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SINCE 1960

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S.E. : Discrete Structures

Notes & GQ

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CONTENTS

Vidyalankar Institute of Technology

S.E.[CMPN/INFT] : Discrete Structures

Syllabus

1. **Set Theory :**
   Sets, Venn Diagram, Set membership of tables.
   Laws of set theory
   Partition of sets.
   Power set.

2. **Logic :**
   Propositions and logical operations.
   Truth tables, Equivalence, Implications.
   Laws of Logic
   Mathematical Induction and Quantifiers.

3. **Relations, Digraphs and Lattice :**
   Relations, paths and diagraphs.
   Properties and types of binary relations.
   Manipulation of relations, closures, Warshall's algorithm.
   Equivalence and Partial ordered relations.
   Posets and Hasse diagram.
   Lattice.

4. **Function and Pigeon Hole Principle :**
   Definition and types of functions : injective, surjective, bijective.
   Composition, identify and inverse.
   Pigeon– hole principle.

5. **Graphs :**
   Definition.
   Paths and circuits ; Eulerian, Hamiltonian.
   Planer Graphs

6. **Groups :**
   Monoids, Semigroups, Groups.
   Products and quotients of algebraic structures.
   Isomorphism, homomorphism, automorphism.
   Normal subgroup.
   Codes and group codes.

7. **Rings and Fields :**
   Rings, integral domains and fields.
   Ring Homomorphism.

8. **Generating Functions and Recurrence Relations :**
   Series and sequences.
   Generating functions.
   Recurrence relations.
   Applications : Solving differential equations, Fibonacci etc.

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Set

A set is a well–defined collection of objects called elements or member of the set.
e.g. : Collection of one legged black birds. Collection of real number between 0 and 1.
Well–defined just means that it is possible to decide if a given object belongs to set.

$A = \{\alpha, \beta, \gamma, \delta\}$, $B = \{\text{Ram, Shyam, Bina, Tina}\}$

Subset

If every element in a set A is the elements of a set B, then A is called subset of B. We also say that A is contained in B or that B contains A. This relationship is denoted by

$A \subseteq B$ or $B \supseteq A$

If A is not a subset of B i.e., if at least one element of A does not belong to B,
we write $A \not\subseteq B$ or $B \not\supseteq A$.

e.g. $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$
$\because$ All the elements of A belongs to B
$\therefore$ $A \subseteq B$ (A is a subset of B)

Universal Set

In any application of the theory of sets, the members of all sets under investigation usually belong to some fixed large set called the universal set.
For example, in a plane geometry, the universal set consists of all points in the plane.
Usually it is denoted by U.

Empty Set

The set with no elements is called empty set or null set and is denoted by ϕ.
e.g. : $A = \{x : x \text{ is a +ve integer}, x < 1\}$
$\Rightarrow$ A has no element.

Note : (i) A is said to be equal to B iff $A \subseteq B$ and $B \supseteq A$.
(ii) For any set A, $\phi \subseteq A$.

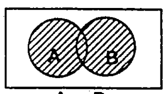
Cardinality of a set : The cardinality of a finite set is numbers of elements in the set. It is denoted by $|A|$ for set A.
e.g., $A = \{1, 2, 3\}$ then $|A| = 3$.

Finite and Infinite Set : A set A is called finite if it has n distinct elements, where $n \in N$. In this case, n is called cardinality of A and is denoted by $|A|$.
A set that is not finite is called infinite set.

Operations on Sets :

(i) Union : Let A and B be two non–empty sets then the union of A and B is a set i.e. the collection of all the elements of either A or B.

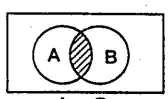
$\therefore$ $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



$A \cup B$

(ii) Intersection : Let A and B be two non–empty sets then the intersection of A and B is a set i.e. the collection of all the elements of A and B both.

$\therefore$ $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

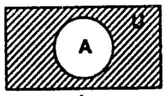


$A \cap B$

Note : $A \cap B = \phi$
Then A and B are called <u>disjoint sets.</u>

(iii) The complement of A : Let A be a non empty subset of a given universal set U then the complement of A is set i.e. the collection of all elements of U but not the elements of A. It is denoted by A' or $\overline{A}$ or A^c.

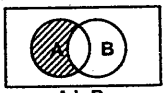
$A' = \{ x \mid x \in U \text{ and } x \notin A\}$ where U is universal set.



A'

(iv) The difference of A and B : Let A and B be two non empty sets then the difference of A and B is a set i.e. the collection of all the elements of A but not the elements of B.

$\therefore\ A \setminus B = \{x \mid x \in A \text{ and } x\ (\ B\}$.



A \ B

(v) Ring sum : Let A and B be two non empty sets then the ring sum of A and B is the set of all the elements which are either in A or in B but not in both.

$A \oplus B = \{ x \mid x \in A \cup B \text{ but } \notin A \cap B\}$

Algebraic Properties of set of operation :

1. Commutative property
   $A \cup B = B \cup A$
   $A \cap B = B \cap A$
2. Associative property
   $A \cup (B \cup C) = (A \cup B) \cup C$
   $A \cap (B \cap C) = (A \cap B) \cap (A$
3. Distributive property
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
4. Idempotent property
   $A \cup A = A$
   $A \cap A = A$
5. Properties of the complement
   $\left(\overline{\overline{A}}\right) = A$
   $A \cup \overline{A} = U$
   $A \cap \overline{A} = \phi$
   $\overline{\phi} = U$
   $\overline{U} = \phi$
   $\left.\begin{array}{l}\overline{A \cup B} = \overline{A} \cap \overline{B} \\ \overline{A \cap B} = \overline{A} \cup \overline{B}\end{array}\right\}$ These are known as De–Morgan's laws.

Examples :

1. Let A, B, C be the subsets of universal set U. Given that $A \cap B = A \cap C$ and $\overline{A} \cap \overline{B} = \overline{A} \cap C$, is it necessary that $B = C$? Justify your answer.

Soln. :.

Yes. B can be expressed as

$$
\begin{aligned}
B &= B \cap U \\
&= B \cap (A \cup \overline{A}) && \because A \cup \overline{A} = U \\
&= (B \cap A) \cup (B \cup \overline{A}) && \text{By distributive law} \\
&= (A \cap B) \cup (\overline{A} \cup B) && \text{Commutative law} \\
&= (A \cap C) \cup (\overline{A} \cap C) && \text{given} \\
&= (A \cup \overline{A}) \cap C && \text{(distributive law)} \\
&= U \cap C \\
&= C && \because \text{U is universal set}
\end{aligned}
$$

2. Prove that $((A \cup B) \cap \overline{A}) \cup (\overline{B \cap A}) = \overline{A \cap B}$

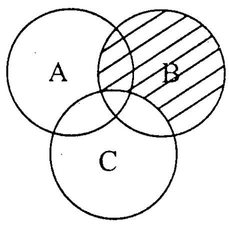
Soln. :.

$$
\begin{aligned}
\text{L.H.S.} &= ((A \cup B) \cap \overline{A}) \cup (\overline{B \cap A}) \\
&= \left[(A \cap \overline{A}) \cup (B \cap \overline{A})\right] \cup (\overline{B \cap A}) && \text{distributive law} \\
&= [\phi \cup B \cap \overline{A}] \cap (\overline{B} \cup \overline{A}) && \overline{B \cap A} = \overline{B} \cup \overline{A} \text{ De Morgan's law \& } A \cap \overline{A} = \phi \\
&= (B \cap \overline{A}) \cup (\overline{B} \cup \overline{A}) \\
&= [B \cup (\overline{B} \cup \overline{A})] \cap [\overline{A} \cup (\overline{B} \cup \overline{A})] && \text{distributive law} \\
&= [(B \cup \overline{B}) \cup \overline{A}] \cap [\overline{A} \cup \overline{B}] && \text{associative law} \\
&= [U \cup \overline{A}] \cap [\overline{B} \cup \overline{A}] && B \cup \overline{B} = U \\
&= U \cap [\overline{A} \cup \overline{B}] && \because \text{ U is universal set} \\
&= \overline{A} \cup \overline{B} \\
&= \overline{A \cap B} && \text{By De Morgan's law}
\end{aligned}
$$

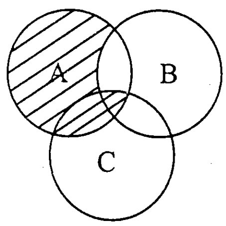
3. Using Venn diagram, prove that

$A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap B \cap C)$

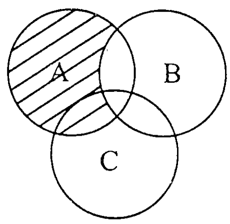
Soln. :.



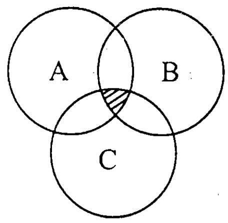
$(B \setminus C)$



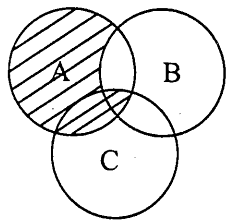
$A \setminus (B \setminus C)$



$A \setminus B$



$A \cap B \cap C$



$(A \setminus B) \cup (A \cap B \cap C)$

$\therefore$ The result

$A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap B \cap C)$ is verified.

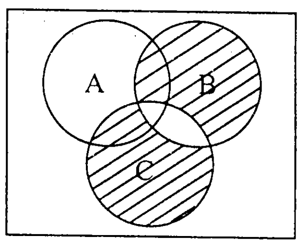
4. Using Venn diagram

(i) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

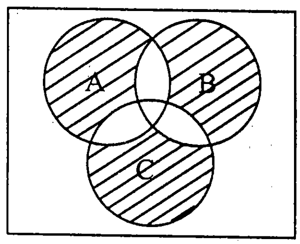
(ii) $A \cap B \cap C = A \setminus [(A \setminus B) \cup (A \setminus C)]$

Soln. :.

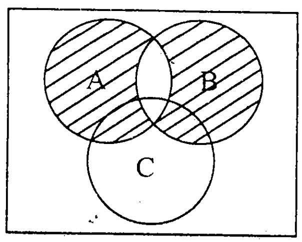
(i)



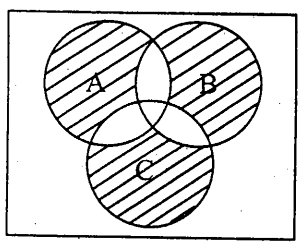
$B \oplus C$



$A \oplus (B \oplus C)$



$A \oplus B$

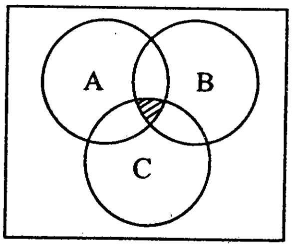


$A \oplus (B \oplus C)$

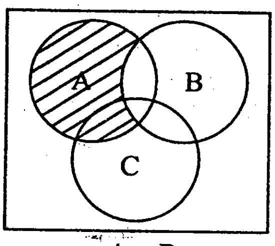
This implies that

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C$$

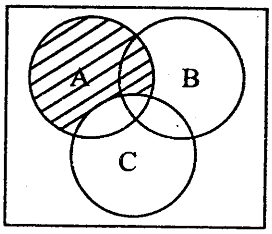
(ii)



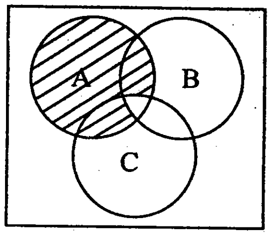
$A \cap B \cap C$



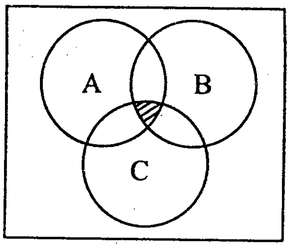
$A - B$



$A - C$



$(A - B) \cup (A - C)$



$A - [(A - B) \cup (A - C)]$

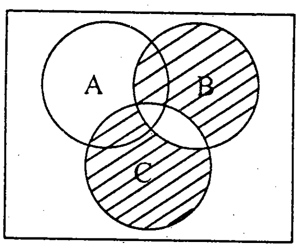
$\therefore\ A \cap B \cap C = A - [(A - B) \cup (A - C)]$

5. Using Venn diagram, prove that

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$

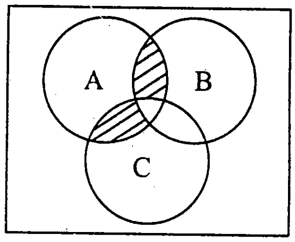
Soln. :.

$B \oplus C =$

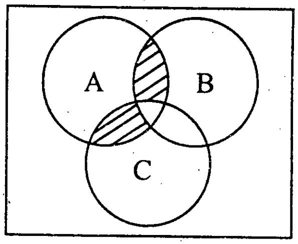


$B \oplus C$

$A \cap (B \oplus C) =$



$(A \cap B) \oplus (A \cap C) =$



$\therefore\ A \cap (B \oplus C) = (A \cap B) \oplus A \cap C$

6. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 4, 6, 8\}$, $B = \{2, 4, 5, 9\}$, $C = \{x / x \text{ is a +ve integer and } x^2 \le 16\} = \{1, 2, 3\}$ and $D = \{7, 8\}$

(i) Compute

(a) $A \cup B = \{1, 2, 4, 5, 6, 8, 9\}$

(b) $A \cup C = \{1, 2, 3, 4, 6, 8\}$

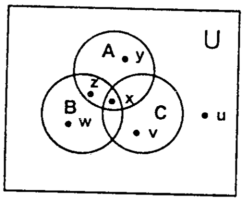
(c) $A \cup D = \{1, 2, 4, 6, 7, 8\}$

(d) $B \cup C = \{1, 2, 3, 4, 5, 9\}$

(e) $A \cap C = \{1, 2\}$
(f) $C \cap D = \phi$
(g) $A \cap D = \{8\}$
(h) $B \cap C = \{2\}$
(i) $A \setminus B = \{1, 6, 8\}$
(j) $B \setminus A = \{5, 9\}$
(k) $C \setminus D = \{1, 2, 3\}$
(l) $\overline{C} = \{4, 5, 6, 7, 8, 9\}$
(m) $\overline{A} = \{3, 5, 7, 9\}$
(n) $A \oplus B = \{x / x \in A \cup B, x \notin A \cap B\}$
$A \cup B = \{1, 2, 4, 5, 6, 8, 9\}$
$A \cap B = \{2, 4\}$
$\therefore\ A \oplus B = \{1, 5, 6, 8, 9\}$
(o) $C \oplus D = \{1, 2, 3, 7, 8\}$
(p) $B \oplus C = \{1, 2, 4, 5, 9\}$

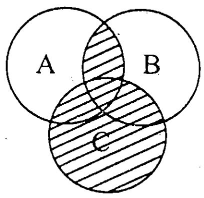
(ii) Compute
(a) $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 8, 9\}$
(b) $A \cap B \cap C = \{2\}$
(c) $A \cap (B \cup C) = \{1, 2, 4\}$
(d) $(A \cup B) \cap D = \{8\}$
(e) $\overline{A \cup B} = \{3, 7\}$
(f) $\overline{A \cap B} = \{1, 3, 5, 6, 7, 8, 9\}$
(g) $B \cup C \cup D = \{1, 2, 3, 4, 5, 7, 8, 9\}$
(h) $A \cup A = A$
(i) $A \cap \overline{A} = \phi$
(j) $A \cup \overline{A} = U$
(k) $A \cap (\overline{C} \cup d) = A \cap \{4, 5, 6, 7, 8, 9\}$
$= \{4, 6, 8\}$

7.



Identify the following as true or false
(a) $y \in A \cap B$ false
(b) $x \in B \cup C$ true
(c) $w \in B \cap C$ false
(d) $u \notin C$ true
(e) $x \in A \cap B \cap C$ true
(f) $y \in A \cup B \cup C$ true
(g) $z \in A \cap C$ false
(h) $v \in B \cap C$ false

8. Describe the shaded region shown in figure using $\cup$ & $\cap$ of the set



$\Rightarrow (A \cap B) \cup C$

Principle of inclusion and exclusion : (Addition principle)

Statement : 1) Let A and B be two finite sets then

$|A \cup B| = |A| + |B| - |A \cap B|$

2) Let A, B and C be finite sets, then

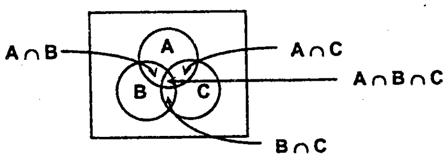
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C|$
$+ |A \cap B \cap C|$

3) Let A, B, C and D be finite sets then

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D|$
$- |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| +$
$|B \cap C \cap D| - |A \cap B \cap C \cap D|$

and so on.

Venn diagram of addition principle



Let A = {a, b, c, d, e}, B = {a, b, e, g, h} and C = {b, d, e, g, h, k, m, n}

We have, $A \cup B \cup C$ = {a, b, c, d, e, g, h, k, m, n}

$A \cap B$ = {a, b, e}

$B \cap C$ = {b, e, g, h}

$A \cap C$ = {b, d, e}

$A \cap B \cap C$ = {b, e}

Now, $|A \cup B \cup C| = 10, |A| = 5, |B| = 5, |C| = 8, |A \cap B| = 3, |A \cap C| = 3,$
$|B \cap C| = 4, |A \cap B \cap C| = 2.$

Thus, $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 5 + 5 + 8 - 3 - 3 - 4 + 2$

$= 10$

$= |A \cup B \cup C|$

Hence addition principle is verified.

9. Let A, B and be finite sets with $|A| = 6, |B| = 8, |C| = 6, |A \cup B \cup C| = 11, |A \cap B| = 3,$ $|A \cap C| = 2$ and $|B \cap C| = 5$, find $|A \cap B \cap C|$.

Soln. :.

By addition principle,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$11 = 6 + 8 + 6 - 3 - 2 - 5 + |A \cap B \cap C|$$
$$11 - 10 = |A \cap B \cap C|$$
$$\text{or } |A \cap B \cap C| = 1$$

10. In a survey of 260 college students, the data were obtained. **[D-03]**

64 had taken a mathematics course.
94 had taken a computer science course.
58 had taken a business course.
28 had taken both a mathematics and business course.
26 had taken both a mathematics and computer science course.
22 had taken both a computer science and business course,
14 had taken all three types of courses.

(a) How many students were surveyed who had taken none of three types of courses ?

(b) Of the students surveyed, how many had taken only a computer science course?

Soln. :.

Let M be a set of mathematics students
C be a set of computer science students
B be a set of business course students

$|M \cup C \cup B|$ = the set of students had taken at least one of these courses.

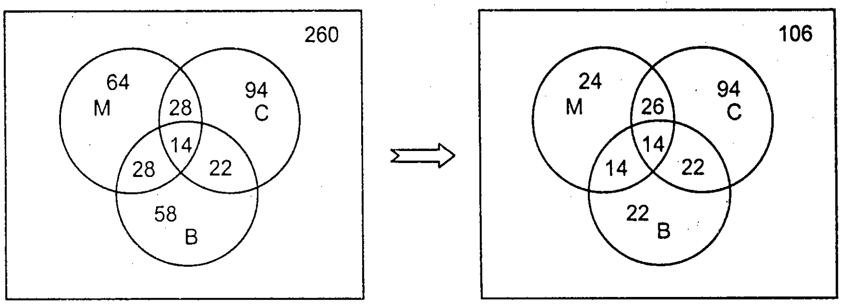
$|M| = 64$
$|C| = 94$
$|B| = 58$
$|M \cap B| = 28$
$|M \cap C| = 26$
$|C \cap B| = 22$
$|M \cap B \cap C| = 14$

By addition principle

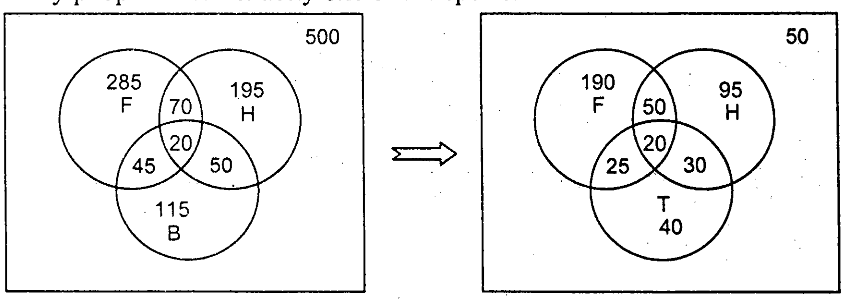
$$|M \cup C \cup B| = |M| + |C| + |B| - |M \cap C| - |M \cap B| - |C \cap B| + |M \cap B \cap C|$$
$$= 64 + 94 + 58 - 28 - 26 - 22 + 14$$
$$= 154.$$

(a) Now number of students had taken none of these types of course
= Total no. of students – No. of students who had taken at least one type of course.
= $260 - 154 = 106$

(b) No. of students had taken only computer science
= $|C| - |M \cap C| - |C \cap B| + |M \cap B \cap C|$
= $94 - 26 - 22 + 14 = 60$



11. A survey of 500 television watchers produced the following information
285 watch football games
195 watch hockey games
115 watch basketball
45 watch football and basketball
70 watch football and hockey
50 watch hockey and basket ball
and 50 do not watch any of three kinds of games.
(a) How many people in the survey watch all three kind of games.
(b) How many people watch exactly one of the sports.



Let F, H and B be the sets of watches of football, hockey and basketball respectively. Here 50 people do not watch any of three kinds of games.
∴ The number of peoples who watch at least one of three kinds of games.
i.e. $|F \cup H \cup B|$ = Total no. of watchers – the no. of people do not watch any kind of games.
$= 500 - 50$
$= 450$

Here $|F| = 285$, $|H| = 195$, $|B| = 115$
$|F \cap B| = 45$, $|F \cap H| = 70$, $|H \cap B| = 50$

(a) The number of people watch all three types of games = $|F \cap H \cap B|$
By addition principle
$$|F \cup B \cup H| = |F| + |B| + |H| - |F \cap B| - |F \cap H| - |B \cap H| + |F \cap B \cap H|$$

$450 = 285 + 195 + 115 - 45 - 70 - 50 + |F \cap B \cap H|$

$\therefore |F \cap B \cap H| = 450 - 430$

$= 20$

(b) The number of peoples who watch only football game

$= |F| - |F \cap B| - |F \cap H| + |F \cap B \cap H|$

$= 285 - 70 - 45 + 20$

$= 190$

The number of peoples who watch only hockey game

$= |H| - |H \cap F| - |H \cap B| + |F \cap H \cap B|$

$= 195 - 70 - 50 + 20$

$= 95$

The number of people who watch only basketball game

$= |B| - |B \cap H| - |B \cap F| + |F \cap H \cap B|$

$= 115 - 45 - 50 + 40$

$= 40$

The number of people watch exactly one of the sport

$= 190 + 95 + 40$

$= 325$

12. Among the integers 1 and 300,

(i) How many of them are divisible by 3, 5 or 7 and are not divisible by 3 nor by 5 nor by 7 ?

(ii) How many of them are divisible by 3 but not by 5 nor by 7 ?

Soln. :.

Let A be a set of integers among 1 and 300 divisible by 3
B be a set of integers among 1 and 300 divisible by 5
C be a set of integers among 1 and 300 divisible by 7

The no. of integers divisible by 3 = $|A| = \dfrac{300}{3} = 100$

The no. of integers divisible by 5 = $|B| = \dfrac{300}{5} = 60$

The no. of integers divisible by 7 = $|C| = \dfrac{300}{7} = 42$

The number of integers divisible by 3 & 5 = $|A \cap B| = \dfrac{300}{3 \times 5} = 20$

The number of integers divisible by 3 & 7 = $|A \cap C| = \dfrac{300}{3 \times 7} = 14$

The number of integers divisible by 5 & 7 = $|B \cap C| = \dfrac{300}{5 \times 7} = 8$

The number of integers divisible by 3, 5 & 7 = $|A \cap B \cap C| = \dfrac{300}{3 \times 5 \times 7} = 2$

(i) Number of integers which are divisible by at least one of them i.e. by 3 or by 5 or by 7.

$= |A \cup B \cup C|$

$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

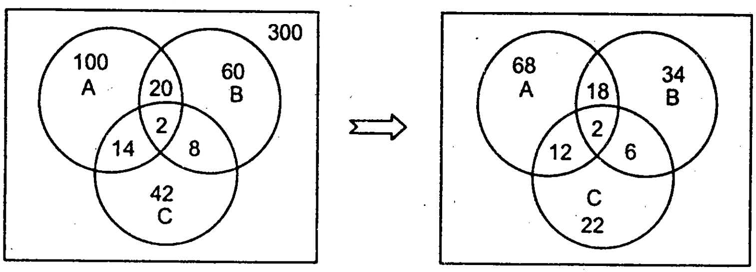
[By addition principle]

$= 100 + 60 + 42 - 20 - 14 - 8 + 2$

$= 162$

No. of integers which are not divisible by 3, nor by 5, nor by 7 = 300 – 162

$= 138$



(ii) Number of integer are divisible by 3 but not by 5, nor by 7

$$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$
$$= 100 - 20 - 14 + 2$$
$$= 68$$

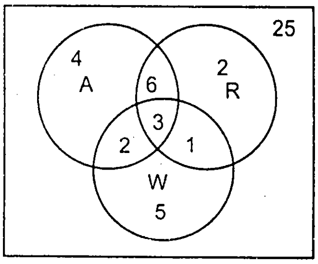
13. A survey on a sample of 25 new cars being sold of a local auto dealer was conducted to see which of three popular options, air conditioning A, radio R, and popular windows W, were already installed. The survey found.

15 had air conditioning
12 had radio
11 had power windows
5 had air conditioning and power window
9 had air conditioning and radio
4 had radio and power windows
5 had all three options

Find the number of cars

(a) only power windows
(b) only air conditioning
(c) only radio
(d) radio and power window but not air conditioning
(e) air conditioning and radio but not power windows
(f) only one of the options
(g) at least one option
(h) none of the options

Here $|A| = 15$, $|R| = 12$, $|W| = 11$
$|A \cap W| = 5$, $|A \cap R| = 9$, $|R \cap W| = 4$
$|A \cap R \cap W| = 3$



By addition principle

$$|A \cup R \cup W| = |A| + |R| + |W| - |A \cap W| - |R \cap W| + |A \cap R \cap W|$$
$$= 15 + 12 + 11 - 5 - 9 - 4 + 3$$
$$= 23$$

(a) The no. of cars had only power windows

$$= |W| - |W \cap A| - |W \cap R| + |W \cap A \cap R|$$
$$= 11 - 5 - 4 + 3$$
$$= 5$$

(b) The no. of cars had only air conditioning

$$= |A| - |A \cap W| - |A \cap R| + |A \cap W \cap R|$$
$$= 15 - 5 - 9 + 3$$
$$= 4$$

(c) The no. of cars had only radio

$$= |R| - |R \cap A| - |R \cap W| + |R \cap W \cap A|$$
$$= 12 - 9 - 4 + 3$$
$$= 2$$

(d) The no. of cars had radio and power windows but not air conditioning

$$= |A \cap W| - |A \cap R \cap W|$$
$$= 4 - 3$$
$$= 1$$

(e) The no. of cars had air conditioning and radio but not power windows

$$= |A \cap R| - |A \cap R \cap W|$$
$$= 9 - 3 = 6$$

(f) The no. of cars had only one option

$= 5 + 4 + 2$

$= 11$

(g) The no. of cars had at least one option = 23

(h) The no. of cars had none of the options = Total no. of options – no. of at least one option

$= 25 - 23$

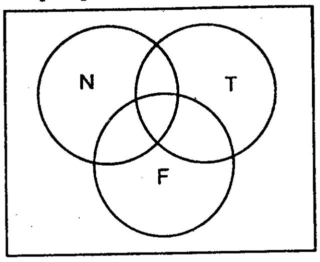
$= 2$

14. In a survey of 60 people, it was found that **[M-98, 02, 06]**

25 read Newsweek magazine
26 read Time
26 read Fortune
9 read both Newsweek and Fortune
11 read both Newsweek and Time
8 read both Time and Fortune
3 read all three magazines

(a) Find the number of people who read at least one of three magazine

(b) Fill in the correct number of people in each of the eight region of the Venn diagram in figure.
Let N, T and F be the sets of people who read Newsweek, Time and Fortune respectively.



(c) Find the number of people who read exactly one magazine.

Soln. :

Here $|N| = 25$, $|T| = 26$, $|F| = 26$, $|N \cap T| = 9$, $|T \cap F| = 11$, $|N \cap F| = 8$, $|N \cap T \cap F| = 3$

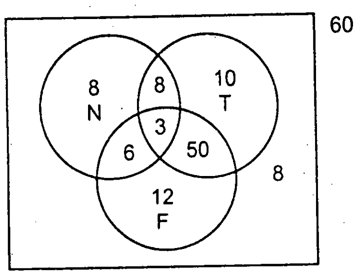
(a) By addition principle,

$|N \cup T \cup F| = |N| + |T| + |F| - |N \cap T| - |T \cap F| - |N \cap F| + |N \cap T \cap F|$

$= 25 + 26 + 26 - 9 - 11 - 8 + 3$

$= 52$

(b)



$\because$ 3 read all three magazine.

$\therefore$ $11 - 3 = 8$ read N & T but not all three magazine

and $9 - 3 = 6$ read N & F but not all three magazines

and $8 - 3 = 5$ read F & T but not all three magazines

No. of people read only N $= |N| - |N \cap F| - |N \cap T| + |N \cap T \cap F|$

$= 25 - 9 - 11 + 3 = 8$

No. of people read only T $= |T| - |T \cap N| - |T \cap F| + |N \cap T \cap F|$

$= 26 - 11 - 8 + 3 = 10$

No. of people read only F $= |F| - |F \cap T| - |F \cap N| + |N \cap T \cap F|$

$= 26 - 9 - 8 + 3 = 12$

No. of people read no magazine at all

$= 60 - 52 = 8$

(c) No. of people read only one magazine

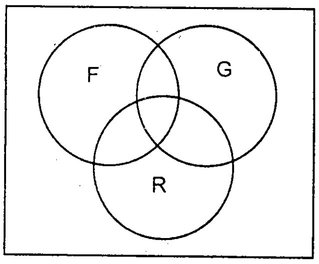
$= 8 + 10 + 12$

$= 30$

15. Consider the following data for 120 mathematics students at a college concerning the languages French, German and Russian.

65 study French
45 study German
42 study Russian
20 study French & German
15 study German & Russian
25 study French & Russian
8 study all three languages

Let F, G and R denote the sets of students studying French, German and Russian respectively. Find the number of students who study at least of three languages and to fill up in the correct number of students in each of eight regions of the Venn diagram shown in figure.



$|F| = 65$, $|G| = 45$, $|R| = 42$
$|F \cap G| = 20$, $|F \cap R| = 25$, $|R \cap G| = 25$
$|F \cap G \cap R| = 8$

By addition principle

$$|F \cup G \cup R| = |F| + |G| + |R| - |F \cap G| - |F \cap R| - |R \cap G| + |F \cap G \cap R|$$
$$= 65 + 45 + 42 - 20 - 25 - 15 + 8$$
$$= 100$$

No. of mathematics students are concerning only French language

$$= |F| - |F \cap G| - |F \cap R| + |F \cap G \cap R|$$
$$= 65 - 20 - 25 + 8 = 28$$

No. of mathematics students are concerning only German language

$$= |G| - |F \cap G| - |G \cap R| + |F \cap G \cap R|$$
$$= 45 - 20 - 15 + 8$$
$$= 18$$

No. of mathematics students are concerning only Russian languages

$$= |R| - |F \cap G| - |G \cap R| + |F \cap G \cap R|$$
$$= 42 - 25 - 15 + 8$$
$$= 10$$

No. of mathematics students are concerning French & German languages

$$= |F \cap G| - |F \cap G \cap R|$$
$$= 20 - 8$$
$$= 12$$

No. of mathematics students are concerning French & Russian languages

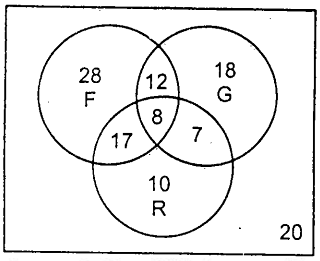
$$= |F \cap G| - |F \cap G \cap R|$$
$$= 25 - 8$$
$$= 17$$

No. of mathematics students are concerning Russian & German languages

$$= |R \cap G| - |F \cap G \cap R|$$
$$= 15 - 8$$
$$= 7$$

No. of mathematics students are concerning French, Russian & German languages

$$= |F \cap G \cap R|$$
$$= 8$$



16. Determine the number of integers between 1 and 250 that are divisible by 2 or 3 or 5 or 7.

Soln. :.

Let A denote the set of integers between 1 and 250, divisible by 2
Similarly B, C & D are sets of integers between 1 and 250, divisible by 3, 5 & 7 respectively.

The no. of integers divisible by 2 = $|A| = \dfrac{250}{2} = 125$

The no. of integers divisible by 3 = $|B| = \dfrac{250}{3} = 83$

The no. of integers divisible by 5 = $|C| = \dfrac{250}{5} = 50$

The no. of integers divisible by 7 = $|D| = \dfrac{250}{7} = 35$

The no. of integers divisible by 2 & 3 = $|A \cap B| = \dfrac{250}{2 \times 3} = 41$

The no. of integers divisible by 2 & 5 = $|A \cap C| = \dfrac{250}{2 \times 5} = 25$

The no. of integers divisible by 2 & 7 = $|A \cap D| = \dfrac{250}{2 \times 7} = 17$

The no. of integers divisible by 3 & 5 = $|B \cap C| = \dfrac{250}{3 \times 5} = 16$

The no. of integers divisible by 3 & 7 = $|B \cap D| = \dfrac{250}{3 \times 7} = 11$

The no. of integers divisible by 5 & 7 = $|C \cap D| = \dfrac{250}{5 \times 7} = 7$

The no. of integers divisible by 2, 3 & 5 = $|A \cap B \cap C| = \dfrac{250}{2 \times 3 \times 5} = 8$

The no. of integers divisible by 2, 3 & 7 = $|A \cap B \cap D| = \dfrac{250}{2 \times 3 \times 7} = 5$

The no. of integers divisible by 2, 5 & 7 = $|A \cap C \cap D| = \dfrac{250}{2 \times 5 \times 7} = 3$

The no. of integers divisible by 2, 3 & 7 = $|B \cap C \cap D| = \dfrac{250}{3 \times 5 \times 7} = 2$

The no. of integers divisible by 2, 3, 5 & 7 = $|A \cap B \cap C \cap D| = \dfrac{250}{2 \times 3 \times 5 \times 7} = 1$

The no. of integers divisible by atleast one of these

$$
\begin{aligned}
&= |A \cup B \cup C \cup D| \\
&= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| \\
&\quad - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap C \cap D| + |A \cap B \cap D| \\
&\quad + |B \cap C \cap D| - |A \cap B \cap C \cap D| \qquad \text{[By addition principle]} \\
&= 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2 - 1 \\
&= 193
\end{aligned}
$$

Cartesian Product : The Cartesian product of two non empty sets A and B is defined as

$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$

e.g. $A = \{1, 2, 3\}$

$B = \{a, b, c\}$

$\therefore\ A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$

17. Prove that

$A \times (B \cap C) = (A \times B) \cap (A \times C)$ **[M-01, 02]**

Soln. :

Let $(x, y) \in A \times (B \cap C)$

$\Rightarrow x \in A \text{ and } y \in B \cap C$

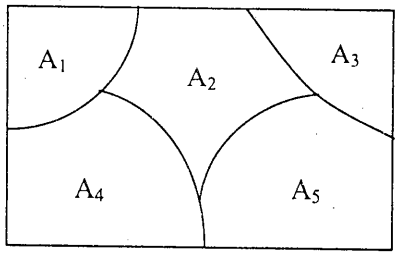
$\Rightarrow x \in A \text{ and } y \in B \text{ and } y \in C$

$\Rightarrow x \in A \text{ and } y \in B, \text{ and } x \in A, y \in C$

$\Rightarrow (x, y) \in A \times B,\ (x, y) \in A \times C$
$\Rightarrow (x, y) \in (A \times B) \cap (A \times C)$
$\therefore\quad A \times (B \cap C) \subseteq (A \times B) \cap (A \times C) \qquad \ldots (1)$
Now let $(x, y) \in (A \times B) \cap (A \times C)$
$\Rightarrow (x, y) \in A \times B\ \&\ (x, y) \in A \times C$
$\Rightarrow x \in A, y \in B\ \&\ x \in A, y \in C$
$\Rightarrow x \in A\ \&\ y \in B$ and $y \in C$
$\Rightarrow x \in A\ \&\ y \in B \cap C$
$\Rightarrow (x, y) \in A \times (B \cap C)$
$\therefore\quad (A \times B) \cap (A \times C) \subseteq A \times (B \cap C) \qquad \ldots (2)$
From (1) and (2) we have
$A \times (B \cap C) \;=\; (A \times B) \cap (A \times C)$

Partition

A partition of S is a collection $\{A_i\}$ of non empty subsets of S such that
(i) Each a in S belongs to one of A_i
(ii) The sets of $\{A_i\}$ are mutually disjoint. i.e. $A_i \neq A_j$ then $A_i \cap A_j = \phi$
The subsets in partition are called cells.

S


$S \;=\; \{A_1, A_2, A_3, A_4, A_5\}$ is a partition.

Briefly it can be expressed as
$S \;=\; \{A_i\}$ such that $\bigcup_i A_i \;=\; S \quad \&\quad A_i \cap A_j = \phi$ when $i \neq j$

18. Let $X = \{1, 2, 3, \ldots . 8, 9\}$. Determine whether or not each of following is a partition
(a) $[\{1, 3, 6, \}, \{2, 8\}, \{5, 7, 9\}]$
(b) $[\{2, 4, 5, 8\}, \{1, 9\}, \{3, 6, 7\}]$
(c) $[\{1, 5, 7\}, \{2, 4, 8, 9\}, \{3, 5, 6\}]$
(d) $[\{1, 2, 7\}, \{3, 5\}, \{4, 6, 8, 9\}, \{3, 5\}]$

Soln. :.
(a) $4 \in X$ but 4 does not belong to any cell.
$\therefore$ it is not a partition of X.
(b) $\because\ \{2, 4, 5, 8\} \cup \{1, 9\} \cup \{3, 6, 7\} = S$
$\&\ \{2, 4, 5, 8\} \cap \{1, 9\} \cap \{3, 6, 7\} = \phi$
i.e. mutually disjoint.
$\therefore$ It is partition of X
(c) $\because\ \{1, 5, 7\} \cap \{3, 5, 6\} = 5$
$\therefore$ it is not a partition of X.
(d) $\because$ 2nd and 4th cells are identical.
$\therefore$ it is not a partition of X.

19. Let $S = \{1, 2, 3, 4, 5, 6\}$. Determine whether or not the following is a partition of S. **[M-02]**
(a) $P_1 = [\{1, 2, 3\}, [1, 4, 5, 6\}]$
(b) $P_2 = [\{1, 2\}, \{3, 5, 6\}]$
(c) $P_3 = [\{1, 3, 5\}, \{2, 4\}, \{6\}]$
(d) $P_4 = [\{1, 3, 5\}, \{2, 4, 6, 7\}\}]$

Soln. :.
(a) 1 is intersection element of two cells
$\therefore$ P_1 is not a partition of S.
(b) $4 \in S$ but is not in any cell
$\therefore$ P_2 is not a partition of S.
(c) $\because\ \{1, 3, 5\} \cup \{2, 4\} \cup \{6\} \;=\; S$
$\&\ \{1, 3, 5\} \cap \{2, 4\} \cap \{6\} \;=\; \phi$
$\therefore$ P_3 is partition of S.

(d) 7 is an element in 2^{nd} cell but $7 \notin S$

$\therefore$ 2^{nd} cell can not be a subset of S.

$\therefore$ It is not a partition of S.

20. Determine whether or not each of the following is a partition of a set N of +ve integers.

(a) $[\{n : n > 5\}, \{n : n < 5\}]$

(b) $[\{n : n > 5\}, \{0\}, \{1, 2, 3, 4, 5\}$

(c) $[\{n : n^2 > 11\}, \{n : n^2 < 11\}$

Soln. :.

(a) 5 is neither the element of 1^{st} cell nor 2^{nd} cell whenever $5 \in N$.

$\therefore$ It is not a partition of N.

(b) $0 \notin N$, but $\{0\}$ is in partition

$\therefore$ It is not a partition of N. $(\{0\} \not\subseteq N)$

(c) $\because$ $\{n : n^2 > 11\} = \{4, 5, 6, \ldots\}$

& $\{n : n^2 < 11\} = \{1, 2, 3\}$

are mutually disjoint and union is the set of natural numbers.

$\therefore$ It is a partition of N.

Power Set

Let A be a given set, then the set of all possible subsets of A is called power set of A and is denoted by P(A).

e.g., $A = \{1, 2, 3\}$ $(|A| = 3)$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

$|P(A)| = 2^{|A|}$. i.e., $|P(A)| = 2^3 = 8$.

21. If $A = \{2^x / x^2 - 5x + 6 = 0\}$, $B = \{2^x / x^3 - 6x^2 + 11x - 6 = 0\}$. Verify that $P(A) \subseteq P(B)$.

Soln. :.

$$x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3.$$

$$x^3 - 6x^2 + 11x - 6 = 0$$

$$(x - 1)(x^2 - 5x + 6) = 0$$

$$(x - 1)(x - 3)(x - 3) = 0$$

$\therefore$ $x = 1, 2, 3$

$\therefore$ $A = \{2^2, 2^3\} = \{4, 8\}$

$B = \{2^1, 2^2, 2^3\} = \{2, 4, 8\}$

$P(A) = \{\phi, \{4\}, \{8\}, \{4, 8\}\}$

$P(B) = \{\phi, \{2\}, \{4\}, \{8\}, \{2, 4\}, \{2, 8\}, \{4, 8\}, \{2, 4, 8\}\}$

$\therefore$ every element of P(A) belong to set P(B)

$\therefore$ $P(A) \subseteq P(B)$

Graded Questions

1. Explain the following terms :

(i)	Set	(ii)	Subset	(iii)	Universal set
(iv)	Empty set	(v)	Disjoint of sets **[N-04]**	(vi)	Complement of a set
(vii)	Finite and infinite sets	(viii)	Cardinality of a set	(ix)	Power set

2. Explain following operations with respect to sets :

(i) Union (ii) Intersection (iii) Complement of a Set

(iv) Difference **[N-04]** (v) Ring sum (vi) Cartesian product of the two sets

3. What is the power set of the empty set ? What is the power set of the set $\{\phi\}$? **[M-04]**

4. Prove the following laws of the algebra of sets.

(a) Indempotent laws (i.e. (i) $A \cup A = A$ (ii) $A \cap A = A$)

(b) Associative laws

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $(A \cap B) \cap C = A \cap (B \cap C)$

(c) Commutative laws

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

(d) Distributive laws

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(e) Identity law

(i) $A \cup \phi = A$ (ii) $A \cap U = A$

(iii) $A \cup U = U$ (iv) $A \cap \phi = \phi$

(f) Involution laws :

$(A^C)^C = A$

(g) Complement laws

(i) $A \cup A^C = U$ (ii) $A \cap A^C = \phi$

(ii) $U^C = \phi$ (iv) $\phi^C = U$

(h) DeMorgan's laws

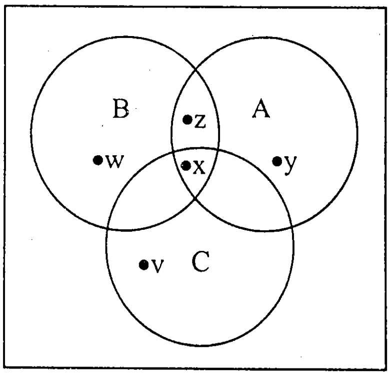
(i) $(A \cup B)^C = A^C \cap B^C$ (ii) $(A \cap B)^C = A^C \cup B^C$

5. Prove the following properties of the symmetric difference :

(i) $A \oplus B = B \oplus A$

(ii) If $A \oplus B = A \oplus C$, then $B = C$

6.



Identify the following statements as true or false

(i) $y \in A \cap B$ (ii) $x \in B \cup C$ (iii) $w \in B \cap C$ (iv) $v \notin C$

(v) $x \in A \cup B \cap C$ (vi) $y \in A \cup B \cup C$ (vii) $z \in A \cap C$ (viii) $v \in B \cap C$

7. Define partition set. Explain it with the suitable example. **[M-00, 01, D-01]**

8. Let $A = \{a, b, c, d, e, f, g, h\}$. Consider following subsets of A. **[M-03]**

$A_1 = \{a, b, c, d\}$, $A_2 = \{a, c, e, f, g, h\}$

$A_3 = \{a, c, e, g\}$, $A_4 = \{b, d\}$, $A_5 = \{f, h\}$.

Determine whether each of the following is partition of A or not. Justify your answer.

(i) $\{A_1, A_2\}$ (ii) $\{A_1, A_5\}$ (iii) $\{A_3, A_4, A_5\}$.

9. List all the partitions of $A = \{1, 2, 3\}$. **[D-03]**

10. Let the universal set be $U = \{1, 2, 3, \ldots 10\}$, let $A = \{2, 4, 7, 9\}$, $B = \{1, 4, 6, 7, 10\}$ and $C = \{3, 5, 7, 9\}$ find :

(i) $A \cup B$ (ii) $A \cap C$ (iii) $B \cap \bar{C}$ (iv) $(A \cap \bar{B}) \cup C$ (v) $\overline{B \cup C} \cap C$

11. Simplify the expression $\overline{\overline{(A \cup B) \cap C \cup \bar{B}}}$ **[M-05]**

12. How many elements are in $A_1 \cup A_2$ if there are 12 elements in a_1 and 18 elements in A_2 and **[M-05]**

(1) $A_1 \cap A_2 = \phi$? (3) $|A_1 \cap A_2| = 6$?

(2) $|A_1 \cap A_2| = 1$? (4) $A_1 \subseteq A_2$?

13. Define the Cartesian product of two sets. **[D-98]**

If $A = \{ x/x$ is real and $-2 \le x \le 3 \}$ and $B = \{ y/y$ is real and $1 \le y \le 5 \}$, sketch the set $A \times B$ in the Cartesian plane.

14. (a) Write the members of $\{a, b\} \times \{1, 2, 3\}$. **[M-99]**

(b) Give examples of sets A, B, C such that $A \cup B = A \cup C$, but $B \ne C$.

(c) Find the power set of the set $A = \{\alpha, \beta, \gamma\}$.

15. If $A = \{a, b\}$ and $B = \{1, 2, 3\}$, find $A \times B$.

16. Let A, B and C are subsets of U (universal set). **[D-03]**
    Prove that : $A \times (B \cup C) = (A \times B) \cup (A \times C)$
17. Prove that : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ **[M-04]**
18. Prove the following **[D-05]**
    $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
19. Prove the following (use laws of set theory) **[M-06]**
    $(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \bar{D}))] = B \cap (A \cup C)$
20. State the principle of inclusion and exclusion (addition principle) for 2, 3 and 4 sets.
21. Verify the theorem
    $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$
    where [a] A = {a, b, c, d, e}
    B = {d, e, f, g, h, i, k}
    C = {a, c, d, e, k, r, s, t}
    [b] A = {4, 5, 9, 15, 36, 70}
    B = {5, 9, 37, 65, 81, 89}
    C = {5, 6, 11, 15, 28, 65, 71, 79, 86}
    [c] A = {1, 2, 3, 4, 5, 6}
    B = {2, 4, 7, 8, 9}
    C = {1, 2, 4, 7, 10, 12}
    [d] A = {x | x is +ve integer $x < 8$ }
    B = {x | x is integers s . 1 $z < x < 4$}
    C = {x | x is an integer s . d $x^2 < 16$ }
    [e] A = {a, b, c, f, g}
    B = {a, c, d, m, n}
    C = {a, d, n, o, p, s}
22. Determine the number of positive integers n where $1 \le n \le 100$ and n is not divisible by 2, 3 or 5. **[D-05]**
23. Consider the set of 11 senators : **[M-99, 01]**
    i. In how many ways a committee of 5 members out of 11 can be selected ?
    ii. In how many ways a committee of 5 members can be selected so that a particular senator, senator A, is always included ?
    iii. In how many ways can we select a committee of five members so that a senator A is always excluded ?
    iv. In how many ways can we select a committee of five members so that at least one of senator A and senator B will be included ?
24. It is known that at the university, 60 percent of the professors play tennis, 50 percent of them play bridge, 70% jog, 20% play tennis and bridge, 30% play tennis and jog and 40% play bridge and jog. If someone claimed that 20% of the professors jog and play bridge and tennis, would you believe this claim ? Why ? **[D-00]**

□□□□□□

Principle of Mathematical Induction

Statement :
Let P(n) be a statement which depends on natural number n such that
1. it is true for $n = n_0$, where n_0 is initial value of n.
2. we assume that it is true for n = k (it is known as Inductive Hypothesis)
3. $P(k) \Rightarrow P(k + 1)$ [It is true for n = k + 1 by Inductive hypothesis.]
   then it is true for all $n \geq n_0$.

Type A

1. P.T. by induction $1 + 2 + 3 + \ldots. + n = \frac{n(n+1)}{2} \quad \forall\, n \in N$

Soln. : When n = 1,

L.H.S. = 1 $\quad$ R.H.S. $= \frac{1(1+1)}{2} = 1$

$\therefore$ the result is true for all n = 1.
Let the result is true for n = k

$1 + 2 + 3 + \ldots. + k = \frac{k(k+1)}{2}$ i.e. P (k)

(This assumption is called inductive hypothesis).
Now we wish to prove that P (k + 1) is true

i.e. $1 + 2 + \ldots. + k + (k + 1) = \frac{(k+1)(k+2)}{2}$

$$\begin{aligned} \text{L.H.S.} &= 1 + 2 + 3 + \ldots. + k + (k + 1) \\ &= \frac{k(k+1)}{2} + k + 1 \qquad \text{by P (k)} \\ &= \frac{k(k+1) + (2k+2)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \text{R.H.S.} \end{aligned}$$

$\therefore$ P (k) $\Rightarrow$ P (k + 1).
$\therefore$ the result is true for n = k + 1.

Hence $1 + 2 + 3 + \ldots. + n = \frac{n(n+1)}{2}$ is true for all n.

2. P.T. by induction $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{n(2n+1)(2n-1)}{3} \forall n \in N$.**[D-98, 00, M-99]**

Soln. :
For n = 1
L.H.S. = $1^2 = 1$

R.H.S. $= \frac{1.(2+1)(1)}{3} = 1$

L.H.S. = R.H.S.
The result is true for n = 1
Let the result is true for n = k

$1^2 + 3^2 + \ldots. + (2k - 1)^2 = \frac{k(2k+1)(2k-1)}{3}$ [i..e P(k)]

Now we have to prove that P(k) $\Rightarrow$ P(k + 1)

Hence $P(k + 1) \Rightarrow 1^2 + 3^2 + \ldots.. + (2k + 1)^2 = \frac{(k+1)(2k+3)(2k+1)}{3}$

L.H.S. $= 1^2 + 3^2 + \ldots.. + (2k - 1)^2 + (2k + 1)^2$

$= \dfrac{k(2k+1)(2k-1)}{3} + (2k+1)^2$ by P(k)

$= (2k+1)\left[\dfrac{k(2k-1)+3(2k+1)}{3}\right]$

$= \dfrac{(2k+1)}{3}[2k^2 - k + 6k + 3]$

$= \dfrac{2k+1}{3}[2k^2 + 5k + 3]$

$= \dfrac{2k+1}{3}[2k^2 + 2k + 3k + 3]$

$= \dfrac{2k+1}{3}[2k(k+1) + 3(k+1)]$

$= \dfrac{(2k+1)}{3}(k+1)(2k+3)$

$= \dfrac{(k+1)(2k+3)(2k+1)}{3}$

= R.H.S.

$\therefore$ P(k) $\Rightarrow$ P(k + 1)

$\therefore$ the result is true n = k + 1

Hence the result is true for all n.

3. $5 + 10 + 15 + \ldots\ldots + 5n = \dfrac{5n(n+1)}{2}$

Soln.:

We put n = 1

L.H.S. = 5 R.H.S. $= \dfrac{5.1(2)}{2} = 5$

L.H.S. = R.H.S.

$\therefore$ the statement is true for n = 1

Let the statement is true for n = k i.e. $5 + 10 + 15 + 5k = \dfrac{5k(k+1)}{2}$

Now we have to prove that P(k) $\Rightarrow$ P(k + 1)

Hence P(k + 1) $\Rightarrow$ $5 + 10 + 15 \ldots\ldots + 5k = \dfrac{5(k+1)(k+2)}{2}$

L.H.S. = 5 + 10 + 15 + …….. + 5k + 5(k + 1)

$= \dfrac{5(k+1)k}{2} + 5(k+1)$ [by P(k)]

$= 5(k+1)\left[\dfrac{k}{2}+1\right]$

$= \dfrac{5(k+1)(k+2)}{2}$

= R.H.S.

$\therefore$ the statement is true for n = k + 1.

Hence the statement is true for all n.

4. $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ **[M-02]**

Soln. :

For n = 1

L.H.S. = 1 R.H.S. $= \dfrac{(1)(2)(3)}{6} = 1$

The result is true for n = 1

Let the result is true for n = k

$1^2 + 2^2 + 3^2 + \ldots\ldots + k^2 = \dfrac{k(k+1)(k+2)}{6}$ [i.e. P(k)]

We have to prove that P(k + 1) is true.

L.H.S. $= 1^2 + 2^2 + 3^2 + \ldots\ldots + k^2 + (k+1)^2$

$= \dfrac{k(k+1)(k+2)}{6} + (k+1)^2$ [P(k)]

$= (k+1)\left[\dfrac{k(2k+1)}{6} + (k+1)\right]$

$= \dfrac{k+1}{6}[2k^2 + k + 6k + 6]$

$= \dfrac{k+1}{6}[2k^2 + 7k + 6]$

$= \dfrac{k+1}{6}[2k^2 + 4k + 3k + 6]$

$= \dfrac{k+1}{6}[2k(k+2) + 3(k+2)]$

$= \dfrac{(k+1)(k+2)(2k+3)}{6}$

= R.H.S.

Hence the result is true for n = k + 1 that means $P(k) \Rightarrow P(k+1)$

$\therefore$ The result is true for all n.

5. Use induction to show that, $1 + 2 + 2^2 + 2^3 + \ldots\ldots\ldots + 2^n = 2^{n+1} - 1$. **[M-04, N-04]**

Soln.:

We put n = 1,

L.H.S. = 3, R.H.S. $= 2^{1+1} - 1 = 3$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ The result is true for n = 1

Let the result is true for n = k

i.e. $1 + 2 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 1$

Now we have to prove that

$P(k) \Rightarrow P(k+1)$

i.e. $1 + 2 + 2^2 + 2^3 + \ldots + 2^{k+1} = 2^{k+2} - 1$

L.H.S. $= 1 + 2^1 + 2^2 + \ldots\ldots + 2^k + 2^{k+1}$

$= 2^{k+1} - 1 + 2^{k+1}$ [by P (k)]

$= 2 \cdot 2^{k+1} - 1$

$= 2^{(k+1)+1} - 1$

$= 2^{k+2} - 1$

= R.H.S.

$\therefore P(k) \Rightarrow P(k+1)$

Hence result is true for n = k + 1

$\therefore$ The result is true for all n.

6. Use induction to show that, $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$

Soln.:

For n = 1.

L.H.S. $= 1^3 = 1$ R.H.S. $= \dfrac{1^2(2)^2}{4} = 1$

Hence result is true for n = 1.

Let for n = k, it is true.

i.e. $1^3 + 2^3 + 3^3 + \ldots + k^3 = \dfrac{k^2(k+1)^2}{4}$ [P (k)]

Now we have to prove that it is true for n = k + 1

1-02]

i.e. $1^3+2^3+3^3+...+(k+1)^3=\dfrac{(k+1)^2(k+2)^2}{4}$

$$\begin{aligned}
\text{L.H.S.} &= 1^3+2^3+3^3+\ .\ .\ .\ +k^3+(k+1)^3 \\
&= \frac{k^2(k+1)^2}{4}+(k+1)^3 \quad \text{[by P (k)]} \\
&= \frac{(k+1)^2}{4}\left[k^2+4(k+1)\right] \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \text{R.H.S.}
\end{aligned}$$

We find the result is true for n = k + 1.
Hence it is true for all n.

7. Use induction to show that, 1 + 5 + 9 + ... + (4n – 3) = n (2n – 1).

Soln.:

For n = 1,
L.H.S. = 1 R.H.S. = 1(1) = 1
⇒ statement is true for n = 1
Let the statement is true for n = k i.e. 1 + 5 + 9 + ...+(4k – 3) = k(2k –1) [P(k)]
Now we have to prove that
P (k) ⇒ P (k + 1)
i.e. 1 + 5 + 9 + ... + (4k + 1) = (k + 1) (2k +1)
Hence the result is true for n = 1

Let it is true for n = k
i.e. 1 + 5 + 9 + ... + (4k – 3) = k (2k + 1)

$$\begin{aligned}
\text{L.H.S.} &= 1+5+9+\ .\ .\ .\ .\ .+(4k-3)(\ 4k+1) \\
&= k\,(2k-1)+(4k+1) \quad \text{[by P (k)]} \\
&= 2k^2-k+4k+1 \\
&= 2k^2+3k+1 \\
&= 2k^2+2k+k+1 \\
&= (k+1)\,(2k+1) \\
&= \text{R.H.S.}
\end{aligned}$$

∴ P (k) ⇒ P (k + 1)
∴ The result is true for n = k + 1

Hence it is true for all n.

8. Use induction to show that, $1+a+a^2+\ ......\ +a^{n-1}=\dfrac{a^n-1}{a-1}$, $a \neq 1$

Soln.:

For n = 1,

L.H.S. = 1 R.H.S. = $\dfrac{a^1-1}{a-1}=1$

∴the result is true for n = 1
Let the result is true for n = k

$1+a+a^2+\ ....\ +a^{k-1}=\dfrac{a^k-1}{a-1}$ [i.e. P(k)]

Now we have to prove that P(k) ⇒ P(k + 1)

i.e. $1+a+a^2+\ ......\ +a^k=\dfrac{a^{k+1}-1}{a-1}$

L.H.S. = $1+a+a^2+\ .....\ +a^{k-1}+a^k$

$=\dfrac{a^k-1}{a-1}+a^k$ [by P(k)]

$$= \frac{a^k - 1 + a^{k+1} + a^k}{a-1}$$

$$= \frac{a^{k+1} - 1}{a-1}$$

= R.H.S.

$\therefore$ P(k) $\Rightarrow$ P(k + 1)

Hence the result is true for n = k + 1

$\therefore$ The result is true for all n.

9. Use induction to show that, $a + ar + ar^2 + \ldots\ldots ar^{n-1} = \frac{a(1-r^n)}{1-r}$ for $r \neq 1$.

Soln.:

For n = 1 we have,

L.H.S. = a R.H.S. = $\frac{a(1-r)}{1-r} = a$

$\therefore$ Statement is true for n = 1

Let the statement is true n = k

$$a + ax + ar^3 + \ldots + ar^{k-1} = \frac{a(1-r^k)}{1-r}$$

Adding ar^k to both sides, we have

$$a + ax + ar^2 + \ldots\ldots + ar^{k-1} + ar^k = \frac{a(1-r^k)}{1-r} + ar^k$$

$$= \frac{a(1-r^k) + ar^k - ar^{k+1}}{1-r}$$

$$= \frac{a - ar^{k+1}}{1-r}$$

$$= \frac{a(1-r^{k+1})}{1-r}$$

The result is true for n = k + 1

$\therefore$ Result is true for all n.

10. Use induction to show that, $1 + 3 + 5 + \ldots\ldots + (2n-1) = n^2$

Soln.:

Let n = 1,

L.H.S. = 1, R.H.S. = $1^2 = 1$

Hence result is true for n = 1

Let the result is true for n = k

$1 + 3 + 5 + \ldots\ldots + (2k-1) = k^2$

Adding 2k + 1 on both sides, we get

$$1 + 3 + 5 + \ldots + (2k-1) + (2k+1) = k^2 + (2k+1)$$

$$= (k+1)^2$$

$\therefore$ Hence the result is true for n = k +1

Hence result is true for all n.

11. Use induction to show that, $2 + 5 + 8 + \ldots\ldots + (3n-1) = \frac{n(3n+1)}{2}$

Soln.:

For n = 1,

L.H.S. = 2 R.H.S. = $\frac{1(3+1)}{2} = 2$

Hence result is true for n = 1

Let the result is true for n = k

$$2 + 5 + 8 + \ldots + (3k-1) = \frac{k(3k+1)}{2}$$

P(k)]

Adding 3k + 2 on both sides

$$2 + 5 + 8 + \ldots.. + (3k-1) + (3k+2) = \frac{k(3k+1)}{2} + (3k+2)$$

$$= \frac{1}{2}(3k^2 + k + 6k + 4)$$

$$= \frac{1}{2}(3k^2 + 3k + 4k + 4)$$

$$= \frac{1}{2}[3k(k+1) + 4(k+1)]$$

$$= \frac{(k+1)(3k+4)}{2}$$

∴ The result is true for n = k + 1

Hence the result is true for n = k + 1.

12. Use induction to show that, $\frac{1}{1.3} + \frac{1}{3.5} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$ for $n \geq 1$

Soln.:

For n = 1,

L.H.S. = $\frac{1}{1.3} = \frac{1}{3}$, R.H.S. = $\frac{1}{3}$

Hence the result is true for n = 1.

Let the result is true for n = k

$$\frac{1}{1.3} + \frac{1}{3.5} + \ldots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$$

For n = k + 1, adding $\frac{1}{(2k+1)(2k+3)}$ on both sides

$$\frac{1}{1.3} + \frac{1}{3.5} + \ldots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{1}{(2k+1)}\left[\frac{k(2k+3)+1}{2k+3}\right] = \frac{1}{(2k+1)}\left[\frac{2k^2+3k+1}{2k+3}\right]$$

$$= \frac{1}{2k+1}\,\frac{(2k+1)(k+1)}{2k+3} = \frac{k+1}{2k+3}$$

Hence the result is true for n = k + 1

∴ It is true for all n.

Type B

1. Show that $5^n - 4n - 1$ is divisible by 16 for $n \geq 1$.

Soln.:

For n = 1

$5^n - 4 \times 1 = 0$ is divisible by 16

Let given statement is true for n = k.

i.e. $5^k - 4k - 1$ is divisible by 16

For n = k + 1, it becomes as

$$5^{k+1} - 4(k+1) - 1 = 5^k . 5 - 4k - 4 - 1$$

$$= 5^k . 5 - 4k - 5$$

$$= 5^k . 5 - 5.4k - 5 . 1 + 16k$$

$$= 5(5^k - 4k - 1) + 16k$$

First part is divisible by 16 by inductive hypothesis and second part is already multiple of 16.

∴ It is divisible by 16

∴ The result is true for n = k + 1

Hence it is true for all n.

2. Show that $7^{2n} + (2^{3n-3})(3^{n-1})$ is divisible by 25 for all natural number n.

Soln. :

For n = 1,

$2^{2n} + (7^{3n-3})(3^{n-1}) \quad = 7^2 + (2^0)(3^0) = 49 + 1$

$= 50$ which is divisible by 25.

$\therefore$ Result is true when n = 1

Now, let the result is true for n = k, we have

$7^{2k} + (2^{3k-3})(3^{k-1})$ is divisible by 25.

For n = k + 1 expression becomes

$7^{2(k+1)} + (2^{3(k+1)-3})(3^{(k+1)-1})$

$= 7^{2k+2} + 2^{3k} . 3^k \quad = \quad 7^{2k} . 7^2 + 2^{3k-3} . 3^{k-1} . 8 . 3$

$= 7^{2k} . 49 + 24 . 2^{3k-3} . 3^{k-1}$

$= (50 - 1)7^{2k} + (25 - 1)2^{3k-3}\, 3^{k-1}$

$= 50 . 7^{2k} + 25 . 3_i^{3k-3} . 3^{k-1} - 7^{2k} - 2^{3k-3} . 3^{k-1}$

$= 50 . 7^{7^{2k}} + 25.3^{3k-3} . 3^{k-1} - (7^{2k} + 2^{3k-3}\, 3^{k-1})$

First and Second term is multiple of 25 and last term is divisible by 25 from inductive hypothesis. Therefore, expression is divisible by 25.

Hence the result is true for n = k + 1. $\quad\therefore$ It is true for all n.

3. Show that $n(n^2 - 1)$ is divisible by 24 where n is an odd integer.

Soln.:

For n = 1,

$n(n^2 - 1) = 1(1 - 1) = 0$ which is divisible by 24.

Let statement is true for n = k

i.e. $k(k^2 - 1)$ is divisible by 24 where k is an odd integer.

For, odd integer next to k is k + 2, expression becomes

$= (k + 2)[(k + 2)^2 - 1] \quad = \quad (k + 2)(k^2 + 4k + 3)$

$= (k + 2)(k^2 - 1 + 4k + 4) \quad = \quad k(k^2 - 1) + 4k(k + 1) + 2(k^2 + 4k + 3)$

$= k(k^2 - 1) + 6k^2 + 12k + 6 \quad = \quad k(k^2 - 1) + 6(k + 1)^2$

$= k(k^2 - 1) + 6(k + 1)^2$

First term is divisible by 24 by inductive hypothesis and second term $6(k + 1)^2$ is divisible by 24 for any odd integer k.

$\therefore$ Statement is true for n = k + 2. Hence the result is true for any odd $n \geq 1$.

4. Show that $n^4 - 4n^2$ is divisible by 3 for $n \geq 2$.

Soln. :

For n = 2,

$n^4 - 4n^2 \quad = \quad 2^4 - 4 . 2^2$

$= \quad 0 \qquad$ which is divisible by 3.

Let it is true for n = k.

i.e. $k^4 - 4k^2$ is divisible by 3.

For n = k + 1, the expression becomes

$$\begin{aligned}(k+1)^4 - 4(k+1)^2 &= k^4 + 4k^2 + 6k^2 + 4k + 1 - 4(k^2 + 2k + 1)\\ &= k^4 + 4k^3 + 6k^2 - 4k - 3 - 4k^2\\ &= k^4 - 4k^2 + 4k^3 + 6k^2 - 4k - 3\\ &= k^4 - 4k^2 + 4k^3 + 6k^2 - 4k - 3\\ &= k^4 - 4k^2 + 4k(k^2 - 1) + 6k^2 - 3\\ &= \left(k^4 - 4k^2\right) + 4(k-1)k(k+1) + 3\left(2k^2 - 1\right)\end{aligned}$$

$\because$ First term is divisible by 3 by inductive hypothesis.

Second term is consecutive product of 3 numbers. therefore one of them is multiple of 3 for any k. Third term is already multiple of 3.

$\therefore$ $(k+1)^4 - 4(k+1)^2$ is divisible by 3. Hence it is true for n = k + 1

$\therefore$ The result is true for all $n \geq 2$.

5. Consider the following function given in psudocode. **[D-03]**

```
FUNCTION SQ (A)
1  C ← O
2  D ← O
3  WHILE (D ≠ A)
   a.   C ← C + A
   b.   D ← D + 1
4. RETURN (C)
END OF FUNCTION SQ
```

Above function computes the square of A. C_n and D_n be the values of variables C and D after passing through while loop n times. Let p(n) be the predicate $C_n = A \times D_n$. Prove by principle of Mathematical Induction that $\forall\, n \geq 0$, p(n) is true.

Soln. :

The name of the function, SQ, suggests that it computes the square of A. Step 3b shows A must be a positive integer if the looping is to end. A few trials with particular values of A will provide evidence that the function does carry out this task. However, suppose we now want to prove that SQ always computes the square of the positive integer A, no matter how large A might be. We shall give a proof by mathematical induction. For each integer $n \geq 0$, let C_n and D_n be the values of the variables C and D, respectively, after passing through the WHILE loop n times. In particular, C_0 and D_0 represent the values of the variables before looping starts. Let P(n) be the predicate $C_n = A \times D_n$. We shall prove by induction that $\forall n \geq 0$ P(n) is true. Here n_0 is 0.

Basis step : P(0) is the statement $C_0 = A \times D_0$, which is true since the value of both C and D, is zero "after" zero passes through the WHILE loop.

Induction step : We must now use

P(k) : $C_k = A \times D_k$

to show that P(k + 1) ; $C_{k+1} = A \times D_{k+1}$. After a pass through the loop, C is increased by A and D is increased by 1, so $C_{k+1} = C_k + A$ and $D_{k+1} = D_k + 1$.

left-hand side of P(k + 1) : $C_{k+1} = C_k + A$

$= A \times D_k + A$ using (2) to replace C_k

$= A \times (D_k + 1)$ factoring

$= A \times D_{k+1}$ right-hand side of P(k + 1)

By the principle of mathematical induction, it follows that as long as looping occurs. $C_n = A \times D_n$. The loop must terminate. (Why ?) When the loop terminates. D = A, so $C = A \times A$, or A^2, and this is the value returned by the function SQ.

Graded Questions

1. State the Principle of Mathematical Induction.
2. Prove that the statement is true by using mathematical induction.

   (i) $\frac{1}{1.4} + \frac{1}{4.7} + ... + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$

   (ii) $1 + 2^n < 3^n \quad n \geq 1$

   (iii) $1(1!) + 2(2!) + ... + n(n!) = (n+1)! - 1$. **[D-05]**

   (iv) $n! \geq 2^{n-1}$ for n = 1, 2, ... **[M-03]**
3. Solve :

   (i) S.T. $n^3 + 2n$ is divisible by 3 for all $n \geq 1$.

   (ii) Let X be a finite set with n elements and P(X) be a powerset. Prove that cordinality of P(X) is 2^n.

   (iii) Prove that $8^n - 3^n$ is a multiple of 5 by mathematical induction for $n \geq 1$.

   (iv) Prove that by induction that sum of cubes of three consecutive integers is divisible by 9.

   (v) Prove that by mathematical induction that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integers.
4. Use induction to show that $5^n - 1$ is divisible by 4 for n = 1, 2, **[M-05]**
5. Use induction to prove that **[M-06]**

   $7^n - 1$ is divisible by 6 for n = 1, 2, 3 .....

Logic

Propositions

Proposition or statement is a declarative sentence which can be true or false but not both.

eg. i) Paris is in France (true)
ii) $1 + 2 = 2$ (false)

Compound propositions :

It is a composite proposition which consists of subpropositions. e.g. "Roses are red and violets are blue".

Note :

The truth value of compound proposition is determined by its sub–propositions and also the way in which they are connected.

e.g. (1) In the statement
"Roses are red and violets are blue both the primitive propositions have to be true for the compound proposition to be true. Since and is the connective

(2) In the statement
"Roses are red or violets are blue" any one of the propositions have to be true for the compound proposition to be true.

Basic Logical Operation

1. Conjunction, $p \wedge q$ (and) :

Any two propositions when combined with the word 'and' to form a compound proposition, it is called a conjunction.

Symbolically,

$$p \wedge q, \ p \,\&\, q, \ p \,.\, q$$

Definition :

If p and q are true, then $p \wedge q$ is true; otherwise $p \wedge q$ is false.

Truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2. Disjunciton, $p \vee q$ (or):

Any two propositions when combined with the word 'or' to form a compound proposition it is called a conjunction

Symbolically,

$$p \vee q, \ p + q$$

Definition :

If p and q are false, then $p \vee q$ is false, otherwise $p \vee q$ is true.

Truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

3. Negation, $\overline{p} / \neg p / \sim p$ (not) :

Given any proposition p, the negation of p can be formed by inserting the word 'not' in p.

Symbolically,

$$\overline{p}, \neg p, \sim p, \ p'$$

Truth table

p	$\overline{p}$
T	F
F	T

Logical Equivalence

Two compound propositions P(p, q, ..) and Q(p, q, …) are said to be logically equivalent, or simply equivalent or equal, if they have identical truth tables. It is denoted by

$P(p, q\ldots.) \equiv Q(p, q\ldots)$

e.g. Consider $\neg(p \wedge q)$ and $\neg p \vee \neg q$

p	Q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

equal

$\therefore \neg(p \wedge q) \equiv \neg p \vee \neg q$

Logical Implications

1) Conditional Statement

If p and q are statements, the compound statement 'if p then q" denoted by '$p \Rightarrow q$' is called conditional statement or implication.

The conditional $p \Rightarrow q$ is false only when first part p is true and second part q is false. Accordingly when p is false, the conditional $p \rightarrow q$ is true regardless of truth value of q.

Truth table

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note :

$p \rightarrow q \equiv \sim p \vee q$

Note : If $p \Rightarrow q$ then its converse is $q \Rightarrow p$ and contra positive is $\sim q \Rightarrow \sim p$.

2) Biconditional Statement

If p and q are statements, the compound statement "p if and only if q" denoted by $p \Leftrightarrow q$, is called an equivalence or biconditional.

The biconditional $p \Leftrightarrow q$ is true whenever p and q have the same truth values

Truth table

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Laws of Logic :

⇒ Idempotent Rules :

1) $p \vee p \equiv p$
2) $p \wedge p \equiv p$

⇒ Associative Laws :

1) $(p \vee q) \vee r \equiv p \vee (q \vee r)$
2) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

⇒ Commutative laws :

1) $p \vee q \equiv q \vee p$
2) $p \wedge q \equiv q \wedge p$

⇒ Distributive laws :

1) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
2) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

⇒ Identity laws :

1) $p \vee F \equiv p$ 3) $p \wedge T \equiv p$
2) $p \vee T \equiv T$ 4) $p \wedge F \equiv F$

⇒ Complement laws :

1) $p \vee \neg p \equiv T$ 3) $p \wedge \neg p \equiv F$
2) $\neg T \equiv F$ 4) $\neg F \equiv T$

⇒ Involution law :

1) $\neg\neg p \equiv p$

DeMorgan's Laws :

1) $\neg(p \vee q) \equiv \neg p \wedge \neg q$
2) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Tautology

A statement which is true for all possible values of its propositional variables.

Contradiction

A statement which is always false is called contradiction.

Solved Problems

1. Write English sentences for following where
   $P(x)$: x is even, $Q(x)$: x is prime, $R(x, y)$: x + y is even
   i) $\exists x \forall y R(x, y)$ **[D-03, M-05]**
   ii) $\forall x \exists y R(x, y)$ **[M-05]**
   iii) $\sim(\exists x P(x))$ **[M-05]**
   iv) $\sim(\forall x Q(x))$ **[M-05]**
   v) $\forall x \exists y R(x, y)$ **[D-05]**
   vi) $\exists x \forall y R(x, y)$ **[D-05]**
   vii) $\forall x (\sim Q(x))$ **[D-05]**
   viii) $\exists y (\sim P(y))$ **[D-05]**
   ix) $\forall x P(x)$ **[D-05]**

Soln. :

i) There exists an x for all y such that x + y is even
ii) For all x there exists a y such that x + y is even.
iii) It is not true that there exists an x such that x is even.
iv) It is not true that for all x, x is prime.
v) For all x there exists y such that x + y is even
vi) There exists an x for all y such that x + y is even.
vii) For all x, it is not true that x is a prime number.
viii) There exists, a y such that it is not even.
ix) For all x, x is even.

2. Construct truth table to determine whether each of the following is tautology, contingency or absurdity. (i) $(q \wedge p) \vee (q \wedge \neg p)$, (ii) $q \rightarrow (q \rightarrow p)$, (iii) $p \rightarrow (q \wedge p)$. **[M-02]**

Soln. :

i) Contingency

p	q	$\bar{p}$	$(q \wedge p)$	$(q \wedge \neg p)$	$(q \wedge p) \vee (q \wedge \neg p)$
T	T	F	T	F	T
T	F	F	F	F	F
F	T	T	F	T	T
F	F	T	F	F	F

ii)

p	q	q − p	q → (q → p)
T	T	T	T
T	F	T	T
F	T	F	F
F	F	T	T

Contingency.

iii)

p	q	q ∧ p	q → (q ∧ p)
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	T

Contingency.

3. Given the truth value of X, Y and Z as T and those of U and V as F, find the truth values of the following : **[M-01]**

$$(X \wedge (Y \vee Z)) \wedge \neg\ ((X \vee Z) \wedge (U \vee V) \wedge Z)$$

Soln. :

$$(X \wedge (Y \vee Z)) \wedge \neg\ ((X \vee Z) \wedge (U \vee V) \wedge Z)$$
$$= (T \wedge (T \wedge T)) \wedge \neg((T \vee T) \wedge (F \vee F) \wedge T))$$
$$= T \wedge \neg (T \wedge F \wedge T)$$
$$= T \wedge \neg (F)$$
$$= T \wedge T$$
$$= T$$

4. Verify that the proposition $p \vee \neg (p \wedge q)$ is a tautology. **[D-00]**

Soln. :

$p \vee \neg (p \wedge q)$

$\Rightarrow p \vee (\bar{p} \vee \bar{q})$

$\Rightarrow (p \vee \bar{p}) \vee p \vee \bar{q}$

$\Rightarrow p \vee \bar{p} \equiv \text{True}$

$\Rightarrow T \vee (p \vee \bar{q})$

$\Rightarrow T \vee p \equiv T$

$\therefore\ p \vee \neg (p \wedge q) = \text{True}$

5. Prove using laws of logic

(a) $\left[(p \vee q) \wedge (p \vee \sim q)\right] \vee q \leftrightarrow p \vee q$ **[D-05]**

Soln. :

$$\text{L.H.S.} = q \vee \left[(p \vee q) \wedge (p \vee \sim q)\right] \quad \text{... Commutative law}$$
$$= \left[(q \vee (p \vee q)) \wedge (q \vee (p \vee \sim q))\right] \quad \text{... Distributive law}$$
$$= [p \vee q] \wedge [p \vee t]$$
$$= [p \vee q] \wedge t \ \text{....} p \vee t = t$$
$$= p \vee q.$$
$$= \text{R.H.S.}$$

(b) PT : $p \vee q \vee (\sim p \wedge \sim q \wedge r) \leftrightarrow p \vee q \vee r$ **[M-04]**

Soln. :

$$\text{L.H.S.} = p \vee q \vee (\sim p \wedge \sim q \wedge r)$$
$$= \left[(p \vee q) \vee (\sim p \wedge \sim q)\right] \wedge \left[(p \vee q) \vee r\right] \text{...Distributive law}$$
$$= \left[(p \vee q) \vee \sim (p \vee q)\right] \wedge [p \vee q \vee r]$$
$$= t \wedge [p \vee q \vee r] \text{...} p \vee \sim p = t$$
$$= p \vee q \vee r$$
$$= \text{R.H.S.}$$

values of [M-01]

[D-00]

[D-05]

[M-04]

(c) PT : $A \to (p \vee c) \leftrightarrow (A \wedge \sim p) \to c$ **[D-00]**

Soln. :

$$\text{L.H.S.} = A \to (p \vee c)$$
$$= \sim A \vee (p \vee c) \ldots p \to q \equiv \sim p \vee q$$
$$= (p \vee c) \vee \sim A \qquad \text{...Commutative law}$$
$$\text{R.H.S.} = (A \wedge \sim p) \to c$$
$$= \sim (A \wedge \sim p) \vee c \ldots p \to q \equiv \sim p \vee q$$
$$= \sim A \vee \sim (\sim p) \vee c \qquad \text{...De Morgan's theorem}$$
$$= \sim A \vee p \vee c$$
$$= p \vee c \vee \sim A$$
$$= \text{L.H.S}$$

(d) $\sim (p \wedge q) \to (\sim p \vee (\sim p \vee q)) \leftrightarrow (\sim p \vee q)$ **[D-00]**

Soln. :

$$\text{L.H.S.} = \sim (p \wedge q) \to (\sim p \vee (\sim p \vee q))$$
$$= (p \wedge q) \vee (\sim p \vee (\sim p \vee q)) \quad \ldots p \to q \equiv \sim p \vee q$$
$$= (p \wedge q) \vee (\sim p \vee q) \ldots p \vee p = p$$
$$= (p \wedge q) \vee \sim p \vee q$$
$$= p \wedge \sim p \wedge (q \vee \sim p) \vee q$$
$$= F \wedge q \vee \sim p \vee q = F \vee (q \vee \sim p) \quad \ldots p \wedge \sim p = F$$
$$= q \vee \sim p$$
$$= \sim p \vee q$$
$$= \text{R.H.S.}$$

(e) $[(\sim p \vee \sim q) \to (p \wedge q \wedge t)] \leftrightarrow p \wedge q$ **[D-03]**

Soln. :

$$\text{L.H.S.} = [(\sim p \vee \sim q) \to (p \wedge q \wedge t)]$$
$$= [\sim (p \wedge q) \to (p \wedge q)] \ldots p \to q \equiv \sim p \vee q$$
$$= [\sim (\sim (p \wedge q)) \vee (p \wedge q)]$$
$$= p \wedge q \vee p \wedge q$$
$$= p \wedge q \qquad \ldots p \vee p = p$$
$$= \text{R.H.S.}$$

(f) $(p \to q) \wedge [\sim q \wedge (t \vee \sim q)] \leftrightarrow \sim (q \vee p)$ **[D-03]**

Soln. :

$$\text{L.H.S.} = (p \to q) \wedge [\sim q \wedge (t \vee \sim q)]$$
$$= (p \to q) \wedge [\sim q \wedge t] \qquad \text{... Identity law}$$
$$= (p \to q) \wedge \sim q \qquad \text{... Identity law}$$
$$= (\sim p \vee q) \wedge \sim q \ldots p \to q = \sim p \vee q$$
$$= \sim q \wedge (\sim p \vee q)$$
$$= (\sim q \wedge \sim p) \vee (\sim q \wedge q)$$
$$= (\sim q \wedge \sim p) \vee f \ldots (\sim p \wedge p = F)$$
$$= \sim q \wedge \sim p \ldots (p \vee F = F)$$
$$= \sim (p \vee q) \qquad \text{... Identity law}$$
$$= \text{R.H.S.}$$

(g) PT : $a \wedge (\bar{a} \vee b) = a \wedge b$ **[M-06]**

Soln. :

$$\text{L.H.S.} = a \wedge (\bar{a} \vee b)$$

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Relation :

Let A and B be two non–empty sets, if $R \subseteq A \times B$ then R is called a relation from A to B.
If $R \subseteq A \times B$ and $(a, b) \in R$, we say a R b (a is related to b)
If a is not related to b we write this negation as $a \not R b$.
If A = B, we say that $R \subseteq A \times A$ is a relation on A, instead from A to A.

Examples :

1. Determine which of the following relations from A = {a, b, c} to B = {1, 2}
   a) $R_1 = \{(a, 1)\ (a, 2)\ (c, 2)\}$
   b) $R_2 = \{(a, 2)\ (b, 1)\ (2, a)\}$
   c) $R_3 = \{(c, 1)\ (c, 2)\ (c, 3)\}$
   d) $R_4 = \{(b, 2)\}$
   e) $R_5 = \phi$
   f) $R_6 = \{a \times b\}$

Soln. :

Here A = {a, b, c} & B = {1, 2}
$\therefore$ $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$
a) $\because$ $R_1 \subseteq A \times B$ $\therefore$ It is a relation.
b) $\because$ $R_2 \not\subseteq A \times B$ $\therefore$ It is not a relation
c) $\because$ $R_3 \subseteq A \times B$ $\therefore$ It is a relation
d) $\because$ $R_4 \subseteq A \times B$ $\therefore$ It is a relation
e) $\because$ $R_5 \subseteq A \times B$ $\therefore$ It is a relation
$R_5 = \phi \subseteq A \times B$, is called an empty relation from A to B.
f) $\because$ $R_6 = A \times B \subseteq A \times B$ $\therefore$ It is a relation, is called universal relation from A to B.

Domain of the relation

Let A and B be two non empty sets and R be a relation from A to B, i.e., $R \subseteq A \times B$ then the domain of R is a subset of A such that the collection of first elements of all ordered pairs of R.
i.e. Domain of $R = \{a \mid (a, b) \in R\}$

Range of the relation

If R is a relation from A to B, then the range of R is a subset of B such that the collection of second element of all ordered pairs of R i.e. Range of $R = \{b \mid (a, b) \in R\}$

Inverse Relation

Let R be a relation from A to B then the inverse relation of R is the collection of all (b, a) such that $(a, b) \in R$. It is denoted by R^{-1}, defined as
$R^{-1} = \{(b, a) \mid (a, b) \in R\}$

2. Let 'R' be the relation on A = {1, 2, 3, 4} defined by "x is less than y"
   i.e. 'R' is the relation <.
   i) Write 'R' as a set of ordered pairs
   ii) Find R^{-1} of the relation R.
   iii) Can R^{-1} be described in words ?

Soln. :

i) Here A = {1, 2, 3, 4}
$R = \{(x, y) \mid x R y \text{ iff } x < y\}$
$= \{(1, 2), (1, 3), (1, 4)), (2, 3), (2, 4), (3, 4)\}$

ii) $R^{-1} = \{(y, x) \mid (x, y) \in R\}$
$= \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$

iii) R^{-1} can be described as a statement "$xR^{-1}y$ iff $x > y$" i.e. "x is greater than y".

3. Let S be the relation on the set N of the +ve integers defined by an equation x + 3y = 13

i.e. $S = \{(x, y) | x + 3y = 13\}$

i) Write S as a set of ordered pairs.

ii) Find the inverse relation S^{-1} of S and describe S^{-1} by an equation.

Soln. :

i) $S = \{(x, y) | x + 3y = 13\}$

$\because$ x + 3y = 13

$\therefore$ x = 13 – 3y $\Rightarrow$ y can not exceed 4.

$\therefore$ y = 1, 2, 3, 4 $\Rightarrow$ corresponding x = 10, 7, 4, 1

$\therefore$ S = { (10, 1), (7, 2), (4, 3), (1, 4) }

ii) $S^{-1} = \{(y, x) \mid (x, y) \in S\}$

$= \{ (1, 10), (2, 7), (3, 4), (4, 1) \}$

S^{-1} can be described as "x is related to y iff 3x + y = 13

i.e. $xS^{-1}y$ iff 3x + y = 13

4. Let R and S be the following relations on A = {1, 2, 3};

R = {(1, 1), (1, 2), (2, 3), (3, 1), (3, 3)}

S = {(2, 3), {1, 3), (2, 1), (3, 3)}

find : $R \cap S, R \cup S, R^C$

Soln. :

Here

R = {(1, 1), (1, 2), (2, 3), (3, 1), (3, 3)}

S = {(2, 3), {1, 3), (2, 1), (3, 3)}

$\therefore R \cap S = \{(x, y) \mid (x, y) \in R \ \& \ (x, y) \in S\}$

$= \{(1, 3), (2, 3), (3, 3)\}$

$R \cup S = \{(x, y) \mid (x, y) \in R \text{ or } (x, y) \in S\}$

$= ((1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3)\}$

Complement of R is denoted by R^C and defined as

$R^C = A \times A \setminus R$

$= \{(x, y) \mid (x, y) \in A \times A \text{ but } (x, y) \notin R\}$

$= \{(1, 3), (2, 1), (2, 2), (3, 2)\}$

Representation of the Relations :

Digraphs :

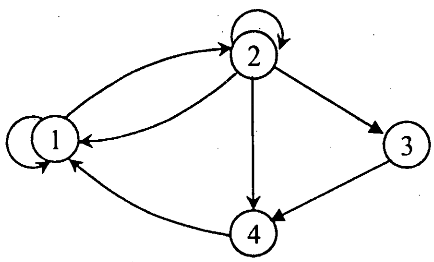
If A is finite set and R is a relation, we can represent R in figure as follows. Draw a small circle for each element of A and label the circle with corresponding element of A.

These circles are known as vertices. Draw an arrow called an edge, from vertex a_i to a_j iff a_i R a_j. The resulting picture of R is called a directed graph or digraph of R.

5. Let A = {1, 2, 3, 4}

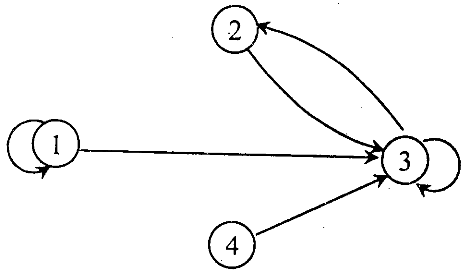
R = {(1, 1), (1, 2), (2, 1), (2, 3), (2, 4), (3, 4), (4, 1)}

Then the digraphs of R is as shown in figure.



' = 13

6. Find the relation determined by digraph.



$\because$ a_i R a_j iff there is an edge from a_i to a_j

$\therefore$ R = {(1, 1), (1, 3), (2, 3), (3, 2), (3, 3), (4, 3)}

The matrix of a relation :

We can represent a relation between two finite sets containing m and n elements i.e.

$$A = \{a_1, a_2, \ldots a_m\}$$
$$B = \{b_1, b_2, \ldots b_n\}$$

in a matrix $M_R = \left[m_{ij}\right]_{m\times n}$

$$\text{where } m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \text{ i.e. } a_i R b_j \\ 0 & \text{if } (a_i, b_j) \notin R \text{ i.e. } a_i \not R b_j \end{cases}$$

then the matrix M_R is called the matrix form of relation R from A to B.

7. Consider the matrix

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$ where R is a relation from A to B.

$\therefore$ $M_R \Rightarrow |A| = 3, \ \& \ |B| = 4$

Let $A = \{a_1, a_2, a_3\}$ & $B = \{b_1, b_2, b_3, b_4\}$

$R = \{(a_i, a_j) \mid (a_i, a_j) \in R \text{ iff } m_{ij} = 1\}$

$\therefore$ $R = \{(a_1,b_1)(a_1,b_4),(a_2,b_2)(a_2,b_3),(a_3,b_1)(a_3,b_3)\}$

Definitions :

In–degrees :

Let R is a relation on A & a $\in$ A then the in–degree of a $\in$ A is the number of b such that b R a i.e. number of b | (b, a) $\in$ R

mall circle

Out–degrees :

i iff a_i R a_j.

Let R is a relation on A & a $\in$ A then the out–degree of a $\in$ A is the number of b such that a R b i.e. number of b | (a, b) $\in$ R.

8. Find matrix relation and relation digraph

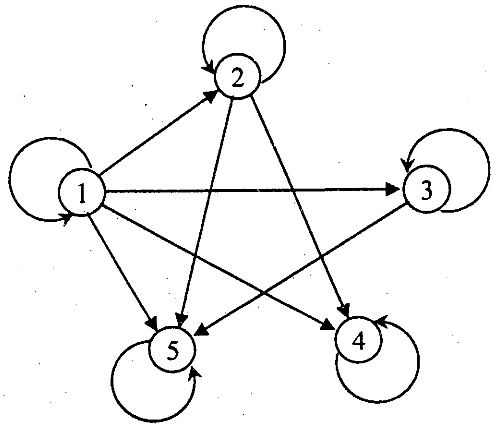
A = {1, 2, 3, 4, 5} = B; a R b iff b is multiple of a.

Find in–degrees & out–degrees of each vertex.

$$M_R = \left[m_{ij}\right]_{5\times5} \text{ where } m_{ij} = \begin{cases} 1 & \text{if } a_i \, R \, a_j \text{ i.e. } a_j \text{ is multiple of } a_i \\ 0 & \text{if } a_i \not R a_j \text{ i.e. } a_j \text{ is not multiple of } a_j \end{cases}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

R can be expressed as the collection of ordered pairs.
R = {(a, b) | a R b} iff b is multiple of a.
R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)}
and digraph of R is

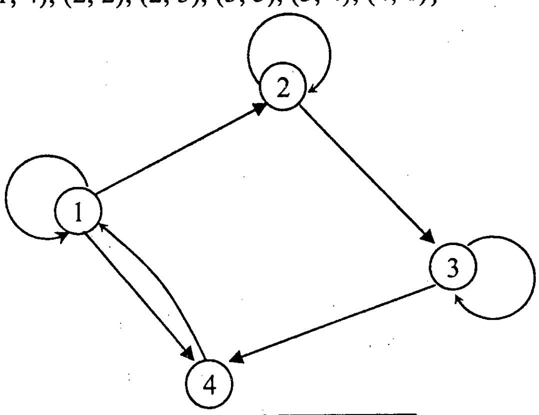


	1	2	3	4	6
In–degrees	1	2	2	3	2
Out–degrees	5	3	2	1	1

9. Give the relation defined on A and its digraph and list in-degress and out-degrees of all vertices.

Let A = {1, 2, 3, 4} and $M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Here R = {(a, b) | a R b}
R = {(1, 1), (1, 2), (1, 4), (2, 2), (2, 3), (3, 3), (3, 4), (4, 1)}
and digraph of R is



	1	2	3	4
In–degrees	2	2	2	2
Out–degrees	3	2	2	1

10. Find the relation R defined on A and its digraph. List the in-degree and out-degree of all vertices. Where A = {a, b, c, d, e} and

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

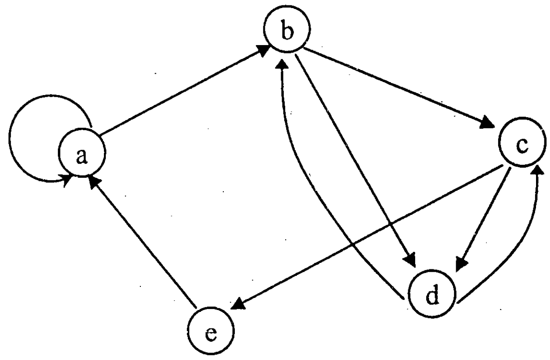
Relation can be found as

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\because m_{ij} = \begin{cases} 1 & \text{if } a_i R a_j \\ 0 & \text{if } a_i a_j \end{cases}$

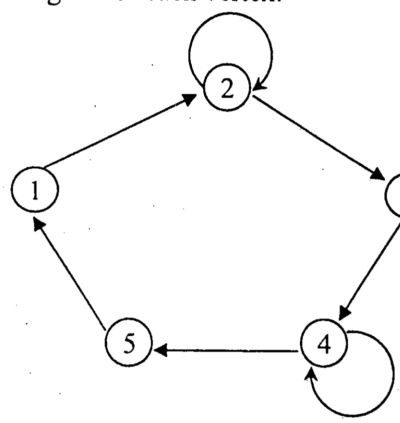
$\therefore R = \{(a_i, a_j) \mid a_i R a_j\}$

$\therefore R = \{(a, a), (a, b), (b, c), (b, d), (c, d), (c, e), (d, b), (d, c), (e, a)\}$ and digraph of given R is



	a	b	c	d	e
In–degrees	2	2	2	2	1
Out–degrees	2	2	2	2	1

11. Find the relation determined by the diagraph and give its matrix. Give the in-degrees and out-degrees of each vertex.



$A = \{1, 2, 3, 4, 5\}$

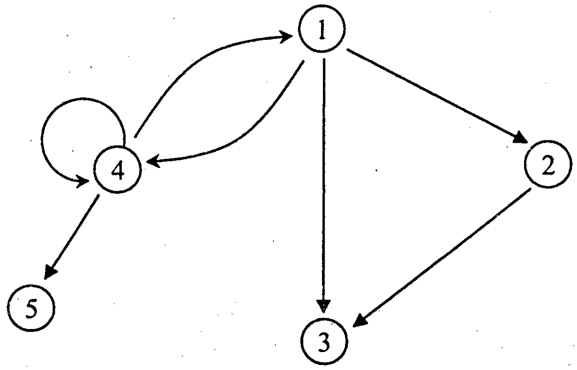
$R = \{(1, 2), (2, 2), (2, 3), (3, 4), (4, 4), (5, 1), (5, 4)\}$

And

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}_{5\times 5} \end{array}$$

	1	2	3	4	5
In–degrees	1	2	1	3	0
Out–degrees	1	2	1	1	2

12. (a) Find the relation determined by the diagraph and give its matrix.
(b) Give the in-degrees and out-degrees of each vertex.



Here,
A = {1, 2, 3, 4, 5}
R = {(1, 2), (1, 3), (1, 4), (2, 3), (4, 1), (4, 4), (4, 5)}

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 5} \end{array}$$

	1	2	3	4	5
In–degrees	1	1	2	2	1
Out–degrees	3	1	0	3	0

Paths in Relations and Digraphs :

Suppose that R is a relation on a set A.
A path of length n in R from a to b is a finite sequence.

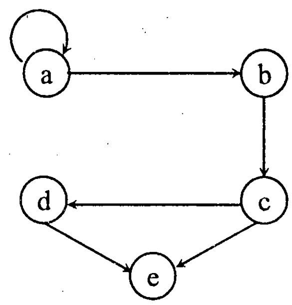
Π : a, x_1, x_2, ……, x_{n-1}, b ; beginning with a and ending with b, such that aRx_1, x_1Rx_2, ……, $x_{n-1}R_b$.

Note : i) A path that begins and end at the same vertex is called cycle.
ii) A path of length n involves n + 1 elements of A, not necessarily distinct.

Connectivity Relation
A relation denoted by R^∞ and defined as x R^∞ y if there is a path of any length from x to y in R, is called connectivity relation for R.

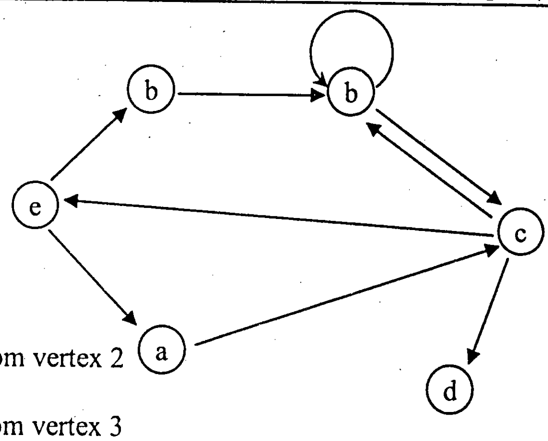
13. Let A = {a, b, c, d, e} and R = {(a, a), (a, b), (b, c), (c, d), (c, e), (d, e)}.
Compute R^2 and R^∞.

Sol. :
The digraph of R is shown as



R^2 = {(a, b) | (a, c) & (c, b) ∈ R for some c}
= {(a, a), (a, b), (a, c), (b, e), (b, d), (c, e)}
R^∞ = {(a, b) | there is a path of any length from a to b}
= {(a, a), (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, d), (c, e), (d, e)}

14. Let R be a relation whose digraph is given in figure.



(i) List all paths of length 1
(ii) (a) List all paths of length 2 starting from vertex 2
(b) List all paths of length 2
(iii)(a) List all paths of length 3 starting from vertex 3
(b) List all paths of length 3
(iv) Find a cycle starting at vertex 2
(v) Find a cycle starting at vertex 6
(vi) Draw a diagraph of R^2
(vii) Find M_{R^2}
(viii)(a) Find R^∞ (b) Find M_{R^∞}

Soln. :

(i) paths of length 1 are :
$1 \to 2, 1 \to 6, 2 \to 3, 3 \to 3, 3 \to 4, 4 \to 3, 4 \to 1, 4 \to 5, 6 \to 4$

(ii) (a) $2R^23$ $\because 2 \to 3 \to 3;$
$2R^24$ $\because 2 \to 3 \to 4;$

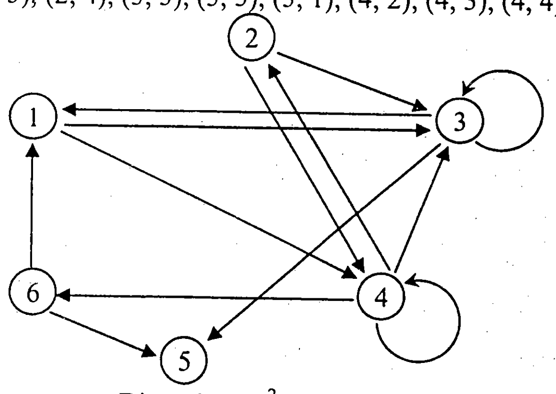
(b) Paths of length 2 are
$1 \to 2 \to 3, 1 \to 6 \to 4, 2 \to 3 \to 3, 2 \to 3 \to 4, 3 \to 3 \to 4, 3 \to 4 \to 3,$
$3 \to 4 \to 1$
$3 \to 4 \to 5, 4 \to 3 \to 3, 4 \to 1 \to 2, 4 \to 3 \to 4, 6 \to 4 \to 1, 6 \to 4 \to 3,$
$6 \to 4 \to 5$

$\therefore R^2 = \{(a, b) \mid a R c \,\&\, c R b \text{ for some } c\}$
$= \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 3), (3, 4), (3, 5), (4, 2), (4, 3), (4, 4), (6, 1), (6, 3), (6, 5)\}$

(iii)(a) $3R^31$ $\because 3 \to 3 \to 4 \to 1$
$3R^32$ $\because 3 \to 4 \to 1 \to 2$
$3R^33$ $\because 3 \to 3 \to 4 \to 3,\quad 3 \to 4 \to 3 \to 3, 3 \to 3 \to 3 \to 3$
$3R^34$ $\because 3 \to 4 \to 3 \to 4,\quad 3 \to 3 \to 3 \to 4$
$3R^35$ $\because 3 \to 3 \to 4 \to 5$
$3R^36$ $\because 3 \to 4 \to 1 \to 6$

(b) Paths of length 3 are
$R^3 = \{(a, b) \mid a R c \,\&\, c R d \,\&\, d R b \text{ for some c and d}\}$
$= \{(1, 1), (1, 3), (1, 4), (1, 5), (2, 1), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 5), (3, 6), (4, 1), (4, 3), (4, 4), (4, 5), (6, 2), (6, 3), (6, 5), (6, 6)\}$

(iv) A cycle starting at vertex 2 is $2 \to 3 \to 4 \to 1 \to 2$
(v) A cycle starting at vertex 6 is $6 \to 4 \to 1 \to 6$
(vi) $R = \{(1, 2), (2, 3), (3, 3), (3, 4), (4, 3), (4, 1), (4, 5), (1, 6), (6, 4)\}$
$R^2 = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 5), (3, 1), (4, 2), (4, 3), (4, 4), (4, 6), (6, 1), (6, 5)\}$



Digraph of R^2

$$M_R = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$M_{R^2} = M_R \times M_R = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$M_{R^3} = M_{R^2} \times M_R = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$$M_{R^4} = M_{R^3} \times M_R = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$M_{R^5} = M_{R^4} \times M_R = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$M_{R^\infty} = M_R \cup M_{R^2} \cup M_{R^3} \cup M_{R^4} \cup M_{R^5}$$

$R^\infty = \{(a, b) \mid \text{there is a path of any length from a to b}\}$

$\therefore R^\infty = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

Composition Relations :

Let A, B and C be the sets, R is a relation from A to B and S is a relation from B to C. We can define a new relation, the composition of R and S, written SoR. The relation SoR from A to C and is defined as follows :
If $a \in A$ and $c \in C$ then a (SoR) c iff for some $b \in B$ we have a R b & b S c.

15. Let A = {1, 2, 3}, B = {a, b, c} and C = {x, y, z}
Consider the following relations R and S from A to B and B to C respectively.
R = {(1, b), (2, a), (2, c)} and S = {(a, y), (b, x), (c, y), (c, z)}
(a) find the composition relation SoR.
(b) find the matrices M_R, M_S and M_{RoS} of the respective relations R, S and SoR and compare M_{SoR} to the product $M_R \cdot M_S$.

Soln :
Here A = {1, 2, 3}, B = {a, b, c} and C = {x, y, z}
R = {(1, b), (2, a), (2, c)} and S = {(a, y), (b, x), (c, y), (c, z)}
(a) SoR = {(a, c) | for some $b \in B$, a R b and b S c }
= {(1, x), (2, y), (2, z) }

(b) $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $M_S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$M_{SoR} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$M_R.M_S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

We find finally $M_{SoR} = M_R \cdot M_S$

16. Let R and S be the relations on A = {1, 2, 3, 4} defined by
$R = \{(1,1),(3,1),(3,4),(4,2),(4,3)\}$
$S = \{(1,3),(2,1),(3,1),(3,2),(4,4)\}$
i) Find the composition relation RoS
ii) Find the composition relation SoR
iii) Find R^2 = RoR & M_{R^2}
iv) Find S^2 = SoS & M_{S^2}
v) Find R^3 = RoRoR
vi) Find R^{-1}
vii) Find RoR^{-1} & R^{-1}o R

Soln :
Here A = {1, 2, 3, 4}
The relation R & S defined on A as
$R = \{(1,1),(3,1),(3,4),(4,2),(4,3)\}$
$S = \{(1,3),(2,1),(3,1),(3,2),(4,4)\}$
i) RoS = {(a, c) | a S b and b R c for some b }
= {(1, 1), (1, 4), (2, 1), (3, 1), (4, 2), (4, 3) }
ii) SoR = {(a, c) | a R b and b S c for some b }
= {(1, 3), (3, 3), (3, 4), (4, 1), (4, 2) }

, 4), (2, 5),
3), (4, 4),

iii) $R^2 = RoR$
$= \{(a, c) \mid a\ R\ b \text{ and } b\ R\ c \text{ for some } b\}$
$= \{(1, 1), (3, 1), (3, 2), (3, 3), (4, 1), (4, 4)\}$

$$M_{R^2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

iv) $S^2 = SoS$
$= \{(a, c) \mid a\ S\ b \text{ and } b\ S\ c \text{ for some } b\}$
$= \{(1, 3), (1, 4), (3, 4), (4, 3)\}$

$$M_{S^2} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

v) $R^3 = RoRoR$
$= R^2 oR = RoR^2$
$= \{(a, c) \mid a\ R\ b \text{ and } b\ R^2\ c \text{ for some } b\}$
$= \{(1, 1), (3, 1), (3, 4), (4, 1), (4, 2), (4, 3)\}$

vi) $R^{-1} = \{(y, x) \mid (x, y) \in R\}$
$= \{(1, 1), (1, 3), (4, 3), (2, 4), (3, 4)\}$

vii) $RoR^{-1} = \{(a, c) \mid a\ R^{-1}\ b\ \&\ b\ R\ c \text{ for some } b\}$
$= \{(1, 1), (1, 4), (2, 2), (2, 3), (3, 2), (3, 3), (4, 1), (4, 4)\}$

$R^{-1}oR = \{(a, c) \mid a\ R\ b\ \&\ b\ R^{-1}\ c \text{ for some } b\}$
$= \{(1, 1), (1, 3), (3, 1), (3, 3), (4, 4)\}$

Note : $RoR^{-1} \neq R^{-1}oR$

Properties of Relations :

i). Reflexive : if $(x, x) \in R\ \forall\ x \in A$; i.e. $x\ R\ x\ \forall\ x \in A$.

ii) Diagonal Relation : $x\ R\ x\ \forall\ x \in A$ but $x \not R\ y$ for any $y \in A$
it is denoted by Δ and defined as
$\Delta = \{(x, x)\ \forall\ x \in A\}$
It is also known as equality relation

iii) Irreflexive : $x \not R\ x\ \forall\ x \in A$
Note : i) A relation R on A is said to be reflexive if $\Delta \subseteq R$.
ii) A relation R on A is said to be irreflexive if $\Delta \cap R = \phi$
iii) A relation R may be either reflexive or irreflexive but cannot be both.

iv) Symmetric : if $(x, y) \in R$ then $(y, x) \in R$; i.e. $x\ R\ y \Rightarrow y\ R\ x$.

v) Not symmetric : $x\ R\ y$ but $y \not R\ x$ for some $(x, y) \in R$

vi) Asymmetric : If $x\ R\ y \not\Rightarrow y\ R\ x\ \forall\ (x, y) \in R$
Note : Asymmetric relation is a particular case of not symmetric.

vii) Anti–Symmetric : If $x\ R\ y\ \&\ y\ R\ x \Rightarrow x = y$
Note : If R is a relation on A. Then
i) R is symmetric iff $R = R^{-1}$
ii) R is asymmetric iff $R \cap R^{-1} = \phi$
iii) R is anti symmetric iff $R \cap R^{-1} \subseteq \Delta$

viii) Transitive : if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.
i.e $x\ R\ y\ \&\ y\ R\ z \Rightarrow x\ R\ z$.

Note :

i) If $M_{R^2} = M_R$ then R is transitive but converse statement need not be true.

ii) If the non zero entries in M_{R^2} are present in M_R at the same position then R is transitive. i.e. $R^2 \subseteq R$.

17. Let A = {1, 2, 3, 4}. Determine whether the relation is reflexive, symmetric, asymmetric, antisymmetric, or transitive.

a) $R = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)\}$

b) $R = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$

Soln :

Here $A = \{1, 2, 3, 4\}$.

a) $R = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)\}$

i) Reflexive : Diagonal relation of A,
$\Delta = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
$\because \Delta \subseteq R$
$\therefore$ R is reflexive.

ii) Symmetric : $R^{-1} = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$
$= R$
$\therefore$ R is symmetric

iii) Asymmetric : $R \cap R^{-1} = R \neq \phi$
$\therefore$ R is not Asymmetric

iv) Antisymmetric: $\because R \cap R^{-1} \not\subseteq \Delta$
$\therefore$ R is not antisymmetric

v) Transitive : $x R y \;\&\; y R z \Rightarrow x R z$
$\therefore$ R is transitive

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$M_{R^2} = M_R \times M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= M_R$$

$\therefore$ R is transitive.

b) $A = \{1, 2, 3, 4\}$

$R = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$

Diagonal relation of A,

$\Delta = \{(x, x) \;\forall\; x \in A\}$
$= \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

i) Reflexive : $\because \Delta \not\subseteq R$
$\therefore$ R is not reflexive.

ii) Irreflexive : $\because \Delta \cap R = \phi$
$\therefore$ R is irreflexive

iii) Symmetric : $R^{-1} = \{(a, b) \mid (b, a) \in R\}$
$= \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$
$\because$ $R^{-1} \neq R$
$\therefore$ R is not symmetric

iv) Asymmetric : $\because$ $R \cap R^{-1} = \phi$
$\therefore$ R is asymmetric

v) Antisymmetric: $\because$ $R \cap R^{-1} = \phi \subseteq \Delta$
$\therefore$ R is antisymmetric

vi) Transitive :

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = M_R \times M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\because$ Non zero entries in M_{R^2} are present in M_R at the same positions.
$\therefore$ $R^2 \subseteq R$
$\therefore$ R is transitive.

(c) $A = \{1, 2, 3, 4\}$
$R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

(i) Reflexive : $x R x \ \forall \ x \in A$
$\therefore$ R is reflexive
$\therefore$ It cannot be irreflexive

(ii) Symmetrix : $\because$ $R^{-1} = R$
$\therefore$ R is symmetric

(iii) Asymmetric : $R \cap R^{-1} \neq \phi$
$\therefore$ R is not asymmetric

(iv) Antisymmetric: $R^{-1} \cap R \subseteq \Delta$
$\therefore$ R is antisymmetric

(v) Transitive : $x R y, y R z \Rightarrow x R z$ (contrapositive)
$\therefore$ R is transitive.

(d) $A = \{1, 2, 3, 4\}$
$R = \phi$

(i) Reflexive : $x \not R x \forall x \in A$
$\therefore$ R is not reflexive
$\therefore$ it is irreflexive

(ii) Symmetric : $\because$ $R^{-1} = R$
$\therefore$ R is symmetric

(iii) Asymmetric : $R \cap R^{-1} = \phi$
$\therefore$ R is asymmetric

(iv) Antisymmetric: $R^{-1} \cap R = \phi \subseteq \Delta$
$\therefore$ R is antisymmetric

(v) Transitive : $x R y, y R z \Rightarrow x R z$ (contrapositive)
$\therefore$ R is transitive.

(e) $A = \{1, 2, 3, 4\}$

$R = A \times A$

(i) Reflexive : $x R x \ \forall \ x \in A$

$\therefore$ R is reflexive

$\therefore$ It cannot be irreflexive.

(ii) Symmetric : $\because R^{-1} = R$

$\therefore$ R is symmetric

(iii)Asymmetric : $R \cap R^{-1} = R \neq \phi$

$\therefore$ R is not asymmetric.

(iv)Antisymmetric: $R^{-1} \cap R = R \not\subseteq \Delta$

$\therefore$ R is not antisymmetric.

(v) Transitive : $x R y, y R z \Rightarrow x R z \ \forall x, y, z \in A$

$\therefore$ R is transitive.

Equivalence Relation :

Let R be a relation on a set A then the relation R is said to be an Equivalence relation, if it is

i) Reflexive : $x R x \ \forall \ x \in A$.

ii) Symmetric : $x R y \Rightarrow y R x$.

iii) Transitive : $x R y \ \& \ y R z \Rightarrow x R z$.

18. Let R be a relation on Z which is defined by as x R y, iff 2x + 3y, is divisible by 5 is an equivalence relation.

Soln. :

Here Z = set of integers

$Z = \{-\infty \ldots -2, -1, 0, 1, 2, \ldots \infty\}$

$Z \times Z = \{ (x, y) \mid x, y \in Z \}$

$R = \{ (x, y) \in Z \times Z \mid 2x + 3y \text{ is divisible by } 5 \}$

i) Reflexive : $\forall \ x \in Z, 2x + 3x = 5x$ and 5x is divisible by 5

$\therefore$ $x R x \quad \forall \ x \in Z$

$\therefore$ R is reflexive

ii) Symmetric : Let x Ry

$\Rightarrow$ 2x + 3y is divisible by 5

$$\Rightarrow \frac{2x + 3y}{5} = k$$

$$\Rightarrow \frac{5x - 3x + 5y - 2y}{5} = k$$

$$\Rightarrow \frac{5(x + y) - (3x + 2y)}{5} = k$$

$$\Rightarrow \frac{2y + 3x}{5} = +(x + y) - k = m(\text{let})$$

$$\Rightarrow \frac{2y + 3x}{5} = m$$

$\Rightarrow$ 2y + 3x is divisible by 5.

$\Rightarrow$ y Rx

$\Rightarrow$ R is symmetric.

iii) Transitive : Let x Ry and y Rz

$\Rightarrow$ 2x + 3y is divisible by 5 and 2y + 3z is divisible by 5.

$$\Rightarrow \frac{2x + 3y}{5} = m \text{ and } \frac{2y + 3z}{5} = n$$

$$\Rightarrow \frac{2x + 5y + 3z}{5} = m + n$$

$$\Rightarrow \frac{2x+3z}{5} + y = m + n$$

$$\Rightarrow \frac{2x+3z}{5} = m + n - y = p(\text{let})$$

$$\Rightarrow \frac{2x+3z}{5} = p$$

$\Rightarrow$ 2x + 3z is divisible by 5.

$\Rightarrow$ (x, z) $\in$ R

$\therefore$ R is transitive

$\therefore$ R is an equivalence relation.

19. Let R be a relation of 'Z' which is defined as x Ry iff 3x + 5y is divisible by 8. Is R an equivalence relation.

Soln. :

Here Z $= \{-\infty \ldots -2, -1, 0, 1, 2, \ldots\ldots \infty\}$

$Z \times Z = \{(x, y) \mid x, y \in Z\}$

& R $= \{(x, y) \in Z \times Z \mid 3x + 5y \text{ is divisible by } 8\}$

$R \subseteq Z \times Z$

(i) Reflexive :

3x + 5x = 8x and 8x is divisible by 8

$\therefore$ (x, x) $\in$ R $\forall$ x $\in$ Z

$\therefore$ R is reflexive

(ii) Symmetric: Let (x, y) $\in$ R

$\Rightarrow$ 3x + 5y is divisible by 8.

$$\Rightarrow \frac{3x+5y}{8} = k$$

$$\Rightarrow \frac{8x-5x+8y-3y}{8} = k$$

$$\Rightarrow \frac{8x+8y-(5x+3y)}{8} = k$$

$$\Rightarrow \frac{8(x+y)}{8} - \frac{5x+3y}{8} = k$$

$$\Rightarrow \frac{3y+5x}{8} = x + y - k$$

$$\Rightarrow \frac{3y+5x}{8} = p(\text{let})$$

$\Rightarrow$ (3y + 5x) is divisible by 8.

$\Rightarrow$ (4, x) $\in$ R

$\therefore$ R is symmetric.

(iii) Transitive : Let (x, y) $\in$ R and (y, z) $\in$ R

$\Rightarrow$ 3x + 5y is divisible by 8 &

3y + 5z is divisible by 8.

$$\Rightarrow \frac{3x+5y}{8} = m_1 \ \& \ \frac{3y+5z}{8} = m_2$$

$$\Rightarrow \frac{3x+5y+3y+5z}{8} = m_1 + m_2$$

$$\Rightarrow \frac{3x+5z}{8} + \frac{8y}{8} = m_1 + m_2$$

$$\Rightarrow \frac{3x+5z}{8} = m_1 + m_2 - 1$$

$\Rightarrow \dfrac{3x+5z}{8} = m(\text{let})$

$\Rightarrow 3x + 5z$ is divisible by 8.

$\therefore$ R is transitive

$\therefore$ R is an equivalence relation.

Equivalence Classes :

Let R be an equivalence relation on a set A then the equivalence class corresponding to $a \in A$ is defined as $\{ x \in A \mid x Ra \}$ and , this set is denoted as $\bar{a}$ or [a] or R(a).

Thus $\bar{a} = \{ x \in A \mid x Ra \}$

R an

Quotient Set

A set denoted by A/R which is the collection of all distinct and disjoint sets of equivalence classes is called quotient set of A induced by an equivalence relation R.

Note : Quotient set is a partition of A

20. Consider the relation "congruence modulo m" given by
x Ry iff x – y is divisible by m over the set of positive integers.
(i) Show that R is an equivalence relation
(ii) Find all equivalence classes
(iii) Show that if x_1 R y_1 and x_2 R y_2 then $(x_1 + x_2)$ R$(y_1 + y_2)$

Soln.:

$R = \{(x, y) \in Z^+ \times Z^+ \mid x - y \text{ is divisible by m}\}$

where Z^+ is the set of positive integers.

(i) (a) Reflexive : $\because x - x = 0$ and 0 is divisible by in

$\therefore (x, x) \in R$ for all $x \in Z^+$

$\therefore$ R is reflexive

(b) Symmetric: Let $(x, y) \in R$

$\Rightarrow x - y$ is divisible by m

$\Rightarrow \dfrac{x-y}{m} = k$

$\Rightarrow \dfrac{y-x}{m} = -k$

$\Rightarrow \dfrac{y-k}{m} = p$ (let) $p = -k$

$\Rightarrow (y, x) \in R$

$\Rightarrow$ R is symmetric.

(c) Transitive : Let $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow x - y$ is divided by m & $y - z$ is divisible by m

$\Rightarrow \dfrac{x-y}{m} = m_1 \ \& \ \dfrac{y-z}{m} = m_2$

$\Rightarrow \dfrac{x-y+y-z}{m} = m_1 + m_2$

$\Rightarrow \dfrac{x-z}{m} = m_1 + m_2 = k(\text{let})$

$\Rightarrow x - 4$ is divisible by m

$\Rightarrow (x, z) \in R$

$\therefore$ R is an equivalence relation.

(ii) $\bar{x} = \{a \mid a R x$ OR $x - a$ is divisible by m i.e. $a \equiv x \pmod m\}$

$\bar{0} = \{\ldots\ldots -2m, -m, 0, m, 2m, \ldots\ldots\}$

$\bar{1} = \{\ldots\ldots -2m+1, -m+1, 1, m+1, 2m+1, \ldots..\}$

$\bar{2} = \{\ldots\ldots -2m+2, -m+2, 2, m+2, 2m+2, \ldots..\}$

$\overline{m-1} = \{\ldots\ldots -m-1, -1, 2m-1, 3m-1, \ldots\ldots\}$

$\therefore$ Quotient set of Z^+ induced by an equivalence relation R is

$Z^+/R = \{\bar{0}, \bar{1}, \bar{2}, \ldots \overline{m-1}\}$

(iii) $x_1 \, R \, y_1$ and $x_2 \, R \, y_2$

$\Rightarrow x_1 - y_1$ is divisible by m & $x_2 - y_2$ is divisible by m.

$\Rightarrow (x_1 - y_1) + (x_2 - y_2)$ is divisible by m.

$\Rightarrow (x_1 + x_2) - (y_1 + y_2)$ is divisible by m.

$\Rightarrow (x_1 + x_2) \, R \, (y_1 + y_2)$

21. Let $A = \{ a, b, c, d, e \}$ and
$P = \{ \{a, b\}, \{c, d\}, \{e\} \}$

i) Is P a partition on A.

ii) Write the equivalence relation and equivalence classes.

Soln. :

i) Since all cells of P are mutually disjoint and $\{a, b\} \cup \{c, d\} \cup \{e\} = A$

$\therefore$ P is a partition on A.

ii) $R = \{ (a, a), (a, b), (b, a), (b, b), (c, d), (d, c), (c, c), (d, d), (e, e) \}$

So by defn.

$\bar{a} = \{x \in A \mid x \, R \, a \} = \{a, b\}$

$\bar{b} = \{x \in A \mid x \, R \, b \} = \{a, b\}$

$\bar{c} = \{x \in A \mid x \, R \, c \} = \{c, d\}$

$\bar{d} = \{x \in A \mid x \, R \, d \} = \{c, d\}$

$\bar{e} = \{x \in A \mid x \, R \, e \} = \{e\}$

22. Let $\{A_1, A_2, \ldots A_k\}$ be the partition of a set A we define a binary relation 'R' on 'A' such that the ordered pair $(a, b) \in R$; iff a and b are in the same block of the partition; show that R is an equivalence relation.

Soln. :

$R \subseteq A \times A$

$R = \{ (a, b) \in A \times A \mid$ a and b both belong to same $A_i \}$

i) $(x, x) \in R$ for $x \in A_i$; since both element in this pair are same, therefore they are in the same block. So therefore 'R' is reflexive.

ii) Let $(x, y) \in R$

$\Rightarrow$ x and $y \in A_i$ for any i

$\Rightarrow (y, x) \in A_i$ $\qquad \therefore$ R is symmetric

iii) Let $(x, y) \in R$, $(y, z) \in R$

$\Rightarrow$ x, y belongs to some one A_i & y, z, belongs to some one A_j

$\Rightarrow y \in A_i$ and A_j both.

$\because A_i \cap A_j = \phi$ if $i \neq j$

$\therefore A_i = A_j$

$\Rightarrow$ x, z belong to same A_i

$\Rightarrow (x, z) \in R$

$\therefore$ R is transitive

$\therefore$ R is an equivalence relation.

Circular Relation :

A relation R on set 'A' is said to be circular if a R b and b R c $\Rightarrow$ c R a.

23. Show that R is reflexive and circular, iff it is an equivalence relation.

Soln. : If part :

Let R be a relation which is reflexive and circular.

$\Rightarrow (x, x) \in R \; \forall \, x$ $\qquad \because$ R is reflexive.

and $(x, y) \in R$ and $(y, z) \in R \Rightarrow (z, x) \in R$ $\qquad \because$ R is circular

Now we have to prove that R is symmetric

Let $(x, y) \in R$

Here $(x, x) \in R$ & $(x, y) \in R$

$\Rightarrow (y, x) \in R$ $\quad\because$ R is circular

$\therefore (x, y) \in R \Rightarrow (y, x) \in R$

Now we will prove that R is transitive

Let $(x, y) \in R$ & $(y, z) \in R$

$\Rightarrow (z, x) \in R$ $\quad\because$ R is circular

$\Rightarrow (x, z) \in R$ $\quad\because$ R is symmetric

$\therefore$ R is transitive.

$\therefore$ R is an equivalence relation.

Only if part :

Let R be an equivalence relation

$\therefore$ R is reflexive

Now we have to prove that R is circular

Let $(x, y) \in R$, $(y, z) \in R$

$$\left.\begin{array}{ll} \Rightarrow (x, z) \in R & \because \text{R is transitive} \\ \Rightarrow (z, x) \in R & \because \text{R is symmetric} \end{array}\right\} \because \text{R is an equivalence relation}$$

Hence R is circular.

24. If N is a set of natural numbers. Let R be the relation on N × N that is defined by (a, b) R (c, d) iff ad = bc. Prove that R is an equivalence relation.

Soln. :

N is a set of natural numbers.

and $R = \{ (a, b) \in N \times N \mid (a, b)\, R\, (c, d) \text{ iff } ad = bc \}$

i) Reflexive : Let (a, b) R (a, b)

$\Rightarrow ab = ba$ $\quad (\because$ multiplication is commutative in N)

$\Rightarrow$ (a, b) R (a, b)

$\Rightarrow$ R is a reflexive

ii) Symmetric : Let (a, b) R (c, d) $\quad \forall\, a, b, c \in N$

$\Rightarrow ad = bc$

$\Rightarrow bc = ad$

$\Rightarrow cb = da$

$\Rightarrow$ (c, d) R (a, b)

$\Rightarrow$ R is symmetric

iii) Transitive : Let (a, b) R (c, d) & (c, d) R (e, f)

$\Rightarrow ad = bc$ & $cf = de$

$\Rightarrow \frac{a}{b} = \frac{c}{d}$ & $\frac{c}{d} = \frac{e}{f}$

$\Rightarrow \frac{a}{b} = \frac{e}{f}$

$\Rightarrow af = be$

$\Rightarrow$ (a, b) R (e, f)

$\Rightarrow$ R is transitive

$\therefore$ R is an equivalence relation.

25. S = {1, 2, 3, 4, 5} and A = S × S. Define the following relation R on A : (a, b) R (a′, b′) iff ab′ = a′b.

(a) Show that R is an equivalence relation

(b) Compute A/R

Soln.:

(i) Reflexive : Let (a, b) R (a, b)

$\Rightarrow ab = ab$

$\Rightarrow$ R is reflexive

(ii) Symmetric: Let (a, b) R (a′, b′)

$\Rightarrow ab' = a'b$

$\Rightarrow a'b = ab'$
$\Rightarrow (a', b')\ R\ (a, b)$
$\Rightarrow$ R is symmetric

(iii)Transitive : Let $(a, b)\ R\ (a', b')$ & $(a', b')\ R\ (a'', b'')$
$\Rightarrow ab' = a'b$ & $a'b'' = a''b'$
$\Rightarrow \dfrac{a}{b} = \dfrac{a'}{b'}$ & $\dfrac{a}{b} = \dfrac{a''}{b''}$
$\Rightarrow \dfrac{a}{b} = \dfrac{a''}{b''}$
$\Rightarrow ab'' = a''b$
$\Rightarrow (a, b)\ R\ (a'', b'')$
$\Rightarrow$ R is transitive
$\therefore$ R is an equivalence relation.

$(a, b)\ R\ (a', b')$ iff $ab' = a'b$ i.e. $\dfrac{a}{b} = \dfrac{a'}{b'}$

$\Rightarrow$ we have to find the collection of all those order pairs which have same ratio.

$$\therefore A/R = \left\{ \begin{array}{l} \{(1,1),(2,2),(3,3),(4,4),(5,5)\},\{(2,1),(4,2)\},\{(3,1)\},\{(4,1)\}, \\ \{(5,1)\},\{(1,2),(2,4)\},\{(1,3)\},\{(1,4)\},\{(1,5)\},\{(2,3)\},\{(3,2)\}, \\ \{(3,4)\},\{(4,3)\},\{(2,5)\},\{(5,2)\},\{(3,5)\},\{(5,3)\},\{(4,5)\},\{(5,4)\} \end{array} \right\}$$

26. Let R be a relation on a set A. Show that R is an equivalence relation iff (a, b) and (a, c) are in R implies that (b, c) is in R.

Soln. :

$\because$ R is a relation on A.
$R = \{(b, c) \mid a\,R\,b \ \&\ a\,R\,c\}$

If Part : Let R be an equivalence relation
Now (a, b) & $(a, c) \in R$
$\Rightarrow (b, a)$ & $(a, c) \in R$ ($\because$ R is symmetric)
$\Rightarrow (b, c) \in R$ ($\because$ R is transitive)

Only if Part :
Let (a, b) & $(a, c) \in R$
$\Rightarrow (b, c) \in R$
We have to prove that R is an equivalence relation.

i) Take $a = b = c$
(a, a) & $(a, a) \in R$
$\Rightarrow (a, a) \in R$ (by definition)
$\Rightarrow$ R is reflexive.

ii) Let $(a, b) \in R$ and $(a, a) \in R$ ($\because$ R is reflexive)
$\Rightarrow (b, a) \in R$ (by definition)
$\Rightarrow$ R is symmetric

iii) Let (a, b) & $(b, c) \in R$
$\Rightarrow (b, a)$ & $(b, c) \in R$ $\because$ R is symmetric
$\Rightarrow (a, c) \in R$ (by definition)
$\Rightarrow$ R is transitive
$\therefore$ R is an equivalence relation.

27. Let $S = \{1, 2, 3, 4\}$ and $A = S \times S$. Define the following relation R on A : $(a, b)\ R\ (a', b')$ iff $a + b = a' + b'$.
(a) Show that R is an equivalence relation.
(b) Compute A/R.

Soln. :

Here $S = \{1, 2, 3, 4\}$
$A = S \times S$
$\because$ R is a relation on A

$\therefore\ R \subseteq A \times A$

Here $S \times S = \{(1, 1) (1, 2) (1, 3) (1, 4) (2, 1) (2, 2,) (2, 3) (2, 4) (3, 1) (3, 2) (3, 3) (4, 1) (4, 2) (4, 3) (4, 4)\}$

i) Reflexive : $\forall\ a, b \in S,\ a + b = a + b$

$\Rightarrow$ (a, b) R (a, b)

$\Rightarrow$ R is reflexive

ii) Symmetric : Let (a, b) R (a′, b′)

$\Rightarrow a + b = a' + b'$

$\Rightarrow a' + b' = a + b$

$\Rightarrow$ (a′, b′) R (a, b)

$\Rightarrow$ R is symmetric.

iii) Transitive: Let (a, b) R (a′, b′) & (a′, b′) R (a″, b″)

$\Rightarrow a + b = a' + b'\ \&\ a' + b' = a'' + b''$

$\Rightarrow a + b = a'' + b''$

$\Rightarrow$ (a, b) R (a″, b″)

$\Rightarrow$ R is transitive.

$\therefore$ R is an equivalence relation.

Now $A/R = \{\{(1, 1)\}, \{(1, 2), (2, 1)\}, \{(1, 3), (2, 2), (3, 1)\} \{(1, 4), (2, 3), (3, 2), (4, 1)\}, \{2, 4), (4, 2), (3, 3)\}, \{(3, 4), (4, 3)\} \{(4, 4)\}\}$

Closure Properties :

Reflexive Closure

Let R be a relation on a set A then a smallest reflexive relation on A is said to be reflexive closure of R if it contains R.

Note :

i) A smallest reflexive relation R_1 containing R can be determined by the formula $R_1 = \Delta \cup R$ where Δ is diagonal relation.

ii) If R is a reflexive relation then reflexive closure of R is itself.

Symmetric Closure.

Let R be a relation on a set A and R is not symmetric then a smallest symmetric relation is said to be symmetric closure of R if it contains R.

Note :

i) A smallest symmetric relation R_1 for a not symmetric relation R can be determined by the formula $R_1 = R \cup R^{-1}$.

ii) If R is a symmetric relation then symmetric closure of R is itself.

28. Let R be the relation on A = {1, 2, 3} defined by

$R = \{(1,1),(1,2),(2,3)\}$

Find i) the reflexive closure of R

ii) the symmetric closure of R.

Soln. :

Here A = {1, 2, 3 } and $R = \{(1,1),(1,2),(2,3)\}$

i) For A diagonal relation Δ = (1,1), (2, 2), (3, 3)

$\because\ \Delta \not\subseteq R$

$\therefore$ R is not reflexive.

$\therefore$ Reflexive closure of R $= \Delta \cup R$

$= \{ (1, 1), (1, 2), (2, 2,), (2, 3), (3, 3) \}$

ii) $R^{-1} = \{(1,1),(2,1),(3,2)\} \neq R$

$\therefore$ R is not symmetric

$\therefore$ Symmetric closure of R $= R \cup R^{-1}$

$= \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 2) \}$

Transitive Closure :

Let R be a relation on a set A and R is not transitive. Then the smallest transitive relation is said to be transitive closure if it contains R.

Note :

i) If R is transitive then the transitive closure of R is itself.
ii) Transitive closure of R = R^∞ i.e. connectivity relation.
iii) Transitive closure of a non transitive relation can be found by Warshall's Algorithm.

Transitive Closure and Warshall's Algorithm

29. Let A = {1, 2, 3, 4} and R = {(1, 2), (2, 1), (2, 2), (4, 3), (3, 1)} **[M-97]**
Find the transitive closure of the relation R.

Soln. :

Since (1, 2), (2, 1) $\in$ R but (1, 1) $\notin$ R
and (4, 3), (3, 1) $\in$ R but (4, 1) $\notin$ R etc.

$\therefore$ The relation R on A is not transitive.

Using Warshall Algorithm :

We have to find transitive closure of R using Warshall's Algorithm.

Consider $w_0 = M_R$

$$w_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_1 \Rightarrow$ 1's are at the position 2 and 3.
$R_1 \Rightarrow$ 1 is at the position 2.

$\therefore$ We have to insert 1 at (2, 2), (3, 2).

$$\therefore\ w_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_2 \Rightarrow$ 1's are at the position 1, 2, 3
$R_2 \Rightarrow$ 1's are at the position 1, 2
$\therefore$ Insert 1 at (1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2).

$$w_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_3 \Rightarrow$ 1 is at the position 4.
$R_3 \Rightarrow$ 1's are at the position 1, 2.
Insert 1 at (4, 1), (4, 2).

$$w_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$C_4 \Rightarrow$ There is no one at any position
$R_4 \Rightarrow$ 1's are at the position 1, 2, 3.
$\therefore$ We will not insert 1's at any place.
$\therefore$ $w_4 = w_3$

$$\therefore \; w_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$\therefore$ Required transitive closure of relation R on A is

$R_{TC} = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$

Which is a smallest transitive relation containing 'R'.

30. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and let R be a relation on A whose matrix is :

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find out transitive closure of R using Warshall's algorithm.

Soln. :

From given matrix form of the relation R,

$R = \{(a_1, a_1), (a_1, a_4), (a_2, a_2), (a_3, a_4), (a_3, a_5), (a_4, a_1), (a_5, a_2), (a_5, a_5)\}$

is not transitive since $(a_3, a_4), (a_4, a_1) \in R$ but $(a_3, a_1) \not\in R$.

By Warshall's algorithm, we consider

$$w_0 = M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

and $C_1 \Rightarrow$ 1's are at the positions 1, 4

$R_1 \Rightarrow$ 1's are at the positions 1, 4

No 1 will insert at any zero position.

$\therefore w_1 = w_0$

$C_2 \Rightarrow$ 1's are at position 2, 5

$R_2 \Rightarrow$ 1 is at the position 2.

No 1 will insert at any zero position.

$\therefore w_2 = w_1 = w_0$

$C_3 \Rightarrow$ There is no 1 at any position

$\therefore$ No 1 will inset at any zero position.

$\therefore w_3 = w_2 = w_1 = w_0$

Again $C_4 \Rightarrow$ 1's are at the position 1, 3, 4

$R_4 \Rightarrow$ 1's are at the position 1, 4

$\therefore$ We insert 1 at the positions (1, 1), (1, 4), (3, 1), (3, 4), (4, 1), (4, 4)

$$\therefore \; w_4 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$C_5 \Rightarrow$ 1's are at positions 3, 5

$R_5 \Rightarrow$ 1's are at positions 2, 5

Insert 1 at position (3, 2), (3, 5), (5, 2), (5, 5)

$$\therefore w_5 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow R_{TC} = \{(a_1, a_1), (a_1, a_4), (a_2, a_2), (a_3, a_1), (a_3, a_2), (a_3, a_4), (a_3, a_5), (a_4, a_1), (a_4, a_4), (a_5, a_2), (a_5, a_5)\}$

It is a transitive relation and containing given relation R.

$\therefore$ R_{TC} is required transitive closure of R.

31. Let A = {1, 2, 3, 4, 5} and let R and S be the equivalence relations on A whose matrices are given below. Compute the matrix of the smallest equivalence relation containing R and S. **[M-01]**

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Soln. :

Note : $M_{R\cup S} = M_R \vee M_S$

$$\therefore \quad M_{R\cup S} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{pmatrix} 0\vee 1 = 1 \\ 1\vee 1 = 1 \\ 0\vee 0 = 0 \end{pmatrix}$$

$R \cup S$ is reflexive and symmetric but not transitive

($\because$ (5, 3), (3, 2) $\in R \cup S$ but (5, 2) $\notin R \cup S$.)

$\therefore$ $R \cup S$ is not an equivalence relation.

To find smallest equivalence relation which contains $R \cup S$, we have to find $(R \cup S)_{TC}$.

By Warshall's algorithm $(R \cup S)_{TC}$ can be found which contains $R \cup S$.

We consider $w_0 = M_{R\cup S}$

$$\therefore \quad w_0 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$C_1 \Rightarrow$ 1's are at the position 1, 2

$R_1 \Rightarrow$ 1's are at the position 1, 2

Here new 1's can not be added at any new position.

$\therefore$ $w_1 = w_0$.

Insert 1's at (1, 3), (1, 4), (3, 1), (4, 1).

$$w_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$C_3 \Rightarrow$ 1's are at the position 1, 2, 3, 4, 5

$R_3 \Rightarrow$ 1's are at the position 1, 2, 3, 4, 5

Now we have to insert 1 at the positions (1, 5), (2, 5), (5, 1), (5, 2)

$$w_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$ is transitive closure. ($\because$ $w_3 = A \times A$ i.e. universal relation)

$(R \cup S)_{TC} = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (4, 4), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)\}$

$\Rightarrow (R \cup S)_{TC} = A \times A.$

i.e., the smallest equivalence relation containing the equivalence relations R & S.

Graded Questions

Relations :

1. Define : i) Relation ii) Domain of Relation iii) Range of Relation iv) Inverse Relation and give suitable example.
2. Determine which of the following are relations from A = {a, b, c} to B = {1, 2}
   a) $R_1 = \{(a,1),(a,2),(c,2)\}$
   b) $R_2 = \{(a,2),(b,1),(2,a)\}$
   c) $R_3 = \{(c,1),(c,2),(c,3)\}$
   d) $R_4 = \{(b,2)\}$
   e) $R_5 = \phi$
   f) $R_6 = A \times B$
3. Let 'R' be the relation on A = {1, 2, 3, 4} defined by "x is less than y" i.e. 'R' is the relation <
   i) Write 'R' as a set of ordered pairs
   ii) Find R^{-1} of the relation R.
   iii) Can R^{-1} be described in words ?
4. Let 'R' be the relation from A = {1, 2, 3, 4} to B = {x, y, z} defined by
   $R = \{(1,y),(1,z),(3,y),(4,x),(y,z)\}$
   a) Determine the domain and range of R.
   b) Find the inverse relation R^{-1} of R.
5. Let S be the relation on the set N of the +ve integers defined by an equation x + 3y = 13 i.e. $S = \{(x,y) | x + 3y = 13\}$
   i) Write S as a set of ordered pairs
   ii) Find the inverse relation S^{-1} of S and describe S^{-1} by an equation.
6. Let S be the relation on the set N of positive integers defined by the equation 3x + 4y = 17. Write S as a set of ordered pairs.
7. Let 'R' be the relation from X = {1, 2, 3, 4} to Y = {a, b, c, d} defined by
   $R = \{(1,a),(1,b),(3,b),(3,d),(4,b)\}$
   Find each of the following subsets of x :
   a) E = {x ; x R b}
   b) F = {x ; x R d}
8. Let R and S be the following relations on A = {1, 2, 3};
   R = {(1, 1), (1, 2), (2, 3), (3, 1), (3, 3)}
   S = {(2, 3), (1, 3), (2, 1), (3, 3)}
   find : $R \cap S$, $R \cup S$, R^C
9. Let A = {2, 5, 9, 13, 16} **[N-04]**
   R = {(2, 5), (2, 13), (16, 5), (16, 13), (9, 13), (5, 16)}
   S = {(2, 9), (2, 13), (5, 13), (9, 16), (5, 16)}
   Compute (i) S^{-1} (ii) $(R \cup S) \cap S^{-1}$ (iii) $\bar{R} \cap S$ (iv) $\bar{R}$ (v) $\bar{S}$ (vi) R^{-1}.

10. Let R and S be the relations from A = $\{1, 2, 3\}$ to B = $\{a, b\}$ defined by

$$R = \{(1,a),(3,a),(2,b),(3,b)\}$$

$$S = \{(1,b),(2,b)\}$$

Find $R \cup S$, $R \cap S$ and R^C.

11. Let R be the relation on the set N of +ve integers defined by the equation $x^2 + 2y = 100$
    i) Write R as a set of ordered pairs
    ii) Find the domain of R
    iii) Find range of R
    iv) Find R^{-1} and describe R^{-1} by an equation.
12. Explain the restriction of relation 'R'.
    Let A = $\{a, b, c, d, e, f\}$
    and R = $\{(a,a),(a,c),(b,c),(a,e),(b,e),(c,e)\}$
    Find the restriction of R to B, where B = $\{a, b, c\}$ is subset of A.

Representation of Relations :

1. Given A = $\{1, 2, 3, 4\}$ and B = $\{x, y, z\}$. Let R be the following relation from A to B.
   R = $\{(1, y), (1, z), (3, y), (4, x), (4, z)\}$
   (a) Determine the matrix of the relation
   (b) Draw the arrow diagram of R
   (c) Find the inverse relation R^{-1} of R
   (d) Determine the domain and range of R

2. Find the digraph of Relation R.
   i) A = $\{1, 2, 3, 4, 8\}$; a R b iff a = b.
   ii) A = $\{1, 2, 3, 4, 6\}$; a R b iff a is multiple of b.
   iii) A = $\{1, 2, 3, 4, 5\}$; a R b iff $a \le b$.
   iv) A = $\{1, 2, 3, 4, 8\}$; a R b iff $a + b \le 9$.

3. Let T be the relation from A = $\{1, 2, 3, 4, 5\}$ to B = {red, white, blue, green} defined by

$$R = \{(1,\text{red}),(1,\text{blue}),(3,\text{blue}),(4,\text{green})\}$$

   i) Draw an arrow diagram of the relation T.
   ii) Find the domain and range of T
   iii) Find the inverse of T^{-1}
   iv) Find matrix M_T and $M_{T^{-1}}$
   v) Find its digraph.

4. Define In–degree and Out–degree of vertex.
   Let A = $\{a, b, c, d\}$ and let 'R' be the relation on A that has the matrix. **[M-02]**

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

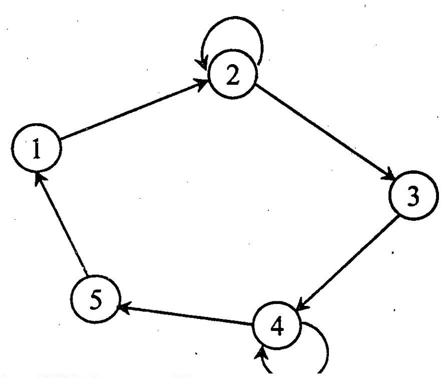
   Construct digraph of R, and list in degrees and out degrees of all vertices.

5. Give the relation defined on A and its digraph and list in–degrees and out–degrees of all vertices. Let A = $\{1, 2, 3, 4\}$

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

6. (a) Find the relation defined by the digraph and write in matrix form.

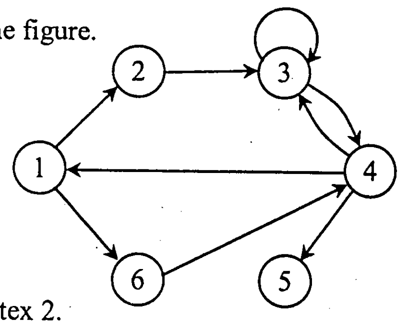
   (b) List in–degrees and out–degrees of each vertex.



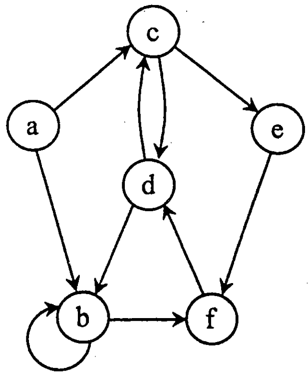
7. Let A = {1, 2, 3, 4, 6} and let R be the relation on A defined by "x divides y", written x/y
   (a) Write R as a set of ordered pairs
   (b) Draw its directed graph
   (c) Find the inverse relation R^{-1} of R. Can R^{-1} be described in words ?

Paths in Relations :

1. Explain the paths in relations and digraphs
   Let R be the relation whose digraph is given in the figure.



   i) List all paths of length 1.
   ii) a) List all paths of length 2 starting from vertex 2.
       b) List all paths of length 2.
   iii) a) List all paths of length 3 starting from vertex 3.
        b) List all paths of length 3.
   iv) Find a cycle starting at vertex 2.
   v) Find a cycle starting at vertex 6.
   vi) Draw the digraph of R^2.
   vii) Find M_{R^2}.
   viii) a) Find R^∞.
         b) Find M_{R^∞}.
2. Let R be the relation whose digraph is given in figure.



   i) List all paths of length 1.
   ii) a) List all paths of length 2 starting from vertex c.
       b) Find all paths of length 2.
   iii) a) List all paths of length 3 starting from vertex a.
        b) Find all paths of length 3.
   iv) Find a cycle starting at vertex c.
   v) Find a cycle starting at vertex d.
   vi) Find a cycle starting at vertex a.
   vii) Draw the digraph of R^2.
   viii) Find M_{R^2}. Is this result consistent with the result of Exercise 15 ?
   ix) a) Find M_{R^∞}
       b) Find R^∞
   x) Let R and S relations on a set A. Show that
      $$M_{R\cup S} = M_R \vee M_S.$$
3. Let R be the relation on set A. Prove that there is a path of length n from a to b if and only if (a, b) $\in R^n$. **[M-03]**

Composition of Relations :

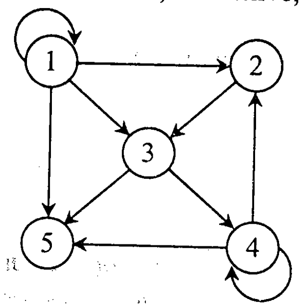
1. Define composition of Relations ?
2. Let A = {1, 2, 3, 4}, B = {a, b, c, d} & C = {x, y, z}
   Consider the relations R from A to B and S from B to C defined by
   $$R = \{(1,a),(2,d),(3,a),(3,b),(3,d)\}$$
   $$S = \{(b,x),(b,z),(c,y),(d,z)\}$$
   i) Find the composition relation SoR.
   ii) Find the matrices M_R, M_S & M_{SoR}, representing the relations R, S & SoR.
   iii) Compare the product $M_R \cdot M_S$ to the matrix M_{SoR}.

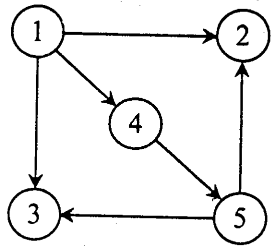
3. Let R and S be the relations on A = {1, 2, 3, 4} defined by
$$R = \{(1,1),(3,1),(3,4),(4,2),(4,3)\}$$
$$S = \{(1,3),(2,1),(3,1),(3,2),(4,4)\}$$
   i) Find the composition relation R O S
   ii) Find S O R
   iii) Find R^2 = R O R & M_{R^2}
   iv) Find S^2 = S O S & M_{S^2}
   v) Find R^3 = R O R O R
   vi) Find R^{-1}
   vii) Find R O R^{-1} & R^{-1} O R

Properties of Relations :

1. Properties of the relation
   i) reflexive
   ii) irreflexive
   iii) symmetric
   iv) asymmetric
   v) antisymmetric [N-04]
   vi) transitive
   vii) equivalence [N-04]
   viii) partial order relation [N-04]
2. Consider the following five relations on the set
   A = {1, 2, 3, 4}
   R_1 = {(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)}
   R_2 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4) }
   R_3 = {(1, 3), (2, 1)}
   R_4 = ϕ, the empty relation
   R_5 = A × A, the universal relation
   Determine which of the relations are reflexive.
3. Let A = {1, 2, 3, 4}. Determine whether the relation is reflexive, symmetric, asymmetric, antisymmetric, or transitive.
   a) $R = \{(1,1,),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)\}$
   b) $R = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$
   c) $R = \{(1,3),(1,1),(3,1),(1,2),(3,3),(4,4)\}$
   d) $R = \{(1,1),(2,2),(3,3)\}$
   e) $R = \phi$
   f) $R = A \times A$
   g) $R = \{(1,2),(1,3),(3,1),(1,1),(3,3),(3,2),(1,4),(4,2),(3,4)\}$
   h) $R = \{(1,3),(4,2),(2,4),(3,1),(2,2)\}$
4. For each of these relations on the set {1, 2, 3, 4}, decide whether it is symmetric, whether it is reflexive, whether it is transitive, and whether it is antisymmetric. (One relation may satisfy more than one property). [M-03]
   (i) {(1, 1) (1, 2) (2, 1) (2, 2) (3, 3) (4, 4) }
   (ii) {(1, 1) (2, 2) (3, 3) (4, 4) }
   (iii) {(1, 3) (1, 4) (2, 3) (2, 4) (3, 1) (3, 4) }
5. Give examples of relations R on A = {1, 2, 3 } having the stated property [M-98]
   (i) R is transitive but not symmetric
   (ii) R is symmetric but not transitive
   (iii) R is both symmetric and anti–symmetric
   (iv) R is neither symmetric nor anti symmetric.
6. A relation 'R' in Z defined as xRy iff $x \le y + 1$. Is R –
   (i) reflexive (ii) symmetric (iii) transitive ?
7. A relation 'R' in Z is defined as xRy iff $x < y + 1$.
   Examine whether 'R' is
   (i) reflexive (ii) symmetric (iii) transitive
8. Let R is a relation on Z^+ (set of positive integers) defined by xRy iff $x \le y$.
   Prove that 'R' is partial order relation.

9. A relation 'R' on Z^+ (set of positive integers) defined by aRb iff a | b i.e. divisibility. Then prove that 'R' is partial order relation.
10. In the figure below, let A = {1, 2, 3, 4, 5}. Determine whether the relation R whose digraph is given is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive.

a) 

b) 

11. Let A = {1, 2, 3, 4} whether the relation R whose matrix M_R is given is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive.

a) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

12. Determine whether the relation R on the set A is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive.
   a) $A = Z$; a R b if and only if $a \le b + 1$
   b) $A = Z^+$; a R b if and only if $|a - b| \le 2$
   c) $A = Z^+$; a R b if and only if $a = b^k$ for some $k \in Z^+$.
   d) $A = Z$; a R b if and only if a + b is even.
   e) $A = Z$; a R b if and only if $|a - b| = 2$
   f) A = the set of real numbers; a R b if and only if $a^2 + b^2 = 4$.
13. Prove that relation R on set A is transitive if and only if $R^n \subseteq R$ for n = 1, 2, 3...... **[D-03, 05]**
14. Let R is relation on set A then prove that if R is reflexive then R^{-1} is also reflexive. **[D-03]**
15. Let R be the relation on set A. Prove that if R is symmetric then R^{-1} and $\bar{R}$ is also symmetric. **[M-05]**
16. Define symmetric and antisymmetric relation. Does there exist a relation which is symmetric and antisymmetric. Justify your answer. **[D-99]**
17. Determine whether the relation R on set of all integers is reflexive, symmetric, antisymmetric and / or transitive. Where $(x, y) \leftarrow R$ if and only if. **[D-05]**
18. Let R and S are relations on set A then prove that **[D-05]**
   (i) If R is reflexive so R^{-1}.
   (ii) If $R \le S$ then $R^{-1} \le S^{-1}$.
19. Let R be the relation represented by the matrix **[M-06]**

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

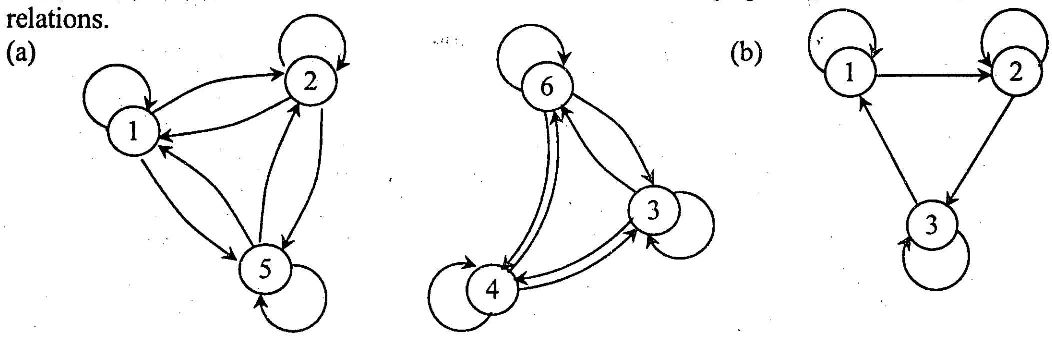
Find the matrix that represents R^4

Equivalence Relations :

1. Define the term Equivalence relation. Give one example. **[N-04]**
2. Let A = {a, b, c}. Determine whether the relation R whose matrix M_R is given is an equivalence relation.

a) $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ b) $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. In figure (a) & (b), determine whether the relation R whose digraph is given is an equivalence relations.



4. Determine whether the relation R on the set A is an equivalence relation.
   a) $A = \{a, b, c, d\}$,
      $R = \{(a,a,),(b,a),(b,b),(c,c),(d,d),(d,c)\}$
   b) $A = \{1, 2, 3, 4, 5\}$,
      $R = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(2,3),(3,3),(4,4),(3,2),(5,5)\}$
   c) $A = \{1, 2, 3, 4\}$,
      $R = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,3),(1,3),(4,1),(4,4)\}$
   d) Consider the set z of integers. Define aRb by $b = a^r$. For some positive integer r. Show that R is a partial order on z, that is , show that R is :
      (a) Reflexive (b) Antisymmetric (c) Transitive
   e) Given a set S = (1, 2, 3, 4, 5), find the equivalence relation on S which generates the partition. {(1, 2), (3), (4, 5)} Draw the graph of the relation. **[D-97]**
5. If $A = \{1, 2, 3\}$
   How many relations are possible on 'A' and how many of them are equivalence.
6. If $A = \{1, 2, 3, 4\}$
   How many relation are there on A and how many of them are equivalence relation.
7. Let $A = \{1, 2, 3, 4, 5, 6\}$ and let R be the equivalence on A defined by
   $R = \{(1,1),(1,5),(2,2),(2,3),(2,6),(3,2),(3,3),(3,6),(4,4),(5,1),(5,5),(6,2),(6,3),(6,6)\}$
   i) Find the partition of A induced by R.
   ii) Find the equivalence classes of R.
8. Show that if a relation on a set A is transitive and irreflexive, then it is asymmetric. **[D-98]**
9. A relation R on a set A is called circular if aRb and bRc imply cRa.
   Show that R is reflexive and circular if and only if it is an equivalence relation. **[D-99, M–00, 04]**
10. a) Let R be a relation on a set of positive integers Z defined by xRy iff 3x + 2y is divisible by 5. Prove that R is an equivalence relation.
    b) Let R be a relation on a set of positive integers Z defined by xRy iff 5x + 3y is divisible by 8. Prove that R is an equivalence relation.
11. Let R be a binary relation Let –
    $S = \{ (a, b) / (a, c) \in R \text{ and } (c, b) \in R \text{ for some } c \}$
    Show that if R is an equivalence relation, then S is also an equivalence relation.
12. Let I be the set of all integers. Let R_1 be a binary relation on $I \times I$ such that the ordered pair ((a, b), (c, d)) is in R_1 if and only if $a - c = b - d$.
    i) What is a geometric interpretation of the binary relation R_1 ?
    ii) Is R_1 an equivalence relation ? **[M-99]**
13. Let $A = \{ 1, 2, 3, \ldots, 13, 14, 15\}$. Let R be the relation on A defined by congruence modulo 4. Find the equivalence classes determined by R.
14. Let $A = \{1, 2, 3, \ldots .\ 10\}$ defined Relation R on $A \times A$ by (a, b) R (c, d) if $a + d = b + c$
    Show R is equivalence relation. **[D-02]**
15. Let R_5 be the equivalence relation on the set Z of integers defined by $x \equiv y(\text{mod}5)$. Find Z/R_5, the induced equivalence classes.
16. Let $A = \{1, 2, 3, \ldots, 14, 15\}$.
    Consider the relation R on $A \times A$ defined by (a, b) R (c, d) iff $a + d = b + c$.
    (i) Prove that R is an equivalence relation (ii) Find the equivalence classes of (2, 7) .

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Closure P

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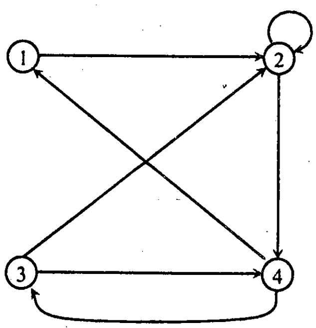
17. Let S = {1,2,3,4,5} and let A = S X S. Define the following relation R on A : (a, b) R(a′,b′) if and only if ab′ = a′b :–
(i) Show that R is an equivalence relation (ii) Compute A/R.
18. Let $\{A_1, A_2, ..., A_k\}$ be a partition of a set A. We define a binary relation R on A such that an ordered paired (a, b) is in R if and only if a and b are in the same block of the partition. Show that R is an equivalence relation.
19. Define equivalence relation on a set. Let R be a relation on the set of integers defined by aRb if and only if a – b is a multiple of 3. Prove that R is an equivalence relation. **[M-01]**
20. Let R be the relation on the set of order pairs of positive integers such that ((a, b), (c, d)) $\in$ R if and only if ad = bc. Show that R is equivalence relation. **[M-05]**
21. Let R be the relation on the set I of integers defined by x R y if x – y is divisible by 4 (i.e., x – y = 4 n for some integer n). Show that R is an equivalence relation and describe the equivalence classes.
22. Consider the relation "congruence modulo m" given by – **[D-99]**
xRy iff x – y is divisible by m
over the set of positive integers.
(i) Show that R is an equivalence relation.
(ii) Find all equivalence classes.
(iii)Show that if x_1 Ry_1 and x_2Ry_2 then $(x_1 + x_2)$ R$(y_1 + y_2)$.
23. Let R be a binary relation on the set of all positive integers such that – **[D-98]**
R = { (a, b) / a – b is an odd positive integer }.
Is R Reflexive ? Symmetric ? Antisymmetric ? Transitive ? An equivalence Relation ? A partial Ordering relation ? Justify.
24. Let A = N × N. Define the following relation R on A : (a, b) R(a′,b′) if and only if ab′ = a′b. Show that R is an equivalent relation. **[M-02]**
OR
Let R be relation on N × N defined as (a, b) R (c, d) if and only if ad = bc. Show that R is an equivalence relation. **[D-01]**
25. Let R be the relation on set of all URLs (or web addresses) such that xRy if and only if the web page at x is same as web page at y. Show that R is equivalence relation. **[M-03]**
26. Let R be the equivalence relation on set A then prove that the following statements are equivalent :–
(i) aRb (ii) [a] = [b] (iii) $[a] \cap [b] \neq \phi$. **[M-03]**
27. Let R and S be equivalence relations on a non–empty set A.
Prove that $R \cap S$ is an equivalence relations but $R \cup S$ need not be an equivalence. Justify.
28. Suppose that A is non empty set, and f is a function that has A as it's domain. Let R be the relation on A consisting of all ordered pairs (x, y) where f (x) = f (y). Show that R is an equivalence relation on A. **[M-04]**
29. An equivalence relation on semigroup (s *) is called congruence relation if a R a′ and b R b′ imply (a, * b) R (a′ * b′). Then prove that following is the congruence relation on semigroup (z, +). aRb if and only if a ≡ b (mod 2) i.e. (2 divides a – b) **[M-05]**
30. Define the relation R on the set Z by aRb if a – b is non – negative even integer. Verify that R defines a partial order for Z. is this partial order a total order ? **[D-05]**
31. Let R be the relation on set of real numbers such that aRb if and only if a –b is an integer. Prove that R is an equivalence relation. **[M-06]**
32. Suppose R and S is the relation from A to B, then prove that **[M-06]**
$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ and $(RUS)^{-1} = R^{-1} \cup S^{-1}$

Closure Properties

1. Define reflexive closure and symmetric closure.
Let R be the relation on A = {1, 2, 3} defined by $R = \{(1,1),(1,2),(2,3)\}$
Find– (i) the reflexive closure of R (ii) the symmetric closure of R.
2. Show that if a set A has three elements, then we can find eight relations on A that all have the same symmetric closure. **[D-99]**
3. Explain the concept of transitive closure.

4. Consider the following relations R defined on set
   S = [a, b, c, d, e]. R is given by
   R = [(a, a), (a, d), (b, b), (c, d), (c, e), (d, e), (e, b), (e, e)]
   Find the transitive closure of R by using the algorithm. **[M-97]**
5. What is a transitive closure ? Consider set
   A = {1, 4, 7, 13} & R{(1, 4), (4, 7), (7, 4), (1, 13)}. Find out the transitive closure of R using Warshall's algorithm. **[M-00, N-04]**
6. Find the transitive closure of the relation R on A = {1, 2, 3, 4} defined by the directed graph shown in figure. **[D-00]**



7. Let A = {1, 2, 3, 4,} and R = {(1,2), (2,3) (3,4),(2,1)}. Find the transitive closure of R using Warshall's algorithm. **[M-01, D-03]**
8. (i) Let A = {1, 2, 3, 4}, for the relation R whose matrix is given, find the matrix of transitive closure by using Warshall's algorithm. **[M-04, M-06]**

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   (ii) Let A = {1, 2, 3, 4} for the relation R whose matrix is given below, find matrix of transitive closure using Warshall algorithm. **[M-05, D-05]**

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

   (iii) $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a matrix for the relation R and let A = { 1, 2, 3, 4}. Find the matrix of the transitive closure by using Warshall's algorithm. **[M-02]**

   (iv) $M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is adjacency matrix of the relation R on {1, 2, 3, 4}. Find transitive closure of R by Warshall's Algorithm. **[D-01]**

9. (i) Explain transitive closure.
   Let A = {a_1, a_2, a_3, a_4, a_5} and let R be a relation on A whose matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$ find out transitive closure of R using Warshall's algorithm

(ii) Let, A = {a, b, c, d, e, f } and R be the relation on A defined by – **[M-02]**

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) Prove R is an equivalence relation.
(ii) Give the partitions of A corresponding to R.

10. What is a transitive closure ? Find transitive closure of R using Warshall's algorithm if– A = {1,2,3,4,5} and R = {(1,2), (4,3), (2,2), (2,4), (3,5), (2,5), (5,1)}. **[D-00]**

11. What is equivalence relation ? Let A = {1, 2, 3, 4, 5} and Let R and S be the equivalence relations on A whose matrices are given below. Compute the matrix of the smallest equivalence relation containing R and S. **[N-04]**

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

12. Let A = {a, b, c, d, e} and let Rand S be the relations on A described by

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Use Warshall's algorithm to compute the transitive closure of $R \cup S$.

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Vidyalankar Institute of Technology

Ch. 4 : Functions and Pigeonhole Principle

Definition :

Let A and B be two non empty sets then a subset 'f' of $A \times B$ is said to be a function from A to B if $\forall x \in A \; \exists$ unique $y \in B$.

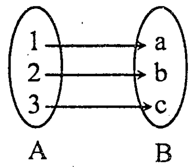
i.e. $f = \{ (x, y) \mid x \in A, y \in B \quad \forall x \in A \; \exists \text{ unique } y \in B \}$

Example $A = \{1, 2, 3\}$

$B = \{a, b, c\}$

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$

1. Consider $f = \{(1, a), (2, b), (3, c)\}$

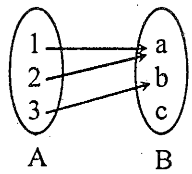


$\because f \subseteq A \times B$

& $\forall x \in A \; \exists$ unique $y \in B$

$\therefore$ f is a function.

2. Consider $f = \{(1, a), (2, a), (3, b)\}$

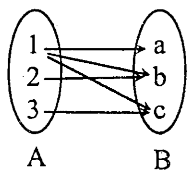


$\because f \subseteq A \times B$

& $\forall x \in A \; \exists$ unique $y \in B$

$\therefore$ f is a function.

3. Consider $f = \{(1, a), (1, b), (1, c), (2, b), (3, c)\}$



$\because f \subseteq A \times B$

for $1 \in A$ there are three y i.e. $a, b, c \in B$

$\therefore$ It does not follow uniqueness.

$\therefore$ f is not a function.

Domain, Co-domain, Image & Range of a function :

Let $f : A \to B$ be a function, then

i) A is called domain and

ii) B is called codomain.

iii) $\because$ f is a function

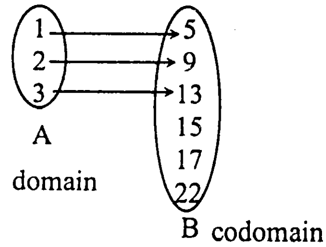
$\therefore \forall x \in A \exists$ unique $y \in B$ then y is called f-image of x.

iv) A subset of co-domain is said to be range of f if it contains all images $\forall x \in A$

Consider, $A = \{1, 2, 3\}$

$B = \{5, 9, 13, 15, 17, 22\}$

and $f = \{(1,5), (2, 9), (3, 13)\}$



$\because$ f $\subseteq$ A × B

$f(1) = 5 = 4 \times 1 + 1$

$f(2) = 9 = 4 \times 2 + 1$

$f(3) = 13 = 4 \times 3 + 1$

$\therefore$ we can define a function $\forall\, x \in A$ as

$f(x) = 4x + 1$

Here f(5) can not be determined $\because$ 5 is not an element of A

$\because$ Range of f is the set of all images $\forall\, x \in A$

$\therefore$ Range (f) $= \{5, 9, 13\}$

Injective Function :

Let $f : A \to B$ be a function then f is said to be injective (one – one)

if $\quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

or

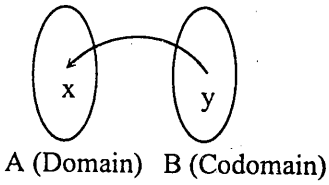
$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Surjective Function :

Let $f : A \to B$ be a function then f is said to be surjective (onto)

if $\forall\, y \in B$ (codomain) $\exists$ pre image $x \in A$ (domain)

or Range of f = codomain.



Bijective Function :

Let $f : A \to B$ be a function then f is said to be bijective if f is injective and surjective both.

Inverse Function :

Let $f : A \to B$ be a function then f is said to be invertible if $f^{-1} : B \to A$ is a function.

NOTE : A function $f : A \to B$ is invertible iff f is bijective.

Examples :

1. A function $f : R - \left\{\frac{7}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ is defined as

$$f(x) = \frac{4x-5}{3x-7}$$

Prove that 'f' is bijective and find the rule for f^{-1}.

Soln. :

$\because$ $f : R - \left\{\frac{7}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$

is defined as $f(x) = \dfrac{4x-5}{3x-7}$

i) Injective

Consider $\quad f(x_1) = f(x_2)$

$$\therefore \quad \frac{4x_1-5}{3x_1-7} = \frac{4x_2-5}{3x_2-7}$$

or $\quad (4x_1-5)(3x_2-7) = (3x_1-7)(4x_2-5)$

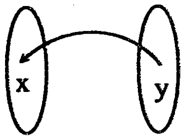
or $\quad 12x_1x_2 - 28x_1 - 15x_2 + 35 = 12x_1x_2 - 28x_1 - 15x_2 + 35$

or $\quad -13x_1 = -13x_2$

or $\quad x_1 = x_2$

$\therefore$ f is injective

ii) **Surjective**



R – {7/3} (domain) R – {4/3} (codomain)

Consider an arbitrary element y in R – {4/3} (codomain)

Let $\quad y = f(x)$

$$y = \frac{4x-5}{3x-7}$$

or $\quad 3xy - 7y = 4x - 5$

or $\quad 3xy - 4x = 7y - 5$

or $\quad x = \dfrac{7y-5}{3y-4}$

$\forall$ y $\in$ R – {4/3} (codomain) $\exists$ pre image x $\in$ R – {7/3} (domain)

$\Rightarrow$ Range of f = codomain

$\Rightarrow$ f is surjective

$\because$ f is injective & surjective both

$\therefore$ f is bijective

$\therefore$ f^{-1} exists.

Let $\quad y = f(x) \quad \Rightarrow x = f^{-1}(y)$

$$y = \frac{4x-5}{3x-7}$$

$\therefore \quad 3xy - 7y = 4x - 5$

$\Rightarrow x = \dfrac{7y-5}{3y-4} = f^{-1}(y)$

$\therefore$ The rule for f^{-1} is

$$f^{-1}(x) = \frac{7x-5}{3x-4}$$

2. $f : R - \left\{\dfrac{2}{5}\right\} \to R - \left\{\dfrac{4}{5}\right\}$ defined by $f(x) = \dfrac{4x+3}{5x-2}$ show that the function is bijective and find rule for f^{-1}. **[D-99, M-01]**

Soln. :

$\because \quad f : R - \left\{\dfrac{2}{5}\right\} \to R - \left\{\dfrac{4}{5}\right\}$ is defined by $f(x) = \dfrac{4x+3}{5x-2}$

i) Injective :

Consider $\quad f(x_1) = f(x_2)$

$$\frac{4x_1+3}{5x_1-2} = \frac{4x_2+3}{5x_2-2}$$

or $\quad (4x_1+3)(5x_2-2) = (4x_2+3)(5x_2-2)$

or $\quad 20x_1x_2 - 8x_1 + 15x_2 - 6 = 20x_1x_2 - 8x_2 + 15x_1 - 6$

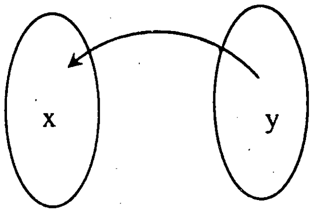
or $\quad -8x_1 - 15x_1 = -8x_2 - 15x_2$

or $\quad -23x_1 = -23x_2$

or $\quad x_1 = x_2$

$\therefore$ f is injective.

ii) surjective :



$R - \{2/5\}$ (domain) $\quad R - \{4/5\}$ (codomain)

Consider an arbitrary element y in $R - \{4/5\}$ (codomain)

Let $\quad y = f(x)$

$$y = \frac{4x+3}{5x-2}$$

or $\quad 5xy - 4x = 4x + 3$

or $\quad 5xy - 4x = 2y + 3$

or $\quad x(5y-4) = 2y + 3$

or $\quad x = \frac{2y+3}{5y-4}$

$\Rightarrow \forall\, y \in R - \left\{\frac{4}{5}\right\}$ (codomain)

$\exists$ pre image $x \in R - \left\{\frac{2}{5}\right\}$ (domain)

$\Rightarrow$ Range of f = codomain

$\Rightarrow$ f is surjective

$\because$ f is injective and surjective both

$\therefore$ f is bijective

$\therefore$ f^{-1} exist

$y = f(x) \Rightarrow x = f^{-1}(y)$

$\Rightarrow x = \frac{2y+3}{5y-4} = f^{-1}(y)$

$\therefore$ The rule for f^{-1} is

$$f^{-1}(x) = \frac{2x+3}{5x-4}$$

3. A function $f : R - \{2\} \to R$ is defined by $f(x) = \frac{1}{x-2}$

Is it i) injective $\quad$ ii) surjective $\quad$ iii) bijective

jective and

-99, M-01]

Soln. :

$\because \quad f : R - \{2\} \to R$ is defined by $f(x) = \frac{1}{x-2}$

i) Injective :

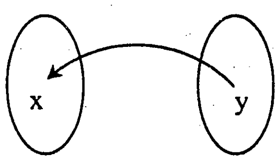
Consider $f(x_1) = f(x_2)$

$$\frac{1}{x_1 - 2} = \frac{1}{x_2 - 2}$$

or $\quad x_2 - 2 = x_1 - 2$

$\therefore \quad x_1 = x_2$

ii) Surjective :



$R - \{2\}$ (domain) $\quad R$ (codomain)

Consider an arbitrary element y in R (codomain)

Let $\quad y = f(x)$

$$y = \frac{1}{x-2}$$

$$\text{or} \quad xy - 2y = 1$$
$$\text{or} \quad xy = 1 + 2y$$
$$\text{or} \quad x = \frac{1}{y} + 2$$

For $y = 0 \in R$ (codomain), x is not defined in $R - \{2\}$ (domain)

$\therefore$ Range of $f \neq$ codomain

$\therefore$ f is not surjective

$\therefore$ f is not bijective.

4. Let $f : R \to R$ be a function defined by $f(x) = 2x - 3$. Prove that it is bijective hence find inverse.

Soln. :

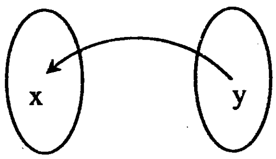
Here $f : R \to R$ is defined by $f(x) = 2x - 3$

i) Injective :

$$\text{Consider} \quad f(x_1) = f(x_2)$$
$$x_1 - 3 = 2x_2 - 3$$
$$\Rightarrow x_1 = x_2$$

$\therefore$ f is injective.

ii) Surjective :



R (domain) R (codomain)

Consider an arbitrary element y in R (codomain)

$$\text{Let} \quad y = f(x)$$
$$y = 2x - 3$$
$$\text{or} \quad y + 3 = 2x$$
$$\text{or} \quad x = \frac{y+3}{2}$$

$\Rightarrow \forall\, y \in R$ (codomain) $\exists$ pre image $x \in R$ (domain)

$\Rightarrow$ Range of f = codomain

$\Rightarrow$ f is surjective

$\because$ f is injective and surjective both

$\therefore$ f is bijective.

$\therefore$ f^{-1} exists.

$$y = f(x) \quad \Rightarrow x = f^{-1}(y)$$
$$y = 2x - 3$$
$$x = \frac{y+3}{2} = f^{-1}(y)$$

$\therefore$ The rule for f^{-1} is

$$f^{-1}(x) = \frac{x+3}{2}$$

5. Let $f : R - \{3\} \to R - \{0\}$ is defined by

$f(x) = \dfrac{1}{x-3}$ then prove that it is bijective and find f^{-1} rule.

Soln. :

Let $f : R - \{3\} \to R - \{0\}$ is defined by $f(x) = \dfrac{1}{x-3}$

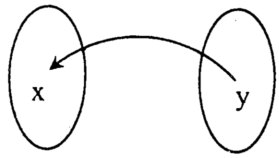
i) Injective :

$$\text{Consider } f(x_1) = f(x_2)$$
$$\frac{1}{x_1 - 3} = \frac{1}{x_2 - 3}$$
$$\text{or} \quad x_2 - 3 = x_1 - 3$$

$\Rightarrow \quad x_2 = x_1$

$\therefore$ f is injective

ii) surjective :



$R - \{3\}$ (domain) $\quad R - \{3\}$ (codomain)

Consider an arbitrary element y in $R - \{3\}$ (codomain)

$$y = f(x)$$

$$y = \frac{1}{x-3}$$

or $\quad xy - 3y = 1$

or $\quad xy = 1 + 3y$

or $\quad x = \frac{1}{y} + 3$

$\Rightarrow \quad y \in R - \{0\}$ (codomain)

$\exists$ pre image $x \in R - \{3\}$ (domain)

$\Rightarrow$ Range of f = codomain

$\Rightarrow$ f is surjective

$\because$ f is injective and surjective both

$\therefore$ f is bijective

$\therefore$ f^{-1} exist

$y = f(x) \quad \Rightarrow \quad x = f^{-1}(y)$

$$y = \frac{1}{x-3}$$

or $\quad xy - 3y = 1$

or $\quad x = \frac{1}{y} + 3 = f^{-1}(y)$

$\therefore$ The rule for f^{-1} is

$$f^{-1}(x) = \frac{1}{x} + 3$$

6. A function $f : R \to R$ is defined by $f(x) = x^2$

Is it i) injective ii) surjective iii) bijective

Soln. :

Let a function $f : R \to R$ is defined by $f(x) = x^2$

i) Injective :

Consider $\quad f(x_1) = f(x_2)$

$\therefore \quad (x_1)^2 = (x_2)^2$

$\therefore \quad x_1 = \pm x_2$

$\Rightarrow \quad x_1 = x_2$ and $x_1 = -x_2$

$\therefore$ f is not injective

Consider $2, -2 \in R$

Let $x_1 = 2, \; x_2 = -2$

We have $x_1 \neq x_2$

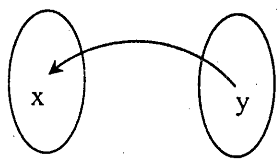
$f(x_1) = f(2) = 4$

$f(x_2) = f(-2) = 4$

$x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$

$\therefore$ It is not injective

ii) Surjective :



R (domain) $\quad$ R (codomain)

Consider an arbitrary element y in R (codomain)

Let $\quad y = f(x)$

$y = x^2$

$x = \pm\sqrt{y}$

if y is –ve, x cannot be found in R (domain)

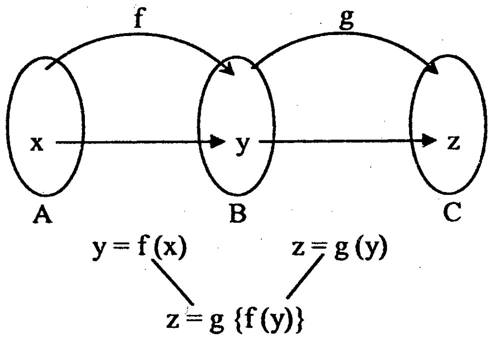
$\therefore$ f is not surjective

It is neither injective nor surjective

$\therefore$ f is not bijective.

Composite Function :

Let $f : A \to B$ and $g : B \to C$ be two functions, then gof is said to be composite function of f and g. if $gof : A \to C$ is a function and is defined as $gof(x) = g\{f(x)\}$.



Note :

$$(fog)^{-1} = g^{-1} \circ f^{-1}$$

Examples :

1. $f : R \to R$ is defined by $f(x) = x^3$ **[M-01, 04, N-04, M-06]**
   $g : R \to R$ is defined by $g(x) = 4x^2 + 1$
   $h : R \to R$ is defined by $h(x) = 7x - 2$
   Find the rule defining
   i) fog ii) gof
   iii) (goh)of iv) go(hof)

Soln. :

i) $(fog)(x) = f\{g(x)\} = f(4x^2 + 1) = (4x^2 + 1)^3$

ii) $(gof)(x) = g\{f(x)\} = g(x^3) = 4(x^3)^2 + 1$

Note : fog $\neq$ gof

iii) $\{(goh)of\}(x) = [(goh)\{f(x)\}] = (goh)(x^3) = g\{h(x^3)\} = g(7x^3 - 2) = 4(7x^3 - 2)^2 + 1$

iv) $\{go(hof)\}(x) = g\{(hof)(x)\} = g[h\{f(x)\}] = g[h(x^3)] = g(7x^3 - 2) = 4(7x^3 - 2)^2 + 1$

Note : (goh)of = go(hof)

2. Let A = B = C = IR and consider, the functions $f : A \to B$ and $g : B \to C$ defined by $f(a) = 2a + 1$ and $g(b) = b/3$ then verify that
   $$(gof)^{-1} = f^{-1} \circ g^{-1}$$

Soln. :

Here A = B = C = IR

$f : A \to B$

$\Rightarrow f : R \to R$ is defined as

$f(a) = 2a + 1$

It is bijective

$\therefore$ f^{-1} exists

Let $f(a) = b \Rightarrow a = f^{-1}(b)$

$\therefore$ $2a + 1 = b$

$\therefore \quad a = \dfrac{b-1}{2} = f^{-1}(b)$

$\therefore$ Rule for $f^{-1}(a) = \dfrac{a-1}{2}$

Similarly, here $g : B \to C$

$\Rightarrow$ $g : R \to R$ is defined as

$g(b) = b/3$. It is bijective

let $g(b) = c$

$\Rightarrow b = g^{-1}(c)$

$\therefore \dfrac{b}{3} = c$

$\therefore b = 3c = g^{-1}(c)$

$\therefore$ Rule for $g^{-1}(a) = 3a$

Here $f(a) = 2a + 1$ and $g(a) = /3$

$$\begin{aligned} \therefore (gof)(a) &= g[f(a)] \\ &= g(2a+1) \\ &= \frac{(2a+1)}{3} \end{aligned}$$

$\because$ f and g are bijective

$\therefore$ gof is bijective

$\therefore$ it is Invertible.

Let, $(gof)(a) = b \Rightarrow a = (gof)^{-1}(b)$

$\therefore \dfrac{2a+1}{3} = b$

$\therefore 2a + 1 = 3b$

$\therefore a = \dfrac{3b-1}{2} = (gof)^{-1}(b)$

$\therefore$ Rule for $(gof)^{-1}$ is

$$(gof)^{-1}(a) = \frac{3a-1}{2} \quad \ldots (1)$$

Now, $\left(f^{-1}og^{-1}\right)(a)$

$$\begin{aligned} &= f^{-1}\left[g^{-1}(a)\right] \\ &= f^{-1}(3a) \\ &= \frac{3a-1}{2} \quad \ldots (2) \end{aligned}$$

from (1) and (2) we get,

$$\therefore (gof)^{-1} = \left(f^{-1}og^{-1}\right)$$

3. Let A = B = R (set of real no.)

let $f : A \to B$ be given by the formula $f(x) = 2x^3 - 1$ and $g : B \to A$ be given by $g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}$, so that f is bijective (i.e. bijective) between A & B & g is bijective between B & A.

[M-03]

Soln. :

Here, A = B = R

$F : A \to B$

$\Rightarrow$ $f : R \to R$ is defined as

$f(x) = 2x^3 - 1$

i) Injective :

Consider $f(x_1) = f(x_2)$

$\therefore\ 2x_1^3 - 1 = 2x_2^3 - 1$

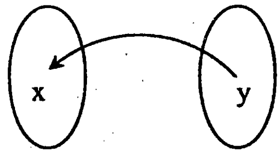
$\therefore\ 2x_1^3 = 2x_2^3$

$\therefore\ x_1^3 = x_2^3$

$\therefore\ x_1 = x_2$

$\therefore$ f is injective

ii) Surjective :



R (domain) R (codomain)

Let $y = f(x)$

$= 2x^3 - 1$

$\therefore\ 2x^3 = y + 1$

$\therefore\ x^3 = \dfrac{y+1}{2}$

$\therefore\ x^3 = \dfrac{y}{2} + \dfrac{1}{2}$

$$x = \sqrt[3]{\frac{y}{2} + \frac{1}{2}} \qquad (1)$$

$\therefore\ \forall\, y \in R$ (codomain)

$\exists$ pre image $x \in R$ (domain)

$\Rightarrow$ 'f' is surjective

iii) Bijective :

$\because$ f is injective and surjective both

$\because$ 'f' is bijective

$\Rightarrow f^{-1}$ exists.

Let $y = f(x) \Rightarrow x = f^{-1}(y)$

$= 2x^3 - 1$

$$\therefore\ x = \sqrt[3]{\frac{y}{2} + \frac{1}{2}} = f^{-1}(y) \qquad \text{(from (1))}$$

$\because$ f is bijective

$\therefore\ f^{-1}$ is bijective

$\because\ f^{-1} = g \qquad [\because f^{-1}(y) = g(y)]$

$\therefore$ g is also bijective

4. Define **[M-97]**

$f : N \to N$ by

$$f(n) = \begin{cases} n + 1 & \text{if } n \text{ is odd} \\ n - 1 & \text{if } n \text{ is even} \end{cases}$$

$g : N \to N$ by

$$g(n) = \begin{cases} n + 1 & \text{if } n \text{ is even} \\ 1 & \text{if } n = 1 \\ n - 1 & \text{if } n \text{ is odd} \end{cases}$$

Prove that f and g are bijection

Soln :

Part I :

Consider the function f.

Let $f(n_1) = f(n_2)$ n_1 & n_2 both odd.

$\Rightarrow n_1 + 1 = n_2 + 1$

$\Rightarrow n_1 = n_2$

$f(n_1) = f(n_2)$ n_1, n_2 both even.

$\Rightarrow n_1 = n_2$

$\therefore$ f is one–one i.e. injective

Let $n \in N$ be an arbitrary that may either even or odd.

Two cases arises

Case (i) : If n is even then $n-1$ is odd

$f(n-1) = n-1+1$
$= n$

$\Rightarrow$ for even n its preimage $(n-1)$ exists in N.

Case (ii) : If n is odd then $n+1$ is even

$f(n+1) = n+1-1$
$= n$

$\Rightarrow \forall$ n is an image of $n+1$

$\Rightarrow$ Range of f = codomain

$\therefore$ f is surjective

Hence f is "Bijection"

Part II :

To prove that g is bijection

$g(1) = 1$ (by def.)

Let $g(n_1) = g(n_2)$, n_1, n_2 both even

$\Rightarrow n_1 + 1 = n_2 + 1$

$\Rightarrow n_1 = n_2$

Let $g(n_1) = g(n_2)$ n_1, n_2 both odd but n_1 and n_2 is not equal to 1.

$\Rightarrow n_1 - 1 = n_2 - 1$

$\Rightarrow n_1 = n_2$

$\therefore$ f is injective.

Consider an arbitrary $n \in N$ that may either even or odd but not equal to 1.

Case (i) : If n is even then $n+1$ is odd. $n > 1$

$\Rightarrow g(n+1) = (n+1) - 1$
$= n$

$\Rightarrow$ n is image of $n + 1 \in N$

Case (ii) : If n is odd then $n-1$ is even $n > 1$

$\Rightarrow g(n-1) = (n-1) + 1$
$= n$

$\Rightarrow$ n is image of $n - 1 \in N$

$\therefore n \in N$ has preimage in N

$\Rightarrow$ g is surjective.

$\because$ g is injective and surjective both.

$\therefore$ g is bijective.

[M-97] **5.** Determine whether the function is one–one, onto [D-98, N-04]

$f: I \to I$

$$f(j) = \begin{cases} j/2 & \text{if } j \text{ is even} \\ (j-1)/2 & \text{if } j \text{ is odd} \end{cases}$$

where I is the set of all integers.

Soln. :

If we take $j = 2$ $f(2) = 2/2 = 1$

$j = 3$ $f(3) = 3-1/2 = 1$

$2 \neq 3 \Rightarrow f(2) = f(3) \Rightarrow$ f is not one–one

Consider an arbitrary $j \in I$

$\Rightarrow$ j is either even or odd

Case (i) : j is even

$\Rightarrow 2j \in Z$

$\because f(2j) = 2j/2$
$= j$

$\therefore$ j has preimage 2j in I.

Case (ii) : j is odd

$\Rightarrow$ 2j + 1 is also odd

$$\because \quad f(2j+1) = \frac{(2j+1)-1}{2}$$
$$= j$$

$\therefore$ j has preimage in I

$\therefore$ Each element of I has preimage in I

$\Rightarrow$ Range = codomain

$\Rightarrow$ f is onto.

Pigeonhole Principle

If 'n' pigeons are assigned to 'm' pigeon holes (m < n) then at least one pigeon hole must have two pigeons.

1. If 13 students are selected from a class, prove that atleast two of them must have their birthday on the same month of a year.

Soln. :

Here, 13 students (13 pigeons) are selected from a class.

There are 12 months (12 pigeonholes) in a year.

Since no. of pigeonholes is less than the no. of pigeons,

$\therefore$ Pigeonhole principle is applicable.

By Pigeonhole principle, atleast one pigeonhole must contain two pigeons.

$\Rightarrow$ At least two students must have their birthday on the same month of the year.

2. Prove that if 8 people are assembled in a room then 2 of them are born on the same day of a week.

Soln. :

There are 7 days (7 pigeonholes) in a week and 8 people (8 pigeon) are assembled in a room.

Since no. of pigeonholes is less than the no. of pigeons,

$\therefore$ Pigeonhole principle is applicable.

By pigeonhole principle, one pigeonhole must contain atleast two pigeons.

$\Rightarrow$ Atleast two people must have borned on the same day of a week.

3. If 11 numbers are chosen from a set = {1, 2, … 20} prove that one of them is multiple of other. **[M-05]**

Soln. :

Any natural number can be expressed $b = 2^k \cdot m$

where 'm' is known as odd part of 'n'. and $k \geq 0$.

By this way we find there are 10 odd parts in the set {1, 2, … 20}

$\because$ there are 10 odd parts (10 pigeon holes) and chosen numbers are 11 (11 pigeon).

$\Rightarrow$ Pigeonhole principle is applicable

By pigeon hole principle one pigeonhole must have atleast two pigeons

$\therefore$ Two natural numbers must have same odd part

Let n_1 & n_2 have same odd part 'n'

$$n_1 = 2^{k_1} m$$
$$n_2 = 2^{k_2} m$$

there are two possibilities

i) $k_1 < k_2$

$\Rightarrow n_1 \mid n_2$

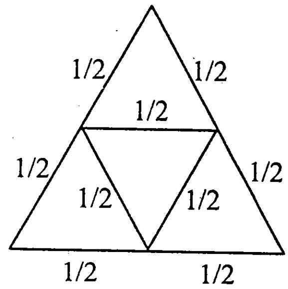
ii) $k_1 > k_2$

$\Rightarrow n_2 \mid n_1$

either way one is multiple of other.

4. Let T is an equilateral triangle of side 1 unit. Prove that if 5 points are chosen inside or on the triangle, then two of them are not more than 1/2 unit apart.

Soln. :



From this figure, we find there are four small equilateral triangles of sides 1/2 unit. Since we are choosing 5 points (5 pigeons) and there are 4 small triangles (4 pigeon holes).

Since no. of pigeonholes is less than the no. of pigeons.

$\therefore$ Pigeonhole principle is applicable.
By pigeonhole principle, one pigeonhole must have atleast two pigeons.

$\therefore$ One small triangle must have two points.

$\because$ Sides of each small triangle is 1/2 unit,

$\therefore$ Two of them can not be more than 1/2 unit apart.

5. If 5 numbers are chosen from a set $\{1, 2, 3, \ldots, 8\}$ then prove that two of them will add up 9.

Soln. :

The numbers of the set $\{1, 2, 3 \ldots 8\}$, can be grouped in pairs such that the sum of each pair $= 9$, i.e., (1, 8), (2, 7), (3, 6), (4, 5).

$\because$ There are four groups (4 pigeonholes) and 5 numbers (5 pigeons) are chosen.

$\Rightarrow$ Pigeonhole principle is applicable.
By pigeonhole principle one pigeonhole must have atleast two pigeons.

$\Rightarrow$ Atleast one group contains 2 numbers.

$\therefore$ Atleast two numbers will add up 9.

6. Shirts numbered consecutively from 1 to 20 are worn by the 20 members of a bowling league. When any 3 of these members are chosen to be a team, the sum of their shirt numbers is used as a code number for the team. Show that if any 8 of the 20 are selected, then from these 8 we may form at least two different teams having the same code number. **[M-02]**

Soln. :

Since the code number of a team is sum of their shirt numbers and shirt numbers are taken form a set $\{1, 2, \ldots 20\}$, therefore one of the lowest code number is $1 + 2 + 3 = 6$ and highest code numbers is $18 + 19 + 20 = 57$. By this way, we find total number of code numbers is equal to 52. (including code numbers 6 and 57).

$\because$ 8 are selected from 20 and a team contains three members, therefore number of teams of three members out of 8 is equal to ${}^8C_3 = 56$.

Since, total code number is less than the number of prepared teams (52 < 56).

$\therefore$ Pigeon hole principle is applicable.
By Pigeonhole principle one pigeonhole must have atleast two pigeons.

$\therefore$ Atleast two different teams have the same code number.

Extended Pigeonhole Principle :

If 'n' pigeons are assigned to 'm' pigeonholes (km < n), then at least one pigeonhole must have $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons.

Note : $\left\lfloor \frac{m}{n} \right\rfloor$ = the largest integer which is less or equal to $\left(\frac{m}{n}\right)$.

e.g., $\left\lfloor \frac{29}{5} \right\rfloor = 5$, $\left\lfloor \frac{35}{7} \right\rfloor = 5$, $\left\lfloor \frac{7}{3} \right\rfloor = 2$.

1. Show that if 30 dictionaries in a library contain a total of 61,327 pages, then one of the dictionaries must have at least 2045 pages.

Soln :

There are 30 dictionaries (30 pigeonholes) and these contain 61,327 pages (61,327 pigeons).

$\because$ 30 << 61,327

$\therefore$ Extended pigeon hole principle is applicable.

By extended pigeon hole principle, one pigeon hole must have atleast $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons.

Here, n = 61327, m = 30.

$\therefore \quad \left\lfloor \frac{n-1}{m} \right\rfloor + 1 = 2044 + 1.$

$= 2045$ pigeons.

$\therefore$ One dictionary must have atleast 2045 pages.

2. If 30 people are assembled in a room. Show that, 5 of them must have their birthday on the same day of a week.

Soln :

There are 7 days (7 pigeonholes) in a week
and 30 people (30 pigeons) are assembled in a room.

$\because$ Number of pigeonholes << number of pigeons
(7 << 30).

$\therefore$ Extended pigeonhole principle is applicable.

Here $\quad n = 30, \; m = 7.$

$\therefore \quad \left\lfloor \frac{n-1}{m} \right\rfloor + 1 = 4 + 1 = 5$ pigeons

$\therefore$ 5 of them must have their birthday on the same day of a week.

3. Prove that if 7 colours are used to paint 50 bicycles, then 8 of them must have same colour : **[M-01, N-04]**

Soln :

There are 50 bicycles (50 pigeons) and 7 colours (7 pigeonholes).

$\because$ Number of pigeonholes << number of pigeons (7 << 50).

$\therefore$ Extended pigeonhole principle is applicable

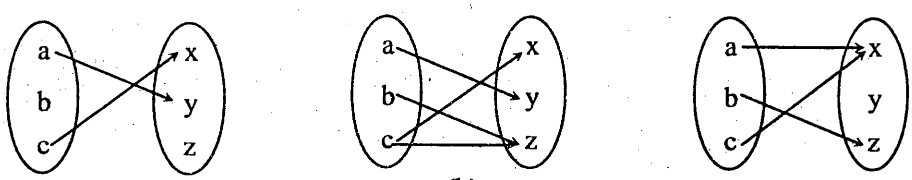
n = 50, m = 7.

$\left\lfloor \frac{50-1}{7} \right\rfloor + 1 = 7 + 1 = 8$ pigeons.

$\therefore$ 8 of them must have same colours.

Graded Questions

1. Define the following terms with example :
   (a) Injective function or one to one function
   (b) Surjective function or onto function
   (c) Bijection / bijective function
   (d) Inverse function
2. State whether or not each diagram in fig. below defines a function from A = {a, b, c} into B = {x, y, z}



(a) (b) (c)

3. Let X = {1, 2, 3, 4}. Determine whether or not each relation below is a function from X into X.
   (a) f = {(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)}
   (b) g = {(3, 1), (4, 2), (1, 1)}
   (c) h = {(2, 1), (3, 4), (1, 4), (2, 1), (4, 4)}

4. If mapping $f : A \to B$ is one to one and onto, the prove that inverse mapping $f^{-1} : B \to A$ is also one to one and onto. **[M-04, 05]**

5. Let $N = \{ 0, 1, 2, \ldots\ldots\ldots n-1 \}$ where n is even. **[M-98]**
   Define $f : N \to N$ by –
   $f(i) = 2i \qquad 0 \le i \le n/2-1$
   $\quad = (2i + 1) \bmod n \qquad n/2 \le i \le n-1$
   Is f a bijection ? Explain.
   If it is, find $f^{-1}(i)$.

6. Let $f : R \to (7/5) \to R - \left(\frac{2}{5}\right)$ be defined by $f(x) = \frac{2x-3}{5x-7}$. **[M-02]**
   Prove that it is a bijection. Hence find f^{-1}.

7. If f and g be the functions from set of integers to set of integers defined by $f(x) = 2x + 3$. $g(x) = 3x + 2$. Find fog and gof. **[D-02]**

8. Let the functions f and g be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Find the formula defining the composition function gof.

9. Show that function $f(x) = ax + b$ from R to R is invertible. where a and b are constants with a $a \neq 0$. Find the inverse of f. **[M-03]**

10. Explain the bijective function with suitable example. **[D-01]**
    Let $f : R \to R$ defined as $f(x) = 2x^3 - 7$
    $g : R \to R$ defined as $g(x) = 3x^2$
    $h : R \to R$ defined as $h(x) = 5x + 4$
    Find the rule of defining (i) (gof) oh (ii) fo(hog).

11. Let $A = B = C = R$ and let $f : A \to B$ and $g : B \to C$ be defined by $f(a) = a - 1$ and $g(b) = b^2$
    find (i) $(gof)_{(x)}$ (ii) $(fof)_{(y)}$ (iii) $(gog)_{(y)}$ **[D-03]**

12. Let $A = B = C = R$ and let $f : A \to B$, $g : B \to C$ be defined by $f(a) = a + 1$ and $g(b) = b^2 + 2$. Find – **[D-01, N-04]**
    (i) (fog) (2) (ii) (go f) (x) (iii) (fog) (x) (iv)(gog) (y) (v) fo(gog) (x)

13. Let $f : A \to B$ be one to one and onto then prove that – **[D-02, 03]**
    $f^{-1} 0 = f = I_A$
    $f 0 f^{-1} = I_B$
    where I_A and I_B are identical mapping on set A and set B.

Pigeonhole Principle

Type I

1. Prove that if any 14 nos. from 1 to 25 are chosen, then one of them is a multiple of another. **[D-01]**
2. Prove that if 6 numbers are chosen from $\{1, 2, \ldots 11\}$ then two of them will add up twelve.
3. If any seven points are chosen from a regular hexagon whose sides are of length 1. Prove that two of them must be no further apart than 1 unit.
4. Show that if any eight positive integers are chosen, two of them will have the same remainder when divided by 7.

Type II

1. If 62 students are selected from a class then atleast 6 of them must have their Birthday on the same month of a year.
2. How many friends must you have to guarantee that at least five of them will have birthdays in the same month? **[M-04]**
3. What is the minimum number of student required in a discrete structure class to be sure that at least six will receive the same grade, if there are five possible grades A, B, C, D and F ? **[D-03]**

□□□□□□

Poset

A relation R on a set A is said to be partial order if it is reflexive, antisymmetric and transitive. If R is a partial order relation on a set A then the set A with the partial order R. i.e. (A, R) is called a partially ordered set or a poset.

Examples

1. Let R be a set of real numbers and a relation $\leq$ defined on R then prove that $(R, \leq)$ is a poset.

Soln. :

i) Reflexive : $\because\ a \leq a \quad \forall\ a \in R$
$\Rightarrow\ \leq$ is reflexive

ii) Antisymmetric : Let $a \leq b,\ b \leq a \ \Rightarrow\ a = b$
$\Rightarrow\ \leq$ is antisymmetric

iii) Transitive : Let $a \leq b,\ \ b \leq c \ \Rightarrow\ a \leq c$
$\Rightarrow\ \leq$ is transitive.
$\therefore\ \leq$ is partial order on R.
$\therefore\ (R, \leq)$ is a poset.

2. Let Z^+ is a set of positive integers and a relation R defined on Z^+ by a R b iff $a \mid b$ then prove that R is a partial order relation and $(Z^+, \mid)$ is a poset.

Soln. :

i) Reflexive : $\because\ a \mid a \quad \forall\ a \in Z^+$
$\therefore\ a\,R\,a \quad \forall\ a \in Z^+$
$\Rightarrow\ \mid$ is reflexive.

ii) Antisymmetric : Let a R b and b R a
$\Rightarrow\ a \mid b\ \&\ b \mid a$
$\Rightarrow\ a = b \quad \forall\ a, b \in Z^+$
$\Rightarrow\ \mid$ is antisymmetric.

iii) Transitive : Let a R b and b R c
$\Rightarrow\ a \mid b\ \ \&\ \ b \mid c$
$\Rightarrow\ a \mid c$
$\Rightarrow\ a\,R\,c$
$\Rightarrow\ \mid$ is transitive.
$\therefore\ \mid$ is partial order relation on Z^+.
$\therefore\ (Z^+, \mid)$ is a poset.

Definition :

Comparable : In a poset $(A, \leq)$, the elements a and b are said to be comparable if $a \leq b$ or $b \leq a$.

Note : In a poset every pair of elements need not be comparable.

Linearly ordered set : If every pair of elements in a poset A is comparable then A is called linearly ordered set or **chain** or **totally ordered set** and partial order is called linear order.

eg.1 : $(R, \leq)$ is linearly ordered set because every pair of real numbers are comparable w.r.t. this partial order.

eg.2 : $(Z^+, \mid)$ is not a linearly ordered set because not all pairs of integers are comparable w.r.t. this partial order (neither 2 divides 3 nor 3 divides 2).

1. If (A, R) is a poset, then (A, R^{-1}) is also a poset (dual poset).

Soln. :

Given R is a partial order on A. To prove that R^{-1} is also a partial order on A.

$a \, R \, a \quad \forall a \in A \quad (\because$ R is reflexive)

$\Rightarrow a \, R^{-1} a \quad \forall \, a \in A$

$\Rightarrow R^{-1}$ is reflexive.

$a \, R \, b, \; b \, R \, a \quad \Rightarrow a = b \quad (\because$ R is antisymmetric)

$\Rightarrow b \, R^{-1} \, a, \; a \, R^{-1} \, b \quad \Rightarrow a = b$

$\Rightarrow R^{-1}$ is antisymmetric.

$a \, R \, b, b \, R \, c \quad \Rightarrow a \, R \, c \quad (\because$ R is transitive)

$\Rightarrow c \, R^{-1} b, b \, R^{-1} \quad \Rightarrow \quad c \, R^{-1} \, a$

$\Rightarrow R^{-1}$ is transitive

$\therefore \; R^{-1}$ is a partial order.

(A, R^{-1}) is a poset.

2. If $(A, \leq)$ and $(B, \leq)$ are posets, then $(A \times B, \leq)$ is a poset with partial order $\leq$ defined by $(a, b) \leq (a', b')$ if $a \leq a'$ in A and $b \leq b'$ in B.

Soln. :

i) Reflexive : Let $(a, b) \in A \times B$

$\because \; a \leq a$ in A & $b \leq b$ in B

$\therefore \; (a, b) \leq (a, b)$

$\therefore \; \leq$ is reflexive.

ii) Antisymmetric : Let $(a, b), (a', b') \in A \times B$ and $(a, b) \leq (a', b')$ & $(a', b') \leq (a, b)$

$\Rightarrow a \leq a', \; b \leq b'$ & $a' \leq a, \; b' \leq b$

$\Rightarrow a \leq a', \; a' \leq a$ & $b \leq b', \; b' \leq b$

$\Rightarrow a = a'$ & $b = b'$

$\therefore \; (a, b) = (a', b')$ in $A \times B$

$\therefore \; \leq$ is antisymmetric

iii) Transitive : Let $(a, b), (a', b'), (a'', b'') \in A \times B$

and $(a, b) \leq (a', b'), (a', b') \leq (a'', b'')$

$\Rightarrow a \leq a'$ & $b \leq b'$ and $a' \leq a''$ & $b' \leq b''$

$\Rightarrow a \leq a''$ & $b \leq b''$

$\Rightarrow (a, b) \leq (a'', b'')$

$\Rightarrow \; \leq$ is transitive.

$\therefore \; \leq$ is a partial order relation on $A \times B$.

$\therefore \; (A \times B, \leq)$ is a poset.

Note : The partial order $\leq$ defined on the Cartesian product is called product partial order.

3. Let $(A, \leq)$ and $(B, \leq')$ be 2 posets. Consider the product set $A \times B$ define a relation $\square$ on $A \times B$ by the following $(a_1, b_1) \; \square \; (a_2, b_2)$ iff $a_1 \leq a_2, b_1 \leq' b_2$. Show that $\square$ is a partial order on $A \times B$. (i.e.,) $(A \times B, \square)$ is a poset. (This is called product poset and the partial order square is called product partial order).

Soln. :

i) Reflexive : $a_1 \leq a_1$, $b_1 \leq' b_1$

$(\because \leq, \leq'$ are reflexive)

$\Rightarrow (a_1, b_1) \; \square \; (a_1, b_1)$

$\Rightarrow \square$ is reflexive.

ii) Antisymmetric :

Let $(a_1, b_1) \; \square \; (a_2, b_2)$ & $(a_2, b_2) \; \square \; (a_1, b_1)$

$\Rightarrow a_1 \leq a_2, \quad b_1 \leq' b_2$ & $a_2 \leq a_1, b_2 \leq' b_1$

$\Rightarrow a_1 \leq a_2, \quad a_2 \leq a_1 \quad \Rightarrow a_1 = a_2$

& $b_1 \leq' b_2, \quad b_2 \leq b_1 \quad \Rightarrow b_1 = b_2$

$\because \; (\leq, \leq'$ is antisymmetric)

$\Rightarrow (a_1, b_1) = (a_2, b_2)$

$\Rightarrow$ □ is antisymmetric.

iii) Transitive

Let (a_1, b_1) □ (a_2, b_2) & (a_2, b_2) □ (a_3, b_3)

$\Rightarrow a_1 \le a_2, b_1 \le' b_2$ & $a_2 \le a_3, b_2 \le' b_3$

$\Rightarrow a_1 \le a_3$ & $b_1 \le' b_3$

$\Rightarrow (a_1, b_1)$ □ (a_3, b_3)

$\Rightarrow$ □ is transitive.

$\therefore$ □ is partial order and $(A \times B,$ □ $)$ is a poset.

Lexicographic Order : If $(A, \le)$ is a poset when $a < b$ if $a \le b$ but $a \ne b$. Let $(A, \le)$ & $(B, \le)$ be the posets then the partial order relation on $A \times B$ denoted by $<$, is defined as

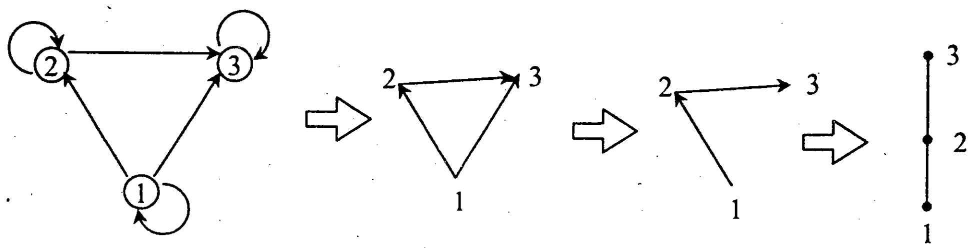
$(a, b) < (a', b')$ if $a < a'$ or if $a = a'$ and $b \le b'$

This ordering is called 'lexicographic' or 'dictionary order'.

Hasse Diagram of a Poset

The diagraph of a partial order relation can be simplified and such simplified graph of a partial order is called Hasse diagram. When the partial order is a total order its Hasse diagram is a straight line and the corresponding poset is called a chain.

Consider $A = \{1, 2, 3\}$, a relation R on A defined by
$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$

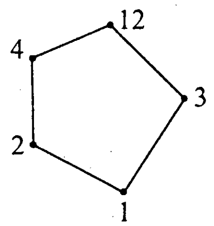


digraph Hasse diagram

1. Draw the Hasse diagram for the following posets.

i) $A = \{1, 2, 3, 4, 12\}$ under partial order $|$.

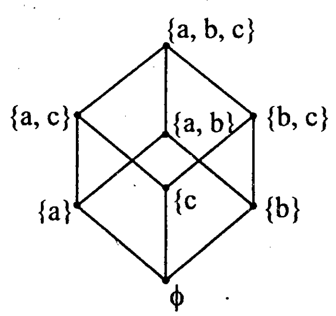
Soln. :



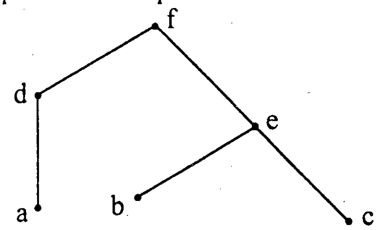
ii) $S = \{a, b, c\}$ & $A = \underline{P}(S)$ under $\subseteq$.
Draw Hasse diagram of poset $(A, \subseteq)$

Soln. :

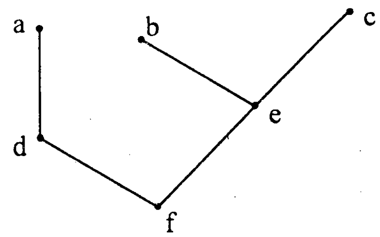
$A = \underline{P}(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$



iii) Draw Hasse diagram of dual poset of the poset of the poset whose Hasse diagram is given.

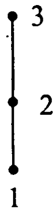


Soln. :



, ≤) & (B, ≤) be

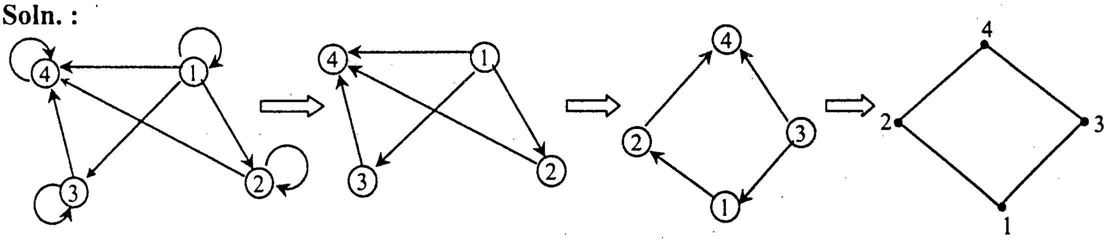
aph of a partial
se diagram is a



2. Determine Hasse diagram of R :

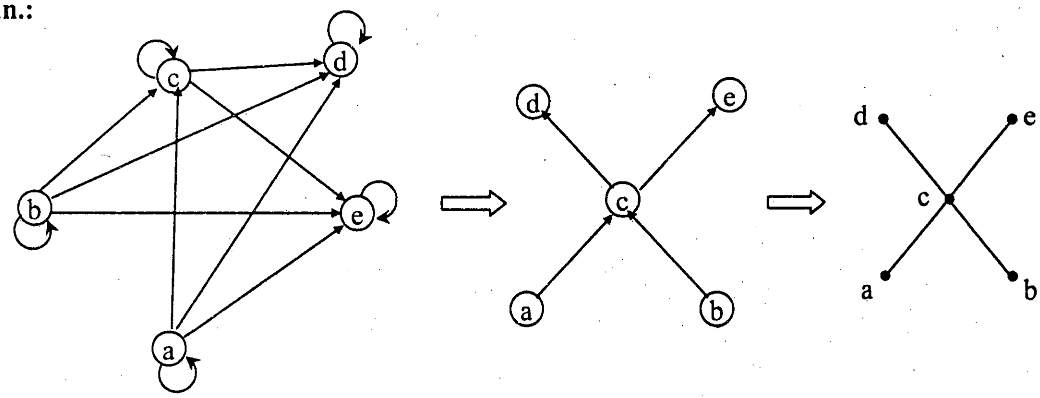
(i) A = {1, 2, 3, 4} and R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (3, 4), (4, 4)}

Soln. :



(ii) A = {a, b, c, d, e},
R = {(a, a), (a, c), (a, d), (a, e), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, d), (e, e)}

Soln.:



3. Describe the ordered pairs in the relation defined by the Hasse diagram on A = {1, 2, 3, 4} in the figures 1, 2.

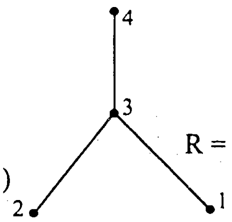


Fig. (1)

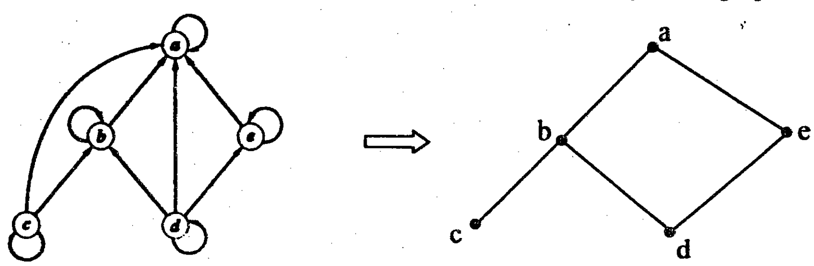
R = {(1, 1), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}

Fig. (2)

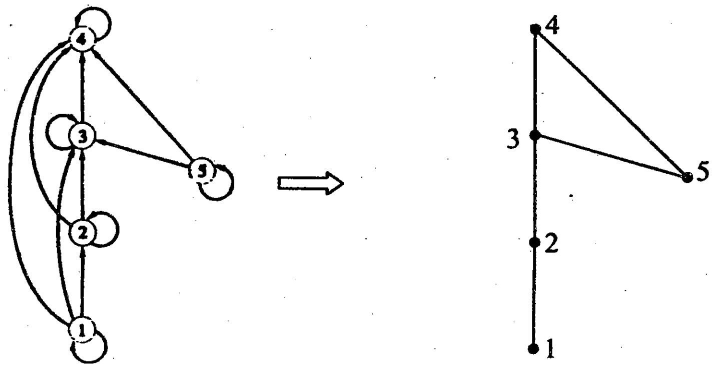
R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4) |

4. **Determine the Hasse diagram of the partial order having the given digraph.**

(a)



(b)



5. **Determine the Hasse diagram of the relation on A = {1, 2, 3, 4, 5}**

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

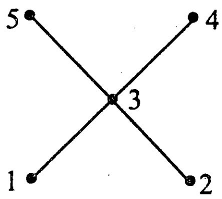
R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (4, 4), (4,5), (5, 5)}

Hasse diagram :

6. Determine the Hasse diagram of the relation on A = {1, 2, 3, 4, 5}

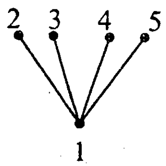
$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hasse diagram :

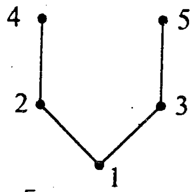


7. Determine matrix of partial order whose Hasse diagrams are given :

(a)



(b)

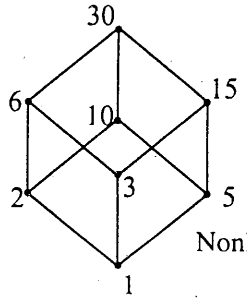


(a) $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

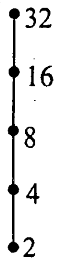
8. Consider the partial order of divisibility on set A.
Draw Hass diagram of the poset and determine which poset are linearly ordered.
(i) A = {1, 2, 3, 5, 6, 10, 15, 30}
(ii) A = {2, 4, 8, 16, 32}
(iii) A = {3, 6, 12, 36, 72}. **[M-03]**
(iv) A = {1, 2, 3, 4, 5, 6, 10, 12, 15, 30, 60}

(i)



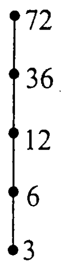
Nonlinearly Ordered

(ii)



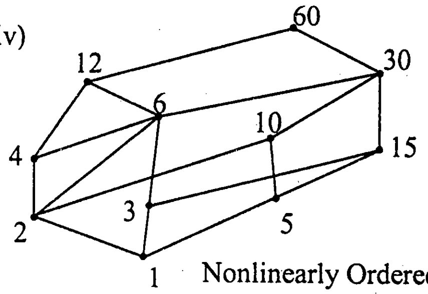
Linearly Ordered

(iii)



Linearly Ordered

(iv)



Nonlinearly Ordered

Note :

Determining whether R is partial order from M_R

Reflexive : Diagonal elements must be non zero.($a_{ii} = 1$)

3), (3, 4), (3, 5),

Antisymmetric : If $a_{ij} = 1$, then $a_{ji} = 0$ ($i \neq j$).

Transitive : $R^2 \subseteq R$

Quasi Order

A relation 'R' on 'A' is called quasi order if it is transitive and irreflexive.

1. Let A = {x | x is a real no. and $-5 \leq x \leq 20$}. S.T. the usual relation '<' is an Quasi order on 'A'.

Irreflexive :

$\because$ $x \not< x$ $\quad \forall x \in A$

$\therefore$ < is irreflexive

4

Transitive : Let $x < y$, $y < z$ $\quad \therefore x < z$

$\therefore$ < is transitive

$\therefore$ < is Quasi order

2

Note : If R is a Quasi order on A then R^{-1} is also a Quasi order on A.

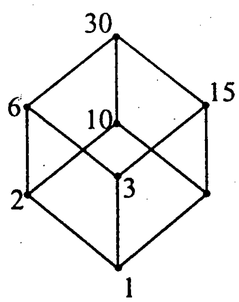
Poset Isomorphism : Let (A, ≤) and (A′, ≤′) be two posets and $f : A \to A'$ be a one to one correspond between A to A′, then 'f' is said to be isomorphic if $\forall$ a, b $\in$ A, $a \leq b \Rightarrow f(a) \leq' f(b)$ in A′.

If $f : A \to A'$ is an isomorphism then (A, ≤) and (A′, ≤) are known as isomorphic posets.

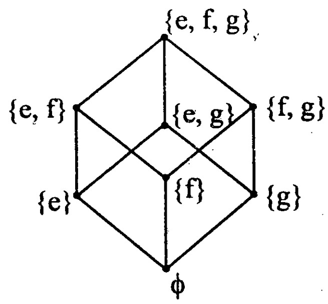
Q. Let A = {1, 2, 3, 5, 6, 10, 15, 30} and consider partial order ≤ of divisibility on A.
i.e. define $a \leq b$ to mean that $a \mid b$. Let A′ = P(S), where S = {e, f, g} be the poset with partial order ⊆. Show that (A, ≤) & (A′, ⊆) are isomorphic.

Ans. :

A = {1, 2, 3, 5, 6, 10, 15, 30}
P(S) = A′ = {ϕ, {e}, {f}, {g}, {e, f}, {f, g}, {e, g}, {e, f, g}}



Hasse diagram of A



Hasse diagram of A′

We define a mapping from $(A, \leq)$ to $(A', \subseteq)$ as :

$$f(1) = \phi$$
$$f(2) = \{e\}$$
$$f(3) = \{f\}$$
$$f(5) = \{g\}$$
$$f(6) = \{e, f\}$$
$$f(10) = \{f, g\}$$
$$f(15) = \{e, g\}$$
$$f(30) = \{e, f, g\}$$

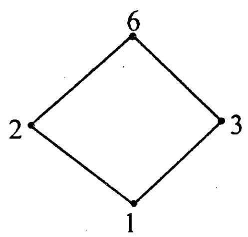
$\because$ $\forall$ a, b $\in$ A, $a \leq b \Rightarrow f(a) \subseteq f(b)$ in A′.

$\therefore$ Posets A and A′ are isomorphic.

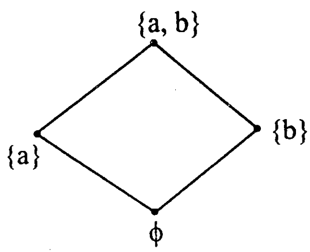
Note : The two posets are isomorphic iff their Hasse diagrams are perfectly identical except for the labels.

e.g., A = {1, 2, 3, 6} under |
A′ = P {a, b} = {ϕ, {a}, {b}, {a, b}} under ⊆. Show that (A, |) is isomorphic to (A′, ⊆).

Soln. :



Hasse diagram of A



Hasse diagram of A′

$f(1) = \phi$, $f(2) = \{a\}$, $f(3) = \{b\}$, $f(6) = \{a, b\}$.

Isomorphism between two posets means
(i) their Hasse Diagrams are absolutely identical.
(ii) there is a bijection among the element of two posets.
(iii) The partial orders are preserved (i.e.,) if $a \leq b$ iff $f(a) \leq f(b)$.

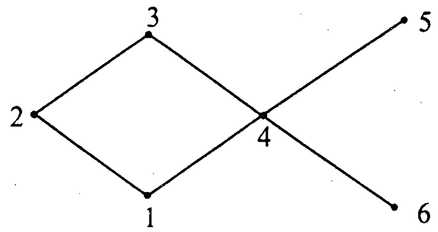
Extremal Elements of Posets :

Maximal Element
Let $(A, \leq)$ be a poset and a $\in$ A, then the element a is said to be maximal element of A if there is no b such that $a \leq b$.

Minimal Element
Let $(A, \leq)$ be a poset, then an element a $\in$ A is said to be minimal element of A if there is no b such that $b \leq a$.

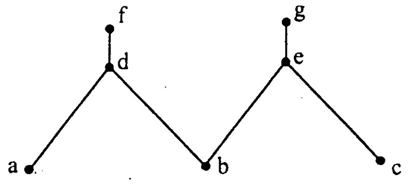
Determine maximal and minimal element of the poset.

1.



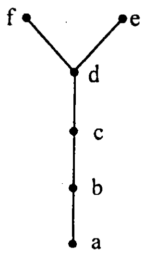
Maximal element = 3, 5
Minimal element = 1, 6

2.



Maximal element = f, g
Minimal element = a, b, c

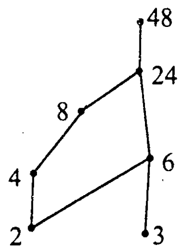
3.



Maximal element = f, g
Minimal element = a

4. A = {x | x is real and $0 \le x < 1$} with usual partial $\le$.
Minimal element = zero
Maximal element = does not exist.

5. A = {2, 3, 4, 6, 8, 24, 48} with partial order of divisibility.

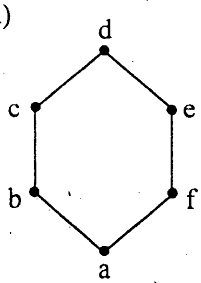


Maximal = 48
Minimal = 2, 3

Greatest element : An element $a \in A$ is said to be greatest element of A if $x \le a \ \forall\ x \in A$.
Least element : An element $a \in A$ is said to be least element of A if $a \le x \ \forall\ x \in A$.

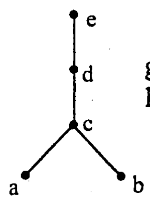
6. Determine the greatest and least elements if they exist in the posets.

(a)



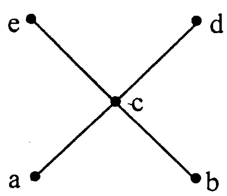
greatest element : d
least element : a

(b)



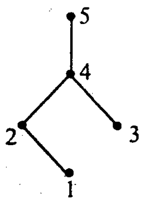
greatest element : e
least element : does not exist

(c)



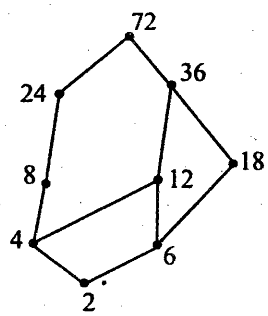
Both greatest and least elements do not exist.

(d)



Greatest element = 5
Least element = does not exist

(e) A = {2, 4, 6, 8, 12, 18, 24, 36, 72} with partial order of divisibility.



Least element = 2 $\quad 2 \le x \quad \forall\, x \in A$
Greatest element = 72 $\quad x \le 72 \quad \forall\, x \in A$

Note :
The greatest element of a poset is denoted by 'I' or '1' is called the unit element.
The least element of a poset is denoted by 0, is called 'zero element'.

Definition :

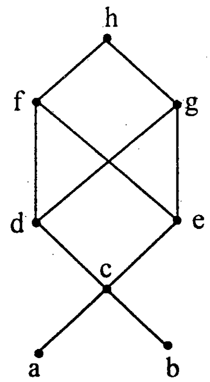
Lower bound : Let $(A, \le)$ be a poset and $B \subseteq A$, then an element $a \in A$ is said to be lower bound of B if $a \le b \;\; \forall\, b \in B$.

Upper bound : Let $(A, \le)$ be a poset and $B \subseteq A$, then an element $a \in A$ is said to be an upper bound of B if $b \le a \;\; \forall\, b \in B$.

Greatest lower bound :
Let $(A, \le)$ be a poset and $B \subseteq A$, then an element $a \in A$ is said to be greatest lower bound of B if it is a lower bound such that $a' \le a$ whenever a' is lower bound of B.

Least upper bound :
Let $(A, \le)$ be a poset and $B \subseteq A$, then an element $a \in A$ is said to be least upper bound of B if it is an upper bound such that $a \le a'$ whenever a' is an upper bound of B.

1. Consider a poset A = {a, b, c, d, e, f, g, h} where Hasse diagram is shown.



Find all upper and lower bounds of subsets of A.
(i) $B_1 = \{a, b\}$ (ii) $B_2 = \{c, d, e\}$

Soln. :

(i) Here A is a poset and $B \subseteq A$.

$\because$ $a \in A$ is said to lower bound of B_1 if $a \le b$ for all $b \in B_1$.

$\therefore$ From figure, we can say that there is no lower bound of subset B_1.

$\therefore$ Greatest lower bound does not exist.

$\because$ $a \in A$ is said to be upper bound of B if $b \leq a$ for all $b \in B_1$.

$\therefore$ The upper bounds are : c, d, e, f, g, h

$\therefore$ Least upper bound : c

(ii) Here A is a poset and $B_2 \subseteq A$

$\because$ $a \in A$ is said to be lower bound of B_2 if $a \leq b$ for all $b \in B_2$.

$\therefore$ The lower bounds are : a, b, c

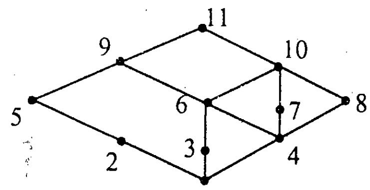
$\therefore$ The greatest lower bound is c.

$\because$ $a \in A$ is said to be upper bound of B_2 if $b \leq a$ for all $b \in B_2$.

$\therefore$ The upper bounds are f, g, h.

$\therefore$ The least upper bound does not exist.

2. Let $(A, \leq)$ be a poset where $A = \{1, 2, \ldots\ldots 11\}$ where Hasse diagram is shown.



Find LUB and GLB of $B = \{7, 8, 10\}$ if there exists.

Here $A = \{1, 2, 3, \ldots\ldots 11\}$

Upper bounds of B are 10, 11

Least upper bound of B is 10

Lower bounds of B are 4, 1

Greatest lower bound of B is 4

Note : Suppose that $(A, \leq)$ & $(A', \leq')$ are isomorphic posets under isomorphic $f : A \to A'$.

(i) If a is maximal or minimal of poset $(A, \leq)$ then f(A) is maximal or minimal of $(A', \leq')$

(ii) If a is greatest or least element of $(A, \leq)$ then f(A) is greatest or least element of $(A', \leq')$.

(iii) If a is upper or lower bound of $(A, \leq)$ then f(A) is upper or lower bound of $(A', \leq')$.

(iv) If a is glb or lub of $(A, \leq)$ then f(A) is glb or lub of $(A', \leq')$.

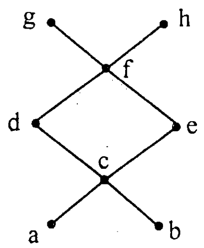
3. (i) Find all upper bounds

(ii) Find all lower bounds

(iii) Least upper bound

(iv) Greatest lower bound

(a)



(i) $B = \{c, d, e\}$

upper bounds of B are f, g, h

least upper bound of B is f

lower bounds of B are a, b, c

greatest lower bound of B is c

(ii) $B = \{b, g, h\}$

There is no upper bound of B.

$\therefore$ there is no least upper bound of B

lower bound of B is b

$\therefore$ greatest lower bound of B is b

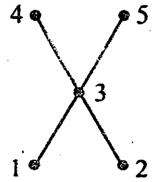
be lower bound

to be an upper

er bound of B if

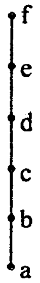
bound of B if it

(b)



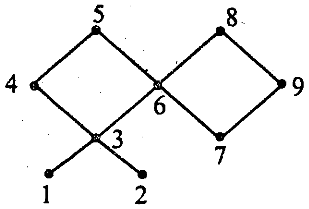
B = {1, 2, 3, 4, 5}
L.B., U.B., G.L.B., L.U.B of B do not exist.

4.



B = {b, c, d}
L.B. = a
G.L.B. = b
U.B. = e, f, d
L.U.B. = d

5.



i) B = {3, 4, 6}
L.B. = 1, 2, 3
G.L.B. = 3
U.B. = 5
L.U.B. = 5

ii) B = {4, 6, 9}
L.B, U.B, G.L.B., L.U.B does not exist.

iii) B = {3, 4, 8}
L.B. = 1, 2, 3
U.B. & L.U.B. does not exist.
G.L.B. = 3

Lattices

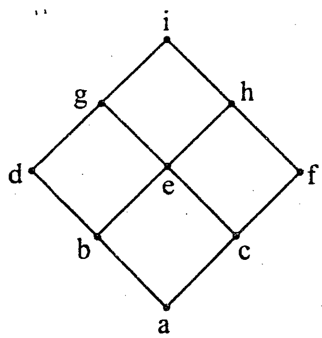
A lattice is a poset (L, $\leq$) in which every subset {a, b} consisting two elements a, b has least upper bound and greatest lower bound.

Note : lub of a and b = $a \vee b$.

glb of a and b = $a \wedge b$.

1. Determine whether the Hasse diagrams represents a lattice

(a)



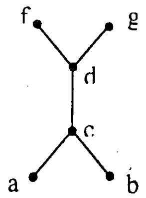
$\vee$	a	b	c	d	e	f	g	h	i
a	a	b	c	d	e	f	g	h	i
b	b	b	e	d	e	h	g	h	i
c	c	e	c	g	e	f	g	h	i
d	d	d	g	d	g	i	g	i	i
e	e	e	e	g	e	h	g	h	i
f	f	h	f	i	h	f	i	h	i
g	g	g	g	g	g	i	g	i	i
h	h	h	h	i	h	h	i	h	i
i	i	i	i	i	i	i	i	i	i

$\wedge$	a	b	c	d	e	f	g	h	i
a	a	a	a	a	a	a	a	a	a
b	a	b	a	b	b	a	b	b	b
c	a	a	c	a	c	c	c	c	c
d	a	b	a	d	b	a	d	b	d
e	a	b	c	b	e	c	e	e	e
f	a	a	c	a	c	f	c	f	f
g	a	b	c	d	e	c	g	e	g
h	a	b	c	b	e	f	e	h	h
i	a	b	c	d	e	f	g	h	i

From these two composition tables, we find that $\forall$ a, b $\in$ A, a $\vee$ b, a $\wedge$ b $\in$ A.

$\therefore$ poset A is a lattice.

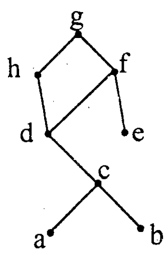
(b)



$\because$ {a, b} has no GLB & {f, g} has no LUB

$\therefore$ This poset cannot be a lattice

(c)



Here c $\not\leq$ e and {a, b} has no GLB.

$\therefore$ given poset is not a lattice.

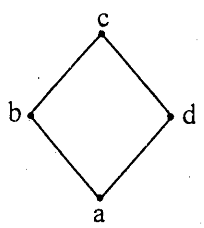
(d)

$\because$ Every pair has GLB and LUB

$\therefore$ This poset is a lattice.

a, b has least

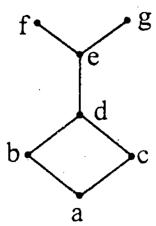
(e)



$\because$ Every pair has GLB and LUB

$\therefore$ This poset is a lattice.

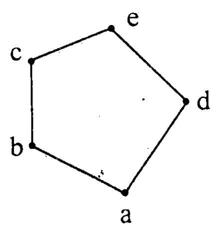
(f)



$\because$ {f, g} has no LUB

$\therefore$ This is not a lattice

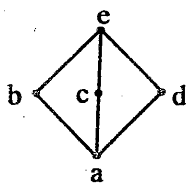
(g)



$\because$ Every pair has GLB and LUB

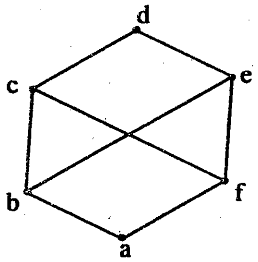
$\therefore$ This poset is a lattice.

(h)



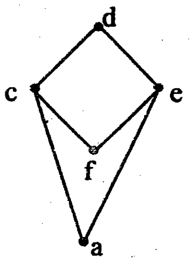
∵ Every pair has GLB and LUB
∴ This poset is a lattice.

(i)



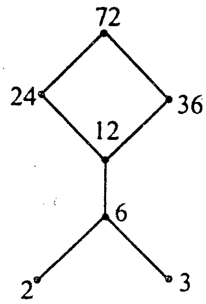
∵ GLB of c & e does not exist.
∴ This is not a lattice

(j)



This is not a lattice.
GLB of {a, f} does not exist.

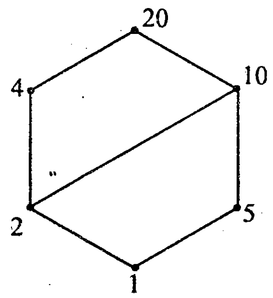
2. If A = {2, 3, 6, 12, 24, 36, 72} under the divisibility relation, a lattice ?



∵ {2, 3} has no GLB
∴ It is not a lattice.

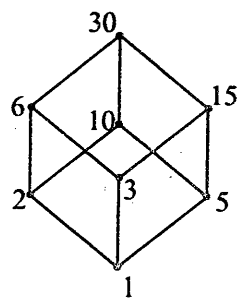
Note : A set D_n is a set of all positive integers which are divisors of n always form a lattice.

3. D_{20} = {1, 2, 4, 5, 10, 20}



This is a lattice
∵ every pair of D_{20} has lub and glb.

4. D_{30} = {1, 2, 3, 5, 6, 10, 15, 30}



Definition

5. If L_1 an
Draw

a

Soln. :

(I_1

(a

Note : Me
(i) a ≤ a ∨
(ii) a ≤ c, b
(iii)a ∧ b ≤
(iv)c ≤ a, c

Theorem

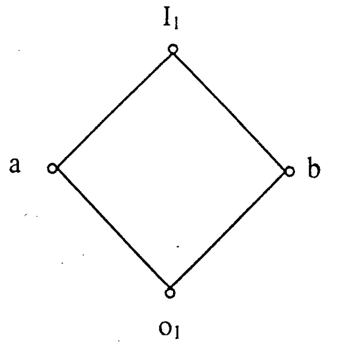
Let L be a
(i) a ∨ b =
(ii) a ∧ b =
(iii)a ∧ b =

Proof :

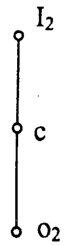
(i) Let a ∨
∵ a ≤
⇒ a ≤
convers
∵ b ≤
∵ a ≤
⇒ a ∨
∴ Fro
a ∨

Definition : If $(L_1, \leq_1)$ and $(L_2, \leq_2)$ are lattices then $(L_1 \times L_2, \leq_3)$ is a lattice, under partially ordered relation $\leq_3$ defined by
$(a, b) \leq_3 (c, d)$ iff $a \leq_1 c$ and $b \leq_2 d$

5. If L_1 and L_2 be two lattices given below.
Draw Hasee diagram of lattice $L_1 \times L_2$.

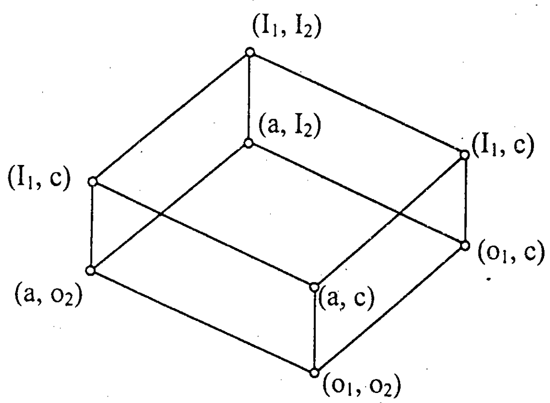


$(L_1 \leq_1)$



$(L_2, \leq_2)$

Soln. :



Hasse diagram of $L_1 \times L_2$.

Note : Meaning of $a \vee b$ & $a \wedge b$.
(i) $a \leq a \vee b$ and $b \leq a \vee b$.
(ii) $a \leq c, b \leq c \Rightarrow a \vee b \leq c$, then $a \vee b$ is LUB of a & b
(iii) $a \wedge b \leq a$, $a \wedge b \leq b$
(iv) $c \leq a$, $c \leq b \Rightarrow c \leq a \wedge b$ then $a \wedge b$ is GLB of a & b.

ttice.

Theorem 1

Let L be a lattice then $\forall$ a, b $\in$ L
(i) $a \vee b = b$ iff $a \leq b$
(ii) $a \wedge b = a$ iff $a \leq b$
(iii) $a \wedge b = a$ iff $a \vee b = b$

Proof :

(i) Let $a \vee b = b$
$\because$ $a \leq a \vee b$ [by definition]
$\Rightarrow$ $a \leq b$ [$\because$ $a \vee b = b$]
conversely let $a \leq b$
$\because$ $b \leq a \vee b$ ... (1) [by definition]
$\because$ $a \leq b$ (given) & $b \leq b$ (by reflexive)
$\Rightarrow$ $a \vee b \leq b$ ... (2) [$\because$ $a \vee b$ = LUB of a & b]
$\therefore$ From (1) & (2),
$a \vee b = b$ (by antisymmetric)

(ii) Let $a \wedge b = a$

By definition, $a \wedge b \leq b$

$\therefore\ a \leq b \quad [\because a \wedge b = a]$

Conversely let $a \leq b$

$\because\ a \wedge b \leq a$ ... (1) [By definition]

$a \leq a$ (by reflexive) and $a \leq b$ (given)

$\Rightarrow\ a \leq a \wedge b$ ... (2) [$\because a \wedge b$ is GLB of a & b]

From (1) & (2),

$a \wedge b = a$ [$\because$ it is antisymmetric]

(iii) Let $a \wedge b = a$

$\Rightarrow\ a \leq b$ from (ii)

$\Rightarrow\ a \vee b = b$ from (i)

Conversely, $a \vee b = b$

$\Rightarrow\ a \leq b$ from (i)

$\Rightarrow\ a \wedge b = a$ from (ii)

Hence the theorem.

Theorem 2

Let L be a lattice then

i) Idempotent properties

(a) $a \vee a = a$

(b) $a \wedge a = a$

ii) Commutative properties

(a) $a \vee a = b \vee a$

(b) $a \wedge b = b \wedge a$

iii) Associative property

(a) $a \vee (b \vee c) = (a \vee b) \vee c$

(b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Proof :

iii) (a) $a \vee (b \vee c) = (a \vee b) \vee c$

$a \leq a \vee (b \vee c)$ and $(b \vee c) \leq a \vee (b \vee c)$

$b \leq b \vee c$ and $c \leq b \vee c$

By transitive property,

$\therefore\ b \leq a \vee (b \vee c)$ and $c \leq a \vee (b \vee c)$

$\because\ a \leq a \vee (b \vee c)$ and $b \leq a \vee (b \vee c)$

$\therefore\ a \vee b \leq a \vee (b \vee c)$ (if $a \leq c, b \leq c \Rightarrow a \vee b \leq c$)

$\Rightarrow\ (a \vee b) \vee c \leq a \vee (b \vee c)$ ... (1)

Similarly, we can find

$a \vee (b \vee c) \leq (a \vee b) \vee c$ ... (2)

By antisymmetric relation

$(a \vee b) \vee c = a \vee (b \vee c)$ from (1) & (2)

Absorption Property

(a) $a \vee (a \wedge b) = a$

(b) $a \wedge (a \vee b) = a$

Proof :

(a) By definition

$a \leq a \vee (a \wedge b)$ ... (1)

$a \leq a$ (by reflexive) $a \wedge b \leq a$ (by definition)

$\Rightarrow\ a \vee (a \wedge b) \leq a$ ... (2) $\because\ a \leq c, b \leq c \Rightarrow a \vee b \leq c$

From (1) & (2)

$a \vee (b \wedge a) = a$ ($\because \leq$ is antisymmetric)

(b) $a \wedge (a \vee b) \leq a$ (by definition) ... (1)

$a \leq (a \vee b)$ (by definition) and $a \leq a$ (by reflexive)

$\Rightarrow\ a \leq a \wedge (a \vee b)$ ... (2) $\because\ c \leq a, c \leq b \Rightarrow c \leq a \wedge b$

From (1) & (2)

$\therefore\ a \wedge (a \vee b) = a$ ... (By antisymmetric relation)

Sub Lattice

Let $(L, \leq)$ be the lattice, a non empty subset S of L is called sublattice of 'L' if $\forall$ a, b $\in$ S, $a \vee b$, $a \wedge b \in S$.

Bounded Lattice

A lattice L is said to be bounded if it has a greatest element 1 and a least element '0'.

Note : If 'L' is bounded lattice then for all $a \in A$, $0 \leq a \leq 1$

$a \vee 0 = a$, $a \wedge 0 = 0$

$a \vee 1 = 1$, $a \wedge 1 = a$

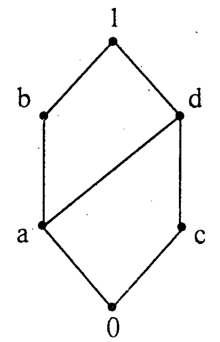
Distributive Lattice

A lattice L is called distributive lattice if $\forall$ a, b, c $\in$ L, it follows distributive properties.

i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

1. Whether the following lattice shown in figure is distributive or not ?



$a, b, c \in L$

$a \vee (b \wedge c) = a \vee 0$

$= a$

$(a \vee b) \wedge (a \vee c) = b \wedge d = a$

$\Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

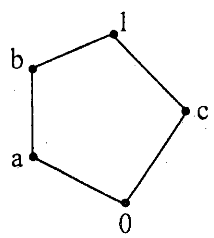
$a \wedge (b \vee c) = a \wedge 1 = a$

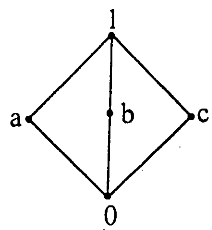
$(a \wedge b) \vee (a \wedge c) = a \vee \ = a$

$\Rightarrow a \wedge (b \vee c) = \wedge b) \vee (a \wedge c)$

$\Rightarrow$ Lattice is distributive lattice

2. Show that the lattices pictured in the figure are non distributive





i) $a \vee (b \wedge c) = a \vee 0 = a$

$(a \vee b) \wedge (a \vee c) = b \wedge 1 = b \qquad \because a \neq b$

$\Rightarrow a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$

$\therefore$ Lattice is not distributive

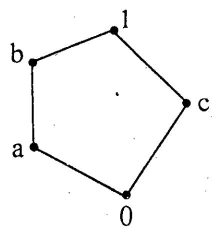
ii) $a \vee (b \wedge c) = a \vee 0 = a$

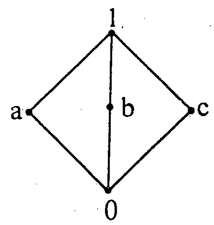
$(a \vee b) \wedge (a \vee c) = 1 \wedge 1 = 1 \qquad \because a \neq 1$

$\therefore a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$

$\therefore$ Lattice is not distributive

Note : A lattice L is non distributive iff it contains a sub lattice which is isomorphic to one of the following.





Complement of an element

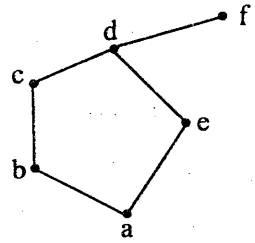
Let L be a bounded lattice with greatest element 1 and least element 0 and $a \in L$. An element $a' \in L$ is said to be complement of a if $a \vee a' = 1$ and $a \wedge a' = 0$

Complemented Lattice

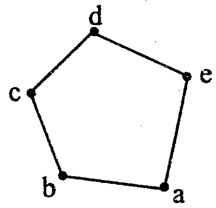
A lattice L is called complemented if it is bounded and every element of L has a complement.

Note : From the definition of complement on an element in a lattice we observe that $0' = 1$ and $1' = 0$

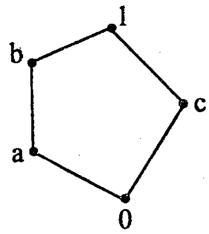
3.
i)



It's one sub lattice



is isomorphic to

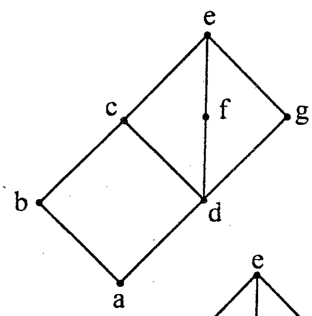


$\therefore$ Given lattice is non distributive.

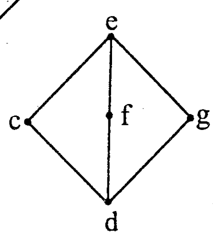
$\because$ For $c \in L$ there is no c' such that $c \vee c' = 1$ & $c \wedge c' = 0$

$\therefore$ It is not complemented lattice.

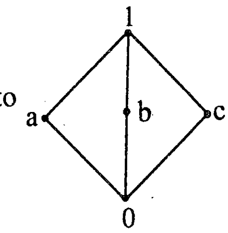
ii)



$\therefore$ Its sub lattice



is isomorphic to

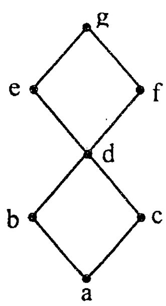


$\therefore$ It is non distributive

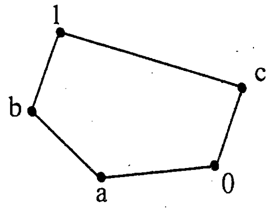
$\because$ For $b \in L$ there is no b' such that $b \vee b' = 1$ & $b \wedge b' = 0$

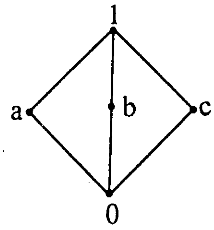
$\therefore$ It is not complemented lattice.

iii)



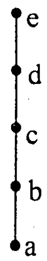
Since there is no sub lattice of this lattice which is isomorphic to anyone of following two



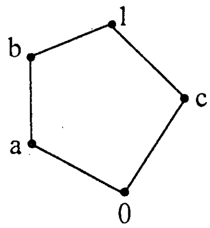


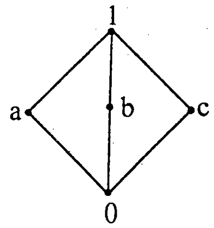
$\therefore$ Given lattice is distributive.

iv)



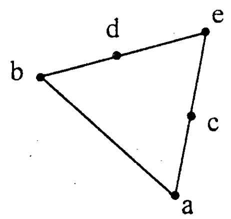
$\because$ This lattice has no sub lattice which is isomorphic to any one of the following lattices





$\therefore$ given lattice is distributive

4. Consider the complemented lattice shown in fig. Give complement of each element



$c' = b$ and d both
$\because$ $c \vee b = e$ and $c \wedge b = a$
$c \vee d = e$ and $c \wedge b = a$
Here a and e are least and greatest elements of L respectively.

Theorem :

If L is a bounded, distributive lattice and a complement exist then it is unique.

Proof :

Let a' and a'' be two complements of $a \in L$ then by definition

$a \vee a' = 1$ and $a \wedge a' = 0$
$a \vee a'' = 1$ and $a \wedge a'' = 0$

$$
\begin{aligned}
a' = a' \vee 0 &= a' \vee (a \wedge a'') && \ldots (a \wedge a'' = 0)\\
&= (a' \vee a) \wedge (a' \vee a'') && \ldots \text{ distributive}\\
&= 1 \wedge (a' \vee a'') && \ldots (a' \vee a = 1)\\
&= a' \vee a'' && \ldots (i)
\end{aligned}
$$

$$
\begin{aligned}
a'' = a'' \vee 0 &= a'' \vee (a \wedge a') && \ldots (a \wedge a') = 0\\
&= (a'' \vee a) \wedge (a'' \vee a') && \\
&= 1 \wedge (a'' \vee a') && \ldots (a \vee a'' = 1)\\
&= a'' \vee a' && \\
&= a' \vee a'' && \ldots (ii)
\end{aligned}
$$

From (i) & (ii),

$a' = a''$

Complement of each element of bounded and distributive lattice is unique.

Modular Lattice

A lattice is said to be modular if for all a, b, c $a \le c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

5. (i) Show that a distributive lattice is modular.

Let L be a distributive lattice $\forall$ a, b, c $\in$ L

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Consider $a \le c$

We have to prove L is modular, for this we require

$(a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c$

L.H.S. = $(a \vee b) \wedge (a \vee c)$

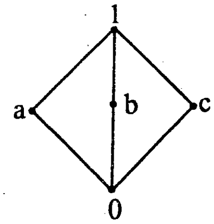
= $(a \vee b) \wedge c$ ... ($\because a \le c$)

= R.H.S.

(ii) Show that the lattice shown in figure is a non distributive lattice that is modular.

We have to prove that given lattice 'L' is modular

$\because$ a, b, 1 $\in$ L and $a \le 1$



$a \vee (b \wedge I) = a \vee b = 1$

$(a \vee b) \wedge 1 = 1 \wedge 1 = 1$

$a \vee (b \wedge 1) = (a \vee b) \wedge 1$

The lattice is a modular lattice.

6. Find complement of D_{42}.

$D_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$

Greatest element = 42

Least element = 1

$1' = 42$	and	$42' = 1$
$2' = 21$	and	$21' = 2$
$3' = 14$	and	$14' = 3$
$6' = 7$	and	$7' = 6$

7. Whether the following lattices are complemented lattice.

$D_{20} = \{1, 2, 4, 5, 10, 20\}$

$1' = 20$; $4' = 5$ but $2'$ does not exist. $\because$ LCM of 2 and 10 is 10

$\therefore$ It is not complemented

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$1' = 30, 2' = 15, 3' = 10, 5' = 6$

$\therefore$ it is complemented

8. P.T. if a and b are elements in a bounded distributive lattice and a has a complement a′, then $a \vee (a' \wedge b) = a \vee b$

$a \wedge (a' \vee b) = a \wedge b$

LHS = $a \vee (a' \wedge b)$

= $(a \vee a') \wedge (a \vee b)$ $\because$ L is distributive

= $1 \wedge (a \vee b)$ $\because$ L is bounded

= $a \vee b$

= RHS

LHS = $a \wedge (a' \vee b)$

= $(a \wedge a') \vee (a \wedge b)$ $\because$ L is distributive

= $1 \vee (a \wedge b)$ $\because$ L is bounded

= $a \wedge b$

= RHS

9. Let L be a distributive lattice, show that if there exist an element a if

$a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ then $x = y$

x = $x \vee (x \wedge a)$ ... by absorption property

= $(x \vee x) \wedge (x \vee a)$ ... by distributive property

= $x \wedge (a \vee y)$ ... given

= $(x \wedge a) \vee (x \wedge y)$ ... by distributive property

= $(a \wedge y) \vee (x \wedge y)$ ... given

= $y \wedge (a \vee x)$ ... by distributive property

$= y \wedge (a \vee y)$ ... given

$= y$ ... by absorption property

Boolean Algebra :

Definition : A Boolean Algebra is a lattice which is bounded, distributive and complemented.

Properties of a Boolean Algebra :

1. Every Boolean Algebra is isomorphic to the Boolean Algebra $(P(S), \subseteq)$ where S is some set.
2. Every Boolean Algebra must have no. of element of the form 2^n. The converse is not true.

e.g. i) $D_{20} = \{1, 2, 4, 5, 10, 20\}$

No. of elements $= 6 \neq 2^n$ for any $n \in N$

$\therefore$ D_{20} cannot be a B.A.

ii) $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

No. of elements $= 8 = 2^3$

The lattice is bounded, complemented and distributive. $\therefore$ It is a B.A.

iii) $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

No. of elements $= 8 = 2^3$

D_{24} has number of elements $8 = 2^3$ but still D_{24} is not a B.A. because it is not complemented. 2, 4, 6, 12 has no complements.

Graded Questions

Poset

1. Define Poset. Explain it with suitable example. Define chain. **[N-04]**
2. Determine whether the relation R is a partial order on the set A.
   (a) $A = Z$, and a R b if and only if $a = 2b$.
   (b) $A = Z$, and a R b if and only if $b^2 \mid a$.
3. Determine whether the relation R is partial order on the set A.
   (a) $A = Z$, and a R b if and only if $a = b^k$ for some $k \in Z^+$. Note that k depend on a and b.
   (b) $A = R$, and a R b if and only if $a \leq b$.
4. Determine whether the relation R is a linear order on the set A.
   (a) $A = R$, and a R b if and only if $a \leq b$.
   (b) $A = R$, and a R b if and only if $a \geq b$.
5. Determine whether the relation R is a linear order on the set A.
   (a) $A = P(S)$, where S is a set. The relation R is set inclusion.
   (b) $A = R \times R$, and (a, b) R (a', b') if and only if $a \leq a'$ and $b \leq b'$, where $\leq$ is the usual partial order on R.
6. On the set $A = \{a, b, c\}$, find all partial orders $\leq$ in which $a \leq b$.
7. Let X be the set N×N where N is set of natural numbers. For $(x_1, y_1), (x_2, y_2) \in X$, define (x_1, y_1) R (x_2, y_2). Iff $x_1 \leq x_2$ & $y_1 \leq y_2$. Prove that R is a partial order on X and (X, R) is a lattice. For $X = A \times A$ where $A = [1, 2, 3]$ draw the Hasse diagram of (X, R).
8. For a given set A, consider the relation $R = \{(x, y) / x \in p(A), y \in p(A), \text{ and } x \subseteq y\}$ where P(A) : denotes power set of A
   (i) Show that R is a partial ordering relation.
   (ii) What is the length of the longest chain in the partially ordered set (P(A), R)

Hasse Diagram

Explain Hasse diagram

1. Determine the Hasse diagram of the relation in R.
   (a) $A = \{1, 2, 3, 4\}$, $R = \{(1, 1), (1, 2), (2, 2), (2, 4), (1, 3), (3, 3), (3, 4), (1, 4), (4, 4)\}$
   (b) $A = \{a, b, c, d, e\}$, $R = \{(a, a), (b, b), (c, c), (a, c), (c, d), (c, e), (a, d), (d, d), (a, e), (b, c), (b, d), (b, e), (e, e)\}$.
2. Determine the Hasse diagram of the partial order having the given digraph.

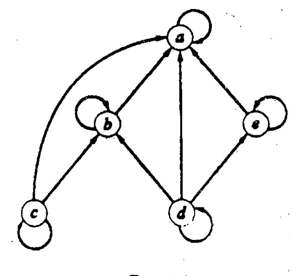


fig.(a)

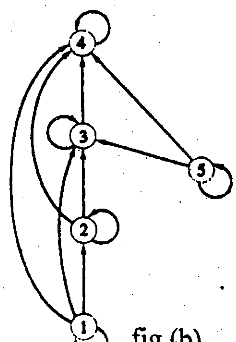
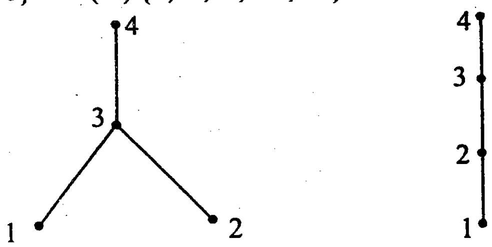


fig.(b)

3. Determine the Hasse diagram of the relation on A = {1, 2, 3, 4, 5} whose matrix is shown.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Draw Hasse diagram of the relation 'divides' on {1, 2, 3, 5, 6, 10, 15, 30}. **[N-04]**
5. Consider the partial order of divisibility on the set A. Draw the Hasse diagram of the poset and determine which posets are linearly ordered.
   (a) A = {2, 4, 8, 16, 32} (b) A = {3, 6, 12, 36, 72} (c) A = {1, 2, 3, 4, 5, 6, 10, 12, 15, 30, 60}
   (d) Describe how to use M_R to determine if R is a partial order.
6. Draw the Hasse diagram of the following sets under partial ordering relation divides and indicate those which are chains.
   i) { 1, 3, 9, 18} (ii) {3, 5, 30} (iii) {1, 2, 5, 10, 20}
7. Describe the ordered pairs in the relation determined by the Hasse diagram on the set A = {1, 2, 3, 4}.

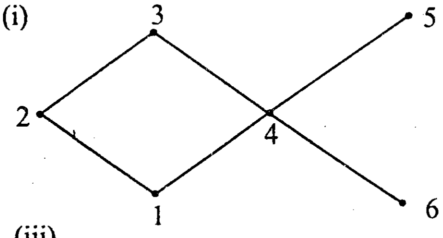


8. Show that there are only five distinct Hasse diagrams for partially ordered sets that contain three elements.
9. Define poset. Draw Hasse diagram represents the partial ordering {(a, b) | a divides b} on {1, 2, 3, 4, 6, 8, 12}
10. Define isomorphism between two posets.
11. Let A = {1, 2, 3, 5, 6, 10, 15, 30} and consider the partial order ≤ of divisibility on A. That is, define a ≤ b to mean that a | b. Let A′ = P(S), where S = {e, f, g}, be the poset with partial order ⊆. Show that (A, ≤) and (A′, ⊆) are isomorphic.
12. Let A = {1, 2, 4, 8} and let ≤ be the partial order of divisibility on A. Let A′ = {0, 1, 2, 3} and let ≤′ be the usual relation "less than or equal to" on integers. Show that (A, ≤) and (A′, ≤′) are isomorphic posets.

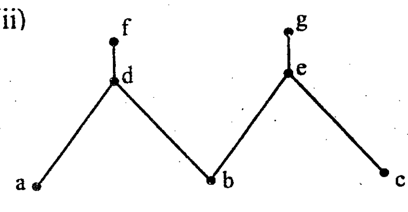
Extremal Elements of Partially Ordered Set

1. Define
   (1) maximal element (2) minimal element (3) greatest element
   (4) least element (5) upper bound (6) lower bound
   (7) greatest lower bound (8) least upper bound
2. Determine all the maximal and minimal elements of the poset.

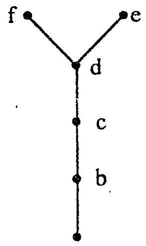
(i)



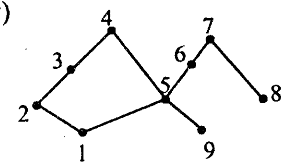
(ii)



(iii)



(iv)



(v) A = {2, 3, 4, 6, 8, 24, 48} with partial order of divisibility.

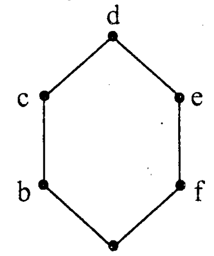
3. Let A = {a, b, c, d } and R be a relation on A whose matrix is **[M-04]**

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

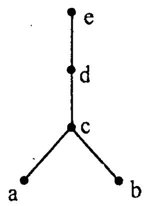
Prove that R is a partial order. Draw the Hasse diagram of R. Is the poset (A, R) is lattice? Justify your answer.

4. Determine the greatest and least elements, if they exist, of the poset.

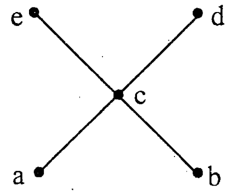
(i)



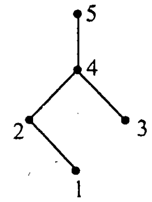
(ii)



(iii)



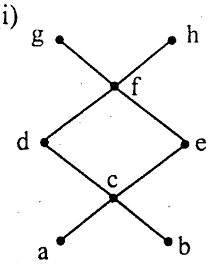
(iv)



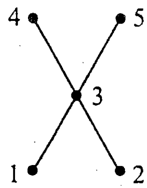
(v) A = {x | x is a real number and $0 < x < 1$} with the usual partial order ≤.
(vi) A = {x | x is a real number and $0 \le x \le 1$} with the usual partial order ≤.
(vii) A = {2, 4, 6, 8, 12, 18, 24, 36, 72} with the partial order of divisibility.
(viii) A = {2, 3, 4, 6, 12, 18, 24, 36} with the partial order of divisibility.

5. In the following exercises find, if they exist, (a) all upper bounds of B; (b) all lower bounds of B; (c) the least upper bound of B; (d) the greatest lower bound of B.

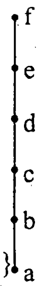
(i)



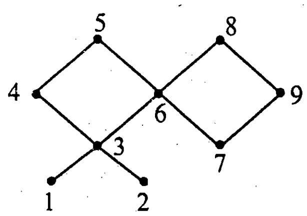
(ii)



(iii)



(iv)

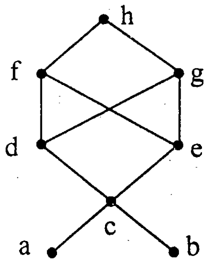


(v) (A, ≤) is the poset in Exercise (i); B = {b, g, h}
(vi)(a) (A, ≤) is the poset in Ex.(iv); B = {4, 6, 9}
(b) (A, ≤) is the poset in Ex.(iv); B = {3, 4, 8}
(vii) A = R and ≤ denotes the usual partial order,
B = {x | x is a real number and $1 < x < 2$}
(viii) A = R and ≤ denotes the usual partial order,
B = {x | x is a real number and $1 \le x < 2$}

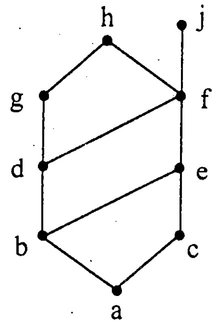
6. Draw the Hasse diagrams of the Lattices having 2^n elements n = 0, 1, 2, 3.

7. Consider the poset A = {a, b, c, d, e, f, g, h}, whose Hasse diagram is shown in figure. Find all upper and lower bounds of the following subsets of A. **[M-04, N-04]**

(1) B_1 = {a, b}
(2) B_2 = {c, d, e}



8. Find lower and upper bounds of the subsets {a, b, c}, {j, h} and {a, c, d, f} in the poset with hasse diagram shown in figure. Also find greatest lower bound and least upper bound of {b, d, g}. **[M-05]**



9. Find the complement of each element in D_{20} and D_{30}. **[M-06]**

Lattice

1. Define the term Lattice. Give one example. **[N-04, D-05]**

2. Determine whether the Hasse diagram represents a lattice.

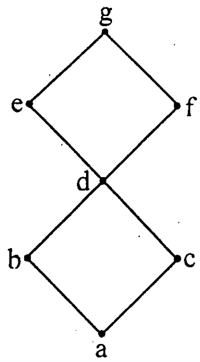


Fig. (i)

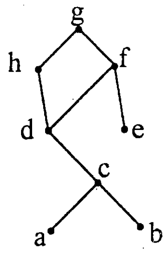
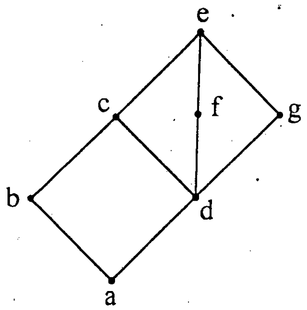


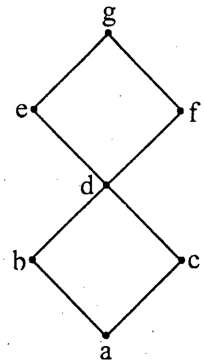
Fig. (ii)

3. Is the poset A = {2, 3, 6, 12, 24, 36, 72} under the relation of divisibility by a lattice ? **[N-04, D-05]**

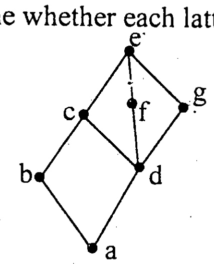
4. Draw the Hasse diagram for divisibility on the set **[M-06]**
   i) {1, 2, 3, 4, 5, 6, 7, 8} ii) {1,2, 3, 5, 7, 11, 13}

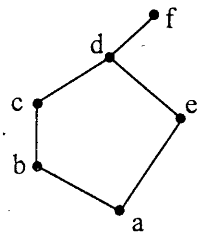
5. Determine whether each lattice is distributive, complemented or both.

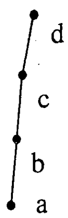




6. Determine whether each lattice is distributive, complemented or both. Justify your answer. **[N-04]**







7. Let L be a lattice. Then for every a, b, ∈ L prove that
   (i) $a \vee b = b$ if and only if $a \leq b$ (ii) $a \wedge b = a$ if and only if $a < b$
   (iii) $a \wedge b = a$ if and only if $a \vee b = b$

in the poset with
it upper bound of
[M-05]

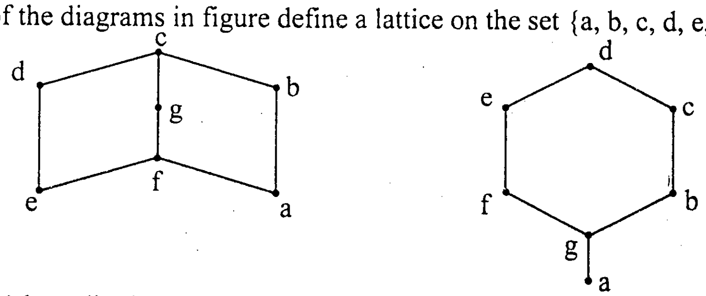
[M-06]

[N-04, D-05]

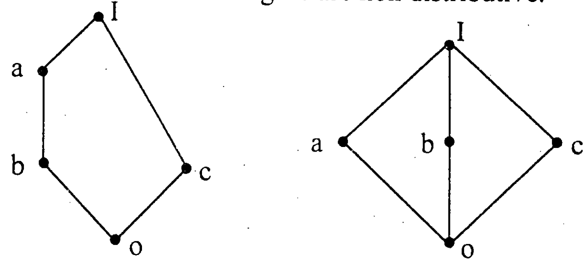
a lattice ?
[N-04, D-05]
[M-06]

your answer.
[N-04]
d

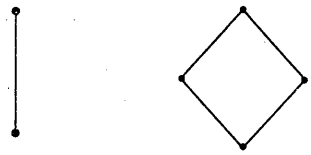
8. Which of the diagrams in figure define a lattice on the set {a, b, c, d, e, f, g}.



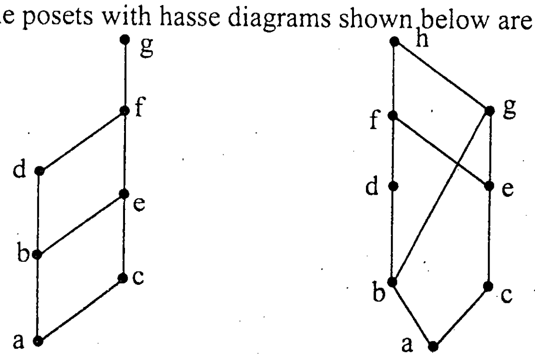
9. Let $(A \le)$ be a distributive lattice. Show that, if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ for some a. Then $x = y$
10. Show that the Lattices shown in figure are non distributive. **[M-04]**



11. Let L_1 and L_2 be the lattices shown below
    Draw the Hasse diagram of $L_1 \times L_2$, with the product partial order.



12. Consider the chains of divisors of 4 and 9 i.e., $L_1 = \{1, 2, 4\}$ and $L_2 = \{1, 3, 9\}$ and partial ordering relation of division on L_1 and L_2. Draw the lattice $L_1 \times L_2$. **[M-06]**
13. Define a complement of an element in lattice
    Find complement of every element in the lattice L = all the positive divisors of 20 with the relation "divisibility".
14. Determine whether the posets with hasse diagrams shown below are lattices or not :
    Justify your answer. **[D-03, M-06]**



15. Draw the Hasse diagram for D_{195}. Is D_{195} under the relation of divisibility a lattice ?
16. Let X = { 1, 5, 7, 35,70, 140, 210, 420 }. Draw the Hasse diagram for (X,R) where R is a relation of divisibility. Whether it is a lattice ? Justify your answer.
17. Define complement of an element in a lattice. Obtain complements of elements in D_{42}. **[M-05]**
18. Let L be a bounded & distributive lattice. Show that if a complement exists, then it is unique. **[N-04, M-05]**
19. Define distributive lattice. Give one example each of distributive and non–distributive lattice.
20. Determine if poset represented by each of the Hasse diagrams in figure are lattices. Justify your answer.

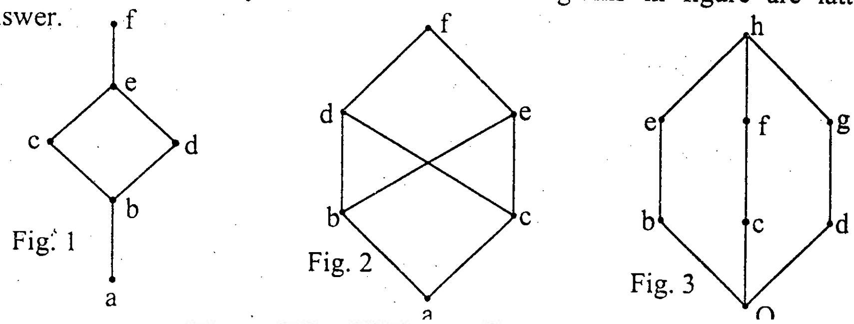


Fig. 1
Fig. 2
Fig. 3

21. Determine whether each lattice shown in figure is distributive, complemented or both

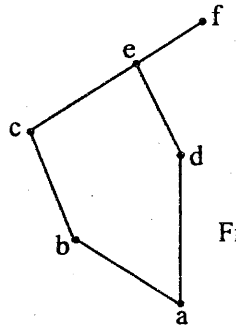


Fig. (i)

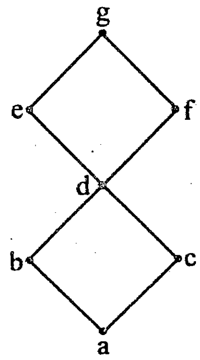
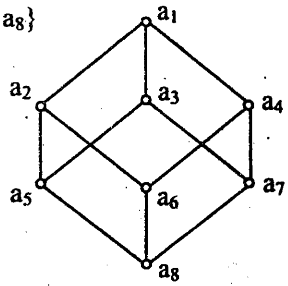


Fig. (ii)

22. Let (L, ≤) be a lattice in which $L = \{a_1, a_2, \ldots\ldots\ldots a_8\}$ diagram of (L, ≤) is given in figure. Let s_1, s_2, and s_3 are subsets of L given by **[D-03]**

$s_1 = \{a_1, a_2, a_4, a_6\}$, $s_2 = \{a_3, a_5, a_7, a_8\}$ $s_3 = \{a_1, a_2, a_4, a_8\}$

Which of the above subsets (s_1 ≤), (s_2 ≤) and (s_3 ≤) are sublattice of L ? Justify your answer.



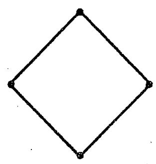
23. Consider the chains of divisors of 4 and 9 i.e. $L_1 = \{1, 2, 4\}$ and $L_2 = \{1, 3, 9\}$ and partial ordering relation of division on L_1 and L_2, draw the lattice $L_1 \times L_2$.

24. Let (A, %) be a partially ordered set. Let % R be a binary relation on A such that for a and b in A, $a \leq_R b$ if and only if $b \leq a$.
   i) Show that $\leq_R$ is a partial ordering relation.
   ii) Show that if (A, ≤) is a lattice, then (A, $\leq_R$) is also a lattice.

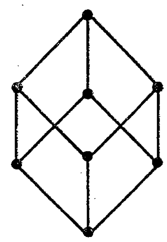
Boolean Algebra

1. Determine whether following Hasse diagrams represents Boolean Algebra.

2. Determine whether the following posets are Boolean algebras. Justify your answers.



3. Determine whether the following posets are Boolean Algebras. Justify your answer.
   (i) A = {1, 2, 3, 6 } with divisibility.
   (ii) D_{20} : divisors of 20 with divisibility.
   (iii) D_{63} is a Boolean Algebra.
4. Determine whether the following Hasse diagrams represents Boolean Algebra. Explain your answer.



5. Show that if n is a positive integer and p^2 divides n, where p is a prime number, then D_n is not a Boolean algebra.
6. Show that : $a \vee (\bar{a} \wedge b) = a \vee b$ $\qquad a \wedge (\bar{a} \vee b) = a \wedge b$
   in a Boolean algebra.

□□□□□□

Graphs :

A graph G consists of parts
(i) A set V = V(G) whose elements are called vertices, points or nodes.
(ii) A collection E = E(G) of unordered pairs of distinct vertices are called edges.
(iii)A function γ that assigns to each edge a subset {v, w} where v & w are vertices (and may be same).
We write G = G (V, E, γ) when we need to name the parts of G.

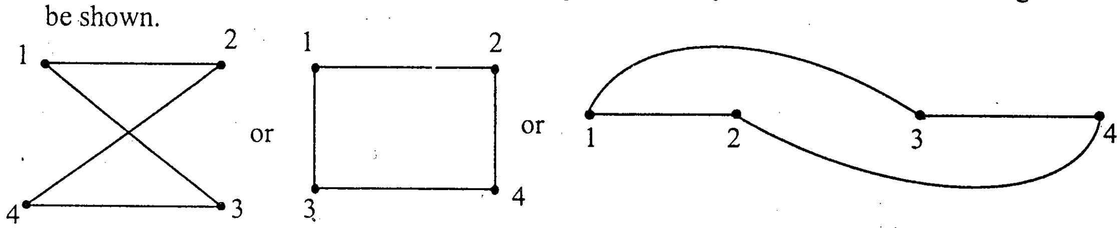
Multigraph :

A multigraph G = G (V, E) also consists of a set V of vertices and set E of edges except that E may contain multiple edges i.e. edges connecting same end points, and E may contain one or more loops as edge whose end points are the same vertex.

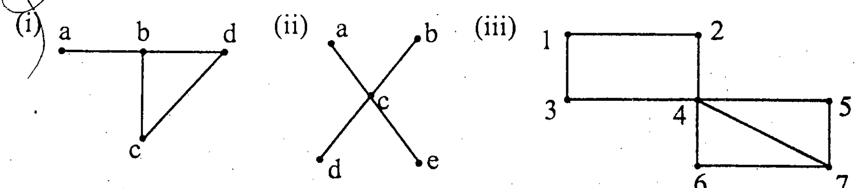
Examples

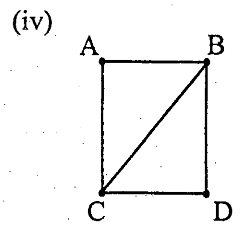
1. Draw a picture of the graph G = G (V, E, γ)
where V = {1, 2, 3, 4}
E = {e_1, e_2, e_3, e_4}
$\gamma(e_1) = \{1, 2\}$, $\gamma(e_2) = \{4, 3\}$, $\gamma(e_3) = \{1, 3\}$, $\gamma(e_4) = \{2, 4\}$.

Soln. :
Graphs are usually represented by pictures using a point for each vertex and line or curve for each edge. We usually omit the names of edges, since they have no intrinsic meaning. It can be shown.



2. Describe formally the following graphs :





Soln. :

(i) Graph G = (V, E, γ)
where V = {a, b, c, d}
E = {e_1, e_2, e_3, e_4}
$\gamma(e_1) = \{a, b\}$, $\gamma(e_2) = \{b, c\}$, $\gamma(e_3) = \{b, d\}$, $\gamma(e_4) = \{c, d\}$.

(ii) Graph G = (V, E, γ)
where V = {a, b, c, d, e}
E = {e_1, e_2, e_3, e_4}
$\gamma(e_1) = (a, c)$, $\gamma(e_2) = (b, c)$, $\gamma(e_3) = \{c, d\}$, $\gamma(e_4) = \{c, e\}$.

(iii)Graph G = (V, E, γ)
where V = (1, 2, 3, 4, 5, 6, 7}.
E = {e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9}.
$\gamma(e_1) = \{1, 2\}$, $\gamma(e_2) = \{1, 3)$, $\gamma(e_3) = \{2, 4\}$.
$\gamma(e_4) = \{3, 4\}$, $\gamma(e_5) = \{4, 5\}$, $\gamma(e_6) = \{4, 6\}$.

$\gamma(e_7) = \{4, 7\}$, $\gamma(e_8) = \{5, 7\}$, $\gamma(e_9) = \{6, 7\}$.

(iv) Graph G = (V, E, γ)

where V = {A, B, C, D}.

E = {e_1, e_2, e_3, e_4, e_5}.

$\gamma(e_1) = \{A, B)$, $\gamma(e_2) = \{A, C)$, $\gamma(e_3) = \{B, C\}$, $\gamma(e_4) = (B, D)$, $\gamma(e_5) = \{C, D\}$.

3. Draw the diagram for each of following graphs.

G = G (V, E)

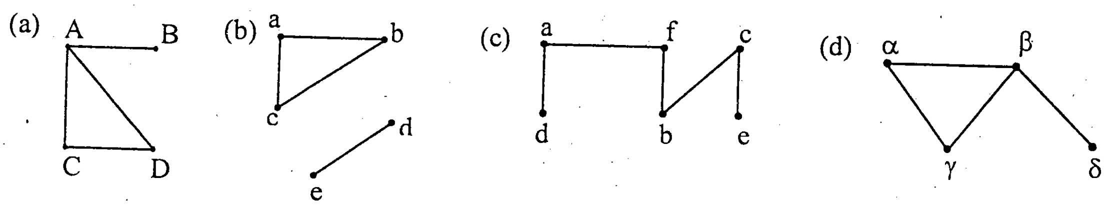
(a) V = {A,B,C,D}
E = [{A,B}, {D,A}, {C,A}, {C, D}]

(b) V = {a, b, c, d, e}
E = [{a, b}, {a, c}, {b, c}, {d, e}]

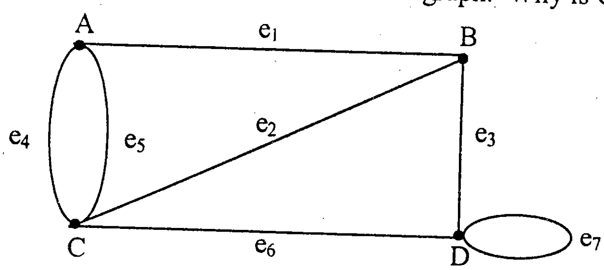
(c) V = {a, b, c, d, e, f}
E = [{a, d}, {a, f}, {b, c}, {b, f}, {c, e}]

(d) V = {α, β, γ, δ }
E = [{α, β}, {α, γ}, {β, γ), {β, δ }]

Soln. :



4. The diagram in the following figure shows a multigraph. Why is G not a graph.



Soln. :

G contains multiple edges, e_4 and e_5, which connect the same two vertices A and C. Also G contains e_7 whose end points are the same vertex D.

$\because$ A graph does not have multiple edges or lops.

$\therefore$ G is a multigraph

5. Draw a diagram of each of the following multigraph G (V, E)

(a) V = {P_1, P_2, P_3, P_4, P_5}
E = [{P_2, P_4}, {P_2, P_3}, {P_3, P_5}, {P_5, P_4}

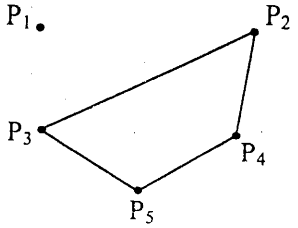
(b) V = {P_1, P_2, P_3, P_4, P_5}
E = [{P_1, P_1}, {P_2, P_3}, {P_2, P_4}, {P_3, P_2}, {P_4, P_2}, {P_4, P_1}, {P_5, P_4}]

(c) V = {P_1, P_2, P_3, P_4, P_5}
E = [{P_1, P_5}, {P_3, P_4}, {P_2, P_3}, {P_2, P_5}, {P_1, P_5}]

(d) V = {P_1, P_2, P_3, P_4, P_5}
E = [{P_2, P_4} {P_2, P_3}, {P_5, P_1}]

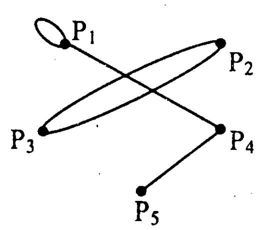
Soln. :

(a)

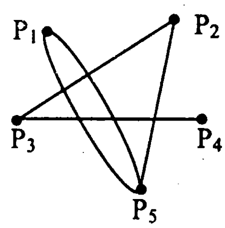


Note : $\because$ there is neither multi edge connecting the two vertices nor a loop connecting same vertex.
$\therefore$ 'G' is a graph, besides being a multigraph.

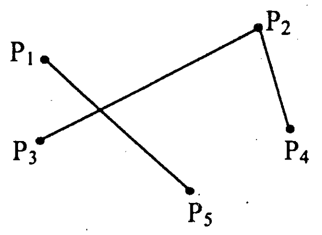
(b)



(c)



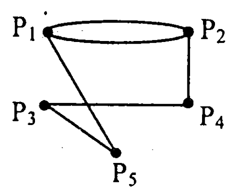
(d)



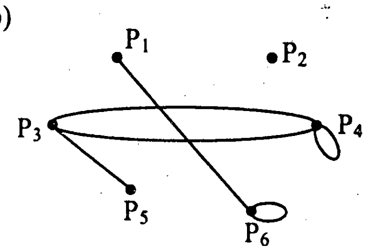
Note : This multigraph is a graph.

6. Describe formally the multigraph shown in the following figures

(a)



(b)

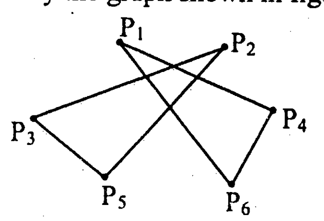


Soln. :

(a) $\because$ There are five vertices
$\therefore V = \{P_1, P_2, P_3, P_4, P_5\}$
$\because$ There are six edges and two of them are multiple edges.
$\therefore E = [\{P_1,P_2\}, \{P_1,P_2\}, \{P_1,P_5\}, \{P_2,P_4\}, \{P_3,P_4\}, \{P_3,P_5\}$

(b) $\because$ There are six vertices
$\therefore V = \{P_1, P_2, P_3, P_4, P_5, P_6\}$
$\because$ There are six edges and two of them are multiple edges having end points P_3 & P_4 and two are loops.
$\therefore E = [\{P_1, P_6\}, \{P_3, P_4\}, \{P_3, P_4\}, \{P_3, P_5\}, \{P_4, P_4\}, \{P_6, P_6\}]$

7. Describe formally the graph shown in figure.



Soln. :

Graph G = (V, E)
$\because$ There are six vertices.
$\therefore V = \{P_1, P_2, P_3, P_4, P_5, P_6\}$
$\because$ There are six edges
$\therefore e = [\{P_1, P_4\}, \{P_1, P_6\}, \{P_2, P_3\}, \{P_2, P_5\}, \{P_3, P_5\}, \{P_4, P_6\}]$

Definition :

Finite Multigraph : A multigraph $G \equiv G(V, E)$ is finite if both V is finite and E is finite.
Note : A graph G with a finite number of vertices V must automatically have a finite number of edges and so must be finite.

Trivial Graph : The trivial graph is a graph with one vertex and no edges.

Empty or Null graph : The empty graph is a graph with no vertex and no edge.

Isolated vertex : A vertex 'V' is isolated if it does not belong to any edge.

Adjacancy and incidence in a graph :

Suppose that e = {u, v} is an edge in G i.e. u and v are end points of e. Then the vertex u is said to be adjacent to the vertex v, and the edge 'e' is said to be incident on u and on v.

Degree of a vertex : The degree of a vertex v in a graph written deg (v), is equal to the number of edges which are incident on v or, in other words, the number of edges which contain v as an end point.

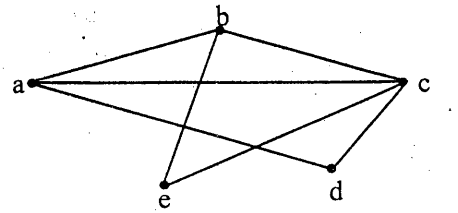
Note : The vertex 'v' is said to be even or odd according as deg (v) is even or odd.

Theorem : The sum of the degrees of the vertices of a graph is equal to twice the number of edges.

Proof : In a graph every edge will be counted twice, since each edge is incident to two vertices. Therefore, sum of degrees of vertices = 2 × No. of edges.

Note : This theorem holds for a multigraph.

8. Consider the graph G = G(V, E) in given figure.
(a) Describe G formally.
(b) Find the degree and parity of each vertex.
(c) Verify that the sum of degrees of the vertices of a graph is equal to twice the number of edges.

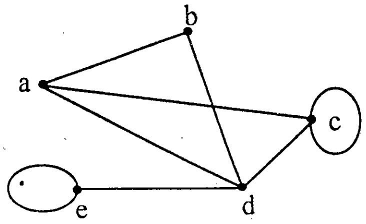


Soln. :
(a) There are five vertices, so V = {a, b, c, d, e}.
There are seven edges.
∴ E = [{a,b}, {a,c}, {a,d}, {b,c},{b,e},{c,d},{c,e}].
(b) The degree of a vertex is equal to number of edges which are incident on it.
i.e. deg (a) = 3 since 'a' belongs to {a, b}, {a, c} and {a, d} or equivalently, there are three edges leaving in the diagram of G. Similarly, deg (b) = 3, deg (c) = 4, deg(d) = 2, deg(e) = 2.
Thus, c, d and e are even vertices and a & b are odd.
(c) The sum of degrees of the vertices = deg (a) + deg (b) + deg (c) + deg (d) + deg (e)
= 3 + 3 + 4 + 2 + 2 =14
Here number of edges = 7.
∴ Sum of degrees of vertices = 2 × number of edges.
Hence the statement.

9. Consider the graph G where
V(G) = {A, B, C, D}
E (G) = [{A, B}, {B, C}, {B, D}, {C, D}]
Find the degree of and parity of each vertex.

Soln. :
To find the degree of vertex, count the number of edges which it belongs.
deg(A) = 1, deg(B) = 3, deg(C) = 2, deg(D) = 2.
Thus, C & D are even and A & B are odd.

10. Find the degree of each vertex in the multigraph.



Soln. :

To find the degree of each vertex, count the number of edges which are incident on it.

i.e. deg (a) = 3, degree (b) = 2, deg(c) = 4

deg (d) = 4, deg(e) = 3.

Thus a & e are odd and b, c & d are even.

11. Find the sum of degrees of the vertices of G where V(G) = {A, B, C, D} and
(a) E(G) = [{A, B}, {A, C}, {B, D}, {C, D}]
(b) E(G) = [{A, B}, {A, C}, {A, D}, {B, A}, {B, B}, {G, B}, {G, D}]

Soln. :

∵ The sum of degrees of vertices = 2(number of edges)

(a) There are four edges.

∴ sum of degrees of vertices = $2 \times 4 = 8$

(b) There are seven edges

∴ sum of degrees of vertices = $2 \times 7 = 14$.

12. Consider a multigraph G where
V(G) = {A, B, C, D}
and E(G) = [{A, C}, {A, D}, {B, B}, {B, C}, {C, A}, {C, B}, {D, B}, {D, D}].
(a) Find degree and parity of each vertex.
(b) Verify that the sum of degrees of vertices of a graph or multigraph is equal to twice the number of edges.

Soln. :

(a) To find the degrees of each vertex, count the number of edges to which it belongs, or equivalently count the number of times each vertex appears in E(G).
i.e. deg (A) = 3 deg (B) = 5, deg (C) = 4, deg (D) = 4.
Thus, A & B are odd and C & D are even.

(b) Sum of degrees of vertices = deg (A) + deg (B) + deg (C) + deg (D)

$= 3 + 5 + 4 + 4$

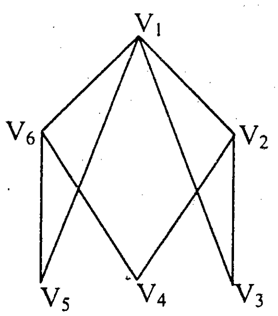
$= 16$

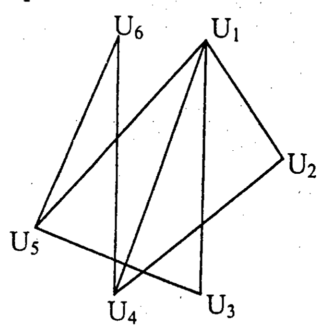
Here number of edges = 8
∴ Sum of degrees of vertices = 2 × No. of edges.
Hence the statement.

Isomorphism between two graphs :

A graph $G_1 = G_1(V_1, E_1)$ is said to be isomorphic to a graph $G_2 = G_2(V_2, E_2)$ if there is a bijective mapping $f : V_1 \to V_2$ such that if $U_1, U_2 \in V_1$, then the number of edges joining U_1, U_2 in G_1 is same as number of edges joining $f(U_1)$ and $f(U_2)$ in G_2. In particular isomorphism of two graphs results adjacency of any two vertices.

13. Show that the following graphs G_1 and G_2 are isomorphic.





Soln. :

A function f : V (G_1) → V (G_2) is defined as

$f(V_1) = U_1$

$f(V_2) = U_4$

$f(V_3) = U_2$

$f(V_4) = U_6$

$f(V_5) = U_3$

$f(V_6) = U_5$

Since i) the function f is a bijection from V (G_1) to V (G_2)

ii) it preserves adjacency of vertices & their adjacency matrices are identical.

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	1	1	0	1	1
V_2	1	0	1	1	0	0
V_3	1	1	0	0	0	0
V_4	0	1	0	0	0	1
V_5	1	0	0	0	0	1
V_6	1	0	0	1	1	0

	U_1	U_4	U_2	U_6	U_3	U_5
U_1	0	1	1	0	1	1
U_4	1	0	1	1	0	0
U_2	1	1	0	0	0	0
U_6	0	1	0	0	0	1
U_3	1	0	0	0	0	1
U_5	1	0	0	1	1	0

Therefore these two graphs are isomorphic.

Necessary conditions for Isomorphism :

If two graphs G_1 and G_2 are isomorphic then

i) The number of vertices in G_1 = number of vertices in G_2

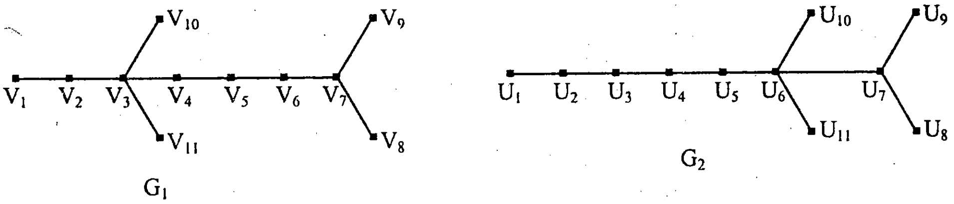
ii) Number of edges in G_1 = number of edges in G_2

iii) Number of vertices of degree k in G_1 = Number of vertices of degree k in G_2.

for k = 0, 1, 2 …

These conditions are only necessary but not sufficient i.e. we may have two graphs satisfying these conditions but still they may not be isomorphic.

14. Show that the following two graphs are not isomorphic even though the necessary conditions are satisfied.



Soln. :

1. No. of vertices in G_1 = 11
   No. of vertices in G_2 = 11
2. No. of edges in G_1 = No. of edges in G_2 = 10
3. No. of vertices of degree 4 in G_1 = No. of vertices of degree 4 in G_2 = 1
   No. of vertices of degree 3 in G_1 = No. of vertices of degree 3 in G_2 = 1
   No. of vertices of degree 2 in G_1 = No. of vertices of degree 2 in G_2 = 4
   No. of vertices of degree 1 in G_1 = No. of vertices of degree 1 in G_2 = 5

The graphs G_1 and G_2 are not isomorphic because under any isomorphism the vertex V_3 of degree 4 must correspond to vertex U_6 of degree 4 and the vertex V_7 of degree 3 must correspond to vertex U_7 of degree 3, but V_3 & V_7 are non–adjacent whereas U_6 & U_7 are adjacent.

∴ Adjacency is not preserved. Hence these are not isomorphic.

Eulerian graph :

A connected graph G is said to be Eulerian if there exists Eulerian circuit in it.

Eulerian path :

A Eulerian path in a graph is a path in which every edge is traveled only once.

Eulerian Circuit :

An Eulerian circuit is an Eulerian path which is closed.

The necessary & sufficient conditions for an Eulerian circuit is that every vertex must have even degree. If there is even one vertex of odd degree, then the circuit cannot be Eulerian. If a graph is connected and contains exactly two vertices of odd degree then there is an Eulerian path in the graph but not a circuit. The path begins with one vertex of odd degree and ends on the other vertex of odd degree.

Hamiltonian graph :

A graph is said to be Hamiltonian if it has a Hamiltonian circuit in it.

Hamiltonian Path :

A Hamilitonian path in a graph is a path in which each vertex is visited only once.

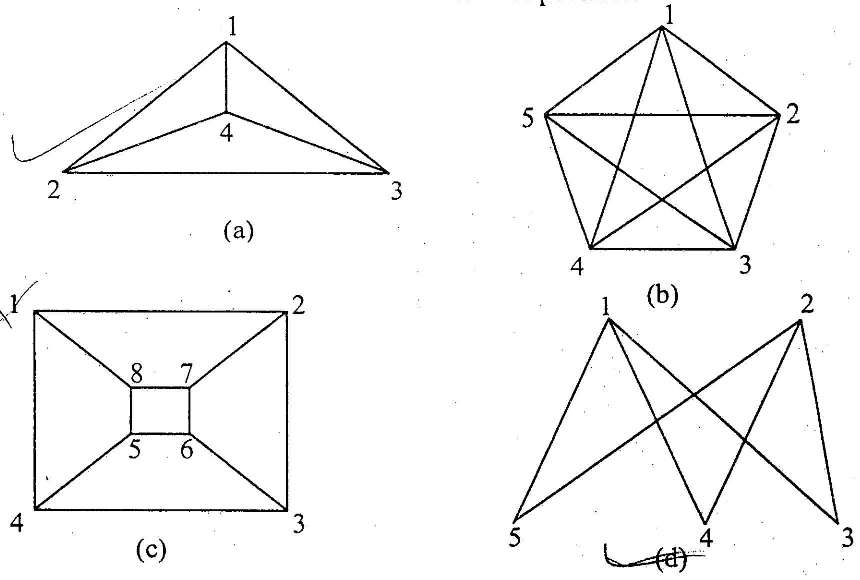
Hamiltonian Circuit :

A Hamiltonian circuit is a Hamiltonian path which is closed (i.e.) Initial vertex is same as final vertex and only this vertex is visited twice.

For a Hamiltonian circuit there are only sufficient conditions but not necessary conditions.
If the sum of the degrees of any two adjacent vertices is greater than or equal to n where n = number of vertices then the graph must be Hamiltonian.
or
If the degree of each vertex is greater than or equal to n/2 where n = number of vertices then the graph must be Hamiltonian.

Note : While searching for Eulerian path, a vertex may be visited more than once. While searching for Hamiltonian path, vertex will be visited once only.

15. Decide which of the following graphs are Eulerian or Hamiltonian or both and write down as Eulerian circuit and Hamiltonian circuit wherever possible.



Soln. :

a) The degree of each vertex is odd.
   $\therefore$ There is no Eulerian circuit and no Eulerian path.
   No. of vertices = 4
   Degree of each vertex = 3
   $\because$ degree of each vertex $> \frac{1}{2} \times$ no. of vertices.
   $\therefore$ There is a Hamiltonian circuit in a graph.
   One Hamiltonian circuit is $1 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 1$
   $\therefore$ Graph is Hamiltonian but not Eulerian.

b) The degree of each vertex is even.

∴ There is an Eulerian circuit in a graph.

One Eulerian circuit is 1→2→3→4→5→3→1→4→2→5→1

∴ The graph is Eulerian.

Degree of each vertex = 4
Number of vertices = 5

∵ Degree of each vertex $> \frac{1}{2} \times$ no. of vertices.

∴ There is an Hamiltonian circuit in a graph.

One Hamiltonian circuit is 1→2→3→4→5→1

Hence the graph is Hamiltonian

Hence the graph is Hamiltonian & Eulerian.

Note : Every complete graph of odd number of vertices greater than or equal to 3 is both Eulerian as well as Hamiltonian.

c) Degree of each vertex is odd.

∴ There is no Eulerian circuit & no Eulerian path. The graph is not Eulerian.

Degree of each vertex = 3

No. of vertices = 8

Degree of each vertex $< \frac{1}{2} \times$ no. of vertices.

∴ Sufficient condition for Hamiltonian graph is not satisfied but still the graph has Hamiltonian circuit. One Hamiltonian circuit is given by 1→2→3→6→7→8→5→4→1

∴ Graph is Hamiltonian but not Eulerian.

d) There are two vertices of odd degree.

∴ There is an Eulerian path in the graph but no Eulerian circuit. Hence the graph is not Eulerian. An Eulerian path may begin with 1 and end on 2 and vice versa. One Eulerian path is 1→5→2→4→1→3→2.

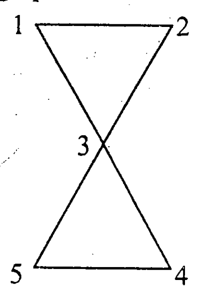
The graph has Hamiltonian path but no Hamiltonian circuit. One Hamiltonian path is 5→1→ 4→ 2→ 3

16. Is every Eulerian graph Hamiltonian.

Soln. :

No, every Eulerian graph is not Hamiltonian.

eg.



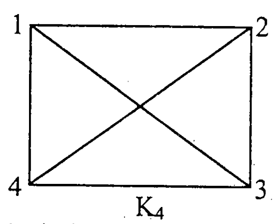
This graph is Eulerian but not Hamiltonian.

17. Is Every Hamiltonian graph Eulerian ?

Soln. :

No, every Hamiltonian graph is not Eulerian.

eg. A complete graph on even number of vertices ≥ 3 is Hamiltonian but not Eulerian.



K_4 is Hamiltonian but not Eulerian.

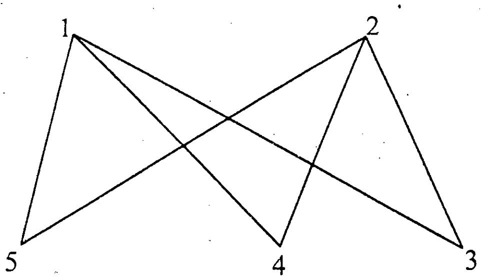
18. Does there exist a graph which is both Eulerian as well as Hamiltonian ?

Soln. :

Yes, every complete graph on odd number of vertices ≥ 3 is both Eulerian as well as Hamiltonian.

19. Give an example of a graph which is neither Eulerian nor Hamiltonian.

Soln. :



This graph is neither Eulerian nor Hamiltonian.

20. A connected planar graph has 9 vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4 and 5. How many edges are there ? **[D-02]**

Soln. :

For any graph G, the sum of degrees of vertices of G is twice of the no. of edges.

$\therefore\ 2+2+2+3+3+3+4+4+5 = 2E$

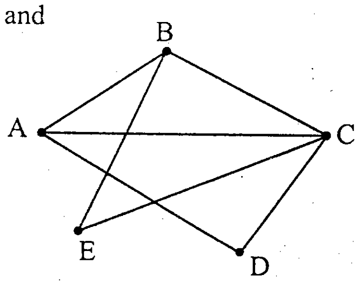
$\therefore\ 2E = 28$

$\therefore\ E = 14$

$\Rightarrow$ No. of edges are 14.

Graded Questions

1. Define planar graphs with the help of an example.
2. Consider the fig. below and

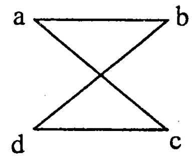


   (a) Describe formally the graph G in the diagram i.e. find the set V(G) of vertices of G and the set E(G) of edges of G.
   (b) Find the degree of each vertex
3. Draw the graph G corresponding to each adjacency matrix

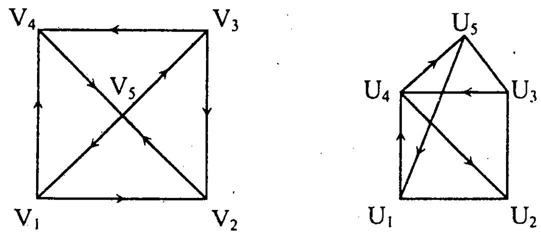
$$(a)\ A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad (b)\ A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

4. A connected planar graph has 9 vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4 and 5. How many edges are there ? **[D-02, M-06]**
5. Define the following terms with one example each : **[M-99]**
   (i) Eulerian circuit (ii) Hamiltonian path (iii) Isomorphic graphs
   (iv) Non–isomorphic graph
6. Prove that a connected graph with n vertices must have at least $n-1$ edges.

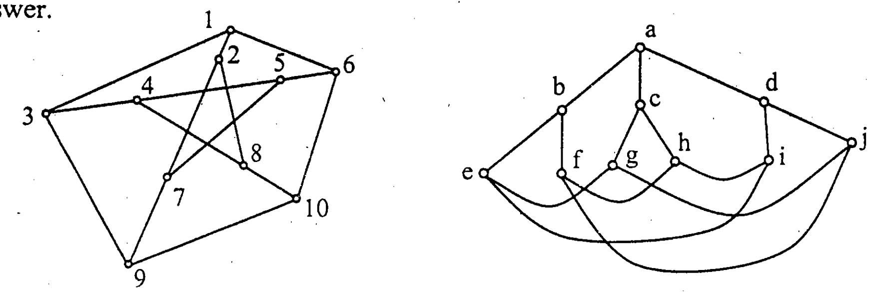
7. How many paths of length 4 are there from a to d in simple graph shown below. **[M-06]**



8. Show that following two graphs are isomorphic **[D-97]**

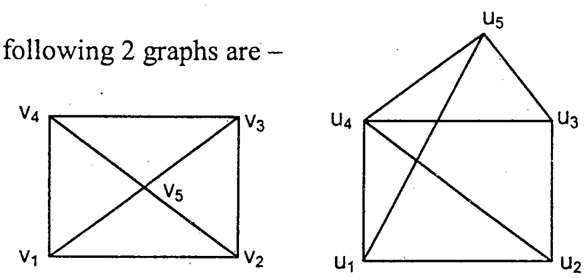


9. Define graph isomorphism. Indicate whether the two graphs below are isomorphic. Justify your answer. **[M-98]**

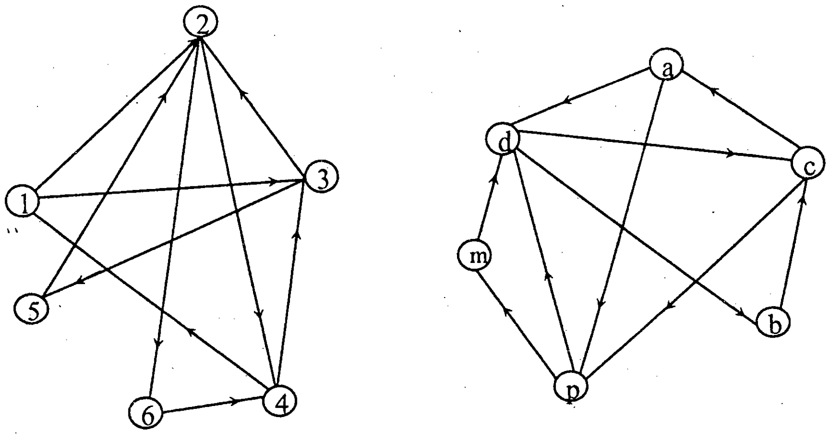


10. Draw all the non–isomorphic directed graphs with three vertices and 4 edges (without loops of length one).

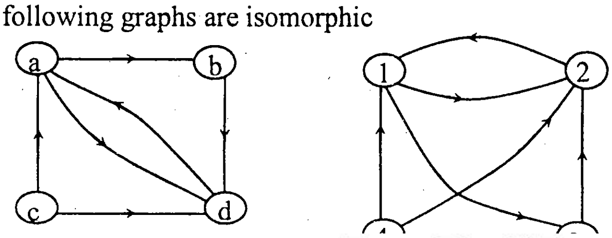
11. Show that the following 2 graphs are – **[M-00]**



12. What are isomorphic graphs ? Show that following 2 graphs are isomorphic. **[D-00, N-04]**



13. Show that the following graphs are isomorphic **[M-01]**



14. Define the

15. What are

16. What are

17. Which of
circuit, o

18. (i) Whic

(ii) Show

ow. [M-06]

[D-97]

morphic. Justify [M-98]

es (without loops

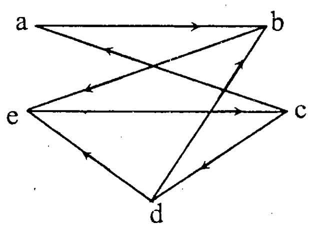
[M-00]

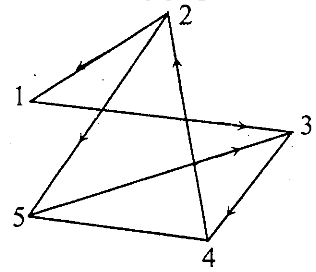
[D-00, N-04]

)

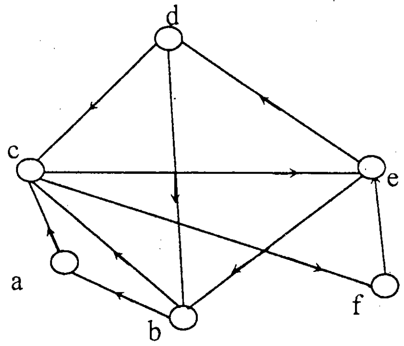
[M-01]

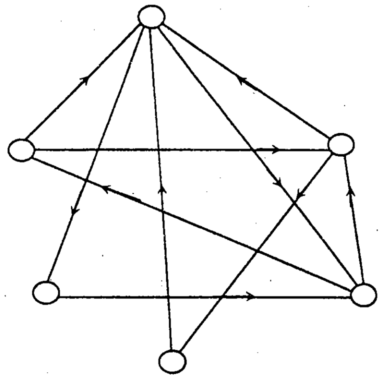
14. Define the isomorphic graphs. Determine whether the following graphs are isomorphic. **[D-01]**



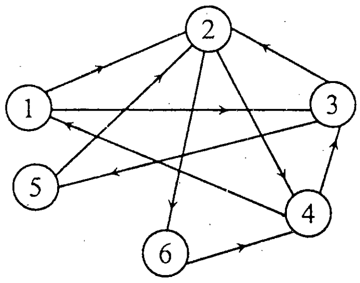


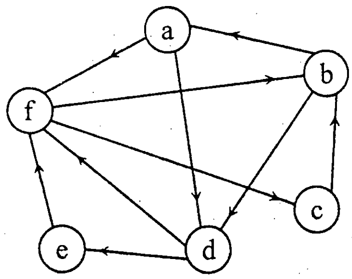
15. What are isomorphics graphs ? Show that the following graphs are isomorphics ? **[D-01]**



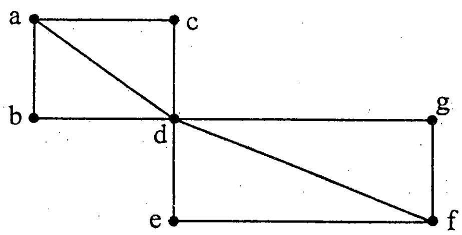


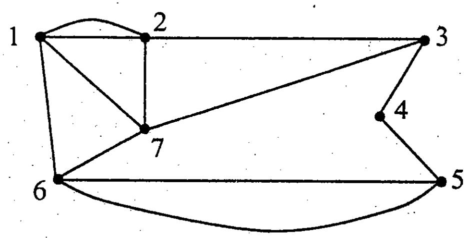
16. What are isomorphic graphs ? Show that following two graphs are isomorphic. **[M-02]**



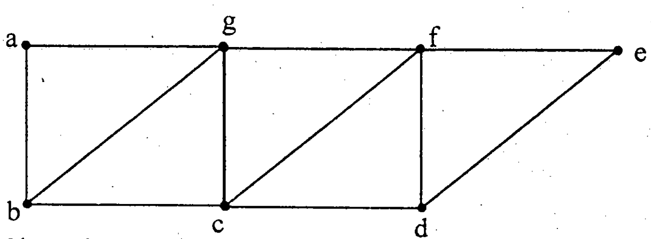


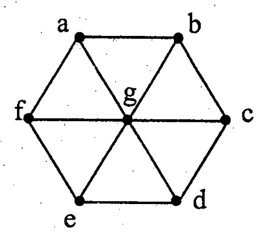
17. Which of the graphs shown in figure below have an Euler circuit, an Euler path but not Euler circuit, or neither ? **[M-04]**



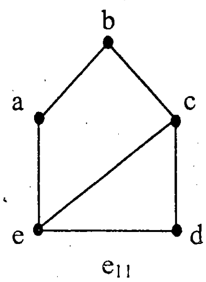


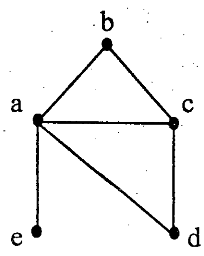
18. (i) Which graph shown in following figure have an Euler path ? Justify your answer. **[M-05]**



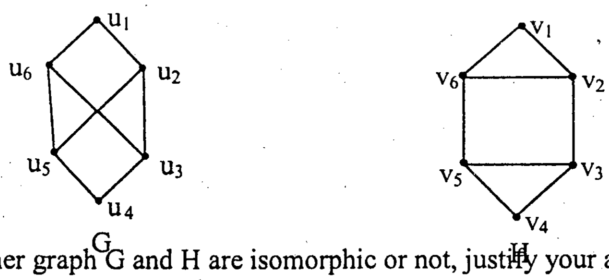


(ii) Show that graphs shown below are not isomorphic. **[M-05]**

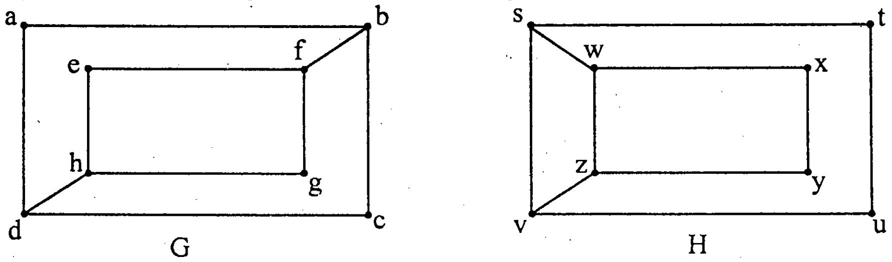




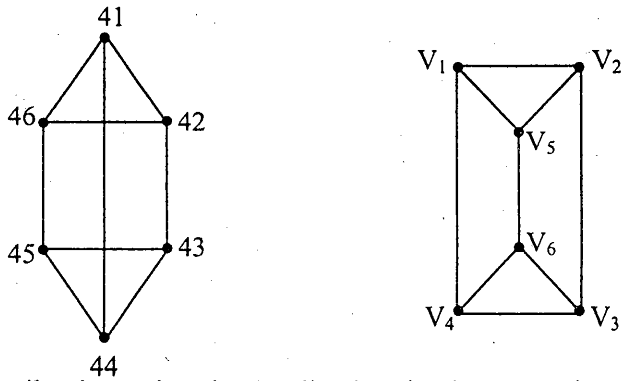
19. Determine graphs G and H shown in figure are isomorphic or not. Justify your answer **[M-03]**



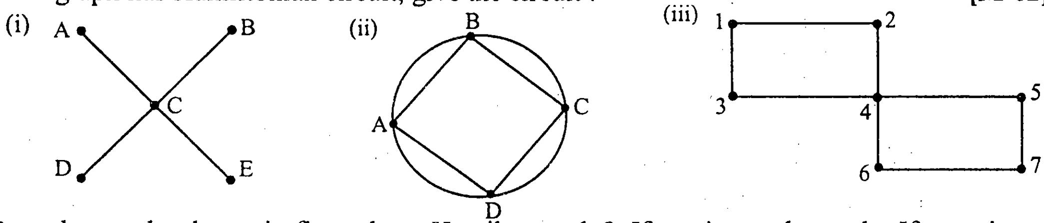
20. Determine whether graph G and H are isomorphic or not, justify your answer. **[D-02]**



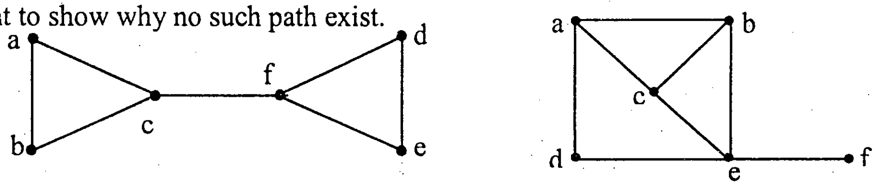
21. Determine graphs shown in **figure** are isomorphic or not justify your answer. **[D-05]**



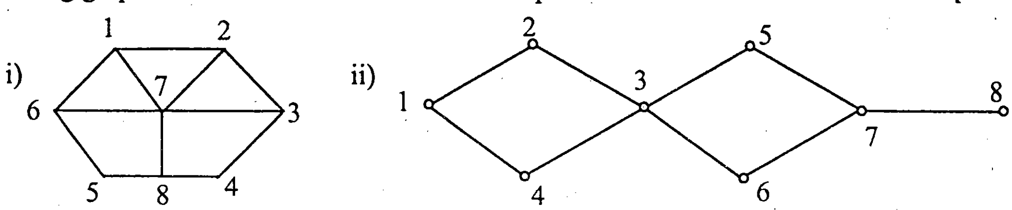
22. Define : A Hamiltonian path and a Hamiltonian circuit. Determine whether the graphs shown below have a Hamiltonian circuit, a Hamiltonian path but no Hamiltonian circuit or neither. If the graph has Hamiltonian circuit, give the circuit : **[M-02]**



23. Does the graphs shown in figure have Hamilton path ? If so give such a path. If not, give an argument to show why no such path exist. **[D-03]**



24. Define Eulerian and Hamiltonian paths. Give one example of each. Also state necessary and sufficient condition for a graph to have Eulerian path. **[D-98, M-04, N-04, M-05]**
25. Explain Eulerian and hamiltonian path with a suitable example. Determine whether following graphs contain hamiltonian or Eulerian path or circuit. **[D-01]**



26. Define Eulerian, Hamilton path and circuit with example. What is the necessary and sufficient condition to exist Euler path, Euler circuit, and Hamilton circuit ? **[D-02, M-06]**
27. Prove that, an undirected graph possesses an Eulerian path if and only if it is connected and has either zero or two vertices of odd degree.

ur answer
[M-03]

r. [D-02]

. [D-05]

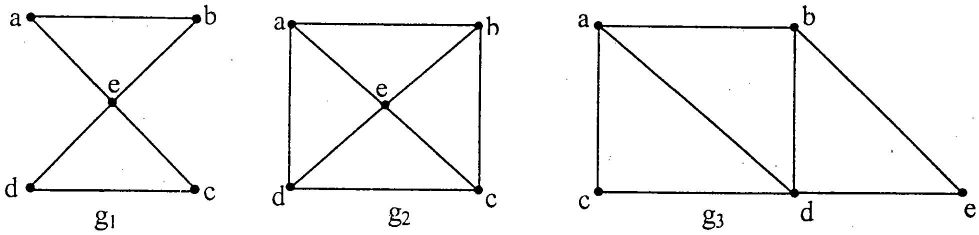
the graphs shown
circuit or neither.
[M-02]

h. If not, give an
[D-03]

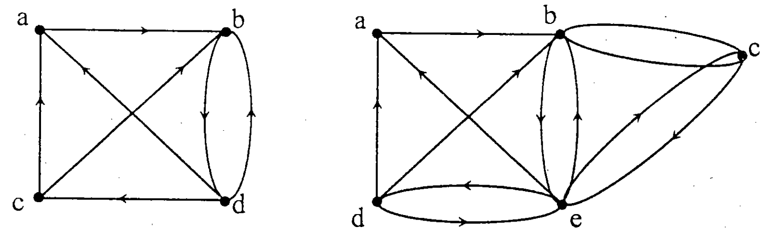
ate necessary and
-04, N-04, M-05]
etermine whether
[D-01]

le necessary and
[D-02, M-06]
is connected and

28. Prove that a finite connected graph G is Eulerian if and only if each vertex has even degree.
29. Which of the undirected graphs in Figure have an Euler circuit ? Of those that do not, which have an Euler path ? Write Euler circuit and path in each of the graph. **[M-03]**



30. Determine whether directed graph shown in **figure** has Euler circuit. Construct Euler circuit if one exists. If no Euler circuit exist, determine whether directed graph has Euler path. Construct Euler path if one exists. **[D-05]**



31. Define tree. Draw all trees with six vertices. **[M-98, D-00]**
32. Define a Forrest. Draw all forests with 6 vertices. **[M-01]**
33. Suppose that a connected planar graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane ? **[D-03]**
34. Define : (a) Tree or tree graph (b) Spanning trees
    (c) Minimum spanning tree (d) Non planar graphs

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Vidyalankar Institute of Technology

Ch.7 : Groups and Applications

Definitions

1. Groupoid :

Let 'G' be a non empty set and '$*$' be a binary operation then the algebraic system $(G, *)$ is called groupoid.

Or

Let 'G' be non empty set, then the system $(G, *)$ is said to be groupoid if it follows closure axiom.

eg. $(N, +)$ $(N, .)$ $(I, +)$
$(I, -)$, $(I, .)$ are the groupoids.

Closure axiom : $\forall$ a, b $\in$ G $\Rightarrow$ a $*$ b $\in$ G

2. Semi Group :

An algebraic system $(G, *)$ is said to be a semi group, if it satisfies
i) Closure axiom
ii) Associativity

Associativity :
$\forall$ a, b, c $\in$ G
$a * (b * c) = (a * b) * c$

eg. $(I, -)$ is a groupoid but $(-)$ is not an associative operator hence it is not a semi group.

3. Monoid :

An algebraic system $(G, *)$ is said to be a monoid if it satisfies
i) closure axiom
ii) associativity
iii) existence of an identity element.

Identity element : $\forall$ a $\in$ G $\exists$ an element 'e' such that $a * e = a = e * a$ then e is called identity element w.r.t. operation '$*$'

eg. $4 \in N$
$4 + e = 4 = e + 4$
$\Rightarrow$ e = 0 (identity element) but 0 is not an element of set of natural number hence $(N, +)$ is not the monoid.
and $(I, +)$ is a monoid
$(I, \cdot)$ is a monoid.

4. Group :

An algebraic system $(G, *)$ is said to be a group if it satisfies
i) closure axiom
ii) associativity
iii) existance of an identity element
iv) existance of inverses.

Inverse element : Let a $\in$ G $\exists$ an element α in G such that $a * \alpha = e = \alpha * a$, where e is an identity element w.r.t. '$*$'. Then α is called inverse of 'a' w.r.t. '$*$'.

eg. $(N, \sqcup)$ is a monoid but not a group.
$(I, +)$ is a group.

5. Abelian Group :

A group $(G, *)$ is said to be an Abelian group or commutative group if it follows commutativity.

Commutativity : $\forall$ a, b $\in$ G $\Rightarrow$ $a * b = b * a$
$(I, +)$, $(R, +)$, $(C, +)$, $(R - \{0\}, \cdot)$ and $(R^+, \cdot)$ are Abelian groups.

6. **Cyclic Group :**

A finite group (G, $*$) is said to be a cyclic group if all the elements of 'G' are generated by a single element of it. This single element is called generator of 'G'.

Questions :

1. Prove that the set $G = \{1, -1, i, -i\}$ is an abelian group w.r.t. multiplication. Is it cyclic ?

Ans. :

STEP 1 :

Composition table

$\bullet$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

STEP 2 :

From composition table we find

i) Closure axiom :

Since all the elements of composition table belongs to G therefore G is closed w.r.t. multiplication.

ii) Associativity :

It is clear that $\forall$ a, b, c $\in$ G

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(Numbers always follows associativity w.r.t. multiplication)

iii) Identity element :

$$\forall a \in G$$

From the table we find $1 \in G$ such that $a \cdot 1 = a = 1 \cdot a$

$\Rightarrow$ 1 is the identity element in G w.r.t. multiplication.

iv) Inverses :

From the table we find all the elements are invertible and their inverses are

$$(1)^{-1} = 1, \quad (-1)^{-1} = -1, \quad (i)^{-1} = -i, \quad (-i)^{-1} = i$$

v) Commutativity :

From table it is clear that

$$\forall (a, b) \in G, \quad a \cdot b = b \cdot a$$

$\therefore$ G is an abelian group w.r.t. multiplication.

Cyclic group : $i^1 = i$ or $(-i)^1 = -i$

$i^2 = -1$ $\quad (-i)^2 = -1$

$i^3 = -i$ $\quad (-i)^3 = i$

$i^4 = 1$ $\quad (-i)^4 = 1$

$\Rightarrow$ 'i' or '$-i$' is a generator of G.

$\Rightarrow$ G is a cyclic group.

2. Prove that a set $G = \{1, \omega, \omega^2\}$ forms an abelian group with respect to multiplication. Is it cyclic ?

Ans. :

Step I :

Preparing composition table

$\bullet$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Step II :

From composition table, we find it follows

i) Closure axiom

ii) Associativity

iii) Identity element exists i.e. $1 \in G$

iv) All the elements are invertible their inverses are

$(1)^{-1} = 1$

$(\omega)^{-1} = \omega^2$

$(\omega^2)^{-1} = \omega$

v) Commutativity :

$\therefore$ (G, $\cdot$) is an Abelian group

Cyclic group :

$\omega^1 = \omega$

$\omega^2 = \omega^2$

$\omega^3 = 1$

$\Rightarrow$ ω is generator of G

$\Rightarrow$ G is a cyclic group

Addition modulo 'm' :

Let 'm' be a positive integer

$\forall$ a, b $\in$ I, $a +_m b = r$

where 'r' is the remainder when (a + b) is divided by 'm'.

eg :

$3 +_4 7 = 2$, $3 +_5 8 = 1$, $3 +_6 3 = 0$,

Multiplication Modulo 'm' :

Let 'm' be a positive integer

$a \times_m b = r$

where 'r' is the remainder when (a × b) is divided by 'm'.

eg. :

$3 \times_3 7 = 0$, $4 \times_3 7 = 3$, $3 \times_4 2 = 2$

Note : Set of integers modulo 'm' is denoted by Z_m and defined as

$Z_m = \{0, 1, 2, \ldots m - 1\}$

Questions :

1. Let G = { 0, 1, 2, 3, 4, 5 } **[M-04, N-04]**
   i) Prepare composition table with respect to '$+_6$'
   ii) Prove that G is an abelian group with respect to '$+_6$'
   iii) Find the inverse of 2, 3 and 5.
   iv) Is it cyclic ?
   v) Find the order of 2, 3 and sub groups generated by these elements.

Soln. :

i) Prepare composition table

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

ii) a) Closure axiom :– $\because$ All the elements of C. T. $\in$ G, $\therefore$ it follows closure axiom.

b) Associativity : – $\forall$ a, b, c $\in$ G

$$a +_6 (b +_6 c) = (a +_6 b) +_6 c \quad \text{(from table)}$$

c) Identity :– From table we find '0' is an identity element of G.

d) Inverses :– All the elements of G are invertible and their inverses are

$(0)^{-1} = 0$ $\quad$ $(3)^{-1} = 3$

$(1)^{-1} = 5$ $\quad$ $(4)^{-1} = 2$

$(2)^{-1} = 4$ $\quad$ $(5)^{-1} = 1$

e) Commutativity : It follows commutative law

$\forall$ a, b, $\in$ G, $\quad a +_6 b = b +_6 a$

$\therefore$ G is an Abelian group with respect to '$+_6$'.

iii) From table $2^{-1} = 4$, $3^{-1} = 3$, $5^{-1} = 1$

iv) $1^1 = 1$

$1^2 = 1 +_6 1 = 2$

$1^3 = 1 +_6 1 +_6 1 = 3$

$1^4 = 1 +_6 1 +_6 1 +_6 1 = 4$

$1^5 = 1 +_6 1 +_6 1 +_6 1 +_6 1 = 5$

$1^6 = 1 +_6 1 +_6 1 +_6 1 +_6 1 +_6 1 = 0$

$\Rightarrow$ 1 is the generator of G

$\Rightarrow$ G is a cyclic group

v) $2^1 = 2$

$2^2 = 2 +_6 2 = 4$

$2^3 = 2 +_6 2 +_6 2 = 0$

$\therefore O(2) = 3$

$3^1 = 3$

$3^2 = 3 +_6 3 = 0$

$\therefore O(3) = 2$

Subgroup generated by 2 is {0, 2, 4}

Subgroup generated by 3 is {0, 3}

2. For $Z_7 - \{0\}$ **[N-04]**

i) Prepare C. T. with respect to '$\times_7$'

ii) Prove that it is an abelian group with respect to '$\times_7$'

iii) Is it cyclic ?

iv) Find the order of 2, & 4 and subgroups generated by these.

Soln. : [M-04, N-04]

i) Composition table :

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

ii) It follows

a) Closure axiom

b) Associativity

c) Identity element exist i.e. 1

d) All the elements of G are invertible inverses are

$(1)^{-1} = 1$ $\quad$ $(4)^{-1} = 2$

$(2)^{-1} = 4$ $\quad$ $(5)^{-1} = 3$

$(3)^{-1} = 5$ $\quad$ $(6)^{-1} = 6$

e) It follows commutativity

$\therefore$ G is an abelian group.

iii) $3^1 = 3$

$3^2 = 3 \times_7 3 = 2$

$3^3 = 3 \times_7 3 \times_7 3 = 6$

$3^4 = 3 \times_7 3 \times_7 3 \times_7 3 = 4$

$3^5 = 3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 = 5$

$3^6 = 3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 = 1$

$\Rightarrow$ 3 is the generator

$\Rightarrow$ G is the cyclic group

iv) $2^1 = 2$

$2^2 = 2 \times_7 2 = 4$

$2^3 = 2 \times_7 2 \times_7 2 = 1$

$\therefore O(2) = 3$ $\quad\therefore$ Subgroup generated by 2 is $H_1 = \{1, 2, 4\}$

$4^1 = 4$

$4^2 = 4 \times_7 3 = 2$

$4^3 = 4 \times_7 4 \times_7 4 = 1$

$O(4) = 3$ $\quad\therefore$ Subgroup generated by 4 is $H_2 = \{1, 2, 4\}$

3. Let Z_4 i.e. $G = \{0, 1, 2, 3\}$
   i) Prepare its composition table with respect to '$\times_4$'
   ii) Is it a group

Soln. :

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\Rightarrow$ It is a monoid but not a group ($\because$ Inverse of 0 does not exist).

4. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$
   i.e. $Z_8 - \{0\}$
   i) Prepare composition table with respect to '$\times_8$'
   ii) Is it a group.

Soln. :

$\times_8$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	0	2	4	6
3	3	6	1	4	7	2	5
4	4	0	4	0	4	0	4
5	5	2	7	4	1	6	3
6	6	4	2	0	6	4	2
7	7	6	5	4	3	2	1

$\because 0 \notin G$ but it is present in composition table.

$\therefore$ It is not closed with respect to '$\times_8$'

$\Rightarrow$ G is not a group.

NOTE :

1. Z_m (set of integers modulo m) always forms an abelian group with respect to addition modulo m.
2. Z_m is not a group with respect to multiplication modulo m since inverse of zero does not exist.
3. $Z_m - \{0\}$ always be an abelian group with respect to multiplication modulo m iff m is a prime integer.

Questions :

1. Prove that set of real numbers is an abelian group with respect to '$*$' where '$*$' is defined as

$$a * b = a + b + 2 \qquad \forall\, a, b \in R$$

Soln. :

i) Closure axiom : $\forall\, a, b \in R$

$\Rightarrow a + b \in R$

$\Rightarrow a + b + 2 \in R$

$\Rightarrow a * b \in R$

$\Rightarrow$ 'R' is closed with respect to $*$

ii) Associativity :–

$\forall\, a, b, c, \in R$

$$\begin{aligned} a * (b * c) &= a * (b + c + 2) && \text{....(by definition)} \\ &= a + (b + c + 2) + 2 \\ &= (a + b + 2) + c + 2 && \because \text{ '+' is associative in 'R'} \\ &= (a + b + 2) * c && \text{....(by definition)} \\ &= (a * b) * c \end{aligned}$$

$\Rightarrow$ '$*$' is an associative operator.

iii) Identity element :–

$\forall\, a \in R\ \exists$ an element $e \in R$ such that

$a * e = a$

$\Rightarrow a + e + 2 = a$

$e + 2 = 0$

$e = -2 \in R$

$\Rightarrow -2$ is an identity element in 'R' with respect to $*$.

iv) Inverses :–

Let $a \in R\ \exists\ \alpha \in R$

such that

$$\begin{aligned} a * \alpha &= -2 \\ a + \alpha + 2 &= -2 \\ \alpha &= -a - 4 \in R \end{aligned}$$

v) Commutativity :–

$\forall\, a, b \in R$

$$\begin{aligned} a * b &= a + b + 2 && \text{by definition} \\ &= b + a + 2 && \text{'+' is commutative in R} \\ \therefore\ a * b &= b * a && \text{by definition} \end{aligned}$$

$\Rightarrow$ '$*$' is commutative

$\Rightarrow (R, *)$ is an abelian group

2. Prove that a set of non–zero real numbers forms an abelian group with respect to binary operation '$*$' where '$*$' is defined as

$\forall\, a, b \in R$

$$a * b = \frac{a.b}{2}$$

Soln. :

i) Closure axiom :–

Let $a, b \in R$

$\Rightarrow a.\, b \in R$, $\frac{1}{2} \in R$

$\Rightarrow \dfrac{a.b}{2} \in R$

$\Rightarrow a * b \in R$

$\Rightarrow$ 'R' is closed with respect to '$*$' operation

ii) Associativity : –

$\forall\, a, b, c \in R$

$a * (b * c) = a * \dfrac{bc}{2}$ (by definition)

:t to addition

zero does not

m iff m is a

$$= \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{\left(\frac{ab}{2}\right)c}{2} = \left(\frac{ab}{2}\right) * c \quad \text{(by definition)}$$

$$= (a * b) * c$$

$\Rightarrow$ '$*$' is an associative operator

iii) Identity element :–

$\forall \; a \in R \; \exists$ an element $e \in R$

such that

$$a * e = a$$

$$\frac{ae}{2} = a$$

$$e = 2 \in R$$

$\Rightarrow$ 2 is an identity element in 'R' with respect to '$*$'.

iv) Inverses :–

Let $a \in R \;\exists\; \alpha \in R$

such that

$$a * \alpha = 2$$

$$\frac{a\alpha}{2} = 2$$

$$a\,\alpha = 4$$

$$\alpha = \frac{4}{a} \in R$$

$\Rightarrow$ All the elements are invertible $\quad \because\; 0 \notin R$

v) Commutativity :–

$\forall$ a, b, $\in$ R

$$a * b = \frac{ab}{2} \quad \text{(by definition)}$$

$$= \frac{ba}{2}$$

$$= b * a \quad \text{(by definition)}$$

$$\therefore\; a * b = b * a$$

$\because$ '$*$' is commutative with respect to R

$\Rightarrow$ '$*$' is commutative

$\Rightarrow$ (R, $*$) forms an abelian group.

3. Prove that G = $\{ x = a + b\sqrt{2}\,;\; a, b \in Q \text{ and } x \neq 0 \}$ forms an abelian group with respect to multiplication.

Soln. :

i) Closure axiom :–

Let $a_1, b_1, a_2, b_2 \in Q$

and $x = a_1 + b_1\sqrt{2}$,

$$y = a_2 + b_2\sqrt{2}$$

$$x \,.\, y = (a_1 + b_1\sqrt{2}) \,.\, (a_2 + b_2\sqrt{2})$$

$$= a_1a_2 + a_1b_2\sqrt{2} + a_2b_1\sqrt{2} + 2\,b_1b_2$$

$$= (a_1a_2 + 2\,b_1b_2) + \sqrt{2}\,(a_1b_2 + a_2b_1)$$

i.e. in the form of $a + \sqrt{2}\,b$ and $(a_1a_2 + 2\,b_1b_2)$ and $(a_1b_2 + b_1a_2) \in Q$.

$\Rightarrow$ 'G' is closed with respect to multiplication.

ii) Associativity : –

$\forall$ x, y, z $\in$ G

($\because\; x = a + b\sqrt{2}$ such that $a, b \in Q \quad \Rightarrow a + b\sqrt{2} \in R$)

$\forall$ x, y, z $\in$ G

nition)

$\Rightarrow$ $(x.y).z = x.(y.z)$
($\because$ product of three real numbers can be taken in any order)
$\Rightarrow$ G holds associativity.

iii) Identity Element :–
$\forall\ x \in G\ \exists$ an element e such that $x.\ e = x$
$\Rightarrow e = 1$
$e = 1 + 0\sqrt{2} \in G$
$\Rightarrow$ Identity exists

iv) Inverses : –
$\forall x \in G$
$x = a + b\sqrt{2} \quad \exists\ \alpha \in G$
$x.\alpha = 1$
$(a + b\sqrt{2})\alpha = 1$

$$\alpha = \frac{1}{a+b\sqrt{2}} \times \frac{(a-b\sqrt{2})}{(a-b\sqrt{2})}$$
$$= \frac{a-b\sqrt{2}}{a^2-2b^2}$$
$$= \frac{a}{a^2-2b^2} + \sqrt{2}\left(\frac{-b}{a^2-2b^2}\right) \in G \qquad \because \frac{a}{a^2-2b^2}, \frac{b}{a^2-2b^2} \in Q$$

$\Rightarrow$ All the elements of G are invertible

v) Commutativity :-
$\forall\ x, y \in G$
$\Rightarrow$ $x, y \in R$ $\quad(\forall\ a, b \in R, a + b\sqrt{2} \in R)$
$\Rightarrow$ $x.\ y = y.\ x$
$\Rightarrow$ G follows commutativity.

OR $x.\ y = (a_1 + b_1\sqrt{2}).(a_2 + b_2\sqrt{2})$
$(a_1, b_1, a_2, b_2 \in Q)$
$= (a_1 a_2 + 2 b_1b_2) + (a_1 b_2 + b_1a_2)\sqrt{2}$
$= (a_2a_1 + 2b_2b_1) + (a_2b_1 + b_2a_1)\sqrt{2}$
$= (a_2 + b_2\sqrt{2}).(a_1 + b_1\sqrt{2}) = y.x$

$\Rightarrow$ it follows commutativity.

$\Rightarrow$ (G, .) is an abelian group.

oup with respect

IMPORTANT THEOREMS

I. State and prove
i) Left cancellation law
ii) Right cancellation law

i) Left cancellation law :–
Statement : In a group $(G, *)\ \forall\ a, b, c \in G$
If $a * b = a * c \Rightarrow b = c$
Proof :
$a \in G \Rightarrow a^{-1} \in G$ $\quad$ ($\because$ inverses exist in a group)
Here $a * b = a * c$
Pre operating a^{-1} on both sides
$a^{-1} * (a * b) = a^{-1} * (a * c)$
$(a^{-1} * a) * b = (a^{-1} * a) * c$ $\quad$ (by associativity)
$e * b = e * c$ ....$(a * a^{-1} = e)$
$\Rightarrow b = c$ $\quad$ ($\because$ e is identity in G)
Hence the statement.

ii) Right cancellation law :–

Statement : In a group $(G, *)$ $\forall$ a, b, c $\in$ G
$b * a = c * a \Rightarrow b = c$

Proof :

$a \in G \Rightarrow a^{-1} \in G$ ($\because$ inverse exists in a group)

Here $b * a = c * a$

Post operating a^{-1} on both sides

$$(b * a) * a^{-1} = (c * a) * a^{-1}$$
$$b * (a * a^{1}) = c * (a * a^{-1}) \quad \text{(by associativity)}$$
$$b * e = c * e \quad ....(a * a^{-1} = e)$$
$$\Rightarrow b = c \quad (\because \text{ e is identity in G})$$

Hence the statement.

II. In a group (G, *), identity element is unique.

Proof :

Let e_1 & e_2 be two identity elements in G

By definition

$$a * e_1 = a = e_1 * a \quad ....(i) \quad \forall a \in G$$
$$a * e_2 = a = e_2 * a \quad ......(ii)$$

From (i) and (ii) we get

$$a * e_1 = a * e_2$$

$\Rightarrow$ $e_1 = e_2$ (By left cancellation law)

$\Rightarrow$ identity is unique

III. Inverses of each element is unique in a group (G, *)

Proof :

Let a $\in$G and its inverse is not unique.

Suppose that $\alpha_1, \alpha_2 \in G$ are two inverses of a.

By definition $a * \alpha_1 = e = \alpha_1 * a$ ....(i)

$a * \alpha_2 = e = \alpha_2 * a$ ....(ii)

From (i) and (ii)

$$a * \alpha_1 = a * \alpha_2$$

$\Rightarrow$ $\alpha_1 = \alpha_2$ (By left cancellation law)

$\Rightarrow$ Inverse of each element in G is unique.

IV. In a group (G, *) prove that $(a^{-1})^{-1} = a \; \forall a \in G$

Proof :

$\because$ G is a group

$\therefore \forall a \in G \Rightarrow a^{-1} \in G \Rightarrow (a^{-1})^{-1} \in G$

By definition

$$a * a^{-1} = e = a^{-1} * a \quad ....(i)$$
$$a^{-1} * (a^{-1})^{-1} = e = (a^{-1})^{-1} * a^{-1} \quad ....(ii)$$

where e is identity element in G.

Consider, $a * a^{-1} = e$

Post operating $(a^{-1})^{-1}$ on both sides we get,

$$a * a^{-1} * (a^{-1})^{-1} = e * (a^{-1})^{-1} = (a^{-1})^{-1}$$
$$a * (a^{-1} * (a^{-1})^{-1}) = (a^{-1})^{-1}$$

($\because$ '*' is associative and 'e' is an identity element)

$$a * e = (a^{-1})^{-1}$$
$$a = (a^{-1})^{-1}$$

Hence the result.

V. In a group (G, *)

$(a * b)^{-1} = b^{-1} * a^{-1}$ $\forall$ a, b, $\in$ G (Reversal law of inverses)

Proof :

$\forall a, b \in G \Rightarrow a * b \in G$ (by closure axiom)

$\forall a, b \in G \Rightarrow a^{-1}, b^{-1} \in G$

$\Rightarrow b^{-1}, a^{-1} \in G$

$\Rightarrow b^{-1} * a^{-1} \in G$ (by closure axiom)

Consider $(a * b) * (b^{-1} * a^{-1})$

$= a * (b * b^{-1}) * a^{-1}$ (by associativity)

$= a * e * a^{-1}$ ($\because b * b^{-1} = e$)

$= a * a^{-1}$ ($\because$ e is identity)

$= e$ ($\because a * a^{-1} = e$)

$\Rightarrow (a * b) * (b^{-1} * a^{-1}) = e$

$\Rightarrow (a * b)^{-1} = b^{-1} * a^{-1}$

Hence the result.

VI. Every cyclic group is an abelian group

Proof :

Let $(G, *)$ be a cyclic group. and 'a' is its generator

Let $b, c \in G$, $b = a^m$, $c = a^n$ for some $m, n \in N$

$b * c = a^m * a^n = a^{(m+n)} = a^{(n+m)}$

$= a^n * a^m = c * b$

$\Rightarrow b * c = c * b$

$\Rightarrow$ G is an abelian group

VII. $\forall a \in G$ prove that $a^{|G|} = e$

or $a^{0(G)} = e$

where e is an identity element of a finite group G.

NOTE :

Lagrange's theorem :–

Let G be a group of order n, (i.e. $O(G) = |G| = n$) and $a \in G$ of order m (i.e. $O(a) = m \Rightarrow a^m = e$)

then $m \mid n$ (m divides n).

Proof :

Let $|G| = O(G) = n$

Let $O(a) = m$ i.e. $a^m = e$

By Lagrange's theorem $m \mid n \Rightarrow n = mk$ for some $k \in N$

Now $a^{|G|} = a^n = a^{km}$

$= (a^m)^k$

$= e^k$

$= e$

$\therefore a^{|G|} = e$

Important Problems :

1. Let (A, *) be an algebraic system, where * is a binary operation such that $\forall a, b \in A$, $a * b = a$.
   (i) Show that * is associative.
   (ii) Is * commutative ?

Soln. :

(i) $\forall a, b, c \in A$

By definition,

$a * (b * c) = a * b = a.$ ... (1)

$(a * b) * c = a * c = a$ ... (2)

From (1) and (2),

$$a * (b * c) = (a * b) * c$$

$\therefore$ * is associative.

(ii) If A has more than one element then

$a * b = a$ & $b * a = b$ (by definition)

$\therefore a * b \neq b * a$

$\therefore$ * is not commutative.

2. Let (A, $*$) be semi group. Let $a \in A$. Consider a binary operation + on A such that, $\forall x, y \in A$. $x + y = x * a * y$. Show that + is an associative operation.

Soln. :

$\forall$ x, y, $\in$ A.

$$\begin{aligned}
\text{Consider } x + (y + z) &= x + (y + z) = x + (y * a * z) && \text{by definition} \\
&= x * a * (y * a * z) && \text{by definition} \\
&= (x * a * y) * a * z && \because * \text{ is associative in A.} \\
&= (x + y) * a * z \\
&= (x + y) + z.
\end{aligned}$$

$\Rightarrow$ + is an associative operation.

3. Show that every element in a group is its own inverse, then the group must be abelian.

Soln. :

Let $x, y \in G$,

$$\begin{aligned}
& x * y \in G && \text{by closure property. } \because \text{ G is a group.} \\
& (x * y)^{-1} = x * y && (\because x^{-1} = x) \\
\Rightarrow\ & y^{-1} * x^{-1} = x * y && \because (a * b)^{-1} = b^{-1} * a^{-1} \\
\Rightarrow\ & y * x = x * y && (x, y \in G, \therefore y^{-1} = y,\ x^{-1} = x)
\end{aligned}$$

$\Rightarrow$ G holds commutativity.

$\Rightarrow$ G is an abelian group.

4. Show that in a group, $\forall a, b \in G$, $(a * b)^2 = a^2 * b^2$, iff (G, *) must be abelian.

Soln. :

If part : $\forall a, b \in G$, $(a * b)^2 = a^2 * b^2$ (given)

$$\begin{aligned}
\Rightarrow\ (a * b) * (a * b) &= (a * a) * (b * b) \\
\Rightarrow\ a * (b * a) * b &= a * (a * b) * b && \text{(by associativity)} \\
\Rightarrow\ b * a &= a * b && \text{by left \& right cancellation law.}
\end{aligned}$$

$\Rightarrow$ * is commutative.

$\Rightarrow$ (G, *) is abelian group.

Only if part :

Let (G, *) be an abelian group.

$\therefore\ a * b = b * a$

We have to prove that

$$\begin{aligned}
(a * b)^2 &= a^2 * b^2 \\
\text{LHS} &= (a * b)^2 \\
&= (a * b) * (a * b) \\
&= a * (b * a) * b && \ldots \text{ by associativity} \\
&= a * (a * b) * b && \ldots \text{ by commutativity} \\
&= (a * a) * (b * b) && \ldots \text{ by associativity} \\
&= a^2 * b^2.
\end{aligned}$$

5. If (A, *) be an abelian group then for all $a, b \in G$, show that $(a * b)^n = a^n * b^n$.

Soln. :

Since (A, *) is an abelian.

$\therefore\ a * b = b * a \qquad \forall\ a, b \in G$

We prove this result by induction method.

$(a * b)^1 = a * b = a^1 * b^1$

Hence, result is true for n = 1.

Let, $(a * b)^k = a^k * b^k$ (inductive hypothesis)

We have to prove that the result is true for n = k + 1

i.e. $(a * b)^{k+1} = a^{k+1} * b^{k+1}$

ɔn A such that,

A.

abelian.

Now L.H.S. = $(a * b)^{k+1}$

$= (a * b)^k * (a * b)^1$

$= (a^k * b^k) * (a * b)$ (by inductive hypothesis)

$= a^k * (b^k * a) * b$ ($\because$ * is associative)

$= a^k * (a * b^k) * b$ ($\because$ * is commutative)

$= (a^k * a) * (b^k * b)$ ($\because$ * is associative)

$= a^{k+1} * b^{k+1}$

= R.H.S.

$\Rightarrow$ the result is true for $n = k + 1$.

Hence the result is true for all n.

6. Let Z_n denote the set of integer $\{0, 1, 2, \ldots. (n-1\}$.
Let * be a binary operation on Z_n such that a * b = the remainder when a b is divided by n.
(i) Construct the table for the operation * for n = 4.
(ii) Show that $(Z_n, *)$ is a semigroup for any n.

Soln. :

Here, $Z_n = \{0, 1, 2, \ldots (n-1)\}$

(i) For n = n, $Z_4 = \{0, 1, 2, 3\}$.

*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

(ii) (a) $\forall\ \alpha, \beta, \in Z_n$, $r \in Z_n$,
where, r = remainder of a . b divided by n.
$\therefore$ Z_n is closed.

(b) Z_n hold associativity as
$a * (b * c) = (a * b) * c \quad \forall\ a, b, c \in Z_n$
$\therefore$ $(Z_n, *)$ is a semigroup.

7. (a) Write all the permutations of the elements of the set {1, 2, 3}
(b) Show that this set of the permutations of the elements of the set {1, 2, 3} forms a group under the composition of permutation.

Soln. :

(a) There are $3P_3 = 3! = 6$ permutation of the elements of the set {1, 2, 3}. These permutations are as follows :

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

The composition of permutation is defined as

$$f_2 \circ f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = f_3$$

$$f_1 \circ f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = f_4$$

$\therefore$ $f_1 \circ f_2 \neq f_2 \circ f_1$

$\Rightarrow$ Composition if permutation is not commutative

(b) Let $S = \{e, f_1, f_2, f_3, f_4, f_5\}$

Composition table :

o	e	f_1	f_2	f_3	f_4	f_5
e	e	f_1	f_2	f_3	f_4	f_5
f_1	f_1	e	f_4	f_5	f_2	f_4
f_2	f_2	f_3	e	f_1	f_5	f_4
f_3	f_3	f_2	f_5	f_4	e	f_1
f_4	f_4	f_5	f_2	e	f_2	f_1
f_5	f_5	f_4	f_3	f_2	f_1	e

From the composition table, we find,

(i) Closure property is satisfied.

(ii) Associativity holds.

i.e., $f \circ (g \circ h) = (f \circ g) \circ h \quad \forall\ f, g, h \in S$

(iii) Identity element $e \in S$ exists.

(iv) All the elements are invertible as

$f_1^{-1} = f_1,\ f_2^{-1} = f_2,\ f_3^{-1} = f_4,\ f_4^{-1} = f_3\quad f_5^{-1} = f_5.$

Hence, (S, o) is a group.

8. Show that the set of matrices $A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a group under matrix multiplication.

Soln. :

Let M = {A cos | θ is real}

(i) Closure property –

Let $A(\theta_1),\ A(\theta_2) \in M$

$$\Rightarrow A(\theta_1) . A(\theta_2) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$= A(\theta_1 + \theta_2) \in M$$

Hence, closure property is satisfied.

(ii) Associativity :

Let $A(\theta_1), A(\theta_2), A(\theta_3) \in M$

$A(\theta_1) . \{A(\theta_2) . A(\theta_3)\}$

$= A(\theta_1) . A(\theta_2 + \theta_3)$ (By closure axiom)

$= A\{\theta_1 + (\theta_2 + \theta_3)\}$ "

$= A\{(\theta_1 + \theta_2) + \theta_3\}$ "

$= A(\theta_1 + \theta_2)\, A(\theta_3)$ "

$= \{A(\theta_1)\, A(\theta_2)\}\, A(\theta_3)$ "

$\Rightarrow$ Associativity hold in M.

(iii) Let $A(0) = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in M$ is an identity element in M.

(iv) $|A(\theta)| = \cos^2\theta + \sin^2\theta$

$= 1$ for any θ.

$\Rightarrow$ Inverse of A(θ) exists.

$\therefore$ (M, .) is a group.

9. Let G is a set containing all matrices of 2×2.

$$G = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad a, b, c, d \in R \ \ \& \ \ |A| \neq 0 \right\}$$

P.T. G is a group under matrix multiplication.

Soln. :

(i) let $A \in G, B \in G \Rightarrow A . B \in G$

$\because$ $|A| \neq 0$, $|B| \neq 0$

$\therefore$ $|A| . |B| \neq 0$

$\therefore$ $|A . B| \neq 0$

$\therefore$ Closure prop. is satisfied under matrix multiplication.

(ii) Matrix multiplication always be associative.

(iii) Consider

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{matrix} 0, 1 \in R \\ |I| = 1 \neq 0 \end{matrix}$$

$\because$ $A . I = A = I . A$

$\therefore$ $I \in G$ is an identity element of G.

(iv) Let $A \in G$

$|A| \neq 0$ (given)

$\Rightarrow$ A^{-1} exists

$\Rightarrow$ $A^{-1} \in G$

$\Rightarrow$ The matrices of G are invertible.

$\therefore$ G is a group under matrix multiplication.

Group Homomorphism :

Consider two non-empty set G and G′ where G and G′ are groups w.r.t. the binary operation O and * respectively, then a function.

$\therefore$ $f : (G, O) \rightarrow (G', *)$ is defined as $f(a \, O \, b) \rightarrow f(a) * f(b)$ $\forall$ a, b $\in$ G. is called group homomorphism.

Group Isomorphism :

Let $f : (G, O) \rightarrow (G', *)$ where (G, O) & (G′, *) be two groups then the function f is said to be group isomorphism if

i) f is homomorphism

i.e. $f(a \, O \, b) = f(a) * f(b) \quad \forall$ a, b, $\in$ G

ii) f is injective (one to one) and

iii) f is surjective (onto).

Group Automorphism :

Let G be a group then a function $f : G \rightarrow G$ is said to be Group Automorphism if it is an Isomorphism.

1. Let the group G be the set of real number w.r.t. t and group G′ be the set of positive real number w.r.t .) Let $f : G \rightarrow G'$ by defined by $f(x) = e^x$. Show that f is an isomorphism.

or

Show that the group G = (R, +) and is isomorphic to G′ = (R^+, .), where R^+ is the set of positive real number.

Soln. :

$G = (R, +)$, $\quad G' = (R^+, .)$

Let $f : G \rightarrow G'$ is defined as $f(x) = e^x$.

i) $f(a + b) = e^{a+b} = e^a . e^b. = f(a) . f(b)$.

$\therefore$ f is group homomorphism.

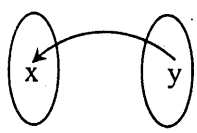
ii) Let $f(x_1) = f(x_2)$

$\Rightarrow$ $e^{x_1} = e^{x_2}$

$\Rightarrow$ $x_1 = x_2$.

$\therefore$ $f(x_1) = f(x_2) \quad \Rightarrow x_1 - x_2$

$\Rightarrow$ f is injective.



R (domain) $\quad R^+$ (codomain)

iii) Let $y \in R^+$ be arbitrary. Let $y = f(x)$

$= e^x$

$\Rightarrow \log y = x$

$\therefore \ \forall\, y \in R^+$ (codomain) $\exists$ preimage x in R (domain)

$\therefore$ Range of f = codomain

$\Rightarrow$ f is subjective.

Hence, f is isomorphism from $G = (R, +)$ to $G' = (R^+, .)$

2. Let $(G, *)$ be a group and $a \in G$. Let $f : G \to G$ defined as

$f(x) = a * x * a^{-1} \quad \forall\, x \in G.$

Show that f is isomorphism.

Soln. :

(i)
$$
\begin{aligned}
f(x * y) &= a * (x * y) * a^{-1} && \text{(by definition)}\\
&= (a * x) * (y * a^{-1}) && \text{(associativity in G)}\\
&= (a * x) * (a^{-1} * a) * (y * a^{-1}) && \because a^{-1} * a = e.\\
&= (a * x * a^{-1}) * (a * y * a^{-1}) && \text{(associativity in G)}\\
&= f(x) * f(y) && \text{(by definition)}
\end{aligned}
$$

$\therefore$ f is homomorphism.

(ii) Let $f(x_1) = f(x_2)$

$\therefore \ a * x_1 * a^{-1} = a * x_2 * a^{-1}$

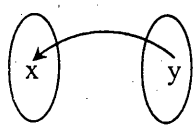
$x_1 = x_2$ by left & right cancellation law.

$\therefore$ f is injective.

(iii) Let $y \in G$ be arbitrary,

$y = f(x)$

$= a * x * a^{-1}$


G (domain) G (codomain)

Pre operating a^{-1} and post operating a on both sides

$$
\begin{aligned}
a^{-1} * y * a &= a^{-1} * (a * x * a^{-1}) * a\\
&= (a^{-1} * a) * x * (a^{-1} * a) \quad \text{by associativity}\\
&= e * x * e\\
&= x.
\end{aligned}
$$

$\therefore \ \forall\, y \in G$ (codomain) $\exists$ preimage x in G (domain)

$\therefore$ Range of f = codomain

$\therefore$ f is surjective.

Hence f is an isomorphism.

3. Prove that additive group of complex numbers $(C, +)$ is isomorphic to multiplicative group of +ve reals $(R^+, .)$ under the mapping $f(a + ib) = 2^a . 3^b$.

Soln. :

Consider the mapping

$f : (c, +) \to (R^+, .)$ defined by

$f(a + ib) = 2^a . 3^b, \quad a + ib \in C$

(i) Let $a + ib,\ c + id \in C$

$f(a + ib) = f(c + id) \Rightarrow 2^a . 3^b = 2^c . 3^d$

$\Rightarrow a = c$ and $b = d$

$\Rightarrow a + ib = c + id$

$\therefore$ f is injective.

(ii) Any positive real number $y \in R^+$ can be expressed as $y = 2^a . 3^b$

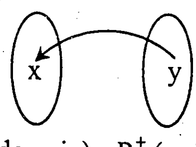
For any $y = 2^a . 3^b \in R^+ \ \exists\ x = a + ib \in C$, such that

$f(a + ib) = 2^a . 3^b$

$\therefore \ \forall\, y \in R^+$ (codomain) $\exists$ preimage x in C (domain)

$\therefore$ Range of f = codomain

$\therefore$ f is surjective.


C (domain) R^+ (codomain)

(iii) Let $a + ib,\ c + id \in C$

$$
\begin{aligned}
f[(a + ib) + (c + id)] &= f[(a + c) + i\,(b + d)]\\
&= 2^{a + c} . 3^{b + d}
\end{aligned}
$$

$$= 2^a . 2^c . 3^b . 3^d$$
$$= (2^a . 3^b) . (2^c . 3^d)$$
$$= f(a + ib) . f(c + id)$$

$\therefore$ f is homomorphism.

Hence f is isomorphic from (C, +) to (R^+ , .).

Graded Questions

1 (i) Define binary operation.
(ii) Find the set of 3 real no. that is closed w.r.t. ordinary addition and multiplication.

2. Explain the following terms :
(i) groupoid (ii) semigroup (iii) monoid (iv) group (v) abelian
(vi) submonoid **[M-05]**

3. Show that
(i) (N, +) is not a group (semigroup)
(ii) (N, .) is not a group (monoid)
(iii) (Z, +) is an abelian group
(iv) (Z, .) is not a group (monoid)
(v) (R, +) is a group

4. Prove that
(i) In a group G identify element is unique
(ii) In a group G inverse of each element is unique **[M-04]**
(iii) Let G be a group $\forall$ a, b, c $\in$ G prove that
(a) $a * b = a * c \Rightarrow b = c$ (left cancellation law)
(b) $b * a = c * a \Rightarrow b = c$ (right cancellation law)
(iv) In a group G prove that
(a) $(a^{-1})^{-1} = a$ for all $a \in G$
(b) $(a * b)^{-1} = b^{-1} * a^{-1}$ for all a, $b \in G$ **[M-04, 06]**
(v) $a^{|G|} = e$ where G is finite
(vi) Show that every cyclic group is abelian

5. (a) Let G = {1, 2, 3, 4, 5, 6} **[M-06]**
(i) Prepare the table for multiplication mod 7.
(ii) Prove that G is an abelian group under multiplication mod 7.
(iii) Find the inverses of 2, 3 & 5.
(iv) Find the order of elements 3 & 4 and find the subgroup generated by these.
(v) Is G is cyclic ?
(b) Consider the group G = {1, 2, 3, 4, 5, 6} under multiplication modulo 7. **[N-04]**
(i) Find multiplication table of G.
(ii) Find $2^{-1}, 3^{-1}, 6^{-1}, 1^{-1}, 4^{-1}, 5^{-1}$.

6. Let G = {1, 2, 3, 4, 5 } = $Z_6 - \{0\}$
(i) prepare the table for multiplication mod 6.
(ii) Is G is a group under multiplication mod 6 ?

7. Define a group and cyclic group. Let A = {0, 3, 6, 9, 12}. Find out the table for addition modulo 15 and multiplication modulo 15. Determine whether (A, $+_{15}$) and (A, $\times_{15}$) are groups ? Are they cyclic groups ? **[D-01]**

8. Let (A, *) be a monoid such that $\forall$ x $\in$ A, x * x = e. where e is the identity element. Show that (A, *) is an abelian group.

9. Determine whether the set together with the binary operation * is a semigroup, monoid or a group. Justify your answer.
(i) set of real numbers with a * b = a + b + 2
(ii) the set of all m $\times$ n matrices under the operation of multiplication.

10. (i) Write permutations of the elements of a set A = {α, β, γ}. Show that a set of all permutations is a group under the composition of permutations.
(ii) Define cyclic group. Is above group is cyclic ?

11. Let s = { x | x is real number and $x \neq 0, x \neq -1$ }. Consider the following functions **[D-03]**
$f_i : s \rightarrow s$, i = 1, 2, …….6.

$$f_1(x) = x,\ f_2(x) = \frac{1}{x},\ f_3(x) = 1-x$$

$$f_4(x) = \frac{x}{1-x},\ f_5(x) = \frac{1}{1-x},\ f_6(x) = \frac{x-1}{x}$$

Show that G = { $f_1, f_2, f_3, f_4, f_5, f_6$ } is a group under operation composition.
Give Multiplication table for G.

12. Prove that G and G′ be abelian groups, the G × G′ is also abelian.

13. Let Q be the set of positive rational numbers which can be expressed in the form $2^a\,3^b$. where a and b are integers. Prove that algebraic structure (Q.) is a group. Where . is multiplication operation. **[M-05, 06]**

14. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4. **[M-06]**

15. Let G be the group of integers under the operation addition, and H be group of all even integers under the operation of addition, show that the function f : G → H is an isomorphism. **[M-06]**

16. Define group and monoid.
Let (S, *) & (T, *) be monoids. Show that S × T is also monoid. Show that identity element of S × T is (e_s, e_r).

17. Define group homomorphism, isomorphism and Automorphism.

18. Let G be the group of integers under operation of addition, and let G′ be the group of all even integers under the operation of addition. Show that the function f : G → G′ defined by f(a) = 2a is an isomorphism. **[D-03]**

19. Let (G, *) and (G′, * ′) be two groups and let f : G → G′ be homomorphism from G to G′ then if e is identity in G and e′ is identity in G′ then prove that f(e) = e′. **[D-03]**

20. Let (G *) and G′, *′) be two groups and let f: G → G′ be a homomorphism from G to G′. Prove that – **[M-05]**
(i) If e is the identity in G and e′ is the identity in G′ then f(e) = e′
(ii) If a ∈ G then $f(a^{-1}) = (f(a))^{-1}$.

21. Show that the group G = (R, +) is isomorphic to G′ = (R^+, ×), where R^+ is a set of positive real numbers.

22. Let G_1, G_2 be groups. Prove that $G_1 \times G_2$ and $G_2 \times G_1$ are isomorphic.

23. Let G_1 and G_2 be two groups, show that function
f: $G_1 \times G_2 \rightarrow G$ defined by f(a, b) = a for $a \in G_1$ and $b \in G_2$ is isomophism.

24. Let Z be a group of the integer under the operation of addition, prove that the function f : Z × Z → Z defined by f(a, b) = a + b

25. If G is a group of all real numbers under addition and G′ is the group of +ve real numbers under multiplication and mapping f : G → G′ is defined by $f(x) = 2^x$, x ∈ G then show that f is homomorphism.

26. Show that the additive group G of integers is isomorphic to the multiplicative group G′ where
$G' = \{ \ldots\ldots. 3^{-3}, 3^{-2}, 3^{-1}, 1, 3^1, 3^2, 3^3, \ldots\ldots \}$

27. Let R^+ be the set of all positive real numbers. Show that the function f : $R^+ \rightarrow R$ defined by f (x) = Inx is an isomorphism of the semigroup (R^+, ×) to the semigroup (R, +) where × and + are ordinary multiplication and addition respectively. **[M-04]**

Subgroups

Let 'H' be a nonempty subset of group 'G'. the subset 'H' is said to be a sub group of a group G if it forms a group w.r.t. binary operation of 'G'.

Cosets

Let H be a subgroup of G and $a \in G$ then the set

$$Ha = \{ha \mid h \in H\}$$

is called right coset of H in G.

Similarly we can define left coset of H in G as

$$aH = \{ ah \mid h \in H\}$$

Normal subgroup :
A subgroup 'H' of G is said to be normal subgroup if all left and right cosets of H are identical.

Quotient group :
Let 'H' be a normal subgroup of 'G' then a set G/H which is the collection of all distinct left or right cosets is said to be Quotient group if it forms a group w.r.t. binary operation of G.

Theorem : A necessary and sufficient condition that a non–empty subset H of a group G to be a subgroup is

$$a \in h, b \in H \;\Rightarrow\; ab^{-1} \in H$$

Proof : Let H be a subgroup of G.

To prove that $a, b \in H \;\Rightarrow\; ab^{-1} \in B$

H is a subgroup $\;\Rightarrow\;$ H is a group.

$a, b \in H \;\Rightarrow\; a, b^{-1} \in H \;\Rightarrow\; ab^{-1} \in H$. by closure axiom.

Conversely, let H be a subset of a group G, such that

$a, b \in H \;\Rightarrow\; ab^{-1} \in H$

To prove that H is a subgroup of G.

For this we have to prove that H is group.

(i) $H \neq \phi \;\Rightarrow\; \exists$ at least an element $a \in H$

taking $b = a$

$a, a \in H \;\Rightarrow\; aa^{-1} \in H \;\Rightarrow\; e \in H$. $\quad (\because aa^{-1} = e$ is identity in G)

Thus, identity element exists in H.

(ii) $e \in H, a \in H \;\Rightarrow\; e\,a^{-1} \in H \;\Rightarrow\; a^{-1} \in H$

i.e., $a \in H \;\Rightarrow\; a^{-1} \in H$

Hence, inverse of each element of H exists.

(iii) $a, b \in H \;\Rightarrow\; a, b^{-1} \in H$

$\Rightarrow\; a(b^{-1})^{-1} \in H \qquad a, b, \in H \;\Rightarrow\; ab^{-1} \in H$

$\Rightarrow\; ab \in H.$

$\Rightarrow$ H is closed w.r.t operation '.'

(iv) $\forall\; a, b, c \in H \;\Rightarrow\; a, b, c \in G$

$\Rightarrow\; a(bc) = (ab)\,c \qquad \because$ G is a group.

$\Rightarrow$ Associative law holds in H.

Hence, H is group.

Therefore H is a subgroup of G.

Theorem : Intersection of two subgroups of a group is also a subgroup.

Proof :

Let H and K are two subgroups of group G, then we have to prove that $H \cap K$ is also a subgroup of G.

Let $x, y \in H \cap K$

$\Rightarrow\; x, y \in H \;\&\; x, y \in K$

$\Rightarrow\; xy^{-1} \in H \;\&\; xy^{-1} \in K \qquad (\because$ H & K are subgroups of G)

$\therefore\; xy^{-1} \in H \cap K$

$\forall\, x, y \in H \cap K,\; xy^{-1} \in H \cap K$

$\therefore\; H \cap K$ is a subgroup of G.

Theorem : The union of two subgroups of a group need not be a subgroup of it. Justify your answer.

Soln. :

Let H and K are subgroups of G, then we have to prove that $H \cup K$ not necessarily a subgroup of G.

$\because$ Z = {.... –2, –1, 0, 1, 2, ... } is a group under addition.
$\therefore$ H = {..... –4, –2, 0, 3, 6, ... }
& K = { ... –6, –3, 0, 3, 6, ... } are subgroups of Z
$H \cup K$ = {.... –6, –4, –3, –2, 0, 2, 3, 4, 6, ... }

Here, $2, 3 \in H \cup K$

But, $2 + 3 = 5 \notin H \cup K$
$\therefore$ $H \cup K$ is not closed.
$\therefore$ $H \cup K$ is not a group.
$\therefore$ $H \cup K$ is not a subgroup.

Theorem : Let G be an abelian group and Na subgroup of G. Prove that G/N is an abelian group.

Proof :

Since G is an abelian group, therefore left and right cosets of N are identical in G.

Let G/N = {aN : a $\in$ G}

$$\begin{aligned}\text{Consider } (aN)(bN) &= a(Nb)N \\ &= a(bN)N \qquad (\because Nb = bN) \\ &= ab(NN) \\ &= abN \\ &= baN \qquad (\because G \text{ is abelian}) \\ &= (bN)(aN).\end{aligned}$$

$\Rightarrow$ G/N is abelian.

Solved Examples

1. Consider G = {1, –1, i, –i} is a group w.r.t. multiplication and H = {1, –1}. Prove that H is a normal subgroup of G and find quotient group.

Soln. :

.	1	–1
1	1	–1
–1	–1	1

H is a group w.r.t. multiplication.
$\Rightarrow$ H is a subgroup of G.

Now the left and right cosets are

$1 \in G$ $\therefore$ 1 . H = {1, –1} = H = H . 1
$-1 \in G$ $\therefore$ –1 . H = {–1, 1} = H = H . (–1)
$i \in G$ $\therefore$ i . H = {i, –i} = H . i
$-i \in G$ $\therefore$ –i . H = {–i, i} = H . (–i)= i . H

$\Rightarrow$ All the left and right cosets of H are identical.

$\Rightarrow$ H is a normal subgroup of G.
G/H = {H, iH}

.	H	iH
H	H	iH
iH	iH	H

$\Rightarrow$ G/H is a group.
$\Rightarrow$ G/H is a quotient group.

Justify your

ecessarily a

s an abelian

e that H is a

2. Let H be a subset of set of integers consisting multiples of 5, then prove that H is a normal subgroup and find quotient group Z/H. **[M-02]**

Soln. :

$Z = \{-\infty, \ldots . -3, -2, -1, 0, 1, 2, 3, \ldots . \infty\}$
$H = \{\ldots \ldots, -10, -5, 0, 5, 10, \ldots \ldots\}$

Given that H is a subgroup of Z

Now left and right cosets are $0 \in Z$; $0 + H = H + 0 = H$

$1 \in Z\,;\, 1 + H = \{-\infty, \ldots -9, -4, 1, 6, 11, \ldots \ldots \infty\} = 6 + H$

$2 \in Z\,;\, 2 + H = \{-\infty \ldots \ldots -8, -3, 2, 7, 12, \ldots . \infty\} = 7 + H$

$3 \in Z\,;\, 3 + H = \{-\infty \ldots \ldots -7, -2, 3, 8, 13, \ldots . \infty\} = 8 + H$

$4 \in Z;\, 4 + H = \{-\infty \ldots . -6, -1, 4, 9, 14, \ldots \ldots \infty\} = 9 + H$

$5 \in Z\,;\, 5 + H = H + 5 = H = 10 + H = H + 10$

$\Rightarrow$ left and right cosets are same.

$\Rightarrow$ H is normal subgroup.

$Z/H = \{H, H + 1, H + 2, H + 3, H + 4\}$

$= \{H, 1 + H, 2 + H, 3 + H, 4 + H\}$

+	H	1 + H	2 + H	3 + H	4 + H
H	H	1 + H	2 + H	3 + H	4 + H
1 + H	1 + H	2 + H	3 + H	4 + H	H
2 + H	2 + H	3 + H	4 + H	H	1 + H
3 + H	3 + H	4 + H	H	1 + H	2 + H
4 + H	4 + H	H	1 + H	2 + H	3 + H

From composition table, we find Z/H is a group.

$\Rightarrow$ Z/H is a quotient group.

3. Let $G = Z_4$. for each of the following subgroup * Hof G determine all the left cosets of H in G.
(a) $H = \{\bar{0}\}$ (b) $H = \{\bar{0}, \bar{2}\}$ (c) $H = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$

Soln. :

$Z_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ is an abelian group under addition modulo 4.

(a) Left cosets of $H = \{\bar{0}\}$ are

$H, \bar{1} + H = \{\bar{1}\}$
$\bar{2} + H = \{\bar{2}\}$
$\bar{3} + H = \{\bar{3}\}$

(b) Left cosets of $H = \{\bar{0}, \bar{2}\}$ are

$\bar{0} + H = \{\bar{0}, \bar{2}\}$

$\bar{1} + H = \{\bar{1}, \bar{3}\}$

$\bar{2} + H = \{\bar{2}, \bar{0}\} = H$

$\bar{3} + H = \{\bar{3}, \bar{1}\} = \bar{1} + H$

$\therefore$ Distinct left cosets are H, $\bar{1} + H$.

(c)

$H = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\} = Z_4 = G$

$\bar{0} + H = \bar{1} + H = \bar{2} + H = \bar{3} + H = H$

$\Rightarrow$ Left coset of H is H only.

4. For the cyclic group of order 8 with generator a, find the quotient group corresponding to the subgroups generated by a^2 and a^4 respectively.

Soln. :

We have, $G = \{0, a^2, a^3, \ldots, a^8 = e\}$

and its subgroups generated by a^2 and a^4 respectively.

$$H = \{a^2, a^4, a^6, a^8 = e\}$$

$$K = \{a^4, a^8 = e\}$$

$\because$ G is cyclic group.

$\therefore$ G is abelian group.

$\Rightarrow$ H and K are normal subgroups of G.

$\therefore$ The right cosets of H are

$$H = \{e, a^2, a^4, a^6\}$$

$$Ha = \{a, a^3, a^5, a^7\}$$

$$Ha^2 = \{a^2, a^4, a^6, a^8 = e\} = Ha^4 = Ha^6 = Ha^8 = H$$

$$Ha^3 = \{a^3, a^5, a^7, a\} = Ha = Ha^5 = Ha^7 = Ha$$

Hence H and Ha are only two distinct right cosets of H in G.

$\therefore$ $G/H = \{H, Ha\}$

Similarly, the cosets of G/K are

$$K = \{e, a^4\}$$

$$Ka = \{a, a^5\}$$

$$Ka^2 = \{a^2, a^6\}$$

$$Ka^3 = \{a^3, a^7\}$$

and $Ka^4 = K, Ka^5 = Ka, Ka^6 = Ka^2, Ka^7 = Ka^3$.

Here, K, Ka, Ka^2, Ka^3 are four cosets of K in G.

$\therefore$ $G/K = \{K, Ka, Ka^2, Ka^3\}$

Graded Questions

1. Define the following terms :
   (i) Subgroup **[D-05]** (ii) Cosets, leftcosets and right cosets
   (iii) Normal subgroup **[M-05]** (iv) Quotient group
2. Let G = Z_8, for each of the following subgroups H of G, determine all the left cosets of H in G.
   (a) H = {0, 4} **[M-05]**
   (b) H = {0, 2, 4, 6}
3. Determine the multiplication table of the quotient group Z/3Z where Z has operation +.
4. Let Z be the group of integers under the operation of addition and let $G = Z \times Z$. Consider the subgroup $H = \{(x, y) / x = y\}$ of G. Determine the left cosets of H in G.
5. Let H be a subgroup of finite group G and suppose that there are only two left cosets of H in G. Prove that H is a normal subgroup of G.
6. Give an example of a normal subgroup of a non–abelian group.
7. Let G be the group and let $H = \{x \mid x \in G \text{ and } xy = yx \text{ for all } y \in G\}$. Prove that H is subgroup of G. **[M-03]**
8. Show that if G is an Abelian group, then every subgroup of G is a normal subgroup. **[M-04]**

□□□□□□

Rings :

Let R be a non-empty set with two binary operations, denoted by $\oplus$ and $\odot$ respectively satisfying the following postulates.

1. $(R, \oplus)$ forms an abelian group.
2. $(R, \odot)$ is semi group.
3. Distributive Law : $\forall$ a, b, c $\in$ R

   $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ left distributive law

   $(b \oplus c) \odot a = (b \odot a) \oplus (c \odot a)$ right distributive law.

Commutative ring :

A ring $(R, \oplus, \odot)$ is called as called commutative ring if the ring follows :

$a \odot b = b \odot a \quad \forall\ a, b \in R$

i.e., commutative law under the composition.

Ring with unit element :

A ring $(R, \oplus, \odot)$ is said to be a ring with unity element if R has an identity w.r.t. $\odot$ i.e., $\forall$ $a \in R$.

Ring with zero divisors :

Let $a \neq 0$, $b \neq 0$ be the elements of R and if $a \odot b = 0$, then we say that elements a and b are zero divisors and the ring R is called ring with zero divisor.

Ring without zero divisors :

If for any a, b $\in$ R $a \odot b = 0 \Rightarrow$ either $a = 0$ or $b = 0$ or $a = 0$ and $b = 0$, then ring R is a ring without zero divisor.

Integral Domain :

A commutative ring $(R, \oplus, \odot)$ is said to be an integral domain if it has no zero divisor.

Fields

A commutative ring $(R, \oplus, \odot)$ with identity elements is said to be a Field if it has no zero divisor and all non zero elements are invertible.

1. Show that the set S = {0, 1, 2, 3, 4} is a ring w.r.t. the operation of addition and multiplication modulo 5. Is it an integral domain or a field or both. **[M-01, D-02]**

Soln. :

i) The composition table w.r.t. $+_5$.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

From composition table we find $(S, +_5)$ is an abelian group.

ii) The composition table w.r.t. $\times_5$.

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

From composition table we find $(S, \times_5)$ is a semigroup.

iii) Distributive Law :
Now, we find, $\forall$ a, b, c $\in$ S
$a \times_5 (b +_5 c) = a \times_5 b +_5 a \times_5 c$ from table (i) and (ii)
$\therefore$ $\times_5$ is distributive over $+_5$.
Hence (S, $+_5$, $\times_5$) is ring.

iv) Commutativity :
$\because$ $a \times_5 b = b \times_5 a$
$\therefore$ $\times_5$ is a commutative operation
$\therefore$ (S, $+_5$, $\times_5$) is a commutative ring.

v) Zero divisor :
From composition table we find,
$a \times_5 b = 0 \Rightarrow a = 0$ or $b = 0$ or $a = b = 0$
$\therefore$ (S, $+_5$, $\times_5$) is an integral domain.

vi) Identity domain :
From composition table we find 1 is the identity element
$\therefore$ (S, $+_5$, $\times_5$) is a commutative ring with identity element and has no zero divisor.

vii) Inverse :
From composition table we find all non zero elements of S are invertible
$\therefore$ $1^{-1} = 1$ $\quad$ $2^{-1} = 3$
$3^{-1} = 2$ $\quad$ $4^{-1} = 4$
$\therefore$ System (S, $+_5$, $\times_5$) is a field.

2. If the addition and multiplication modulo 10 is defined on the set of integers, R = {0, 2, 4, 6, 8} then show that the system is a ring with unity. Is it an integral domain or a field or both ? **[M-99, 05]**

Soln. :

1. The composition table :

$+_{10}$	0	2	4	6	8
0	0	2	4	6	8
2	2	4	6	8	0
4	4	6	8	0	2
6	6	8	0	2	4
8	8	0	2	4	6

From table,
(R_1 $+_{10}$) is an abelian group.

2. The composition table w.r.t. x_{10}.

$\times_{10}$	0	2	4	6	8
0	0	0	0	0	0
2	0	4	8	2	6
4	0	8	6	4	2
6	0	2	4	6	8
8	0	6	2	8	4

From composition table, (R, $\times_{10}$) is a semi group.

3. Distributive Law : From composition table (1) and (2)
$\times_{10}$ is distributive over $+_{10}$.
Hence, (R, $+_{10}$, $\times_{10}$) is a ring.

4. Commutativity :
Now, $\times_{10}$ is commutative in R,
If a, b $\in$ R
$a \times_{10} b = b \times_{10} a$ from table (2).
$\therefore$ (R, $+_{10}$, $\times_{10}$) is a commutative ring.

5. Zero divisor :
   From table (2) $a \times_{10} b = 0 \Rightarrow$ either $a = 0$ or $b = 0$ or $a = b = 0$
   $\therefore$ R has no zero divisor.
   $\therefore$ $(R, +_{10}, \times_{10})$ is commutative ring without zero divisor.
   $\therefore$ $(R, +_{10}, \times_{10})$ is an **integral domain.**
6. Identity Element :
   From table (2) we find $\forall\ a \in R$
   $a \times_{10} 6 = 6 \times_{10} a = a$
   $\Rightarrow$ 6 is identity w.r.t. x_{10}
   $\Rightarrow$ $(R, +_{10}, \times_{10})$ is commutative ring with unity has no zero divisor.
7. Inverses :
   From table (2) all the non zero elements of R are invertible. i.e. $2^{-1} = 8, 4^{-1} = 4, 6^{-1} = 6, 8^{-1} = 2.$
   $\therefore$ Algebraic system $(R, +_{10}, \times_{10})$ is a **field.**

ior.

0, 2, 4, 6, 8}
both ?
[M-99, 05]

3. Show that $(I, \oplus, \odot)$ is a commutative ring with unity where the operations $\oplus$ and $\odot$ are defined as follows **[M-99, 01, D-00, 03, N-04]**

$a \oplus b = a + b - 1$
$a \odot b = a + b - ab$
$\forall\ a, b \in I$ i.e. a set of integer.

Soln. :

1. $(R, \oplus)$ is an abelian group.

(i) Closure prop. :
Let $a, b, \in I$
$\Rightarrow a + b \in I\ \&\ -1 \in I$
$\therefore\ a + b, -1 \in I$
$\therefore\ a \oplus b \in I$
Hence I is closed w.r.t. $\oplus$.

(ii) Associativity –
For $\forall\ a, b, c \in I$

$$\left.\begin{aligned} a \oplus (b \oplus c) &= a \oplus (b + c - 1) \\ &= a + (b + c - 1) - 1 \end{aligned}\right\} \text{by definition.}$$
$$= (a + b - 1) + c - 1$$
$$\left.\begin{aligned} &= (a \oplus b) + c - 1 \\ &= (a \oplus b) \oplus c \end{aligned}\right\} \text{by definition.}$$

$\therefore$ $\oplus$ is associative.

(iii) Identity element :
For any $a \in I$
$e \in I$ such that
$a \oplus e = a$
$\Rightarrow\ a + e - 1 = a$
$\Rightarrow\ e - 1 = 0$
$\Rightarrow\ e = 1.$
$\therefore$ 1 is identity in I w.r.t. $\oplus$.

(iv) Inverse :
For $a \in I \quad \exists\ \alpha \in I$ such that
$a \oplus \alpha = 1$ (identity element)
$a + \alpha - 1 = 1$
$\alpha = 2 - a \quad \in I$
$\therefore$ all the elements of I are invertible w.r.t. $\oplus$.

(v) Commutativity :
$$\begin{aligned} a \oplus b &= a + b - 1 \\ &= b + a - 1 \\ &= b \oplus a \end{aligned}$$
$\therefore$ $\oplus$ is commutativity.
$\therefore$ $(I, \oplus)$ is an abelian group.

2. Consider (I, ⊙)

   (i) Closure prop. :

   Let a, b, ∈ I
   $a + b \in I$ & $ab \in I$ (∵ ordinary multiplication & addition are binary operations in I)
   $a + b - ab \in I$
   $\Rightarrow a \odot b \in I$
   ∴ Closure prop. is satisfied.

   (ii) Associativity :
   Let a, b, c ∈ I

$$\begin{aligned} a \odot (b \odot c) &= a \odot (b + c - bc) && \text{by def.} \\ &= a + (b + c - bc) - a(b + c - bc) && \text{by def.} \\ &= a + b + c - abc - ab - ac + abc \\ &= (a + b - ab) + c - c(a + b - ab) \\ &= (a \odot b) + c - c\,(a \odot b) && \text{by def.} \\ &= (a \odot b) \odot c && \text{by def.} \end{aligned}$$

   ∴ ⊙ is associative.
   ∴ (I, ⊙) is semi group.

3. Distributive Law :

$$\begin{aligned} a \odot (b \oplus c) &= a \odot (b + c - 1) && \text{by def.} \\ &= a + (b + c - 1) - a\,(b + c - 1) && \text{by def.} \\ &= 2a + b + c - ab - ac - 1 && (1) \end{aligned}$$

$$\begin{aligned} \therefore\ (a \odot b) \oplus (a \odot c) &= (a + b - ab) \oplus (a + c - ac) && \text{by def.} \\ &= 2a + b + c - ab - ac - 1 && \text{by def.} \quad (2) \end{aligned}$$

   From (1) and (2), we get,
   $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$

4. Commutativity –

$$\begin{aligned} a \odot b &= a + b - ab \\ &= b + a - ba \\ &= b \odot a \end{aligned}$$

   ∴ ⊙ is commutative.

5. Identity w.r.t. ⊙ .

   $\forall\ a \in I\ \exists$ e such that

$$\begin{aligned} a \odot e &= a \\ a + e - ae &= a \\ e - ae &= 0 \\ e\,(1 - a) &= 0 \\ e &= 0. \end{aligned}$$

   ⇒ 0 is identity w.r.t. ⊙.
   ∴ (I, ⊕, ⊙) is commutative ring with identity.

4. Show that $R = \{a + b\sqrt{2} : a, b \in I\}$ is an integral domain. **[M-98, D-00]**

Soln. :

1. Consider (R, +)

   (i) Closure prop.

$$\left(a_1 + b_1\sqrt{2}\right), \left(a_2 + b_2\sqrt{2}\right) \in R \qquad \text{for all } a_1, b_1, a_2, b_2 \in I$$

$$\left(a_1 + b_1\sqrt{2}\right) + \left(a_2 + b_2\sqrt{2}\right) = (a_1 + a_2) + (b_1 + b_2)\sqrt{2} \in R \quad \because a_1 + a_2, b_1 + b_2 \in II$$

   (ii) $\forall$ a, b ∈ I $a + b\sqrt{2} \in R$ i.e. a real number.

   ∵ R is subset of real number set
   ∴ sum of elements of R can be taken in any order.
   ∴ Addition is associative.

(iii) Identity element :

$\because\ 0 \in I. \qquad \therefore\ 0 + 0\sqrt{2} = 0 \in R.$

for all, $a + b\sqrt{2} \in R \Rightarrow \left(a + b\sqrt{2}\right) + 0 = a + b\sqrt{2} = 0 + \left(a + b\sqrt{2}\right)$

$\therefore$ Identity exists.

(iv) Inverses :

$\forall a, b \in I, \qquad -a, -b \in I$

$a + b\sqrt{2} \in R, -a + (-b)\sqrt{2} \in R$

$$\because\ (a + b\sqrt{2}) + (-a - b\sqrt{2})$$
$$= [a + (-a)] + [(b + (-b))]\sqrt{2}$$
$$= 0 + 0\sqrt{2} = 0.$$

$\therefore$ Inverses exist.

(v) Commutativity :

For all $a_1 + b_1\sqrt{2},\ a_2 + b_2\sqrt{2} \in R$

$$\left(a_1 + b_1\sqrt{2}\right) + \left(a_2 + b_2\sqrt{2}\right)$$
$$= \left(a_1 + a_2\right) + \left(b_1 + b_2\right)\sqrt{2}$$
$$= \left(a_2 + a_1\right) + \left(b_2 + b_1\right)\sqrt{2}$$
$$= \left(a_2 + b_2\sqrt{2}\right) + \left(a_1 + b_2\sqrt{2}\right)$$

$\Rightarrow$ R holds commutativity w.r.t. +.

Hence, (R, +) is an abelian group.

2. Consider (R, .)
   (i) Closure property :

   Let $a_1 + b_1\sqrt{2},\ a_2 + b_2\sqrt{2} \in R$

$$\left(a_1 + b_1\sqrt{2}\right).\left(a_2 + b_2\sqrt{2}\right)$$
$$= \left(a_1a_2 + 2b_1b_2\right) + \left(a_1b_2 + b_1b_2\right)\sqrt{2} \in R \left(\because a_1a_2 + 2b_1b_2, a_1b_2 + b_1a_2 \in R\right)$$

   $\therefore$ it holds closure property.

   (ii) Associativity :

   In fact $a + b\sqrt{2}$ is a real number $\forall$ a, b $\in$ I
   and $R = \{ a + b\sqrt{2} \mid a, b \in I\}$ $\because$ product of real numbers can be taken in any order

   $\therefore$ multiplication is associative in R.
   $\therefore$ (R, .) is a semi group.

3. Distributive Law :

   Since multiplication always be distributive over addition in number system & R is a subset of number system

   $\therefore$ R holds Distributive law.
   $\therefore$ (R, +, .) is a ring.

4. Commutativity :

   $a_1 + b_1\sqrt{2},\ a_2 + b_2\sqrt{2} \in R$

$$\left(a_1 + b_1\sqrt{2}\right).\left(a_2 + b_2\sqrt{2}\right) = \left(a_1a_2 + 2b_1b_2\right) + \left(a_{12} + b_1a_2\right)\sqrt{2} \qquad (1)$$
$$\left(a_2 + \sqrt{2}b_2\right).\left(a_1 + b_1\sqrt{2}\right) = \left(a_2a_1 + 2b_2b_1\right) + \left(a_2b_1 + b_2a_1\right)\sqrt{2}$$
$$= \left(a_1a_2 + 2b_1b_2\right) + \left(a_1b_2 + b_1a_2\right)\sqrt{2} \qquad (2)$$

From (1) and (2),

$$\left(a_1 + b_1\sqrt{2}\right).\left(a_2 + b_2\sqrt{2}\right) = \left(a_2 + b_2\sqrt{3}\right).\left(a_1 + b_1\sqrt{2}\right)$$

$\therefore$ multiplication is commutative in R.

$\therefore$ R is a commutative ring.

5. Identity :

$\because$ $0, 1 \in I$

$\therefore$ $1 + 0\sqrt{2} = 1 \in R$

$$\left(a + b\sqrt{2}\right)\cdot 1 = a + b\sqrt{2} = 1\cdot\left(a + b\sqrt{2}\right) \quad \forall\ a + b\sqrt{2} \in R$$

$\therefore$ $1 = 1 + 0\sqrt{2}$ is an identity element in R.

$\therefore$ (R, .) is commutative ring with unity.

5. (R, +, .) is a ring and that $a^2 = a$, $\forall$ $a \in R$ then show that,
(i) $a + a = 0$, $\forall$ $a \in R$ i.e., each element its own inverse.
(ii) $a + b = 0 \Rightarrow a = b$
(iii) R is commutative ring i.e., $a . b = b . a \quad \forall\ a, b, \in R$.

Soln. :

(i) Let $a \in R$, then $a + a \in R$, by given condition, we have,

$$\begin{aligned}
(a + a)^2 &= a + a \\
\text{or} \quad (a + a) . (a + a) &= a + a \\
(a + a)\, a + (a + a)\, a &= a + a && \text{by distributive law} \\
(a . a + a . a) + (a . a + a . a) &= a + a \\
(a^2 + a^2) + (a^2 + a^2) &= a + a \\
(a + a) + (a + a) &= (a + a) + 0 && \text{by left cancellation law} \\
a + a &= 0
\end{aligned}$$

(ii) Let $a + b = 0$.

$\therefore$ $a + b = a + a \qquad \because a + a = 0$

$\therefore$ $b = a$. $\qquad$ by left cancellation law.

(iii) Let $a, b \in R$

$$\begin{aligned}
(a + b)^2 &= a + b && \because a^2 = a \\
(a + b) . (a + b) &= a + b \\
(a + b) .a + (a + b) .b &= a + b && \text{by distributive law} \\
(a + b) . (a + b) &= a + b \\
(a + b) . a + (a + b) .b &= a + b && \text{by distributive law} \\
a^2 + b . a + a . b + b^2 &= a + b \\
a + b . a + a . b + b &= a + b && \text{by given condition} \\
b . a + a . b &= 0 && \text{by cancellation law} \\
\Rightarrow b . a &= a . b && \text{by case (ii)}
\end{aligned}$$

$\therefore$ R is commutative ring.

Graded Questions

1. Define the following with examples :
   (i) ring (ii) commutative ring (iii) ring with unity
   (iv) ring with zero divisor (v) ring without zero divisor (vi) integral domain
   (vii) field **[D-05]**
2. Define integral domain and commutative ring.
   Let A = {0, 1, 2, 3, 4, 5}. Find the table for addition module 6 and multiplication modulo 6. Verify whether it is an integral domain or field or both. **[M-00]**
3. Prove that if (f, +, .) is a field then it is an integral domain. **[D-03, M-05, M-06]**
4. Show that the set of complex numbers with the ordinary addition and multiplication operations is a field. **[D-98]**
5. Let R be a ring with unity element 1. Show that the set R* of units in R is a group under multiplication. **[M-02]**

6. Define an integral domain and a field. Show that the ring Z_{29} of integers modulo 29 is an integral domain whereas ring Z_{105} of the integers modulus 105 is not an integral domain.**[M-02]**
7. In an integral domain D, show that if $ab = ac$ with $a \neq 0$, the $b = c$.
8. Do the following sets from integral domain with respect to addition and multiplication
   i) the set of even integers ii) the set of +ve integers
9. Show that, the matrices of the form s $\begin{bmatrix} a & b \\ 2b & a \end{bmatrix} | a, b \in Z$ forms a field under matrix addition and multiplication.
10. Let R be a non–empty set with two binary operations addition and multiplication denoted by + and $\bullet$. Then explain when the algebraic structure (R, +, $\bullet$) becomes a ring and show that the set R consisting of single elements 0 with two binary operations defined by $0 + 0 = 0$ and $0.0 = 0$ is a ring.
11. Prove that the set M_n of all $n \times n$ matrices is a ring with respect to addition and multiplication of matrices. **[M-03]**
12. Show that the set R of ordered pairs (a, b) of real is field under addition and multiplication as follows : **[M-03]**
    (i) $(a, b) + (c, d) = (a + c, b + d)$ (ii) $(a , b) . (c, d) = (ac - bd, bc + ad)$
13. If $f : (R, +, \bullet) \rightarrow (S, \oplus, \square)$ is a ring homomorphism then prove that **[M-04]**
    $f(-a) = -f(a)$ for all $a \in R$;
14. Prove that in any ring (R + ·) the additive inverse of each ring element is unique. **[D-05, M-06]**
15. (i) Define ring homomorphism. **[M-05, D-05]**
    (ii) Define ring Isomorphism & Automorphism.
    (iii)Show that the ring normal of integers under ordinary addition and multiplication is isomorphic to the ring normal of even integers with respect to ordinary addition and another operation * is a and b defined by a * b = a . b/2 where a . b is ordinary multiplication of two even integers.

□□□□□□

Encoding Function :

Let m < n (m, n are +ve integers) then an one to one function $e : B^m \to B^n$ is called encoding function.

Weight : Let $x \in B^n$, then the weight of x is denoted by |x| and it defines the number of 1's.

eg. i) $x = 001110 \in B^6 \quad \therefore \ |x| = 3$

ii) $x = 1111111 \in B^7 \quad \therefore \ |x| = 7$

iii) $x = 00000 \in B^5 \quad \therefore \ |x| = 0$

Hamming distance :

Let $x, y \in B^n$, the hamming distance between x and y is denoted by $\delta(x, y)$ and defined as $\delta(x, y) = |x \oplus y|$

eg. $x = 11111, \ y = 01110 \in B^5$
$\delta(x, y) = |10001| = 2$

$x = 01010, \ y = 01010 \in B^5$
$\delta(x, y) = |00000| = 0$

Minimum distance :

Let m < n and $e : B^m \to B^n$ is an encoding function the minimum distance of an encoding function e is the minimum of hamming distances between all distinct pairs of code words. That means minimum distance of e is equal to minimum of

$\{ \delta(ex, ey) \text{ such that } x, y \in B^m \}$

Note :

1. If m < n, $e : B^m \to B^n$ is an encoding function then it can detect k or fewer error if the minimum distance of e is (k + 1).
2. If m < n, $e : B^m \to B^n$ is an encoding function then it can correct k or fewer error if the minimum distance is (2k + 1)

Examples

1. Consider (2, 5) encoding function e defined as i.e. $e : B^2 \to B^5$ **[M-04]**

e (00) = 00000
e (01) = 01110
e (10) = 00111
e (11) = 11111

i) Find minimum distance
ii) How many errors can 'e' detect and
iii) How many errors will 'e' correct ?

Soln. :

(2 < 5) and $e : B^2 \to B^5$ is an encoding function defined as

$e(00) = 00000 = x_0$
$e(01) = 01110 = x_1$
$e(10) = 00111 = x_2$
$e(11) = 11111 = x_3$

i) $|x_0 \oplus x_1| = |01110| = 3$

$|x_0 \oplus x_2| = |00111| = 3$

$|x_0 \oplus x_3| = |11111| = 5$

$|x_1 \oplus x_2| = |01001| = 2$

$|x_1 \oplus x_3| = |10001| = 2$

$|x_2 \oplus x_3| = |11000| = 2$

$\Rightarrow$ Minimum distance of e = 2.

ii) $\because$ An (m, n) encoding function can detect k or fewer error if the minimum distance is (k + 1)

$k + 1 = 2$

$k = 1$

$\Rightarrow$ the function can detect 1 or fewer error.

iii) $\because$ An (m, n) encoding function can correct k or fewer error if the minimum distance is (2k + 1)

$2k + 1 = 2$

$$k = \frac{1}{2} \approx 0$$

$\Rightarrow$ e can correct 0 error.

2. Consider (2, 6) encoding function $e : B^2 \rightarrow B^6$ defined as

e (00) = 000000
e (01) = 011110
e (10) = 101010
e (11) = 111000

(i) Find the minimum distance (ii) How many error can 'e' detect ?

Soln. :

(2 < 6) and $e : B^2 \rightarrow B^6$ is an encoding function defined as

e (00) = 000000 = x_0
e (01) = 011110 = x_1
e (10) = 101010 = x_2
e (11) = 111000 = x_3

i) $|x_0 \oplus x_1| = |011110| = 4$, $|x_0 \oplus x_2| = |101010| = 3$

$|x_0 \oplus x_3| = |111000| = 3$, $|x_1 \oplus x_2| = |110100| = 3$

$|x_1 \oplus x_3| = |100110| = 3$, $|x_2 \oplus x_3| = |010010| = 2$

$\Rightarrow$ Minimum distance = 2

ii) $\because$ An (m, n) encoding function can detect k or fewer error if the minimum distance is (k + 1)

$k + 1 = 2$

$k = 1$

$\Rightarrow$ e can detect 1 or fewer error.

3. Consider (3, 8) an encoding function $e : B^3 \rightarrow B^8$ defined as **[M-99, 06]**

e (000) = 00000000
e (001) = 10111000
e (010) = 00101101
e (011) = 10010101
e (100) = 10100100
e (101) = 10001001
e (110) = 00011100
e (111) = 00110001

How many error can 'e' detect ?

Soln. :

e (000) = 00000000 = x_0 e (001) = 10111000 = x_1
e (010) = 00101101 = x_2 e (011) = 10010101 = x_3
e (100) = 10100100 = x_4 e (101) = 10001001 = x_5
e (110) = 00011100 = x_6 e (111) = 00110001 = x_7

i) $|x_0 \oplus x_1| = |10111000| = 4$, $|x_0 \oplus x_2| = |00101101| = 4$

$|x_0 \oplus x_3| = |10010101| = 4$, $|x_0 \oplus x_4| = |10100100| = 3$

$|x_0 \oplus x_5| = |10001001| = 3$, $|x_0 \oplus x_6| = |00011100| = 3$

$|x_0 \oplus x_7| = |00110001| = 3$, $|x_1 \oplus x_2| = |10010101| = 4$

$|x_1 \oplus x_3| = |00100101| = 3$, $|x_1 \oplus x_4| = |00011100| = 3$

$|x_1 \oplus x_5| = |00110001| = 3$, $|x_1 \oplus x_6| = |10100001| = 3$

$|x_1 \oplus x_7| = |10001001| = 3$, $|x_2 \oplus x_3| = |10111000| = 4$

$|x_2 \oplus x_4| = |10001001| = 3$, $|x_2 \oplus x_5| = |10100100| = 3$

$|x_2 \oplus x_6| = |00110001| = 3$, $|x_2 \oplus x_7| = |00011100| = 3$

$|x_3 \oplus x_4| = |00110001| = 3$, $|x_3 \oplus x_5| = |00011100| = 3$

$|x_3 \oplus x_6| = |10001001| = 3$, $|x_3 \oplus x_7| = |10100100| = 3$

$|x_4 \oplus x_5| = |00101101| = 4$, $|x_4 \oplus x_6| = |10111000| = 4$

$|x_4 \oplus x_7| = |10010101| = 4$, $|x_5 \oplus x_6| = |10010101| = 4$

$|x_5 \oplus x_7| = |10111000| = 4$, $|x_6 \oplus x_7| = |00101101| = 4$

$\Rightarrow$ Minimum distance of e = 3

ii) $\because$ An (m, n) encoding function detect k of fewer error if the minimum distance is (k + 1)

$k + 1 = 3$

$k = 2$

$\Rightarrow$ the function can detect 2 or fewer errors.

Group Codes

Let $m < n$ and $e : B^m \to B^n$ be an encoding function, then the encoding function 'e' is said to be group code if range of 'e' is a subgroup of B^n.

1. Show that encoding function $e : B^2 \to B^5$ defined by **[N-04]**

e (00) = 00000

e (01) = 01110

e (10) = 10101

e (11) = 11011 is a group code.

Soln. :

We have range of e

e = $\{x_0 = 00000,\ x_1 = 01110,\ x_2 = 10101,\ x_4 = 11011\}$

Now preparing a composition table

$\oplus$	x_0	x_1	x_2	x_3
x_0	x_0	x_1	x_2	x_3
x_1	x_1	x_0	x_3	x_2
x_2	x_2	x_3	x_0	x_1
x_3	x_3	x_2	x_1	x_0

From this table we find range of 'e' is a group

and range of 'e' $\subseteq B^5$ $\Rightarrow$ range of e is a subgroup of B^5

$\therefore$ $e : B^2 \to B^5$ is a group code.

2. (a) Define Group code. Show that (3, 7) encoding function **[M-99, D-05]**

$e : B^2 \to B^5$ defined by

e (000) = 0000000

e (001) = 0010110

e (010) = 0101000

e (011) = 0111110

e (100) = 1000101

e (101) = 1010011

e (110) = 1101101

e (111) = 1111011 is a group code.

(b) Find the minimum distance of encoding function.

(c) How many errors will 'e' detect 2.

Soln. :

(a) We have range of e defined as

$e(000) = 0000000 = x_0$
$e(001) = 0010110 = x_1$
$e(010) = 0101000 = x_2$
$e(011) = 0111110 = x_3$
$e(100) = 1000101 = x_4$
$e(101) = 1010011 = x_5$
$e(110) = 1101101 = x_6$
$e(111) = 1111011 = x_7$

$\oplus$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_0	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	x_1	x_0	x_3	x_2	x_5	x_4	x_7	x_6
x_2	x_2	x_3	x_0	x_1	x_6	x_7	x_4	x_5
x_3	x_3	x_2	x_1	x_0	x_7	x_6	x_5	x_4
x_4	x_4	x_5	x_6	x_7	x_0	x_1	x_2	x_3
x_5	x_5	x_4	x_7	x_6	x_1	x_0	x_3	x_2
x_6	x_6	x_7	x_4	x_5	x_2	x_3	x_0	x_1
x_7	x_7	x_6	x_5	x_4	x_3	x_2	x_1	x_0

From this table we find range of 'e' is a group and range of e is subset of $B^7 \Rightarrow$ range of e is subgroup of B^7

$\therefore\ e : B^3 \to B^7$ is a group code.

(b) **Note :** If an encoding function is a group code the minimum distance of an encoding function is the minimum of weight of non–zero code words.

$|x_1| = 3$
$|x_2| = 2$
$|x_3| = 5$
$|x_4| = 3$
$|x_5| = 4$
$|x_6| = 5$
$|x_7| = 6$

$\Rightarrow$ minimum distance of $e = 2$.

(c) An (m, n) encoding function can detect k or fewer error if minimum distance is $(k + 1)$

$k + 1 = 2$
$k = 1$

hence 'e' can detect 1 or fewer error.

Procedure for generating group code :

Parity check matrix :

Let $m < n$ and $r = n - m$

A Boolean matrix of order $n \times r$

$$H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1r} \\ h_{21} & h_{22} & \cdots & h_{2r} \\ \vdots & \vdots & \cdots & \vdots \\ h_{m1} & h_{m2} & \cdots & h_{mr} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

whose last 'r' rows is an identity matrix of order r, is called parity check matrix.

We use the parity check matrix to define a group code $e_H : B^m \to B^n$.

Examples

1. Let m = 2, n = 5, and

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ Determine group code $e_H : B^2 \to B^5$

Soln. :

We have $B^2 = \{ 00, 01, 10, 11 \}$

Let $e(00) = 00\, x_1\, x_2\, x_3$

$$[x_1\, x_2\, x_3] = [0\ 0] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [0\ 0\ 0]$$

$\Rightarrow \quad x_1 = x_2 = x_3 = 0$

$e(01) = 01\, x_1\, x_2\, x_3$

$$[x_1\, x_2\, x_3] = [0\ 1] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [0\ 1\ 1]$$

$\Rightarrow \quad x_1 = 0, x_2 = 1, x_3 = 1$

$e(10) = 10\, x_1\, x_2\, x_3$

$$[x_1\, x_2\, x_3] = [1\ 0] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [1\ 1\ 0]$$

$\Rightarrow \quad x_1 = 1, x_2 = 1, x_3 = 0$

$e(11) = 11\, x_1\, x_2\, x_3$

$$[x_1\, x_2\, x_3] = [1\ 1] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [1\ 0\ 1]$$

$\Rightarrow \quad x_1 = 1, x_2 = 0, x_3 = 1$

Hence group code $e_H : B^2 \to B^5$ is defined as

$e(00) = 00000$
$e(01) = 01011$
$e(10) = 10110$
$e(11) = 11101$

2. Let $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be a parity check matrix. Determine (3, 6) group code $e_H : B^3 \to B^6$

[D-02, M-04, N-04]

Soln. :

We have $B^3 = \{ 000, 001, 010, 011, 100, 101, 110, 111 \}$

Let $e(000) = 000\, x_1\, x_2\, x_3$

$$[x_1\, x_2\, x_3] = [0\ 0\ 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = [0\ 0\ 0]$$

$$\Rightarrow \quad x_1 = x_2 = x_3 = 0$$

$$e(001) = 001\, x_1\, x_2\, x_3$$

$$[x_1\, x_2\, x_3] = [0\,0\,1]\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

$$= [1\,1\,1]$$

$$\Rightarrow \quad x_1 = x_2 = x_3 = 1$$

$$e(010) = 010\, x_1\, x_2\, x_3$$

$$[x_1\, x_2\, x_3] = [0\,1\,0]\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

$$= [0\,1\,1]$$

$$\Rightarrow \quad x_1 = 0, x_2 = 1, x_3 = 1$$

$$e(011) = 011\, x_1\, x_2\, x_3$$

$$[x_1\, x_2\, x_3] = [0\,1\,1]\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

$$= [1\,0\,0]$$

$$\Rightarrow \quad x_1 = 1, x_2 = 0, x_3 = 0$$

$$e(100) = 100\, x_1\, x_2\, x_3$$

$$[x_1\, x_2\, x_3] = [1\,0\,0]\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

$$= [1\,0\,0]$$

$$\Rightarrow \quad x_1 = 1, x_2 = 0, x_3 = 0$$

$$e(101) = 101\, x_1\, x_2\, x_3$$

$$[x_1\, x_2\, x_3] = [1\,0\,1]\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

$$= [0\,1\,1]$$

$$\Rightarrow \quad x_1 = 0, x_2 = 1, x_3 = 1$$

$$e(110) = 110\, x_1\, x_2\, x_3$$

$$[x_1\, x_2\, x_3] = [1\,1\,0]\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

$$= [1\,1\,1]$$

$$\Rightarrow x_1 = x_2 = x_3 = 1$$

$$e(111) = 111\, x_1\, x_2\, x_3$$

$$[x_1\, x_2\, x_3] = [1\,1\,1]\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

$$= [0\,0\,0]$$

$$\Rightarrow \quad x_1 = x_2 = x_3 = 0$$

Hence e : $B^3 \to B^6$ is defined as

$e(000) = 000000$
$e(001) = 001111$
$e(010) = 010011$
$e(011) = 011100$
$e(100) = 100100$
$e(101) = 101011$
$e(110) = 110111$
$e(111) = 111000$

Decoding Function : (Maximum Likelihood Technique)

3. Let $H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

be a parity check matrix. Decode the following word related to maximum likelihood technique (Decoding function) associated with e_H. Decode the following.

(i) 10100 (ii) 01101 (iii) 11011

Soln. :

Here, $m = 2, n = 5$

$\therefore$ We have $B^2 = \{00, 01, 10, 11\}$

$e(00) = 00\, x_1\, x_2\, x_3$

$$[x_1\, x_2\, x_3] = [0\ 0]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = [0\,0\,0]$$

$\Rightarrow x_1 = x_2 = x_3 = 0$

$e(01) = 01\, x_1\, x_2\, x_3$

$$[x_1\, x_2\, x_3] = [0\ 1]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = [1\,0\,1]$$

$\Rightarrow x_1 = 1, x_2 = 0, x_3 = 1$

$e(10) = 10\, x_1\, x_2\, x_3$

$$[x_1\, x_2\, x_3] = [1\ 0]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = [0\,1\,1]$$

$\Rightarrow x_1 = 0, x_2 = 1, x_3 = 1$

$e(11) = 11\, x_1\, x_2\, x_3$

$$[x_1\, x_2\, x_3] = [1\ 1]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = [1\,1\,0]$$

$\Rightarrow x_1 = 1, x_2 = 1, x_3 = 0$

Hence, $e_H : B^2 \to B^5$ is defined as

$e(00) = 00000 = x_0$
$e(01) = 01101 = x_1$
$e(10) = 10011 = x_2$
$e(11) = 11110 = x_3$

i) let $x_t = 10100$

$|x_0 \oplus x_t| = |x_t| = 2$

$|x_1 \oplus x_t| = |11001| = 3$

$|x_2 \oplus x_t| = |00111| = 3$

$|x_3 \oplus x_t| = |01010| = 2$

$\Rightarrow$ Minimum distance is not unique.

Note :
If minimum distance for x_t is not unique, then we see on priority basis which one comes first.

The required nearer word to x_t
$d(x_t) = d(x_0) = 00$

$\therefore$ decode word for 10100 is 00

ii) $x_t = 01101$

$|x_0 \oplus x_t| = |x_t| = 3$

$|x_1 \oplus x_t| = |00000| = 0$

$|x_2 \oplus x_t| = |11110| = 4$

$|x_3 \oplus x_t| = |10011| = 3$

$d(x_t) = d(x_1) = 01$

ikelihood

$\therefore$ decode word for 01101 is 01.

iii) $11011 = x_t$

$|x_0 \oplus x_t| = |x_t| = 4$

$|x_1 \oplus x_t| = |10110| = 3$

$|x_2 \oplus x_t| = |01000| = 1$

$|x_3 \oplus x_t| = |00101| = 2$

$d(x_t) = d(x_2) = 10$

$\therefore$ decode word for 11011 is 10.

Graded Questions

1. (a) Define
   (i) Weight (ii) Hamming distance
   (iii) Minimum distance **[M-04]** (iv) Group code **[D-05]**
   (b) What is the HAMMING DISTANCE between two binary words x and y ? How is error detection done with Group Code ? **[D-00]**
   (c) Define the Hamming Distance of a code. How will you find out error correcting and detecting capability of the code ? **[N-04]**
2. Consider the (2, 4) encoding function e. How many errors will e detect ?
   e (00) = 0000
   e (01) = 1011
   e (10) = 0110
   e (11) = 1100.
3. Consider (3, 9) encoding function e **[D-02]**
   e(000) = 000000000
   e(001) = 011100101
   e(010) = 010101000
   e(011) = 110010001
   e(100) = 010011010
   e(101) = 111101011
   e(110) = 001011000
   e(111) = 110000111
   (i) Find the minimum distance.
   (ii) How many errors will e detect ?
4. Find minimum distance of the (3, 8) encoding function e shown below : **[M-05]**
   e(000) = 00000000
   e(001) = 01110010
   e(010) = 10011100

e(011) = 01110001
e(100) = 01100101
e(101) = 10110000
e(110) = 11110000
e(111) = 00001111

how many errors will e detects ?

5. Consider the (3, 6) encoding function e as follows :–

e (000) = 000000	e (100) = 100101
e (001) = 000110	e (101) = 100011
e (010) = 010010	e (110) = 110111
e (011) = 010100	e (111) = 110001.

(i) Show that the encoding function e is a group code.

(ii) Decode the following words with maximum likelihood technique :
101101, 011011.

6. Let $H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be a parity check matrix. Decode the following words relative to a maximum likelihood decoding function associated with c_H

(i) 110010 (ii) 101001 (iii) 111010

7. Let $H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. **[M-05]**

be a parity check matrix. Decode the following words relative to maximum likelihood decoding function associated with eH.

(1) 011001 (2) 101011 (3) 111010.

8. Let

$$H = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

be a parity check matrix. Decode the following words relative to maximum likelihood decoding function. **[M-02, D-03]**

(i) 0101 (ii) 1010 (iii)1101

9. Consider the parity check matrix H given by–

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) Determine the group code $e_H : B^2 \rightarrow B^5$

(ii) Decode the following words relative to a maximum likelihood decoding function associated with e_H . 01110, 11101, 00001, 11000.

10. Consider the (3, 5) group encoding function $e : B^3 \rightarrow B^5$ defined by – **[M-01]**

e(000) = 00000	e(100) = 10011
e(001) = 00110	e(101) = 10101
e(010) = 01001	e(110) = 11010
e(011) = 01111	e(111) = 11100

Decode the following words relative to a maximum likelihood decoding function

(i) 11001

(ii) 01010

(iii)00111.

11. (a) Let $H = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ be a parity check matrix. Decode the following words

(i) 10100 (ii) 11011. **[D-01]**

(b) Prove that, for a group G, identity element is unique.

12. Decode the following words relative to a maximum likelihood decoding function – **[D-01]**

(i) 011110 (ii) 110010

if (3, 6) group encoding function $e : B^3 \rightarrow B^6$ defined by –

e(000) = 000000
e(001) = 000110
e(010) = 010010
e(011) = 010100
e(100) = 100101
e(101) = 100011
e(110) = 110111
e(111) = 110001

13. Consider the encoding function:

E : $B^2 \rightarrow B^6$ defined as follows:

E(00) = 001000
E(01) = 010100
E(10) = 100010
E(11) = 110001

i) How many errors can this code detect and correct?

ii) If the received word is 000000. What is most likely to be the transmitted word ?

14. Consider (3, 8) encoding function $e : B^3 \rightarrow B^8$ defined by **[M-03]**

e(000) = 00000000	e(100) = 10100100
e(001) = 10111000	e(101) = 10001001
e(010) = 00101101	e(110) = 00011100
e(011) = 10010101	e(111) = 00110001

and let d be (8, 3) maximum likelihood decoding function associated with e. How many errors can (e, d) correct ?

❑❑❑❑❑❑

Type I

1. Determine the sequence a_n whose recurrence relation is $a_n = a_{n-1} + 3$ with initial condition $a_1 = 2$.

Soln. :

This is linear, non–homogenous recurrence relation.

$\therefore$ We use back tracking method

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \qquad \text{with } a_1 = 2 \\
a_n &= (a_{n-2}+3)+3 \\
&= a_{n-2}+3+3 \\
&= (a_{n-3}+3)+3+3 \\
&= a_{n-3}+3+3+3 \\
& - \; - \; - \; - \; - \; - \\
&= a_{n-(n-1)}+3+3+3+\ldots \text{ to } (n-1) \\
&= a_1 + 3(n-1) \\
&= 3n-1
\end{aligned}
$$

$\therefore$ Sequence is 2, 5, 8, 11, …

2. Determine the sequence b_n whose recurrence relation is $b_n = 2b_{n-1} + 1$ with initial condition $b_1 = 7$.

Soln. :

The recurrence relation is linear & non–homogenous

$\therefore$ We use back tracking method

$$
\begin{aligned}
b_n &= 2b_{n-1} + 1, \quad b_1 = 7 \\
b_n &= 2(2b_{n-2} + 1) + 1 \\
&= 2^2 b_{n-2} + 2 + 1 \\
&= 2^2(2b_{n-3}+1)+2+1 \\
&= 2^3 b_{n-3} + 2^2 + 2 + 1 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= 2^{n-1} b_{n-(n-1)} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \\
&= 2^{n-1} \cdot b_1 + (1 + 2 + 2^2 + \ldots + 2^{n-2}) \\
&= 2^{n-1} \cdot 7 + \frac{1 \cdot (2^{n-1}-1)}{2-1} \\
&= 7 \cdot 2^{n-1} + 2^{n-1} - 1 \\
b_n &= 8 \cdot 2^{n-1} - 1 = 2^{n+2} - 1
\end{aligned}
$$

$\therefore$ The sequence is 7, 15, 31, 63, …

Type II

Homogenous, Linear Recurrence Relation

Note :

(i) If the quadratic equation $ax^2 + bx + c = 0$ has distinct roots, s_1 & s_2, then the formula for the sequence is given by $a_n = us_1^n + vs_2^n$ where u and v are constants depending on initial conditions.

(ii) If the quadratic equation $ax^2 + bx + c = 0$ has equal roots each = s, then the formula for the sequence is given by $a_n = us^n + vns^n$ where u and v are constants depending on initial conditions.

3. Determine the sequence whose recurrence relation is given by $C_n = 3C_{n-1} - 2C_{n-2}$ with initial conditions $C_1 = 5$, $C_2 = 3$.

Soln. :

The quadratic equation associated with this recurrence relation is $x^2 = 3x - 2$ with every linear homogenous recurrence relation there is associated a quadratic equation.

$$\therefore \quad C_n = 2C_{n-1} - 2C_{n-2}$$

$$\therefore \quad x^2 = 3x - 2$$

$$\therefore \quad x^2 - 3x + 2 = 0$$

$$\therefore \quad (x-1)(x-2) = 0$$

$$\Rightarrow x = 1, 2$$

$$\therefore \quad C_n = u(1)^n + v(2)^n$$

$$\therefore \quad C_n = u + v(2)^n$$

Put $n = 1$

$$\therefore \quad C_1 = u + 2v$$

$$\therefore \quad u + 2v = 5 \qquad \ldots (1)$$

Put $n = 2$

$$\therefore \quad C_2 = u + 4v$$

$$\therefore \quad u + 4v = 3 \qquad \ldots (2)$$

$$\therefore \quad v = -1 \;\; \& \;\; u = 7$$

$$\therefore \quad C_n = 7 - (2)^n$$

$\therefore$ Sequence is 5, 3, –1, –9, ...

4. Determine the sequence whose recurrence relation is $a_n = 2a_{n-1} - a_{n-2}$ with initial conditions $a_1 = 1.5$, $a_2 = 3$.

Soln. :

$a_n = 2a_{n-1} - a_{n-2}$ & the associated quadratic equation is $x^2 = 2x - 1$

$$\therefore \quad x^2 - 2x + 1 = 0$$

$$\therefore \quad (x-1)^2 = 0$$

$$\therefore \quad x = 1, 1$$

$\therefore$ Formula for sequence is $a_n = u(1)^n + vn(1)^n$

$$\therefore \quad a_n = u + vn$$

Put $n = 1$

$$\therefore \quad a_1 = u + v \quad \Rightarrow u + v = 1.5 \qquad \ldots (1)$$

Put $n = 2$

$$\therefore \quad a_2 = u + 2v \quad \Rightarrow u + 2v = 3 \qquad \ldots (2)$$

$$\therefore \quad v = 1.5 \;\; \& \;\; u = 0$$

$$\therefore \quad a_n = (1.5)\, n$$

The sequence is 1.5, 3, 4.5, 6, ...

5. Determine the sequence whose recurrence relation is $a_n = 4a_{n-1} + 5a_{n-2}$ with $a_1 = 2$ & $a_2 = 6$.

Soln. :

$$x^2 = 4x + 5 \quad \Rightarrow x^2 - 4x - 5 = 0$$

$$x^2 - 5x + x - 5 = 0$$

$$x(x-5) + 1(x-5) = 0 \qquad (x+1)(x-5) = 0$$

$$x = -1 \;\&\; x = 5$$

$$\therefore \quad a_n = u(-1)^n + v(5)^n$$

Put $n = 1$

$$\therefore \quad a_1 = -u + 5v \quad \Rightarrow -u + 5v = 2 \qquad \ldots (1)$$

Put n = 2

$$\therefore \quad a_2 = u + 25v \quad \Rightarrow u + 25v = 6 \qquad \dots (2)$$

$$30\,v = 8$$

$$\therefore \quad v = 4/15$$

$$\therefore \quad -u + 4/3 = 2$$

$$\therefore \quad u = \frac{4}{3} - 2 = -\frac{2}{3}$$

$$a_n = \left(-\frac{2}{3}\right)(-1)^n + \left(\frac{4}{15}\right)(5)^n$$

Fibonacci Sequence

1, 1, 2, 3, 5, 8, ...

$f_n = f_{n-1} + f_{n-2}$

The recurrence relation is given by $f_n = f_{n-1} + f_{n-2}$ which is homogenous & linear. The quadratic equation is $x^2 = x + 1$, $f_1 = f_2 = 1$

$$\therefore \quad x^2 - x - 1 = 0$$

$$\therefore \quad x = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\therefore \quad x = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore \quad x_1 = \frac{1+\sqrt{5}}{2}; \quad x_2 = \frac{1-\sqrt{5}}{2}$$

$$\therefore \quad f_n = us_1^n + vs_2^n$$

$$\therefore \quad f_n = u \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + v \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Put n = 1

$$\therefore \quad f_1 = u \cdot \left(\frac{1+\sqrt{5}}{2}\right) + v \cdot \left(\frac{1-\sqrt{5}}{2}\right)$$

$$\therefore \quad u \cdot \left(\frac{1+\sqrt{5}}{2}\right) + v \cdot \left(\frac{1-\sqrt{5}}{2}\right) = 1 \qquad \dots (1)$$

Put n = 2

$$\therefore \quad f_2 = u \cdot \left(\frac{1+\sqrt{5}}{2}\right)^2 + v \cdot \left(\frac{1-\sqrt{5}}{2}\right)^2$$

$$\therefore \quad u \cdot \left(\frac{1+\sqrt{5}}{2}\right)^2 + v \cdot \left(\frac{1-\sqrt{5}}{2}\right)^2 = 1 \qquad \dots (2)$$

Solving (1) & (2) simultaneous $u = \frac{1}{\sqrt{5}}$ & $v = -\frac{1}{\sqrt{5}}$

$$\therefore \quad f_n = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Graded Questions

1. What is the solution of the recurrence relation. **[M-04]**

   $$a_n = 6a_{n-1} - 9a_{n-2}$$

   with initial conditions $a_0 = 1$ and $a_1 = 6$?

2. Solve the recurrence relation **[D-05]**

   $d_n = 4\,(d_{n-1} - d_{n-2})$

   Subject to the initial conditions $d_0 = 1 = d_1$.

adratic

3. Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial condition $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$. **[M-03, 05]**
4. Determine whether the sequence $\{a_n\}$ is solution of recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$ where $a_n = 3n$ for every nonnegative integer n. Answer the same question for $a_n = 5$. **[M-03]**
5. Find the solution of $a_{n+2} + 2a_{n+1} - 3a_n = 0$ that satisfies $a_0 = 1$, $a_1 = 2$.
6. Find the solution of Fibonacci relation $a_n = a_{n-1} + a_{n-2}$ with the initial conditions $a_0 = 0$, $a_1 = 1$. **[D-02]**
7. Find the solution of recurrence relation. **[M-06]**
   $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$
8. Solve the recurrence relation **[D-05[**
   $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$
   with $a_0 = 5$, $A_1 = -9$, $a_2 = 15$
9. Find the solution to the recurrence relation **[D-03]**
   $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$
   with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.
10. Given that $a_0 = 0, a_1 = 1, a_2 = 4$ and $a_3 = 12$ satisfy the recurrence relation
    $a_r + C_1 a_{r-1} + C_2 a_{r-2} = 0$
    Determine a_r.
11. Solve the following recurrence relations :
    (i) $a_r - 7a_{r-1} + 10a_{r-2} = 0$, given that $a_0 = 0$ & $a_1 = 3$
    (ii) $a_r - 4a_{r-1} + 4a_{r-2} = 0$, given that $a_0 = 1$ & $a_1 = 6$
    (iii) $a_r + 6a_{r-1} + 9a_{r-2} = 3$, given that $a_0 = 0$ & $a_1 = 1$
    (iv) $a_r + a_{r-1} + a_{r-2} = 0$, given that $a_0 = 0$ & $a_1 = 2$
    (v) $a_r - 2a_{r-1} + 2a_{r-2} - a_{r-3} = 0$ given that $a_0 = 2, a_1 = 1$ & $a_2 = 1$
12. Find the particular solution of the following :
    (i) $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$ (ii) $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1$
    (iii) $a_r - 5a_{r-1} + 6a_{r-2} = 1$ (iv) $a_r + a_{r-1} = 3r\ 2^r$
    (v) $a_r - 2a_{r-1} = 3.2^r$ (vi) $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r$
13. Find the homogeneous solution of the following :
    (i) $a_r + 6a_{r-1} + 12a_{r-2} + 8a_{r-3} = 0$
    (ii) $4a_r - 20a_{r-1} + 17a_{r-2} - 4a_{r-3} = 0$

Generating Functions

Definition :

Let $a_0, a_1, a_2, a_3 \ldots$ be a sequence of real numbers, then the function.
$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$
is the generating function for the sequence. $\{a_n\}$.

Generating functions for a finite sequence $a_0, a_{n_1}, \ldots, a_n$ can also be defined by letting ai = 0 for i > n. Thus,, $g(x) = a_0 + a_1x + \ldots + a_nx^n$ is the generating function for the finite sequence $a_0, a_1, \ldots, a_n$

e.g.
$g(x) = 1 + 2x + 3x^2 + \ldots + (n+1)x^n + \ldots$
is the generating function for positive integers.

-04]

$g(x) = 1 + x + x^2 + x^3 + \ldots + x^{n-1}$ is the generating function for the sequence of n ones.

-05]

$\because\ g(x) = 1 + x + x^2 + x^3 + \ldots + x^{n-1} = \dfrac{x^n - 1}{x - 1}$

$\therefore g(x) = \dfrac{x^n - 1}{x - 1}$

Equality of Generating functions :

Two generating functions are said to be equal if $a_n = b_n$ for every $n \geq 0$, where

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n x^n$$

Addition and Multiplication

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ & $g(x) = \sum_{n=0}^{\infty} b_n x^n$

be two generating functions

Then, $f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$

and

$$f(x)g(x) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i} \right) x^n.$$

Uses

- To solve Linear Homogenous Recurrence Relations With Constant Co–efficients (LHRRWCC).
- To solve combinatorial problems.
- Abraham De Moivre, their inventor, used them to solve the Fibonacci recurrence relation.

Shifting Properties

1. If $G(x) = \sum_{n=0} a_n x^n$ generates (a_0, a_1, a_2 ......)

   then x G(x) generates (0, a_0, a_1 ......), x^2 G(x) generates (0, 0, a_0, a_1, ...), in general x^k G(x) generates (0, 0, ...0, a_0, a_1, a_2 ....) where there are K zeros before a_0.

2. If $G(x) = \sum_{n=0}^{\infty} a_n x^n$ generates (a_0, a_1, a_2......)

   then $G(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n$ generates (0, a_1, a_2......)

   then $G(x) - a_0 - a_1 x = \sum_{n=2}^{\infty} a_n x^n$ generates (0, 0, ...0, a_k, a_{k+1}......), where there are k zeros before a_k.

3. Dividing by powers of x shifts require to left. For instance, $\left(G(x) - a_0\right)/x = \sum_{n=1}^{\infty} a_n x^{n-1}$ generates sequence (a_1, a_2, a_3 ......); in general for $k \geq 1$, $(G(x) - a_0 - a_1 x \text{......} - a_{k-1} x^k) / x^k$ generates (a_k, a_{k+1}, a_{k+2}).

Solved Examples :

1. Find generating function for sequence 1, a, a^2 ...... where a is a fixed constant.

Soln. :

Let $G(x) = 1 + ax + a^2x^2 + a^3x^3 + \text{......}$

So, $G(x) - 1 = ax + a^2x^2 + a^3x^3 + \text{......}$

$$\therefore \frac{G(x)-1}{ax} = 1 + ax + a^2x^2 + \text{......}$$

$$\therefore \frac{G(x)-1}{ax} = G(x)$$

$\Rightarrow \; G(x) = \dfrac{1}{1-ax}$

$\therefore$ required generating function is $\dfrac{1}{1-ax}$

2. Use generating function $a_n = 3a_{n-1} + 2, \quad a_0 = 1$

Soln. :

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ where G(x) is general function for sequence $\{a_n\}$.

Multiply each term by x^n, and summing from 1 to ∞,

$$\sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1} x^n + 2\sum_{n=1}^{\infty} x^n$$

$$G(x) - a_0 = 3x\,G(x) + 2\left[\frac{1}{1-x} - 1\right]$$

$$\left(\because x\,G(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n\right)$$

$$\therefore G(x) = 3x\,G(x) = 1 + \frac{2x}{1-x} \quad (\because a_0 = 1)$$

$$\therefore G(x) = \frac{1+x}{(1-x)(1-3x)}$$

$$= \frac{2}{1-3x} - \frac{1}{1-x} \qquad \text{......by partial fractions}$$

$$\therefore \sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} x^n$$

Hence $a_n = 2.3^n - 1$

3. Find formula for sequences with following first five terms. **[M-04]**

(a) 1, 1/2, 1/4, 1/8, 1/16

(b) 1, 3, 5, 7, 9

(c) 1, −1, 1, −1, 1

Soln. :

To find formula,

(a) $\dfrac{1}{1}, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}$

$$= \frac{1}{2^0}, \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}$$

$$= \sum_{n=0}^{4} \frac{1}{2^n}$$

(b) 1, 3, 5, 7, 9

This is arithmetic progression,

Here a = 1, d = 2

Now, $t_n = a + (n-1)d$

$= 1 + (n-1)\,2 = 2n - 1$

$$\therefore t_n = \sum_{n=1}^{5} (2n-1)$$

(c) 1, −1, 1, −1, 1

alternate positive, negative term exist,

$$\therefore t = \sum_{r=0}^{4} (-1)^r.$$

4. Let G(x) be generating function for sequence $\{a_k\}$. What is generating function for following sequence ? **[M-03]**
   (i) 0, 0, 0, a_3, a_4, a_5 ......
   (ii) 0, 0, 0, 0, a_0, a_1, a_2 ......

Soln. :

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

This generates a_0, a_1, a_2 ......

Now, $x\,G(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$ ......multiply 'x' both sides.

This generates 0, a_0, a_1, a_2 ......

also $x^2 G(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$ ......multiply 'x' both sides

this generates 0, 0, a_0, a_1, a_2 ......

$\therefore$ generating functions for

0, 0, 0, a_3, a_4, a_5 ......

is

$G(x) - a_0 - a_1 x - a_2 x^2 = \sum_{n=3}^{\infty} a_n x^n$

This generates 0, 0, 0, a_3, a_4, a_5 ......

also,

$G(x) - a_1 x - a_2 x^2 - a_3 x^3 = \sum_{n=4}^{\infty} a_n x^n$

this generates 0, 0, 0, 0, a_0, a_1, a_2, a_3 ......

Graded Questions

1. What are the generating functions for the following sequences ? **[D-03, M-05]**
   i) 1, 1, 1, 1, 1, 1
   ii) 1, 1, 1, 1, ...

2. Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are
   1, 7, 25, 79, 241, 727, 2185, 65559, 19687, 59047 **[M-06]**

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Vidyalankar Institute of Technology

S.E. [INFT /CMPN] : Discrete Structures – May '06

Time : 3Hrs.] **[Marks : 100**

N.B. : (1) Question number 1 is **compulsory**.
(2) Attempt any **four** out of remaining **six** questions.
(3) Assumptions made should be clearly stated.
(4) **Figures** to the **right** indicate **full** marks.
(5) Assume suitable data wherever required but justify the same.

1. (a) (i) Let Q be the set of positive rational numbers which can be expressed in the form $2^a 3^b$. where a and b are integers. Prove that algebraic structure (Q.) is a group. Where . is multiplication operation. **[6]**

(ii) Prove the following (use laws of set theory) **[6]**
$(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \bar{D}))] = B \cap (A \cup C)$

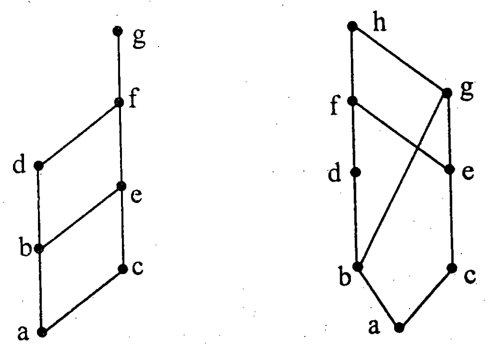
(b) (i) Let G be the group and let a and b are elements of G, then verify that $(ab)^{-1} = b^{-1}a^{-1}$ **[4]**

(ii) Let R be the relation represented by the matrix **[4]**

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find the matrix that represents R^4

2. (a) (i) Determine whether the poset with the following Hasse diagrams are lattices or not. **[6]**
Justify your answer.



(ii) Use induction to prove that **[6]**
$7^n - 1$ is divisible by 6 for n = 1, 2, 3 .....

(b) (i) Let A = {1, 2, 3, 4} for the relation R whose matrix is given below **[4]**
Find the matrix of transitive closure using Warshall algorithm.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Let R be the relation on set of real numbers such that aRb if and only if a –b is an integer. Prove that R is an equivalence relation. **[4]**

3. (a) (i) Find the solution of recurrence relation. **[6]**
$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

(ii) Suppose R and S is the relation from A to B, then prove that **[6]**
$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ and $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

(b) (i) $f : R \to R$ is defined as $f(x) = x^3$ **[4]**
$g : R \to R$ is defined as $h(x) = 7x - 1$
find the rule of defining $(h \circ g) \circ f$, $g \circ (h \circ f)$

(ii) Consider the chains of divisors of 4 and 9 i.e., $L_1 = \{1, 2, 4\}$ and $L_2 = \{1, 3, 9\}$ and partial ordering relation of division on L_1 and L_2. Draw the lattice $L_1 \times L_2$. **[4]**

4. (a) (i) Show that

$((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R)) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge$

is tautology. (use laws of logic)

(ii) Prove that if (F, +, .) is a field then it is an integral domain.

(b) (i) Show that among any group of five (not necessarily consecutive) integers, the two with the same remainder when divided by 4.

(ii) Define Eulerian, Hamilton path and circuit with example. What is the necessary and sufficient condition for euler path and circuit ? **[4]**

5. (a) (i) In a survey of 60 people, it was found that **[6]**

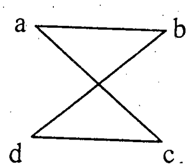
25 read Business India.
26 reads India Today.
26 reads Times of India.
11 read both Business India and India Today.
09 read both Business India and times of India.
08 read both India Today and Times of India.
08 read none of the three.

i) How many read all three ?
ii) How many read exactly one ?

(ii) Prove that the set {1, 2, 3, 4, 5, 6} is group under multiplication modulo 7 **[6]**

(b) (i) Define universal and existential quantifier with suitable example. **[4]**

(ii) How many paths of length 4 are there from a to d in simple graph shown below. **[4]**



6. (a) (i) Draw the Hasse diagram for divisibility on the set **[6]**

i) {1, 2, 3, 4, 5, 6, 7, 8} ii) {1,2, 3, 5, 7, 11, 13}

(ii) Consider (3, 8) encoding function e: $B^3 \to B^8$ defined by **[6]**

e(000) = 00000000	e(100) = 10100100
e(001) = 10111000	e(101) = 10001001
e(010) = 00101101	e(110) = 00011100
e(011) = 10010101	e(111) = 00110001

and let d be the (8, 3) maximum likelihood decoding function associated with e. How many errors can (e, d) correct ?

(b) (i) Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 65559, 19687, 59047 **[4]**

(ii) Prove that in any ring (R+.) the additive inverse of each ring element is unique. **[4]**

7. (a) (i) Find the complement of each element in D_{20} and D_{30}. **[6]**

(ii) Let G be the group of integers under the operation addition, and H be group of all even integers under the operation of addition, show that the function f : G → H is an isomorphism. **[6]**

(b) (i) A connected planar graph has a 9 vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4, 5. How many edges are there ? **[4]**

(ii) Define with example Reflexive closure and symmetric closure. **[4]**

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