Generalised Rule

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$$

Module 1:- LAPLACE TRANSFORM

Def^n:- If $f(t)$ be a function defined for $t>0$, then the definite integration $\int_0^\infty e^{-st} f(t)\,dt$ is called laplace transform of $f(t)$, ~~then the~~ and is denoted by $L\{f(t)\}$ or $\phi(s)$

$$L\{f(t)\} = \int_0^\infty \underbrace{e^{-st}}_{\text{Kernel}} f(t)\,dt = \phi(s)$$

parameter

Q Find the Laplace Transform of $f(t)$

1) where $f(t) = t$, $0<t<4$ & $f(t)=5$, $t>4$

→ By def^n

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^4 e^{-st}\, t\,dt + \int_4^\infty e^{-st}\, 5\,dt$$

$$= \left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^4 + 5\left[\frac{e^{-st}}{-s}\right]_4^\infty$$

$$= \frac{4}{-s}e^{-4s} - \frac{e^{-4s}}{s^2} - 0 + \frac{1}{s^2} + 5\left(0 + \frac{e^{-4s}}{s}\right)$$

$$L\{f(t)\} = \frac{e^{-4s}}{s} + \frac{1}{s^2}\left(1 - e^{-4s}\right)$$

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2) Where $f(t) = \cos t$ for $0 < t < \pi$ &
$f(t) = \sin t$ for $t > \pi$

→ By def^n

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} \cos t\, dt + \int_\pi^\infty e^{-st} \sin t\, dt$$

formula

$$\int e^{ax} \sin(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$$

$$\int e^{ax} \cos(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c) + b\sin(bx+c)\right]$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\cos t + 1.\sin t)\right]_0^\pi + \left[\frac{e^{-st}}{s^2+1}(-s\sin t - 1.\cos t)\right]_\pi^\infty$$

$$= \frac{e^{-\pi s}}{s^2+1}(s+0) - \frac{1}{s^2+1}(-s+0) + \left[0 - \frac{e^{-\pi s}}{s^2+1}(0+1)\right]$$

$$= \cancel{\frac{e^{-\pi s}}{s^2+1}\left[s\right.}$$

$$= \frac{1}{s^2+1}\left[s e^{-\pi s} + s - e^{-\pi s}\right.$$

$$\Gamma n = \int_0^\infty e^{-t} \cdot t^{n-1} \cdot dx$$

* Linearity Property

If $f_1(t), f_2(t), \ldots\ldots, f_n(t)$ be functions, then

$$L\{c_1 f_1(t) \pm c_2 f_2(t) \pm \ldots \pm c_n f_n(t)\}$$

$$= c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\} + \ldots + c_n L\{f_n(t)\}$$

where $c_1, c_2, \ldots, c_n$ are constants

Inverse Laplace Transform

If $L\{f(t)\} = \phi(s)$, then $f(t)$ is called inverse laplace Transform of $\phi(s)$ and denoted by $L^{-1}\{\phi(s)\}$

Thus, $\boxed{f(t) = L^{-1}\{\phi(s)\}}$

Std. results :- (Use def^n to find)

L.T.	I.L.T.
1) $L\{1\} = \frac{1}{s}$	1) $L^{-1}\{\frac{1}{s}\} = 1$
2) $L\{e^{at}\} = \frac{1}{s-a}$	2) $L^{-1}\{\frac{1}{s-a}\} = e^{at}$
3) $L\{t^n\} = \frac{\Gamma n+1}{s^{n+1}}$ or $\frac{n!}{s^{n+1}}$	3) $L\{\frac{1}{s^n}\} = \frac{t^{n-1}}{(n-1)!}$ or $\frac{t^{n-1}}{\Gamma n}$

$$L\{t^{n-1}\} = \frac{\Gamma n}{s^n} \text{ or } \frac{(n-1)!}{s^n}$$

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4) $L(\sin at) = \frac{a}{s^2+a^2}$ 4) $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$

5) $L(\cos at) = \frac{s}{s^2+a^2}$ 5) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

6) $L(\sinh at) = \frac{a}{s^2-a^2}$ 6) $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{\sinh at}{a}$

7) $L(\cosh at) = \frac{s}{s^2-a^2}$ 7) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

Q) Find the Laplace Transform of the following functions.

1) $\cos t \cos 2t . \cos 3t$

$\rightarrow L\{\cos t . \cos 2t . \cos 3t\}$

$= \frac{1}{2} L\left\{\left(\frac{\cos 3t + \cos t}{2}\right) . \cos 3t\right\}$

$= \frac{1}{2} L\{\cos^2 3t + \cos t . \cos 3t\}$

$= \frac{1}{2} L\left\{\frac{1+\cos 6t}{2} + \frac{\cos 4t + \cos 2t}{2}\right\}$

$= \frac{1}{4} L\{1 + \cos 6t + \cos 4t + \cos 2t\}$

$= \frac{1}{4}\left\{\frac{1}{s} + \frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4}\right\}$

HW:- $\sinh^5 t$

2) $\sin^5 t$

$\sin t = \frac{e^{it} - e^{-it}}{2i}$

$\sin^5 t = \frac{(e^{it} - e^{-it})^5}{2^5 i^5}$

$= \frac{1}{32i}\left[e^{it} - e^{-it}\right]^5$

$= \frac{1}{32i}\left[e^{5it} - 5e^{4it} . e^{-it} + 10 e^{3it} . e^{-2it} - 10 e^{2it} . e^{-3it} + 5 e^{it} . e^{-4it} - e^{-5it}\right]$

$= \frac{1}{32i}\left[(e^{5it} - e^{-5it}) - 5(e^{3it} - e^{-3it}) + 10(e^{it} - e^{-it})\right]$

$= \frac{1}{16}\left[\left(\frac{e^{5it} - e^{-5it}}{2i}\right) - 5\left(\frac{e^{3it} - e^{-3it}}{2i}\right) + 10\left(\frac{e^{it} - e^{-it}}{2i}\right)\right]$

$\sin^5 t = \frac{1}{16}\{\sin 5t - 5\sin 3t + 10 \sin t\}$.

$L\{\sin^5 t\} = L\left\{\frac{1}{16}(\sin 5t - 5\sin 3t + 10 \sin t)\right\}$

$= \frac{1}{16}\left\{\frac{5}{s^2+25} - 5 \times \frac{3}{s^2+9} + 10 \times \frac{1}{s^2+1}\right\}$

$= \frac{5}{16}\left\{\frac{1}{s^2+25} - \frac{3}{s^2+9} + \frac{2}{s^2+1}\right\}$

$= \frac{5}{16}\left\{\frac{-2s^2-66}{(s^2+25)(s^2+9)} + \frac{2}{s^2+1}\right\}$

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$$= \frac{5}{8}\left\{\frac{-s^2-33}{(s^2+25)(s^2+9)} + \frac{1}{(s^2+1)}\right\}$$

$$= \frac{5}{8}\left\{\frac{-s^4-34s^2-33+s^4+34s^2+225}{(s^2+25)(s^2+9)(s^2+1)}\right\}$$

$$= \frac{5}{8} \times \frac{192}{(s^2+25)(s^2+9)(s^2+1)}$$

$$= \frac{5}{(s^2+25)(s^2+9)(s^2+1)}$$

3) $\sqrt{1+\sin t}$

$\rightarrow L\{\sqrt{1+\sin t}\}$

$$= L\left\{\sqrt{\cos^2\frac{t}{2} + \sin^2\frac{t}{2} + 2\sin\frac{t}{2}\cos\frac{t}{2}}\right\}$$

$$= L\left\{\cos\frac{t}{2} + \sin\frac{t}{2}\right\}$$

$$= \frac{s}{s^2+\frac{1}{4}} + \frac{1/2}{s^2+\frac{1}{4}}$$

$$= \frac{\left(s+\frac{1}{2}\right)}{s^2+\frac{1}{4}} = \frac{\frac{2s+1}{2}}{\frac{4s^2+1}{4}} = \frac{2(2s+1)}{4s^2+1}$$

Properties

If $L\{f(t)\} = \phi(s)$

1) First Shifting Property

then $L\{e^{at} f(t)\} = \phi(s-a)$

2) Change of Scale Property

then $L\{f(at)\} = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

3) Effect of Multiplication Property

then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\phi(s)$

4) Effect of Division

then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \phi(s)\, ds$

5) Laplace of Integral

then $L\left\{\int_0^t f(u)\, du\right\} = \frac{1}{s}\phi(s)$

★ 6) Laplace Transform of Derivative

$L\{f'(t)\} = s\,\phi(s) - f(0)$

$L\{f''(t)\} = s^2\,\phi(s) - s f(0) - f'(0)$

$L\{f'''(t)\} = s^3\,\phi(s) - s^2 f(0) - s f'(0) - f''(0)$

$L\{f^{iv}(t)\} = s^4\,\phi(s) - s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0)$

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HW $\sinh^5 t$

$$\sinh t = \frac{e^{t} - e^{-t}}{2i}$$

$$\sinh^5 t = \left(\frac{e^{t} - e^{-t}}{2i}\right)^5$$

$$= \frac{1}{32i}\left[e^{5t} - 5e^{4t}\cdot e^{-t} + 10e^{3t}\cdot e^{-2t} - 10e^{2t}\cdot e^{-3t} + 5e^{t}\cdot e^{-4t} - e^{-5t}\right]$$

$$= \frac{1}{32i}\left[(e^{5t} - e^{-5t}) - 5(e^{3t} - e^{-3t}) + 10(e^{t} - e^{-t})\right]$$

$$= \frac{1}{16}\left[\frac{(e^{5t} - e^{-5t})}{2i} - 5\frac{(e^{3t} - e^{-3t})}{2i} + 10\left(\frac{e^{t} - e^{-t}}{2i}\right)\right]$$

$$= \frac{1}{16}\left[\sinh 5t - 5\sinh 3t + 10\sinh t\right]$$

$$L\{\sinh^5 t\} = \frac{1}{16}L\{\sinh 5t - 5\sinh 3t + 10\sinh t\}$$

$$= \frac{1}{16}\left\{\frac{5}{s^2-25} - \frac{5\times 3}{s^2-9} + \frac{10}{s^2-1}\right\}$$

$$= \frac{5}{16}\left\{\frac{1}{s^2-25} - \frac{3}{s^2-9} + \frac{2}{s^2-1}\right\}$$

Find the Laplace Transform of

1) $\frac{\cos 2t \sin t}{e^{t}}$

$$= L\left\{\frac{\cos 2t \sin t}{e^{t}}\right\}$$

$$= L\{e^{-t}\cos 2t \sin t\}$$

Now,

$$L\{\cos 2t \sin t\} = L\left\{\frac{\sin 3t - \sin t}{2}\right\}$$

$$= \frac{1}{2}L\{\sin 3t - \sin t\}$$

$$= \frac{1}{2}\left\{\frac{3}{s^2+9} - \frac{1}{s^2+1}\right\}$$

Applying First Shifting property

$$L\{e^{-t}\cos 2t \sin t\} = \frac{1}{2}\left\{\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right\}$$

2) $e^{-2t}\cosh t \sin t$

$$L\{e^{-2t}\cosh t \sin t\}$$

$$= L\left\{e^{-2t}\left(\frac{e^{t}+e^{-t}}{2}\right)\sin t\right\}$$

$$= \frac{1}{2}L\{(e^{-t} + e^{-3t})\sin t\}$$

$$= \frac{1}{2}L\{e^{-t}\sin t\} + \frac{1}{2}L\{e^{-3t}\sin t\}$$

$$L(\sin t) = \frac{1}{s^2+1}$$ ~~and~~

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Applying First shifting prop.

$$L\{e^{-2t}\cosh t \sin t\} = \frac{1}{2}\times\frac{1}{(s+1)^2+1} + \frac{1}{2}\times\frac{1}{(s+3)^2+1}$$

$$= \frac{1}{2}\left\{\frac{1}{s^2+2s+2} + \frac{1}{s^2+6s+10}\right\}$$

$$= \frac{1}{2}\left\{\frac{2s^2+8s+12}{(s^2+2s+2)(s^2+6s+10)}\right\}$$

$$= \left\{\frac{s^2+4s+6}{(s^2+2s+2)(s^2+6s+10)}\right\}$$

3) $te^{-t}\sin 2t \cos t$

$$L\{te^{-t}\sin 2t \cos t\}$$

Now,

$$L\{\sin 2t \cos t\} = L\left\{\frac{\sin 3t + \sin t}{2}\right\}$$

$$= \frac{1}{2}\left\{\frac{3}{s^2+9} + \frac{1}{s^2+1}\right\}$$

$$L\{t \sin 2t \cos t\} = (-1)^1 \frac{d}{ds}\left\{\frac{1}{2}\left(\frac{3}{s^2+9} + \frac{1}{s^2+1}\right)\right\}$$

$$= \frac{-1}{2}\left[\frac{-3(2s)}{(s^2+9)^2} - \frac{2s}{(s^2+1)^2}\right]$$

$$= \frac{2}{2}\left[\frac{3s}{(s^2+9)^2} + \frac{s}{(s^2+1)^2}\right]$$

$$L\{t \sin 2t \cos t\} = \frac{3s}{(s^2+9)^2} + \frac{s}{(s^2+1)^2}$$

Now,

$$L\{e^{-t}\, t \sin 2t \cos t\} = \frac{3(s-1)}{((s-1)^2+9)^2} + \frac{(s-1)}{[(s-1)^2+1]^2}$$

4) If $L\{f(t)\} = \frac{8(s-3)}{(s^2-6s+25)^2}$, find $L\{f(2t)\}$

By Change of Scale Property

$$L\{f(2t)\} = \frac{1}{2}\times\frac{8\left[\frac{s}{2}-3\right]}{\left[\left(\frac{s}{2}\right)^2 - 6\left(\frac{s}{2}\right)+25\right]^2}$$

$$= 4\frac{(s/2-3)}{[(s/2)^2-3s+25]^2}$$

$$= \frac{32(s-6)}{(s^2-12s+100)^2}$$

5) If $L\{f(t)\} = \frac{1}{s(s^2+1)}$, find $L\{e^{-t} f(2t)\}$

By change of scale prop

$$L\{f(2t)\} = \frac{1}{2}\times\frac{1}{\frac{s}{2}\left[\left(\frac{s}{2}\right)^2+1\right]}$$

$$= \frac{1}{s}\times\frac{1}{\frac{s^2+4}{4}} = \frac{4}{s(s^2+4)}$$

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$$L\{e^{-t} f(2t)\} = \frac{4}{(s+1)\{(s+1)^2+4\}} = \frac{4}{(s+1)[s^2+2s+5]}$$

6) $t.e^{-4t} \sin 3t$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{t \sin 3t\} = (-1)^1 \frac{d}{ds} \frac{3}{s^2+9}$$

$$= (-1)(3) \frac{-2s}{(s^2+9)^2}$$

$$L\{t.\sin 3t\} = \frac{6s}{(s^2+9)^2}$$

$$L\{e^{-4t}\, t.\sin 3t\} = \frac{6(s+4)}{[(s+4)^2+9]^2}$$

$$= \frac{6(s+4)}{(s^2+8s+25)^2}$$

7) $t \sin^3 t$

$$L\{\sin^3 t\} = L\left\{\frac{3\sin\theta - \sin 3\theta}{4}\right\}$$

$$= \frac{1}{4}\left\{\frac{3}{s^2+1} - \frac{3}{s^2+9}\right\}$$

$$= \frac{3}{4}\left\{\frac{1}{s^2+1} - \frac{1}{s^2+9}\right\}$$

$$\therefore L\{t.\sin^3 t\} = (-1)^1 \frac{d}{ds}\left\{\frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]\right\}$$

$$= (-1)\left(\frac{3}{4}\right)\left[\frac{2s}{(s^2+1)^2} - \frac{2s}{(s^2+9)^2}\right]$$

$$= -\frac{6s}{4}\left[\frac{-1}{(s^2+1)^2} + \frac{1}{(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{-1}{(s^2+9)^2} + \frac{1}{(s^2+1)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{1}{(s^2+1)^2} - \frac{1}{(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{1}{s^4+2s^2+1} - \frac{1}{s^2+18s^2+81}\right]$$

$$= \frac{3s}{2}\left[\frac{16s^2+80}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= 24s\left[\frac{s^2+5}{(s^2+1)^2(s^2+9)^2}\right]$$

8) If $L\{\operatorname{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s^2+1}}$, find $L\{t \operatorname{erf} 2\sqrt{t}\}$

$$L\{\operatorname{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}$$

$$L\{\operatorname{erf}\sqrt{4t}\} = \frac{1}{4}\phi\left(\frac{s}{4}\right) = \frac{1}{4} \times \frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}}$$

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$$= \frac{1}{s} \times \frac{2}{\sqrt{s+4}} = \frac{2}{s\sqrt{s+4}}$$

Now,

$$L\{t \operatorname{erf} 2\sqrt{t}\} = (-1)^1 \frac{d}{ds}\left(\frac{2}{s\sqrt{s+4}}\right)$$

$$= -2 \frac{d}{ds} \frac{1}{(s^3+4s^2)^{1/2}}$$

$$= -2 \frac{d}{ds} (s^3+4s^2)^{-1/2}$$

$$= -2 \times \left(\frac{-1}{2}\right)(s^3+4s^2)^{-3/2} \cdot [3s^2+8s]$$

$$= (s^3+4s^2)^{-3/2} \cdot (3s^2+8s)$$

$$= \frac{s(3s+8)}{s^3(s+4)^{3/2}}$$

$$= \frac{3s+8}{s^2(s+4)^{3/2}}$$

9) $\frac{1}{t} e^{-t} \sin t$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \phi(s)\, ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\, ds$$

$$= \left[\tan^{-1}(s)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

$$= \cot^{-1}(s)$$

$$\therefore L\left\{e^{-t} \cdot \frac{\sin t}{t}\right\} = \cot^{-1}(s+1)$$

10) $\frac{\sin t \cdot \sin 5t}{t}$

$$\rightarrow L\{\sin t \cdot \sin 5t\} = L\left\{\frac{\cos 4t - \cos 6t}{2}\right\}$$

$$= \frac{1}{2}\left\{\frac{s}{s^2+16} - \frac{s}{s^2+36}\right\}$$

$$= \frac{s}{2}\left\{\frac{1}{s^2+16} - \frac{1}{s^2+36}\right\}$$

$$L\left\{\frac{\sin t \cdot \sin 5t}{t}\right\} = \frac{1}{4}\int_s^\infty \left\{\frac{2s}{s^2+16} - \frac{2s}{s^2+36}\right\}$$

$$= \frac{1}{4}\left[\log(s^2+16) - \log(s^2+36)\right]_s^\infty$$

$$= \frac{1}{4}\left\{\log\left[\frac{s^2+16}{s^2+36}\right]\right\}_s^\infty$$

$$= \frac{1}{4}\left\{\log\left[\frac{1+16/s^2}{1+36/s^2}\right]_s^\infty\right\}$$

$$= \frac{1}{4}\left\{0 - \log\left(\frac{s^2+16}{s^2+36}\right)\right\}$$

$$= \frac{1}{4}\log\left[\frac{s^2+36}{s^2+16}\right]$$

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11) Given $f(t) = t \quad 0 \le t \le 3$ & $f(t) = 6$ for $t > 3$ Find $L\{f(t)\}$ & $L\{f'(t)\}$

→ By defn,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^3 e^{-st}\, t.\, dt + \int_3^\infty e^{-st}.\, 6\, dt$$

$$= \left[t\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right)\right]_0^3 + 6\left\{\frac{e^{-st}}{-s}\right\}_3^\infty$$

$$= \frac{3}{-s}e^{-3s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + 6\left(0 + \frac{e^{-3s}}{s}\right)$$

$$L\{f(t)\} = \frac{3e^{-3s}}{s} + \frac{1}{s^2}(1 + e^{-3s})$$

$$L\{f'(t)\} = s\,\phi(s) - f(0)$$

*But $f(0) = 0$

$$= s\left\{\frac{3e^{-3s}}{s} + \frac{1}{s^2}(1 - e^{-3s})\right\} - 0$$

$$= 3e^{-3s} + \frac{1}{s}(1 - e^{-3s})$$

12) $\frac{d}{dt}\left(\frac{\sin t}{t}\right)$

Here $f(t) = \frac{\sin t}{t}$

$$L\left\{\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right\} = L\{f'(t)\} = s\,\phi(s) - f(0)$$

Now

$$L\{\sin t\} = \frac{1}{s^2 + a^2}$$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2 + a^2} = (\tan^{-1} s)_s^\infty = \cot^{-1}(s)$$

But $f(0) = 1$ (by L'Hopital)

$$\therefore L\left\{\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right\} = s\cot^{-1}(s) - 1.$$

13) $\int_0^t u\cosh u\, du$

Here $f(t) = t\cosh t$

$$L\{\cosh t\} = \frac{s}{s^2 - a^2}$$

$$L\{t\cosh t\} = (-1)^1 \frac{d}{ds}\frac{s}{(s^2 - a^2)}$$

$$= -\frac{(s^2 - a^2).(1) - s(2s)}{(s^2 - 1)^2}$$

$$= \frac{s^2 + 1}{(s^2 - 1)^2}$$

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$$L\left[\int_0^t u\cosh u\,du\right] = \frac{1}{s}\phi(s) = \frac{1}{s}\left[\frac{s^2+1}{(s^2-1)^2}\right]$$

14) $\int_0^t u\cos^2 u\,du$

Here, $f(t) = t\cos^2 t$

$$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\}$$

$$= \frac{1}{2}\left[L(1) + L(\cos 2t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

$$L\{t.\cos^2 t\} = \frac{1}{2}(-1)^1 \frac{d}{ds}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} + \frac{(s^2+4) - s(2s)}{(s^2+4)^2}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} - \frac{s^2-4}{(s^2+4)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{(s^2-4)}{(s^2+4)^2}\right]$$

$$L\left[\int_0^t u\cos^2 u\,du\right] = \frac{1}{2s}\left\{\frac{1}{s^2} + \frac{s^2-4}{(s^2+4)^2}\right\}$$

15) $e^{-4t}\int_0^t u\sin 3u\,du$

Here $f(t) = t\sin 3t$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{t\sin 3t\} = 3(-1)\frac{d}{ds}\frac{1}{(s^2+9)}$$

$$= (-3)\frac{[-(2s)]}{(s^2+9)^2}$$

$$L\{t\sin 3t\} = \frac{6s}{(s^2+9)^2}$$

$$\therefore L\left[\int_0^t u\sin 3u\,du\right] = \frac{1}{s}\times\frac{6s}{(s^2+9)^2} = \frac{6}{(s^2+9)^2}$$

$$L\left[e^{-4t}\frac{6}{(s^2+9)^2}\right] = \frac{6}{[(s+4)^2+9]^2}$$

$$= \frac{6}{(s^2+8s+25)^2}$$

HW i) $\int_0^t u e^{-3u}\sinh u\,du$, ii) $\int_0^t e^u \frac{\sin 4u}{u}\,du$

iii) $t\sqrt{1+\sin 2t}$ iv) $(t\sinh 2t)^2$

v) $e^t\sin 4t\cos 2t$ vi) $\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t$.

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i) $\int_0^t u.e^{-3u} \sinh u \, du$

Here $f(t) = t.e^{-3u} \sinh t$

$L\{\sinh t\} = \frac{1}{s^2-1}$

$L\{t.\sinh t\} = (-1)^1 \frac{d}{ds}\left(\frac{1}{s^2-1}\right)$

$= (-1) \times \frac{-1}{(s^2-1)^2} \times 2s$

$L\{t.\sinh t\} = \frac{2s}{(s^2-1)^2}$

$L\{e^{-3u} t \sinh t\} = \frac{2(s+3)}{[(s+3)^2-1]^2}$

$= \frac{2(s+3)}{(s^2+6s+8)}$

$\therefore L\left\{\int_0^t u.e^{-3u} \sinh u \, du\right\} = \frac{1}{s} \times \frac{2(s+3)}{(s^2+6s+8)}$

$= \frac{2(s+3)}{(s^3+6s^2+8s)}$

ii) $\int_0^t e^u \frac{\sin 4u}{u} . du$

Here $f(t) = e^t . \frac{\sin 4t}{t}$

$L\{\sin 4t\} = \frac{4}{s^2+16}$

$L\left\{\frac{\sin 4t}{t}\right\} = \int_s^\infty \frac{4}{s^2+16} . ds$

$= 4 \int_s^\infty \frac{1}{s^2+(4)^2} . ds$

$= \frac{4}{4}\left[\tan^{-1}\left(\frac{s}{4}\right)\right]_s^\infty$

$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{4}\right)$

$L\left\{\frac{\sin 4t}{t}\right\} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{4}\right)$

$L\left\{e^t . \frac{\sin 4t}{t}\right\} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s-1}{4}\right)$

$L\left\{\int_0^t e^u . \frac{\sin 4u}{u} . du\right\} = \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s-1}{4}\right)\right]$

$= \frac{\pi}{2s} - \frac{1}{s}\tan^{-1}\left(\frac{s-1}{4}\right)$.

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iii) $t\sqrt{1+\sin 2t}$

$= t\sqrt{\cos^2 t + \sin^2 t + \sin 2t}$

$= t\sqrt{(\cos t + \sin t)^2}$

$= t(\cos t + \sin t)$.

$L\{\cos t + \sin t\} = \frac{s}{s^2+1} + \frac{1}{s^2+1}$

$= \frac{s+1}{s^2+1}$

$L\{t.(\cos t + \sin t)\} = (-1)\frac{d}{ds}\left(\frac{s+1}{s^2+1}\right)$

$= (-1)\left[\frac{(s^2+1)(1) - (s+1)(2s)}{(s^2+1)^2}\right]$

$= (-1)\left[\frac{s^2+1-2s^2-2s}{(s^2+1)^2}\right]$

$= (-1)\left[\frac{-s^2-2s+1}{(s^2+1)^2}\right]$

$= \frac{s^2+2s-1}{(s^2+1)^2}$

$\cancel{= \frac{(s+1)^2}{(s^2+1)^2}}$

$\cancel{= \left[\frac{s+1}{s^2+1}\right]^2}$

iv) $(t\sinh 2t)^2$

$= t^2\sinh^2 2t$.

$\sinh^2 2t = \left[\frac{e^{2t}-e^{-2t}}{2}\right]^2$

$= \frac{1}{4}\left[e^{4t} - 2 + e^{-4t}\right]$

$L\{(\sinh 2t)^2\} = \frac{1}{4}L\{e^{4t} - 2 + e^{-4t}\}$

$= \frac{1}{4}\left[\frac{1}{s-4} - \frac{2}{s} + \frac{1}{s+4}\right]$

$L\{t^2.(\sinh 2t)^2\} = \frac{1}{4}(-1)^2\frac{d}{ds}\left[\frac{1}{s-4} - \frac{2}{s} + \frac{1}{s+4}\right]$

$=$

take double derivative

$= \frac{1}{4}\left[\frac{-1}{(s-4)^2} + \frac{2}{s^2} + \frac{(-1)}{(s+4)^2}\right]$

$= \frac{-1}{4}\left[\frac{1}{(s-4)^2} + \frac{1}{(s+4)^2} - \frac{2}{s^2}\right]$

v) $e^t.\sin 4t.\cos 2t$

$\sin 4t.\cos 2t = \frac{\sin 6t + \sin 2t}{2}$

$\therefore L\{\sin 4t.\cos 2t\} = \frac{1}{2}\left[\frac{6}{s^2+36} + \frac{2}{s^2+4}\right]$

$L\{e^t(\sin 4t.\cos 2t)\} = \frac{1}{2}\left[\frac{6}{(s-1)^2+36} + \frac{2}{(s-1)^2+4}\right]$

$= \frac{3}{(s^2-2s+37)} + \frac{1}{(s^2-2s+5)}$

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vi) $\sinh\frac{t}{2} \cdot \sin\frac{\sqrt{3}t}{2}$

$$= \left(\frac{e^{t/2} - e^{-t/2}}{2}\right) \cdot \sin\frac{\sqrt{3}t}{2}$$

$$L\left\{\sin\frac{\sqrt{3}t}{2}\right\} = \frac{\frac{\sqrt{3}}{2}}{s^2 + \frac{3}{4}} = \frac{\sqrt{3}}{2} \times \frac{4}{4s^2+3}$$

$$L\left\{\sin\frac{\sqrt{3}t}{2}\right\} = \frac{2\sqrt{3}}{4s^2+3}$$

$$\therefore \frac{1}{2} L\left\{\left(e^{t/2} - e^{-t/2}\right) \sin\frac{\sqrt{3}t}{2}\right\}$$

$$= \frac{1}{2}\left[L\left\{e^{t/2} \sin\frac{\sqrt{3}}{2}\right\} - L\left\{e^{-t/2} \sin\frac{\sqrt{3}t}{2}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{2\sqrt{3}}{4\left(s-\frac{1}{2}\right)^2+3}\right] - \frac{1}{2}\left[\frac{2\sqrt{3}}{4\left(s+\frac{1}{2}\right)^2+3}\right]$$

$$= \frac{1}{2}\left[\frac{2\sqrt{3}}{4\left[s^2 - s + \frac{1}{4}\right]+3}\right] - \frac{1}{2}\left[\frac{2\sqrt{3}}{4\left(s^2+s+\frac{1}{4}\right)+3}\right]$$

$$= \frac{\sqrt{3}}{4s^2-4s+4} - \frac{\sqrt{3}}{4s^2+4s+4}$$

$$= \frac{\sqrt{3}}{4}\left[\frac{1}{(s^2-s+1)} - \frac{1}{(s^2+s+1)}\right]$$

⇒ Evaluation of Integral :-

1) Evaluate $\int_0^\infty e^{-3t} \cdot t \cdot \cos t \, dt$.

Here $f(t) = t\cos t$

$$L\{\cos t\} = \frac{s}{s^2+1}$$

$$L\{t\cos t\} = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= (-1)\left[\frac{(s^2+1)(1) - (s)(2s)}{(s^2+1)^2}\right]$$

$$= (-1)\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$L\{t\cos t\} = \frac{s^2-1}{(s^2+1)^2}$$

Applying defn on LHS

$$\int_0^\infty e^{-st} \cdot t\cos t \, dt = \frac{s^2-1}{(s^2+1)^2}$$

Put $s = 3$

$$\therefore \int_0^\infty e^{-3t} \cdot t \cdot \cos t \, dt = \frac{9-1}{(9+1)^2} = \frac{8}{100} = 0.08.$$

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2) P.T. $\int_0^\infty e^{-t} \frac{\sin^2 t}{t}\, dt = \frac{1}{4} \log 5$.

Here $f(t) = \frac{\sin^2 t}{t}$

$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$L\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds$

$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$

$= \frac{1}{2} \times \frac{1}{2}\left[\log s^2 - \log(s^2+4)\right]_s^\infty$

$= \frac{1}{4}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty$

$= \frac{1}{4}\left[0 - \log\left(\frac{s^2}{s^2+4}\right)\right]$

$L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$

Applying def^n

$\int_0^\infty e^{-st} \frac{\sin^2 t}{t}\, dt = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$

Put $s = 1$

$\therefore \int_0^\infty e^{-t} \frac{\sin^2 t}{t}\, dt = \frac{1}{4}\log\left(\frac{1+4}{1}\right) = \frac{1}{4}\log 5$

3) Find $\int_0^\infty e^{-t}\left(\int_0^t u^2 \sinh u \cosh u\, du\right)$

Here $f(t) = \int_0^t u^2 \sinh u \cosh u\, du$

$L\{\sinh t \cosh t\} = L\left\{\frac{1}{2}\sinh 2t\right\}$

$= \frac{1}{2} \times \frac{2}{s^2-4}$

$= \frac{1}{s^2-4}$

$L\{u^2 \sinh u \cosh u\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s^2-4}\right)$

$= \frac{d}{ds}\left[\frac{2s}{(s^2-4)^2}\right]$

$= \frac{(s^2-4)^2(2) - (2s)2(s^2-4)(2s)}{(s^2-4)^4}$

$= \frac{2(s^2-4)^2 - 8s^2(s^2-4)}{(s^2-4)^4}$

$= -2\frac{(s^2-4-4s^2)}{(s^2-4)^4}$

$L\{t^2 \sinh t \cosh t\} = \frac{2(3s^2+4)}{(s^2-4)^3}$

$L\left\{\int_0^t u^2 \sinh t \cosh t\right\} = \frac{2}{s}\frac{(3s^2+4)}{(s^2-4)^3}$

$\int_0^\infty e^{-st}\left(\int_0^t u^2 \sinh t \cosh t\right) dt = \frac{2(3s^2+4)}{s(s^2-4)}$

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Put $s=1$

$$\int_0^\infty e^{-t}\left(\int_0^t u^2 \sinh u \cosh u \, du\right) dt = \frac{2(3+4)}{1(1-4)^3} = -\frac{14}{27}$$

4) Evaluate $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt$

Here $f(t) = \frac{\cos 6t - \cos 4t}{t}$

$$L\{\cos 6t - \cos 4t\} = \frac{s}{s^2+36} - \frac{s}{s^2+16}$$

$$L\left\{\frac{\cos 6t - \cos 4t}{t}\right\} = \int_s^\infty \frac{s}{s^2+36} - \frac{s}{s^2+16} \, ds$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+36}{s^2+16}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$$

Applying def^n,

$$\int_0^\infty e^{-st}\frac{(\cos 6t - \cos 4t)}{t} = \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$$

Put $s=0$

$$\int_0^\infty \frac{\cos 6t - \cos 4t}{t} = \frac{1}{2}\log\left(\frac{16}{36}\right) = \log\left(\frac{4}{6}\right)$$

$$= \log(2/3)$$

* Methods of finding Inverse Laplace Transform:-

1) Using First Shifting Property:-

$$L^{-1}\{\phi(s-a)\} = e^{at} L^{-1}\{\phi(s)\}$$

Q Find the I.L.T. of:-

1) $\frac{4s+12}{s^2+8s+12}$

$$\rightarrow L^{-1}\left\{\frac{4s+12}{s^2+8s+16-4}\right\}$$

$$= L^{-1}\left\{\frac{4s+12}{(s+4)^2-4}\right\}$$

$$= L^{-1}\left\{\frac{4(s+4)-4}{(s+4)^2-4}\right\}$$

$$= e^{-4t} L^{-1}\left\{\frac{4s-4}{s^2-4}\right\}$$

$$= 4e^{-4t}\left[L^{-1}\left(\frac{4s}{s^2-4}\right) - L^{-1}\left(\frac{4}{s^2-4}\right)\right]$$

$$= e^{-4t}\left(4\cosh 4t - 4\frac{\sinh 2t}{2}\right)$$

$$= e^{-4t}\left(4\cosh 4t - 2\sinh 2t\right).$$

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2) $\dfrac{3s+7}{s^2-2s-3}$

$\rightarrow L^{-1}\left\{\dfrac{3s+7}{s^2-2s+1-3-1}\right\}$

$= L^{-1}\left\{\dfrac{3s+7}{(s-1)^2-4}\right\}$

$= L^{-1}\left\{\dfrac{3(s-1)+10}{(s-1)^2-4}\right\}$

$= e^{t} L^{-1}\left\{\dfrac{3s+10}{s^2-4}\right\}$

$= e^{t}\left[L^{-1}\left(\dfrac{3s}{s^2-2^2}\right) + L^{-1}\left(\dfrac{10}{s^2-2^2}\right)\right]$

$= e^{t}\left[3\cosh 2t + 10\dfrac{\sinh 2t}{2}\right]$

$= e^{t}\left[3\cosh 2t + 5\sinh 2t\right]$

3) $\dfrac{s}{(2s+1)^2}$

$= L^{-1}\left(\dfrac{s}{(2s+1)^2}\right)$

$= L^{-1}\left\{\dfrac{s}{4(s+1/2)^2}\right\}$

$= \dfrac{1}{4} L^{-1}\left\{\dfrac{s}{(s+1/2)^2}\right\}$

$= \dfrac{1}{4} L^{-1}\left\{\dfrac{(s+1/2)-1/2}{(s+1/2)^2}\right\}$

Applying first shifting,

$= \dfrac{1}{4} e^{-\frac{1}{2}t} L^{-1}\left\{\dfrac{s-1/2}{s^2}\right\}$

$= \dfrac{1}{4} e^{-\frac{1}{2}t}\left[L^{-1}\left\{\dfrac{1}{s}\right\} - \dfrac{1}{2} L^{-1}\left\{\dfrac{1}{s^2}\right\}\right]$

$= \dfrac{1}{4} e^{-\frac{1}{2}t}\left[1 - \dfrac{1}{2}t\right]$

$= \dfrac{e^{-\frac{1}{2}t}}{8}\left[2-t\right]$

$= \dfrac{1}{8}(2-t)e^{-\frac{1}{2}t}$

HW 1) $\dfrac{2s+3}{s^2+2s+2}$

2) $\dfrac{s+2}{s^2+4s+7}$

3) $\dfrac{s}{s^2+2s+2}$

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2) Using Partial Fraction

Q) Find the I.L.T. of :

1) $\dfrac{5s^2+8s-1}{(s+3)(s^2+1)}$

→ By Partial Fraction,

$$\frac{5s^2+8s-1}{(s+3)(s^2+1)} = \frac{A}{(s+3)} + \frac{Bs+C}{(s^2+1)}$$

$$5s^2+8s-1 = A(s^2+1) + [B(s)+C](s+3)$$

Put $s+3=0$ ie $s=-3$.

$$\therefore 5(-3)^2+8(-3)-1 = A[9+1] + (Bs+C)(0)$$

$$5\times 9-24-1 = A(10)$$

$$45-25 = 10A$$

$$A = \frac{20}{10} = 2$$

Put $s=0$

$$\therefore 5(0)^2+8(0)-1 = 2(0+1)+B(0)+C(0+3)$$

$$-1 = 2+3C$$

$$3C = -3$$

$$\boxed{C=-1}$$

Equating co-effients of s^2

$$5 = A+B$$

$$5 = 2+B$$

$$\boxed{B=3}$$

$$\therefore \frac{5s^2+8s-1}{(s+3)(s^2+1)} = \frac{2}{s+3} + \frac{3s-1}{(s^2+1)}$$

$$\therefore L^{-1}\left\{\frac{2}{s+3} + \frac{3s-1}{(s^2+1)}\right\}$$

$$= L^{-1}\left\{\frac{2}{s+3}\right\} + 3\left[L^{-1}\left\{\frac{s}{s^2+1}\right\}\right] - \left[L^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$$= 2e^{-3t} + 3[\cos t] - [\sin t]$$

$$= 2e^{-3t} + 3\cos t - \sin t$$

2) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{As+B}{(s^2+a^2)} + \frac{Cs+D}{(s^2+b^2)}$$

$$\therefore s^2 = (As+B)(s^2+b^2) + (Cs+D)(s^2+a^2)$$

Cannot solve by this method

Take $s^2 = x$.

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{x}{(x+a^2)(x+b^2)}$$

By Partial fraction,

$$\frac{x}{(x+a^2)(x+b^2)} = \frac{A}{(x+a^2)} + \frac{B}{(x+b^2)}$$

$$x = A(x+b^2) + B(x+a^2)$$

Put $x+a^2=0$ ie $x=-a^2$.

$$-a^2 = A(-a^2+b^2)$$

$$\therefore \boxed{A = \frac{a^2}{a^2-b^2}}$$

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Put $x+b^2=0$ ie $x=-b^2$

$$-b^2 = A(0)+B(-b^2+a^2)$$

$$\boxed{B = \frac{b^2}{b^2-a^2} = \frac{-b^2}{a^2-b^2}}$$

$$\frac{x}{(x+a^2)(x+b^2)} = \frac{a^2}{a^2-b^2}\frac{1}{(x+a^2)} - \frac{b^2}{(a^2-b^2)}\frac{1}{(x+b^2)}$$

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{a^2}{a^2-b^2}\times\frac{1}{(s^2+a^2)} - \frac{b^2}{a^2-b^2}\times\frac{1}{(s^2+b^2)}$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{a^2}{a^2-b^2}L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - \frac{b^2}{a^2-b^2}L^{-1}\left\{\frac{1}{s^2+b^2}\right\}$$

$$= \frac{a^2}{a^2-b^2}\frac{\sin at}{a} - \frac{b^2}{a^2-b^2}\frac{\sin bt}{b}$$

$$= \frac{a\sin at}{a^2-b^2} - \frac{b\sin bt}{a^2-b^2}$$

$$x = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

3) $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$

→ Consider $s^2+2s = x$

$$\therefore \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \frac{x+3}{(x+5)(x+2)}$$

By P.F

$$\frac{x+3}{(x+5)(x+2)} = \frac{A}{(x+5)} + \frac{B}{(x+2)}$$

$$(x+3) = A(x+2)+B(x+5)$$

Put $x=-2$

$\therefore 1 = B(5-2)$

$\boxed{B = 1/3}$

Put $x=-5$

$-2 = A(-5+2)$

$\therefore 2 = A(3)$

$\boxed{A = 2/3}$

$$\frac{x+3}{(x+5)(x+2)} = \frac{2}{3(x+5)} + \frac{1}{3(x+2)}$$

$$\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \frac{2}{3(s^2+2s+5)} + \frac{1}{3}\times\frac{1}{s^2+2s+2}$$

$$\rightarrow L^{-1}\left\{\frac{2}{3}\times\frac{1}{(s^2+2s+5)}\right\} + L^{-1}\left\{\frac{1}{3}\times\frac{1}{s^2+2s+2}\right\}$$

$$= \frac{2}{3}L^{-1}\left\{\frac{1}{(s^2+2s+1)+4}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{(s^2+2s+1)+1}\right\}$$

$$= \frac{2}{3}L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\}$$

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$$= \frac{2}{3} e^{-t} L^{-1}\left\{\frac{1}{s^2+2^2}\right\} + \frac{1}{3} e^{-t} L^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

$$= \frac{2}{3} e^{-t} \frac{\sin 2t}{2} + \frac{1}{3} e^{-t} \sin t$$

$$= \frac{e^{-t} \sin 2t}{3} + \frac{e^{-t} \sin t}{3}$$

4) $\frac{s}{s^4+4a^4}$

$$\rightarrow \frac{s}{s^4+4a^4} = \frac{s}{s^4+4a^2s^2+4a^2-4a^2s^2}$$

$$= \frac{s}{(s^2+2a^2)^2-(2as)^2}$$

$$= \frac{s}{(s^2+2a^2-2as)(s^2+2a^2+2as)}$$

By P.F.,

$$\frac{s}{(s^2-2as+2a^2)(s^2+2as+2a^2)} = \frac{As+B}{(s^2-2as+2a^2)} + \frac{Cs+D}{(s^2+2as+2a^2)}$$

$$s = (As+B)(s^2+2as+2a^2) + (Cs+D)(s^2-2as+2a^2)$$

Equating co-effient of like terms.

$0 = A+C$ ..... ①

$0 = B+2aA-2aC+D$ .. ②

$1 = 2a^2A+2aB-2aD+2a^2C$

$= 2a^2(A+C)+2a(B-D)$ .. (From ① $A+C=0$)

$1 = 2a(B-D)$ ... ③

$0 = 2a^2B+2a^2D$

$0 = 2a^2(B+D)$

$\therefore (B+D) = 0$ .. ④

Adding ③ & ④

$$\begin{aligned} B+D &= 0 \\ + \quad B-D &= 1/2a \\ \hline 2B &= +\frac{1}{2}a \end{aligned}$$

$$\boxed{B = \frac{1}{4a}}$$

$$\therefore \boxed{D = -\frac{1}{4a}}$$

In eqn ②

$(B+D)+2a(A-C) = 0$

$0 + 2a(A-C) = 0$

$\therefore A-C = 0$ ... ⑤

$\therefore A = C$

From ① & ⑤

$A = C = 0$

Thus,

$$\frac{s}{s^4+4a^2} = \frac{1/(4a)}{(s^2-2as+2a^2)} - \frac{1/(4a)}{(s^2+2as+2a^2)}$$

$$= \frac{1}{4a} L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\} - \frac{1}{4a} L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\}$$

$$= \frac{1}{4a} e^{at} L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - \frac{1}{4a} e^{-at} L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$= \frac{1}{4a} e^{at} \frac{\sin at}{a} - \frac{1}{4a} e^{-at} \frac{\sin at}{a}$$

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$= \frac{1}{2a^2} \sin at \sinh at$

⇒ Using Convolution Theorem:-

Statement:

If $L^{-1}\{\phi_1(s)\} = f_1(t)$, $L^{-1}\{\phi_2(s)\} = f_2(t)$,

then $L^{-1}\{\phi_1(s) \cdot \phi_2(s)\} = \int_0^t f_1(u) \cdot f_2(t-u)\, du$

Eg) Q) Using Convolution Theorem, find I.L.T of:-

1) $\frac{s^2}{(s^2+a^2)^2}$

Let $\phi_1(s) = \frac{s}{s^2+a^2}$, $\phi_2(s) = \frac{s}{s^2+a^2}$

$f_1(t) = L^{-1}\{\phi_1(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

$f_2(t) = L^{-1}\{\phi_2(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

By Convolution Theorem,

$L^{-1}\{\phi_1(s) \cdot \phi_2(s)\} = \int_0^t f_1(u) \cdot f_2(t-u) \cdot du$

$= \int_0^t \cos au \cdot \cos a(t-u)\, du$

$= \cancel{\frac{1}{2} \int_0^t S}$

$= \frac{1}{2} \int_0^t \{\cos at + \cos(2au - at)\}$

$= \frac{1}{2} \left[u \cos at + \frac{\sin(2au - at)}{2a}\right]_0^t$

$= \frac{1}{2} \left[t \cos at + \frac{\sin(2at - at)}{2a}\right\} + \frac{\sin at}{2a}$

$= \frac{1}{2} \left[t \cos at + \frac{\sin at}{a}\right\}$

$= \frac{1}{2a} \{at \cos at + \sin at\}$

2) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

Let $\phi_1(s) = \frac{s}{s^2+a^2}$, $\phi_2(s) = \frac{s}{s^2+b^2}$

$f_1(t) = L^{-1}\{\phi_1(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

$f_2(t) = L^{-1}\{\phi_2(s)\} = L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt$

By Convolution Theorem,

$L^{-1}\{\phi_1(s) \cdot \phi_2(s)\} = \int_0^t f_1(u) \cdot f_2(t-u)\, du$

$= \int_0^t [\cos au \cdot \cos b(t-u)]\, du$

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$$= \frac{1}{2}\int_0^t \left[\cos\{(a-b)u+bt\} + \cos\{(a+b)u-bt\}\right] du$$

$$= \frac{1}{2}\left[\frac{\sin[(a-b)u+bt]}{(a-b)} + \frac{\sin[(a+b)u-bt]}{(a+b)}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at}{(a-b)} + \frac{\sin at}{(a+b)} - \frac{\sin bt}{(a-b)} + \frac{\sin bt}{(a+b)}\right]$$

$$= \frac{1}{2}\left\{\sin at\left[\frac{1}{(a-b)} + \frac{1}{a+b}\right] - \sin bt\left[\frac{1}{(a-b)} - \frac{1}{a+b}\right]\right\}$$

$$= \frac{1}{2}\left\{\sin at\left[\frac{2a}{a^2-b^2}\right] - \sin bt\left[\frac{2b}{a^2-b^2}\right]\right\}$$

$$= \frac{1}{a^2-b^2}\{a\sin at - b\sin bt\}$$

3) $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$

$$\rightarrow L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right\}$$

$$= L^{-1}\left\{\frac{(s+1)^2+2}{\{(s+1)^2+4\}\{(s+1)^2+1\}}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{s^2+2}{(s^2+4)(s^2+1)}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+1)}\right\} + 2e^{-t}L^{-1}\left\{\frac{1}{(s^2+4)(s^2+1)}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+1)}\right\} - \frac{2}{3}e^{-t}L^{-1}\left\{\frac{1}{s^2+2^2} - \frac{1}{s^2+1^2}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{s}{s^2+2^2} - \frac{s}{s^2+1}\right\} - \frac{2}{3}e^{-t}L^{-1}\left\{\frac{1}{s^2+2^2} - \frac{1}{s^2+1}\right\}$$

$$\phi_1(s_1) = \frac{s}{s^2+2^2}, \quad \phi_2(s_2) = \frac{s}{s^2+1^2}$$

$$f(t_1) = L^{-1}\{\phi_1(s)\} = L^{-1}\left\{\frac{s}{s^2+2^2}\right\} = \cos 2t$$

$$f(t_2) = L^{-1}\{\phi_2(s)\} = L^{-1}\left\{\frac{s}{s^2+1^2}\right\} = \cos t$$

$$L^{-1}\{\phi_1(s)\,\phi_2(s)\} = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$= \int_0^t \cos 2t \,\cos(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \left[\cos(3t-u) + \cos(t+u)\right] du$$

$$= \frac{1}{2}\left[\sin(3t-u)\right.$$

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4) $\dfrac{1}{(s+3)(s-2)^4}$

Let $\phi_1(s) = \dfrac{1}{(s-2)^4}$, $\phi_2(s) = \dfrac{1}{s+3}$

$$f_1(t) = L^{-1}\left\{\frac{1}{(s-2)^4}\right\} = e^{2t} L^{-1}\left\{\frac{1}{s^4}\right\} = e^{2t}\,\frac{t^3}{3!}$$

$$f_2(t) = L^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

By convolution

$$L^{-1}\{\phi_1(s).\,\phi_2(s)\} = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$= \int_0^t e^{2u}\,\frac{u^3}{3!}\,.\,e^{-3(t-u)}$$

$$= \frac{1}{6} e^{-3t} \int_0^t u^3 . e^{5u}\, du$$

$$= \frac{1}{6} e^{-3t}\left\{u^3 \frac{e^{5u}}{5} - 3u^2\left(\frac{e^{5u}}{25}\right) + 6u\left(\frac{e^{5u}}{125}\right) - 6\left(\frac{e^{5u}}{625}\right)\right\}_0^t$$

$$= \frac{1}{6} e^{-3t}\left\{\frac{t^5}{5} e^{5t} - \frac{3t^2}{25} e^{5t} + \frac{6t}{125} e^{5t} - \frac{6}{625} e^{5t}\right\}$$

4) Using differentiation of $\phi(s)$:

$$L^{-1}\{\phi(s)\} = -\frac{1}{t} L^{-1}\{\phi'(s)\}$$

1) $\log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)$

$$\rightarrow L^{-1}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left(\log(s^2+a^2) - \log(s^2+b^2)\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right\}$$

$$= -\frac{2}{t} L^{-1}\left\{\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right\}$$

$$= -\frac{2}{t}(\cos at - \cos bt)$$

$$= \frac{2}{t}(\cos bt - \cos at)$$

2) $2\tanh^{-1}(s)$

$$\rightarrow L^{-1}\{2\tanh^{-1}(s)\}$$

$$= 2L^{-1}\{\tanh^{-1}(s)\}$$

$$= -\frac{2}{t} L^{-1}\left\{\frac{d}{ds}\tanh^{-1}(s)\right\}$$

$$= -\frac{2}{t} L^{-1}\left\{\frac{1}{1-s^2}\right\} = \frac{2}{t} L^{-1}\left\{\frac{1}{s^2-1}\right\}$$

$$= \frac{2}{t}\sinh t$$

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3) $s\log\left(\frac{s+1}{s-1}\right)$

$\rightarrow L^{-1}\left\{s\log\left(\frac{s+1}{s-1}\right)\right\}$

$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds} s\log\left(\frac{s+1}{s-1}\right)\right\}$

$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[s\log(s+1) - s\log(s-1)\right]\right\}$

$= -\frac{1}{t} L^{-1}\left\{\frac{s}{s+1} + \log(s+1) - \frac{s}{s-1} - \log(s-1)\right\}$

$= -\frac{1}{t} L^{-1}\left\{\frac{s}{s+1} - \frac{s}{s-1}\right\} - \frac{1}{t} L^{-1}\left\{\log(s+1) - \log(s-1)\right\}$

$= -\frac{1}{t} L^{-1}\left\{\frac{-2s}{s^2-1}\right\} + \frac{1}{t^2} L^{-1}\left\{\frac{d}{ds}\left(\log(s+1) - \log(s-1)\right)\right\}$

$= \frac{2}{t} L^{-1}\left\{\frac{s}{s^2-1}\right\} + \frac{1}{t^2} L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s-1}\right\}$

$= \frac{2}{t}\cosh t + \frac{1}{t^2}\left(e^{-t} - e^{t}\right)$

$= \frac{2}{t}\cosh t - \frac{2}{t^2}\left(\frac{e^{t} - e^{-t}}{2}\right)$

$= \frac{2}{t^2}\left[t\cosh t - \sinh t\right]$

4) $L^{-1}\left\{\cot^{-1}(as)\right\}$

$\rightarrow L^{-1}\left\{\cot^{-1}(as)\right\}$

$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\cot^{-1}(as)\right\}$

$= -\frac{1}{t} L^{-1}\left\{\frac{-a}{1+(as)^2}\right\}$

$= \frac{a}{t} L^{-1}\left\{\frac{1}{a^2\left[s^2 + \left(\frac{1}{a}\right)^2\right]}\right\}$

$= \frac{1}{at} \frac{\sin\left(\frac{1}{a}t\right)}{1/a}$

$= \frac{1}{t}\sin\left(\frac{t}{a}\right)$

5) $\tan^{-1}\left(\frac{2}{s^2}\right)$

$\rightarrow L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$

$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$

$= -\frac{1}{t} L^{-1}\left\{\frac{1}{1+4/s^4}\left(\frac{-4}{s^3}\right)\right\}$

$= \frac{4}{t} L^{-1}\left\{\frac{s}{s^4+4}\right\}$

$\vdots$

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5) Using differentiation of $f(t)$:

$$L^{-1}\{s.\phi(s)\} = f'(t)$$

$$L^{-1}\{s^2.\phi(s)\} = f''(t)$$

1) $\dfrac{s^2}{(s+a)^3}$

Here $\phi(s) = \dfrac{1}{(s+a)^3}$

$$f(t) = L^{-1}\{\phi(s)\}$$

$$= L^{-1}\left\{\frac{1}{(s+a)^3}\right\}$$

$$= e^{-at} L^{-1}\left(\frac{1}{s^3}\right)$$

$$= e^{-at} \cdot \frac{t^2}{2!}$$

$$L^{-1}\left\{s^2 \cdot \frac{1}{(s+a)^3}\right\} = f''(t)$$

$$= \frac{d^2}{dt^2}\left(\frac{1}{2} t^2 e^{-at}\right)$$

* Special Function :-

1) Periodic Function:

Def^n:- $f(t)$ is said to periodic function with period a, if

$$\boxed{f(t) = f(t+a) = f(t+2a) = f(t+3a) = \ldots}$$

eg:- $\sin\theta$, $\cos\theta$ are periodic functions with period 2π.

If $f(t)$ be a periodic function with period a, then

$$L\{f(t)\} = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\, dt.$$

① Find LT. of $f(t) = 1$, $0 \le t < a$ & $f(-1)$, $a < t < 2a$ and $f(t)$ is periodic with period $2a$

→ Since $f(t)$ is periodic with period $2a$

$$L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t).\, dt$$

$$= \frac{1}{1-e^{-2as}} \left\{ \int_0^a e^{-st}(1)\, dt + \int_a^{2a} e^{-st}(-1)\, dt \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left(\frac{e^{-st}}{-s}\right)_0^a - \left(\frac{e^{-st}}{-s}\right)_a^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \frac{e^{-as}}{-s} + \frac{1}{s} + \frac{e^{-2as}}{s} - \frac{e^{-as}}{s} \right\}$$

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$$= \frac{1}{s}\frac{1}{1-e^{-2as}}\left\{1-2e^{-as}+\left(e^{-as}\right)^2\right\}$$

$$= \frac{1}{s}\frac{1}{1-e^{-2as}}\left(1-e^{-as}\right)^2$$

$$= \frac{1}{s}\frac{1-e^{-as}}{1+e^{as}}$$

$$= \frac{1}{s}\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}$$

$$= \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$

2) $f(t) = |\sin pt|$, $t \geq 0$.

→ Here $f(t) = |\sin pt|$

$$f\left(t+\frac{\pi}{p}\right) = \left|\sin p\left(t+\frac{\pi}{p}\right)\right| = \left|\sin(pt+\pi)\right|$$

$$= |\sin pt| = f(t).$$

$f(t)$ is a periodic function with periodic $\frac{\pi}{p}$

$$L\{f(t)\} = \frac{1}{1-e^{-\frac{\pi}{p}s}}\int_0^{\pi/p} e^{-st}\sin pt\,dt \quad \left(\text{for } 0 \leq t \leq \frac{\pi}{p},\ \sin pt \geq 0\right)$$

$$= \frac{1}{1-e^{-\frac{\pi}{p}s}}\left\{\frac{e^{-st}}{s^2+p^2}\left(-s\sin pt - p\cos pt\right)\right\}_0^{\pi/p}$$

$$= \frac{1}{1-e^{-\pi s/p}}\left\{\frac{e^{-\pi s/p}}{s^2+p^2}(0+p) - \frac{1}{s^2+p^2}(0-p)\right\}$$

$$= \frac{1}{1-e^{-\pi s/p}}\frac{p}{s^2+p^2}\left(1+e^{-\pi s/p}\right)$$

$$= \frac{p}{s^2+p^2}\frac{1+e^{-\pi s/p}}{1-e^{-\pi s/p}}$$

$$= \frac{p}{s^2+p^2}\frac{e^{\pi s/2p}+e^{-\pi s/2p}}{e^{\pi s/2p}-e^{-\pi s/2p}}$$

$$= \frac{p}{s^2+p^2}\coth\left(\frac{\pi s}{2p}\right)$$

2) Heavisides unit step function:-

→ Displaced Heaviside function

A function defined as

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

is called displaced heaviside function

→ Heavisides Unit Step function

It is defined as,

$$H(t) \text{ or } U(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

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Results :-

* 1) $L\{H(t-a)\} = \frac{1}{s} e^{-as}$

2) $L\{H(t)\} = \frac{1}{s}$

* 3) $L\{f(t)H(t-a)\} = e^{-as} L\{f(t+a)\}$

4) $L\{f(t-a)H(t-a)\} = e^{-as} L\{f(t)\} = e^{-as}\phi(s)$

* 5) $L\{e^{-as}\phi(s)\} = f(t-a)H(t-a)$

Find the laplace transform of

1) $\sin t\, H\left(t-\frac{\pi}{2}\right) - H\left(t-\frac{3\pi}{2}\right)$

$\rightarrow L\left\{\sin t\, H\left(t-\frac{\pi}{2}\right) - H\left(t-\frac{3\pi}{2}\right)\right\}$

$= L\left\{\sin t\, H\left(t-\frac{\pi}{2}\right)\right\} - L\left\{H\left(t-\frac{3\pi}{2}\right)\right\}$

$= e^{-\frac{\pi}{2}s} L\left\{\sin\left(t+\frac{\pi}{2}\right)\right\} - \frac{L}{s} e^{-\frac{3\pi}{2}s}$

$= e^{-\frac{\pi s}{2}} L\{\cos t\} - \frac{L}{s} e^{-\frac{3\pi s}{2}}$

$= e^{-\pi s/2} \frac{s}{s^2+1} - \frac{e^{-3\pi s/2}}{s}$

2) $(1+2t-3t^2+4t^3)\, H(t-2)$

$\rightarrow L\{(1+2t-3t^2+4t^3)\, H(t-2)\}$

$= e^{-2s} L\{1+2(t+2)-3(t+2)^2+4(t+2)^3\}$

$= e^{-2s} L\{1+2t+4-3(t^2+4t+4)+4(t^3+3t^2(2) + 3t(2)^2+(2)^3\}$

$= e^{-2s} L\{1+2t+4-3t^2-12t-12+4t^3+24t^2+12t+32\}$

$= e^{-2s} L\{25+38t+21t^2+4t^3\}$

$= e^{-2s}\left\{\frac{25}{s}+\frac{38}{s^2}+\frac{21(2!)}{s^3}+\frac{4(3!)}{s^4}\right\}$

$= e^{-2s}\left\{\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right\}$

3) Find $L^{-1}\left\{\frac{e^{-2s}}{s^2+8s+25}\right\}$

$\rightarrow L^{-1}\{e^{-as}\phi(s)\} = f(t-a)H(t-a)$

$L^{-1}\left\{e^{-as}\frac{1}{s^2+8s+25}\right\} = f(t-2)H(t-2)$

Now,

$f(t) = L^{-1}\{\phi(s)\}$

$= L^{-1}\left\{\frac{1}{s^2+8s+25}\right\}$

$= L^{-1}\left\{\frac{1}{(s+4)^2+9}\right\}$

$= e^{-4t} L^{-1}\left\{\frac{1}{s^2+9}\right\}$

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$$f(t) = e^{-4t}\,\frac{\sin 3t}{3}$$

Thus,

$$L^{-1}\left\{\frac{e^{-2s}}{s^2+8s+25}\right\} = e^{-4(t-2)}\,\frac{\sin 3(t-2)}{3}\,H(t-2)$$

4) Find $L^{-1}\left\{\frac{e^{4-3s}}{(s+b)^{5/2}}\right\}$

$$\rightarrow L^{-1}\left\{\frac{e^{4-3s}}{(s+b)^{5/2}}\right\}$$

$$= e^{4}\,L^{-1}\left\{\frac{e^{-3s}}{(s+b)^{5/2}}\right\}$$

$$= e^{4} f(t-3)\,H(t-3)$$

Now,

$$f(t) = L^{-1}\left\{\frac{1}{(s+b)^{5/2}}\right\}$$

$$= e^{-bt}\,L^{-1}\left\{\frac{1}{s^{5/2}}\right\}$$

$$= e^{-bt}\,\frac{t^{3/2}}{\Gamma\frac{5}{2}}$$

$$= e^{-bt}\,\frac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}$$

$$f(t) = \frac{4}{3\sqrt{\pi}}\,e^{-bt}\,t^{3/2}$$

Thus,

$$L^{-1}\left\{\frac{e^{4-3s}}{(s+b)^{5/2}}\right\} = \frac{e^{4}\cdot 4}{3\sqrt{\pi}}\,e^{-b(t-3)}\,(t-3)^{3/2}\,H(t-3)$$

* Expression of Function in terms of Heaviside function :

If

$$f(t) = \begin{cases} f_1(t), & 0<t<a \\ f_2(t), & a<t<b \\ f_3(t), & b<t<c \\ f_4(t), & t>c \end{cases}$$

then $f(t)$ can be expressed as,

$$f(t) = f_1(t)\{H(t-a) - H(t)\} + f_2(t)\{H(t-b) - H(t-a)\} + f_3(t)\{H(t-c) - H(t-b)\} + f_4(t)\,H(t-c)$$

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Ex→ Express the following functions as Heavisides unit step functions & find their Laplace Transforms.

1) $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \cos 2t & \pi < t < 2\pi \\ \cos 3t & t > 2\pi \end{cases}$

→ Heaviside's expression is :-

$$f(t) = \cos t \{H(t-\pi) - H(t)\}$$
$$+ \cos 2t \{H(t-2\pi) - H(t-\pi)\}$$
$$+ \cos 3t \{H(t-2\pi)\}$$

$$f(t) = -\cos t\, H(t) + (\cos t - \cos 2t) H(t-\pi) + (\cos 2t + \cos 3t) H(t-2\pi)$$

$$L\{f(t)\} = -L\{\cos t\, H(t)\} + L\{(\cos t - \cos 2t) H(t-\pi)\} + L\{(\cos 2t + \cos 3t) H(t-2\pi)\}$$

$$= -L\{\cos t\} + L\{(\cos t - \cos 2t) H(t-\pi)\} + L\{(\cos 2t + \cos 3t) H(t-2\pi)\}$$

$$= -L\{\cos t\} + e^{-\pi s} L\{\cos(t+\pi) - \cos 2(t+\pi)\} + e^{-2\pi s} L\{\cos 2(t+2\pi) + \cos 3(t+2\pi)\}$$

$$= -\frac{s}{s^2+1} + e^{-\pi s} L\{-\cos t - \cos 2t\} + e^{-2\pi s} L\{\cos 2t + \cos 3t\}$$

$$= \frac{-s}{s^2+1} + e^{-\pi s}\left\{\frac{-s}{s^2+1} - \frac{s}{s^2+4}\right\} + e^{-2\pi s}\left\{\frac{s}{s^2+4} + \frac{s}{s^2+9}\right\}$$

3) Dirac-delta function (Unit Impulse Function)

Def^n :- It is denoted by $\delta(t-a)$ and is defined as

$$\delta(t-a) = \lim_{\epsilon \to 0} F(t)$$

where

$$F(t) = \begin{cases} 0 & , t < a \\ 1/\epsilon & , a < t < a+\epsilon \\ 0 & , t > \epsilon \end{cases}$$

and, $\int_0^\infty F(t)\, dt = 1 \quad , \forall \epsilon$

* Results :-

1) $L\{\delta(t-a)\} = e^{-as}$

2) $L^{-1}\{e^{-as}\} = \delta(t-a)$

3) $L\{\delta(t)\} = 1$

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4) $L^{-1}(1) = \delta(t)$

5) $L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$

① $t\,H(t-4) + t^2\,\delta(t-4)$

$\rightarrow L\{t\,H(t-4) + t^2\,\delta(t-4)\}$

$= L\{t\,H(t-4)\} + L\{t^2\,\delta(t-4)\}$

$= e^{-4s} L\{t+4\} + e^{-4s}(4)^2$

$= e^{-4s}\left\{\frac{1}{s^2} + \frac{4}{s}\right\} + 16\,e^{-4s}$

$= e^{-4s}\left\{\frac{1}{s^2} + \frac{4}{s} + 16\right\}$

② Find $L\{t^2 H(t-2) - \cosh t\;\delta(t-4)\}$

$\rightarrow L\{t^2 H(t-2) - \cosh t\;\delta(t-4)\}$

$= e^{-2s} L\{(t+2)^2\} - e^{-4s}\cosh 4$

$= e^{-2s} L\{t^2 + 4t + 4\} - e^{-4s}\cosh(4)$

$= e^{-2s}\left\{\frac{2!}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right\} - e^{-4s}\cosh(4)$

Application of Laplace Transform.

$L\{y'(t)\} = s\bar{y} - y(0)$

$L\{y''(t)\} = s^2\bar{y} - s\,y(0) - y'(0)$

$L\{y'''(t)\} = s^3\bar{y} - s^2 y(0) - s\,y'(0) - y''(0)$

① Solve $3\frac{dy}{dt} + 2y = e^{3t}$, when $y=1$ at $t=0$

Given, $3y' + 2y = e^{3t}$, $y = 1$ at $t = 0$

$3L\{y'\} + 2L\{y\} = L\{e^{3t}\}$

$3\{s\bar{y} - y(0)\} + 2\bar{y} = \frac{1}{s-3}$

$3(s\bar{y} - 1) + 2\bar{y} = \frac{1}{s-3}$

$(3s+2)\bar{y} = \frac{1}{s-3} + 3$

$\bar{y} = \frac{1 + 3(s-3)}{(3s+2)(s-3)}$

$y = L^{-1}(\bar{y})$

$= L^{-1}\left\{\frac{3s-8}{(3s+2)(s-3)}\right\}$

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By Partial Fraction,

$$\frac{3s-8}{(s-3)(3s+2)} = \frac{A}{s-3} + \frac{B}{(3s+2)}$$

$$3s-8 = A(3s+2) + B(s-3)$$

$$\therefore A = \frac{1}{11}, \quad B = \frac{30}{11}$$

$$\therefore y = L^{-1}\left\{\frac{\frac{1}{11}}{s-3} + \frac{\frac{30}{11}}{3s+2}\right\}$$

$$y = \frac{1}{11} L^{-1}\left\{\frac{1}{s-3}\right\} + \frac{30}{11} L^{-1}\left\{\frac{1}{3s+2}\right\}$$

$$= \frac{1}{11} e^{3t} + \frac{30}{11} \times \frac{1}{3} L^{-1}\left\{\frac{1}{s+\frac{2}{3}}\right\}$$

$$= \frac{e^{3t}}{11} + \frac{10}{11} e^{-\frac{2}{3}t}$$

2) Solve $(D^2-3D+2)y = 4e^{2t}$, with $y(0)=3$
$y'(0)=5$

Given, $y''-3y'+2y = 4e^{2t}$, $y(0)=3$, $y'(0)=5$

$$L(y'')-3L(y')+2L(y) = 4L\{e^{2t}\}$$

$$s^2\bar{y} - sy(0) - y'(0) - 3\{s\bar{y} - y(0)\} + 2\bar{y} = \frac{4}{s-2}$$

$$s^2\bar{y} - s(3) - 5 - 3(s\bar{y} - 3) + 2\bar{y} = \frac{4}{s-2}$$

$$(s^2-3s+2)\bar{y} = \frac{4}{s-2} + 3s - 4$$

$$\bar{y} = \frac{3s^2-10s+12}{(s-2)(s^2-3s+2)}$$

$$y = L^{-1}(\bar{y})$$

$$= L^{-1}\left\{\frac{3s^2-10s+12}{(s-1)(s-2)^2}\right\}$$

By Partial Fraction.

$$\frac{3s^2-10s+12}{(s-1)(s-2)^2} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2}$$

$$3s^2-10s+12 = A(s-2)^2 + B(s-1)(s-2) + C(s-1)$$

Put $s-2=0$, $\therefore s=2$.

$\therefore 3\times4 - 10\times2 + 12 = C(2-1) = C$

$C = 12-20+12 = 4$

Put $s-1=0$. $\therefore s=1$

$\therefore 3(1)^2 - 10(1) + 12 = A(-1)^2$

$3-10+12 = A(1)$

$\boxed{A = 5}$

Put $s=0$

$12 = 5(-2)^2 + B(-1)(-2) + 4(-1)$

$12 = 5(4) + 2B - 4$

$16 - 20 = 2B$

$\boxed{B = -2}$

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$$y = L^{-1}\left\{\frac{5}{s-1} - \frac{2}{s-2} + \frac{4}{(s-2)^2}\right\}$$

$$= 5L^{-1}\left\{\frac{1}{s-1}\right\} - 2L^{-1}\left\{\frac{1}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$y = 5e^{t} - 2e^{2t} + 4e^{2t}(t)$$

3) Solve $\frac{d^2y}{dt^2} + 9y = 18(t)$, given that $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 0$

→ Given, $y'' + 9y = 18(t)$, $y(0) = 0 = y\left(\frac{\pi}{2}\right)$.

$$L(y'') + 9L(y) = 18L(t)$$

$$s^2\bar{y} - s\,y(0) - y'(0) + 9\bar{y} = \frac{18}{s^2}$$

$$s^2\bar{y} - s(0) - k + 9\bar{y} = \frac{18}{s^2}$$

$$(s^2+9)\bar{y} = \frac{18}{s^2} + k$$

$$\bar{y} = \frac{18}{s^2(s^2+9)} + \frac{k}{s^2+9}$$

$$y = L^{-1}\{\bar{y}\}$$

$$= L^{-1}\left\{\frac{18}{s^2(s^2+9)}\right\} + kL^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= 2L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+9}\right\} + \frac{k}{3}\sin 3t$$

$$y = 2t + \frac{(k-2)}{3}\sin 3t$$

But $y(\pi/2) = 0$

$$\therefore\ 0 = y\left(\frac{\pi}{2}\right) = 2\left(\frac{\pi}{2}\right) + \frac{k-2}{3}\sin\frac{3\pi}{2}$$

$$\therefore\ -\pi = \frac{k-2}{3}(-1)$$

$$\frac{k-2}{3} = \pi$$

$$y = 2t + \pi\sin 3t$$

4) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = t\,\delta(t-1)$, with $y(0) = y'(0) = 1$

→ Given

$$y'' + 3y' + 2y = t\,\delta(t-1),\quad y(0) = y'(0) = 1$$

$$L(y'') + 3L(y') + 2L(y) = L\{t\,\delta(t-1)\}$$

$$s^2\bar{y} - s\,y(0) - y'(0) + 3\{s\bar{y} - y(0)\} + 2\bar{y} = e^{-s}(1)$$

$$s^2\bar{y} - s(0) - 0 + 3(s\bar{y} - 0) + 2\bar{y} = e^{-s}$$

$$(s^2+3s+2)\bar{y} = e^{-s}$$

$$\bar{y} = \frac{e^{-s}}{s^2+3s+2}$$

$$y = L^{-1}\{\bar{y}\} = L^{-1}\left\{\frac{e^{-s}}{s^2+3s+2}\right\}$$

$$= f(t-1)\,H(t-1)$$

$$f(t) = L^{-1}\left\{\frac{1}{s^2+3s+2}\right\}$$

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$$= L^{-1}\left\{\frac{1}{s^2+3s+\frac{9}{4}-\frac{9}{4}+2}\right\}$$

$$= L^{-1}\left\{\frac{1}{\left(s+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right\}$$

$$= e^{\frac{-3t}{2}} L^{-1}\left\{\frac{1}{s^2-(1/2)^2}\right\}$$

$$f(t) = e^{\frac{-3t}{2}}\, 2\sinh\left(\frac{t}{2}\right)$$

$$y = e^{-3t/2}(2)\sinh\frac{t}{2}$$

$$y = e^{-3(t-1)/2}(2)\sinh\frac{(t-1)}{2}\, H(t-1)$$

26-7-18

FOURIER SERIES

Definition :-

Let $f(x)$ be a periodic function defined on $(c, c+2l)$ then $f(x)$ can be expressed as :-

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

This is known as Fourier series of $f(x)$, where,

$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

a_0, a_n, b_n are fourier/Euler's constants

Drichlit's condition :-

A function $f(x)$ defined in the interval can be expressed as a Fourier series if :-

i) $f(x)$ & its integrals $\int f(x)$ are finite & single valued.

ii) $f(x)$ has finite no. of discontinuities.

iii) $f(x)$ has finite no. of maxima & minima.

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Eg :- $\tan x$ $[0, 2\pi]$ cannot be expressed as a Fourier series in $[0, 2\pi]$ because

$$\boxed{\tan \frac{\pi}{2} = \infty}$$

Problems

1) Find a Fourier series to represent $f(x) = x^2$ in $(0, 2\pi)$ and hence deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

→ Let,

$$f(x) = x^2 = a_0 + \sum a_n \cos nx + \sum b_n \sin nx \quad (1)$$

Now,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$a_0 = \frac{1}{2\pi} \times \frac{1}{3} \times 8\pi^3 = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[x^2 \frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(0 + \frac{4\pi}{n^2} + 0\right) - (0 + 0 - 0)\right]$$

$$a_n = \frac{1}{\pi}\left(\frac{4\pi}{n^2}\right) = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left\{x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right\}_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(0 + \cancel{2x(2\pi)}\right)\right.$$

$$= \frac{1}{\pi}\left[\left(\frac{4\pi^2(-1)}{n} + 0 + \frac{2}{n^3}\right) - \left(0 - 0 + \frac{2}{n^3}\right)\right]$$

$$= \frac{1}{\pi} \times \frac{(-4\pi^2)}{n}$$

$$= -\frac{4\pi}{n}$$

$$\boxed{\therefore b_n = -\frac{4\pi}{n}}$$

Putting above values in (1),

$$x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{-4\pi}{n} \sin nx$$

Put $x = \pi$

$$\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n + 0$$

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$$-\frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n$$

$$-\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} \cdots$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} \cdots$$

⇒ Parseval's Identity :-

Let $f(x)$ be a function defined on $(c, c+2l)$, then the identity

$$\frac{1}{2l}\int_c^{c+2l} \{f(x)\}^2 dx = a_0^2 + \frac{1}{2}\sum (a_n^2 + b_n^2)$$

is called Parseval's Identity.

2) ⇒ Obtain Fourier expansion of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $0 \leq x \leq 2\pi$ and $f(x) = f(x+2\pi)$

i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

iv) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Now,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 dx$$

$$= \frac{1}{8\pi}\left[\frac{(\pi - x)^3}{(-3)}\right]_0^{2\pi} = \left[\frac{(\pi - x)^3}{-8\pi}\right]_0^{2\pi}$$

$$= \frac{1}{-8\pi}\left[(\pi - 2\pi)^3 - (\pi - 0)^3\right]$$

$$= \frac{-1}{8\pi}\left[(-\pi)^3 - \pi^3\right]$$

$$= \frac{-1}{8\pi}\left[-2\pi^3\right] = \frac{\pi^2}{4} \times \frac{1}{3} = \frac{\pi^2}{12}$$

$$a_n = \frac{1}{l}\int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{l}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \cos nx \, dx$$

$$= \frac{1}{4l}\int_0^{2\pi} (\pi - x)^2 \cdot \cos nx \, dx$$

$$= \frac{1}{4l}\left[(\pi - x)^2 \frac{\sin nx}{n} - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

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$$= \frac{1}{4\pi}\left\{\left(0 + \frac{2\pi}{n^2} - 0\right) - \left(0 - \frac{2\pi}{n^2} - 0\right)\right\}$$

$$= \frac{1}{4\pi}\left[\frac{4\pi}{n^2}\right] = \frac{1}{n^2}$$

$$\boxed{\therefore a_n = \frac{1}{n^2}}$$

$$b_n = \frac{1}{l}\int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \sin nx \, dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx \, dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n} + 0 + \frac{2}{n^3}\right) - \left(-\frac{\pi^2}{n} + 0 + \frac{2}{n^3}\right)\right]$$

$$= \frac{1}{4\pi}(0) = 0$$

$$\boxed{\therefore b_n = 0}$$

Putting above values in (1)

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx.$$

i) put $x = 0$

$$\therefore \frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

ii) put $x = \pi$

$$0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n$$

$$-\frac{\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \dots$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \dots$$

iii) Adding results of (i) & (ii);

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right]$$

$$\frac{2\pi^2 + \pi}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right]$$

$$\therefore \frac{3\pi^2}{24} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right]$$

$$\therefore \frac{\pi^2}{8} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right]$$

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iv) By Parseval's Identity :-

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum(a_n^2+b_n^2)$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^4 dx = \frac{\pi^4}{144}+\frac{1}{2}\sum\frac{1}{n^4}$$

$$\frac{1}{32\pi}\left[\frac{(\pi-x)^5}{(-5)}\right]_0^{2\pi} = \frac{\pi^4}{144}+\frac{1}{2}\sum\frac{1}{n^4}$$

$$-\frac{1}{160\pi}\left[-\pi^5-\pi^5\right] = \frac{\pi^4}{144}+\frac{1}{2}\sum\frac{1}{n^4}$$

$$\frac{\pi^4}{80} = \frac{\pi^4}{144}+\frac{1}{2}\sum\frac{1}{n^4}$$

$$2\left(\frac{\pi^4}{80}-\frac{\pi^4}{144}\right) = \sum\frac{1}{n^4}$$

$$\frac{2\pi^4}{180} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\dots = \frac{\pi^4}{90}$$

3) Find Fourier Series for $f(x)$ in $(0,2\pi)$ where $f(x) = x$, $0<x\le\pi$
$= 2\pi - x$, $\pi\le x<2\pi$

Hence deduce $\frac{\pi^4}{96} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots$

→ Let,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi-x)\,dx\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi}+\left[2\pi x-\frac{x^2}{2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2}+\left[2\pi(2\pi-\pi)-\frac{(2\pi)^2-\pi^2}{2}\right]\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2}+\left[2\pi^2-\frac{3\pi^2}{2}\right]\right]$$

$$= \frac{1}{2\pi}\left[-\pi^2+2\pi^2\right] = \frac{1}{2\pi}\left[\pi^2\right] = \frac{\pi}{2}$$

$$\boxed{a_0 = \frac{\pi}{2}}$$

$$a_n = \frac{1}{l}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi-x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\left(\frac{x\sin nx}{n}-(1)\left(-\frac{\cos nx}{n^2}\right)\right)\right]_0^{\pi}+\left[\frac{(2n-x)\sin nx}{n}-(-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right.$$

$$= \frac{1}{\pi}\left\{0+\frac{(-1)^n}{n^2}-\left(0+\frac{1}{n^2}\right)\right\}+\frac{1}{n}\left\{\left(0-\frac{1}{n^2}\right)-\left(0-\frac{(-1)^n}{n^2}\right)\right.$$

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$$= \frac{1}{\pi}\left\{\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right\}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$= 0$, if n is even

$= \frac{-4}{\pi n^2}$, if n is odd

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x \sin nx\, dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\, dx\right.$$

$$= \frac{1}{\pi}\left\{x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right\}_0^{\pi}$$

$$+ \frac{1}{\pi}\left\{(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right\}_{\pi}^{2\pi}$$

2) → Obtain Fourier expansion of $|\cos x|$ in $(-\pi, \pi)$

→ Here,

$$f(-x) = |\cos(-x)| = |\cos x| = f(x)$$

$\therefore f(x)$ is even function

$\therefore b_n = 0$

Let, $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ ... (1)

Now, $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx$

$$= \frac{1}{\pi}\int_0^{\pi} |\cos x|\, dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} -\cos x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{(1-0) - (0-1)\right\}$$

$$a_0 = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx\, dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} \cos x \cos nx\, dx + \int_{\pi/2}^{\pi} -\cos x \cos nx\, dx\right\}$$

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$$= \frac{1}{\pi}\left[\int_0^{\pi/2}\{\cos(n+1)x + \cos(n-1)x\}\,dx - \int_{\pi/2}^{\pi}\{\cos(n+1)x + \cos(n-1)x\}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_0^{\pi/2} - \left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} - 0 - 0 - 0 - 0 + \frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{2\sin(n-1)\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\left(\frac{-2}{n^2-1}\right)\cos\frac{n\pi}{2}$$

$$a_n = \frac{-4\cos\frac{n\pi}{2}}{\pi(n^2-1)}$$

$$\therefore a_n = \begin{cases} 0, & \text{if } n \text{ is odd } \& \ n \neq 1 \\ \dfrac{-4\cos n\pi/2}{\pi(n^2-1)}, & \text{if } n \text{ is even} \end{cases}$$

If $n = 1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi}|\cos x|\cos x\,dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2}\cos^2 x\,dx - \int_{\pi/2}^{\pi}\cos^2 x\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2}\frac{1+\cos 2x}{2}\,dx - \int_{\pi/2}^{\pi}\frac{1+\cos 2x}{2}\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2}(1+\cos 2x)\,dx - \int_{\pi/2}^{\pi}(1+\cos 2x)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left(x + \frac{\sin 2x}{2}\right)_0^{\pi/2} - \left(x + \frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left(\frac{\pi}{2} + 0 - 0 + 0\right) - \left(\pi - 0 - \frac{\pi}{2} - 0\right)\right\}$$

$$= \frac{1}{\pi}(0)$$

$$\boxed{\therefore a_1 = 0}$$

putting above values in ①

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\left\{\frac{-1}{2^2-1}\cos 2x + \frac{1}{4^2-1}\cos 4x + \frac{-1}{6^2-1}\cos 6x + \dots\right\}$$

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left\{\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x \dots\right\}$$

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3) Obtain Fourier series of $x\cos x$ in $(-\pi,\pi)$

→ Here,

$$f(-x) = (-x)\cos(-x) = -x\cos x = -f(x)$$

$$\therefore a_0 = a_n = 0$$

Let,

$$f(x) = x\cos x = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{①}$$

Now,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos x.\sin nx$$

$$= \frac{1}{\pi}\int_0^{\pi} x.(2\sin nx.\cos x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\{\sin(n+1)x + \sin(n-1)x\}dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\sin(n+1)x\,dx + \int_0^{\pi} x\sin(n-1)x\,dx\right\}$$

$$= \frac{1}{\pi}\left[\left\{x\left(\frac{-\cos(n+1)x}{n+1}\right) - 1\left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right\}_0^{\pi}\right.$$

$$\left. + \left\{x\left(\frac{-\cos(n-1)x}{(n-1)}\right) - 1\left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left\{\frac{-\pi\cos(n+1)\pi}{n+1} + 0 + 0 - 0 - \frac{\pi\cos(n-1)\pi}{(n-1)}\right\}$$

$$= \frac{-\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{(n-1)}$$

$$= \frac{-\cos(n\pi+\pi)}{n+1} - \frac{\cos(n\pi-\pi)}{n-1}$$

$$= \frac{\cos n\pi}{n+1} + \frac{\cos n\pi}{n-1}$$

$$= \left(\frac{1}{n+1} + \frac{1}{n-1}\right)(-1)^n$$

$$b_n = \frac{2n(-1)^n}{n^2-1}, \text{ if } n \neq 1$$

for $n = 1$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x(2\sin x\cos x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left\{x\left(\frac{-\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{4}\right)\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{-\frac{\pi}{2} - 0 + 0 - 0\right]$$

$$b_1 = \frac{-1}{2}$$

$$\therefore x\cos x = \frac{-1}{2}\sin x + \sum_{n=2}^{\infty}\frac{2n(-1)^n}{n^2-1}\sin nx$$

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4) Find $f(x) = x^2$ in $(0, a)$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

→ Compare $(0, a)$ with $(0, 2l)$

$\therefore 2l = a \Rightarrow \boxed{l = a/2}$

Let,

$$f(x) = x^2 = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum a_n \cos\frac{2n\pi x}{a} + \sum b_n \sin\frac{2n\pi x}{a} \quad (1)$$

Now,

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{a}\int_0^{a} x^2\,dx$$

$$= \frac{1}{a}\left\{\frac{x^3}{3}\right\}_0^a$$

$$= \frac{1}{a}\left\{\frac{a^3}{3}\right\} = \frac{a^2}{3}$$

$$a_0 = \frac{a^2}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{a}\int_0^{a} x^2\cos\frac{2n\pi x}{a}\,dx$$

$$= \frac{2}{a}\left\{x^2\left(\frac{-a}{2n\pi}\cos\frac{2n\pi x}{a}\right) - 2x\left(\frac{-a^2}{4n^2\pi^2}\sin\frac{2n\pi x}{a}\right) + 2\left(\frac{a^3}{8n^3\pi^3}\cos\frac{2n\pi x}{a}\right)\right\}_0^a$$

$$= \cancel{\frac{2}{a}\left\{\left(\frac{-a^3}{2n\pi} + 0 + \frac{a^3}{4n^3\pi^3}\right) - \left(0 + 0 + \frac{a^3}{4n^3\pi^3}\right)\right\}}$$

$$= \frac{2}{a}\left\{\left(0 + \frac{a^3}{2n^2\pi^2} + 0\right) - (0 + 0 + 0)\right\}$$

$$a_n = \frac{a^2}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{a}\int_0^{a} x^2\sin\frac{2n\pi x}{a}\,dx$$

$$= \frac{2}{a}\left\{x^2\left(\frac{-a}{2n\pi}\cos\frac{2n\pi x}{a}\right) - 2x\left(\frac{-a^2}{4n^2\pi^2}\sin\frac{2n\pi x}{a}\right) + 2\left(\frac{a^3}{8n^3\pi^3}\cos\frac{2n\pi x}{a}\right)\right\}_0^a$$

$$= \frac{2}{a}\left\{\left(\frac{-a^3}{2n\pi} + 0 + 0 + \frac{a^3}{4n^3\pi^3}\right) - \left(0 + 0 + \frac{a^3}{4n^3\pi^3}\right)\right\}$$

$$= \frac{2}{a}\left(\frac{-a^3}{2n\pi}\right)$$

$$b_n = \frac{-a^2}{n\pi}$$

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Putting value in ①,

$$x^2 = \frac{a^2}{3} + \sum_{n=1}^{\infty} \frac{a^2}{n^2\pi^2} \cos\frac{2n\pi x}{a} + \sum_{n=1}^{\infty} \frac{-a^2}{n\pi} \sin\frac{2n\pi x}{a}$$

Put $x = 0$

$$0 = \frac{a^2}{3} + \sum_{n=1}^{\infty} \frac{a^2}{n^2\pi^2}$$

$$\frac{-a^2}{3} = \frac{a^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{-\pi^2}{3} = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad ②$$

Put $x = a$

$$a^2 = \frac{a^2}{3} + \sum_{n=1}^{\infty} \frac{a^2}{n^2\pi^2}$$

$$\frac{2a^2}{3} = \frac{a^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad ③$$

Adding ② & ③

$$\frac{\pi^2}{3} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Ex ⇒ $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$

S.T. :- $f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)\pi x$

→ Compare $(0,2)$ with $(0,2l)$

$2l = 2$

$l = 1$

Let,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \quad ①$$

$$a_0 = \frac{1}{2l} \int_0^{2l} f(x)\,dx$$

$$= \frac{1}{2} \int_0^2 f(x)\,dx$$

$$= \frac{1}{2} \left\{ \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx \right\}$$

$$= \frac{1}{2} \left\{ \pi \left(\frac{x^2}{2}\right)_0^1 + \pi \left(2x - \frac{x^2}{2}\right)_1^2 \right\}$$

$$= \frac{1}{2} \left\{ \pi\left(\frac{1}{2} - 0\right) + \pi\left(4 - 2 - 2 + \frac{1}{2}\right) \right\}$$

$$a_0 = \frac{\pi}{2}$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1} \int_0^2 f(x) \cos n\pi x\,dx$$

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$$= \int_0^1 \pi x \cos n\pi x \, dx + \int_1^2 \pi (2-x) \cos n\pi x \, dx$$

$$= \pi \left[\left\{ x \left(\frac{\sin n\pi x}{n\pi} \right) - 1 \left(\frac{-\cos n\pi x}{(n\pi)^2} \right) \right\}_0^1 + \left\{ (2-x) \left(\frac{\sin n\pi x}{n\pi} \right) - (-1) \left(\frac{-\cos n\pi x}{n^2\pi^2} \right) \right\}_1^2 \right]$$

$$= \pi \left\{ 0 + \frac{(-1)^n}{n^2\pi^2} - 0 - \frac{1}{n^2\pi^2} + 0 - \frac{1}{n^2\pi^2} + 0 + \frac{(-1)^n}{n^2\pi^2} \right\}$$

$$= \frac{\pi}{n^2\pi^2} \left\{ 2(-1)^n - 2 \right\}$$

$$a_n = \frac{2}{n^2\pi} \left\{ (-1)^n - (1) \right\}$$

$$\cancel{a_n = \frac{2}{n^2\pi}}$$

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ -\dfrac{4}{n^2\pi}, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{1} \int_0^2 f(x) \sin n\pi x \, dx$$

$$= \int_0^1 f(x) \sin n\pi x \, dx + \int_1^2 \pi (2-x) \sin n\pi x \, dx$$

$$= \pi \left\{ x \left(\frac{-\cos n\pi x}{n\pi} \right) - 1 \left(\frac{-\sin n\pi x}{n^2\pi^2} \right) \right\}_0^1 + \pi \left\{ (2-x) \left(\frac{-\cos n\pi x}{n\pi} \right) - (-1) \left(\frac{-\sin n\pi x}{n^2\pi^2} \right) \right\}_1^2$$

$$= \pi \left\{ -\frac{(-1)^n}{n\pi} + 0 + 0 + 0 \right\} + \pi \left\{ 0 - 0 + \frac{(-1)^n}{n\pi} + 0 \right\}$$

$$b_n = 0$$

Putting above values in ①

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left\{ \frac{1}{1^2} \cos \pi x + \frac{1}{3^2} \cos 3\pi x + \frac{1}{5^2} \cos 5\pi x + \dots \right\}$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos (2n+1)\pi x$$

Ex ⇒ $f(x) = 1 - x^2$ in $(-1, 1)$

→ Compare $(-1, 1)$ with $(-l, l)$

$$\boxed{\therefore l = 1}$$

Also,

$$f(-x) = 1 - (-x)^2 = 1 - x^2 = f(x)$$

$\therefore f(x)$ is an even function

$$\boxed{\therefore b_n = 0}$$

Let,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x \quad \text{①}$$

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$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$= \frac{1}{1}\int_0^1 (1-x^2)\,dx$$

$$a_0 = \frac{2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{1}\int_0^1 (1-x^2)\cos n\pi x\,dx$$

$$= 2\left\{(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right\}_0^1$$

$$= 2\left\{\left(0 - \frac{2(-1)^n}{n^2\pi^2} + 0\right) - (0+0+0)\right\}$$

$$a_n = -\frac{4(-1)^n}{n^2\pi}$$

Putting in ①

$$1-x^2 = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n^2\pi^2}\cos n\pi x$$

Ex ⇒ $f(x) = \begin{cases} 0, & -2 < x < -1 \\ 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$

→ Compare $(-2, 2)$ with $(-l, l)$

$\therefore l = 2$

Also,

$$f(-x) = \begin{cases} 0, & -2 < -x < -1 \\ 1-x, & -1 < -x < 0 \\ 1+x, & 0 < -x < 1 \\ 0, & 1 < -x < 2 \end{cases}$$

$$= \begin{cases} 0, & 2 > x > 1 \text{ ie } 1 < x < 2 \\ 1-x, & 1 > x > 0 \quad 0 < x < 1 \\ 1+x, & 0 > x > -1 \quad -1 < x < 0 \\ 0, & -1 > x > -2 \quad -2 < x < -1 \end{cases}$$

$\therefore f(-x) = f(x)$

$\therefore f(x)$ is even function,

$\therefore b_n = 0$

Let,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \quad \text{.... ①}$$

Now,

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$= \frac{1}{2}\int_0^2 f(x)\,dx$$

$$= \frac{1}{2}\left\{\int_0^1 (1-x)\,dx + \int_1^2 0\,dx\right\}$$

$$= \frac{1}{2}\left\{x - \frac{x^2}{2}\right\}_0^1 \qquad \boxed{\therefore a_0 = \frac{1}{4}}$$

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$$\cancel{a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 (1-x)\cos\frac{n\pi x}{2}\,dx + \int_1^2 0.\cos\frac{n\pi x}{2}\,dx$$

$$= \left\{(1-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (-1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}$$

$$= 0 - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - 0 + \frac{4}{n^2\pi^2}$$

$$\therefore a_n = \frac{4}{n^2\pi^2}\left[1-\cos\frac{n\pi}{2}\right]$$

Putting above value in ①,

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left\{1-\cos\frac{n\pi}{2}\right\}\cos\frac{n\pi x}{2}$$

Obtain half-range sine series for

1) $f(x)$, when

$$f(x) = \begin{cases} x & , 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \pi \end{cases}$$

Hence find the sum, $\sum_{2n-1}^{\infty}\frac{1}{n^4}$

and deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

→ Let,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots\ldots (1)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi-x)\sin nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right\}_0^{\pi/2}$$

$$+ \frac{2}{\pi}\left\{(\pi-x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right\}_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left\{-\frac{\pi\cos n\pi/2}{2n} + \frac{\sin n\pi/2}{n^2} - 0 - 0\right\}$$

$$+ \frac{2}{\pi}\left\{0 - 0 + \frac{\pi\cos n\pi/2}{2n} + \frac{\sin n\pi/2}{n^2}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{2\sin(n\pi/2)}{n^2}\right\}$$

$$= \frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)$$

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$$b_n = \begin{cases} \frac{4}{n^2\pi} \sin\frac{n\pi}{2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Putting above value in ①

$$f(x) = \frac{4}{\pi}\left\{\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \dots\right\}$$

By Parseval's Identity,

$$\frac{1}{\pi}\int_0^{\pi}\{f(x)\}^2\,dx = \frac{1}{2}\sum b_n^2$$

$$\frac{1}{\pi}\left\{\int_0^{\pi/2} x^2\,dx + \int_{\pi/2}^{\pi}(\pi-x)^2\,dx\right\} = \frac{1}{2}\,\frac{16}{\pi^2}\left\{\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots\right\}$$

$$= \frac{8}{\pi^2}\left\{\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots\right\}$$

$$= \frac{8}{\pi^2}\sum_{2n-1}^{\infty}\frac{1}{n^4}$$

LHS:- $\frac{1}{\pi}\left\{\int_0^{\pi/2} x^2\,dx + \int_{\pi/2}^{\pi}(\pi-x)^2\,dx\right\}$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi/2} + \frac{1}{\pi}\left[\frac{(\pi-x)^3}{-3}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{3\pi}\times\left(\frac{\pi}{2}\right)^3 + \frac{1}{-3\pi}\left[0-\left(\pi-\frac{\pi}{2}\right)^3\right]$$

$$= \frac{\pi^2}{24} - \frac{1}{3\pi}\left[\frac{-\pi^3}{8}\right]$$

$$= \frac{\pi^2}{24} + \frac{\pi^2}{24} = \frac{\pi^2}{12}$$

$$\therefore \frac{\pi^2}{12} = \frac{8}{\pi^2}\sum_{2n-1}^{\infty}\frac{1}{n^4}$$

$$\therefore \sum_{2n-1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96}$$

For deduction put $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \frac{4}{\pi}\left\{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right\}$$

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-} x = \frac{\pi}{2}$$

$$\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^+}(\pi-x) = \frac{\pi}{2}$$

$$\therefore f\left(\frac{\pi}{2}\right) = \frac{1}{2}\left\{\frac{\pi}{2}+\frac{\pi}{2}\right\} = \frac{\pi}{2}$$

Thus

$$\frac{\pi}{2} = \frac{4}{\pi}\left\{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right\}$$

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Expand $f(x) = lx - x^2$ in $0 < x < l$ in a half range
i) cosine series
ii) sine series
Hence from sine series deduce that

i) $\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$

ii) $\frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots$

iii) $\frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots$

→ Here, $f(x) = lx - x^2$ on $(0, l)$

i) For half range cosine series,

Let,

$$f(x) = lx - x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \cdots (1)$$

$$a_0 = \frac{1}{l} \int_0^l f(x)\, dx$$

$$= \frac{1}{l} \int_0^l (lx - x^2)\, dx$$

$$= \frac{1}{l} \left\{ l\frac{x^2}{2} - \frac{x^3}{3} \right\}_0^l$$

$$= \frac{1}{l}\left(\frac{l^3}{6}\right) - \frac{l^2}{6}$$

$$\boxed{a_0 = \frac{l^2}{6}}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l} \int_0^l (lx - x^2) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l} \left\{ (lx - x^2)\left(\frac{l}{n\pi} \sin \frac{n\pi x}{l}\right) - (l - 2x)\left(\frac{-l^2}{n^2\pi^2} \cos \frac{n\pi x}{l}\right) + \cancel{(-2)\left(\frac{-l^3}{n^3\pi^3}(-1)^n\right)} + (-2)\left(\frac{-l^2}{n^3\pi^3} \sin \frac{n\pi x}{l}\right) \right\}_0^l$$

$$= \frac{2}{l} \left\{ \left(0 - \frac{l^3}{n^2\pi^2}(-1)^n + 0\right) - \left(0 + \frac{l^3}{n^2\pi^2} + 0\right) \right\}$$

$$= \frac{2l^2}{n^2\pi^2} \left\{ 1 + (-1)^n \right\}$$

$$a_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{-4l^2}{n^2\pi^2}, & \text{if } n \text{ is even} \end{cases}$$

putting above values in (1),

$$lx - x^2 = \frac{l^2}{6} - \frac{4l^2}{\pi^2} \left\{ \frac{1}{2^2} \cos \frac{2\pi x}{l} + \frac{1}{4^2} \cos \frac{4\pi x}{l} + \frac{1}{6^2} \cos \frac{6\pi x}{l} + \cdots \right\}$$

ii) For half range sine series:

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \cdots (2)$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

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$$= \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left\{(lx-x^2)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right\}_0^l$$

$$= \frac{2}{l}\left\{\left(0+0-\frac{2l^3}{n^3\pi^3}(-1)^n\right) - \left(0+0-\frac{2l^3}{n^3\pi^3}\right)\right\}$$

$$= \frac{4l^2}{n^3\pi^3}\left\{1-(-1)^n\right\}$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{8l^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \end{cases}$$

Putting b_n value in eqn ②

$$lx - x^2 = \frac{8l^2}{\pi^3}\left\{\frac{1}{1^3}\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \dots\right\}$$

Put $x = l/2$

$$\frac{l^2}{2} - \frac{l^2}{4} = \frac{8l^2}{\pi}\left\{\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots\right\}$$

ii) By Parseval's identity:

$$\frac{1}{l}\int_0^l \{f(x)\}^2 dx = \frac{1}{2}\Sigma b_n^2$$

$$\frac{1}{l}\int_0^l (lx-x^2)^2 dx = \frac{1}{2}\times\frac{64l^4}{\pi^6}\left\{\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\dots\right\}$$

$$= \frac{32l^4}{\pi^6}\left\{\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\dots\right\}$$

LHS:- $\frac{1}{l}\int_0^l (lx-x^2)^2 dx$

$$= \frac{1}{l}\int_0^l (l^2x^2 - 2lx^3 + x^4)\,dx$$

$$= \frac{1}{l}\left[l^2\frac{x^3}{3} - 2l\frac{x^4}{4} + \frac{x^5}{5}\right]_0^l$$

$$= \frac{1}{l}\left[\frac{l^5}{3} - \frac{l^5}{2} + \frac{l^5}{5}\right]$$

$$= \frac{1}{l}\left[\frac{10-15+6}{30}\right]l^5$$

$$= \frac{l^4}{30}$$

$$\therefore \frac{l^4}{30} = \frac{32}{\pi^6}l^4\left\{\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\dots\right\}$$

$$\frac{\pi^6}{960} = \left\{\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\dots\right\}$$

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iii) Let,

$$x = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots$$

$$= \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \cdots\right) + \left(\frac{1}{2^6} + \frac{1}{(2\times2)^6} + \frac{1}{(2\times3)^6} + \cdots\right)$$

$$x = \frac{\pi^6}{960} + \frac{1}{2^6}\left\{\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} \cdots\right\}$$

$$x = \frac{\pi^6}{960} + \frac{1}{64}x$$

$$\frac{63}{64}x = \frac{\pi^6}{960}$$

$$\therefore x = \frac{\pi^6}{960} \times \frac{64}{63} = \frac{\pi^6}{945}$$

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots = \frac{\pi^6}{945}$$

3) Prove that, in the interval $0 < x < \pi$

$$\frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\left\{\frac{\sin x}{a^2+1} - \frac{2\sin 2x}{a^2+4} + \frac{3\sin 3x}{a^2+9} \cdots\right\}$$

→ Let $f(x) = \dfrac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}}$ on $(0, \pi)$

Let,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \cdots (1)$$

Now,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} \sin nx \, dx$$

$$= \frac{2}{\pi(e^{a\pi} - e^{-a\pi})}\left[\int_0^{\pi} e^{ax} \sin nx \, dx - \int_0^{\pi} e^{-ax} \sin nx \, dx\right]$$

$$= \frac{1}{\pi \sinh a\pi}\left[\left\{\frac{e^{ax}}{a^2+n^2}(a \sin nx - n \cos nx)\right\} - \left\{\frac{e^{-ax}}{a^2+n^2}(-a \sin nx - n \cos nx)\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi \sinh a\pi}\left\{\frac{e^{a\pi}}{a^2+n^2}(0 - n(-1)^n) - \frac{1}{a^2+n^2}(-n)\right\}$$

$$- \frac{1}{\pi \sinh a\pi}\left\{\frac{e^{-a\pi}}{a^2+n^2}(0 - n(-1)^n) + \frac{1}{a^2+n^2}(0 - n)\right\}$$

$$= \frac{-n(-1)^n}{\pi(a^2+n^2)\sinh a\pi} \sinh a\pi$$

$$b_n = \frac{-2n(-1)^n}{\pi(a^2+n^2)}$$

Putting b_n value in (1)

$$f(x) = \sum_{n=1}^{\infty} \frac{-2n(-1)^n}{\pi(a^2+n^2)} \sin nx$$

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Orthogonal & Orthonormal Functions

⇒ Orthogonal

Def^n - A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ defined on (a, b), then the set of functions is said to be orthogonal if

$$\int_a^b f_n(x)\, f_m(x)\, dx = \begin{cases} 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$$

⇒ Orthonormal

A set of functions $\{f_n(x)\}_{n=1}^{\infty}$ defined on (a, b) is said to be orthonormal set if

$$\int_a^b f_n(x)\, f_m(x)\, dx = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$$

Every orthonormal set is ~~or~~ orthogonal but the converse may or may not be true.

Eg 1) Show that the set of functions $\sin(2n+1)x$ where $n = 0, 1, 2, \ldots$ is orthogonal over $\left[0, \frac{\pi}{2}\right]$. Hence construct orthonormal set of function.

→ Here, $f_n(x) = \sin(2n+1)x$

Consider

$$\int_0^{\pi/2} \sin(2n+1)x \, \sin(2m+1)x$$

$$= \frac{1}{2}\int_0^{\pi/2} \left\{\cos(2n-2m)x - \cos(2n+2m+2)x\right\} dx$$

$$= \frac{1}{2}\left\{\frac{\sin(2n-2m)x}{2n-2m} - \frac{\sin(2n+2m+2)x}{2n+2m+2}\right\}_0^{\pi/2}$$

Case i) If $m \neq n$

$$\int_0^{\pi/2} \sin(2n+1)x \, \sin(2m+1)x \, dx = 0.$$

Case ii) If $m = n$

$$\int_0^{\pi/2} \sin^2(2n+1)x \, dx = \int_0^{\pi/2} \frac{1 - \cos 2(2n+1)x}{2} dx$$

$$= \frac{1}{2}\left\{x - \frac{\sin(2(2n+1)x)}{2(2n+1)}\right\}_0^{\pi/2}$$

$$= \frac{1}{2}\left\{\frac{\pi}{2} - 0 - 0 + 0\right\} = \frac{\pi}{4}$$

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Thus,

$$\int_0^{\pi/2} \sin(2n+1)x \cdot \sin(2m+1)x \, dx = \begin{cases} 0 & , \text{if } m \neq n \\ \frac{\pi}{4} \neq 0 & , \text{if } m = n \end{cases}$$

$\therefore$ Set of functions $\sin(2n+1)x$, $n = 0, 1, 2, \ldots$ is orthogonal.

Dividing both the sides by $\frac{\pi}{4}$,

$$\frac{4}{\pi} \int_0^{\pi/2} \sin(2n+1)x \cdot \sin(2m+1)x \, dx = \begin{cases} 0 & , \text{if } m \neq n \\ 1 & , \text{if } m = n \end{cases}$$

$$\therefore \int_0^{\pi/2} \frac{2}{\sqrt{\pi}} \sin(2n+1)x \cdot \frac{2}{\sqrt{\pi}} \sin(2m+1)x \, dx = \begin{cases} 0 & , \text{if } m \neq n \\ 1 & , \text{if } m = n \end{cases}$$

Hence $\frac{2}{\sqrt{\pi}} \sin(2n+1)x$, $n = 0, 1, 2, \ldots$

2) A set of functions $f_n(x) = \cos nx$, where $n = 1, 2, \ldots$, on $(0, 2\pi)$

$\rightarrow$ Here, $f_n(x) = \cos nx$

$$\int_0^{2\pi} f_n(x) \cdot f_m(x) \, dx = \int_0^{2\pi} \cos nx \cdot \cos mx \, dx$$

$$= \frac{1}{2} \int_0^{2\pi} \{\cos(n+m)x + \cos(n-m)x\} \, dx$$

$$= \frac{1}{2} \left\{ \frac{\sin(n+m)x}{n+m} + \frac{\sin(n-m)x}{n-m} \right\}_0^{2\pi}$$

Case i) if $m \neq n$

$$\int_0^{2\pi} f_n(x) \cdot f_m(x) = 0$$

Case ii) If $m = n$

$$\int_0^{2\pi} f_n^2(x) \, dx = \int_0^{2\pi} \cos^2 nx \, dx$$

$$= \int_0^{2\pi} \frac{1 + \cos 2nx}{2} \, dx$$

$$= \frac{1}{2} \left\{ x + \frac{\sin 2nx}{2n} \right\}_0^{2\pi}$$

$$= \frac{1}{2} \{2\pi + 0\} = \pi \neq 0$$

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Thus,

$$\int_0^{2\pi} f_n(x) \cdot f_m(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi \neq 0, & \text{if } m = n \end{cases}$$

$\cos nx$ is orthogonal.

Q3) Prove that $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = \frac{3x^2-1}{2}$ over $(-1, 1)$

→ Consider,

$$\int_{-1}^{1} f_1(x)\, f_2(x)\,dx = \int_{-1}^{1} 1 \cdot x\,dx = \left\{\frac{x^2}{2}\right\}_{-1}^{1} = \frac{1}{2}\left[1-1\right] = 0$$

$$\int_{-1}^{1} f_2(x)\, f_3(x)\,dx = \int_{-1}^{1} x\left(\frac{3x^2-1}{2}\right)dx$$

$$= \frac{1}{2}\int_{-1}^{1}(3x^3 - x)\,dx$$

$$= \frac{1}{2}\left[\frac{3x^4}{4} - \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{3}{4} - \frac{1}{2} - \frac{3}{4} + \frac{1}{2}\right] = 0$$

$$\int_{-1}^{1} f_1(x)\, f_3(x)\,dx = \int_{-1}^{1} 1\left(\frac{3x^2-1}{2}\right)dx$$

$$= \frac{1}{2}\left[x^3 - x\right]_{-1}^{1} = \frac{1}{2}\left[1-1+1-1\right] = 0$$

$$\int_{-1}^{1} f_1^2(x)\,dx = \int_{-1}^{1} 1^2\,dx = \{x\}_{-1}^{1} = 1+1 = 2 \neq 0$$

$$\int_{-1}^{1} f_2^2(x)\,dx = \int_{-1}^{1} x^2\,dx = \left(\frac{x^3}{3}\right)_{-1}^{1} = \frac{1}{3}(1+1) = \frac{2}{3} \neq 0.$$

$$\int_{-1}^{1} f_3^2(x)\,dx = \int_{-1}^{1}\left(\frac{3x^2-1}{2}\right)^2 dx = \frac{1}{4}\int_{-1}^{1}(9x^4 - 6x^2 + 1)\,dx$$

$$= \frac{1}{4}\int_{-1}^{1}(9x^4 - 6x^2 + 1)\,dx$$

$$= \frac{1}{4}\left[\frac{9x^5}{5} - \frac{6x^3}{3} + x\right]_{-1}^{1}$$

$$= \frac{1}{4}\left[\frac{9}{5} - 2 + 1 + \frac{9}{5} - 2 + 1\right] = \frac{2}{5} \neq 0$$

Hence $f_1(x), f_2(x), f_3(x)$ are orthogonal functions.

Q4) If $f(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x)$, where c_1, c_2, c_3 are constants and $\phi_1(x), \phi_2(x), \phi_3(x)$ are orthonormal sets on (a,b). Show that,

$$\int_a^b \{f(x)\}^2\,dx = c_1^2 + c_2^2 + c_3^2$$

→ Since $\phi_1(x), \phi_2(x), \phi_3(x)$ are orthonormal on (a,b)

$$\int_a^b \{\phi_1(x)\}^2\,dx = \int_a^b \{\phi_2(x)\}^2\,dx = \int_a^b \{\phi_3(x)\}^2\,dx = 1$$

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and $\int_a^b \phi_1(x)\,\phi_2(x)\,dx = \int_a^b \phi_2(x)\,\phi_3(x)\,dx$

$$= \int_a^b \phi_1(x)\,\phi_3(x)\,dx = 0$$

Now,

$$\int_a^b \{f(x)\}^2 dx = \int_a^b \{c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x)\}^2 dx$$

$$= \int_a^b \{c_1^2\phi_1^2(x) + c_2^2\phi_2^2(x) + c_3^2\phi_3^2(x)$$
$$+ 2c_1c_2\phi_1(x)\phi_2(x)$$
$$+ 2c_2c_3\phi_2(x)\phi_3(x)$$
$$+ 2c_1c_3\phi_1(x)\phi_3(x)\}dx$$

$$= c_1^2\int_a^b \phi_1^2(x)dx + c_2^2\int_a^b \phi_2^2(x)dx + c_3^2\int_a^b \phi_3^2(x)dx$$
$$+ 2c_1c_2\int_a^b \phi_1(x)\,\phi_2(x) + 2c_2c_3\int_a^b \phi_2(x)\phi_3(x)dx$$
$$+ 2c_1c_3\int_a^b \phi_1(x)\,\phi_3(x)\,dx$$

$$= c_1^2(1) + c_2^2(1) + c_3^2(1) + 2c_1c_2(0)$$
$$+ 2c_2c_3(0) + 2c_1c_3(0)$$

$$\therefore \int_a^b \{f(x)\}^2 dx = c_1^2 + c_2^2 + c_3^2$$

Q5) $f_1(x) = 1$, $f_2(x) = x$ on $(-1, 1)$. Determine the constants a & b such that the function $f_3(x) = -1 + ax + bx^2$

→ Since $f_3(x)$ is orthogonal to $f_1(x)$ & $f_2(x)$

$$\int_{-1}^{1} f_1(x)\,f_3(x)\,dx = 0$$

$$\int_{-1}^{1} (1)(-1 + ax + bx^2)\,dx = 0$$

$$\left[-x + \frac{ax^2}{2} + \frac{bx^3}{3}\right]_{-1}^{1} = 0$$

$$\left[-1 + \frac{a}{2} + \frac{b}{3} - \left(1 + \frac{a}{2} - \frac{b}{3}\right)\right] = 0$$

$$-1 + \frac{a}{2} + \frac{b}{3} - 1 - \frac{a}{2} + \frac{b}{3} = 0$$

$$-2 + \frac{2b}{3} = 0$$

$$\frac{b}{3} = 1$$

$$\boxed{b = 3}$$

Also,

$$\int_{-1}^{1} f_2(x)\,f_3(x)\,dx = 0$$

$$\int_{-1}^{1} x(-1 + ax + bx^2)\,dx = 0$$

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$$\int_{-1}^{1}(-x+ax^2+bx^3)\,dx = 0$$

$$\left\{-\frac{x^2}{2}+\frac{ax^3}{3}+\frac{bx^4}{4}\right\}_{-1}^{1} = 0$$

$$-\frac{1}{2}+\frac{a}{3}+\frac{3}{4}+\frac{1}{2}+\frac{a}{3}-\frac{3}{4} = 0$$

$$\frac{2a}{3} = 0$$

$$\boxed{a = 0}$$

Complex form of Fourier series

Def^n:- Let $f(x)$ be a function defined on $(c, c+2l)$, then its complex form of Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

where $c_n = \frac{1}{2l}\int_{c}^{c+2l} f(x)\, e^{-in\pi x/l}\, dx$

⇒ Obtain complex form of Fourier series of $f(x) = e^{ax}$ in $(-l, l)$

→ Let,

$$f(x) = e^{ax} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \quad (1)$$

Now,

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\pi x/l}\, dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{ax}\, e^{-in\pi x/l}\, dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{(al-in\pi)x/l}\, dx$$

$$= \frac{1}{2l}\left\{\frac{l}{(al-in\pi)}\, e^{(al-in\pi)x/l}\right\}_{-l}^{l}$$

$$= \frac{1}{2(al-in\pi)}\left\{e^{al-in\pi} - e^{-(al-in\pi)}\right\}$$

$$= \frac{1}{2(al-in\pi)}\left\{e^{al}\, e^{-in\pi} - e^{-al}\, e^{in\pi}\right\}$$

But, $e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = (-1)^n$

$$c_n = \frac{1}{2(al-in\pi)}\left\{e^{al}(-1)^n - e^{-al}(-1)^n\right\}$$

$$= \frac{(-1)^n}{al-in\pi}\left(\frac{e^{al}-e^{-al}}{2}\right)$$

$$= \frac{(-1)^n}{al-in\pi}\sinh al$$

$$c_n = \frac{(-1)^n(al+in\pi)}{a^2l^2+n^2\pi^2}\sinh al$$

Putting val in (1),

$$e^{ax} = \sum_{-\infty}^{\infty}\frac{(-1)^n(al+in\pi)}{a^2l^2+n^2\pi^2}\sinh al\, e^{in\pi x/l}$$

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Find the complex Fourier Series of

i) $f(x) = e^{-ax}$ in $(-l, l)$

ii) $f(x) = e^{ax}$ in $(-\pi, \pi)$

iii) $f(x) = e^{-ax}$ in $(-\pi, \pi)$

iv) $f(x) = \sinh ax$ in $(-l, l)$

$= \frac{1}{2}(e^{ax} - e^{-ax})$ in $(-l, l)$

v) $f(x) = \cosh ax$ in $(-l, l)$

$= \frac{1}{2}(e^{ax} + e^{-ax})$ in $(-l, l)$

vi) $f(x) = \sinh ax$ in $(-\pi, \pi)$

$= \frac{1}{2}(e^{ax} - e^{-ax})$ in $(-\pi, \pi)$

vii) $f(x) = \cosh ax$ in $(-\pi, \pi)$

$= \frac{1}{2}(e^{ax} + e^{-ax})$ in $(-\pi, \pi)$

viii) $f(x) = \cosh ax + \sinh ax$ in $(-l, l)$

$= \frac{e^{ax} + e^{-ax}}{2} + \frac{e^{ax} - e^{-ax}}{2}$

$= e^{ax}$ in $(-l, l)$

ix) $f(x) = \cosh ax + \sinh ax$ in $(-\pi, \pi)$

$= e^{ax}$ in $(-\pi, \pi)$

x) $f(x) = \cosh 3x + \sinh 3x$ in $(-3, 3)$

Put $a = 3$ in (ix)

xi) $f(x) = \sin ax$ in $(-\pi, \pi)$

$= \frac{1}{2i}(e^{iax} - e^{-iax})$ in $(-\pi, \pi)$.

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COMPLEX VARIABLE

Analytic function:-

* A function $f(z) = u+iv$ is said to be analytic function (regular or holomorphic) at z_0 if it is differentiable at z_0 and in some neighbourhood of z_0.

Theorem:-

$f(z) = u+iv \longrightarrow$ analytic

The necessary & sufficient condition for a continuous & single valued function $f(z) = u+iv$ will be analytic if

i) u_x, u_y, v_x, v_y are continuous fns.

ii) $u_x = v_y$ & $u_y = -v_x$

$$\therefore \left.\begin{aligned} u_x - v_y &= 0 \\ u_y + v_x &= 0 \end{aligned}\right\} \text{Cauchy-Riemann condition}$$

Equations in condition (ii) are called Cauchy-Riemann Equations or briefly C-R equations

Note:-

1) If $f(z)$ is analytic, then $f'(z) = u_x + i v_x$

2) If $f(z)$ is analytic fn in z then it can be differentiated & integrated in usual manner.

If function is analytic the u & v are conjugates of each other

⇒ Cauchy-Riemann Eqns in polar form:

If $f(z) = u(r,\theta) + i v(r,\theta)$, then CR-eqns will be

$$u_r = \frac{1}{r} v_\theta$$

$$u_\theta = -r v_r$$

* A function $\phi(x,y)$ is said to be Harmonic function if it satisfies Laplace's eqn

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

⇒ If $f(z)$ be an analytic function, then u & v are harmonic functions called Harmonic conjugates.

Note:-

* Laplace equation in polar form

If $\phi(r,\theta)$ be a f^n, then its Laplace equation will be

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

⇒ Orthogonal trajectories

By orthogonal trajectory we mean a curve perpendicular to every member of the family of curves

Note:
If $f(z) = u + iv$ is analytic function then the curves $u(x,y) = c_1$ & $v(x,y) = c_2$ are orthogonal trajectories to each other.

→ Show the following functions are analytic & find their derivatives.

$f(z) = u + iv$

i) $\sinh z$

Here $f(z) = \sinh z$

$= \sinh(x+iy)$

$= \sinh x \cosh iy + \cosh x \sinh iy$

$= \sinh x \cos y + i \cosh x \sin y$

$= u + iv$

$u = \sinh x \cos y$

$v = \cosh x \sin y$

$u_x = \cosh x \cos y$

$u_y = -\sinh x \sin y$

$v = \cosh x \sin y$

$v_x = \sinh x \sin y$

$v_y = \cosh x \cos y$

Thus,

$u_x = \cosh x \cos y$, $v_y = \cosh x \cos y$

$u_y = -\sinh x \sin y$, $v_x = \sinh x \sin x$

$\therefore u_x = v_y$ & $u_y = -v_x$

This satisfies CR eqn

ii) ze^z

Here, $f(z) = ze^z$

$= (x+iy)e^{x+iy}$

$= e^x(x+iy)e^{iy}$

$= e^x(x+iy)(\cos y + i\sin y)$

$= e^x\{(x\cos y - y\sin y) + i(y\cos y + x\sin y)\}$

$= e^x(x\cos y - y\sin y) + i(e^x\{y\cos y + x\sin y\})$

$= u + iv$

$u = e^x(x\cos y - y\sin y)$

$v = e^x(y\cos y + x\sin y)$

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$\therefore u_x = e^x(x\cos y - y\sin y) + e^x\cos y$

$u_y = e^x(-x\sin y - \sin y - y\cos y)$

$v_x = e^x(y\cos y + x\sin y) + e^x\sin y$

$v_y = e^x(\cos y - y\sin y + x\cos y)$

$\therefore u_x = e^x(x\cos y - y\sin y + \cos y) = v_y$

$\therefore u_y = -e^x(x\sin y + \sin y + y\cos y) = -v_x$

$\therefore$ CR equations are satisfied.

$\therefore f(z) = ze^z$ is analytic function

$\Rightarrow$ Find the constants a, b, c, d, e if $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is analytic.

$\rightarrow f(z) = u + iv$

$u = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2)$

$v = 4x^3y - exy^3 + 4xy$

$u_x = 4ax^3 + 2bxy^2 + 2dx$

$u_y = 2bx^2y + 4cy^3 - 4y$

And $v_x = 12x^2y - ey^3 + 4y$

$v_y = 4x^3 - 3exy^2 + 4x$

By C-R eqn,

$u_x = v_y$

$\therefore 4ax^3 + 2bxy^2 + 2dx = 4x^3 - 3exy^2 + 4x$

Equating co-effients

$4a = 4$, $2b = -3e$, $2d = 4$

$a = 1$, $d = 2$

Also $u_y = -v_x$

$\therefore 2bx^2y + 4cy^3 - 4y = -(12x^2y - ey^3 + 4y)$

$\therefore 2b = -12$

$b = -6$

$4c = +e$ $\quad e = 4$

$c = \frac{e}{4} = \frac{4}{4} = 1$

$c = 1$

$\Rightarrow$ Find k s.t, $\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{kx}{y}\right)$ is analytic.

$\rightarrow$ Given, $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{kx}{y}\right)$

$= u + iv$

$u = \frac{1}{2}\log(x^2+y^2)$ & $v = \tan^{-1}\left(\frac{kx}{y}\right)$

$u_x = \frac{1}{2}\frac{1}{(x^2+y^2)}2x = \frac{x}{x^2+y^2}$

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Similarly, $u_y = \frac{y}{x^2+y^2}$

$$v_x = \frac{1}{1+\left(\frac{kx}{y}\right)^2}\,\frac{k}{y} = \frac{ky}{y^2+k^2x^2}$$

$$v_y = \frac{1}{1+\left(\frac{kx}{y}\right)^2}\left(\frac{-kx}{y^2}\right) = \frac{-kx}{y^2+k^2x^2}$$

Since $f(z)$ is analytic
By CR eqn

$u_x = v_y$ & $u_y = -v_x$

$\therefore \frac{x}{x^2+y^2} = \frac{-kx}{y^2+k^2x^2}$ & $\frac{y}{x^2+y^2} = \frac{-ky}{y^2+k^2x}$

On comparing, we get,

$k = -1$

⇒ Find p if $f(z) = r^2\cos 2\theta + i r^2 \sin p\theta$ is analytic.

→ Given :-

$f(z) = r^2\cos 2\theta + i r^2 \sin p\theta$
$= u + iv$

$u = r^2\cos 2\theta$, $v = r^2 \sin p\theta$

$u_r = 2r\cos 2\theta$

$u_\theta = r^2(-\sin 2\theta)\,2 = -2r^2\sin 2\theta$

$v_r = 2r\sin p\theta$

$v_\theta = r^2\cos p\theta \cdot p = pr^2\cos p\theta$

$\because f(z)$ is analytic, by CR-equation,

$u_r = \frac{1}{r}v_\theta$ & $u_\theta = -r v_r$

$\therefore 2r\cos 2\theta = \frac{1}{r}(pr^2\cos p\theta)$

& $-2r^2\sin 2\theta = -r(2r\sin p\theta)$

$\therefore 2r^2\cos 2\theta = pr^2\cos p\theta$ & $-2r^2\sin 2\theta = -2r^2\sin p\theta$

On comparing, we get,

$\boxed{p = 2}$

⇒ If $f(z)$ & $\overline{f(z)}$ both are analytic, P.T. both are constants.

→ Let $f(z) = u + iv$ be an analytic
By CR - equations

$u_x = v_y$ & $u_y = -v_x$ ①

& $\overline{f(z)} = u - iv = u + i(-v)$ is also analytic.
By CR - equations

$u_x = (-v)_y = -v_y$ & $u_y = -(-v)_x = v_x$ ②

Adding results of ① & ②,

$2u_x = 0$ & $2u_y = 0$
$\therefore u_x = 0$ $\therefore u_y = 0$

$\therefore u(x,y)$ is constant.

Subtracting results of ① & ②

$0 = 2v_y$ & $0 = 2v_x$
$v_y = 0$ & $v_x = 0$.

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$\therefore v(x,y) = \text{constant}$

$\therefore f(z) = u + iv = \text{constant}$

$\Rightarrow$ If $f(z)$ is analytic and $|f(z)|$ is constant then PT $f(z)$ is constant

$\rightarrow$ Since $f(z) = u+iv$ is analytic
By CR equation

$u_x = v_y$ & $u_y = -v_x$

Also, $|f(z)| = \text{constant} = c$ (say)

$\sqrt{u^2+v^2} = c$

$u^2+v^2 = c^2$

Diff wrt x & y partially,

$2uu_x + 2vv_x = 0$

$\therefore uu_x + vv_x = 0$ ..... ①

$2uu_y + 2vv_y = 0$

$uu_y + vv_y = 0$ ..... ②

Using CR-equations in ②

$\therefore -uv_x + vu_x = 0$ ..... ③

$u \times ① \Rightarrow u^2u_x + uvv_x = 0$

$+ \quad v \times ③ \Rightarrow -uvv_x + v^2u_x = 0$

$u^2u_x + v^2u_x = 0$

$(u^2+v^2)u_x = 0$

$c^2u_x = 0$

$u_x = 0$

Put $u_x = 0$ in ③, $-uv_x = 0$

$\therefore v_x = 0$

Now, $f'(z) = u_x + iv_x = 0 + i0 = 0$

$\therefore f(z)$ is constant.

HW $\Rightarrow$ 1) If $f(z)$ is analytic and if the amplitude (argument) of $f(z)$ is constant PT $f(z)$ is constant

2) P.T. an analytic function with its derivative 0 is constant.

$\Rightarrow$ If $u(x,y)$ is a harmonic function, PT $f(z) = u_x - iu_y$ is an analytic function.

$\rightarrow$ Since $u(x,y)$ is a harmonic function

$u_{xx} + u_{yy} = 0$ ①

Now, let $f(z) = u_x - iu_y = u + iv$

where $u = u_x$

& $v = -u_y$

$\therefore u_x = u_{xx}$, $u_y = u_{xy}$

& $v = -u_y$

$v_x = -u_{xy}$, $v_y = -u_{yy} = u_{xx}$ (from ①)

Thus, $u_x = u_{xx} = v_y$ & $v_x = -u_{xy} = -u_y$

CR eqns are satisfied by u & v

$f(z) = u + iv = u_x - iu_y$ is an analytic function.

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⇒ Method of Finding Analytic Function:-

→ Follow the following steps to obtain an analytic fn.

Steps:-

1) Find u_x, u_y or (v_x, v_y).
2) Use $f'(z) = u_x + iv_x$
3) Apply Milne-Thompson's method, ie put $x = z$, $y = 0$
4) Integrate

⇒ Find an analytic function $f(z) = u + iv$ where $u = \frac{x}{2}\log(x^2+y^2) - y\tan^{-1}\left(\frac{y}{x}\right) + \sin x \cosh y$

→ Given, $u = \frac{x}{2}\log(x^2+y^2) - y\tan^{-1}\left(\frac{y}{x}\right) + \sin x \cosh y$

$$u_x = \frac{1}{2}\log(x^2+y^2) + \frac{x}{2}\frac{2x}{x^2+y^2} - y\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) + \cos x \cosh y$$

$$= \frac{1}{2}\log(x^2+y^2) + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + \cos x \cosh y$$

$$u_x = \frac{1}{2}\log(x^2+y^2) + 1 + \cos x \cosh y$$

And,

$$u_y = \frac{x}{2}\frac{2y}{x^2+y^2} - \tan^{-1}\left(\frac{y}{x}\right) - y\frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x} + \sin x \sinh y$$

$$= \frac{xy}{x^2+y^2} - \tan^{-1}\left(\frac{y}{x}\right) - \frac{xy}{x^2+y^2} + \sin x \sinh y$$

$$= -\tan^{-1}\left(\frac{y}{x}\right) + \sin x \sinh y$$

Step 2:-

$$f(z) = u_x + iv_x$$
$$= u_x - iu_y$$
$$= \left(\frac{1}{2}\log(x^2+y^2) + 1 + \cos x \cosh y\right) - i\left(-\tan^{-1}\left(\frac{y}{x}\right) + \sin x \sinh y\right)$$

Step 3:- By Milne-Thompson's Method

Put $x = z$, $y = 0$.

$$f'(z) = \frac{1}{2}\log$$

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⇒ Find the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y + \cos 2x}$

→ Given,

$$u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$$

1) $$u_x = \frac{(\cosh 2y + \cos 2x)(2\cos 2x) - (\sin 2x)(-2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x + 2\cos^2 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x + 2}{(\cosh 2y + \cos 2x)^2}$$

$$u_y = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$$

2) $$f'(z) = u_x + i v_x$$
$$= u_x - i u_y$$
$$= \frac{2\cosh 2y \cos 2x + 2}{(\cosh 2y + \cos 2x)^2} - i \frac{-2\sin 2x \sin 2y}{(\cosh 2y + \cos 2x)^2}$$

3) By Milne-Thompson's Method put $x = z$, $y = 0$

$$\therefore f'(z) = \frac{2\cos 2z + 2}{(1 + \cos 2z)^2} + i(0)$$

$$= \frac{2}{1 + \cos 2z}$$

$$= \frac{2}{2\cos^2 z} = \sec^2 z.$$

4) Integrating

$$f(z) = \int \sec^2 z \, dz + c$$

$$f(z) = \tan z + c$$

⇒ Find the imaginary part of analytic function whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$. Also verify that v is harmonic

→ Given,

$$u = e^{2x}(x\cos 2y - y\sin 2y)$$

1) $$u_x = 2e^{2x}(x\cos 2y - y\sin 2y) + e^{2x}(\cos 2y)$$

$$u_y = e^{2x}(x\, 2\sin 2y(-1) - \sin 2y - 2y\cos 2y)$$

2) $$f'(z) = u_x + i v_x$$
$$= u_x - i u_y$$
$$= \{2e^{2x}(x\cos 2y - y\sin 2y) + e^{2x}\cos 2y\}$$
$$- i\{e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y)\}$$

3) By Milne-Thompson's method put $x = z$, $y = 0$

$$f'(z) = \{2e^{2z}(z - 0) + e^{2z}\} - i\{e^{2z}(0 - 0 + 0)\}$$

$$f'(z) = (2z + 1)e^{2z}$$

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4) Integrating

$$f(z) = \int (2z+1)\, e^{2z}\, dz + c$$

$$= (2z+1)\frac{e^{2z}}{2} - 2\,\frac{e^{2z}}{4} + ic$$

$$= (x+iy)\, e^{2(x+iy)} + ic$$

$$= (x+iy)\, e^{2x} e^{2iy} + ic$$

$$= e^{2x}(x+iy)(\cos 2y + i\sin 2y) + ic$$

$$= e^{2x}(x\cos 2y + ix\sin 2y + iy\cos 2y - y\sin 2y) + ic$$

$$= e^{2x}(x\cos 2y - y\sin 2y) + i\{e^{2x}(x\sin 2y + y\cos 2y) + c\}$$

Imaginary part $= e^{2x}(x\sin 2y + y\cos 2y) + c$

$$V_x = 2e^{2x}(x\sin 2y + y\cos 2y) + e^{2x}\sin 2y$$

$$V_{xx} = 4e^{2x}(x\sin 2y + y\cos 2y) + 2e^{2x}\sin 2y + 2e^{2x}\sin 2y$$

$$= 4e^{2x}(x\sin 2y + y\cos 2y + \sin 2y)$$

Now

$$V_y = e^{2x}(2x\cos 2y - \cos 2y - 2y\sin 2y)$$

$$V_{yy} = e^{2x}(-4x\sin 2y - 2\sin 2y - 2\sin 2y - 4y\cos 2y)$$

$$= -4e^{2x}(x\sin 2y + \sin 2y + y\cos 2y)$$

$\therefore V_{xx} + V_{yy} = 0$

$\therefore V$ is harmonic

Ex:- Find an analytic function whose imaginary part is $e^{-x}(y\sin y + x\cos y)$

$\rightarrow$ Given $V = e^{-x}(y\sin y + x\cos y)$

1) $V_x = -e^{-x}(y\sin y + x\cos y) + e^{-x}\cos y$

$\therefore V_y = e^{-x}(y\cos y + \sin y - x\sin y)$

2) $f'(z) = u_x + iV_x$

$= V_y + iV_x$

$= e^{-x}(y\cos y + \sin y - x\sin y) + i(-e^{-x}\{y\sin y + x\cos y\} + e^{-x}\cos y)$

3) By Milne Thompson's method put $x = z$, $y = 0$

$$\therefore f'(z) = 0 + i\{-e^{-z}(0+z) + e^{-z}\}$$

$$f'(z) = i\{e^{-z}(1-z)\}$$

4) Integrating

$$f(z) = i\int e^{-z}(1-z)\, dz + c$$

$$= i\{(1-z)(-e^{-z}) + e^{-z}\} + c$$

$$= i\{-(1-z)e^{-z} + e^{-z}\} + c$$

$$= i\{e^{-z}[1-1+z]\} + c$$

$$\boxed{f(z) = ie^{-z}z + c}$$

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Ex ⇒ If the imaginary part of the analytic function $w = f(z)$ is $v = x^2 - y^2 + \frac{x}{x^2+y^2}$

Find an analytic function, $f(z) = u + iv$ And hence find u.

→ Given,

$$v = x^2 - y^2 + \frac{x}{x^2+y^2}$$

1) $$v_x = 2x + \frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2}$$

$$= 2x + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$= 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v_y = -2y - \frac{2xy}{(x^2+y^2)^2}$$

2) $$f'(z) = u_x + i v_x$$
$$= v_y + i v_x$$

$$= -2y - \frac{2xy}{(x^2+y^2)^2} + i\left\{2x + \frac{y^2-x^2}{(x^2+y^2)^2}\right\}$$

3) By Milne-Thompson

$x = z$, $y = 0$

$$\therefore f'(z) = 0 + i\left\{2z - \frac{z^2}{z^4}\right\}$$

$$= 0 + i\left\{2z - \frac{1}{z^2}\right\} = i\left\{2z - \frac{1}{z^2}\right\}$$

4) Intergrating,

$$f(z) = i\int\left(2z - \frac{1}{z^2}\right)dz + c$$

$$f(z) = i\left\{z^2 + \frac{1}{z}\right\} + c$$

$$f(z) = i\left\{(x+iy)^2 + \frac{1}{x+iy}\right\} + c$$

$$= i\left\{x^2 + 2ixy - y^2 + \frac{x-iy}{x^2+y^2}\right\}$$

$$= \left(-2xy + \frac{y}{x^2+y^2} + c\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right)$$

$$\boxed{\therefore u = -2xy + \frac{y}{x^2+y^2} + c}$$

Ex ⇒ Find an analytic function $f(z) = u + iv$, where $u + v = e^x(\cos y + \sin y)$

1) $$u + v = e^x(\cos y + \sin y)$$

$$\therefore u_x + v_x = \cancel{e^x \sin y}\ e^x(\cos y + \sin y) \quad ①$$

$$u_y + v_y = -e^x(\sin y - \cos y) \quad ②$$

By CR equations, $u_y = -v_x$ & $v_y = u_x$.

$$\therefore \cancel{u_x - v_x = e^x \sin y + \cos y} \quad ③$$

$$u_x - v_x = e^x(\cos y - \sin y)$$

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Adding ① & ③

$$2u_x = 2e^x \cos y$$
$$u_x = e^x \cos y$$

Putting u_x value in ①,

$$e^x \cos y + v_x = e^x \cos y + e^x \sin y$$
$$v_x = e^x \sin y$$

2) $f'(z) = u_x + i v_x$
$= e^x \cos y + i e^x \sin y$

3) By Milne-Thompson's Method,

Put $x = z$, $y = 0$

$\therefore f'(z) = e^z + i(0) = e^z$

4) Integrating

$f(z) = \int e^z + c$

$= e^z + c$

Ex ⇒ $3u + 2v = y^2 - x^2 + 16xy$

→ Given,

1) $3u + 2v = y^2 - x^2 + 16xy$

$3u_x + 2v_x = -2x + 16y$ ①

$3u_y + 2v_y = 2y + 16x$

$-3v_x + 2u_x = 2y + 16x$ ②

$2u_x - 3v_x = 2y + 16x$ ③

$2 \times$ ① $\quad 6u_x + 4v_x = -4x + 32y$

$3 \times$ ② $- \quad 6u_x - 9v_x = 6y + 48x$

$- \quad + \quad - \quad -$

$13v_x = -4x + 32y - 6y - 48x$

$13v_x = 26y - 52x$

$\boxed{v_x = 2y - 4x}$

Putting v_x in ①,

$3u_x + 2(2y - 4x) = -2x + 16y$

$3u_x + 4y - 8x = -2x + 16y$

$3u_x = 12y + 6x$

$u_x = 4y + 2x$

$\boxed{u_x = 2x + 4y}$

② $f'(z) = u_x + i v_x$
$= (2x + 4y) + i(2y - 4x)$

③ By Milne-Thompson, $x = z$, $y = 0$.

$f'(z) = 2z + i(-4z) = 2z - i4z$
$= 2z(1 - 2i)$

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4) Integrating,

$$f(z) = \int 2z(1-2i)\,dz + c$$

$$= (1-2i)(2)\int z + c$$

$$= 2(1-2i)\frac{z^2}{2} + c$$

$$\boxed{f(z) = (1-2i)z^2 + c}$$

⇒ State True or False with proper justification.

"There doesnot exist an analytic function whose real part is $x^3 - 3x^2y - y^3$"

→ Given, $u = x^3 - 3x^2y - y^3$

$$u_x = 3x^2 - 6xy$$
$$u_{xx} = 6x - 6y$$
$$u_y = -3x^2 - 3y^2$$
$$u_{yy} = -6y$$

Consider

$$u_{xx} + u_{yy} = 6x - 6y - 6y \neq 0$$

u does not satisfy Laplace's equatio
ie, u is not a harmonic function.
Hence, it can't be real part of an analytic function.
∴ Statement is true.

⇒ Verify Laplace equation for $u = \left(r + \frac{a^2}{r}\right)\cos\theta$.
Also find v & f(z).

→ Given, $u = \left(r + \frac{a^2}{r}\right)\cos\theta$

$$\frac{\partial u}{\partial r} = \left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{2a^2}{r^3}\cos\theta$$

Also,

$$\frac{\partial u}{\partial \theta} = -\left(r + \frac{a^2}{r}\right)\sin\theta$$

$$\frac{\partial^2 u}{\partial \theta^2} = -\left(r + \frac{a^2}{r}\right)\cos\theta$$

Consider,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \frac{2a^2}{r^3}\cos\theta + \frac{1}{r}\left(1 - \frac{a^2}{r^2}\right)\cos\theta + \frac{1}{r^2}\left\{-\left(r + \frac{a^2}{r}\right)\cos\theta\right\}$$

$$= \frac{2a^2}{r^3}\cos\theta + \frac{1}{r}\cos\theta - \frac{a^2}{r^3}\cos\theta - \frac{1}{r}\cos\theta - \frac{a^2}{r^3}\cos\theta$$

$$= 0$$

∴ u satisfies laplace equation.

$$u_r = \frac{1}{r}v_\theta$$

$$\therefore \left(1 - \frac{a^2}{r^2}\right)\cos\theta = \frac{1}{r}v_\theta$$

$$v_\theta = \left(r - \frac{a^2}{r}\right)\cos\theta$$

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$$v = \int \left(r - \frac{a^2}{r}\right) \cos\theta \, d\theta + ic$$

$$v = \left(r - \frac{a^2}{r}\right) \sin\theta + ic$$

Now,

$$f(z) = u + iv$$

$$= \left(r + \frac{a^2}{r}\right)\cos\theta + i\left(r - \frac{a^2}{r}\right)\sin\theta - c$$

⇒ Find the orthogonal trajectory of the family of curves given by

$$2x - x^3 + 3y^2 x = a$$

→ Let $u = 2x - x^3 + 3y^2 x = a$

1) $u_x = 2 - 3x^2 + 3y^2$

$u_y = 6xy$

2) $f'(z) = u_x + i v_x$

$= u_x - i u_y$

$= (2 - 3x^2 + 3y^2) - i(6xy)$

3) $x = z$, $y = 0$,

$f'(z) = (2 - 3z^2) - i(0)$

4) Integrating,

$$f(z) = \int (2 - 3z^2)\, dz + ic$$

$$= 2z - z^3 + ic$$

$$f(z) = 2(x+iy) - (x+iy)^3 + ic$$

$$= 2x + 2iy - x^3 - 3x^2 iy - 3x(iy)^2 - (iy)^3 + ic$$

$$f(z) = (2x - x^3 + 3xy^2) + i(2y - 3x^2 y + y^3 + c)$$

∴ Orthogonal trajectory to $u = 2x - x^3 + 3xy^2$ is $v = 2y - 3x^2 y + y^3 + c$.

* ⇒ Mapping

~~Def~~

Ex 1) Determine region D' in w plane corresponding to the region D in z-plane given by $x = 0$, $y = 0$, $x = 1$, $y = 1$ under the transformation $w = z + (2 - i)$.

→ Given $w = z + (2 - i)$

$= (x + iy) + (2 - i)$

$$u + iv = (x + 2) + i(y - 1)$$

$u = x + 2$

$v = y - 1$

when $x = 0$, then $u = 2$

when $y = 0$, then $v = -1$

when $x = 1$, then $u = 3$

when $y = 1$, then $v = 0$

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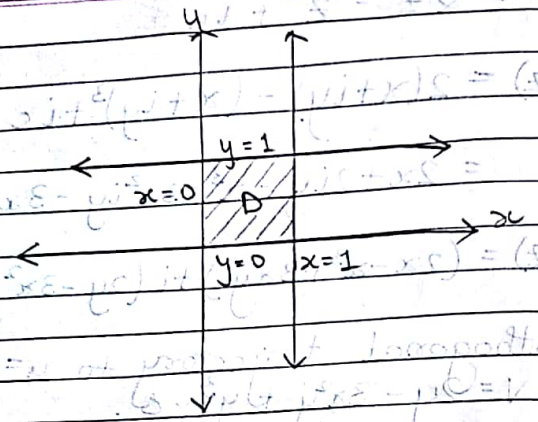
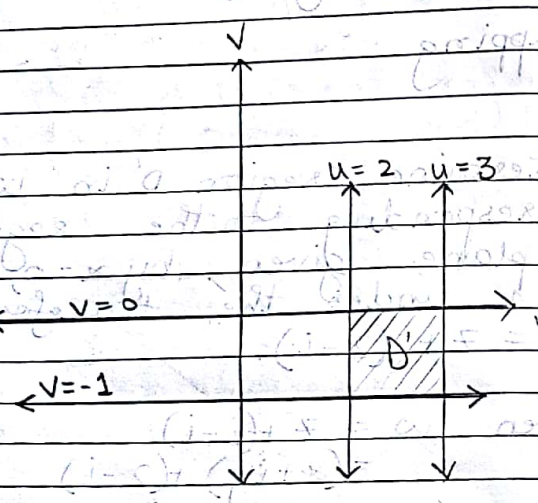




2) $w = (1+i)z - (1-i)$

$u + iv = (1+i)z - (1-i)$

$= (1+i)(x+iy) - (1-i)$

$= x + ix + iy - y - 1 + i$

$u + iv = (x-y-1) + i(x+y+1)$

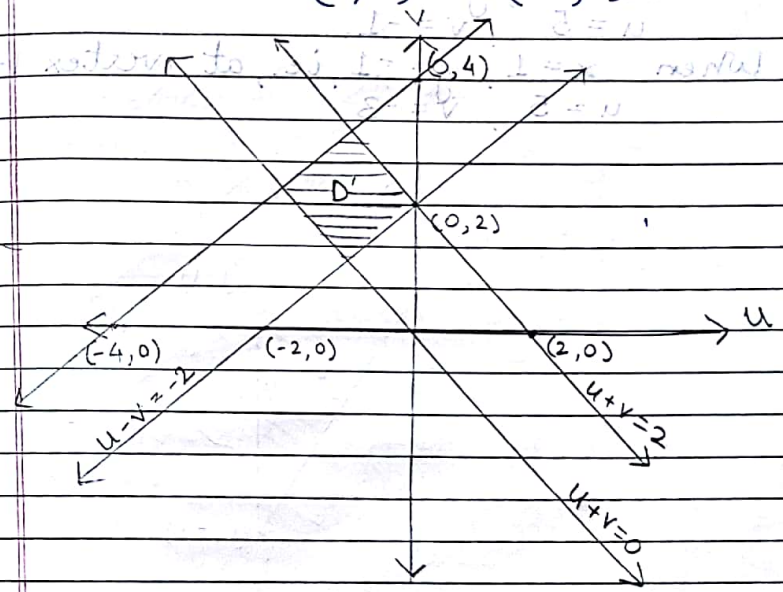
$u = x-y-1 \quad , \quad v = x+y+1$

When $x=0$, then $u = -y-1$, $v = y+1$.
Adding $u+v=0$, ie $v=-u$.

When $y=0$, then $u = x-1$, $v = x+1$.
Subtracting $u-v = -2$.
$(0,2)$ & $(-2,0)$.

When $x=1$, then $u = -y$, $v = y+2$.
Adding $u+v = 2$
$(2,0)$ & $(0,2)$

When $y=1$, then $u = x-2$, $v = x+2$
Sub $u-v = -4$.
$(0,4)$ & $(-4,0)$.



u-v plane.

4) $i, 1+i, 1-i$

→ Given,

$$w = z+4-2i$$

$$u+iv = x+iy+4-2i$$

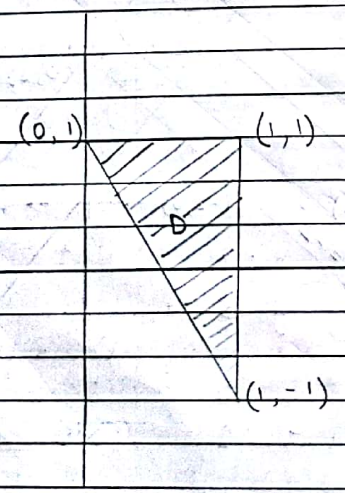
$$= (x+4)+i(y-2)$$

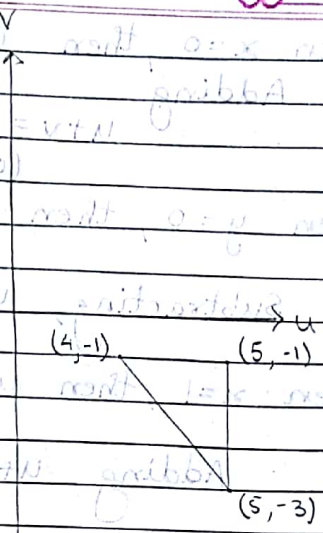
$$u = x+4$$
$$v = y-2$$

When $x=0$, $y=1$ ie, at vertex i, then
$u=4$, $v=-1$

When $x=1$, $y=1$ ie, at vertex $1+i$, then
$u=5$, $v=-1$

When $x=1$, $y=-1$, ie, at vertex $1-i$, then
$u=5$, $v=-3$





6) $w = (1+i)z+2-i$

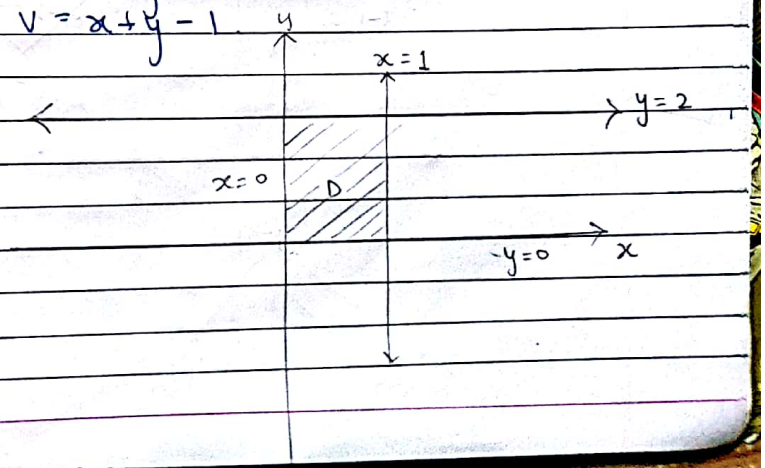
$u+iv = (z+2)+i(z-1)$

$x=0$, $y=0$,
$x=1$, $y=2$.

$\cancel{u = z+2}$
$\cancel{v = z-1}$

$$u+iv = (x+iy+2)+i(x+iy-1)$$
$$= x+iy+2+ix-y-i$$
$$= (x-y+2)+i(x+y-1)$$

$\therefore$ $u = x-y+2$
$v = x+y-1$



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When $x=0$, then $u=-y+2$, $v=y-1$

Adding $u+v=1$

$(0,1);(1,0)$

When $y=0$, then $u=x+2$, $v=x-1$

Subtracting, $u-v=3$

$(3,0);(0,-3)$

When $x=1$, then $u=-y+3$, $v=y$

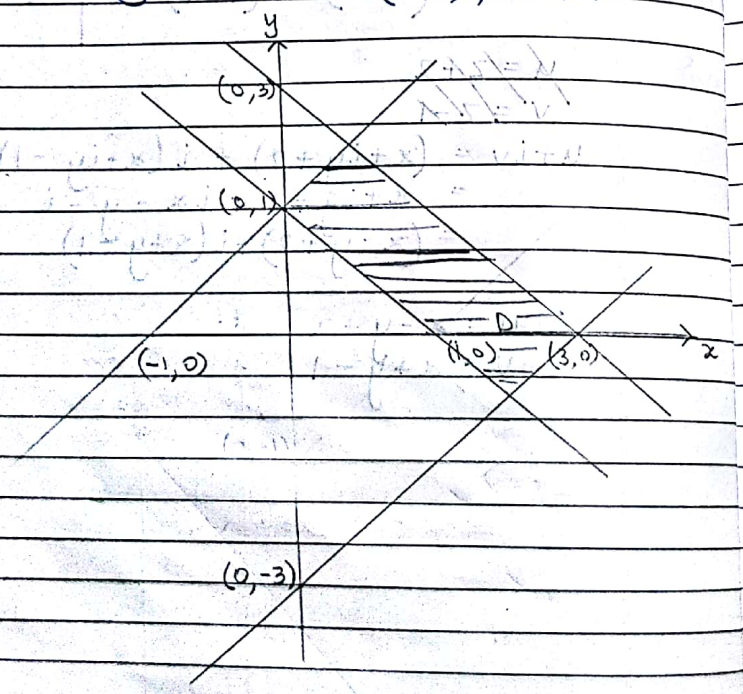
Adding $u+v=3$

$(3,0)\ (0,3)$

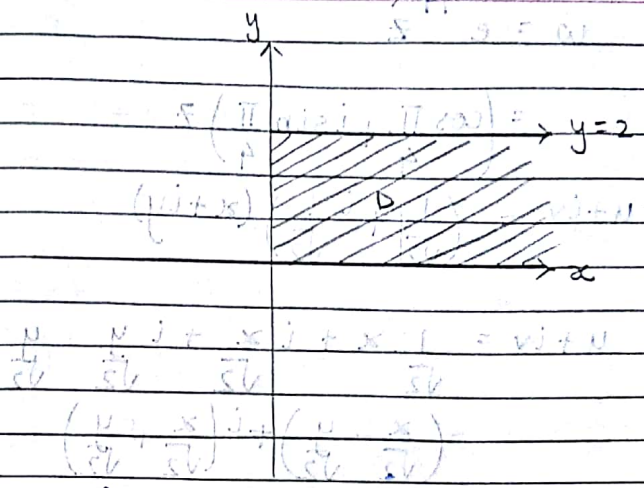
When $y=2$, the $u=x$, $v=x+1$

Subtracting $u-v=-1$

$(0,1);(-1,0)$



8)



$w = iz+1$

$u+iv = i(x+iy)+1$

$u+iv = ix-y+1$

$u+iv = (1-y)+ix$

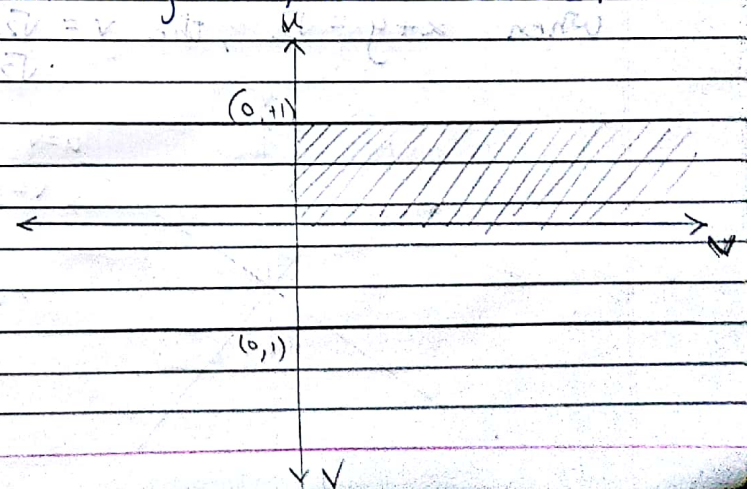
$u=1-y$, $v=x$

When $x>0$, $v>0$

When $y=0$, then $u=1$

$y=2$, then $u=-1$

$\therefore$ When $0<y<2$, then $-1<u<1$



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9) $w = e^{i\pi/4} z$

$= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) z$

$u+iv = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)(x+iy)$

$u+iv = \frac{1}{\sqrt{2}}x + i\frac{x}{\sqrt{2}} + i\frac{y}{\sqrt{2}} - \frac{y}{\sqrt{2}}$

$= \left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right) + i\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)$

$u = \frac{x-y}{\sqrt{2}}, \quad v = \frac{x+y}{\sqrt{2}}$

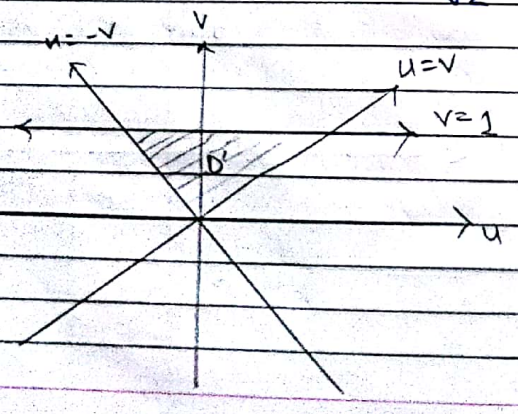
When $x=0$, then $u = \frac{-y}{\sqrt{2}}, \quad v = \frac{y}{\sqrt{2}}$

Adding, $u+v = 0$

When $y=0$, the $u = \frac{x}{\sqrt{2}}, \quad v = \frac{x}{\sqrt{2}}$

Sub, $u-v = 0$.

When $x+y = \sqrt{2}$, the $v = \frac{\sqrt{2}}{\sqrt{2}} = 1$



$|z - z_0| = 3$ .. eqn of circle

12) $|z| = 2$

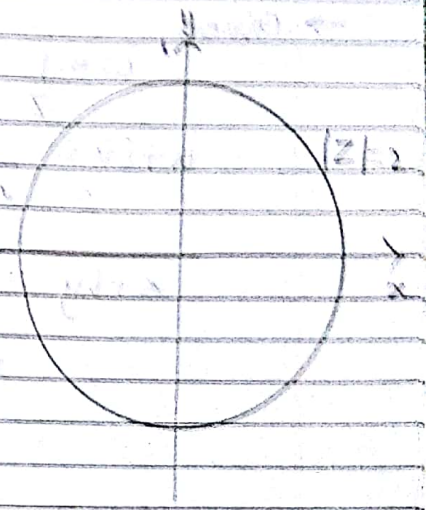
$\rightarrow w = z+3+2i$

$= x+iy+3+2i$

$u+iv = (x+3)+i(y+2)$

$u = x+3$

$v = y+2$

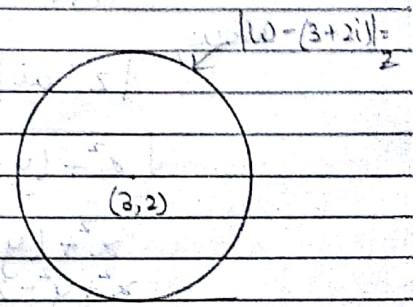


Now, $|z| = 2$

$\sqrt{x^2+y^2} = 2$

$x^2+y^2 = 4$

$(u-3)^2 + (v-2)^2 = 4 = (2)^2$



11) $|z - ai| = a$, $w = \frac{1}{z}$

→ Given,

$$w = \frac{1}{z}$$

$$u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2}$$

$$x + iy = \frac{1}{u+iv}$$

$$= \frac{u-iv}{u^2+v^2}$$

$$= \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

Now,

$$|z - ai| = a$$

$$\sqrt{x^2 + (y-a)^2} = a$$

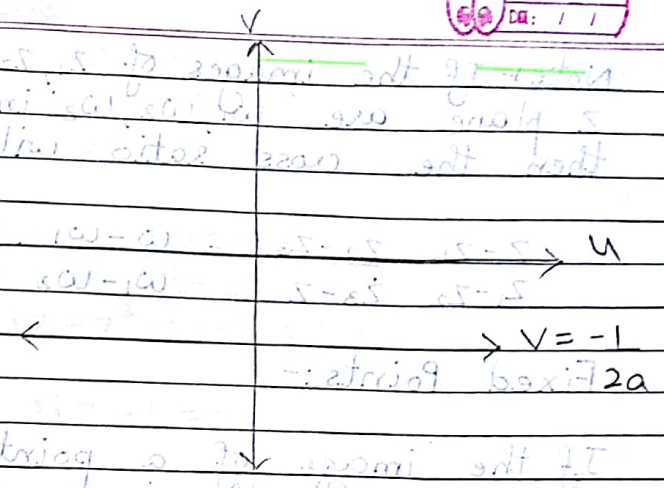
$$x^2 + (y-a)^2 = a^2$$

$$x^2 + y^2 - 2ay + a^2 = a^2$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} = -2a\frac{v}{u^2+v^2}$$

$$\frac{u^2+v^2}{(u^2+v^2)^2} = \frac{-2av}{(u^2+v^2)}$$

$$v = -\frac{1}{2a}$$



⇒ Bilinear Transformation :-

A transformation of the type $w = \frac{az+b}{cz+d}$ where a, b, c, d are complex constants & $ad - cb \neq 0$, is called a Bilinear Transformation.

⇒ Cross Ratio :-

If z_1, z_2, z_3, z_4 be four complex nos. then the ratio of the type

$$\frac{z_1 - z_2}{z_2 - z_3} \cdot \frac{z_3 - z_4}{z_4 - z_1}$$ is called cross ratio

and denoted by

Cross-Ratio preservation property :-

If the images of z_1, z_2, z_3, z_4 under bilinear transformation are w_1, w_2, w_3, w_4 then their cross ratio is preserved.

ie $$\left(\frac{z_1 - z_2}{z_2 - z_3}\right)\left(\frac{z_3 - z_4}{z_4 - z_1}\right) = \left(\frac{w_1 - w_2}{w_2 - w_3}\right)\left(\frac{w_3 - w_4}{w_4 - w_1}\right)$$

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Note:- If the images of z_1, z_2, z_3 from z plane are w_1, w_2, w_3 in w plane, then the cross ratio will be

$$\frac{z-z_1}{z_1-z_2} \cdot \frac{z_2-z_3}{z_3-z} = \frac{w-w_1}{w_1-w_2} \cdot \frac{w_2-w_3}{w_3-w}$$

Fixed Points:-

If the image of a point in z plane coincides with the image of a point in w plane under bilinear transformation, then the point is said to be fixed point.

Fixed points will be obtained on putting $w = z$ in bilinear transformation.

Parabolic

A bilinear transformation is said to be parabolic if it has only one fixed point

⇒ Find the fixed points of binear transformation.

i) $w = \frac{1+3iz}{i+z}$

ii) $w = \frac{z-1}{z+1}$

iii) $w = \frac{z-4}{2z-5}$

iv) $w = \frac{2z+1}{z+2}$

Given:- $w = \frac{1+3iz}{i+z}$

Put $w = z$

$\therefore z = \frac{1+3iz}{i+z}$

$iz + z^2 = 1+3iz$

$z^2 - 2iz - 1 = 0$

$z^2 - 2iz + i^2 = 0$

$(z-i)^2 = 0$

$z = i, i$

It is parabolic transformation

iv) Put $w = z$

$\therefore z = \frac{2z+1}{z+2}$

$z^2 + 2z = 2z+1$

$z^2 = 1$

$z = \pm 1$

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Examples:-

1) Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points ~~onto th~~ $w = i, 0, -i$. Hence find fixed points of the transformation and the image of $|z| < 1$.

→ Let, $w = \dfrac{az+b}{cz+d}$ ..... ①

When $z = 1$, $w = i$

$\therefore\ i = \dfrac{a+b}{c+d}$

$ic + id = a + b$ ... ②

When $z = i$, $w = 0$

$\therefore\ 0 = \dfrac{ai+b}{ci+d}$

$ia + b = 0$

$b = -ia$ ... ③

When $z = -1$, $w = -i$

$\therefore\ -i = \dfrac{-a+b}{-c+d}$

$ic - id = -a + b$ ... ④

Adding ② & ④

$2ic = 2b$

$c = -ib$

$= -i(-ia)$

$\boxed{c = -a}$

Put values of b & c in ②

$\therefore\ -ia + id = a - ia$

$id = a$

$\therefore\ d = -ia$

Putting values of b, c, d in ①,

$w = \dfrac{az - ia}{-az - ia}$

$w = \dfrac{z - i}{-z - i}$ $\qquad \therefore\ a = 1,\ b = -1,\ c = -1,\ d = -1$

For fixed point, put $w = z$

$\therefore\ z = \dfrac{az+b}{cz+d}$

$z = \dfrac{z - i}{-z - i}$

$\therefore\ z(-z - i) = z - i$

$\therefore\ -z^2 - iz = z - i$

$\therefore\ z^2 + (1+i)z - i = 0$

$z = \dfrac{-(1+i) \pm \sqrt{(1+i)^2 - 4(1)(-i)}}{2 \times 1}$

$= \dfrac{-(1+i) \pm \sqrt{1 + 2i - 1 + 4i}}{2}$

$= \dfrac{-(1+i) \pm \sqrt{6i}}{2}$

Now,

$w = -\dfrac{z-i}{z+i}$

$\therefore\ wz + iw = -z + i$

$\therefore\ wz + z = i(1 - w)$

$z(1+w) = i(1-w)$

$\therefore\ z = i\left(\dfrac{1-w}{1+w}\right)$

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$|z| < 1$

$\therefore \left| i \frac{1-w}{1+w} \right| < 1$

$\therefore \frac{|1-w|}{|1+w|} < 1$

$\therefore |1-w| < |1+w|$

$|1-(u+iv)| < |1+(u+iv)|$

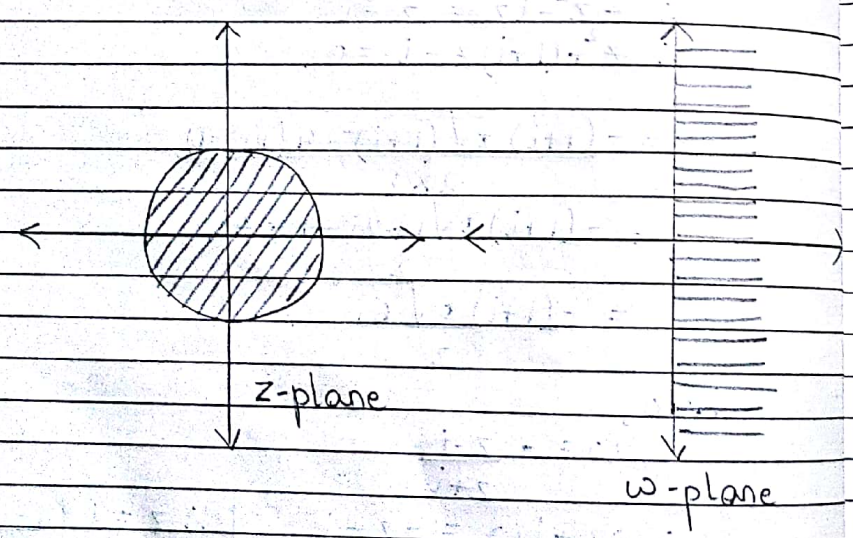
$\sqrt{(1-u)^2+v^2} < \sqrt{(1+u)^2+v^2}$

$\therefore 1-2u+u^2+v^2 < 1+2u+u^2+v^2$

$4u > 0$

$u > 0$

which is half of the w-plane towards the right of v-axis.



2) Find the bilinear transformation under which the points $z = 1, i, -1$ are mapped onto the points $w = 0, 1, \infty$. Further show that under this transformation the unit circle in w-plane ($|w| = 1$) is mapped onto straight line in z plane. Write the name of this line.

→ Let, $w = \frac{az+b}{cz+d}$ ①

Put $z = 1$ & $w = 0$

$\therefore 0 = \frac{a+b}{c+d}$

$\therefore a+b = 0$

$\therefore a = -b$ ②

Put $z = i$, & $w = 1$

$\therefore 1 = \frac{ia+b}{ic+d}$

$ic+d = ia+b$ ③

Put $z = -1$ & $w = \infty$

$\infty = \frac{-a+b}{-c+d}$

$\therefore -c+d = 0$

$\therefore d = c$

Put $a = -b$ & $c = d$ in ③

$\therefore id+d = -ib+b$

$\therefore \cancel{d(i}$

$(1+i)d = (1-i)b$

$\therefore d = \frac{1-i}{1+i} b$

$d = \frac{(1-i)^2}{1+1} b$

$\therefore d = \frac{1-2i-1}{2} b = -ib$

$\therefore d = -ib$

$\therefore c = d = -ib$

Putting values of a, c, d in ①

$$\omega = \frac{-bz+b}{-ibz-ib}$$

$$= \frac{-b(z-1)}{-ib(z+1)}$$

$$= \frac{1(z-1)}{i(z+1)}$$

$$\omega = \frac{z-1}{iz+i}$$

Now,

$$|\omega| = 1$$

$$\therefore \left|\frac{z-1}{iz+i}\right| = 1$$

$$\frac{|z-1|}{|i(z+1)|} = 1$$

$$\therefore \frac{|z-1|}{|z+1|} = 1$$

$$\therefore |z-1| = |z+1|$$

$$\therefore |x+iy-1| = |x+iy+1|$$

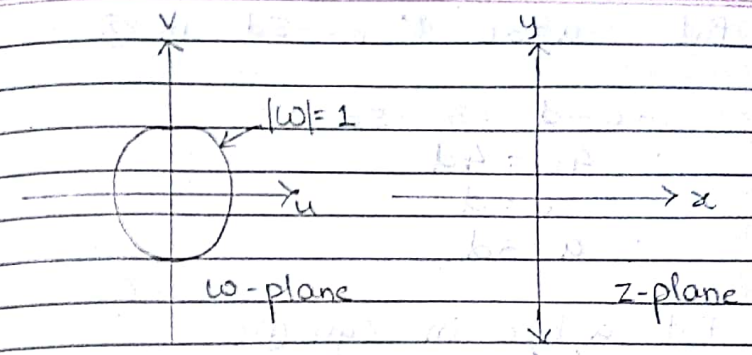
$$\therefore \sqrt{(x-1)^2+y^2} = \sqrt{(x+1)^2+y^2}$$

$$\therefore x^2-2x+1+y^2 = x^2+2x+1+y^2$$

$$\therefore 4x = 0$$

$$\therefore x = 0$$

which is y-axis.



3) → Find bilinear transformation which maps the points $z = 0, 1, \infty$ onto the points $\omega = -5, -1, 3$

→ Let,

$$\omega = \frac{az+b}{cz+d} \quad ①$$

When $z = 0$, $\omega = -5$.

$$\therefore -5 = \frac{b}{d}$$

$$b = -5d$$

When $z = 1$, $\omega = -1$

$$-1 = \frac{a+b}{c+d}$$

$$a+b = -c-d \quad ②$$

When $z = \infty$, $\omega = 3$.

$$\therefore 3 = \omega = \lim_{z\to\infty} \frac{az+b}{cz+d} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{z\to\infty} \frac{a}{c} \quad \text{(By L'Hopital)}$$

$$\therefore 3 = \frac{a}{c}$$

$$a = 3c$$

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Put $a = 3c$ & $b = -5d$ in ②

$$-c - d = 3c - 5d$$
$$\therefore 4c = 4d$$
$$c = d$$
$$\therefore a = 3d$$

Put a, b, c in eqn ①.

$$\therefore w = \frac{3dz - 5d}{dz + d}$$

$$\boxed{w = \frac{3z - 5}{z + 1}}$$

4) Find the bilinear transformation which maps the points $z = \infty, i, 0$ onto the points $w = 0, i, \infty$ by using cross ratio.

$$\frac{z - z_1}{z_1 - z_2} \cdot \frac{z_2 - z_3}{z_3 - z} = \frac{w - w_1}{w_1 - w_2} \cdot \frac{w_2 - w_3}{w_3 - w}$$

$$\frac{\left(\frac{z}{z_1} - 1\right) \cdot (z_2 - z_3)}{\left(1 - \frac{z_2}{z_1}\right)(z_3 - z)} = \frac{(w - w_1)\left(\frac{w_2}{w_3} - 1\right)}{(w_1 - w_2)\left(1 - \frac{w}{w_3}\right)}$$

$$\frac{(0 - 1) \cdot (i - 0)}{(1 - 0) \cdot (0 - z)} = \frac{(w - 0)(0 - 1)}{(0 - i)(1 - 0)}$$

$$\frac{-i}{-z} = \frac{-w}{-i} = \frac{w}{i}$$

$$\therefore \frac{i^2}{z} = w$$

$$\therefore w = \frac{-1}{z}$$

5) Find Bilinear Transformation which maps the points $z = 1, i, -1$ onto $w = i, 0, -i$. Hence find image of $|z| < 1$ onto w-plane

Let,

$$w = \frac{az + b}{cz + d}$$

When $z = 1$, $w = i$.

$$\therefore i = \frac{a + b}{c + d}$$

$$ic + id = a + b$$

⊛ Same as Question No ①.

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Module 06

Correlation, Regression, Curve Fitting

1) Correlation :-

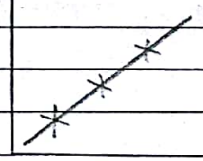
Def^n:- If as one variable changes because of change in other variable either in same direction or in opposite direction, then the variables are said to be correlated.

Types of Correlation

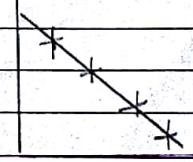
i) Positive correlation:-
If the change in variables is in the same direction then the correlation is +ve correlation.

ii) Negative correlation:-
If the change in variables is in the opposite direction then the correlation is -ve correlation.

iii) Linear & Non-linear Correlation:-
- If the graph of correlated variables in a scatter diagram is linear, then the correlation is said to be linear correlation.
- And if the curve is not linear then the correlation is said to be Non-Linear or curvi-linear correlation



perfect +ve correlation



perfect -ve correlation

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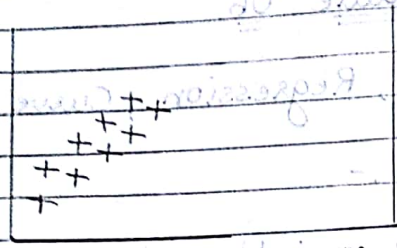
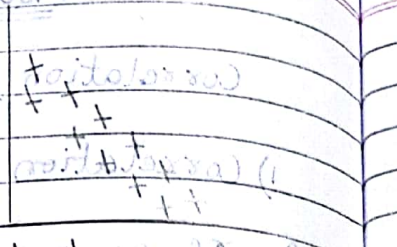
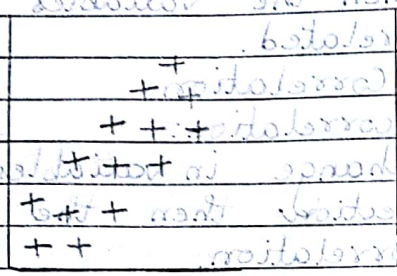


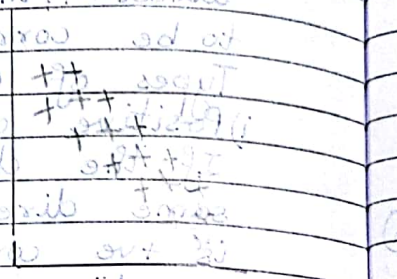
High degree of +ve correlation



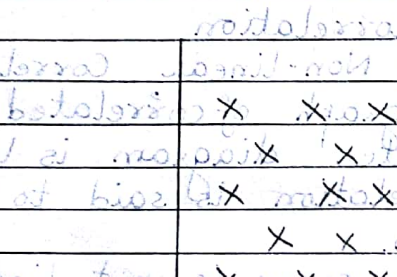
High degree of -ve correlation



Low degree of +ve correlation



Low degree of -ve correlation



No correlation

1. Karl-Pearson's Co-efficient of Correlation.

The correlation co-efficient suggested by Karl Pearson is denoted by r and defined as

$$r = \frac{cov(x,y)}{\sigma_x \sigma_y} \quad \dots (1)$$

where $\sigma_x^2 = \frac{\Sigma(x-\bar{x})^2}{n}$ & $\sigma_y^2 = \frac{\Sigma(y-\bar{y})^2}{n}$ are standard deviations in x & y

and $cov(x,y) = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n}$ is covariance between x & y

$$\therefore \quad r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2 \Sigma(y-\bar{y})^2}} \quad \dots (2)$$

Take $X = x - \bar{x}$ & $Y = y - \bar{y}$

$$\therefore \quad r = \frac{\Sigma XY}{\sqrt{\Sigma X^2 \Sigma Y^2}} \quad \dots (3)$$

From eqn (2),

$$\Sigma(x-\bar{x})(y-\bar{y}) = \Sigma(xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y})$$
$$= \Sigma xy - \bar{y}\Sigma x - \bar{x}\Sigma y + \bar{x}\bar{y}\Sigma 1$$
$$= \Sigma xy - \bar{y}n\bar{x} - \bar{x}n\bar{y} + n\bar{x}\bar{y}$$
$$= \Sigma xy - n\bar{x}\bar{y}$$

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$$r = \frac{\Sigma xy - n\bar{x}\bar{y}}{\sqrt{(\Sigma x^2 - n\bar{x}^2)(\Sigma y^2 - n\bar{y}^2)}} \quad (4)$$

$$r = \frac{\Sigma xy - \frac{1}{n}\Sigma x \Sigma y}{\sqrt{\left(\Sigma x^2 - \frac{1}{n}(\Sigma x)^2\right)\left(\Sigma y^2 - \frac{1}{n}(\Sigma y)^2\right)}} \quad (5)$$

If A and B are assumed means of x & y, then $d_x = x - A$, $d_y = y - B$

$$r = \frac{\Sigma d_x d_y - \frac{1}{n}\Sigma d_x \Sigma d_y}{\sqrt{\left[\Sigma d_x^2 - \frac{1}{n}(\Sigma d_x)^2\right]\left[\Sigma d_y^2 - \frac{1}{n}(\Sigma d_y)^2\right]}} \quad (6)$$

For single digit → 4, 5
For double & triple digit → 2, 3, 6.

Results :-

1) $-1 \leq r \leq 1$
2) $r = +1$ (Perfect Correlation)
3) If $r > 0.95$, then there is a high degree of correlation
4) $0.75 < r < 0.85$, probably a definite relation
5) $0.4 < r < 0.6$, there is some relation
6) $r < 0.35$, poor correlation
7) If $r = 0$, no relation.

Ex 1) Calculate Karl-Pearson's co-effient of correlation for the following bivariate series.

X : 28, 45, 40, 38, 35, 33, 40, 32, 36, 33

Y : 23, 34, 33, 34, 30, 26, 28, 31, 36, 35

→ Here, $n = 10$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{360}{10} = 36$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{310}{10} = 31$$

X	Y	$x = X - \bar{X}$	$y = Y - \bar{Y}$	x^2	y^2	xy
28	23	-8	-8	64	64	64
45	34	9	3	81	9	27
40	33	4	2	16	4	8
38	34	2	3	4	9	6
35	30	-1	-1	1	1	1
33	26	-3	-5	9	25	15
40	28	4	-3	16	9	-12
32	31	-4	0	16	0	0
36	36	0	5	0	25	0
33	35	-3	4	9	16	-12
				$\Sigma x^2 = 216$	$\Sigma y^2 = 162$	$\Sigma xy = 97$

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \Sigma y^2}}$$

$$= \frac{97}{\sqrt{216 \times 162}} = 0.52$$

There is some relation betⁿ X & Y.

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2) Calculate co-efficient of correlation from the data :-

X : 100, 200, 300, 400, 500

Y : 30, 40, 50, 60, 70

→ Here $n = 5$

$$\bar{X} = \frac{\Sigma X}{n} = \frac{1500}{5} = 300$$

$$\bar{Y} = \frac{\Sigma Y}{n} = \frac{250}{5} = 50$$

X	Y	$x = \frac{X-\bar{X}}{100}$	x^2	$y = \frac{Y-\bar{Y}}{10}$	y^2	xy
100	30	-2	4	-2	4	4
200	40	-1	1	-1	1	1
300	50	0	0	0	0	0
400	60	1	1	1	1	1
500	70	2	4	2	4	4
			$\Sigma x^2 = 10$		$\Sigma y^2 = 10$	$\Sigma xy = 10$

$$\therefore r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \Sigma y^2}} = \frac{10}{\sqrt{10 \times 10}} = \frac{10}{10} = 1$$

∴ There is a perfect +ve correlation between x & y.

3) Calculate the co-efficient of correlation from the data :-

X : 23, 27, 28, 29, 30, 31, 33, 35, 36, 39

Y : 18, 22, 23, 24, 25, 26, 28, 29, 30, 32

→ Here $n = 10$

$$\bar{X} = \frac{\Sigma X}{n} = \frac{311}{10} = 31.1$$

$$\bar{Y} = \frac{\Sigma Y}{n} = \frac{257}{10} = 25.7$$

Let $A = 31$ & $B = 26$ be assumed means of X and Y

X	Y	$dx = X - A$	$dy = Y - B$	dx^2	dy^2	$dxdy$
23	18	-8	-8	64	64	64
27	22	-4	-4	16	16	16
28	23	-3	-3	9	9	9
29	24	-2	-2	4	4	4
30	25	-1	-1	1	1	1
31	26	0	0	0	0	0
33	28	2	2	4	4	4
35	29	4	3	16	9	12
36	30	5	4	25	16	20
39	32	8	6	64	36	48
		$\Sigma dx = 1$	$\Sigma dy = -3$	$\Sigma dx^2 = 203$	$\Sigma dy^2 = 159$	$\Sigma dxdy = 178$

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$$r = \frac{\Sigma dx\,dy - \frac{1}{n}\Sigma dx\,\Sigma dy}{\sqrt{\left[\Sigma dx^2 - \frac{1}{n}(\Sigma dx)^2\right]\left[\Sigma dy^2 - \frac{1}{n}(\Sigma dy)^2\right]}}$$

$$= \frac{178 - \frac{1}{10}(1)(-3)}{\sqrt{\left(203 - \frac{(1)^2}{10}\right)\left(159 - \frac{(-3)^2}{10}\right)}}$$

$$= \frac{178 + 0.3}{\sqrt{(203-0.1)(159-0.9)}}$$

$$= \frac{178.3}{\sqrt{(202.9)(158.1)}}$$

$$= 0.9955$$

∴ there is a high degree of correlation.

2. Spearmans Rank coefficient

Spearmans Rank coefficient is denoted by R and defined as

$$R = 1 - \frac{6\Sigma D^2}{n^3 - n}$$

$$= 1 - \frac{6}{n^3-n}\left\{\Sigma D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \dots\right\}$$

where D = $R_1 \sim R_2$, R_1 & R_2 are ranks of X & Y

Note:- $-1 \leq R \leq 1$

Ex: 1) Calculate Rank correlation co-efficient (R) from the following data relating to the ranks of 10 students in English & Maths.

Data:-

Stud. No.	1	2	3	4	5	6	7	8	9	10
Rank in Eng	1	3	7	5	4	6	2	10	9	8
Rank in Math	3	1	4	5	6	9	7	8	10	2

→ Here, n = 10

R_1	R_2	$D^2 = (R_1 - R_2)^2$
1	3	4
3	1	4
7	4	9
5	5	0
4	6	4
6	9	9
2	7	25
10	8	4
9	10	1
8	2	36
		$\Sigma D^2 = 96$

$$R = 1 - \frac{6\Sigma D^2}{n^3 - n}$$

$$= 1 - \frac{6 \times 96}{1000 - 10} = 1 - \frac{576}{990} = 0.4182$$

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2) Obtain the Rank Correlation Co-efficient from the following data

X: 10, 12, 18, 18, 15, 40,

Y: 12, 18, 25, 25, 50, 25

→ Here, $n = 6$.

X	Y	R_1	R_2	D	D^2
10	12	1	1	0	0
12	18	2	2	0	0
18	25	4.5	4	0.5	0.25
18	25	4.5	4	0.5	0.25
15	50	3	6	-3	9
40	25	6	4	2	4
					$\Sigma D^2 = 13.5$

Here $m_1 = 2$, $m_2 = 3$.

$$\therefore R = 1 - \frac{6}{n^3 - n}\left\{\Sigma D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2)\right\}$$

$$= 1 - \frac{6}{216-6}\left\{13.5 + \frac{1}{12}(8-2) + \frac{1}{12}(27-3)\right\}$$

$$= 1 - \frac{\cancel{6}^{2}}{\cancel{210}_{70}}\left\{13.5 + \frac{6}{12} + \frac{24}{12}\right\}$$

$$= \frac{1}{1} - \frac{1}{35}\{14+2\} = 1 - \frac{16}{35} = \frac{19}{35}$$

$$= 0.5428$$

3) Calculate R & r from the following data

X: 12, 17, 22, 27, 32

Y: 113, 119, 117, 115, 121.

Interpret your result

→ Here $n = 5$.

$$\bar{X} = \frac{\Sigma x}{n} = \frac{110}{5} = 22$$

$$\bar{Y} = \frac{\Sigma y}{n} = \frac{585}{5} = 117$$

X	Y	$x = X - \bar{X}$	$y = Y - \bar{Y}$	x^2	y^2	xy
12	113	-10	-4	100	16	40
17	119	-5	2	25	4	-10
22	117	0	0	0	0	0
27	115	5	-2	25	4	-10
32	121	10	4	100	16	40
				$\Sigma x^2 = 250$	$\Sigma y^2 = 40$	$\Sigma xy = 60$

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \cdot \Sigma y^2}} = \frac{60}{\sqrt{250 \times 40}} = 0.6$$

$r = 0.6$

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X	Y	R_1	R_2	$D = R_1 - R_2$	D^2
12	113	1	1	0	0
17	119	2	4	-2	4
22	117	3	3	0	0
27	115	4	2	2	4
32	121	5	5	0	0
					$\Sigma D^2 = 8$

$$R = 1 - \frac{6}{n^3 - n} \Sigma D^2$$

$$= 1 - \frac{6}{125 - 5} \times 8$$

$$= 1 - \frac{\cancel{6} \times 8}{\cancel{120}_{20}}$$

$$= \frac{12}{20} = 0.6$$

$$\boxed{R = 0.6}$$

$$\therefore \boxed{r = R = 0.6}$$

4) $r_{xy} = 0.4$, $cov(x,y) = 1.6$, $\sigma^2_y = 25$
Find σ_x.

→ We know

$$r = \frac{cov(x,y)}{\cancel{\sigma_x \sigma_y}\ \sigma_x \sigma_y}$$

$$0.4 = \frac{1.6}{\sigma_x \times 5}$$

$$\sigma_x = \frac{1.6}{0.4 \times 5} = \frac{4}{5} = 0.8$$

Regression :-

→ Method of finding 1 variable when other variable is given and both the variables are correlated.

Def^n :- The method of estimating the value of one variable when that of the other is known and when the variables are correlated is called Regression.

→ Line of Regression of Y on X.
A regression line used to obtain Y when X is given is called line of regression of Y on X and is represented as $y = a + bx$ OR $y - \bar{y} = b_{yx}(x - \bar{x})$

→ Line of Regression of X on Y
A regression line used to obtain X when Y is given is called line of regression of X on Y and is represented as $x = a + by$ OR $x - \bar{x} = b_{xy}(y - \bar{y})$.

Note: 1) The line of regression always passes through the mean point.
2) The slopes b_{yx} and b_{xy} are called regression co-efficients and are given by

$$b_{yx} = r \frac{\sigma_y}{\sigma_x} \qquad b_{xy} = r \frac{\sigma_x}{\sigma_y}$$

$$b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\ \sigma_x^2} \qquad b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\ \sigma_y^2}$$

$$b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} \qquad b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$$

$$b_{yx} = \frac{\Sigma XY}{\Sigma X^2} \qquad b_{xy} = \frac{\Sigma XY}{\Sigma Y^2}$$

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$$b_{yx} = \frac{\Sigma xy - \frac{1}{n}\Sigma x \Sigma y}{\Sigma x^2 - \frac{1}{n}(\Sigma x)^2} \qquad b_{xy} = \frac{\Sigma xy - \frac{1}{n}\Sigma x \Sigma y}{\Sigma y^2 - \frac{1}{n}(\Sigma y)^2}$$

$$= \frac{\Sigma dx\,dy - \frac{1}{n}\Sigma dx \Sigma dy}{\Sigma dx^2 - \frac{1}{n}(\Sigma dx)^2} \qquad = \frac{\Sigma dx\,dy - \frac{1}{n}\Sigma dx \Sigma dy}{\Sigma dy^2 - \frac{1}{n}(\Sigma dy)^2}$$

Note:- 1) $b_{xy}\, b_{yx} = r^2$

- Regression co-efficient must be of same sign

2) Angle between Regression Lines

If θ is the angle betn two regression lines, then it is given by

$$\tan\theta = \frac{1-r^2}{r}\left(\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\right)$$

Q) State True or False with justification

1) If two regression lines are $x+3y-5=0$ & $4x+3y-8=0$ then the ~~regress~~ correlation co-effiient is 0.5

→ Let $x+3y-5=0$ & $4x+3y-8=0$ be line of regressions of y on x & x on y respectively

Thus, $x+3y-5=0$

$$y = \frac{5}{3} - \frac{1}{3}x$$

$$\therefore b_{yx} = -\frac{1}{3}$$

Now,

$4x+3y-8=0$

$$x = \frac{8}{4} - \frac{3}{4}y = 2 - \frac{3}{4}y$$

$$\boxed{\therefore b_{xy} = -\frac{3}{4}}$$

$$r = \sqrt{\left(-\frac{1}{3}\right)\times\left(-\frac{3}{4}\right)} = \sqrt{\frac{1}{4}} = \frac{1}{2} = 0.5$$

Since b_{xy} & b_{yx} are -ve

$$\boxed{r = -0.5}$$

Q Find the equation of the lines of regression for the data

x:	5	6	7	8	9	10	11
y:	11	14	14	15	12	17	16

Here, $n = 7$

$$\bar{x} = \frac{\Sigma x}{n} = 8 \qquad \bar{y} = \frac{\Sigma y}{n} = 14.14$$

$$b_{yx} = \frac{\Sigma dx\,dy - \frac{1}{n}\Sigma dx \Sigma dy}{\Sigma dx^2 - \frac{1}{n}(\Sigma dx)^2}$$

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x	y	$dx = x-8$	$dy = y-14$	dx^2	dy^2	$dx\,dy$
5	11	-3	-3	9	9	9
6	14	-2	0	4	0	0
7	14	-1	0	1	0	0
8	15	0	1	0	1	0
9	12	1	-2	1	4	-2
10	17	2	3	4	9	6
11	16	3	2	9	4	6

$\Sigma dx = 0$ $\Sigma dy = 1$ $\Sigma dx^2 = 28$ $\Sigma dy^2 = 27$

$\Sigma dx\,dy = 19.$

$$b_{yx} = \frac{\Sigma dx\,dy - \frac{1}{n}\Sigma dx\,\Sigma dy}{\Sigma dx^2 - \frac{1}{n}(\Sigma dx)^2} \qquad b_{xy} = \frac{\Sigma dx\,dy - \frac{1}{n}\Sigma dx\,\Sigma dy}{\Sigma dy^2 - \frac{1}{n}(\Sigma dy)^2}$$

$$= \frac{19 - \frac{1}{7}(0)(1)}{28 - \frac{1}{7}(0)^2} \qquad = \frac{19 - \frac{1}{7}(0)(1)}{27 - \frac{1}{7}(1)^2}$$

$$= \frac{19}{28} \qquad = \frac{19 - \frac{1}{7}}{27 - \frac{1}{7}} = 0.707$$

$= 0.678$

$\therefore$ Line of regression of y on x is,

$y - \bar{y} = b_{yx}(x - \bar{x})$

$y - 14.14 = 0.678(x - 8)$

$y = 0.678x + 8.716$

And line of regression of x on y is $x - \bar{x} = b_{xy}(y - \bar{y})$

$x - 8 = 0.707(y - 14.14)$

$x = 0.707y - 7.$

Q) A chemical engineer is investigating the effect of process operating temperature x on product yield Y. The study results in the following data.

X: 100, 110, 120, 130, 140, 150, 160, 170, 180, 190

Y: 45, 51, 54, 61, 66, 70, 74, 78, 85, 89.

Find the equation of least square line which enable to predict yield for a particular temperation. Find also the degree of relationship between temp. & the yield. Also verify that the sum of co-effients of regression is greater than 2r.

→ Here $n = 10$.

$\bar{X} = \frac{\Sigma X}{n} = 145$ $\qquad \bar{Y} = \frac{\Sigma Y}{n} = 67.3.$

~~Let A = 140~~

Let $A = 140$ & $B = 70$.

X	Y	$dx = X - A$	dx^2	$dy = Y - B$	$dx\,dy$
100	45	-40	1600	-25	1000
110	51	-30	900	-19	570
120	54	-20	400	-16	320
130	61	-10	100	-9	90
140	66	0	0	-4	0
150	70	10	100	0	0
160	74	20	400	4	80
170	78	30	900	8	240
180	85	40	1600	15	600
190	89	50	2500	19	950

$\Sigma dx = 50$

$\Sigma dx^2 = 8500$

$\Sigma dy = -27$

$\Sigma dy^2 = 2005$

$\Sigma dx\,dy = 3850.$

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$$b_{yx} = \frac{\Sigma dx\,dy - \frac{1}{n}\Sigma dx\,\Sigma dy}{\Sigma dx^2 - \frac{1}{n}(\Sigma dx)^2}$$

$$= \frac{3850 - \frac{1}{10}(50)(-27)}{8500 - \frac{1}{10}(50)^2}$$

$b_{yx} = 0.483$

And

$$b_{xy} = \frac{\Sigma dx\,dy - \frac{1}{n}\Sigma dx\,\Sigma dy}{\Sigma dy^2 - \frac{1}{n}(\Sigma dy)^2}$$

$$= \frac{3850 - \frac{1}{10}(50)(-27)}{2005 - \frac{1}{10}(-27)^2}$$

$b_{xy} = 2.062$

Line of regression of y on x

$y - \bar{y} = b_{yx}(x - \bar{x})$

$y - 67.3 = 0.483(x - 145)$

$y = -2.735 + 0.483x$.

Also, $r = \sqrt{b_{yx}\, b_{xy}}$

$= \sqrt{0.483 \times 2.062}$

$r = 0.9979$

$\therefore$ both are +ve. $\therefore$ r is +ve.

$\therefore$ There is a high degree of ~~B~~ relation bet^n yield & temp.

$2r = 2(0.999) = 1.98$

$b_{yx} + b_{xy} = 0.483 + 2.062 = 2.54$.

$\therefore b_{xy} + b_{yx} > 2r$.

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⇒ Curve fitting :-

⇒ 1) Fitting a straight line :-

Let $y = a + bx$ be a straight line where a, b are constants and determine using following normal eqns,

$$\Sigma y = na + b\Sigma x$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

Similarly for $x = a + by$ (x on y line)

This line $y = a + bx$ is line of regression of y on x.

Ex ⇒ 1) Fit a straight line $y = a + bx$ to the following data

x :	0	5	10	15	20	25
y :	12	15	17	22	24	30

→ Let $y = a + bx$ be a straight line

Normal equations are :-

$$\Sigma y = na + b\Sigma x$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

x	y	x^2	xy
0	12	0	0
5	15	25	75
10	17	100	170
15	22	225	330
20	24	400	480
25	30	625	750

$\Sigma x = 75$ $\Sigma y = 120$ $\Sigma x^2 = 1375$ $\Sigma xy = 1805$.

Thus, using tabular values in normal equations

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

$$\therefore 1805 = a(75) + b(1375)$$

$$75a + 1375b = 1805 \quad \ldots (1)$$

Also,

$$120 = (6)a + b(75)$$

$$120 = 6a + 75b \quad \ldots (2)$$

Solving eqn (1) & (2),

$$a = 11.285$$

$$b = 0.697$$

Thus reqd line is,

$$\boxed{y = 11.285 + 0.697x}$$

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2) Fit a straight line to the following data

Year x :	1951	1961	1971	1981	1991
Production y: (1000 tons)	10	12	8	10	15

Also estimate the production in 1987

$\therefore y$ on x.

→ Let the straight line be $y = a + bx$

Here $n = 5$.

$\bar{x} = \frac{\Sigma x}{n} = 1971$, $\bar{y} = \frac{\Sigma y}{n} = 11$

x	y	$X = x - \bar{x}$	$Y = y - \bar{y}$	X^2	XY
1951	10	-20	-1	400	20
1961	12	-10	1	100	10
1971	8	0	-3	0	0
1981	10	10	-1	100	-10
1991	15	20	4	400	80
		$\Sigma X = 0$	$\Sigma Y = 0$	$\Sigma X^2 = 1000$	$\Sigma XY = 100$

Normal equations are,

$\Sigma Y = na + b\Sigma X$

$\Sigma XY = a\Sigma X + b\Sigma X^2$

$0 = 5a + b(0)$

$\boxed{a = 0}$

$80 = a(0) + b(1000)$

$b = \frac{80}{1000} = 0.08$.

Eqn of st. line is,

$Y = a + bx$

$Y = 0 + 0.08X$

$\therefore y - 11 = 0.08(x - 1971)$

$y = -146.6 + 0.08x$.

when $x = 1987$

$y = -146.6 + 0.08(1987)$

$= 12.28$ (1000 tonnes).

$\therefore$ 12280 tones was the production in the year 1987.

3) Fit a straight line to the following data, with x as independent variable

x:	1965	1966	1967	1968	1969
y:	125	140	165	195	1200

→ Here $n = 5$.

$\bar{x} = \frac{\Sigma x}{n} = 1967$ $\bar{y} = \frac{\Sigma Y}{n} = 165$.

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Let the equation of straight line be
$y = a + bx$.

x	y	$X = x - \bar{x}$	$Y = y - \bar{y}$	X^2	XY
1965	125	-2	-40	4	80
1966	140	-1	-25	1	25
1967	165	0	0	0	0
1968	195	1	30	1	30
1969	200	2	35	4	70
		$\Sigma X = 0$	$\Sigma Y = 0$	$\Sigma X^2 = 10$	$\Sigma XY = 205$

Normal eqns are,

$$\Sigma Y = na + b\Sigma x$$
$$\Sigma XY = a\Sigma X + b\Sigma X^2$$

$$0 = 5a + b(0)$$
$$\boxed{\therefore a = 0}$$

$$205 = 0 + b(10)$$
$$\boxed{b = 20.5}$$

Equation of straight line is,

$$Y = a + bX =$$
$$\therefore Y = 0 + 20.5(x - \bar{x})$$
$$\therefore (y - \bar{y}) = 20.5(x - \bar{x})$$
$$\therefore y - 165 = 20.5(x - 1967)$$
$$y = -40158.5 + 20.5x$$

⇒ 2) Fitting a parabolic curve :-

Let the parabolic curve be $y = a + bx + cx^2$ where a, b, c are unknowns to be determined using the following normal equations

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Ex → 1) Fit a second degree parabolic curve to the following data

x:	1	2	3	4	5	6	7	8	9
y:	2	6	7	8	10	11	11	10	9

$$\bar{x} = \frac{\Sigma x}{n} = \frac{45}{9} = 5 \qquad \bar{y} = \frac{\Sigma y}{n} = \frac{74}{9} = 8.22$$

X	$y = Y$	$X = x - 5$	X^2	X^3	XY	X^4	X^2Y
1	2	-4	16	-64	-8	256	32
2	6	-3	9	-27	-18	81	54
3	7	-2	4	-8	-14	16	28
4	8	-1	1	-1	-8	1	8
5	10	0	0	0	0	0	0
6	11	1	1	1	11	1	11
7	11	2	4	8	22	16	44
8	10	3	9	27	30	81	90
9	9	4	16	64	36	256	144
	$\Sigma y = 74$	$\Sigma X = 0$	$\Sigma X^2 = 60$	$\Sigma X^3 = 0$	$\boxed{\Sigma XY = 51}$	$\Sigma X^4 = 768$	$\Sigma X^2 Y = 411$

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Normal equations are :-

$\Sigma y = na + b\Sigma x + c\Sigma x^2$
$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$
$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$

$\therefore 74 = 9a + b(0) + c(60)$
$74 = 9a + 60c$ ... ①

$\therefore 51 = 0 + b(60) + 0$
$b = \frac{51}{60} = \frac{\cancel{5.10}^{0.85}}{\cancel{6}} = 0.85$

$\boxed{\therefore b = 0.85}$

$411 = 60a + b(0) + c(708)$

$411 = 60a + 708c$ ... ②

Solving ① & ②,

$a = 10.0043$
$c = -0.27$

$y = a + bx + cx^2$
$y = 10 + 0.85x - 0.27x^2$
$\therefore y = 10 + 0.85(x-5) - 0.27(x-5)^2$

$\therefore y = 10 + 0.85x - 4.25 - 0.27(x^2 - 10x + 25)$

$= 5.75 + 0.85x - 0.27x^2 + 2.7x - 6.75$
$y = -1 + 3.55x - 0.27x^2$

2) Fit a parabola to the following data & estimate y when x = 10.

x : 10	20	30	40	50	60	70
y : 20	60	70	80	90	100	100

→ Here $n = 7$.

$\bar{x} = \frac{\Sigma x}{n} = 40$ $\qquad \bar{y} = \frac{\Sigma y}{n} = 74.28$

x	y	$X = \frac{x-\bar{x}}{10}$	$Y = \frac{y-\bar{y}}{10}$	X^2	X^3	X^4	XY	X^2Y
10	20	-3	2	9	-27	81	-6	18
20	60	-2	6	4	-8	16	-12	24
30	70	-1	7	1	-1	1	-7	7
40	80	0	8	0	0	0	0	0
50	90	1	9	1	1	1	9	9
60	100	2	10	4	8	16	20	40
70	100	3	10	9	27	81	30	90
		0	52	28	0	196	34	188

$\Sigma Y = na + b\Sigma X + c\Sigma X^2$
$\therefore 52 = 7(a) + 0 + 28c$ ... ①

$\Sigma XY = a\Sigma X + b\Sigma X^2 + c\Sigma X^3$
$\therefore 34 = 0 + b(28) + 0$
$b = \frac{34}{28} = 1.214$

$\Sigma X^2 Y = a\Sigma X^2 + b\Sigma X^3 + c\Sigma X^4$
$188 = a(28) + 0 + c(196)$ ... ②

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Solving ① & ②,

$a = 8.38$

$c = -0.238$

$Y = a + bx + cX^2$

$\therefore \frac{y}{10} = a + b\left(\frac{x-\bar{x}}{10}\right) + c\left(\frac{x-\bar{x}}{10}\right)^2$

$\therefore \frac{y}{10} = 8.38 + \frac{1.214}{10}(x-40) - 0.238\left(\frac{x-40}{10}\right)^2$

$\therefore \frac{y}{10} = 8.38 + \frac{1.214}{10}(x-40) - \frac{0.238}{100}(x-40)^2$

$\therefore 10y = 838 + 12.14(x-40) - 0.238(x^2 - 80x + 1600)$

$10y = 838 + 12.14x - 485.6 - 0.238x^2 + 19.04x - 380.8$

$\therefore 10y = -28.4 + 31.18x - 0.238x^2$

$y = -2.84 + 3.118x - 0.0238x^2$

Fitting exponential curve of the type $y = ab^x$ or $y = ax^b$ or $y = ae^x$.

→ Consider,

$y = ab^x$

$\log y = \log a + x \log b$

$\boxed{Y = A + BX}$

where $Y = \log y$, $A = \log A$, $B = \log b$, $X = x$

Q) Find law of the form $y = ab^x$ to the following data

x:	1	2	3	4	5	6	7	8
y:	1	1.2	1.8	2.5	3.6	4.7	6.6	9.1

→ Given to fit $y = ab^x$

$\log y = \log a + x \log b$

$Y = A + XB$

	$x = X$	y	$Y = \log y$	X^2	XY
	1	1	0	1	0
	2	1.2	0.1823	4	0.3646
	3	1.8	0.5877	9	1.7631
	4	2.5	0.9162	16	3.6648
	5	3.6	1.2809	25	6.4045
	6	4.7	1.5475	36	9.285
	7	6.6	1.887	49	13.209
	8	9.1	2.2082	64	17.6656
Σ	36		8.6098	204	52.347

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Normal eqns are:-

$\Sigma Y = NA + B\Sigma X$

$\therefore 8.6098 = 8(A) + B(36)$ ... ①

$\Sigma XY = A\Sigma X + B\Sigma X^2$

$\therefore 52.347 = 36A + 204B$ .. ②

$A = -0.3822$
$B = 0.3241$

$\therefore A = \log a$
$\therefore a = e^A = e^{-0.3822} = 0.682$

$B = \log b$
$b = e^B = e^{0.3241} = 1.38$

Thus reqd exponential curve is,

$$\boxed{y = (0.68)(1.38)^x}$$

Z-TRANSFORM

⇒ If $f(k) = \{\ldots, f(-2), f(-1), f(0), f(1), f(2) \ldots\}$ be a sequence of terms, where k varies from $-\infty$ to $+\infty$ and $z = x + iy$ be a complex no, then the sequence $\sum_{k=-\infty}^{\infty} f(k) z^{-k}$ is called z transform of $f(k)$ and denoted by $Z\{f(k)\}$ or $F(z)$

Therefore

$$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$\therefore f(k) = Z^{-1}\{F(z)\}$$

Note:- $Z\{f(k)\}$ is a function of complex variable z and is defined only if the sum is finite ie if infinite series $\sum_{k=-\infty}^{\infty} f(k) z^{-k}$ is convergent

⇒ If $F(z) = Z\{f(k)\}$, then $f(k)$ is called inverse z transform of $f(z)$ and denoted by $f(k) = Z^{-1}\{F(z)\}$.

Thus

$$f(k) = Z^{-1}\{F(z)\}$$

The region of convergence is the region for which $\sum_{k=-\infty}^{\infty} f(k) z^{-k}$ is convergent

It is abbreviated as ROC.

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Ex ⇒ 1) If $f(k) = \{9, 6, \underset{\uparrow}{3}, 0, -3, -6, -9\}$, find $Z\{f(k)\}$, where $\uparrow$ denotes the element corresponding to $k=0$.

→ Given,

$f(-2) = 9$, $f(-1) = 6$, $f(0) = 3$, $f(1) = 0$, $f(2) = -3$, $f(3) = -6$, $f(4) = -9$

$$\therefore Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k).z^{-k}$$

$$= f(-2)z^{2} + f(-1)z^{1} + f(0).z^{0} + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + f(4)z^{-4}$$

$$= 9z^2 + 6z + 3 + 0 + (-3)z^{-2} + (-6)z^{-3} + (-9)z^{-4}$$

$$Z\{f(k)\} = 9z^2 + 6z + 3 - 3z^{-2} - 6z^{-3} - 9z^{-4}$$

2) ⇒ Find the z-transform of the sequence,

$$f(k) = \begin{cases} 5^k, & \text{for } k < 0 \\ 3^k, & \text{for } k \geq 0 \end{cases}$$

Also show ROC.

→ We have,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k).z^{-k} + \sum_{k=0}^{\infty} f(k).z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 5^k z^{-k} + \sum_{k=0}^{\infty} 3^k.z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \left(\frac{5}{z}\right)^k + \sum_{k=0}^{\infty} \left(\frac{3}{z}\right)^k$$

$$= \sum_{n=1}^{\infty} \left(\frac{5}{z}\right)^{-n} + \sum_{k=0}^{\infty} \left(\frac{3}{z}\right)^k, \text{ using } n=-k.$$

$$= \sum_{n=1}^{\infty} \left(\frac{z}{5}\right)^n + \sum_{k=0}^{\infty} \left(\frac{3}{z}\right)^k$$

$$= \frac{z}{5} + \left(\frac{z}{5}\right)^2 + \left(\frac{z}{5}\right)^3 + \dots + 1 + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \dots$$

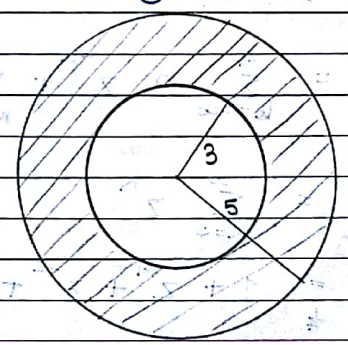
$$= \frac{z}{5}\left\{1 + \frac{z}{5} + \left(\frac{z}{5}\right)^2 + \left(\frac{z}{5}\right)^3 + \dots\right\} + \left\{1 + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \dots\right\}$$

$$= \frac{z}{5} \frac{1}{1 - \left(\frac{z}{5}\right)} + \frac{1}{1 - \left(\frac{3}{z}\right)}$$

$$= \frac{z}{5} \times \frac{5}{5-z} + \frac{z}{z-3}$$

$$= \frac{z}{5-z} + \frac{z}{z-3} \quad \text{if } |z| < 5 \ \& \ 3 < |z|$$

ROC is ~~is~~ $3 < |z| < 5$ an annulus or annular region.



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⇒ z-Transform of some std. functions

1) ⇒ Find the z-Transform of unit impulse function $\delta(k)$

$$\delta(k) = \begin{cases} 1 & , \text{for } k=0 \\ 0 & , \text{otherwise} \end{cases}$$

$$Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} \delta(k) z^{-k}$$

$$= \ldots + \delta(-2) z^{2} + \delta(-1) z^{1} + \delta(0) z^{0} + \delta(1) z^{-1} + \delta(2) z^{-2} + \ldots$$

$$= \ldots + 0 z^{2} + 0 z + (1)(1) + 0 z^{-1} + 0 z^{-2} + \ldots$$

$\therefore Z\{\delta(k)\} = 1$

and ROC is the whole z-plane.

2) ⇒ Find the z-transform of discrete unit step function $u(k)$

$$u(k) = \begin{cases} 1 & , \text{for } k \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$\rightarrow Z\{u(k)\} = \sum_{k=-\infty}^{\infty} f(z) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 \, z^{-k} + \sum_{k=0}^{\infty} (1) z^{-k}$$

$$= \sum_{k=0}^{\infty} z^{-k}$$

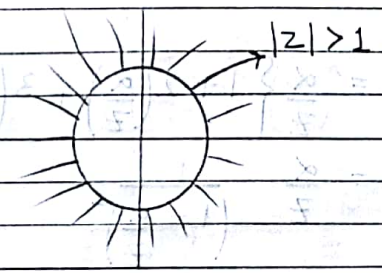
$$= 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \ldots$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ldots$$

$$= \frac{1}{1-\left(\frac{1}{z}\right)} = \frac{z}{z-1} \quad , \text{if } \left|\frac{1}{z}\right| < 1.$$

$$= \frac{z}{z-1} \quad , \text{if } |z| > 1$$

ROC is an external part of unit circle with centre origin ie $|z| > 1$



3) ⇒ Find the z-transform of $f(k) = k\alpha^{k}$, $k \geq 0$

$$\rightarrow Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k) z^{-k} + \sum_{k=0}^{\infty} f(z) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 \, z^{-k} + \sum_{k=0}^{\infty} k \alpha^{k} z^{-k}$$

$$= \sum_{k=0}^{\infty} k \left(\frac{\alpha}{z}\right)^{k}$$

$$= 0\left(\frac{\alpha}{z}\right)^{0} + 1\left(\frac{\alpha}{z}\right)^{1} + 2\left(\frac{\alpha}{z}\right)^{2} + 3\left(\frac{\alpha}{z}\right)^{3} + \ldots$$

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$$= \frac{\alpha}{z} + 2\left(\frac{\alpha}{z}\right)^2 + 3\left(\frac{\alpha}{z}\right)^3 + \cdots$$

1) $\frac{1}{(1-x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

2) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

3) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

$$= \frac{\alpha}{z}\left\{1 + 2\left(\frac{\alpha}{z}\right) + 3\left(\frac{\alpha}{z}\right)^2 + 4\left(\frac{\alpha}{z}\right)^3 + \cdots\right\}$$

$$= \frac{\alpha}{z} \frac{1}{\left(1 - \frac{\alpha}{z}\right)^2}$$

$$Z\{k\alpha^k\} = \frac{\alpha z}{(z-\alpha)^2}, \text{ if } \left|\frac{\alpha}{z}\right| < 1 \text{ ie } |z| > |\alpha|$$

Note:- If $\alpha = 1$

$$Z\{k\} = \frac{z}{(z-1)^2}, \text{ if } |z| > 1$$

Find z-transform of

a) $f(k) = k2^k \quad k \geq 0$

b) $f(k) = k2^k + k3^k, \quad k \geq 0$

c) $f(k) = \alpha^k, \quad k \geq 1$

$$\rightarrow Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{0} 0 \cdot z^{-k} + \sum_{k=1}^{\infty} \alpha^k \cdot z^{-k} = \sum_{k=1}^{\infty} \left(\frac{\alpha}{z}\right)^k$$

$$= \frac{\alpha}{z} + \left(\frac{\alpha}{z}\right)^2 + \left(\frac{\alpha}{z}\right)^3 + \left(\frac{\alpha}{z}\right)^4 + \cdots$$

$$= \frac{\alpha}{z}\left[1 + \frac{\alpha}{z} + \left(\frac{\alpha}{z}\right)^2 + \left(\frac{\alpha}{z}\right)^3 + \cdots\right]$$

$$= \frac{\alpha}{z}\left[\frac{1}{1 - \frac{\alpha}{z}}\right] = \frac{\alpha}{z-\alpha} \quad \text{if } \left|\frac{\alpha}{z}\right| < 1$$

$$= \frac{\alpha}{z-\alpha} \qquad |z| > |\alpha|$$

$$\cancel{Z\{\alpha^k\} = \frac{z}{z-a}}$$

$$\Rightarrow f(k) = a^k, \quad k \geq 0$$

$$\rightarrow Z\{f(k)\} = \sum_{k=0}^{\infty} a^k \cdot z^{-k} = \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

$$= \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a} \quad \text{if } \left|\frac{a}{z}\right| < 1$$

$$\therefore |z| > |a|$$

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$\Rightarrow f(k) = b^k$, for $k < 0$.

$\rightarrow Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$

$$= \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} b^k z^{-k} + \sum_{k=0}^{\infty} 0 \cdot z^{-k}$$

$$= \sum_{n=1}^{\infty} b^{-n} z^{n} \quad (\text{Taking } k = -n).$$

$$= \sum_{n=1}^{\infty} \left(\frac{z}{b}\right)^n$$

$$= \frac{z}{b} + \left(\frac{z}{b}\right)^2 + \left(\frac{z}{b}\right)^3 + \left(\frac{z}{b}\right)^4 + \cdots$$

$$= \frac{z}{b}\left[\frac{1}{1-\left(\frac{z}{b}\right)}\right]$$

$$= \frac{z}{b}\left[\frac{b}{b-z}\right] \quad \text{if } \left|\frac{z}{b}\right| < 1$$

$$= \frac{z}{b-z} \quad \text{if } |z| < |b|$$

$$Z\{b^k\} = \frac{z}{b-z}, \quad \text{if } |z| < |b|, \; k < 0.$$

$\Rightarrow$ Find $Z\{f(k)\}$, if $f(k) = \frac{\alpha^k}{k}$, $k \geq 1$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) \cdot z^{-k}$$

$$= \sum_{k=-\infty}^{0} f(k) \cdot z^{-k} + \sum_{k=1}^{\infty} f(k) \cdot z^{-k}$$

$$= \sum_{k=-\infty}^{0} 0 \cdot z^{-k} + \sum_{k=1}^{\infty} \frac{\alpha^k}{k} z^{-k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{\alpha}{z}\right)^k$$

$$= \frac{\alpha}{z} + \frac{1}{2}\left(\frac{\alpha}{z}\right)^2 + \frac{1}{3}\left(\frac{\alpha}{z}\right)^3 + \frac{1}{4}\left(\frac{\alpha}{z}\right)^4 + \cdots$$

$$= -\log\left(1 - \frac{\alpha}{z}\right), \quad \text{if } \left|\frac{\alpha}{z}\right| < 1$$

$$= -\log\left(\frac{z-\alpha}{z}\right)$$

$$= \log\left(\frac{z}{z-\alpha}\right)$$

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Properties :-

1) Linearity :-

If $f(k)$ & $g(k)$ be two sequences, then

$$Z\{c_1 f(k) + c_2 g(k)\} = c_1 Z\{f(k)\} + c_2 Z\{g(k)\}$$

2) Change of scale property

If $Z\{f(k)\} = F(z)$
then $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

If ROC of $Z\{f(k)\}$ is $R_1 < Z < R_2$, then ROC of $Z\{a^k f(k)\}$ will be $|a|R_1 < Z < |a|R_2$

3) Shifting Property :-

If $Z\{f(k)\} = F(z)$
then $Z\{f(k+n)\} = z^n F(z)$

or

$$Z\{f(k-n)\} = z^{-n} F(z)$$

4) Multiplication by k :

If $Z\{f(k)\} = F(z)$

then $Z\{k^n f(k)\} = \left(-z \frac{d}{dz}\right)^n F(z)$

5) Initial Value :-

If $Z\{f(k)\} = F(z)$, $k \geq 0$,

then $f(0) = \lim_{z \to \infty} F(z)$

6) Final Value :-

If $Z\{f(k)\} = F(z)$

$$\lim_{k \to \infty} f(k) = \lim_{z \to 1} (z-1) F(z)$$

⇒ Convolution Theorem :-

If $f(k)$ & $g(k)$,

$$Z\{f(k) * g(k)\} = Z\{f(k)\} * Z\{g(k)\}$$

Ex ⇒ Find z-transform of $\{a^{|k|}\}$

$$\rightarrow Z\{a^{|k|}\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} a^{|k|} z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^{k} z^{-k}$$

$$= \sum_{n=1}^{\infty} a^{n} z^{n} + \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k \text{ (Using } n=-k\text{)}$$

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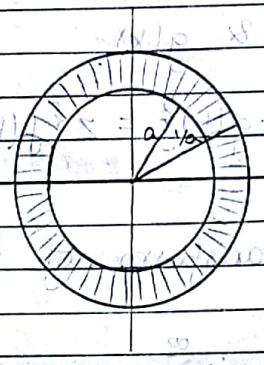
$$= \{az + (az)^2 + (az)^3 + \dots\}$$
$$+ \left\{1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots\right\}$$

$$= \frac{az}{1-az} + \frac{1}{1-\frac{a}{z}}, \text{ if } |az| < 1 \ \& \ \left|\frac{a}{z}\right| < 1$$

$$= \frac{az}{1-az} + \frac{z}{z-a}, \text{ if } |z| < \frac{1}{|a|} \ \& \ |z| > |a|$$

$$= \frac{z(1-a^2)}{(1-az)(z-a)}, \text{ if } |a| < |z| < \frac{1}{|a|}$$



2) $\Rightarrow Z\{c^k \cos \alpha k\}$, $k \geq 0$ and α is real

$$\rightarrow Z\{\cos \alpha k\} = \sum_{k=-\infty}^{\infty} \cos \alpha k \, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 \, z^{-k} + \sum_{k=0}^{\infty} \cos \alpha k \, z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^{i\alpha k} + e^{-i\alpha k}}{2}\right) z^{-k}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} e^{i\alpha k} z^{-k} + \frac{1}{2} \sum_{k=0}^{\infty} e^{-i\alpha k} z^{-k}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{e^{i\alpha}}{z}\right)^k + \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{1}{e^{i\alpha} z}\right)^k$$

$$= \frac{1}{2}\left\{1 + \frac{e^{i\alpha}}{z} + \left(\frac{e^{i\alpha}}{z}\right)^2 + \left(\frac{e^{i\alpha}}{z}\right)^3 + \dots\right\}$$
$$+ \frac{1}{2}\left\{1 + \frac{1}{e^{i\alpha} z} + \left(\frac{1}{e^{i\alpha} z}\right)^2 + \left(\frac{1}{e^{i\alpha} z}\right)^3 + \dots\right\}$$

$$= \frac{1}{2} \frac{1}{1 - \frac{e^{i\alpha}}{z}} + \frac{1}{2} \frac{1}{1 - \frac{1}{e^{i\alpha} z}}, \text{ if } \left|\frac{e^{i\alpha}}{z}\right| < 1 \ \& \ \left|\frac{1}{e^{i\alpha} z}\right| < 1$$

$$= \frac{z}{2(z - e^{i\alpha})} + \frac{e^{i\alpha} z}{2(e^{i\alpha} z - 1)}, \text{ if } |z| > |e^{i\alpha}|$$

$$= \frac{z}{2}\left\{\frac{z e^{i\alpha} - 1 + (z - e^{i\alpha}) e^{i\alpha}}{(z - e^{i\alpha})(z e^{i\alpha} - 1)}\right\}, \text{ if } |z| > 1$$

$$= \frac{z}{2}\left\{\frac{2 z e^{i\alpha} - 1 - e^{2i\alpha}}{z^2 e^{i\alpha} - z - z e^{2i\alpha} + e^{i\alpha}}\right\}$$

$$= \frac{z(2z - e^{-i\alpha} - e^{i\alpha})}{2(z^2 - z e^{-i\alpha} - z e^{i\alpha} + 1)}$$

$$= \frac{z(2z - 2\cos\alpha)}{2(z^2 - z(2\cos\alpha) + 1)}$$

$$Z\{\cos \alpha k\} = \frac{z(z - \cos\alpha)}{z^2 - 2z\cos\alpha + 1}$$

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$$Z\{c^k \cos\alpha k\} = \frac{\frac{z}{c}\left(\frac{z}{c} - \cos\alpha\right)}{\left(\frac{z^2}{c^2} - 2\frac{z}{c}\cos\alpha + 1\right)}$$

$$= \frac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}$$

3) Find z transform of 1 and hence deduce that $Z\{a^k\}$, $k \geq 0$.

$$\rightarrow Z\{1\} = \sum_{k=-\infty}^{\infty} (1)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} (1) \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^k$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots$$

$$= \frac{1}{1 - \frac{1}{z}}, \text{ if } \left|\frac{1}{z}\right| < 1$$

$$\therefore Z\{1\} = \frac{z}{z-1}, \text{ if } |z| > 1$$

Now, by change of scale property,

$$Z\{a^k\} = Z\{a^k \cdot 1\} = F\left(\frac{z}{a}\right)$$

$$= \frac{z/a}{z/a - 1}, \text{ if } \left|\frac{z}{a}\right| > 1$$

$$Z\{a^k\} = \frac{z}{z-a}, \quad |z| > |a|$$

4) $Z\left\{\frac{1}{k+1}\right\}$, $k \geq 1$

$$\rightarrow Z\left\{\frac{1}{k+1}\right\} = z^{-1} Z\left\{\frac{1}{k}\right\}$$

Now,

$$Z\left\{\frac{1}{k}\right\} = \sum_{k=-\infty}^{\infty} \frac{1}{k} z^{-k}$$

$$= \sum_{k=-\infty}^{0} 0 \cdot z^{-k} + \sum_{k=1}^{\infty} \frac{1}{k} z^{-k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{z^k}$$

$$= 1 \cdot \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3}\frac{1}{z^3} + \dots$$

$$= -\log\left(1 - \frac{1}{z}\right)$$

$$= \log\left(\frac{z}{z-1}\right), \text{ if } |z| > 1$$

$$Z\left\{\frac{1}{k}\right\} = \log\left(\frac{z}{z-1}\right), \text{ if } |z| > 1$$

$$Z\left\{\frac{1}{k+1}\right\} = z\log\left(\frac{z}{z-1}\right), \text{ if } |z| > 1$$

ROC is exterior of unit circle

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5) Find $Z\{k^2\}$

$\rightarrow Z\{1\} = \dfrac{z}{z-1}$

$$Z\{k^2\} = Z\{k^2 \cdot 1\}$$

$$= \left(-z\frac{d}{dz}\right)^2 F(z)$$

$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right)$$

$$= \left(z\frac{d}{dz}\right) z\left(\frac{z-1-z}{(z-1)^2}\right)$$

$$= z\frac{d}{dz}z\left(\frac{-1}{(z-1)^2}\right)$$

$$= -z\frac{d}{dz}\frac{z}{(z-1)^2}$$

$$= (-z)\left\{\frac{(z-1)^2(1) - z.z(z-1)}{(z-1)^4}\right\}$$

$$= -z\left\{\frac{z-1-2z}{(z-1)^3}\right\}$$

$$Z\{k^2\} = \frac{z(z+1)}{(z-1)^3}$$

6) If $f(k) = u(k)$ & $g(k) = 2^k u(k)$ find $Z\{f(k) * g(k)\}$

$$Z\{f(k)\} = Z\{u(k)\} = \frac{z}{z-1}$$

$$Z\{g(k)\} = Z\{2^k u(k)\}$$

$$= \frac{(z/2)}{(z/2)-1}$$

$$Z\{f(k) * g(k)\} = Z\{u(k)\} * Z\{2^k u(k)\}$$

$$= \frac{z}{z-1} \cdot \frac{z}{z-2}$$

$$= \frac{z^2}{(z-1)(z-2)}$$

$$= \frac{z^2}{z^2-3z+2}$$

7) Find $Z\{k^2 a^{k-1} u(k-1)\}$

$\rightarrow Z\{u(k-1)\} = \dfrac{z}{z-1}$

$$Z\{a^{k-1} u(k-1)\} = \frac{z/a}{\frac{z}{a}-1} = \frac{z}{z-a}$$

$$Z\{a^{k-1} u(k-1)\} = z^{-1} F(z)$$

$$= z^{-1}\frac{z}{z-a} = \frac{1}{z-a}.$$

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$$Z\{k^2 a^{k-1} u(k-1)\} = \left(-z\frac{d}{dz}\right)^2\left(\frac{1}{z-a}\right)$$

$$= \left(-z\frac{d}{dz}\right)\left[-z\frac{d}{dz}\left(\frac{1}{z-a}\right)\right]$$

$$= +z\frac{d}{dz}\left\{\frac{-z}{(z-a)^2}\right\}$$

$$= -z\left\{\frac{(z-a)^2(1) - z\,\cancel{z(z-a)}}{(z-a)^4}\right\}$$

$$= -z\left\{\frac{z-a-2z}{(z-a)^3}\right\}$$

$$= \frac{z(z+a)}{(z-a)^3}$$

⇒ Inverse z - Transform.

$$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)\, z^{-k}$$

$$Z^{-1}\{F(z)\} = f(k)$$

① $\frac{z}{z-a}$, $|z| > a$.

$$= \frac{1}{1-\left(\frac{a}{z}\right)}$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \left(\frac{a}{z}\right)^4 + \cdots \frac{a^k}{z^k}$$

$$= 1 + a z^{-1} + a^2 z^{-2} + \cdots + a^k z^{-k}$$

co-eff of $z^{-k} = f(k) = a^k$.

$$\therefore Z^{-1}\left\{\frac{z}{z-a}\right\} = a^k \quad , \quad k \geq 0.$$

② $F(z) = \frac{1}{z-a}$ for i) $|z| < |a|$ ii) $|z| > |a|$

ii) $$= \frac{1}{z\left(1-\frac{a}{z}\right)}$$

$$= \frac{1}{z}\left[\frac{1}{1-\left(\frac{a}{z}\right)}\right] \quad \because \frac{|a|}{|z|} < 1 \quad \therefore |a| < |z|$$

$$= \frac{1}{z}\left[1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots + \left(\frac{a}{z}\right)^k z^{-k}\right]$$

Coefficient of $z^{-k} = \frac{a^k}{z} = f(k)$, $k \geq 0$.

$$\therefore Z^{-1}\left\{\frac{1}{z-a}\right\} = \frac{a^k}{z} = a^{k-1} \quad , \quad k \geq 1$$

i) For $|z| < |a|$

$$\frac{|z|}{|a|} < 1$$

$$\therefore F(z) = \frac{1}{z-a} = \frac{1}{-a\left(1-\frac{z}{a}\right)}$$

$$= \frac{-1}{a}\left[1 + \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \cdots + \left(\frac{z}{a}\right)^k\right]$$

$$= \frac{-1}{a} - \frac{z}{a^2} - \frac{z^2}{a^3} - \cdots - \frac{z^k}{a^{k+1}}$$

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co-efficient of $z^{k} = -\frac{1}{a^{k+1}}$, $k \geq 0$.

co-efficient of $z^{-k} = \frac{-1}{a^{-k+1}}$, $-k \geq 0$

$= -a^{k-1}$, $k \leq 0$.

$\therefore z^{-1}\{F(z)\} = z^{-1}\left\{\frac{1}{z-a}\right\} = -a^{k-1}$, $k \leq 0$.

3) Find Inverse z transform of $\frac{1}{(z-5)^3}$,

if (i) $|z| > 5$, (ii) $|z| < 5$

→ i) When ~~$|z| < 5$~~ $|z| > 5$.

$\therefore \frac{5}{|z|} < 1$

$$F(z) = \frac{1}{(z-5)^3}$$

$$= \frac{1}{z^3}\frac{1}{\left\{\left(1-\frac{5}{z}\right)\right\}^3}$$

$$= \frac{1}{z^3}\left\{1-\left(\frac{5}{z}\right)\right\}^{-3}$$

$$= \frac{1}{z^3}\left\{1-(-3)\left(\frac{5}{z}\right)+\frac{(-3)(-4)}{2!}\left(\frac{5}{z}\right)^2 - \frac{(-3)(-4)(-5)}{3!}\left(\frac{5}{z}\right)^3 + \cdots\right\}$$

$$= \frac{1}{z^3}\left\{1+3\left(\frac{5}{z}\right)+6\left(\frac{5}{z}\right)^2+10\left(\frac{5}{z}\right)^3+\cdots+\frac{(n+1)(n+2)}{2}\left(\frac{5}{z}\right)^n+\cdots\right\}$$

$$= \left\{\frac{1}{z^3}+3\cdot\frac{5}{z^4}+6\frac{5^2}{z^5}+10\frac{5^3}{z^6}+\cdots+\frac{(n+1)(n+2)}{2}\frac{5^n}{z^{n+3}}\right\}$$

co-eff of $z^{-(n+3)} = \frac{(n+1)(n+2)}{2}\cdot(5)^n$, $n \geq 0$.

co-eff. of $z^{-k} = \frac{(k-3+1)(k-3+2)}{2}(5)^{k-3}$, $k-3 \geq 0$.

$= \frac{(k-2)(k-1)}{2}(5)^{k-3}$, $k \geq 3$

$z^{-1}\{f(z)\} = \left\{\frac{(k-1)(k-2)}{2}5^{k-3}\right\}$, $k \geq 3$.

4) Find inverse z transform of $\frac{1}{(z-3)(z-2)}$

if ROC is i) $|z| < 2$
ii) $2 < |z| < 3$
iii) $|z| > 3$.

→ $F(z) = \frac{1}{(z-3)(z-2)} = \frac{(z-2)-(z-3)}{(z-3)(z-2)}$

$F(z) = \frac{1}{(z-3)} - \frac{1}{(z-2)}$

i) $|z| < 2$ & $|z| < 3$

$\therefore \frac{|z|}{2} < 1$. $\therefore \frac{|z|}{3} < 1$.

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$$F(z) = -\frac{1}{3}\,\frac{1}{1-\frac{z}{3}} + \frac{1}{2}\,\frac{1}{1-\frac{z}{2}}$$

$$= \left\{-\frac{1}{3}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\dots+\left(\frac{z}{3}\right)^k+\dots\right]+\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots+\left(\frac{z}{2}\right)^k+\dots\right]\right\}$$

$$= \left[-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\dots-\frac{z^k}{3^{k+1}}-\dots\right] + \left[\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\dots+\frac{z^k}{2^{k+1}}\right]$$

From first series,

co-eff of $z^k = -\frac{1}{3^{k+1}}$, $k \geq 0$

co-eff of $z^{-k} = -\frac{1}{3^{-k+1}}$, $-k \geq 0$

co-eff of $z^{-k} = -3^{k-1}$, $k \leq 0$

From second series,

co-eff of $z^k = \frac{1}{2^{k+1}}$ $k \geq 0$

co-eff of $z^{-k} = \frac{1}{2^{-k+1}}$, $-k \geq 0$

co-eff of $z^{-k} = 2^{k-1}$, $k \leq 0$

$\therefore z^{-1}\{F(z)\} = -3^{k-1} + 2^{k-1}$

ii) $2 < |z| < 3$.

ie. $2 < |z|$ & $|z| < 3$

ie $\frac{2}{|z|} < 1$ & $\frac{|z|}{3} < 1$

$$F(z) = -\frac{1}{3}\,\frac{1}{1-\left(\frac{z}{3}\right)} - \frac{1}{z}\,\frac{1}{1-\left(\frac{2}{z}\right)}$$

$$= -\frac{1}{3}\left\{1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\dots+\left(\frac{z}{3}\right)^k+\dots\right\} - \frac{1}{z}\left\{1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\dots+\left(\frac{2}{z}\right)^k\right\}$$

$$= \left\{-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\dots-\frac{z^k}{3^{k+1}}-\dots\right\} + \left\{-\frac{1}{z}-\frac{2}{z^2}-\frac{2^2}{z^3}-\dots-\frac{2^k}{z^{k+1}}\right\}$$

From first series,

co-eff of $z^k = -\frac{1}{3^{k+1}}$ $k \geq 0$.

co-eff of $z^{-k} = -\frac{1}{3^{-k+1}} = -3^{k-1}$

From second series,

co-eff of $z^{-(k+1)} = -2^k$ $k \geq 0$

co-eff of $z^{-k} = -2^{+(k-1)}$, $k \geq 1$.

$$z^{-1}\{F(z)\} = \begin{cases} -3^{k-1}, & k \leq 0 \\ -2^{k-1}, & k \geq 1 \end{cases}$$

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