Evaluation of Integral (circle).

Equation of circle in a complex plane is

$$|z - z_0| = r$$

where $z_0 = a + ib = (a, b) \Rightarrow$ centre of circle

and r is radius.

$$z = x + iy$$

$$|z - z_0| = r$$

$$|x + iy - (a + ib)| = r$$

$$(x-a)^2 + (y-b)^2 = r^2$$

To solve $\int_C f(z)\,dz$

where C is circle $|z - z_0| = r$

Put $z - z_0 = re^{i\theta}$

① Evaluate $\int_C \frac{dz}{(z-z_0)^{n+1}}$ where n is integer

& C is circle $|z - z_0| = r$.

Put $z - z_0 = re^{i\theta}$

$$dz - 0 = re^{i\theta}\, i\,d\theta.$$

$$dz = re^{i\theta}.\, i\,d\theta$$

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θ varies from 0 to 2π.

$$I = \int_C \frac{dz}{(z-z_0)^{n+1}}$$

$$= \int_0^{2\pi} \frac{re^{i\theta}\, i\,d\theta}{(re^{i\theta})^{n+1}}$$

$$= \frac{ri}{r^{n+1}} \int_0^{2\pi} \frac{e^{i\theta}}{e^{i(n+1)\theta}}\, d\theta$$

$$= \frac{\cancel{r}i}{r^n \cdot \cancel{r}} \int_0^{2\pi} e^{i\theta} \cdot e^{-i(n\theta+\theta)}\, d\theta.$$

$$= \frac{i}{r^n} \int_0^{2\pi} e^{i(\cancel{\theta}-n\theta-\cancel{\theta})}\, d\theta.$$

$$= \frac{i}{r^n} \int_0^{2\pi} e^{-in\theta}\, d\theta.$$

$$= \frac{i}{r^n} \left[\frac{e^{-in\theta}}{-in} \right]_0^{2\pi} \Rightarrow n \neq 0.$$

$$= \frac{i}{r^n} \left[\frac{e^{-in2\pi}}{-in} - \left(\frac{e^{-in(0)}}{-in} \right) \right]$$

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$$= \frac{i}{r^n}\left[\frac{1}{in}\left(-e^{-2in\pi}+1\right)\right]$$

$$= \frac{\cancel{i}}{r^n \cancel{i} n}\left[1-e^{-2in\pi}\right]$$

$$\left[e^{-i\theta} = \cos\theta - i\sin\theta\right]$$

$$e^{-2in\pi} = \cos 2n\pi - i\sin 2n\pi$$

$$= (1) - i(0)$$

$$= 1.$$

$$I = \frac{1}{nr^n}\left[1-1\right]$$

$$I = 0. \quad \text{when } n \neq 0$$

<u>for</u> $\underline{n=0}$

$$I = \int_C \frac{dz}{(z-z_0)^{n+1}}$$

$$= \frac{i}{r^n}\int_0^{2\pi} e^{-in\theta}\, d\theta. \quad \text{for } n=0$$

$$= i\int_0^{2\pi} d\theta.$$

$$= i\left[\theta\right]_0^{2\pi}$$

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$= i[2\pi - 0]$

$I = 2\pi i$ when $n = 0$.

(2) Evaluate $\int_C \frac{2z+3}{z} dz$ where C is

i) upper half of a circle $|z| = 2$

ii) lower half of a circle $|z| = 2$

iii) whole circle $|z| = 2$ in anti clockwise direction.

Soln:-

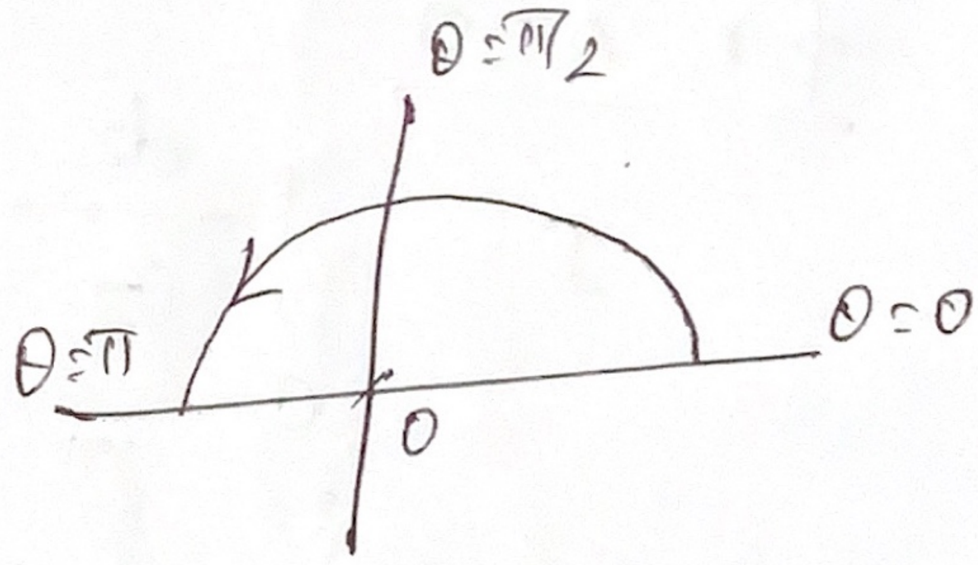
Given circle is $|z| = 2$

Put $z = 2e^{i\theta}$

$dz = 2e^{i\theta} i d\theta$

i) upper half of $|z| = 2$

θ varies from 0 to π.



$I = \int_C \frac{2z+3}{z} dz$.

$= \int_0^{\pi} \frac{2(2e^{i\theta})+3}{\cancel{2e^{i\theta}}} \cancel{2e^{i\theta}} i d\theta$.

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$$= \int_0^{\pi} (4e^{i\theta} + 3)\, i\, d\theta$$

$$= i \int_0^{\pi} (4e^{i\theta} + 3)\, d\theta$$

$$= i \left[\frac{4e^{i\theta}}{i} + 3\theta \right]_0^{\pi}$$

$$= i \left[\frac{4e^{i\pi}}{i} + 3\pi - \left[\frac{4e^{i(0)}}{i} + 3(0) \right] \right]$$

$$= i \left[\frac{4e^{i\pi}}{i} + 3\pi - \frac{4}{i} \right]$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$= (-1) + i(0)$$

$$= -1$$

$$= i \left[\frac{4(-1)}{i} + 3\pi - \frac{4}{i} \right]$$

$$= i \left[\frac{-4}{i} + 3\pi - \frac{4}{i} \right]$$

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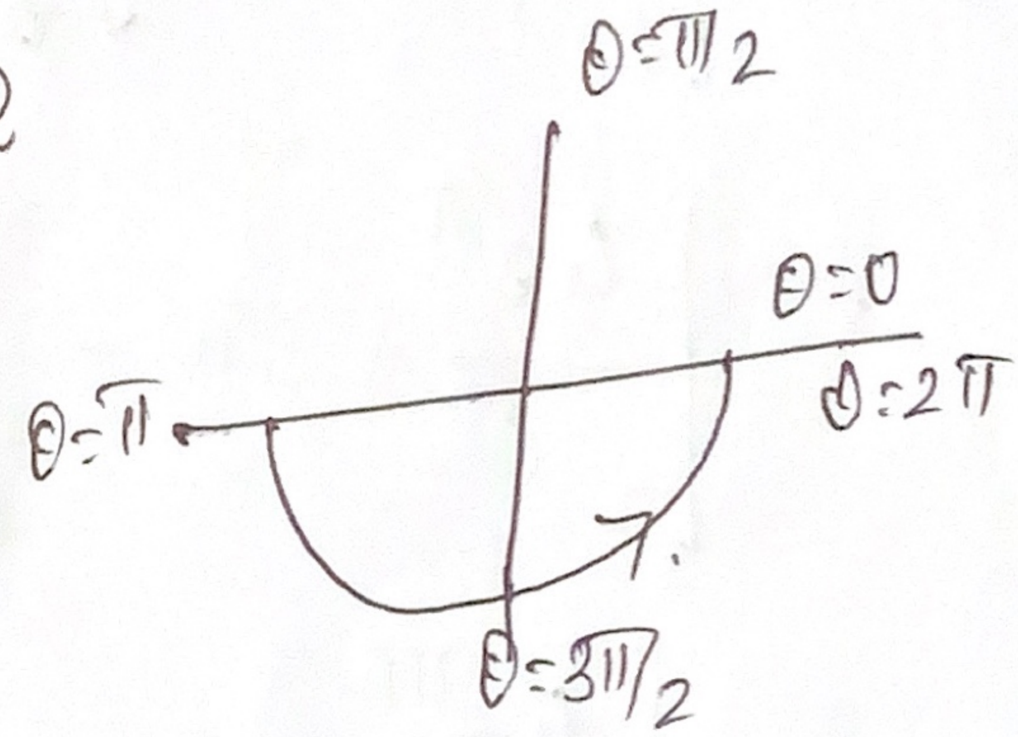
$$I = i\left[\frac{-8}{i} + 3\pi\right]$$

$$= \frac{-8i}{i} + 3\pi i$$

$$I = 3\pi i - 8.$$

ii) lower half of $|z| = 2$

θ varies from π to 2π.



$$I = \int_C \frac{2z+3}{z}\, dz$$

$$= \int_{\pi}^{2\pi} \frac{2(2e^{i\theta})+3}{2e^{i\theta}}\, 2e^{i\theta}\, i\, d\theta.$$

$$= i\int_{\pi}^{2\pi} (4e^{i\theta}+3)\, d\theta.$$

$$= i\left[\frac{4e^{i\theta}}{i} + 3\theta\right]_{\pi}^{2\pi}$$

$$= i\left[\frac{4e^{i2\pi}}{i} + 3(2\pi) - \frac{4e^{i\pi}}{i} - 3\pi\right]$$

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$e^{2i\pi} = \cos 2\pi + i\sin 2\pi$, $e^{i\pi} = \cos\pi + i\sin\pi$

$= 1 + i(0)$ $\qquad\qquad$ $= -1 + i(0)$

$= 1$ $\qquad\qquad$ $= -1$

$$I = i\left[\frac{4}{i} + 6\pi - \frac{4}{i}(-1) - 3\pi\right]$$

$$= \cancel{i\left[\frac{4}{i} + 6\pi - \frac{4}{i} - 3\pi\right]} = i\left[\frac{4}{i} + 6\pi + \frac{4}{i} - 3\pi\right]$$

$$= \cancel{i(3\pi)}$$

$$\cancel{I = 3\pi i}$$

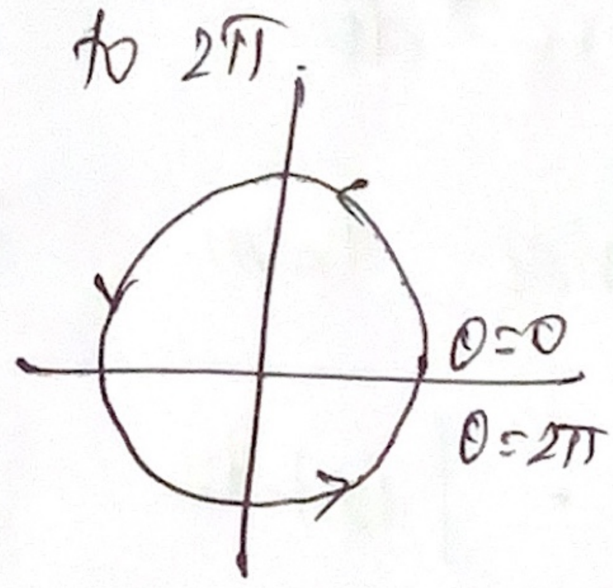
$$I = i\left[\frac{8}{i} + 3\pi\right]$$

$$I = 8 + 3\pi i$$

iii) whole circle of $|z| = 2$

θ varies from 0 to 2π.

$$I = \int_C \frac{2z+3}{z}\,dz$$



$$= i\int_0^{2\pi} (4e^{i\theta} + 3)\,d\theta$$

$$= i\left[\frac{4e^{i\theta}}{i} + 3\theta\right]_0^{2\pi}$$

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$$= i\left[\frac{4e^{i2\pi}}{i} + 3(2\pi) - \frac{4e^{i(0)}}{i} - 3(0)\right]$$

$$= i\left[\cancel{\frac{4}{i}} + 6\pi - \cancel{\frac{4}{i}}\right]$$

$$I = 6\pi i$$

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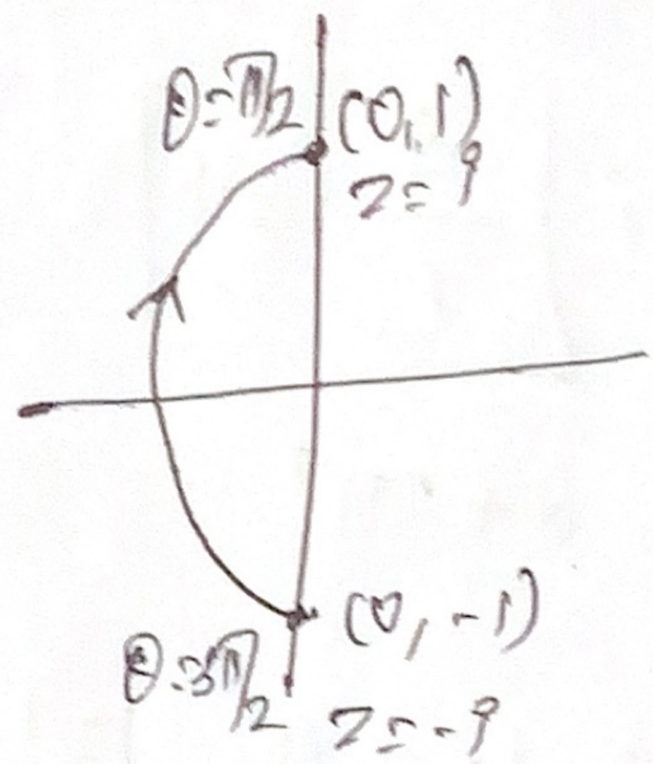
③ Evaluate $\int_C |z|\,dz$ where C is left half of a unit circle from $z=-i$ to i

$z = -i$
$= (0,-1)$

$z = i$
$= (0,1)$



Given z is a unit circle

$\Rightarrow |z| = 1.$

Put $z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$

θ varies from $3\pi/2$ to $\pi/2$

$$I = \int_C |z|\,dz$$

$$= \int_{3\pi/2}^{\pi/2} 1 \cdot i e^{i\theta}\,d\theta. \quad [|z| = 1]$$

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$$= i\left[\frac{e^{i\theta}}{i}\right]_{3\pi/2}^{\pi/2}$$

$$I = \left[e^{i\pi/2} - e^{i3\pi/2}\right]$$

$$e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2$$

$$= 0 + i(1)$$

$$e^{i\pi/2} = i$$

$$e^{i3\pi/2} = \cos 3\pi/2 + i\sin 3\pi/2$$

$$= 0 + i(-1)$$

$$e^{i3\pi/2} = -i$$

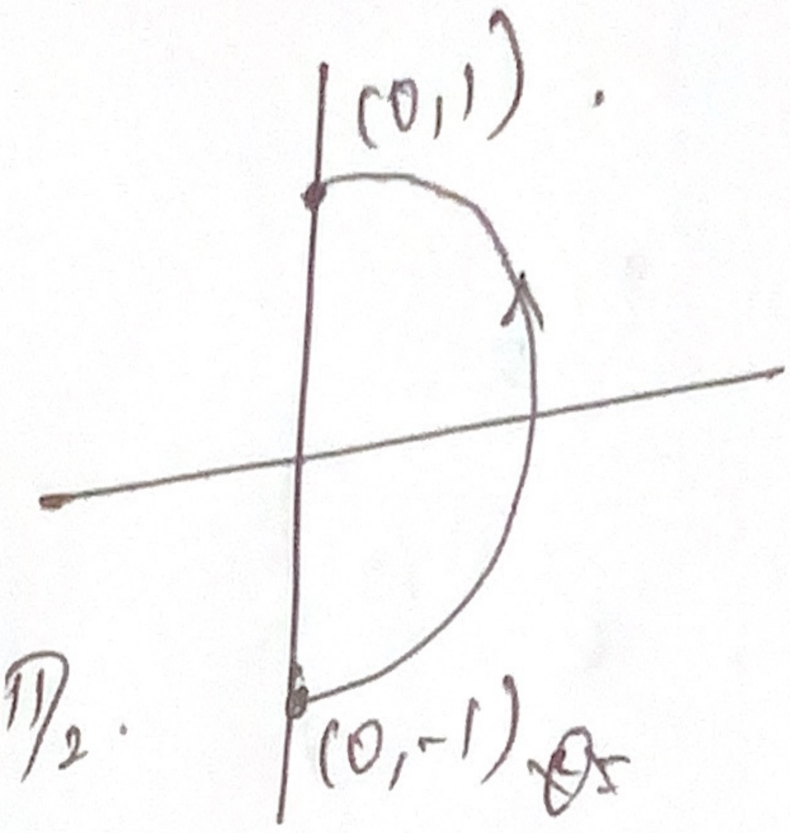
$$I = [i - (-i)]$$

$$\boxed{I = 2i}$$

(4) Evaluate $\int_C |z|\,dz$ where C is right half of a unit circle from $z = -i$ to i

$z = -i \quad (0,-1)$.

$z = i \quad (0,1)$



θ varies from $-\pi/2$ to $\pi/2$.

$$I = \int_C |z|\,dz$$

$$= \int_{-\pi/2}^{\pi/2} 1 \cdot ie^{i\theta}\,d\theta \qquad [|z| = 1]$$

$$= i \int_{-\pi/2}^{\pi/2} e^{i\theta}\,d\theta.$$

$$= i \left[\frac{e^{i\theta}}{i}\right]_{-\pi/2}^{\pi/2}$$

$$I = i\left[\frac{e^{i\pi/2}}{i} - \frac{e^{-i\pi/2}}{i}\right]$$

$$e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2$$

$$= 0 + i(1) = i$$

$$e^{-i\pi/2} = \cos\pi/2 - i\sin\pi/2$$

$$= 0 - i(1) = -i.$$

$$I = \frac{i}{i}\left[i - (-i)\right]$$

$$I = 2i$$

5) Show that $\int_C \log z\,dz = 2\pi i$ where C is $|z| = 1$

Soln:-

Put $z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$.

θ varies from 0 to 2π

$\log z = \log e^{i\theta}$

$= i\theta \log e$

$\log z = i\theta$.

$$I = \int_C \log z\,dz$$

$$= \int_0^{2\pi} i\theta\,(i e^{i\theta} d\theta)$$

$$= i^2 \int_0^{2\pi} \underset{u}{\theta} \cdot \underset{v}{e^{i\theta}}\, d\theta$$

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$$\left[i^2 = -1 \right]$$

$$I = -\left[\theta \left(\frac{e^{i\theta}}{i} \right) - (1) \frac{e^{i\theta}}{i^2} \right]_0^{2\pi}$$

$$= -\left[2\pi \frac{e^{2i\pi}}{i} - \frac{e^{i2\pi}}{i^2} - \left[0 - \frac{e^{i(0)}}{-1} \right] \right]$$

$$= -\left[\frac{2\pi (1)}{i} - \frac{1}{(-1)} - 1 \right]$$

$$= -\left[\frac{2\pi}{i} + 1 - 1 \right]$$

$$I = -\frac{2\pi}{i}$$

$$= \frac{-2\pi}{i} \times \frac{i}{i}$$

$$= \frac{-2\pi i}{i^2} \qquad \left[i^2 = -1 \right]$$

$$= \frac{-2\pi i}{-1}$$

$$\boxed{I = 2\pi i}$$

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Exercise

1. Evaluate $\int_C \left(z^2 + \frac{3}{z^4}\right) dz$ where C is upper half of unit circle from $(1,0)$ to $(-1,0)$.

2. Evaluate $\int_C (z - z^2)\, dz$ where C is lower half of a unit circle.

3. Evaluate $\int_C \bar{z}\, dz$ where C is

   i) $|z| = 1$

   ii) $|z-2| = 3$

Hint: ii) $|z-2| = 3$

$$z - 2 = 3e^{i\theta}$$

$$dz = 3e^{i\theta} i\, d\theta$$

$$\theta : 0 \text{ to } 2\pi.$$

$$z = 2 + 3e^{i\theta}$$

$$\bar{z} = 2 + 3e^{-i\theta}$$

$$\bar{z} = 2 + 3(\cos\theta - i\sin\theta)$$

$$\bar{z} = 2 + 3\cos\theta - 3i\sin\theta$$

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Exercise (1) Solution.

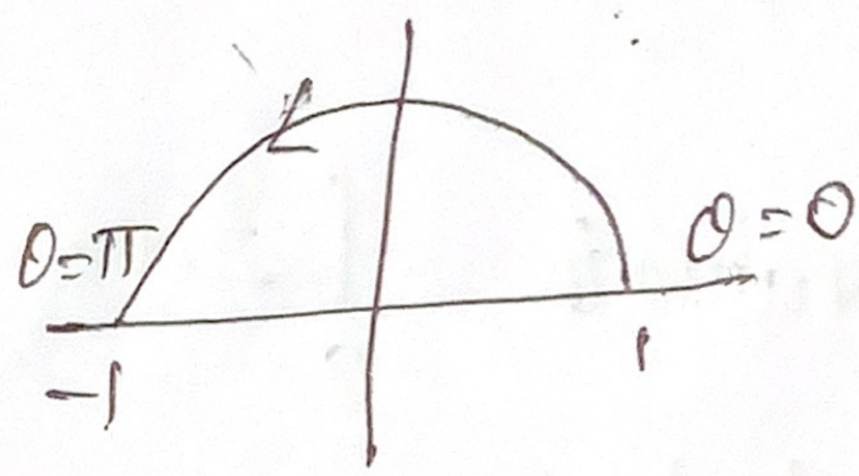
$$\int_C \left(z^2 + \frac{3}{z^4}\right) dz$$

where C is upper half of unit circle from $(1,0)$ to $(-1,0)$.

Solution

Gn: upper half of unit circle

(i) $|z| = 1$



Put $z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$

θ varies from 0 to π.

$$I = \int_C \left(z^2 + \frac{3}{z^4}\right) dz$$

$$= \int_0^{\pi} \left[(e^{i\theta})^2 + \frac{3}{(e^{i\theta})^4} \right] e^{i\theta}\, i\, d\theta$$

$$= \int_0^{\pi} \left[e^{2i\theta} + \frac{3}{e^{4i\theta}} \right] i\, e^{i\theta}\, d\theta$$

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$$I = i\int_0^{\pi}\left[e^{2i\theta}\cdot e^{i\theta} + 3e^{-4i\theta}e^{i\theta}\right]d\theta$$

$$= i\int_0^{\pi}\left[e^{3i\theta} + 3e^{-3i\theta}\right]d\theta$$

$$= i\left[\frac{e^{3i\theta}}{3i} + \cancel{3}\frac{e^{-3i\theta}}{-\cancel{3}i}\right]_0^{\pi}$$

$$= i\left[\frac{e^{3i\theta}}{3i} - \frac{e^{-3i\theta}}{i}\right]_0^{\pi}$$

$$= i\left[\frac{e^{3i\pi}}{3i} - \frac{e^{-3i\pi}}{i} - \left[\frac{e^{3i(0)}}{3i} - \frac{e^{-3i(0)}}{i}\right]\right]$$

$$e^{3i\pi} = \cos 3\pi + i\sin 3\pi$$

$$= (-1) + i(0)$$

$$e^{3i\pi} = -1$$

$$e^{-3i\pi} = \cos 3\pi - i\sin 3\pi$$

$$= (-1) - i(0)$$

$$e^{-3i\pi} = -1$$

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$$= i\left[\frac{-1}{3i} - \frac{(-1)}{i} - \frac{1}{3i} + \frac{1}{i}\right]$$

$$= i\left[\frac{-1}{3i} + \frac{1}{i} - \frac{1}{3i} + \frac{1}{i}\right]$$

$$= i\left[\frac{-2}{3i} + \frac{2}{i}\right]$$

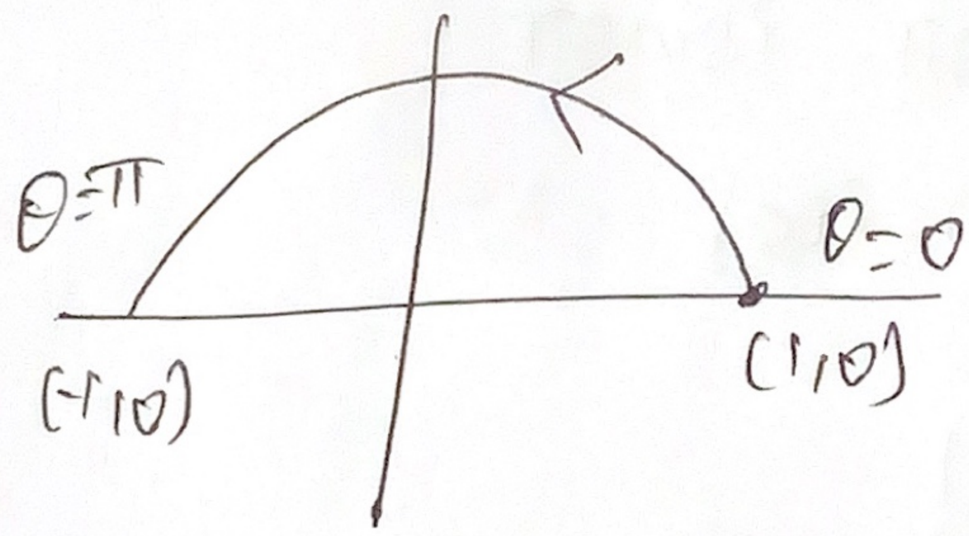
$$= \cancel{i} \times \frac{2}{\cancel{i}}\left[\frac{-1}{3} + 1\right]$$

$$= 2\left(\frac{2}{3}\right)$$

$$\boxed{I = 4/3}$$

Evaluate $\int_C \left(\frac{z^2+3}{z^4}\right) dz$ where C is upper half of unit circle from $(1,0)$ to $(-1,0)$.

Soln:-



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Given: upper half of unit circle

i.e. $|z| = 1$

Put $z = e^{i\theta}$

$dz = e^{i\theta} \cdot i\, d\theta$.

θ varies from 0 to π.

$$I = \int_C \left(\frac{z^2+3}{z^4} \right) dz$$

$$= \int_0^{\pi} \left(\frac{(e^{i\theta})^2+3}{(e^{i\theta})^4} \right) i e^{i\theta}\, d\theta.$$

$$I = \int_0^{\pi} \left(\frac{e^{2i\theta}+3}{e^{4i\theta}} \right) i e^{i\theta}\, d\theta.$$

$$= i \int_0^{\pi} \frac{e^{2i\theta}+3}{e^{4i\theta} \cdot e^{-i\theta}}\, d\theta.$$

$$= i \int_0^{\pi} \left(\frac{e^{2i\theta}+3}{e^{3i\theta}} \right) d\theta.$$

$$= i \int_0^{\pi} \left[\frac{e^{2i\theta}}{e^{3i\theta}} + \frac{3}{e^{3i\theta}} \right] d\theta$$

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$$= i \int_0^{\pi} \left(\frac{1}{e^{i\theta}} + \frac{3}{e^{3i\theta}} \right) d\theta$$

$$= i \int_0^{\pi} \left(e^{-i\theta} + 3 e^{-3i\theta} \right) d\theta .$$

$$= i \left[\frac{e^{-i\theta}}{-i} + \cancel{3} \frac{e^{-3i\theta}}{-\cancel{3}i} \right]_0^{\pi}$$

$$= i \left[\frac{e^{-i\pi}}{-i} - \frac{e^{-3i\pi}}{i} - \left[\frac{e^{-i(0)}}{-i} - \frac{e^{-3i(0)}}{i} \right] \right]$$

$$= i \left[\frac{e^{-i\pi}}{-i} - \frac{e^{-3i\pi}}{i} + \frac{1}{i} + \frac{1}{i} \right]$$

$$= i \quad e^{-i\pi} = \cos\pi - i\sin\pi$$

$$= (-1) - i(0)$$

$$e^{-i\pi} = -1$$

$$e^{-3i\pi} = \cos 3\pi - i\sin 3\pi$$

$$= -1 - i(0)$$

$$e^{-3i\pi} = -1$$

$$\cancel{= i \left[\frac{e^{-i\pi}}{-i} \right] -}$$

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$$I = \int_C (z - z^2)\,dz$$

$$= \int_{\pi}^{2\pi} \left[e^{i\theta} - (e^{i\theta})^2\right] e^{i\theta} \cdot i\,d\theta.$$

$$= \int_{\pi}^{2\pi} \left[e^{i\theta} - e^{2i\theta}\right] i e^{i\theta}\,d\theta.$$

$$= i\int_{\pi}^{2\pi} \left[e^{i\theta}\cdot e^{i\theta} - e^{2i\theta} e^{i\theta}\right] d\theta.$$

$$= i\int_{\pi}^{2\pi} \left[e^{2i\theta} - e^{3i\theta}\right] d\theta.$$

$$= i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_{\pi}^{2\pi}$$

$$= i\left[\frac{e^{2i(2\pi)}}{2i} - \frac{e^{3i(2\pi)}}{3i} - \left[\frac{e^{2i\pi}}{2i} - \frac{e^{3i\pi}}{3i}\right]\right]$$

$$= i\left[\frac{e^{4i\pi}}{2i} - \frac{e^{6i\pi}}{3i} - \frac{e^{2i\pi}}{2i} + \frac{e^{3i\pi}}{3i}\right]$$

$$e^{4i\pi} = \cos 4\pi + i\sin 4\pi = 1 + i(0)$$

$$= 1$$

$$e^{6i\pi} = \cos 6\pi + i\sin 6\pi = 1 + i(0)$$

$$= 1$$

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$$= i\left[\frac{-1}{-i} - \frac{(-1)}{i} + \frac{2}{i}\right]$$

$$= i\left[\frac{+1}{i} + \frac{1}{i} + \frac{2}{i}\right]$$

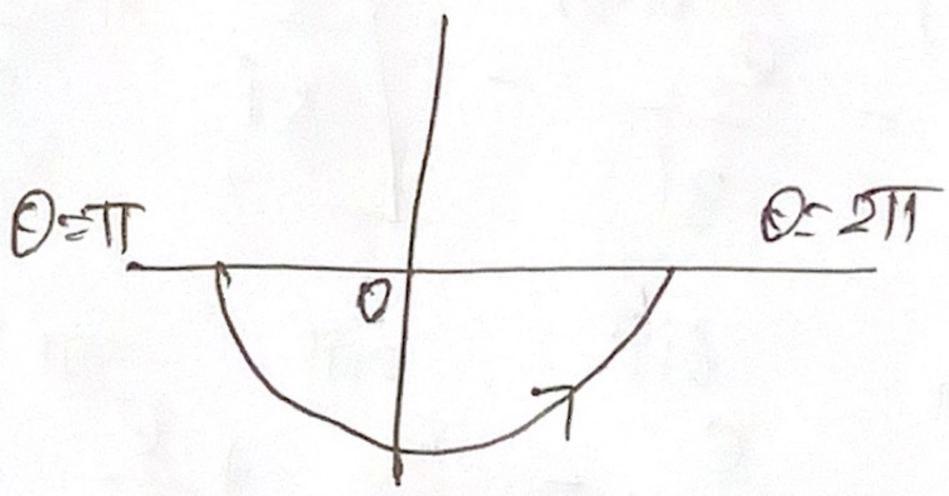
$$= i\left[\frac{4}{i}\right]$$

$$\boxed{I = 4}$$

Exercise 2 Solution

$\int_C (z - z^2)\, dz$ where C is lower half a unit circle

Soln:-



Given: lower half of a unit circle

i.e) $|z| = 1$

$\theta : \pi$ to 2π.

Put $z = e^{i\theta}$

$dz = e^{i\theta}\, i\, d\theta$

$$e^{2i\pi} = \cos 2\pi + i\sin 2\pi$$

$$= 1 + i(0) = 1.$$

$$e^{3i\pi} = \cos 3\pi + i\sin 3\pi$$

$$= -1 + i(0) = -1$$

$$I = i\left[\frac{1}{2i} - \frac{1}{3i} - \frac{1}{2i} + \frac{(-1)}{3i}\right]$$

$$= i\left[\frac{1}{2i} - \frac{1}{3i} - \frac{1}{2i} - \frac{1}{3i}\right]$$

$$= \cancel{i}\left[\frac{-2}{3\cancel{i}}\right]$$

$$\boxed{I = -2/3}$$