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{bmatrix} = 0$$

$$\lambda^3 - \lambda^2 + \lambda[(21-32) + (-63+64) + (-27+32)] - 3 = 0$$

$$\lambda^3 - \lambda^2 - 5\lambda^2 - 3 = 0 \qquad (\lambda+1)^2(\lambda-3) = 0$$

$$\therefore \lambda = -1, -1, 3 \qquad AM(-1) = 2,\ AM(3) = 1$$

The eigen values are $-1, -1, 3$

For $\lambda = -1$

$$[A - \lambda I]X = 0\ ;$$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As only one equation

$$-8x_1 + 4x_2 + 4x_3 = 0$$

$$\text{i.e. } 2x_1 - x_2 - x_3 = 0$$

Case (i)

Let $x_3 = 0$ $\therefore 2x_1 - x_2 = 0$ $\therefore x_2 = 2x_1$

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Case (ii)

Let $x_2 = 0$ $\therefore 2x_1 - x_3 = 0$ $\therefore x_3 = 2x_1$

$$\therefore x_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \therefore GM(-1) = 2$$

For $\lambda = 3$, $[A - \lambda I]\, X = 0$

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Crammers Rule,

$$\frac{x_1}{\begin{vmatrix} 0 & 4 \\ 8 & 4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -8 & 4 \\ -16 & 4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -8 & 0 \\ -16 & -8 \end{vmatrix}}$$

$$\frac{x_1}{-32} = \frac{-x_2}{32} = \frac{x_3}{-64} \qquad \therefore X_3 = \begin{bmatrix}1\\1\\2\end{bmatrix}$$

$\therefore$ GM(3) = 1 Since AM(λi) = GM(λi) for all λi

$\therefore$ A is diagnolisable.

$$\text{Diagonal matrix D} = \begin{bmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 3\end{bmatrix} \text{and}$$

$$\text{transforming matrix is } M = \begin{bmatrix}1 & 1 & 1\\2 & 0 & x\\0 & 2 & 2\end{bmatrix}$$

Q. 10 Find the eigen values of $A^2 + 2I$ where A $\begin{bmatrix}1 & 0 & 0\\2 & -2 & 0\\3 & 5 & 3\end{bmatrix}$ and I is identity matrix of orders 3. May 2016

Ans. : As, $A = \begin{bmatrix}1 & 0 & 0\\2 & -2 & 0\\3 & 5 & 3\end{bmatrix}$

consider characteristic equation of A

$$|A - \lambda I| = 0$$

$$\begin{bmatrix}1-\lambda & 0 & 0\\2 & -2-\lambda & 0\\3 & 5 & 3-\lambda\end{bmatrix} = 0$$

$$\therefore (\lambda - 1)(\lambda + 2)(\lambda - 3) = 0$$

$$\therefore \lambda = 1, -2, 3$$

The eigen values of $A^2 + 2I$ are $(1)^2 + 2(1)$, $(-2)^2 + 2(1)$, $(3)^2 + 2(1)$

i.e. The eigen values of $A^2 + 2I$ are 3, 6, 11

Q. 11 Is the following matrix derogatory ? Justify

$$\begin{bmatrix}5 & -6 & -6\\-1 & 4 & 2\\3 & -6 & -4\end{bmatrix}$$

May 2016

Ans. :

Let, $A = \begin{bmatrix}5 & -6 & -6\\-1 & 4 & 2\\3 & -6 & -4\end{bmatrix}$ be the matrix

Consider characteristics equation of A

$$|A - \lambda I| = 0$$

$$\begin{bmatrix}5-\lambda & -6 & -6\\-1 & 4-\lambda & 2\\3 & -6 & -4-\lambda\end{bmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$(\lambda - 2)^2 (\lambda - 1) = 0$$

$$\therefore \lambda = 2, 2, 1$$

The eigen values are 2, 2, 1

Consider $f(x) = (x-1)(x-2) = x^2 - 3x + 2$

As

$$A^2 = AA = \begin{bmatrix}5 & -6 & -6\\-1 & 4 & 2\\3 & -6 & -4\end{bmatrix}\begin{bmatrix}5 & -6 & -6\\-1 & 4 & 2\\3 & -6 & -4\end{bmatrix}$$

$$= \begin{bmatrix}13 & -18 & -18\\-3 & 10 & 6\\9 & -18 & -14\end{bmatrix}$$

Consider $A^2 - 3A + 2I$

$$= \begin{bmatrix}13 & -18 & -18\\-3 & 10 & 6\\9 & -18 & -14\end{bmatrix} - 3\begin{bmatrix}5 & -6 & -6\\-1 & 4 & 2\\3 & -6 & -4\end{bmatrix} + 2\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$\because$ $f(A) = A^2 - 3A + 2I = 0$,

$\therefore$ $f(x) = x^2 - 3x + 2$ annnihilates of A

$\therefore$ f(x) is minimal polynominal of A

$\therefore$ degree f(x) = 2 < 3 = Order of A

$\therefore$ A is derogatory matrix.

Q. 12 Is the following matrix diagonalizable ?

If yes, find the transforming matrix and the diagonal matrix

$$\begin{bmatrix}8 & -8 & -2\\4 & -3 & -2\\3 & -4 & 1\end{bmatrix}$$

May 2016

Ans. : Let, $A = \begin{bmatrix}8 & -8 & -2\\4 & -3 & -2\\3 & -4 & 1\end{bmatrix}$

As $|A| = \begin{bmatrix}8 & -8 & -2\\4 & -3 & -2\\3 & -4 & 1\end{bmatrix} = 6$

Consider the charactristic Equation of A as,

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3-6\lambda^2+11\lambda-6 = 0$$

$$(\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$\lambda = 1, 2, 3$. The eigen values are 1, 2, 3

$AM(\lambda_i) = 1$ for all $i = 1, 2, 3$

for $\lambda = 1$, $[A-\lambda I]X = 0$

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Crammer's Rule,

$$\frac{x_1}{4} = \frac{-x_2}{-3} = \frac{x_3}{2}$$

$$\therefore \quad X_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

For $\lambda = 2$ $\quad [A-\lambda I]X = 0$

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Crammer's Rule,

$$\frac{x_1}{3} = \frac{x_3}{1} \quad \therefore X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

For $\lambda = 3$, $\quad [A-\lambda I]X = 0$

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -2 & -2 \\ 3 & -4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Crammer's Rule, $\frac{x_1}{2} = \frac{-x_2}{-1} = \frac{x_3}{1} \quad \therefore X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$$AM(\lambda_1) = GM(\lambda_i) \text{ for all } i = 1,2,3$$

$\therefore$ The matrix A is diagonalizable where diagonal matrix is

Where diagonal matrix is

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ and transforming matrix x is

$$M = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Q. 13 **Find the eigen values and eigen vectors of the matrix.**

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Dec 2016

Ans. :

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Characteristic Equation of A

$$|A-\lambda I| = 0$$

$$\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)^3 = 0$$

$$\lambda = 2, 2, 2$$

$$\text{For } \lambda = 2$$

$$[A-\lambda I]X = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Crammer's Rule

$$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\therefore x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ is the only eigen vector for $\lambda = 2$.

Q. 14 **Show that the matrix**

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$ **is non-derogatory.**

Dec 2016

Ans. : Let, $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Consider the characteristic Equation of A

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix} = 0$$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 1)(\lambda + 2)(\lambda - 3) = 0$$

$\therefore \lambda = 1, -2, 3.$

By Cayley Hamilton Theorem

$$A^3 - 2A^2 - 5A + 6I = 0$$

$$\therefore \quad f(x) = x^3 - 2x^2 - 5x + 6$$

Amihilates the matrix A.

$\therefore$ f(x) is minimal polynomial of A

$\therefore$ degree f(x) = 3 = order of A

$\therefore$ A is non-derogatory matrix.

Q. 15 Find the relative maximum or minimum (if any) of the function

$$\mathbf{z = x_1^2 + x_2^2 + x_3^2 - 4x_1 - 8x_2 - 12x_3 + 100.}$$

Dec. 2016

Ans. : Consider, $f = x_1^2 + x_2^2 + x_3^2 - 4x_1 - 8x_2 - 12x_3 + 100$

The stationary points are given by

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \quad \frac{\partial f}{\partial x_3} = 0$$

$$\therefore \quad \frac{\partial f}{\partial x_1} = 2x_1 - 4 = 0 \quad \therefore x_1 = 2$$

$$\frac{\partial f}{\partial x_2} = 2x_2 - 8 = 0 \quad \therefore x_2 = 4$$

$$\frac{\partial f}{\partial x_3} = 2x_3 - 12 = 0 \quad \therefore x_3 = 6$$

$\therefore x_0 (2, 4, 6)$ is a stationary point.

Consider $\dfrac{\partial^2 f}{\partial x_1^2} = 2, \dfrac{\partial^2 f}{\partial x_1 \partial x_2} = 0, \dfrac{\partial^2 f}{\partial x_1 \partial x_3} = 0$

$$\frac{\partial^2 f}{\partial x_4 \partial x_1} = 0, \quad \frac{\partial^2 f}{\partial x_1^2} = 2, \quad \frac{\partial^2 f}{\partial x_2 \partial x_3} = 0, \quad \frac{\partial^2 f}{\partial x_3 \partial x_1} = 0$$

$$\frac{\partial^2 f}{\partial x_3 \partial x_2} = 0, \quad \frac{\partial^2 f}{\partial x_3^2} = 2$$

The Hessian matrix at $x_0 (2, 4, 6)$ as

$$H = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_3} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_3} \\ \dfrac{\partial^2 f}{\partial x_3 \partial x_1} & \dfrac{\partial^2 f}{\partial x_3 \partial x_2} & \dfrac{\partial^2 f}{\partial x_3^2} \end{bmatrix},$$

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The principal minors of H are

$$|2| = 2, \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 8$$

The values are 2, 4, 8.

Since all the values are positive.

$\therefore$ z is minimum at $x_0 (2, 4, 6)$.

$\therefore \quad \mathbf{Z_{min} = 44}$

Q. 16 If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ find A^{50}

Dec. 2016

Ans. :

$$\text{Let } A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Characteristic equation of A

$$|A - \lambda I| = 0 \therefore \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0 \quad \therefore (4 - 4\lambda + \lambda^2) - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0 \quad \therefore (\lambda - 1)(\lambda - 3) = 0$$

$$\therefore \lambda = 1, 3$$

The eigen values are 1, 3.

$$\text{For } \lambda = 1$$

$$[A - \lambda I] X = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As

$x_1 + x_2 = 0 \qquad \therefore x_2 = -x_1$

$$x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda = 3$ $\quad \therefore [A - \lambda I] X = 0$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As

$$x_1 - x_2 = 0 \qquad \therefore x_1 = x_2$$

$$\therefore x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$|M| = 2 \qquad \therefore M^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Consider

$$f(A) = M\, f(D)\, M^{-1}$$

$$\therefore A^{50} = M D^{50} M^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^{50} & 0 \\ 0 & 3^{50} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3^{50} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 3^{50} \\ -1 & 3^{50} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$$

$$\therefore A^{50} = \frac{1}{2}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$$

Chapter 3 : Probability

Q. 1 **The daily consumption of electric power (in millions of kwh) is r.v. X with PDF $f(x) = k\, x e^{-x/3}$, $x > 0$. Find k and the probability that on a given day the electricity consumption is more than expected electricity consumption.** May 2014

Ans. :

For any probability density function,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\therefore \int_0^{\infty} k\, x\, e^{-x/3}\, dx = 0$$

$$\therefore k\left[\frac{x\, e^{-x/3}}{-1/3} - 1 \cdot \frac{e^{-x/3}}{(-1/3)^2}\right]_0^{\infty} = 1$$

$$\therefore k\left[(0 \cdot 0) - \left(0 - \frac{e^0}{(1/9)}\right)\right] = 1$$

$$\therefore k \times 9 = 1$$

$$\therefore k = \frac{1}{9}$$

$$\therefore f(x) = \frac{1}{9} x\, e^{-x/3}$$

Expected electric consumption = E(x)

$$= \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{\infty} x \frac{1}{9} x\, e^{-x/3}\, dx$$

$$= \frac{1}{9}\int_0^{\infty} x^2 e^{-x/3}\, dx$$

$$= \frac{1}{9}\left[x^2 \frac{e^{-x/3}}{-1/3} - 2x \frac{e^{-x/3}}{(-1/3)^2} + 2\frac{e^{-x/3}}{(-1/3)^3}\right]_0^{\infty}$$

$$= \frac{1}{9}\left\{(0 - 0 + 0) - \left[0 - 0 + 2\frac{e^0}{(-1/27)}\right]\right\}$$

$$= \frac{1}{9}\left\{2 \times \frac{27}{1}\right\} = 6$$

P (consumption is more than the expected value)

$$= P(X > 6) = \int_6^{\infty} \frac{1}{9} x\, e^{-x/3}\, dx = \frac{1}{9}\left[x \frac{e^{-x/3}}{-1/3} - 1\frac{e^{-x/3}}{(-1/3)^2}\right]_6^{\infty}$$

$$= \frac{1}{9}\left[(0 - 0) - \left[6 \cdot \frac{e^{-2}}{-1/3} - \frac{e^{-2}}{(1/9)}\right]\right]$$

$$= \frac{1}{9} \times (6 \times 3 e^{-2} + 9 e^{-2}) = \frac{1}{9} \times 27 e^{-2} = 3 e^{-2} = 0.406$$

Q. 2 **Find the moment generating function of Poisson distribution and hence find mean and variance.** May 2014

Ans. :

For poission distribution $P(X = x) = \dfrac{e^{-m}\, m^x}{X\,!}$ where m is the poission parameter by definition

$$\text{Moment} = E(e^{tx}) = \sum_{X=0}^{\infty} P_t\, e^{tx}$$

$$= \sum_{X=0}^{\infty} \frac{e^{-m}\, m^{x}}{X!} e^{tx} = e^{-x} \sum_{X=0}^{\infty} \frac{(m e^{t})^{x}}{X!}$$

$$= e^{-m} \cdot e^{-me^{t}} \qquad \left[\because \sum_{X=0}^{\infty} \frac{a^{x}}{x!} = e^{a} \right]$$

$$= e^{-m + me^{t}} = e^{-m(1-e^{t})}$$

Now $\mu_r' = \left[\frac{d^r}{dt^r} M_0(t) \right]_{t=0} = \left[\frac{d^r}{dt^r} e^{-m(1-e^{t})} \right]_{t=0}$

$\therefore \mu_1' = \left[\frac{d}{dt} e^{-m(1-e^{t})} \right]_{t=0} = [e^{-m(1-e^{t})} \cdot - m(0-e^{t})]_{t=0}$

$= e^{-m(1-1)} \cdot (-m)(0-1) = e^{\circ} \cdot m = m$

$\mu_2' = \left[\frac{d^2}{dt^2} e^{-m(1-e^{t})} \right]_{t=0} = \left[\frac{d}{dt} e^{-m(1-e^{t})} \cdot me^{t} \right]_{t=0}$

$= \{ m \cdot [e^{-m(1-e^{t})} \cdot e^{t} \cdot e^{-m(1-e^{t})} - m \cdot (0-e^{t})] \}_{t=0}$

$= m \cdot [e^{-m(1-1)} \cdot 1 + 1 \cdot e^{-m(1-1)} \cdot m \cdot 1]$

$= m[1+m] = m + m^2$

Mean $\mu_1' = m$

$\therefore$ Variance $= \mu_2 - \mu_1'^2 = (m+m^2) - m^2 = m$

Q. 3 **Average mark scored by 32 boys is 72 with standard deviation of 8 while that for 36 girls is 70 with standard deviation of 6. Test at 1% LoS whether the boys perform better than the girls.** May 2014

Ans. :

$n_1 = 32$ and $n_2 = 36$ (> 30, it is large sample)

$\bar{x}_1 = 72$; $\bar{x}_2 = 70$; $s_1 = 8$; $s_2 = 6$

Step 1 :

Null Hypothesis (H_0) : $\mu_1 = \mu_2$ (i.e. performance of boys and girls is equal).

Alternative Hypothesis (H_a) : $\mu_1 > \mu_2$ (i.e. boys perform better than the girls) (One tailed test)

Step 2 :

LOS = 1% (Two tailed test)

$\therefore$ LOS = 2% (One tailed test)

$\therefore$ Critical value (z_α) = 2.33

Step 3 : Since samples are large,

$$S.E. = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{8^2}{32} + \frac{6^2}{36}} = 1.732$$

Step 4 : Test statistic

$$z_{cal} = \frac{\bar{x}_1 - \bar{x}_2}{S.E.} = \frac{72-70}{1.732} = 1.1547$$

Step 5 : Decision

Since $|z_{cal}| < z_\alpha$, H_0 is accepted.

$\therefore$ Boys do not perform better than the girls.

Q. 4 **In a distribution exactly normal 7 % of items are under 35 and 89 % of the items are under 63. Find the probability that an item selected at random lies between 45 and 56.** May 2015

Ans. :

Let the mean and standard deviation be 'm' and 'σ'.

Let SNV corresponding to x = 35 be z_1.

$P(x < 35) = 7\%$

$\therefore P(z < z_1) = 0.07$

$\therefore$ 0.5 – Area between 'z = 0' to 'z = – z_1' is 0.43

Form z-table $-z_1 = 1.4758$

$\therefore z_1 = -1.4758$

But $z = \frac{x-m}{\sigma}$

$\therefore z_1 = \frac{35-m}{\sigma}$

$\therefore -1.4758 = \frac{35-m}{\sigma}$

$\therefore m - 1.4758\,\sigma = 35$ ...(1)

Let SNV corresponding to x = 63 be z_2

$P(x < 63) = 89\%$

$\therefore p(z < z_2) = 0.89$

$\therefore$ 0.5 + Area between 'z = 0' to 'z = z_2' is 0.39

$\therefore$ Area between 'z = 0' to 'z = z_2' is 0.39

From z-table, $z_2 = 1.2265$

But $z_2 = \frac{63-m}{\sigma}$

$\therefore 1.2265 = \frac{63-m}{\sigma}$

$\therefore m + 1.2265\,\sigma = 63$ ...(2)

Solving Equation (1) and (2) simultaneously,

$m = 50.2916$ and $\sigma = 10.3615$

Now, probability that an item selected at random lies between 45 and 56 = P (45 < x < 56)

$= P\left(\frac{45-50.2916}{10.3615} < \frac{x-m}{\sigma} < \frac{56-50.2916}{10.3615}\right)$

$= P(-0.5106 < z < 0.5509)$

$=$ Area between 'z = 0' to 'z = – 0.5106'

+ Area between 'z = 0' to 'z = 0.5509'

$= 0.1952 + 0.2092 = 0.4044$

Mean = 50.2916, standard deviation = 10.3615

Probability that an item selected at random lies between 45 and 56 = 0.4044

Q. 5 A continuous random variable has probability density function $f(x) = 6(x - x)^2$, $0 \le x \le l$. Find (i) mean (ii) variance.

May 2015

Ans. :

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^1 x \cdot 6(x - x^2)\,dx$$

$$= 6\int_0^1 (x^2 - x^3)\,dx$$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left\{\left[\frac{1^3}{3} - \frac{1^4}{4}\right] - [0-0]\right\} = 0.5$$

Consider, $$E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^1 x^2 \cdot 6(x - x^2)\,dx$$

$$= 6\int_0^1 (x^3 - x^4)\,dx = 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$$

$$= 6\left\{\left[\frac{1^4}{4} - \frac{1^5}{5}\right] - [0-0]\right\} = 0.3$$

$\therefore$ Variance $= E(x^2) - [E(x)]^2 = 0.3 - 0.5^2 = 0.05$

Hence mean = 0.5, Variance = 0.05

Q. 6 Find the moment generating function of Binomial distribution and hence find mean and variance.

May 2015

Ans. :

For binomial distribution, $P(X = x) = {}^nC_x\, p^x q^{n-x}$

By definition moment generating function about origin $M_0(t) = E(e^{tx})$

$$= \sum_{x=0}^{n} p_t e^{tx} = \sum_{x=0}^{n} {}^nC_x\, p^x q^{n-x} e^{tx} = \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}$$

$$= (q + pe^t)^n \qquad \ldots\left[\because \sum_{x=0}^{n} {}^nC_x\, a^x b^{n-x} = (a+b)^n\right]$$

Now $$\mu_r' = \left[\frac{d^r}{dt^r} M_0(t)\right]_{t=0}$$

$$\therefore \mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = \left[\frac{d}{dt}(q + pe^t)^n\right]_{t=0}$$

$$= [n(q + pe^t)^{n-1} \cdot pe^t]_{t=0}$$

$$= [n(q + pe^0)^{n-1} pe^0] = [n(q+p)^{n-1} p]$$

$$= [n \cdot 1^{n-1} \cdot p] \qquad (\because q + p = 1)$$

$$= np$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = \left[\frac{d^2}{dt^2}(q + pe^t)^n\right]_{t=0}$$

$$= \left[\frac{d}{dt} np(q + pe^t)^{n-1} e^t\right]_{t=0}$$

$$[np\{q + pe^t\}^{n-1} \cdot e^t + e^t \cdot (n-1)(q + e^t)^{n-2} pe^t]_{t=0}$$

$$= np\{(q+p)^{n-1} \cdot 1 + 1 \cdot (n-1)(q+p)^{n-2} p \cdot 1\}$$

$$= np\{(1)^{n-1} + (n-1)(1)^{n-2} p\} \qquad \ldots(\because q + p = 1)$$

$$= np\{1 + np - p\} = np\{(q + np)\} \qquad \ldots(\because 1 - p = q)$$

$$= npq + n^2p^2$$

$\therefore$ Mean $= \mu_1' = np$

$\therefore$ Variance $\mu_2 = \mu_2' - \mu_1'^2 = (npq + n^2p^2) - (np)^2 = npq$

Q. 7 A die was thrown 132 times and the following frequencies were observed.

No. obtained :	1	2	3	4	5	6	Total
Frequency :	15	20	25	15	29	28	132

May 2015

Ans. :

Binomial distribution is applied only in those experiments which have exactly two out-comes viz success and failure. A dice has six equally likely customers. Assuming 'x' to be a binomial variate then its first value should be zero and not one. This problem is solved just for the sake of solving.

x	f	fx	Theoretical frequency
0	0	0	0.31 ≈ 0

x	f	fx	Theoretical frequency
1	15	15	$3.25 \approx 3$
2	20	40	$14.16 \approx 14$
3	25	75	$32.86 \approx 33$
4	15	60	$42.89 \approx 43$
5	29	145	$29.86 \approx 30$
6	28	168	$8.66 \approx 9$
Total	132	503	132

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{503}{132} = 3.8106$$

But for binomial distribution, mean = np

$$\therefore\ 38106 = 6p$$

$$\therefore\ p = 0.6351$$

$$\therefore\ q = 1 - p = 1 - 0.6351 = 0.3649$$

$$n = 6 \text{ and } N = 132$$

$$\therefore\ P(X = x) = {}^nC_x\, p^x q^{n-x} = {}^6C_x \times (0.6351)^x \times (0.3649)^{6-x}$$

$$\text{Theoretical frequency} = N \times P(X = x)$$

$$= 132 \times {}^6C_x \times (0.6351)^x \times (0.3649)^{6-x}$$

Q. 8 A random sample of 50 items gives the mean 6.2 and standard deviation 10.24, can it be regarded as drawn from a normal population with mean 5.4 at 5 % level of significance ? May 2015

Ans. :

n = 50 (> 30, so it is large sample)

$x = 6.2$; $\sigma = 10.24$

Step 1 :

Null hypothesis (H_0) : $\mu = 5.4$ (i.e. sample belongs to the population with mean 5.4). Alternative hypothesis (H_a) : $\mu \neq 5.4$ (i.e. sample does not belong to population with mean 5.4) (two tailed test)

Step 2 :

LOS 5 % (Two tailed test)

$\therefore$ Critical value $(Z_\alpha) = 1.96$

Step 3 :

Since sample is large,

$$\text{S. E.} = \frac{\sigma}{\sqrt{n}} = \frac{10.24}{\sqrt{50}} = 1.4482$$

Step 4 : Test Statistic

$$Z_{cal} = \frac{\bar{x} - \mu}{\text{S. E.}} = \frac{6.2 - 5.4}{1.4482} = 0.5524$$

Step 5 : Decision

Since $Z_{cal} < z_\alpha$, H_0 is accepted

$\therefore$ Sample can be regarded as drawn from a normal population with mean 5.4 at 5 % LOS

Q. 9 Find the M. G. F. of the following distribution.

X :	– 2	3	1
P (X = x)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Hence find first four central moments. May 2015

Ans. :

By definition, Mean $\bar{x} = E(X)$

$$= \sum x_i p_i = -2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{2} + 1 \cdot \frac{1}{6} = 1$$

By definition, moment generating function (M. G. F.) about mean $M_{\bar{x}}(t) = E = \left[e^{t(x - \bar{x})} \right]$

$$= \sum e^{t(x - \bar{x})} p_i = \sum e^{t(x-1)} p_i = e^{-3t} \cdot \frac{1}{3} + e^{2t} \cdot \frac{1}{2} + e^{0t} \cdot \frac{1}{6}$$

$$= \frac{1}{3} e^{-3t} + \frac{1}{2} e^{2t} + \frac{1}{6}$$

Now, central moments are given by

$$\mu_r = \left[\frac{d^r}{dt^r} M_{\bar{x}}(t) \right]_{t=0}$$

$$\therefore \text{ First central moment } \mu_1 = \left[\frac{d}{dt} M_{\bar{x}} \right] (t))_{t=0}$$

$$= \left[\frac{d}{dt} \left(\frac{1}{3} e^{-3t} + \frac{1}{2} e^{2t} \right) + \frac{1}{6} \right]_{t=0}$$

$$= \left[\frac{1}{3} e^{-3t} \cdot -3 + \frac{1}{2} e^{2t} \cdot 2 + 0 \right]_{t=0} \qquad \text{...(1)}$$

$$= -1 + 1 + 0 = 0$$

$$\therefore \text{ Second central moment } \mu_2' = \left[\frac{d^2}{dt^2} M_{\bar{x}}(t) \right]_{t=0}$$

$$= \left[\frac{d^2}{dt^2} \left(\frac{1}{3} e^{-3t} + \frac{1}{2} e^{2t} \right) + \frac{1}{6} \right]_{t=0}$$

$$= \left[\frac{d}{dt} (e^{-3t} + e^{2t}) \right]_{t=0} \qquad \text{...(From Equation (1))}$$

$$= [e^{-3t} \cdot -3 + e^{2t} \cdot 2]_{t=0}\ ) \qquad \text{...(2)}$$

$= 3 + 2 = 5$

$\therefore$ Third central moment $\mu_3' = \left[\frac{d^3}{dt^3} M_{\bar{x}}(t)\right]_{t=0}$

$= \left[\frac{d^3}{dt^3}\left(\frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t}\right) + \frac{1}{6}\right]_{t=0}$

$= \left[\frac{d}{dt}(3e^{-3t} + 2e^{2t})\right]_{t=0}$ ...(From Equation (2))

$= [3e^{-3t} \cdot -3 + 2e^{2t} \cdot 2]_{t=0}$ ...(3)

$= -9 + 4 = -5$

$\therefore$ Fourth central moment $\mu_4' = \left[\frac{d^4}{dt^4} M_{\bar{x}}(t)\right]_{t=0}$

$= \left[\frac{d^4}{dt^4}\left(\frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t}\right) + \frac{1}{6}\right]_{t=0}$

$= \left[\frac{d}{dt}(-9e^{-3t} + 4e^{2t})\right]_{t=0}$ ...(From Equation (3))

$= [-9e^{-3t} \cdot -3 + 4e^{2t} \cdot 2]_{t=0} = 27 + 8 = 35$

Q. 10 The probability density function of a random variable x is.

x	− 2	− 1	0	1	2	3
p(x)	0.1	k	0.2	2k	0.3	k

Find (i) k (ii) mean (iii) variance

Dec. 2015

Ans. : Consider the probability density function of a random variable 'x' as

x	− 2	− 1	0	1	2	3
P(x)	0.1	k	0.2	2k	0.3	k

Consider, $\sum p_i S= 1$

$0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$0.6 + 4k = 1$

$\therefore 4k = 1 - 0.6 = 0.4$

$4k = 0.4$

$\therefore$ **k = 0.1**

Now,

Mean $= E(X) = \sum x_i p_i = (-2)(0.1) + (-1)(0.1) + 0 + 1(0.2) + 2(0.3) + 3(0.1)$

$= -0.2 - 0.1 + 0.2 + 0.6 + 0.3 = 0.8$

$\therefore$ **Mean = E(X) = 0.8**

Consider $E(X^2) = \sum x_i^2 p_i = 0.4 + 0.1 + 0.2 + 1.2 + 0.9$

$E(X^2) = 2.8$

$\therefore$ Variance $= E(X^2) - [E(X)]^2 = 2.8 - [0.8]^2 = 2.8 - 0.64 = 2.16$

$\therefore$ **Variance = 2.16**

Q. 11 If the height of 500 students is normally distributed with mean 68 inches and standard deviation of 4 inches, estimate the number of students having heights (i) less than 62 inches (ii) between 65 and 71 inches.

Dec. 2015

Ans. : Let x be a normal variate with $\mu = 68$ inches, $\sigma = 4$ inches, N = 500

As $Z = \frac{x - \mu}{\sigma} = \frac{x - 68}{4}$

(i) Consider

$P(x < 62) = P\left(Z < \frac{62 - 68}{4}\right) = P(Z < -1.5)$

= Area between $Z = -\infty$ to $Z = -1.5$

= 0.5 − Area between Z = 0 and

Z = 1.5 = 0.5 − 0.4332

$P(x < 62) = 0.0668$

$\therefore$ No.of students with height less than 62 inches.

$= N\,P(X < 62) = 500\,(0.0668) = 33.4 \cong 33$

(ii) Now consider

$P(65 < x < 71) = P(-0.75 < Z < 0.75)$

= 2 (Area between Z = 0 to Z = 0.75) = 2(0.2734)

$P(65 < x < 71) = 0.5468$

No. of students with height between 65 and 71 inches.

$= N\,P(65 < x < 71) = 500\,(0.5468) = 273.4 \cong 273$

$\therefore$ No. of students with height between 65 inches and 71 inches = 273

Q. 12 Fit a Poisson distribution to the following data

x	0	1	2	3	4	5	6	7	8
f	56	156	132	92	37	22	4	0	1

Dec. 2015

Ans. : Let fm be the mean of Poisson distribution. The probability function is given by

$P(X = x) = \frac{e^{-m} \cdot m^x}{x!}$

Consider the following table.

x	f	$x_i f_i$	$P(X = x_i) = P_i$	Expected frequency = N · Pi
0	56	0	0.1392	$69.6 \cong 70$
1	156	156	0.2745	$137.2 \cong 137$
2	132	264	0.2707	$135.3 \cong 135$
3.	92	276	0.1779	$88.9 \cong 87$
4.	37	148	0.0877	$43.8 \cong 44$
5.	22	110	0.0346	$17.2 \cong 17$
6.	4	24	0.0114	$5.6 \cong 6$
7	0	0	0.0032	$1.6 \cong 2$
8.	1	8	0.0008	$0.3 \cong 0$

$$m = \frac{\sum x_i f_i}{\sum f_i} = \frac{986}{500} = 1.972$$

As, $P(X = x) = \dfrac{e^{-1.972} \cdot (1.972)^x}{x!}$

Q. 13 **If x is a continuous random variable with the probability density function given by**

$$f(x) = \begin{cases} k(x - x^3) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) k (ii) the mean of distribution. May 2016

Ans. : Let x be a continuous random variable with probability density function.

$$f(x) = \begin{cases} k(x - x^3) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

As, $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 k(x - x^3)\,dx = 1$$

$$k\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 1$$

$$\therefore\ k\left[\frac{1}{2} - \frac{1}{4}\right] = 1 \qquad \therefore\ \frac{K}{4} = 1 \qquad \therefore\ K = 4$$

As

$$\text{Mean} = \int_0^1 x\,f(x)\,dx = \int_0^1 4 \cdot x\,(x - x^3)\,dx$$

$$= 4\int_0^1 (x^2 - x^4)\,dx = 4\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= 4\left[\frac{1}{3} - \frac{1}{5}\right] = 4\left[\frac{2}{15}\right] = \frac{8}{15}$$

$$\textbf{Mean} = \frac{8}{15}$$

Q. 14 **The marks of 1000 students in an examination are found to be normally distributed with mean 70 and standard deviation 5, estimate the number of students. Whose marks will be (i) between 60 and 75 (ii) more than 75.** May 2016

Ans. : Here $N = 1000,\ \mu = 70,\ \sigma = 5$

As standard normal variable as

$$Z = \frac{x - \mu}{\sigma} = \frac{x - 70}{5}$$

(i) Now consider the probability for 60 and 70

$\therefore$ For $x = 60$ $\quad z = \dfrac{60 - 70}{5} = -2$

$x = 75$ $\quad z = \dfrac{75 - 70}{5} = 1$

As,

$P(60 < x < 75) = P(-2 \le z \le 1)$

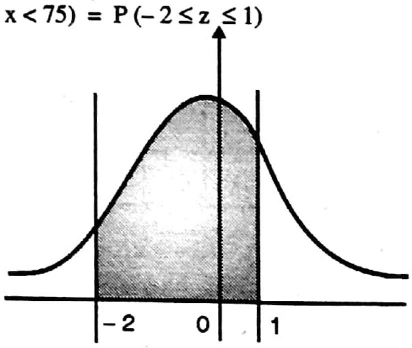

Fig. 3.1

= Area from $z = -2$ to $z = 0$ + Area from $z = 0$ to $z = 1$

$= 0.4772 + 0.3413 = 0.8185$

$\therefore$ Numbers of students between 60 and 75

$= 1000\ P(60 < x < 75)$

$= 1000\,(0.8185) = 818.5 \cong 819$

$= 819$

(ii) Numbers of students more than 75

For $x = 75$, $\therefore\ z = 1$

$P(x > 75) = P(z > 1)$

$= 0.5 - \text{Area from } z = 0 \text{ to } z = 1$

$= 0.5 - 0.3413 = 0.1587$

$P(X > 75) = 0.1587$

Numbers of students more than 75 = NP (X > 75)

$= 0.1587 \times 1000 = 158.7 \cong 159$

$\therefore$ Numbers of students more than 75 marks = 159

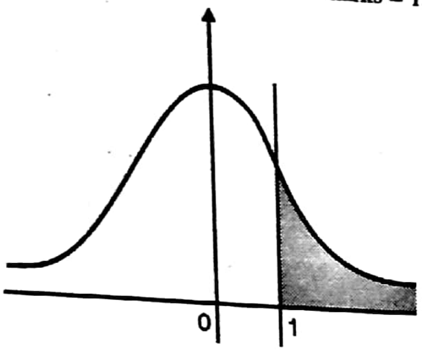

Fig. 3.2

Q. 15 Fit a Binomial distribution to the following data.

x	0	1	2	3	4	5	6
f	5	18	28	12	7	6	4

May 2016

Ans. : Let x be a binomial variate. Here n = 6.

The probability distribution function is given as and N = 80

$$P(X = x) = {}^6C_x\, P^x\, q^{6-x}$$

Consider the following table

x	f	$x_i f_i$	$P(x = x_i) = P_i$	Expected frequency = N P_i
0	5	0	0.0467	$3.7325 \cong 4$
1	18	18	0.1866	$14.9299 \cong 15$
2	28	56	0.3110	$24.8832 \cong 25$
3	12	36	0.2765	$22.1184 \cong 22$
4	7	28	0.1382	$11.0592 \cong 11$
5	6	30	0.0369	$2.9491 \cong 3$
6	4	24	0.0041	$0.3277 \cong 0$

As, $\bar{x} = \dfrac{\sum xi\, fi}{N} = \dfrac{192}{80} = 2.4$

and $\bar{x} = np \quad \therefore np = 2.4 \quad \therefore P = \dfrac{2.4}{6} = 0.4$

$P = 0.4 \; ; \quad q = 1 - P = 1 - 0.4$

$\therefore q = 0.6$

Consider $P(X = x) = {}^6C_X (0.4)^x (0.6)^{6-x}$

$P(X = 0) = 0.0467$ Expected frequency = NP_i

$P(X = 1) = 0.1866 \qquad P(X = 2) = 0.3110$

$P(X = 3) = 0.2765 \qquad P(X = 4) = 0.1382$

$P(X = 5) = 0.0369 \qquad P(X = 6) = 0.0041$

Q. 16 If a random variable X follows the poission distribution such that,

P (X = 1) = 2 P (X = 2)

Find mean, the variance and the distribution and P (X = 3)

May 2016

Ans. : Let m be the parameter of poission distribution and x be a poisons variable

$\therefore$ As

$$P(X = x) = \frac{e^{-m} \cdot m^x}{x!}$$

Consider, $P(X = 1) = 2P(X = 2)$

$$\frac{e^{-m} \cdot m^1}{1!} = \frac{2e^{-m} \cdot m^2}{2!}$$

$\therefore \mathbf{m = 1}$

$\therefore$ The mean = variance = 1

Now

$$P(X = 3) = \frac{e^{-m} \cdot m^x}{x!} = \frac{e^{-1} \cdot 1^3}{3!} = 0.0613$$

$\therefore \mathbf{P(X = 3) = 0.0613}$

Q. 17 If x is a normal variate with mean 10 and standard deviation 4 find (I) p(|x – 14| < 1) (II) p(5 ≤ x ≤ 18) (III) p (x ≤ 12).

Dec. 2016

Ans. :

Let, $\mu = 10, \sigma = 4$, define standard normal vitiate z as

$$z = \frac{x - \mu}{\sigma} = \frac{x - 10}{4}$$

(i)when $x = 14, z = \dfrac{14 - 10}{4} = 1$

Therefore,

$p(|x - 14| \le 1) = p(|z| \le 1)$ = Area between z = – 1 and z = 1

= 2 area between z = 0 and z = 1

$= 2(0.3413) = 0.6826$

$\therefore \mathbf{p(|x - 14| \le 1) = 0.6826}$

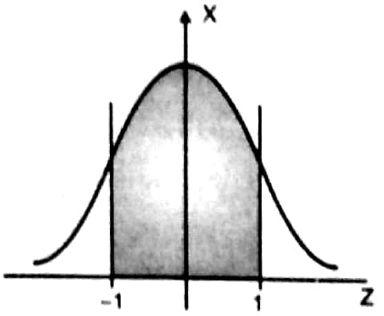

Fig. 3.3

(ii) When $x = 5,\ z = \frac{5-10}{4} = -1.25$

$$x = 18,\ z = \frac{18-10}{4} = 2$$

$$\therefore p(5 \le x \le 18) = p(-1.25 \le z \le 2)$$

$$= \text{Area between } z = -1.25 \text{ and } z = 2$$

$$= \text{Area between } z = 0 \text{ and}$$

$$z = 1.25 + \text{Area between } z = 0 \text{ and } z = 2.$$

$$P(5 \le x \le 18) = 0.3944 + 0.4772 = 0.8716$$

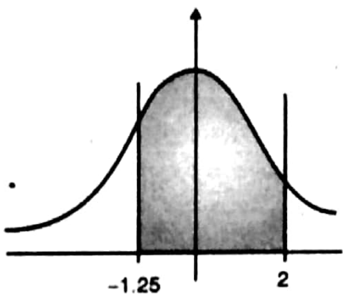

Fig. 3.4

(iii) When $x = 12$,

$$z = \frac{12-10}{4} = 0.5$$

$$p(x \le 12) = p(z < 0.5)$$

$$= \text{Area between } -\infty \text{ to } 0.5$$

$$= (\text{Area between } z = -\infty \text{ to } z = 0)$$

$$+ (\text{Area between } z = 0 \text{ to } z = 0.5)$$

$$= 0.5 + 0.1915 = 0.6915$$

$$p(x \le 12) = 0.6915$$

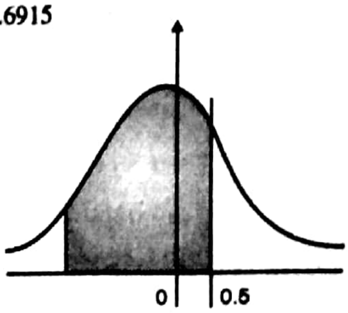

Fig. 3.5

Q. 18 If x is Binomial distributed with E(x) = 2 and $V(x) = \frac{4}{3}$. Find the probability distribution of x. Dec. 2016

Ans. : Let x is Binomial distributed with

$$E(x) = np = 2 \text{ and } \quad Var(x) = npq = \frac{4}{3}$$

As
$$\frac{np}{npq} = \frac{\frac{4}{3}}{2} = \frac{3}{2}$$

$$\therefore \frac{1}{q} = \frac{3}{2} \qquad \therefore q = \frac{2}{3}$$

$$\therefore p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3} \therefore \quad p = \frac{1}{3}$$

Since $np = 2$

$$\therefore n \cdot \frac{1}{3} = 2 \qquad \therefore n = 6$$

The distribution is given as

$$p(X = x) = {}^nC_x\, p^x \cdot q^{n-x}$$

$$p(X = x) = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

where x = 0, 1, 2, 3, …6.

Hence the following probability distribution is given as

x	0	1	2	3	4	5	6
p(X = x)	$\frac{64}{729}$	$\frac{192}{729}$	$\frac{240}{729}$	$\frac{160}{729}$	$\frac{60}{729}$	$\frac{12}{729}$	$\frac{1}{729}$

Q. 19 If a random variable x follows Poisson distribution such that p(x = 1) = 2 p(x = 2) Find the mean and variance of the distribution. Also find p(x = 3). Dec. 2016

Ans. : Let x follows Poisson Distribution. The probability function is given as

$$P(X = x) = \frac{e^{-m} m^x}{x!} \text{ where } m = \text{mean}$$

As, $p(X = 1) = 2P(X = 2)$

$$\frac{e^{-m} m^1}{1!} = 2\frac{e^{-m} m^2}{2!} \qquad \therefore m = 1$$

$$\therefore \text{Mean} = m = 1$$

Now

$$p(X = 3) = \frac{e^{-m} m^3}{3!} = \frac{e^{-1} \cdot m^1}{3!} = 0.0613$$

$$\therefore p(X = 3) = 0.0613$$

Q. 1 **In a competitive examination, the top 15% of the students appeared will get grade 'A', while the bottom 20 % will be declared fail. If the grades are normally distributed with mean % of marks 75 and S.D. 10, determine the lowest % of marks to receive grade A and the lowest % of marks that passes.**

May 2016

Ans. :

Mean (m) = 75, standard deviation (σ) = 10

Let X denote marks obtained by a student on 100

(i) Let x_1 be the lowest marks of the top 15% (Grade A)

Let Z_1 be corresponding SNV

Proportion of the top 15 % = 0.15

$\therefore$ area between '$Z = 0$' to '$Z = Z_1$' = $0.5 - 0.15 = 0.35$

$\therefore$ from Z – table $Z_1 = 1.0364$

$$\therefore \quad \frac{X_1 - m}{6} = 1.0364$$

$$\therefore \quad \frac{X_1 - 75}{10} = 1.0364$$

$$\therefore \quad X_1 = 75 + 10 \times 1.0364 = 85.3643$$

$$X_1 = 85$$

Hence, the lowest marks to receive grade

A = 85 %

(ii) Let X_2 be the highest marks of the bottom 20 % or the lowest % of marks that passes.

Let Z_2 be corresponding SNV

Proportion of the bottom 20% = 0.20

$\therefore$ Area between '$Z = 0$' to $Z = Z_2' = 0.5 - 0.2 = 0.3$

$\therefore$ From Z table $Z_2 = -0.8416$

$$\therefore \quad \frac{X_2 - m}{6} = -0.8416$$

$$\therefore \quad \frac{X_2 - 75}{10} = -0.8416$$

$$X_2 = 75 + 10 \times -0.8416 = 66.5838$$

$$X_2 = 67$$

Hence the lowest % of marks that passes = 67 %.

Q. 2 **A sample of 8 students of 16 years each shown up a mean systolic blood pressure of 118.4 mm of Hg with S.D. of 12.17 mm. While a sample of 10 students of 17 years each showed the mean systolic BP of 121.0 mm with S.D. of 12.88 mm during in investigation. The investigator feels that the systolic BP is related to age. Do you think that the data provides enough reasons to support investigator's feeling at 5% LoS ? Assume the distribution 'of systolic BP to be normal.**

May 201[illegible]

Ans. : $n_1 = 8$ and $n_2 = 10$ (< 30, so it is small sample)

Step 1 :

Null Hypothesis (H_0) : $\mu_1 = \mu_2$ (i.e. There is no relation between the systolic b.p. and the age) Alternative Hypothesis (H_a) : $\mu_1 \neq \mu_2$ (i.e. There is a relation between the systolic b.p. and the age).

Step 2 :

LOS = 5% (Two tailed test)

Degree of freedom= $n_1 + n_2 - 2 = 8 + 10 - 2 = 16$

$\therefore$ Critical value (t_α) = 2.12

Step 3 :

Given : $\bar{x}_1 = 118.4$; $\bar{x}_2 = 121$;

$s_1 = 12.17$; $s_2 = 12.88$

Since sample is small,

$$S_p = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}}$$

$$= \sqrt{\frac{8 \times 12.17^2 + 10 \times 12.88^2}{8 + 10 - 2}} = 13.3319$$

$$\text{S.E.} = S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 13.3319 \sqrt{\frac{1}{8} + \frac{1}{10}} = 6.3239$$

Step 4 : Test statistic :

$$t_{cal} = \frac{\bar{x}_1 - \bar{x}_2}{\text{S.E.}} = \frac{118.4 - 121}{6.3239} = -0.4111$$

Step 5 : Decision :

Since $|t_{cal}| < t_\alpha$, H_0 is accepted.

$\therefore$ There is no relation between the systolic b.p. and the age.

Hence, the data does not support investigators feelings.

Q. 3 **A. total of 3759 individuals were interviewed in a public opinion survey on a political proposal of them,1872 were men and the rest were women. A total of 2257 individuals were in favour of the proposal and 917 were opposed to it. A total of 243 men were undecided and 442 women were opposed to the proposal. Do you justify on the hypothesis that there is no association between sex and attitude, at 5 % LoS.**

May 2014

Ans. :

Attitude	Observed Frequency			Expected Frequency		
	Sex			Sex		
	Men	Women	Total	Men	Women	Total
In favour	1154	1103	2257	$\frac{1872 \times 2257}{3759} \approx 1124$	1133	2257
Not in favour	475	442	917	$\frac{1872 \times 917}{3759} = 456.67 \approx 457$	460	917
Undecided	243	342	585	$1872 - (1124 + 457) = 291$	294	585
Total	1872	1887	3759	1872	1887	3759

Step 1 :

Null Hypothesis (H_0) : There is no association between the sex and attitude.

Alternative Hypothesis (H_a) : There is association between the sex and attitude.

Step 2 :

$$\text{LOS} = 5\% \text{ (Two tailed test)}$$

$$\text{Degree of freedom} = (r-1)(c-1) = (3-1)(2-1) = 2$$

$$\therefore \text{ Critical value } (\chi_a^2) = 5.9915$$

Step 3 : Test statistic.

Observed Frequency (O)	Expected Frequency (E)	$\chi^2 = \frac{(O-E)^2}{E}$
1154	1124	0.80
1103	1133	0.79
475	457	0.71
442	460	0.71
243	291	7.92
342	294	7.84
	Total	**18.77**

$$\chi_{cal}^2 = \sum \frac{(O-E)^2}{E} = 18.77$$

Step 4 : Decision

Since $\chi_{cal}^2 > \chi_a^2$, H_0 is rejected.

$\therefore$ There is association between the sex and attitude.

Q. 4 **A machine is claimed to produce nails of mean length 5 cm. and standard deviation of 0.45 cm. A random sample of 100 nails gave 5.1 cm as average length. Does the performance of the machine, justify the claim ? Mention the level of significance you apply.**

Dec. 2014

Ans. :

Expected mean thickness $= \mu = 5$

Standard deviation $= \sigma = 0.45$

Number of samples $= n = 100$

Actual mean of samples $= \bar{x} = 5.1$

H_0 : Machine fulfils the requirement.

$$\therefore \quad t = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{5.1-5}{\frac{0.45}{\sqrt{100}}}$$

$$\therefore \quad t = \frac{0.1}{\frac{0.45}{10}} = \frac{1}{0.45} = 2.2223 = 2$$

$$\therefore \quad t_{calculated} = 2$$

$$\text{Level of significance} = 5\%$$

$$\therefore \quad t_{0.05\ (tabulated)} = 1.96$$

$$\therefore \quad t_{calculated} > t_{tabulated}$$

$\therefore$ we reject the null hypothesis H_0.

i.e. machine does not fulfil the requirement.

Q. 5 **The mean inside diameter of a sample of 200 washers produced by a machine is 0.502 cm and the standard deviation is 0.005 cm. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 0.496 to 0.508 cm, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machine, assuming the diameters are normally distributed.**

Dec. 2014

Ans. : $Z_1 = \dfrac{x-\mu}{\sigma} = \dfrac{0.496-0.502}{0.005} = -1.2$

$$Z_2 = \frac{x-\mu}{\sigma} = \frac{0.508-0.502}{0.005} = +1.2$$

Hence the area for non-defective washers.

$= $ Area between $Z = -1.2$ and $Z = +1.2$

$= 2$ (Area between $z = 0$ and $z = 1.2$)

$= 2 \times (0.3849) = 0.7698 = 76.98\ \%$

Then the percentage of defective washers

$= 100 - 76.98 = 23.02\ \%$

Q. 6 **The means of two random samples of size 9 and 7 are 196.42 and 198.82 respectively. The sums of the squares of the deviation from the means are 26.94 and 18.73 respectively. Can the samples be considered to have been drawn from the same population ?** **Dec. 2014**

Ans. : $n_1 = 9$; $n_2 = 7$

$\bar{x}_1 = 196.42$; $\bar{x}_2 = 198.82$

$S_1^2 = 26.94$; $S_2^2 = 18.73$

For 95% confidence level; $Z_\alpha = 1.96$

$$\bar{x}_1 - \bar{x}_2 = 196.42 - 198.82 = -2.4$$

$$S = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} = \sqrt{\frac{26.94}{9} + \frac{18.73}{7}}$$

$$= \sqrt{2.99 + 2.675} = 2.38$$

$$\therefore\ Z = \frac{\bar{x}_1 - \bar{x}_2}{S} = \frac{-2.4}{2.38} = -1.008$$

$|Z| = 1.008$; $Z_\alpha = 1.96$

$|Z| < Z_\alpha$

$\therefore$ The hypothesis that, the two samples are drawn from same population, is accepted.

Q. 7 **For special security in a certain-protected area, it was decided to put three lighting bulbs on each pole. If each bulb has a probability p of burning out in the first 100 hours of service, calculate the probability that at least one of them is still good after 100 hours.**

If p = 0.3, how many, bulbs would be needed on each pole to ensure 99% safety that atleast one is good after 100 hours ? **Dec. 2014**

Ans. :

Probability of burning out in the first 100 hours of service is

$$p = 0.3 \rightarrow q = 1 - p = 1 - 0.3 = 0.7$$

Then the probability that atleast one of them is still good after 100 hours

$$= p(X=1) + p(X=2) + p(X=3)$$

$$= {}^3C_0 q^2 p^1 + {}^3C_2 q^1 p^2 + {}^3C_3 q^0 p^3$$

$$= \left[{}^3C_0 q^3 p^0 + {}^3C_1 q^2 p^1 + {}^3C_2 q^1 p^2 + {}^3C_3 q^0 p^3\right] - {}^3C_0 q^3 p^0$$

$$= 1 - {}^3C_0 q^3 p^0 = 1 - q^3 = 1 - (0.7)^3 = 0.657$$

Second part : Probability $= 99\ \% = 0.99$

Let the number of bulbs required be n. Now p (at least one bulb is good) $= 1 - q^n$

$$0.99 = 1 - (0.7)^n$$

$$(0.7)^n = 1 - 0.99 = 0.01$$

$$\log (0.7)^n = \log (0.01)$$

$$n \log (0.7) = \log (0.01)$$

$$n = \frac{\log (0.01)}{\log (0.7)}$$

$$n = \frac{2.000}{1.8401} = \frac{-2}{-0.1549} = 13$$

Q. 8 **The number of car accidents in a metropolitan city was found to be 20, 17, 12, 6, 7, 15, 8, 5, 16 and 14 per month respectively. Use χ^2 test to check whether these frequencies are in agreement with the belief that occurrence of accidents was the same during 10 months period. Test at 5% level of Significance.** **Dec. 2014**

Ans. :

Given : $\chi_{0.05,9} = 16.92$

H_0 : The accident conditions are same during 10-month period.

$$\therefore\ \text{Mean accidents} = \frac{20+17+12+6+7+15+8+5+16+14}{10}$$

$$= \frac{110}{10} = 101$$

$\therefore\ e_i = 10$ for $i = 1, 2, \ldots\ 10$.

o_i	:	20	17	12	6	7	15	8	5	16	14
e_i	:	10	10	10	10	10	10	10	10	10	10
$o_i - e_i$	:	10	7	2	−4	−3	5	−2	−5	6	4

$$\chi^2 = \frac{(10)^2}{10} + \frac{(7)^2}{10} + \frac{(2)^2}{10} + \frac{(-4)^2}{10} + \frac{(-3)^2}{10} + \frac{(5)^2}{10} + \frac{(-2)^2}{10} + \frac{(-5)^2}{10} + \frac{(6)^2}{10} + \frac{(4)^2}{10}$$

$$= \frac{100 + 49 + 4 + 16 + 9 + 25 + 4 + 25 + 36 + 16}{10}$$

$$= \frac{274}{10}$$

$$\chi^2 = 27.4$$

Now degree of freedom = $10 - 1 = 9$

($\because$ one degree of freedom lost due to $\Sigma o_i = \Sigma e_i$)

$$\chi^2_{0.05,9} = 16.92 \text{ (from table)}$$

$\therefore$ As calculated value of χ^2 is greater than $\chi^2_{0.05,9}$

$\Rightarrow H_0$ is rejected.

Q. 9 **Ten individuals are chosen at random from a population and their heights are found to be 63, 63, 64, 65, 66, 69, 69, 70, 70, 71 inches. Discuss the suggestion that the mean height of the universe is 65 inches.** May 2015

Ans. : $n = 10$ (< 30, so it is small sample)

Step 1 : Null Hypothesis (H_0) : $\mu = 65$ (i.e. the mean height of the universe is 65 inches)

Alternative hypothesis (H_a) : $\mu_1 \neq 65$ (i.e. the mean height of the universe is not 65 inches) (Two tailed test).

Step 2 :

LOS = 5 % (Two tailed test). Degree of Freedom = $n - 1 = 10 - 1 = 9$

$\therefore$ Critical value (t_α) = 2.2622

Step 3 :

Values (x_i)	$d_i = x_i - 67$	d_i^2
63	– 4	16
63	– 4	16
64	– 3	9
65	– 2	4
66	– 1	1
69	2	4
69	2	4
70	3	9
70	3	9
71	4	16
Total	0	88

$$\bar{d} = \frac{\Sigma d_i}{n} = \frac{0}{10} = 0$$

$$\therefore \bar{x} = a + \bar{d} = 67 + 0 = 67$$

Since sample is small,

$$s = \sqrt{\frac{\Sigma d_i^2}{1} - \left(\frac{\Sigma d_i}{n}\right)^2} = \sqrt{\frac{88}{10} - \left(\frac{0}{10}\right)^2} = 2.9965$$

$$\text{S. E.} = \frac{s}{\sqrt{n-1}} = \frac{2.9965}{\sqrt{9}} = 0.9888$$

Step 4 : Test statistic,

$$t_{cal} = \frac{\bar{x} - \mu}{\text{S. E.}} = \frac{67 - 65}{0.9888} = 2.0227$$

Step 5 : Decision : Since $| t_{cal} | < t_\alpha$, H_0 is accepted.

$\therefore$ The mean height of the universe is 65 inches.

Q. 10 **In sampling a large number of parts manufactured by a machine, the mean number of defective is a sample of 20 is 2. Out of 100 such samples, how many would you expect to contain 3 defective (i) using the Binomial distribution. (ii) Poisson distribution.** Dec. 2015

Ans. :

Let 2 out of 20 are defective parts.

$$\therefore P = \frac{2}{20} = 0.1, \qquad q = 1 - p = 0.9, N = 100$$

By Binomial Distribution

$$P(X = x) = {}^nC_x\, p^x \cdot q^{n-x}$$

As $n = 20, x = 3$

$$P(X = 3) = {}^{20}C_3\,(0.1)^3 \cdot (0.9)^{17} = 0.1901$$

No. of samples with 3 defective parts

$$= N\,P(X = 3) = 100\,(0.1901) = 19.01 \cong 19$$

For Poisson's distribution

$$m = np = 20\,(0.1) = 2$$

Here $$P(X = x) = \frac{e^{-m} \cdot m^x}{x!}$$

$$\therefore P(X = 3) = \frac{e^{-2} \cdot 2^3}{3!} = 0.1804$$

No. of samples with 3 defective parts

$$= N\,P(X = 3) = 100\,(0.1804) = 18.04 \cong 18$$

Q. 11 The average of marks scored by 32 boys is 72 with standard deviation 8 while that of 36 girls is 70 with standard deviation 6. Test at 1% level of significance whether the boys perform better than girls. Dec. 2015

Ans. :

Consider the hypothesis

(i) Null Hypothesis $H_0 : \mu_1 = \mu_2$

Alternative Hypothesis $H_a : \mu_1 \neq \mu_2$

(ii) Calculation of statistic

$$\bar{x}_1 - \bar{x}_2 = 72 - 70 = 2$$

As $S_1^2 = 64,\ S_2^2 = 36,$

$n_1 = 32,\ n_2 = 36$

Consider, $$\text{S.E.} = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} = \sqrt{\frac{64}{32} + \frac{36}{36}} = \sqrt{3}$$

$$\text{Now } Z = \frac{\bar{x}_1 - \bar{x}_2}{\text{S.E}} = \frac{2}{\sqrt{3}} = 1.15$$

(iii) Level of significance $\therefore \alpha = 1\%$

(iv) **Critical value :** The critical value of Z_α at 1% level of significance from the table is 2.58.

(v) **Decision :** Since the computed value of $|Z| = 1.15$ is less than the critical value $Z_\alpha = 2.58$.

The hypothesis is accepted.

$\therefore$ Boys do not perform better than girls.

Q. 12 In an experiment of immunization of cattle from Tuberculosis, the following result where obtained

	Affected	Not affected	Total
Inoculated	267	27	294
Not inoculated	757	155	912
Total	1024	182	1206

Use X^2 Test to determine the efficiency of Vaccine in preventing truberculosis. Dec. 2015

Ans. :

(i) Consider Null hypothesis H_0 : Vaccine is not effective.

Alternative Hypothesis H_a : Vaccine is effective.

(ii) Calculation of statistic : Consider the following table

	Affected	Not affected	Total
Inoculated	267	27	294
Not inoculated	757	155	912
Total	1024	182	1206

On the basis of this hypothesis the number in there first cell = $\frac{A \times B}{N}$

Where A = no. of affected cattle = Total of first column

B = no. of inoculated cattle=total of first row

N = total no.s of observations

Thus the following table of calculated frequencies is

	Affected	Not affected	Total
Inoculated	249.6 = 250	44.6 = 44	294
Not inoculated	774.36 = 774	137.6 = 138	912
Total	1024	182	1206

Calculation of $\frac{(0-E)^2}{E}$

0	E	$\frac{(0-E)^2}{E}$
267	250	1.156
27	44	6.568
757	774	0.373
155	138	1.864
		9.961

$\therefore X^2 = 9.961$

Degree of freedom $= (2-1)(2-1) = 1$

Level of significance $= \alpha = 5\%$

$X^2_{tab} = 3.84$ Since, $X^2_{cal} = 9.96 \leq X^2_{tab} = 3.84$

$\therefore$ We reject null hypothesis i.e. Vaccine is effective.

Q. 13 The regression lines of a sample are $x + 6y = 6$ and $3x + 2y = 10$. Find (a) sample means $\bar{x}$ and $\bar{y}$ (b) coefficients of correlation between x and y (II) If two independent random samples of sizes 15 and 8 have respective means and population standard deviations as $\bar{x}_1 = 980$, $\bar{x}_2 = 1012$, $\sigma_1 = 75$, $\sigma_2 = 80$. Test the hypothesis that $\mu_1 = \mu_2$ at 5%level of significance. Dec. 2015

Ans. :

(i) Let the line of regression of y on x be

$$x + 6y = 6 \quad \ldots(1)$$

$$\therefore y = -\frac{1}{6}x + 6 \qquad \therefore byx = -\frac{1}{6}$$

Consider the line of regression of x on y.

$$3x + 2y = 10 \quad \ldots(2)$$

$$\therefore x = -\frac{2}{3}y + 10 \qquad \therefore bxy = -\frac{2}{3}$$

For coefficient of correlation,

$$r^2 = bxy \cdot byx = \left(-\frac{1}{6}\right)\left(-\frac{2}{3}\right) = \frac{1}{9}$$

$$\therefore r = +\frac{1}{3}$$

Since bxy, byx ≤ 0 $\quad \therefore$ $\mathbf{r = -\frac{1}{3}}$

For means, solving Equations (1) and (2) for x and y $\bar{x} = 3$ and $\bar{y} = \frac{1}{2}$

(ii) Consider $\bar{x}_1 = 980, \bar{x}_2 = 1012, \sigma_1 = 75, \sigma_2 = 80$

Null Hypothesis $H_0 : \mu_1 = \mu_2$

Alternative Hypothesis $H_a : \mu_1 \neq \mu_2$

Here $n_1 = 15, n_2 = 8$

Calculation of test statistics

$$S_t = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{75^2}{15} + \frac{80^2}{8}}$$

$$\text{S.E.} = 34.28$$

$$\therefore Z = \frac{\bar{x}_1 - \bar{x}_2}{\text{S.E}} = \frac{980 - 1012}{34.28} = -0.93$$

$$|Z| = 0.93$$

As level of significance $= \alpha = 0.05$

$$\therefore Z_\alpha = 1.96 \qquad \text{since } |Z| = 0.93 < 1.96 = Z_\alpha$$

$\therefore$ Null hypothesis is accepted.

i.e. Population means are equal $\mu_1 = \mu_2$

Q. 14 A certain stimulus administered to each of 12 patients resulted in the following change in blood pressure 5, 2, 8, – 1, 3, 0, – 2, 1, 5, 0, 4, 6

Can it be concluded that the stimulus will increase the blood pressure (at 5% level of significance) ? May 2016

Ans. :

Consider the following table

As, a = 3

x	:	5	2	8	–1	3	0	6	–2	1	5	0	4
$d_i = x - 3$	:	2	–1	5	–4	0	–3	3	–5	–2	2	–3	1
$d_i^2 = (x-3)^2$	:	4	1	25	16	0	9	9	25	4	4	9	1

$$\bar{x} = a + \frac{\sum d_i}{n} = 3 + \left(\frac{-5}{12}\right) = 2.5833 = 2.58$$

$$\bar{x} = 2.58$$

$$\sum(x_i - \bar{x}) = \sum d_i^2 - \frac{\left(\sum d_i\right)^2}{n} = 107 - \frac{25}{12}$$

$$\sum(x_i - \bar{x}) = 104.92$$

Consider Null Hypothesis $H_0 : \mu = 0$

Alternative Hypothesis $H_a : \mu > 0$

As $$S = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n}} = 2.96$$

$$\therefore t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{2.58 - 0}{2.96/\sqrt{11}} = \frac{2.58}{0.892}$$

$$\therefore t = \mathbf{2.89}$$

and degree of freedom = n – 1 = 11 with 5% level of significance $t_\alpha = 2.201$

Since $t_{cal} > t_\alpha$

$\therefore$ Null hypothesis is rejected.

i.e. There is increase in blood pressure.

Q. 15 In an experiment pea breading, the following frequencies of seeds were obtained.

Round and yellow	Wrinkled and yellow	Round and green	Wrinkled and green	Total
315	101	108	32	556

Theory preluts that the frequencies should be in the proportions 9 : 3 : 3: 1. Examine the correspondence between theory and experiment using Chi-square Test. May 2016

Ans. :

Following are the result of pea breeding.

Round and yellow (Ry)	Wrinkled and yellow (Wy)	Round and green (Re)	Wrinkled and green (WG)	Total
315	101	108	32	556

Theory predicts that the frequencies should be in proportions 9 : 3 : 3 : 1.

Calculate the expected frequencies using these ratios.

Expected frequency of Ry $= \frac{9}{16} \times 556 = 312.75 = 313$

Expected frequency of Wy $= \frac{3}{16} \times 556 = 104.25 = 104$

Expected frequency of RG $= \frac{3}{16} \times 556 = 104$

Expected frequency of WG $= \frac{1}{16} \times 556 = 34.75 = 35$

Consider the table for X^2 test.

O	E	$\frac{(O-E)^2}{E}$
315	313	0.0128
101	104	0.0865
108	104	0.1538
32	35	0.2571
		0.5102

Here,

Null Hypotheses Ho : Frequencies are in proportions 9 : 3 : 3 : 1

Alternate Hypothesis Ha : Frequencies are not is proportion

Since, $X^2 = 0.5102$

Level of significance $\alpha = 0.05$

Degree of freedom $n - 1 = 4 - 1 = 3$

For 3 degree of freedom at 5 % level of significance, the table value of X^2 is 7.81

Since, $X^2_{cal} < X^2_{tab}$

$\therefore$ Null Hypothesis is accepted.

$\therefore$ The frequencies are its proportion 9 : 3 : 3 : 1

Q. 16 **The average of marks scored by 32 boys is 72 with standard deviation 8 while that of 36 girls is 70 with standard deviation 6. Test at 1% level of significance whether the boys perform better than girls.** Dec 2016

Ans. :

Consider the hypotheses

(i) Null hypothesis : $H_0 : \mu_1 = \mu_2$

Alternative hypothesis $H_1 : \mu_1 \neq \mu_2$

(ii) Calculation of statistic $\bar{x}_1 - \bar{x}_2 = 72 - 70 = 2$

$$\text{As } S_1^2 = 64, S_2^2 = 36,$$

$$M = 32, n_2 = 36$$

Consider

$$S.E. = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} = \sqrt{\frac{64}{32} + \frac{36}{36}} = \sqrt{3}$$

$$S.E. = \sqrt{3}$$

$$\text{Now } z = \frac{\bar{x}_1 - \bar{x}_2}{S.E.} = \frac{2}{\sqrt{3}} = 1.15$$

(iii) Level of significance : $\alpha = 1$ %

(iv) Critical value : The critical value of z_α at 1 % level of significance from the table is 2.58.

(v) Decision : Since the computed value of $|z| = 1.15$ is less than the critical value $z_\alpha = 2.58$. The hypotheses is accepted.

$\therefore$ Boys do not perform better than girls.

Q. 17 **The regression lines of a sample are $x + 6y = 6$ and $3x + 2y = 10$. Find**

(i) Samples means $\bar{x}$ and $\bar{y}$

(ii) Correlation coefficient between x and y.

Also estimate y when x = 12. Dec 2016

Ans. : Consider the line of regression of y on x as

$$x + 6y = 6 \quad \text{...(1)}$$

$$\therefore y = -\frac{1}{6}x + 1$$

$$\therefore b_{yx} = \frac{-1}{6}$$

Also consider the line of regression of x on y as

$$3x + 2y = 10 \quad \text{...(2)}$$

$$\therefore x = -\frac{2}{3}y + \frac{10}{3}$$

$\therefore \; b_{xy} = \frac{-2}{3}$

To find means $\bar{x}$ and $\bar{y}$, we solve the Equations (1) and (2) as,

$$3x + 18y = 18$$
$$3x + 2y = 10$$
$$- \quad - \quad -$$
$$16y = 8 \qquad \therefore y = \frac{1}{2} \text{ and } x = 3$$

The sample means are $\bar{y} = \frac{1}{2}$ and $\bar{x} = 3$.

As $r^2 = b_{xy} \cdot b_{yx}$

$$r^2 = \left(\frac{-2}{3}\right)\left(\frac{-1}{6}\right) = \frac{1}{9}$$

$$\therefore r = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

Since bxy and byx are negative

$$\therefore r = \frac{-1}{3}$$

When $x = 12$, from Equation (1),

$$y = \frac{-1}{6}(12) + 1 = -2 + 1 = -1$$

$$\therefore y = -1$$

Q. 18 A die was theorem 132 times and the following frequencies were observed.

No. obtained	1	2	3	4	5	6	Total
Frequency	15	20	25	15	29	28	132

Using x^2 – test. Examine the hypothesis that the die is unbiased. Dec. 2016

Ans. :

(i) Consider null hypothesis : H_0 : The die is unbiased.

Alternative hypothesis Ha : The die is not unbiased.

(ii) Calculation test statistic :

$$\text{Expected frequency} = \frac{132}{6} = 22$$

The following table shows frequencies

No.	O	E	$(O-E)^2$
1	15	22	49
2	20	22	4
3	25	22	9
4	15	22	49
5	29	22	49
6	28	22	36
			196

$$X^2 = \frac{\sum (O-E)^2}{E} = \frac{196}{22} = 8.91$$

(iii) Level of significance : $\alpha = 0.05$

Degree of freedom : $n - 1 = 6 - 1 = 5$

(iv) **Critical value :** For 5 df at 5% level of significance, the table value of X^2 is 11.07.

(v) **Decision :** Since the calculated value is less than table value ; the hypothesis is accepted.

i.e. the die is unbiased.

Chapter 5 : Mathematical Programming

Q. 1 Find dual of following LP model

$\max z = 2x_1 + 3x_2 + 5x_3$

Subject to

$x_1 + x_2 - x_3 \geq -5$

$x_1 + x_2 + 4x_3 = 10$

$-6x_1 + 7x_2 - 9x_3 \leq 4$

and x_1, x_2 0 and x_3 is unrestricted. May 2014

Ans:

Since X_3 is unrestricted, let $x_3 = x_3' - x_3''$. since objective is a maximization type we convert all constraints to $\leq$ type

Primal :

$$\text{Maximize } z = 2x_1 + 2x_2 + 5(x_3' - x_3'')$$

Constraints

1. $x_1 + x_2 - x_3 \geq -5$

$$\therefore \; -x_1 - x_2 + (x_3' - x_3'') \leq 5$$

$$\therefore \; -x_1 - x_2 + x_3' - x_3'' \leq 5$$

2. $x_1 + x_2 + 4x_3 = 10$

$\therefore$ $x_1 + x_2 + 4x_3 \leq 10$ and $x_1 + x_2 + 4x_3 \geq 10$

$$\therefore \; x_1 + x_2 + 4(x_3' - x_3'') \leq 10$$

$$\text{and} \quad -x_1 - x_2 - 4(x_3' - x_3'') \leq -10$$

$$\therefore \; x_1 + x_2 + 4x_3' - 4x_3'' \leq 10$$

$$\text{and} \quad -x_1 - x_2 - 4x_3' + x_3'' \leq -10$$

3. $-6x_1 + 7x_2 - 9x_3 \leq 4$

$$\therefore \; -6x_1 + 7x_2 - 9(x_3' - x_3'') \leq 4$$

$$\therefore \; -6x_1 + 7x_2 - 9x_3' + 9x_3'' \leq 4$$

4. $x_1, x_2, x_3', x_3'' \geq 0$

The dual of above primal is objective

Minimize $W = 5y_1 + 10y_2' - 10y_2'' + 4y_3$ i.e.

$$W = 5y_1 + 10(y_2' - y_2'') + 4y_3$$

Constraints

1. $$-1y_1 + 1y_2' - 1y_2'' - 6y_3 \geq 2$$

$$\text{i.e.} \; -y_1 + (y_2' - y_2'') - 6y_3 \geq 2$$

2. $$-1y_1 + 1y_2' - 1y_2'' - 7y_3 \geq 3$$

$$\text{i.e.} \; -y_1 + (y_2' - y_2'') - 7y_3 \geq 3$$

3. $$1y_1 + 4y_2' - 4y_2'' - 9y_3 \geq 5$$

$$\text{i.e.} \; y_1 + 4(y_2' - y_2'') - 9y_3 \geq 5$$

Put $y_2 = y_2' - y_2''$ where y_2 is unrestricted

The dual is

$\therefore$ minimize $W = 5y_1 + 10y_2 + 4y_3$

Constraints

1. $-y_1 + y_2 - 6y_3 \geq 2$
2. $-y_1 + y_2 + 7y_3 \geq 3$
3. $y_1 + 4y_2 - 9y_3 = 5$

$y_1, y_3 \geq 0$, y_2 is unrestricted

Q. 2 **Using Simplex method, solve the following LPP**

max $z = 15x_1 + 6x_2 + 9x_3 + 2x_4$

S.T. $2x_1 + x_2 + 5x_3 + 6x_4 \leq 20$

$3x_1 + x_2 + 3x_3 + 25x_4 \leq 24$

$7x_1 + x_4 \leq 70$

and $x_1, x_2, x_3, x_4 \geq 0$ **May 2014**

Ans. :

In the standard from

Maximize $Z = 15x_1 + 6x_2 + 9x_3 + 2x_4 + 0S_1 + 0S_2 + 0S_3$

Constraints :

$$2x_1 + 1x_2 + 5x_3 + 6x_4 + 1S_1 + 0S_2 + 0S_3 = 20$$

$$3x_1 + 1x_2 + 3x_3 + 25x_4 + 0S_1 + 1S_2 + 0S_3 = 24$$

$$7x_1 + 0X_2 + 0x_3 + 1x_4 + 0S_1 + 0S_2 + 0S_3 = 70$$

$$x_1, x_2, x_3, x_4, S_1, S_2, S_3, S_4 \geq 0$$

In the table from

	C_B	Z		x_1	x_2	x_3	x_4	S_1	S_2	S_3	RHS	Ratio RHS/kc
		B_V	C_J	15	6	9	2	0	0	0		
x_1 entres	0	S_1		2	1	5	6	1	0	0	20	10
S_2 leaves	0	S_2		3	1	3	25	0	1	0	24	8
	0	S_3		7	0	0	1	0	0	1	70	10
		$Z_j = \Sigma C_B \times B$		0	0	0	0	0	0	0	0	
		$\Delta_j = C_j - Z_j$		15	6	9	2	0	0	0		

	C_B	Z		x_1	x_2	x_3	x_4	S_1	S_2	S_3	RHS	Ratio RHS/kc
		B_V	C_J	15	6	9	2	0	0	0		
X_2 entres	0	S_1		0	1/3	3	– 32/3	1	– 2/3	0	4	12
S_1 leaves	15	S_2		1	1/3	1	25/3	0	1/3	0	8	24
	0	S_3		0	– 7/3	– 7	– 172/3	0	– 7/3	1	14	–
				15	5	15	125	0	5	0	120	
				0	1	– 6	– 123	0	– 5	0		

C_B	z: B_V	C_j	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS	Ratio RHS/kc
			15	6	9	2	0	0	0		
6	x_2		0	1	9	– 32	3	– 2	0	12	–
15	x_1		1	0	– 2	19	– 1	1	0	4	–
0	S_3		0	0	14	– 132	7	– 7	1	42	–
			15	6	24	93	3	3	0	132	
			0	0	– 15	– 91	– 3	– 3	0		

For maximization $x_1 = 4$, $x_2 = 12$, $x_3 = 0$, $x_4 = 0$, $Z_{max} = 132$

Q. 3 Using dual simplex method, solve

$$\max z = -2x_1 - x_3$$

s.t. $x_1 + x_2 - x_3 \geq 5$

$x_1 - 2x_2 + 4x_3 \geq 8$

and $x_1, x_2, x_3 \geq 0$ **May 2014**

Ans. :

In dual simplex, we write the objective function as maximization type and constraints in "≤" type.

Objective : Maximize $z = -2x_1 + 0x_2 - 1x_3$;

Constraints : Multiplying each constraints by "– 1",

1. $-1x_1 - 1x_2 + 1x_3 \leq -5$;
2. $-1x_1 + 2x_2 - 4x_3 \leq -8$;

Converting each constraints into equality by adding slack variable,

Maximize : $z = -2x_1 + 0x_2 - 1x_3 + 0s_1 + 0s_2$;

Constraints :

$$-1x_1 - 1x_2 + 1x_3 + 1s_1 + 0s_2 = -5;$$

$$-1x_1 + 2x_2 - 4x_3 + 0s_1 + 1s_2 = -8;$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

In the table form,

	C_B	z: ↓ BV	C_j →	x_1	x_2	x_3	s_1	s_2	RHS
				– 2	0	– 1	0	0	
x_3 enters	0	s_1		– 1	– 1	1	1	0	– 5
s_2 leaves	0	s_2		– 1	2	– 4	0	1	– 8 KR
		$z_j = \Sigma C_B X_B$		0	0	0	0	0	0
		$\Delta_j = C_j - z_j$		– 2	0	– 1	0	0	
Ratio =		Δ_j / KR		2	-	0.25	-	-	

↑ KC

	C_B	z: ↓ BV	C_j →	x_1	x_2	x_3	s_1	s_2	RHS
				– 2	0	– 1	0	0	
x_2 enters	0	s_1		– 5/4	– 1/2	0	1	1/4	– 7 KR
s_1 leaves	0	x_3		1/4	– 1/2	1	0	– 1/4	2
		$z_j = \Sigma C_B X_B$		– 1/4	1/2	– 1	0	1/4	– 2
		$\Delta_j = C_j - z_j$		– 7/4	– 1/2	0	0	– 1/4	
Ratio =		Δ_j / KR		1.4	1	-	-	-	

↑ KC

C_B	z: ↓ BV	C_j →	x_1	x_2	x_3	s_1	s_2	RHS
			– 2	0	– 1	0	0	
0	x_2		5/2	1	0	– 2	– 1/2	14
– 1	x_3		3/2	0	1	– 1	– 1/2	9
	$z_j = \Sigma C_B X_B$		– 3/2	0	– 1	1	1/2	– 9
	$\Delta_j = C_j - z_j$		– 1/2	0	0	– 1	– 1/2	

For maximization, $x_1 = 0$; $x_2 = 14$; $x_3 = 9$; $z_{max} = -9$;

Q. 4 Using Kuhn Tucker's method solve

Maximize $Z = 2x_1^2 + 12x_1x_2 - 7x_2^2$

Subject to the constraints $2x_1 + 5x_2 \leq 98$ **and** $x_1, x_2 \geq 0$. **May 2014**

Ans. :

Let $f(x_1, x_2) = 2x_1^2 - 7x_2^2 + 12x_1x_2$;

$h = 2x_1 + 5x_2 - 98$ and λ be Lagrangian multiplier.

∴ Lagrangian function $L = f - \lambda h$

$$\therefore \quad L = (2x_1^2 - 7x_2^2 + 12x_1x_2) - \lambda(2x_1 + 5x_2 - 98)$$

$$= 2x_1^2 - 7x_2^2 + 12x_1x_2 - 2\lambda x_1 - 5\lambda x_2 + 98\lambda$$

Kuhn-Tucker (K-H) conditions are :

1. $\frac{\partial L}{\partial x_1} = 0 \Rightarrow 4x_1 + 12x_2 - 2\lambda = 0$
2. $\frac{\partial L}{\partial x_2} = 0 \Rightarrow -14x_2 + 12x_1 - 5\lambda = 0$
3. $\lambda h(x_1, x_2) = 0 \Rightarrow \lambda(2x_1 + 5x_2 - 98) = 0$
4. $h(x_1, x_2) \leq 0 \Rightarrow 2x_1 + 5x_2 - 98 \leq 0$
5. $\lambda, x_1, x_2 \geq 0$

From K-H condition (3), following two cases arises :

Case 1 : $\lambda = 0$

From Equation (1), $4x_1 + 12x_2 = 0$

From Equation (2), $-14x_2 + 12x_1 = 0$

On solving simultaneously, $x_1 = 0; \quad x_2 = 0$

Substitute x_1 and x_2 in Equation (4)

$$\text{LHS} = 2(0) + 5(0) - 98 = -98$$

Values of x_1 and x_2 satisfies K-H condition (4) and (5).

$\therefore$ Stationary point $X_0 = (0, 0)$

Optimal solution at $X_0 = z_{max}$

$$= 2(0)^2 - 7(0)^2 + 12(0) + (0)$$

$$= 0$$

Case 2 : $\lambda \neq 0$

From Equation (3), $2x_1 + 5x_2 - 98 = 0$

$$\therefore \quad 2x_1 + 5x_2 = 98 \quad \ldots(6)$$

Solving Equations (1), (2) and (6), simultaneously,

$x_1 = 44; \quad x_2 = 2; \quad \lambda = 100$

Values of x_1 and x_2 satisfies K-H condition (4) and (5)

$\therefore$ Stationary point $X_0 = (44, 2)$

Optimal solution at $X_0 = z_{max}$

$$= 2(44)^2 - 7(2)^2 + 12(44)(2) = 4900$$

Hence, Stationary point $X_0 = (44, 2)$ and $z_{max} = 4900$.

Q. 5 Solve the following LPP by Simplex method :

Max $z = x_1 + 4x_2$

Subject to $2x_1 + x_2 \le 3$

$3x_1 + 5x_2 \le 9$

$x_1 + 3x_2 \le 5$

$x_1, x_2 \ge 0$

Dec. 2011

Ans. : $Z = x_1 + 4x_2$ ......Maximize

Subject to : $2x_1 + x_2 \le 3 \quad 3x_1 + 5x_2 \le 9$

$x_1 + 3x_2 \le 5 \quad x_1, x_2 \ge 0$

$$\therefore \quad Z = x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3$$

$$\therefore \quad Z - x_1 - 4x_2 + 0s_1 + 0s_2 + 0s_3 = 0$$

$$2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 3$$

$$3x_1 + 5x_2 + 0s_1 + s_2 + 0s_3 = 9$$

$$x_1 + 3x_2 + 0s_1 + 0s_2 + s_3 = 5$$

	x_1	x_2	s_1	s_2	s_3	RHS
Z	–1	–4	0	0	0	0
s_1	2	1	1	0	0	3
s_2	3	5	0	1	0	9
s_3	1	3	0	0	1	5
Z	1/3	0	0	0	4/3	20/3
s_1	5/3	0	1	0	– 1/3	4/3
s_2	4/3	0	0	0	5/3	2/3
x_2	1/3	1	0	0	1/3	5/3

$$\therefore \quad Z_{max} = \frac{20}{3} \quad \text{at} \quad x_1 = 0 \quad \text{and} \quad x_2 = \frac{5}{3}$$

Q. 6 Use Duality to solve the following LPP :

Max $Z = 2x_1 + x_2$

Subject to $-x_1 + 2x_2 \le 2$

$x_1 + x_2 \le 4$

$x_1 \le 3$

$x_1, x_2 \ge 0$

Dec. 2011

Ans. :

To solve the problem from its Primal write it in the standard form

Maximize $z = 2x_1 + x_2 - 0s_1 - 0s_2 - 0s_3$

i.e. $z - 2x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

Subject to $-x_1 + 2x_2 + s_1 - 0s_2 + 0s_3 = 2$

$$x_1 + x_2 + 0s_1 + s_2 + 0s_3 = 4$$

$$x_1 + 0x_2 + 0s_1 + 0s_2 + s_3 = 3$$

The dual of the above problem clearly is

Minimize $w = 2y_1 + 4y_2 + 3y_3$

Subject to $-y_1 + y_2 + y_3 \ge 2$

$$2y_1 + y_2 \ge 1$$

$$y_1, y_2, y_3 \ge 0.$$

The dual in the standard form will be

$$\text{Maximize } w = -w = -2y_1 - 4y_2 - 3y_3 - 0s_1 - 0s_2 - MA_1 - MA_2 \quad \ldots(1)$$

$$\text{Subject to } -y_1 + y_2 + y_3 - s_1 + 0s_2 + A_1 - 0A_2 = 2 \quad \ldots(2)$$

$$2y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1 \quad \ldots(3)$$

Multiply (2) and (3) M and add to (1).

Maximize $w' = -2y_1 - 4y_2 - 3y_3 + My_1 + 2My_2 + My_3 - Ms_1 - Ms_2 - 0A_1 - 0A_2 - 3M$

i.e. $w' + (2-M)y_1 + (4-2M)y_2 + (3-M)y_3 + Ms_1 + Ms_2 + 0A_1 + 0A_2 = -3M$

Subject to $-y_1 + y_2 + y_3 - s_1 - 0s_2 + A_1 - 0A_2 = 2$

$2y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1$

For Simplex Table (Primal).

$\therefore x_1 = 3, x_2 = 1, z_{max} = 7.$

Simplex Table (Primal)

Iteration Number	Basic Variables	x_1	x_2	s_1	s_2	s_3	R.H.S. Solution	Raito
0	z	–2	–1	0	0	0	0	
s_3 leaves	A_1	–1	2	1	0	0	2	–
x_1 enters	A_2	1	1	0	1	0	4	4
	A_3	1*	0	0	0	1	3	3←
		↑						
1	z	0	–1	0	0	2	6	
s_2 leaves	A_1	0	2	1	0	1	5	2.5
x_2 enters	x_3	0	1*	0	1	–1	1	2←
	A_3	1	0	0	0	1	3	–
			↑					
2	z	0	0	0	1	3	7	
	x_2	0	0	1	–2	3	7	
	x_3	0	1	0	1	–1	1	
	A_3	1	0	0	0	1	3	

Deleting M from the coefficients A_1 and A_2 in the final table and then changing the signs of these coefficients we see that A_1 and A_2 corresponding to 3 and 1.

These are the values of x_1 and x_2 in the primal. Hence, $x_1 = 3$, $x_2 = 1$ and $w'_{Max} = -7$. $w_{Min} = 7$

Now, once again read (3) of § 5.

Q. 7 Use Kuhn Tucker method to solve the NLPP :

Max $Z = -x_1^2 - x_2^2 - x_3^3 + 4x_1 + 6x_2$

St $x_1 + x_2 \le 2$

$2x_1 + 3x_2 \le 12$

$x_1, x_2 \ge 0$

Dec 2014

Ans. : The given problem is modified as,

Maximize $Z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

Subject to

$$x_1 + x_2 - 2 \le 0$$

$$2x_1 + 3x_2 - 12 \le 0$$

$$x_1, x_2 \ge 0$$

Then lagrangian function is given by,

$$L = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2 - \lambda_1(x_1 + x_2 - 2) - \lambda_2(2x_1 + 3x_2 - 12) \quad ...(1)$$

The four Kuhn-Tucker condition are as given below.

1. $\lambda_i \ge 0, \quad i = 1, 2$

2. $$\frac{\partial L}{\partial x_1} = -2x_1 + 4 - \lambda_1 - 2\lambda_2 = 0$$

$$\therefore -2x_1 - \lambda_1 - 2\lambda_2 + 4 = 0 \quad ...(2)$$

$$\frac{\partial L}{\partial x_2} = -2x_2 + 6 - \lambda_1 - 3\lambda_2 = 0$$

$$\therefore -2x_2 - \lambda_1 - 3\lambda_2 + 6 = 0 \quad ...(3)$$

$$\frac{\partial L}{\partial x_3} = -2x_3 = 0$$

$$x_3 = 0$$

3. $$\lambda_1 (x_1 + x_2 - 2) = 0 \quad ...(4)$$

And $$\lambda_2 (2x_1 + 3x_2 - 12) = 0 \quad ...(5)$$

4. $$x_1 + x_2 - 2 \le 0$$

$$2x_1 + 3x_2 - 12 \le 0$$

From Equation (4), $x_1 + x_2 = 2$ ...(6)

From Equation (5), $2x_1 + 3x_2 = 12$ ...(7)

$2 \times$ Equation (6) – Equation (7)

$$\begin{aligned} \cancel{2x_1} + 2x_2 &= 4 \\ \cancel{2x_1} + 3x_2 &= 12 \\ - \quad - \quad &\; - \\ \hline -x_2 &= -3 \end{aligned}$$

$\therefore x_2 = 3$

Put $x_2 = 3$ in Equation (6)

$$x_1 + x_2 = 2$$

$$x_1 + 3 = 2$$

$$x_1 = -1$$

Put $x_1 = -1$, $x_2 = 3$ in Equations (2) and (3),

$$2 - \lambda_1 - 2\lambda_2 + 4 = 0$$

$$\lambda_1 + 2\lambda_2 - 6 = 0 \quad ...(8)$$

$$\lambda_1 + 2\lambda_2 = 6$$

$$-6 - \lambda_1 - 3\lambda_2 + 6 = 0$$

$$\lambda_1 + 3\lambda_2 = 0 \quad ...(9)$$

Equation (8) – Equation (9)

$$\begin{aligned} \cancel{\lambda_1} + 2\lambda_2 &= 6 \\ \cancel{\lambda_1} + 3\lambda_2 &= 0 \\ - \quad - \quad &\; - \\ \hline -\lambda_2 &= 6 \end{aligned}$$

$$\therefore \lambda_2 = -6$$

Put $\lambda_2 = -6$ in Equation (9)

$$\lambda_1 = -3 \times -6 = 18$$

The solution $x_1 = -1$, $x_2 = 3$, $x_3 = 0$, $\lambda_1 = 18$, $\lambda_2 = -6$ gives maximum value of the objective function.

$$Z_{max} = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2 = 1 - 9 - 4 + 18 = 6$$

Q. 8 Find the dual of the following L.P.P. maximize $Z = 2x_1 - x_2 + 3x_3$; Subject to :

$x_1 - 2x_2 + x_3 \ge 4$; $2x_1 + x_3 \le 10$; $x_1 + x_2 + 3x_3 = 20$; $x_1 \cdot x_3 \ge 0$; x_2 is unrestricted. May 2015

Ans. :

Since x_2 is unrestricted, let $x_2 = x_2' - x_2''$

Since objective is of maximization type, convert all constants to "≤" type.

Maximize $Z = 2x_1 - (x_2' - x_2'') + 3x_3$

Constraints :

1. $x_1 - 2x_2 + x_3 \ge 4$

   $\therefore -x_1 + 2x_2 - x_3 \le -4$

   $\therefore -x_1 + 2(x_2' - x_2'') - x_3 \le -4$

2. $2x_1 + 0x_2 + x_3 \le 10$

   $\therefore 2x_1 + 0(x_2' - x_2'') + x_3 \le 10$

3. $x_1 + x_2 + 3x_3 = 20$

   It can be written as :

   $x_1 + x_2 + 3x_3 \le 20$ and $x_1 + x_2 + 3x_3 \ge 20$

   $\therefore x_1 + x_2 + 3x_3 \le 20$ and $-x_1 - x_2 - 3x_3 \le -20$

   $\therefore x_1 + (x_2' - x_2'') + 3x_3 \le 20$ and

   $-x_1 - (x_2' - x_2'') - 3x_3 \le -20$

4. $x_1, x_2', x_2'', x_3 \ge 0$

So the primal is,

Maximum $Z = 2x_1 - 1x_2' + 1x_2'' + 3x_3$

Constraints :

1. $-1x_1 + 2x_2' - 2x_2'' - 1x_3 \le -4$
2. $2x_1 + 0x_2' - 0x_2'' + 1x_3 \le 10$
3. $1x_1 + 1x_2' - 1x_2'' + 3x_3 \le 20$

   $-1x_1 - 1x_2' + 1x_2'' - 3x_3 \le -20$
4. $x_1, x_2', x_2'', x_3 \ge 0$

The dual of above primal is :

Minimize $w = -4y_1 + 10y_2 + 20\,y_3' - 20\,y_3''$

i.e. $w = -4y_1 + 10y_2 + 20\left(y_3' - y_3''\right)$

Constraints :

1. $-1y_1 + 2y_2 + 1y_3' - 1y_3'' \geq 2$

   i.e. $-y_1 + 2y_2 + \left(y_3' - y_3''\right) \geq 2$

2. $2\,y_1 + 0y_2 + 1y_3' - 1y_3'' \geq -1$

   i.e. $-2y_1 - 1y_3' + 1y_3'' \leq 1$

   i.e. $-2y_1 - \left(y_3' - y_3''\right) \geq 1$

3. $-2y_1 - 0y_2 - 1y_3' + 1y_3'' \geq 1$

   i.e. $-2y_1 - \left(y_3' - y_3''\right) \geq 1$

4. $-1y_1 + 1y_2 + 3y_3' - 3y_3'' \geq 3$

   i.e. $-y_1 + y_2 + 3\left(y_3' - y_3''\right) \geq 1$

5. $y_1, y_2, y_3', y_3'' \geq 0$

Put $y_3' - y_3'' = y_3$, where y_3 is unrestricted

$\therefore$ The dual is :

Minimize $w = -4y_1 + 10y_2 + 20y_3$

Constraints :

1. $-1y_1 + 2y_2 + y_3 \geq 2$
2. $-2y_1 - y_3 \leq 1$
3. $-2y_1 - y_3 \geq 1.$
4. $-y_1 + y_2 + 3y_3 \geq 3$
5. $y_1, y_2 \geq 0$, y_3 is unrestricted.

$\therefore$ Combining (2) and (3) constraints, the dual is :

Minimize $w = -4y_1 + 10y_2 + 20y_3$

Constraints :

1. $-y_1 + 2y_2 + y_3 \geq 2$
2. $-2y_1 - y_3 = 1$
3. $-y_1 + y_2 + 3\,y_3 \geq 3$
4. $y_1, y_2 \geq 0$, y_3 is unrestricted.

Q. 9 **Solve the following L. P. P. by simplex method : Maximize $Z = 4x_1 + 3x_2 + 6x_3$; subject to $2x_1 + 3x_2 + 2x_3 \leq 440$; $4x_1 + 3x_3 \leq 470$; $2x_1 + 5x_2 \leq 430$; $x_1, x_2, x_3 \geq 0$.**

May 2015

Ans. :

In the standard form, Maximize

$$Z = 4x_1 + 3x_2 + 6x_3 + 0s_1 + 0s_2 + 0s_3$$

Constraints :

$$2x_1 + 3x_2 + 2x_3 + 1s_1 + 0s_2 + 0s_3 = 440$$

$$4x_1 + 0x_2 + 3x_3 + 0s_1 + 1s_2 + 0s_3 = 470$$

$$2x_1 + 5x_2 + 0x_3 + 0s_1 + 0s_2 + 1s_3 = 430$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

In the table form

	C_B	Z		x_1	x_2	x_3	s_1	s_2	s_3	RHS	Ratio = RHS/KC
		↓ BV	$C_j \rightarrow$	4	3	6	0	0	0		
x_3 enters	0	s_1		2	3	2	1	0	0	440	440/2 = 220
s_2 leaves	0	s_2		4	0	$\boxed{3}$	0	1	0	470	470/3 = 156.67 KR
	0	s_3		2	5	0	0	0	1	430	430/0 =
		$z_j = \sum C_B X_B$		0	0	0	0	0		0	
		$\Delta_j = C_j - z_j$		4	3	6	0	0	0		

↑KC

Iteration : 1										
x_2 enters	0	s_1	$-2/3$	$\boxed{3}$	0	1	$-2/3$	0	380/3	$\frac{380/3}{3} = 42.22$ KR
s_1 leaves	6	x_3	4/3	0	1	0	1/3	0	470/3	$\frac{470/3}{0} =$
	0	s_3	2	5	0	0	0	1	430	$430/5 = 86$
		$z_j = \sum C_B X_B$	8	0	6	0	2	0	940	
		$\Delta_j = C_j - z_j$	-4	3	0	0	-2	0		
Iteration : 2				↑KC						
	3	x_2	$-2/9$	1	0	1/3	$-2/9$	0	380/9	
s_1 leaves	6	x_3	4/3	0	1	0	1/3	0	470/3	
	0	s_3	28/9	0	0	$-5/3$	10/9	1	1970/9	
		$z_j = \sum C_B X_B$	22/3	3	6	1	4/3	0	3200/3	
		$\Delta_j = C_j - z_j$	$-10/3$	0	0	-1	$-4/3$	0		

For maximization, $x_1 = 0,\ x_2 = \frac{380}{9} = 42.2222$

$x_3 = \frac{470}{3} = 156.6667 \quad Z_{max} = \frac{3200}{3} = 1066.6667$

Q. 10 Use Kuhn-Tucker conditions to solve the following N. L. P. P.

Maximise $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

subject to $3x_1 + 2x_2 \le 6$; $x_1, x_2 \ge 0$. May 2015

Ans. : Let $f(x_1, x_2) = 8x_1 + 10x_2 - x_1^2 - x_2^2$; $h = 3x_1 + 2x_2 - 6$ and λ be lagrangian multiplier

$\therefore$ Lagrangian function $L = f - \lambda h$

$$\therefore L = (8x_1 + 10x_2 - x_1^2 - x_2^2) - \lambda(3x_1 + 2x_2 - 6)$$

$$= 8x_1 + 10x_2 - x_1^2 - x_2^2 - 3\lambda x_1 + 2\lambda x_2 - 6\lambda$$

Kuhn-Tucker (K-H) condition are :

1. $\frac{\partial L}{\partial x_1} = 0 \Rightarrow 8 - 2x_1 - 3\lambda = 0 \quad \therefore 2x_1 + 0x_2 + 3\lambda = 8$
2. $\frac{\partial L}{\partial x_2} = 0 \Rightarrow 10 - 2x_2 - 2\lambda = 0 \quad \therefore 0x_1 + 2x_2 + 2\lambda = 10$
3. $\lambda h(x_1, x_2) = 0 \Rightarrow \lambda(3x_1 + 2x_2 - 6) = 0$
4. $h(x_1, x_2) \le 0 \Rightarrow 3x_1 + 2x_2 - 6 \le 0$
5. $\lambda, x_1, x_2 \ge 0$

From K-H condition (3), following two cases arises,

Case I : $\lambda = 0$

From (1), $2x_1 = 8 \quad \therefore x_1 = 4$

From (2), $2x_2 = 10 \quad \therefore x_2 = 5$

Substitute x_1 and x_2 in (4)

LHS $= 3(4) + 2(5) - 6 = 16 \not< 0$

Values of x_1 and x_2 do not satisfy K-H condition (4)

$\therefore$ Reject this case.

Case II : $\lambda \ne 0$

From (3), $3x_1 + 2x_2 - 6 = 0 \quad \therefore 3x_1 + 2x_2 + 0\lambda = 6$ ...(6)

Solving (1), (2) and (6) simultaneously,

$$x_1 = \frac{4}{13},\ x_2 = \frac{33}{13};\ \lambda = \frac{32}{13}$$

Values of x_1 and x_2 satisfy K-H conditions (4) and (5)

$\therefore$ stationary point $X_0 = \left(\frac{4}{13}, \frac{33}{13}\right)$

Since $\lambda > 0$, z is maximum at X_0

Maximum value at $X_0 = z_{max}$

$$= 8 \times \frac{4}{13} + 10 \times \frac{33}{13} - \left(\frac{4}{13}\right)^2 - \left(\frac{32}{13}\right)^2 = \frac{277}{13} = 21.3077$$

Q. 11 **Use duality solve the following L.P.P.**

Maximise $Z = 5x_1 - 2x_2 + 3x_3$

Subject to $2x_1 + 2x_2 - x_3 \geq 2$; $3x_1 - 4x_2 \leq 3$; $x_1 + 3x_3 \leq 5$; $x_1, x_2, x_3 \geq 0$. May 2015

Ans. : When primal is of maximization type, the constraints should be of "≤" type. So multiplying first constraints by '– 1'; it becomes, $-2x_1 - 2x_2 + x_3 \leq -2$.

So the primal is, Maximize $Z = 5x_1 - 2x_2 + 3x_3$

Constraints :

$$-2x_1 - 2x_2 + 1x_3 \leq -2 \quad 3x_1 - 4x_2 + 0x_3 \leq 3$$

$$1x_1 + 0x_2 + 3x_3 \leq 5 \quad x_1, x_2, x_3 \geq 0$$

The dual of the given primal is,

Maximize $w = -2y_1 + 3y_2 + 5y_3$

Constraints :

$$-2y_1 + 3y_2 + 1y_3 \geq 5$$

$$-2y_1 - 4y_2 + 0y_3 \geq -2 \quad \text{i.e. } 2y_1 + 4y_2 - 0y_3 \leq 2$$

$$1y_1 + 0y_2 + 3y_3 \geq 3$$

$$y_1, y_2, y_3 \geq 0$$

Dual in standard form is,

Maximize $w' = -w = 2y_1 - 3y_2 - 5y_3$

i.e. $w' = 2y_1 - 3y_2 - 5y_3 + 0s_1 + 0s_2 + 0s_3 - MA_1 - MA_3$

Constraints :

$$-2y_1 + 3y_2 + 1y_3 - 1s_1 + 0s_2 + 0s_3 + 1A_1 + 0A_3 = 5$$

$$2y_1 + 4y_2 - 0y_3 + 0s_1 + 1s_2 + 0s_3 + 0A_1 + 0A_3 = 2$$

$$1y_1 + 0y_2 + 3y_3 + 0s_1 + 0s_2 - 1s_3 + 0A_1 + 1A_3 = 3$$

$$y_1, y_2, y_3, s_1, s_2, s_3, A_1, A_3 \geq 0$$

In the table form,

	C_B	w' ↓BV	$C_j \rightarrow$	y_1	y_2	y_3	s_1	s_2	s_3	A_1	A_3	RHS	Ratio = RHS/KC
				2	– 3	– 5	0	0	0	– M	– M		
y_3 enters	– M	A_1		– 2	3	1	– 1	0	0	1	0	5	5/1 = 5
A_3 leaves	0	s_2		2	4	0	0	1	0	0	0	2	2/0 = …
	– M	A_3		1	0	$\boxed{3}$	0	0	– 1	0	1	3	3/3 = 1 KR
		$w'_j = \Sigma C_B Y_B$		M	– 3M	– 4M	M	0	M	– M	– M	– 8 M	
		$\Delta_j = C_j - w'_j$		2 – M	3M – 3	4M – 5	– M	0	– M	0	0		

↑KC

	C_B	w' ↓BV	$C_j \rightarrow$	y_1	y_2	y_3	s_1	s_2	s_3	A_1	A_3	RHS	Ratio = RHS/KC
				2	– 3	– 5	0	0	0	– M	– M		
y_2 enters	– M	A_1		– 7/3	3	0	– 1	0	1/3	1		4	4/3 = 1.3
s_2 leaves	0	s_2		2	$\boxed{4}$	0	0	1	0	0		2	2/4 =0.5
	– 5	y_3		1/3	0	1	0	0	– 1/3	0		1	1/0 =
		$w'_j = \Sigma C_B Y_B$		$\frac{7M-5}{3}$	– 3M	– 5	M	0	$\frac{5-M}{3}$	– M		– 4 M	
		$\Delta_j = C_j - w'_j$		$\frac{11-7M}{3}$	3M – 3	0	– M	0	$\frac{M-5}{3}$	0		– 5	

↑KC

	CB	w^s ↓ BV	$C_j \rightarrow$	y_1 3	y_2 −3	y_3 −5	s_1 0	s_2 0	s_3 0	A_1 −M	A_3 −M	RHS	Ratio = RHS/KC
y_2 enters	−M	A_1		−23/6	0	0	−1	−3/4	1/3	1		5/2	
s_1 leaves	−3	y_2		1/3	1	0	0	1/4	0	0		1/2	
	−5	y_3		1/3	0	1	0	0	−1/3	0		1	
		$w_j^s = \Sigma C_B Y_B$		$\frac{23M-19}{6}$	−3	−5	M	$\frac{3M-3}{4}$	$\frac{5-M}{3}$	−M		$\frac{5M+13}{-2}$	
		$\Delta_j = C_j - w_j^s$		$\frac{31-23M}{6}$	0	0	−M	$\frac{3-3M}{4}$	$\frac{M-5}{3}$	0			

Since all values of Δ_j are negative or zero and artificial variable (and penalty M) remains in the solution, the dual of the given LPP has infeasible solution.

So the primal of the given LPP has unbounded solution.

Q. 12 **Find all basic solutions to the following problems.**

Maximize $Z = x_1 + 3x_2 + 3x_3$

Subject to $x_1 + 2x_2 + 3x_3 = 4$

$2x_1 + 3x_2 + 5x_3 = 7$ and $x_1, x_2, x_3 \geq 0$ **Dec 2015**

Ans. : Consider the following LPP

Maximize $Z = x_1 + 3x_2 + 3x_3$

Subject to $x_1 + 2x_2 + 3x_3 = 4$

$2x_1 + 3x_2 + 5x_3 = 7$ and $x_1, x_2, x_3 \geq 0$

Here 2 equations and 3 variables. For basic solution, we can consider at the most 2 variable we set one variable equal to zero each time.

Consider the following table.

No. of solutions	Non-basic variable	Basic variables	Equations and solutions	Is solutions feasible ?	Is solution degenerate ?	Value of Z	Is solutions optimal ?
1.	$x_3 = 0$	x_1, x_2	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $\therefore x_1 = 2, x_2 = 1$	Yes	No	5	Yes
2.	$x_2 = 0$	x_1, x_3	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $\therefore x_1 = 1, x_3 = 1$	Yes	No	4	No
3.	$x_1 = 0$	x_2, x_3	$2x_2 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$ $\therefore x_2 = -1, x_2 = 2$	No	No	3	No

Q. 13 Solve the following LPP using Simplex method Maximize $Z = 6x_1 - 2x_2 + 3x_3$

Subject to $2x_1 - x_2 + 2x_3 \le 2$

$x_1 + 4x_3 \le 4$

$x_1, x_2, x_3 \ge 0$

Dec. 2015

Ans. : Consider the LPP

$$\text{Max } Z = 6x_1 - 2x_2 + 3x_3$$

Subject to $2x_1 - x_2 + 2x_3 \le 2$

$x_1 + 4x_3 \le 4$ and $x_1, x_2, x_3 \ge 0$

As standard form of LPP is,

$$\text{Max } Z = 6x_1 - 2x_2 + 3x_3 + 0.S_1 + 0.S_2$$

Subject to

$$2x_1 - x_2 + 2x_3 + S_1 = 2$$

$$x_1 + 4x_3 + S_2 = 4$$

and $s_1, s_2, x_1, x_2, x_3 \ge 0$

where S_1, S_2 are slack variable.

Here 5 variables and 2 equations. For basic solution, we set $5 - 2 = 3$ variables equal to zero. We set $x_1 = x_2 = x_3 = 0$

$\therefore$ IBFS $S_1 = 2$, $S_2 = 4$

Consider the simplex table.

	C_j	6	-2	3	0	0		Ratio
C_B	B_v	x_1	x_2	x_3	s_1	s_2	b_i	$\theta = \frac{b_i}{a_{ij}}$
0	S_1	$\boxed{2}$	-1	2	1	0	2	$1 \rightarrow$ leaving variable
0	S_2	1	0	4	0	1	4	4
Z_j		0	0	0	0	0		
$C_j - z_j$		6	-2	3	0	0		
		$\uparrow$						

Entering Variable

$\frac{1}{2}R_1$, $R_2 - \frac{1}{2}R_1$

	C_j	6	-2	3	0	0		Ratio
C_B	B_v	x_1	x_2	x_3	s_1	s_2	b_i	$\theta = \frac{b_i}{a_{ij}}$
6	x_1	1	$\frac{-1}{2}$	1	$\frac{1}{2}$	0	1	
0	S_2	0	$\boxed{\frac{1}{2}}$	3	$\frac{-1}{2}$	1	3	$6 \rightarrow$ Leaving variable
Z_j		6	-3	6	3	0		
$C_j - z_j$		0	1	-3	-3	0		
			$\uparrow$					

Entering value $2R_2$, $R_1 + R_2$

	C_j	6	-2	3	0	0		Ratio
C_B	B_v	x_1	x_2	x_3	s_1	s_2	b_i	$\theta = \frac{b_i}{a_{ij}}$
6	x_1	1	0	4	0	1	4	
-2	x_2	0	1	6	-1	2	6	
Z_j		6	-2	12	2	2		
$C_j - z_j$		0	0	-9	-2	-2		

Since $C_i - Z_j \le 0$ for all j. hence the above solution is optimal i.e. the optimal solution is $x_1 = 4, x_2 = 6, x_3 = 0$

$$\therefore \; Z_{max} = 12$$

Q. 14 Solve the following LPP using Dual Simplex method Minimize $Z = 2x_1 + 2x_2 + 4x_3$

$2x_1 + 3x_2 + 5x_3 \ge 2$

$3x_1 + x_2 + 7x_3 \le 3$

$x_1 + 4x_2 + 6x_3 \le 5$

$x_1, x_2, x_3 \ge 0$

Dec. 2015

Ans. :

Consider the following LPP

$$\text{Min } Z = 2x_1 + 2x_2 + 4x_3$$

Subject to $2x_1 + 3x_2 + 5x_3 \ge 2$

$3x_1 + x_2 + 7x_3 \le 3$

$x_1 + 4x_2 + 6x_3 \le 5$

$x_1, x_2, x_3 \ge 0$

let Max $Z^1 = -Z$ are consider the standard form.

$$Z^1 = -2x_1 - 2x_2 - 4x_3 + 0.S_1 + 0.S_2 + 0.S_3$$

Subject to $2x_1 + 3x_2 + 5x_3 - S_1 = 2$

$3x_1 + x_2 + 7x_3 + S_2 = 3$

$x_1 + 4x_2 + 6x_3 + S_3 = 5$

where $S_1, S_2, S_3, x_1, x_2, x_3$ and S_1 is surplus variable, S_2, S_3 are slack variable.

Here 6 variables and 3 equations. For basic solution $6 - 3 = 3$ variables equal to zero.

Put $x_1 = x_2 = x_3 = 0$, basic solution as $S_1 = -2, S_2 = 3, S_3 = 5$

Consider the dual simplex table, multiply first constraint by -1,

	C_j	-2	-2	-4	0	0	0	
C_B	B_v	x_1	x_2	x_3	s_1	s_2	s_3	b_i
0	S_1	-2	$\boxed{-3}$	-5	1	0	0	$-2 \leftarrow$ leaving variable
0	S_2	3	1	7	0	1	0	3
0	S_3	1	4	6	0	0	1	5
Z_j		0	0	0	0	0	0	
$C_j - z_j$		-2	-2	-4	0	0	0	
Ratio $\frac{C_j - z_j}{a_{ij}}$		1	$\frac{2}{3}$	$\frac{4}{5}$	0	0	0	
			$\uparrow$ Entering variable					

$\frac{-1}{3}R_1,\ R_2 + \frac{1}{3}R_1,\ R_3 + \frac{4}{3}R_1$

	C_j	-2	-2	-4	0	0	0	
C_B	B_v	x_1	x_2	x_3	s_1	s_2	s_3	b_i
-2	x_2	$\frac{2}{3}$	1	$\frac{5}{3}$	$\frac{-1}{3}$	0	0	$\frac{2}{3}$
0	S_2	$\frac{7}{3}$	0	$\frac{16}{3}$	$\frac{1}{3}$	1	0	$\frac{7}{3}$
0	S_3	$\frac{5}{3}$	0	$\frac{-2}{3}$	$\frac{4}{3}$	0	1	$\frac{7}{3}$

Since all b_i's ≥ 0 hence the above.

Solution is optimal. i.e. the optimal solution is

$$x_1 = 0,\ x_2 = \frac{2}{3},\ x_3 = 0$$

$$\text{Max } Z^1 = \frac{-4}{3} \qquad \therefore \text{Min } Z = -\text{Max } Z^1$$

$\therefore$ **Min Z** $= \frac{4}{3}$ with optimal solution $x_1 = 0,\ x_2 = \frac{2}{3},\ x_3 = 0$

Q. 15 **Solve the following NLPP using Kuhn-Tueker conditions Maximize**

$Z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

subject to $2x_1 + x_2 \leq 5$ and $x_1, x_2 \geq 0$.

Dec. 2015

Ans. :

Consider the NLPP as

$$\text{Max} \quad Z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$$

Subject to $2x_1 + x_2 \leq 5,\ x_1,\ x_2 \geq 0$

As $\quad f = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

$\quad h = 2x_1 + x_2 - 5$

Now, Kuhn-Tueker conditions are

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial f}{\partial x_1} = 0, \qquad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial f}{\partial x_2} = 0,$$

$$\lambda h(x_1, x_2) = 0, \qquad h(x_1, x_2) \leq 0, \lambda \geq 0$$

$$\therefore\ 10 - 4x_1 - 2\lambda = 0 \quad \text{...(1)}$$

$$4 - 2x_2 - \lambda = 0 \quad \text{...(2)}$$

$$\lambda(2x_1 + x_2 - 5) = 0 \quad \text{...(3)}$$

$$2x_1 + x_2 - 5 \leq 0 \quad \text{...(4)}$$

$$x_1,\ x_2, \lambda \geq 0 \quad \text{...(5)}$$

from Equation (3), $\lambda = 0$ or $2x_1 + x_2 - 5 = 0$

Case (i) If $\lambda = 0$, from (1) and (2)

$$x_1 = 5 \text{ and } x_2 = 2.$$

But this solution does not satisfy the given constraint hence rejected.

Case (ii) If $\lambda \neq 0$, hence $2x_1 + x_2 = 5$ ...(6)

Solving Equation (1), (2) and (6)

$x_1 = \frac{11}{6},\ x_2 = \frac{4}{3},\ \lambda = \frac{4}{3}$ which is an optimal solution.

$$Z_{max} = \frac{91}{6}$$

Q. 16 **Solve the following non-linear programming problem with Kuhn Tueker conditions maximize $z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$**

Subject to $2x_1 + x_2 \leq 5$ and $x_1, x_2 \geq 0$

May 2016

Ans. : Consider NLPP as

Maximum $z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

Subject to $2x_1 + x_2 \leq 5$

$\quad x_1,\ x_2 \geq 0$

As $\quad f = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

$\quad h = 2x_1 + x_2 - 5$

Kuhn-Tueker conditions are

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\lambda h(x_1, x_2) = 0, \quad h(x_1, x_2) \leq 0, \quad \lambda \geq 0$$

Therefore

$$10 - 4x_1 - 2\lambda = 0 \quad \text{...(1)}$$

$$4 - 2x_2 - \lambda = 0 \qquad \ldots(2)$$

$$\lambda(2x_1 + x_2 - 5) = 0 \qquad \ldots(3)$$

$$2x_1 + x_2 - 5 \le 0 \qquad \ldots(4)$$

$$x_1,\ x_2,\ \lambda \ge 0 \qquad \ldots(5)$$

From Equation (3), $\lambda = 0$ or $2x_1 + x_2 - 5 = 0$

Case 1 : If $\lambda = 0$ from (1) and (2),

$4x_1 = 10$ i.e. $x_1 = \frac{10}{4} = 5/2$, $x_2 = 2$

The solution $x_1 = \frac{5}{2}$, $x_2 = 2$ does not satisfy equation (4)

Hence, $\lambda = 0$ does not yield a fessible solution.

Case 2 : If $\lambda \ne 0$, $2x_1 + x_2 - 5 = 0$ ...(6)

Solving the equations (1), (2) and (6)

$$\therefore \quad x_1 = \frac{11}{6}, \quad x_2 = \frac{4}{3}, \quad \lambda = \frac{4}{3}$$

The solution $\lambda = \frac{4}{3}$, $x_1 = \frac{11}{6}$, $x_2 = \frac{4}{3}$ is an optimal solution.

$$\therefore \quad z_{max} = 10\left(\frac{11}{6}\right) + 4\left(\frac{4}{3}\right) - 2\left(\frac{11}{6}\right)^2 - \left(\frac{4}{3}\right)^2$$

$$\therefore \quad z_{max} = \frac{91}{6}$$

Q. 17 Solve the following LPP using simplex method

Maximize $z = 4x_1 + x_2 + 3x_3 + 5x_4$

Subject to

$$-4x_1 + 6x_2 + 5x_3 + 4x_4 \le 20$$
$$-3x_1 - 2x_2 + 4x_3 + x_4 \le 10$$
$$-8x_1 - 3x_2 + 3x_3 + 2x_4 \le 20$$
$$x_1,\ x_2,\ x_3,\ x_4 \ge 0$$

May 2016

Ans. : Consider the LPP as

Maximum $z = 4x_1 + x_2 + 3x_3 + 5x_4$

Subject to

$$-4x_1 + 6x_2 + 5x_3 + 4x_3 \le 20$$
$$-3x_1 - 2x_2 + 4x_3 + x_4 \le 10$$
$$-8x_1 - 3x_2 + 3x_3 + 2x_4 \le 20$$

and $x_1,\ x_2,\ x_3,\ x_4 \ge 0$

Consider the standard form,

Maximum $z = 4x_1 + x_2 + 3x_3 + 5x_4 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3$

Subject to

$$-4x_1 + 6x_2 + 5x_3 + 4x_4 + s_1 = 20$$
$$-3x_1 - 2x_2 + 4x_3 + x_4 + s_2 = 10$$
$$-8x_1 - 3x_2 + 3x_3 + 2x_4 + s_3 = 20$$

and $x_1, x_2, x_3, x_4, s_1, s_2, s_3 \ge 0$ where s_1, s_2, s_3 are slack variables.

Here 3 equations and 7 variables. Hence we set $7 - 3 = 4$ variables equal to zero.

Put $x_1 = x_2 = x_3 = x_4 = 0$

$\therefore$ IBFS is $s_1 = 20$, $s_2 = 10$, $s_3 = 20$

Consider the simplex table

	C_j	4	1	3	5	0	0	0		Ratio,
C_B	B_v	x_1	x_2	x_3	x_4	s_1	s_2	s_3	b_i	$\theta = b_i / a_{ij}$
0	s_1	−4	6	5	4	1	0	0	20	5 → leaving variable
0	s_2	−3	−2	4	1	0	1	0	10	10 (Min + ve)
0	s_3	−8	−3	3	2	0	0	1	20	10
z_j		0	0	0	0	0	0	0		
$C_j - z_j$		4	1	3	5	0	0	0		
					↑ Entering variable (Max + ve)					

$$\frac{1}{4}R_1, \quad R_2 - \frac{1}{4}R_1, \quad R_3 = \frac{1}{2}R_1$$

	C_j	4	1	3	5	0	0	0		Ratio,
C_B	B_v	x_1	x_2	x_3	x_4	s_1	s_2	s_3	b_i	$\theta = b_i / a_{ij}$
5	x_4	−1	3/2	5/4	1	1/4	0	0	5	−5
0	s_2	−2	−7/2	−11/4	0	−1/4	1	0	5	−5/2
0	s_3	−6	−6	1/2	0	−1/2	0	1	10	−5/3
z_j		−5	15/2	25/4	5	5/4	0	0		
$C_j - z_j$		9	−13/2	−13/4	0	−5/4	0	0		
		↑ Entering variable								

Since all entries in the column of ratio, are negative. Hence the problem has unbounded solution.

Q. 18 Since the following LPP using Dual simplex method

Maximum $z = -3x_1 - 2x_2$

Subject to

$$x_1 + x_2 \ge 1$$
$$x_1 + x_2 \le 7$$
$$x_1 + 2x_2 \ge 10$$
$$x_2 \le 3$$
$$x_1, x_2 \ge 0$$

May 2016

Ans. : Consider the given LPP as

Maximum $z = -3x_1 - 2x_2$

Subject to
$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \geq 10$$
$$x_2 \leq 3 \text{ and } x_1, x_2 \geq 0$$

Consider the standard form

Maximum $z = -3x_4 - 2x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 + 0 \cdot s_3 + 0 \cdot s_4$

Subject to,

$x_1 + x_2 - s_1 = 1$, $\quad x_1 + x_2 + s_2 = 7$

$x_1 + x_2 - s_3 = 10$, $\quad x_2 + s_4 = 3$

and $s_1, s_2, s_3, s_4, x_1, x_2 \geq 0$ where s_2, s_4 are slack variables and s_1, s_3 are surplus variables.

Here 6 variables and 4 equations, hence put $x_1 = x_2 = 0$

$\therefore$ initial solution as

$s_1 = -1$, $s_2 = 7$, $s_3 = -10$, $s_3 = 3$

Consider the dual simplex table

	c_j	-3	-2	0	0	0	0	
C_B	B_v	x_1	x_2	s_1	s_2	s_3	s_4	b_i
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	-2	0	0	1	0	-10 ←leawing variable
0	s_4	0	1	0	0	0	1	3 [Maximum-ve]
z_j		0	0	0	0	0	0	
$c_j - z_j$		-3	-2	0	0	0	0	
$\frac{c_j - z_j}{a_j}$		3	1					
		Entering variable ↑ (Minimum Positive)						

$\frac{-1}{2} R_3$, $\quad R_1 - \frac{1}{2} R_3$, $\quad R_2 + \frac{1}{2} R_3$, $\quad R_4 + \frac{1}{2} R_3$

	c_j	-3	-2	0	0	0	0	b_i
C_B	B_v	x_1	x_2	s_1	s_2	s_3	s_4	
0	s_1	-1/2	0	1	0	-1/2	0	4
0	s_2	1/2	0	0	1	1/2	0	2
-2	x_2	1/2	1	0	0	-1/2	0	5
0	s_4	$\boxed{-1/2}$	0	0	0	½	1	-2←leaving variable
z_j		-1	-2	0	0	1	0	
$c_j - z_j$		-2	0	0	0	-1	0	
$\frac{c_j - z_j}{a_j}$		4	0	0	0	-2	0	
		↑ Entering variable						

$-2R_4$, $R_1 - R_4$, $R_3 + R_4$, $R_2 + R_4$

	C_j	-3	-2	0	0	0	0	b_i
C_B	B_v	x_1	x_2	s_1	s_2	s_3	s_4	
0	s_1	0	0	1	0	-1	-1	6
0	s_2	0	0	0	1	1	1	0
-2	x_2	0	1	0	0	0	1	3
-3	x_1	1	0	0	0	-1	-2	4
z_j		-3	-2	0	0	3	4	
$c_j - z_j$		0	0	0	0	-3	-4	

Since the solution is feasible and $C_j - z_j \leq 0$ the above solution is optimal

$\therefore$ The optimal solution is,

$x_1 = 4, x_2 = 3$

$\therefore$ $\mathbf{Z_{max} = -18}$

Q. 19 Solve the following LPP by simplex method

Minimize $z = 3x_1 + 2x_2$

Subject to $3x_1 + 2x_2 \leq 18$

$$\mathbf{0 \leq x_1 \leq 4}$$
$$\mathbf{0 \leq x_2 \leq 6}$$
$$\mathbf{x_1, x_2 \geq 0}$$

Dec. 2016

Ans. : The following LPP is

Min $z = 3x_1 + 2x_2$

Subject to $3x_1 + 2x_2 \leq 18 \quad \therefore \quad x_1 \leq 4$

$x_2 \leq 6 \quad$ and $\quad x_1, x_2 \geq 0$

consider the standard form

min $z = 3x_1 + 2x_2 \quad \therefore \quad z' = -z = -3x_1 - 2x_2$

$\therefore$ Consider max z' = min z

$\therefore$ max $z' = -3x_1 - 2x_2 + 0.s_1 + 0.s_2 + 0.s_3$

Subject to

$$3x_1 + 2x_2 + s_1 = 18$$
$$x_1 + s_2 = 4$$
$$x_2 + s_3 = 6 \text{ and } s_1, s_2, s_3, x_1, x_2 \geq 0.$$

Where s_1, s_2, s_3 are slack variables.

Here 5 variables and 3 equations. Hence it is required to set $5 - 3 = 2$ variables equal to zero (non-basic variables).

Put $x_1 = x_2 = 0$

$\therefore$ IBFS as $s_1 = 18, s_2 = 4, s_3 = 6.$

Consider the simplex table

	C_j	-3	-2	0	0	0		Ratio
C_B	B_v	x_1	x_2	s_1	s_2	s_3	b_i	$\theta = \frac{b_i}{a_{ij}}$
0	s_1	3	2	1	0	0	18	
0	s_2	1	0	0	1	0	4	
0	s_3	0	1	0	0	1	6	
	z_j	0	0	0	0	0		
	$C_j - z_j$	-3	-2	0	0	0		

Since $C_j - z_j \le 0$, the above solution is optimal solution.

i.e. $x_1 = 0, \quad x_2 = 0$

$\therefore \; z'_{max} = 0 \qquad \therefore z_{min} = -z'_{max}$

$\therefore \; z_{min} = 0$

Q. 20 Use Penalty method to solve the following : LPP minimize $z = 2x_1 + 3x_2$ Subject to $x_1 + x_2 \ge 5; x_1 + 2x_2 \ge 6; \; x_1, x_2 \ge 0$ Dec. 2016

Ans. : Consider the LPP

$$\text{Min } z = 2x_1 + 3x_2$$

Subject to $x_1 + x_2 \ge 5 \qquad x_1 + 2x_2 \ge 6$

$x_1, x_2 \ge 0$

The standard form is

$$\text{Min} \quad z = 2x_1 + 3x_2 - 0.S_1 + 0.S_2 + MA_1 + MA_2$$

Subject to $x_1 + x_2 - S_1 + A_1 = 5$

$x_1 + 2x_2 - S_2 + A_2 = 6$

Where $S_1, S_2, A_1, A_2, x_1, x_2 \ge 0$ and

S_1, S_2 are surplus variables and A_1, A_2 are artificial variables.

Here 2 equations and 6 variables. Hence it is required to $6 - 2 = 4$ variables equal to zero.

Put $x_1 = x_2 = S_1 = S_2 = 0,$

$\therefore$ IBFS as, $A_1 = 5 \;;\; A_2 = 6$

Consider the simplex table

	Cj	2	3	0	0	M	M		Ratio $Q = b/a_{ij}$
C_B	B_v	x_1	X_2	S_1	S_1	A_1	A_2	bi	
M	A_1	1	1	-1	0	1	0	5	5/1 = 5 leaving
M	A_2	1	$\boxed{2}$	0	-1	0	1	6	6/2 = 3 → variable
	z_j	2M	3M	$-M$	$-M$	M	M		
	$C_j - z_j$	$2-2M$	$3-3M$	M	M	0	0		
			↑ Entering variable (Min-negative)						

$\frac{1}{2} R_2, R_1 - \frac{1}{2} R_2,$

	C_j	2	3	0	0	M	M	b_i	Ratio
C_B	B_r	x_1	x_2	S_1	S_2	A_1	A_2		$\theta = b/a_{ij}$
M	A_1	$\boxed{1/2}$	0	-1	1/2	1	$-1/2$	2	$\frac{2}{1/2} = 4 \rightarrow$ leaving variable
3	x_2	1/2	1	0	$-1/2$	0	1/2	3	$\frac{3}{1/2} = 6$
	z_j	$\frac{3+M}{2}$	3	$-M$	$\frac{M-3}{2}$	M	$\frac{3-M}{2}$		
	$C_j - z_j$	$\frac{-1-M}{2}$	0	M	$\frac{3-M}{2}$	0	$\frac{M-3}{2}$		
		↑ Entering variable (Min negative)							

$2R_1, R_2 - R_1$

	C_j	2	3	0	0	M	M	b_i
C_B	B_r	x_1	x_2	S_1	S_2	A_1	A_2	
2	x_1	1	0	-2	1	2	-1	4
3	x_2	0	1	1	-1	-1	0	1
	z_j	2	3	-1	-1	1	-2	
	$C_j - z_j$	0	0	1	1	$M-1$	$M+2$	

Since all $C_j - z_j \ge 0$, Hence the above solution is optimal.

$\therefore$ The optimal solution is $x_1 = 4, x_2 = 1$

$\therefore \; z_{min} = 2(4) + 3(1) = 8 + 3 = 11$

$\therefore \; z_{min} = 11$