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CHAPTER 1

Complex Integration

1. Introduction

We have already studied differentiability of complex functions. Now, we shall take up the reverse process *i.e.* integration of complex functions. As in real variables, in complex variables also we have definite and indefinite integrals. An indefinite integral of a complex variable is a function whose derivative is equal to a given analytic function in a region. The indefinite integrals of many elementary functions can be obtained by mere inversion of known derivatives. However, the theory of definite integral of real variables cannot be used straight way for complex variables. The definite integral of a complex variable may depend upon the path of integration in the complex plane.

2. Path of Integration

In the case of real variables, the path of integration of $\int_a^b f(x)\,dx$ is always along the real axis from $x = a$ to $x = b$. But in the case of complex variables the path of the definite integral $\int_a^b f(z)\,dz$ may be any curve joining the points $z = a$ and $z = b$. Generally, the value of this integration depends upon the path. However, as we shall see later the value remains the same in some special cases. Our approach to this topic for obvious reasons will be practical rather than analytical.

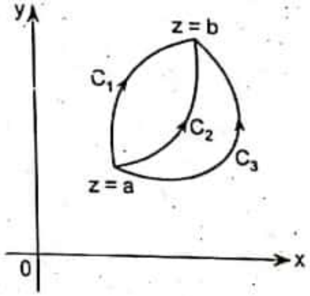


Fig. 1.1

3. Definition

Let $f(z)$ be a continuous function of the complex variable $z = x + iy$ defined at every point of a curve C whose end points are A and B. Let us divide the curve C into n parts by points

$$A = P_0(z_0),\ P_1(z_1),\ P_2(z_2) \ldots\ldots P_i(z_i) \ldots\ldots P_n(z_n) = B.$$

Let $\delta z_i = z_i - z_{i-1}$ and let ξ_i be a point on the arc $P_{i-1} - P_i$. Then the limit of the sum $\sum_{i=1}^{n} f(\xi_i)\,\delta z_i$ as $n \to \infty$ in such a way that each $\delta z_i \to 0$, if it exits is called the **line Integral** of $f(z)$ along C and is denoted by

$$\int_C f(z)\,dz.$$

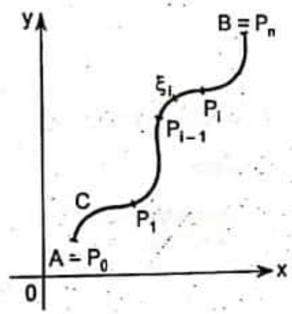


Fig. 1.2

If C is a closed curve *i.e.* if P_0 and P_n coincide the integral is called the **contour Integral** and is denoted by

$$\oint_C f(z)\,dz.$$

4. Evaluation of Line Integral

In practice, the evaluation of a line integral is reduced to the evaluation of two real line integrals as follows.

Since $z = x + iy,\ dz = dx + i\,dy.$

If $f(z) = u + iv.$

$$\int_C f(z)\,dz = \int_C (u + iv)(dx + i\,dy)$$

$$\boxed{\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)}$$

Thus, the integral on the l.h.s. is converted into two integrals on r.h.s.

(a) When the contour is a circle

Example 1 : Evaluate $\int_C \frac{dz}{(z-z_0)^{n+1}}$, where n is an integer and C is a circle $|z - z_0| = r$. **(M.U. 2003)**

Sol. : Let $z - z_0 = r\,e^{i\theta}$, so that θ varies from 0 to 2π as z describes the circle C.

$$\therefore\quad dz = r\,e^{i\theta} i\,d\theta$$

$$\therefore\quad I = \int_0^{2\pi} \frac{r\,e^{i\theta} i\,d\theta}{(r\,e^{i\theta})^{n+1}} = \frac{i}{r^n}\int_0^{2\pi} e^{-in\theta}\,d\theta$$

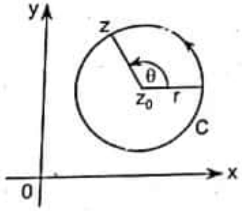


Fig. 1.3

Case I : Let $n = 0$, then $I = i\int_0^{2\pi} d\theta = 2\pi i$

Case II : Let $n \neq 0$, then $I = \frac{i}{r^n}\int_0^{2\pi} (\cos n\theta - i \sin n\theta)\,d\theta$

$$\therefore\quad I = \frac{i}{nr^n}\left[\sin n\theta + i \cos n\theta\right]_0^{2\pi} = 0$$

Example 2 : Evaluate $\int_C |z|\,dz$, where C is the left half of unit circle $|z| = 1$ from $z = -i$ to $z = i$. **(M.U. 1993, 2001, 03, 05, 06)**

Sol. : Since the contour is a semi-circle and $f(z) = |z| = \sqrt{x^2 + y^2}$, we use polar coordinates. $x = r\cos\theta,\ \ y = r\sin\theta.$

$$z = x + iy = re^{i\theta}\quad \therefore\ \sqrt{x^2 + y^2} = r^2$$

Since, $|z| = 1,\ r = 1$

$$\therefore\quad z = e^{i\theta}\quad \therefore\ dz = i\,e^{i\theta} d\theta.$$

$$\text{Hence, } \int_C |z|\,dz = \int_{3\pi/2}^{\pi/2} 1 \cdot i\,e^{i\theta} d\theta = \left[e^{i\theta}\right]_{3\pi/2}^{\pi/2}$$

$$= e^{i\pi/2} - e^{3i\pi/2}$$

$$= [\cos(\pi/2) + i\sin(\pi/2)] - [\cos(3\pi/2) + i\sin(3\pi/2)]$$

$$= (0 + i) - (0 - i) = 2i$$

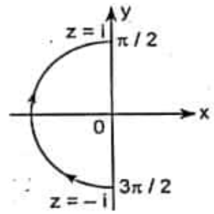


Fig. 1.4

Remark ....

When the contour is a circle it is better to use polar form $z = r\,e^{i\theta}$.

Note ....

We shall often need the curves represented by

(i) $|z| = r$, (ii) $|z - z_0| = r$, (iii) $|z - c| + |z + c| = k$.

(i) Since $z = x + iy,\ |z| = \sqrt{x^2 + y^2}$.

$$\therefore\ |z| = r \text{ gives } \sqrt{x^2 + y^2} = r\ \ i.e.\ \ x^2 + y^2 = r^2.$$

Thus, $|z| = r$ represents a circle with centre at the origin and radius r.

(ii) Since $z - z_0 = (x + iy) - (x_0 + iy_0) = (x - x_0) + i(y - y_0)$.

$$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

$$\therefore\ |z - z_0| = r \text{ gives } = \sqrt{(x - x_0)^2 + (y - y_0)^2} = r$$

$$i.e.\quad (x - x_0)^2 + (y - y_0)^2 = r^2.$$

Thus, $|z - z_0| = r$ represents a circle with centre at (x_0, y_0) and radius r.

Further, since parametric equations of the circle with centre at the origin and radius r are $x = r\cos\theta,\ y = r\sin\theta$, we have

$$z = x + iy = r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta) = r\,e^{i\theta}$$

Thus, $z = r\,e^{i\theta},\ 0 \le \theta \le \pi$ represents the circle with centre at the origin and radius r, in polar form.

Similarly, $(z - z_0) = r\,e^{i\theta}$, represents a circle with centre at $z_0\ (x_0, y_0)$ and radius r in polar form. This is so because $|z - z_0| = r$ is the circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ as seen above.

(iii) If we denote points (x, y) by P, $(c, 0)$ by A and $(-c, 0)$ by B, then $|z - c| + |z + c| = k$ means $l(PA) + l(PB) = k$. But by definition of ellipse, this is an ellipse with foci at $A(c, 0)$ and $B(-c, 0)$ and major axis equal to k.

Now, for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we know that the foci are at $(\pm ae, 0)$ and also the semi-minor axis is given by $b^2 = a^2(1-e^2) = a^2 - a^2 e^2$. Hence, for the above ellipse semi-minor axis is given by $b^2 = \left(\frac{k}{2}\right)^2 - c^2$.

For example, $|z-2|+|z+2| = 6$ represents an ellipse with foci at (2, 0) and (−2, 0) and major axis equal to 6 *i.e.* semi-major axis equal to 3. The semi-minor axis is given by

$$b^2 = \left(\frac{k}{2}\right)^2 - c^2 = \left(\frac{6}{2}\right)^2 - (2)^2 = 9 - 4 = 5.$$

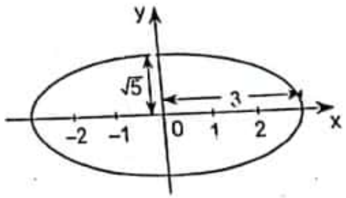


Fig. 1.5

The ellipse is shown in the adjoining figure. Thus, the ellipse $|z-2|+|z+2| = 6$ is $\frac{x^2}{9} + \frac{y^2}{5} = 1$.

Example 3 : Evaluate $\int_C \bar{z}\, dz$, where C is the upper half of the circle $r = 1$. **(M.U. 2014)**

Sol. : Let us put $z = r e^{i\theta}$.

Since $r = 1$, $z = e^{i\theta}$, $dz = e^{i\theta} \cdot i \cdot d\theta$.

$$\therefore \int_C \bar{z}\, dz = \int_1^{-1} \bar{z}\, dz = \int_0^{\pi} e^{-i\theta} \cdot e^{i\theta} \cdot i \cdot d\theta$$

$$= \int_0^{\pi} i\, d\theta = i[\theta]_0^{\pi} = i\pi.$$

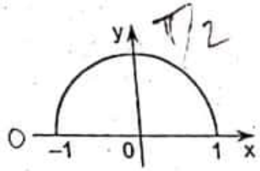


Fig. 1.6

3π/2

Example 4 : Evaluate $\int_C (\bar{z} + 2z)\, dz$ along the circle $x^2 + y^2 = 1$. **(M.U. 2014, 16)**

Sol. : Let us put $z = r e^{i\theta}$.

$\because\ r = 1,\ z = e^{i\theta},\ \bar{z} = e^{-i\theta} \qquad \therefore\ dz = i e^{i\theta}\, d\theta$

$$\therefore \int_C (\bar{z} + 2z)\, dz = \int_0^{2\pi} (e^{-i\theta} + 2e^{i\theta})\, i \cdot e^{i\theta}\, d\theta$$

$$= \int_0^{2\pi} i\, d\theta + 2i \int_0^{2\pi} e^{2i\theta}\, d\theta = i[\theta]_0^{2\pi} + 2i\left[\frac{e^{2i\theta}}{2i}\right]_0^{2\pi}$$

$$= 2i\pi + e^{4i\pi} - 1$$

Example 5 : Evaluate $\int_C \frac{2z+3}{z}\, dz$, where C is

(i) the upper half of the circle $|z| = 2$.

(ii) the lower half of the circle $|z| = 2$.

(ii) the whole circle in anticlock-wise direction. **(M.U. 2005)**

Sol. : Let us put $z = 2e^{i\theta}$.

$$\therefore\ dz = 2i e^{i\theta}\, d\theta \qquad \therefore\ \frac{2z+3}{z} = 2 + \frac{3}{z} = 2 + 3e^{-i\theta}$$

(i) For integral over the upper half

$$\int_C f(z)\, dz = \int_C \frac{2z+3}{z}\, dz = \int_0^{\pi} (2 + 3e^{-i\theta})\, 2i e^{i\theta}\, d\theta$$

$$= 2i \int_0^{\pi} (2e^{i\theta} + 3)\, d\theta = 2i\left[\frac{2e^{i\theta}}{i} + 3\theta\right]_0^{\pi}$$

$$= 2i\left[\frac{2e^{i\pi}}{i} + 3\pi - \frac{2}{i}\right] = 2\left[2e^{i\pi} + 3\pi i - 2\right]$$

$$= 2\left[2(\cos\pi + i\sin\pi) + 3\pi i - 2\right]$$

$$= 2\left[-2 + 3\pi i - 2\right] = 2(3\pi i - 4).$$

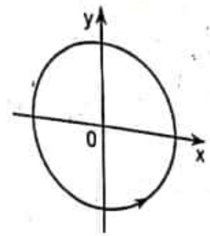


Fig. 1.7

(ii) For integral over the lower half.

$$\int_C f(z)\, dz = 2i \int_{\pi}^{0} (2e^{i\theta} + 3)\, d\theta = 2i\left[\frac{2e^{i\theta}}{i} + 3\theta\right]_{\pi}^{0} = 2i\left[\frac{2}{i} - \frac{2e^{i\pi}}{i} - 3\pi\right]$$

$$= 2\left[2 - 2(\cos\pi + i\sin\pi) - 3\pi i\right] = 2\left[4 - 3\pi i\right]$$

π (iii) For the whole circle

$$\int_C f(z)\, dz = 2i\left[\frac{2e^{i\theta}}{i} + 3\theta\right]_0^{2\pi} = 2i\left[\frac{2}{i}e^{2i\pi} + 6\pi - \frac{2}{i}\right] = 2\left[2e^{2i\pi} + 6\pi i - 2\right]$$

$$= 2\left[2(\cos 2\pi + i\sin 2\pi) + 6\pi i - 2\right] = 2\left[6\pi i\right] = 12.$$

Example 6 : Show that $\int_C \log z\, dz = 2\pi i$, where C is the unit circle in the z-plane. **(M.U. 2000, 06, 17)**

Sol. : Since the contour is a circle we use polar coordinates.

We put $z = e^{i\theta}$ $\because\ r = 1$. $\therefore\ dz = i e^{i\theta} d\theta$; θ varies from 0 to 2π.

$$\therefore\ I = \int_0^{2\pi} (\log e^{i\theta}) \cdot i e^{i\theta} d\theta = i\int_0^{2\pi} i\theta e^{i\theta} d\theta = -\int_0^{2\pi} e^{i\theta}\theta\, d\theta$$

$$= -\left[\theta \cdot \frac{e^{i\theta}}{i} - \int \frac{e^{i\theta}}{i} \cdot 1 \cdot d\theta\right]_0^{2\pi} = -\left[\theta \cdot \frac{e^{i\theta}}{i} - \frac{e^{i\theta}}{-1}\right]_0^{2\pi} = -\left[\theta \cdot \frac{e^{i\theta}}{i} + e^{i\theta}\right]_0^{2\pi}$$

$$= -\left[\frac{2\pi e^{2i\pi}}{i} + e^{2i\pi} - 0 - 1\right] = -\left[\frac{2\pi}{i} + 1 - 1\right] \qquad [\because e^{2i\pi} = 1]$$

$$= -\frac{2\pi i}{i^2} = 2\pi i.$$

Example 7 : Evaluate $\int_C z^2 dz$, where C is the circle $x = r\cos\theta$, $y = r\sin\theta$, from $\theta = 0$ to $\theta = \pi/3$. **(M.U. 2005, 06)**

Sol. : As above

$$I = \int_C z^2\,dz = \int_0^{\pi/3} r^2 e^{2i\theta} \cdot r e^{i\theta} \cdot i\,d\theta = r^3 i \int_0^{\pi/3} e^{3i\theta}\,d\theta = r^3 i\left[\frac{e^{3i\theta}}{3i}\right]_0^{\pi/3}$$

$$= \frac{r^3}{3}\left[e^{i\pi} - 1\right] = \frac{r^3}{3}\left[\cos\pi + i\sin\pi - 1\right] = -\frac{2r^3}{3}.$$

Example 8 : Evaluate $\int_C (z - z^2)\,dz$, where C is the upper half of the circle $|z| = 1$. What is the value of the integral for the lower half of the same circle ? **(M.U. 1997, 2004, 06)**

Sol. : Let us put $z = e^{i\theta}$ $\therefore dz = e^{i\theta} d\theta$. And θ varies from 0 to π.

$$\therefore \int_C (z - z^2)\,dz = \int_0^{\pi} (e^{i\theta} - e^{2i\theta})\,e^{i\theta} \cdot i\,d\theta = i\int_0^{\pi} (e^{2i\theta} - e^{3i\theta})\,d\theta$$

$$= i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_0^{\pi} = \left[\frac{e^{2i\pi}}{2} - \frac{e^{3i\pi}}{3} - \frac{1}{2} + \frac{1}{3}\right]$$

$$= \left[\frac{1}{2}(\cos 2\pi + i\sin 2\pi) - \frac{1}{3}(\cos 3\pi + i\sin 3\pi) - \frac{1}{2} + \frac{1}{3}\right]$$

$$= \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{2} + \frac{1}{3}\right] = \frac{2}{3}.$$

The value of the integral for the lower half of the same circle in the same positive direction *i.e.* when θ varies from π to 2π.

$$= \int_C (z - z^2)\,dz = i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_{\pi}^{2\pi} = i\left[\frac{e^{4i\pi}}{2i} - \frac{e^{6i\pi}}{3i} - \frac{e^{2i\pi}}{2i} + \frac{e^{3i\pi}}{3i}\right]$$

$$= \left[\frac{\cos 4\pi + i\sin 4\pi}{2} - \frac{\cos 6\pi + i\sin 6\pi}{3} - \frac{\cos 2\pi + i\sin 2\pi}{2} + \frac{\cos 3\pi + i\sin 3\pi}{2}\right]$$

$$= \left[\frac{1}{2} - \frac{1}{3} - \frac{1}{2} - \frac{1}{3}\right] = -\frac{2}{3}.$$

Alternatively : Since for the closed curve C, the total integral is zero, for the remaining part of the circle the integral $= -2/3$.

(b) When the contour is a straight line or a parabola

Example 1 : Evaluate the integral $\int_0^{1+i} (x - y + ix^2)\,dz$.

(i) along the line from $z = 0$ to $z = 1 + i$. **(M.U. 2002, 05)**

(ii) along the real axis from $z = 0$ to $z = 1$ and then along the line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$. **(M.U. 1999)**

(iii) along the imaginary axis from $z = 0$ to $z = i$ and then along the line parallel to the real axis from $z = i$ to $z = i + 1$.

(iv) along the parabola $y^2 = x$. **(M.U. 1990)**

Sol. : (i) Let OA be the line from $z = 0$ *i.e.*, (0, 0) to $z = 1 + i$, *i.e.*, (1, 1). Equation of the line OA is $y = x$ *i.e.*, on OA, $y = x$ $\therefore$ $dy = dx$.

$\therefore$ $dz = dx + i\,dy = dx + i\,dx = (1 + i)\,dx$

and x varies from 0 to 1.

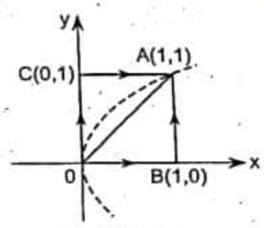


Fig. 1.8

$$\therefore I = \int_0^{1+i} (x - y + ix^2)\,dz$$

$$= \int_0^1 (x - x + ix^2)(1 + i)\,dx = \int_0^1 ix^2(1 + i)\,dx$$

$$= \int_0^1 (i - 1)\,x^2\,dx = (i - 1)\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}(i - 1)$$

(ii) Here, the contour is first the segment OB and then the segment BA.

On OB, $y = 0$. $\therefore$ $dy = 0$ $\therefore$ $dz = dx + i\,dy = dx$ and x varies from 0 to 1.

$$\therefore \int_{OB} (x - y + ix^2)\,dz = \int_0^1 (x + ix^2)\,dx = \left[\frac{x^2}{2} + i\frac{x^3}{3}\right]_0^1 = \frac{1}{2} + i\frac{1}{3}$$

On BA, $x = 1$ $\therefore$ $dx = 0$ $\therefore$ $dz = dx + i\,dy = i\,dy$ and y varies from 0 to 1.

$$\therefore \int_{BA} (x - y + ix^2)\,dz = \int_0^1 (1 - y + i)\,i\,dy = \int_0^1 [(-1 + i) - iy]\,dy$$

$$\therefore \int_{BA} (x - y + ix^2)\,dz = \left[(-1 + i)y - \frac{iy^2}{2}\right]_0^1 = -1 + \frac{i}{2}$$

Hence, adding the two results.

$$I = \frac{1}{2} + i\frac{1}{3} - 1 + \frac{i}{2} = -\frac{1}{2} + \frac{5}{6}i$$

(iii) Here, the contour is first the segment OC and then the segment CA.

On OC, $x = 0$ $\therefore$ $dx = 0$ $\therefore$ $dz = dx + i\,dy = i\,dy$ and y varies from 0 to 1.

$$\therefore \int_{OC} (x - y + ix^2)\,dz = \int_0^1 (-y)\,i\,dy = -i\left[\frac{y^2}{2}\right]_0^1 = -\frac{i}{2}$$

On CA, $y = 1$ $\therefore$ $dy = 0$ $\therefore$ $dz = dx + i\,dy = dx$ and x varies from 0 to 1.

$$\therefore \int_{CA} (x - y + ix^2)\,dz = \int_0^1 (x - 1 + ix^2)\,dx = \left[\frac{x^2}{2} - x + i\frac{x^3}{3}\right]_0^1 = -\frac{1}{2} + \frac{i}{3}$$

Hence, adding the two results.

$$I = -\frac{i}{2} - \frac{1}{2} + \frac{i}{3} = -\frac{1}{2} - \frac{i}{6}.$$

(iv) Here, the contour is the arc OA of the parabola $y^2 = x$.

Hence, $z = x + iy = y^2 + iy$

$\therefore\ dz = 2y\,dy + i\,dy = (2y + i)\,dy$ and y varies from 0 to 1.

$$\therefore\ I = \int_0^{1+i}(x - y + ix^2)\,dz = \int_0^1 (y^2 - y + iy^4)(2y + i)\,dy$$

$$= \int_0^1 (2y^3 - 2y^2 + i2y^5 + iy^2 - iy - y^4)\,dy$$

$$= \left[\frac{1}{2}y^4 - \frac{2y^3}{3} + i\frac{1}{3}y^6 + \frac{iy^3}{3} - \frac{iy^2}{2} - \frac{y^5}{5}\right]_0^1$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{i}{3} + \frac{i}{3} - \frac{i}{2} - \frac{1}{5} = -\frac{11}{30} + \frac{i}{6}.$$

Note ....

Note that the line integral depends upon the path.

Example 2 : Evaluate $\int_{1-i}^{2+i}(2x + iy + 1)\,dz$, along (i) the straight line joining $(1 - i)$ to $(2 + i)$, (ii) $x = t + 1$, $y = 2t^2 - 1$ a parabola. **(M.U. 1993, 2005)**

Sol. : (i) We first find the equation of the line through the given points $(1, -1)$ and $(2, 1)$.

The equation is $\dfrac{y+1}{-1-1} = \dfrac{x-1}{1-2}$ i.e. $\dfrac{y+1}{2} = \dfrac{x-1}{1}$

$\therefore\ y + 1 = 2x - 2$ $\quad\therefore\ y = 2x - 3$

$\therefore\ dy = 2\,dx$ $\quad\therefore\ dz = dx + i\,dy = (1 + 2i)\,dx$

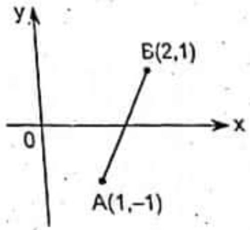


Fig. 1.9

Hence, the integral becomes

$$\int_{1-i}^{2+i}(2x + iy + 1)\,dz = \int_1^2 [2x + i(2x - 3) + 1](1 + 2i)\,dx$$

$$= (1 + 2i)\left[x^2 + i(x^2 - 3x) + x\right]_1^2$$

$$= (1 + 2i)\left[\{4 + i(4 - 6) + 2\} - \{1 + i(1 - 3) + 1\}\right]$$

$$= (1 + 2i)\left[(6 - 2i) - (2 - 2i)\right] = 4(1 + 2i)$$

(ii) If $x = t + 1$, $y = 2t^2 - 1$, $z = (t + 1) + i(2t^2 - 1)$. $\quad\therefore\ dz = (1 + 4it)\,dt$.

When $z = 1 - i$, $t = 0$ and when $z = 2 + i$, $t = 1$.

$$\therefore\ I = \int_0^1 \left[2(t + 1) + i(2t^2 - 1) + 1\right](1 + 4it)\,dt$$

$$= \int_0^1 \left\{2(t + 1) + i(2t^2 - 1) + 1\right\} + \left\{8i(t^2 + t) - 4(2t^3 - t) + 4it\right\}dt$$

$$\therefore\ I = \left[\left\{2\left(\frac{t^2}{2} + t\right) + i\left(\frac{2t^3}{3} - t\right) + t\right\} + \left\{8i\left(\frac{t^3}{3} + \frac{t^2}{2}\right) - 4\left(\frac{2t^4}{4} - \frac{t^2}{2}\right) + 2it^2\right\}\right]_0^1$$

$$= \left[2\left(\frac{3}{2}\right) + i\left(-\frac{1}{3}\right) + 1 + 8i\left(\frac{5}{6}\right) - 4(0) + 2i\right]$$

$$= 4 + \frac{25}{3}i.$$

Example 3 : Evaluate $\int_0^{1+i} z^2\,dz$, along (i) the line $y = x$, (ii) the parabola $x = y^2$. Is the line integral independent of the path ? Explain. **(M.U. 1992, 98, 2013, 16)**

Sol. : Let OA be the line from $z = 0$ to $z = 1 + i$.

(i) On the line OA i.e. $y = x$, $dy = dx$

$\therefore\ dz = dx + i\,dy = (1 + i)\,dx$

and x varies from 0 to 1.

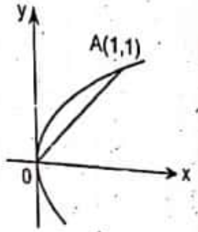


Fig. 1.10

$$\therefore\ I = \int_0^{1+i}(x + iy)^2\,dz = \int_0^1 (x^2 - y^2 + 2ixy)(1 + i)\,dx$$

$$= \int_0^1 (x^2 - x^2 + 2ix^2)(1 + i)\,dx \quad [\because y = x]$$

$$= 2i(1 + i)\int_0^1 x^2\,dx = 2i(1 + i)\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3}i(1 + i) = \frac{2}{3}(i - 1)$$

(ii) On the arc OA of the parabola $x = y^2$, $dx = 2y\,dy$

$\therefore\ dz = dx + i\,dy = (2y + i)\,dy$

$$\therefore\ I = \int_0^1 (x^2 - y^2 + 2ixy)(2y + i)\,dy$$

$$= \int_0^1 (y^4 - y^2 + 2iy^3)(2y + i)\,dy \quad [\because x = y^2]$$

$$= \int_0^1 (2y^5 - 2y^3 + 4iy^4 + iy^4 - iy^2 - 2y^3)\,dy$$

$$= \int_0^1 \left[(2y^5 - 4y^3) + i(5y^4 - y^2)\right]dy$$

$$= \left[\left(\frac{y^6}{3} - y^4\right) + i\left(y^5 - \frac{y^3}{3}\right)\right]_0^1 = \left[\left(\frac{1}{3} - 1\right) + i\left(1 - \frac{1}{3}\right)\right]$$

$$= -\frac{2}{3} + \frac{2}{3}i = \frac{2}{3}(i - 1).$$

The two integrals are equal *i.e.* the integral is independent of path because $f(z) = z^2$ is **an analytic function**. (See the corollary given on page 2-3)

Example 4 : Integrate the function $f(z) = x^2 + i\,xy$ from $A(1, 1)$ to $B(2, 4)$ along the curve $x = t$, $y = t^2$. **(M.U. 1993, 98, 2003, 04)**

(Or integrate xz along the line from $A(1, 1)$ to $B(2, 4)$ in the complex plane.) **(M.U. 2011)**

Sol. : Putting $x = t$, $y = t^2$ in $f(z)$, we get $f(z) = x^2 + i\,xy = t^2 + i\,t^3$

and $dz = dx + i\,dy = dt + 2i\,t\,dt = (1 + 2i\,t)\,dt$. And t varies from 1 to 2.

$$\therefore \int_A^B f(z)\,dz = \int_1^2 (t^2 + it^3)(1 + 2it)\,dt = \int_1^2 (t^2 + 2it^3 + it^3 - 2t^4)\,dt$$

$$= \int_1^2 \left[(t^2 - 2t^4) + 3it^3\right] dt = \left[\frac{t^3}{3} - \frac{2t^5}{5} + 3i \cdot \frac{t^4}{4}\right]_1^2$$

$$= \left[\left(\frac{8}{3} - \frac{64}{5} + 3i \cdot \frac{16}{4}\right) - \left(\frac{1}{3} - \frac{2}{5} + \frac{3i}{4}\right)\right] = -\frac{151}{15} + i\,\frac{45}{4}.$$

Example 5 : Evaluate $\int \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve $z = t^2 + it$. **(M.U. 1993)**

Sol. : The curve $z = t^2 + it$ can be given by $x = t^2$, $y = t$. When z varies from $z = 0$ to $z = 4 + 2i$, t varies from $t = 0$ to $t = 2$.

Further $\bar{z} = t^2 - it$ and $dz = (2t + i)\,dt$ since $z = t^2 + i\,t$.

$$\therefore \int_A^B \bar{z}\,dz = \int_0^2 (t^2 - it)(2t + i)\,dt = \int_0^2 (2t^3 + it^2 - 2it^2 + t)\,dt$$

$$= \int_0^2 \left[(2t^3 + t) - it^2\right] dt = \left[\frac{t^4}{2} + \frac{t^2}{2} - \frac{it^3}{3}\right]_0^2$$

$$= \left[8 + 2 - i \cdot \frac{8}{3}\right] = \frac{30 - 8i}{3}.$$

Example 6 : If O is the origin, L is the point $z = 3$, M is the point $z = 3 + i$, evaluate $\int z^2 dz$ along (i) the path OM, (ii) the path OLM, (iii) the path $OLMO$. **(M.U. 1992, 94)**

Sol. : Now $z^2 = (x + iy)^2 = x^2 - y^2 + 2i\,xy$

(i) Along the path *OM* : The equation of the line OM is

$$\frac{y - 1}{1 - 0} = \frac{x - 3}{3 - 0} \quad i.e.\ \ y = \frac{x}{3}$$

Also $dz = dx + i\,dy = dx + \frac{i}{3}dx$.

Along OM, x varies from 0 to 3.

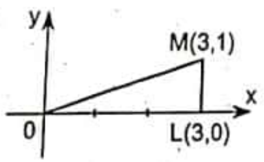


Fig. 1.11

$$\therefore \int z^2 dz = \int (x^2 - y^2 + 2i\,xy)(dx + i\,dy)$$

$$= \int_0^3 \left(x^2 - \frac{x^2}{9} + 2i\,x \cdot \frac{x}{3}\right)\left(1 + \frac{i}{3}\right) dx = \left(1 + \frac{i}{3}\right)\int_0^3 \left(\frac{8x^2}{9} + \frac{2i}{3}x^2\right) dx$$

$$= \left(1 + \frac{i}{3}\right)\left[\frac{8x^3}{27} + \frac{2i}{9}x^3\right]_0^3 = \left(1 + \frac{i}{3}\right)[8 + 6i]$$

$$= 8 + \frac{8i}{3} + 6i - \frac{6}{3} = \frac{18 + 26i}{3}$$

(ii) Along the path *OLM* : Along OL, $y = 0$

$\therefore$ $z^2 = x^2$, $dy = 0$ and $dz = dx$, x varies from 0 to 3.

$$\int z^2 dz = \int_0^3 x^2 dx = \left[\frac{x^3}{3}\right]_0^3 = 9$$

Along LM, $x = 3$, $\therefore$ $z^2 = 9 - y^2 + 6iy$, $dx = 0$ $\therefore$ $dz = i\,dy$ and y varies from 0 to 1.

$$\therefore \int z^2 dz = \int_0^1 (9 - y^2 + 6iy)\,i\,dy = i\left[9y - \frac{y^3}{3} + 3iy^2\right]_0^1 = i\left(9 - \frac{1}{3} + 3i\right)$$

$$= \left(\frac{26}{3} + 3i\right)i = -3 + \frac{26}{3}i.$$

$\therefore$ Integral along OLM = Integral along OL + integral along LM

$$= 9 + \left(-3 + \frac{26}{3}i\right) = 6 + \frac{26i}{3}$$

(iii) Along the path *OLMO*

Integral along $OLMO$ = integral along OL + integral along LM + integral along MO.

$$= 9 + \left(-3 + \frac{26i}{3}\right) + \left[-\left(\frac{18 + 26i}{3}\right)\right] = 0$$

Example 7 : Find $\int \text{Im}(z)\,dz$ along (i) the unit circle described once in positive direction from $z = 1$ to $z = 1$. (ii) the straight line from $P(z_1)$ to $Q(z_2)$. **(M.U. 1991)**

Sol. : Since $z = x + iy$, $\text{Im}(z) = y$ and $dz = dx + i\,dy$

(i) Along the unit circle $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$

$$\therefore I = \int \text{Im}(z)\,dz = \int_C y(dx + i\,dy) = \int_0^{2\pi} \sin\theta(-\sin\theta + i\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} (-\sin^2\theta + i\sin\theta\cos\theta)\,d\theta = \int_0^{2\pi}\left[-\left\{\frac{1 - \cos 2\theta}{2}\right\} + i\sin\theta\cos\theta\right] d\theta$$

$$= \left[-\frac{1}{2}\left\{\theta - \frac{\sin 2\theta}{2}\right\} + i\,\frac{\sin^2\theta}{2}\right]_0^{2\pi} = -\frac{1}{2}(2\pi) = -\pi.$$

(ii) The equation of the line PQ is $\dfrac{y-y_1}{y_1-y_2}=\dfrac{x-x_1}{x_1-x_2}$

$\therefore\ (x_1-x_2)(y-y_1)=(y_1-y_2)(x-x_1)$

$\therefore\ (x_1-x_2)\,dy=(y_1-y_2)\,dx$

$$\therefore\ I=\int_C \operatorname{Im}(z)\,dz=\int_{y_1}^{y_2} y(dx+i\,dy)=\int_{y_1}^{y_2}(y\,dx+iy\,dy)$$

$$=\int_{y_1}^{y_2}\left[\left(\frac{x_1-x_2}{y_1-y_2}\right)y+iy\right]dy=\frac{x_1-x_2}{y_1-y_2}\left\{\frac{y^2}{2}\right\}_{y_1}^{y_2}+i\left\{\frac{y^2}{2}\right\}_{y_1}^{y_2}$$

$$=\frac{x_1-x_2}{y_1-y_2}\cdot\frac{(y_2^2-y_1^2)}{2}+i\,\frac{(y_2^2-y_1^2)}{2}$$

$$=\frac{(x_2-x_1)(y_2+y_1)}{2}+i\,\frac{(y_2+y_1)(y_2-y_1)}{2}$$

$$=\left[\frac{(x_2-x_1)+i(y_2-y_1)}{2}\right](y_2+y_1)$$

$$=\frac{1}{2}(z_2-z_1)\operatorname{Im}(z_2+z_1)$$

Example 8 : Evaluate $\displaystyle\int_0^{1+i}(x^2+iy)\,dz$, along the path (i) $y=x$, (ii) $y=x^2$.

Is the line integral independent of the path ? **(M.U. 1996, 2009, 14)**

Sol. : (i) Along the path $y=x$: $\because\ y=x,\ dy=dx$

$\therefore\ dz=dx+i\,dy=dx+i\,dx=(1+i)\,dx$. And x varies from 0 to 1.

$$\int_0^{1+i}(x^2+iy)\,dz=\int_0^1(x^2+ix)(1+i)\,dx=(1+i)\left[\frac{x^3}{3}+\frac{ix^2}{2}\right]_0^1$$

$$=(1+i)\left(\frac{1}{3}+\frac{i}{2}\right)=(1+i)\frac{(2+3i)}{6}$$

$$=\frac{(2+2i+3i-3)}{6}=\frac{-1+5i}{6}$$

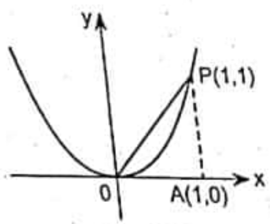


Fig. 1.12

(ii) Along the path $y=x^2$: $y=x^2,\ dy=2x\,dx$

$$\int_0^1(x^2+ix^2)(dx+2ix\,dx)=(1+i)\int_0^1 x^2(1+2ix)\,dx$$

$$=(1+i)\int_0^1(x^2+2ix^3)\,dx$$

$$=(1+i)\left[\frac{x^3}{3}+i\cdot\frac{x^4}{2}\right]_0^1=(1+i)\left(\frac{1}{3}+\frac{i}{2}\right)=\frac{-1+5i}{6}$$

The two line integrals are equal.

(iii) Now, consider the integral along a third path, say, along OA and then along AP. Along OA, x varies from 0 to 1 and $y=0$

$\therefore\ dy=0\qquad \therefore\ dz=dx.$

$$\therefore\ \int_{OA}(x^2+iy)\,dz=\int_0^1 x^2\,dx=\left[\frac{x^3}{3}\right]_0^1=\frac{1}{3}$$

Along AP, $x=1$ $\therefore\ dx=0$ and y varies from 0 to 1. $\therefore\ dz=i\,dy.$

$$\therefore\ \int_{AP}(x^2+iy)\,dz=\int_0^1(1+iy)\,i\,dy=i\left[y+\frac{iy^2}{2}\right]_0^1=i\left(1+\frac{i}{2}\right)=i-\frac{1}{2}$$

$$\therefore\ \int_0^{1+i}(x^2+iy)\,dz=\frac{1}{3}+i-\frac{1}{2}=-\frac{1}{6}+i$$

Thus, the third integral is not equal to the first two. Hence, the integral is not independent of the path.

(iv) Again let $f(z)=x^2+iy=u+iv$. $\therefore\ u=x^2,\ v=y.$

$\therefore\ u_x=2x,\ u_y=0,\ v_x=0,\ v_y=1.$

Hence, Cauchy-Rieman equations $u_x=v_y$ and $u_y=-v_x$ are not satisfied. $f(z)$ is not analytic and hence, integral is not independent of the path.

(See the corollary given on page 2-3)

Example 9 : Evaluate $\int f(z)\,dz$ along the parabola $y=2x^2$ from $z=0$ to $z=3+18i$ where $f(z)=x^2-2iy$. **(M.U. 1996, 2015)**

Sol. : $\because\ y=2x^2\qquad \therefore\ dy=4x\,dx$

$\therefore\ dz=dx+i\,dy=dx+i\,4x\,dx=(1+4ix)\,dx$

$$\therefore\ \int_C f(z)\,dz=\int_0^3(x^2-2i\cdot 2x^2)(1+4ix)\,dx=\int_0^3(x^2-4ix^2+4ix^3+16x^3)\,dx$$

$$=\left[\frac{x^3}{3}-4i\cdot\frac{x^3}{3}+4i\cdot\frac{x^4}{4}+16\cdot\frac{x^4}{4}\right]_0^3$$

$$=[9-4i\cdot 9+i\cdot 81+4\cdot 81]=333+45i.$$

Example 10 : Evaluate $\int f(z)\,dz$, along the parabola $y=2x^2$ from $z=0$ to $z=3+18i$ where $f(z)=x^2-2i\,xy$. **(M.U. 1996, 2014)**

Sol. : $\because\ y=2x^2,\ dy=4x\,dx$

$\therefore\ dz=dx+i\,dy=dx+i\,4x\,dx=(1+4ix)\,dx$

$$\therefore\ \int_C f(z)\,dz=\int_0^3(x^2-2ix\cdot 2x^2)(1+4ix)\,dx=\int_0^3(x^2-4ix^3+4ix^3+16x^4)\,dx$$

$$=\int_0^3(x^2+16x^4)\,dx=\left[\frac{x^3}{3}+\frac{16x^5}{5}\right]_0^3=\left[9+16\cdot\frac{243}{5}\right]=\frac{3933}{5}.$$

Example 11 : State true or false with proper justification.

If $f(z) = (x^2 + 2x - y^2) + 2iy\,(x+1)$ then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$, where C_1 and C_2 are respectively $y^2 = x^3$ and $y = x^2$ joining the points (0, 0) and (1, 1). (M.U. 1996)

Sol. : Let $f(z) = u + iv$ where $u = x^2 + 2x - y^2$, $v = 2y\,(x+1) = 2xy + 2y$

$\therefore\ u_x = 2x + 2,\quad u_y = -2y$

$v_x = 2y,\quad v_y = 2x + 2$

$\therefore\ u_x = v_y$ and $u_y = -v_x$

Cauchy-Rieman equations are satisfied. Hence, $f(z)$ is analytic.

Hence, $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ because for an analytic function the integral is independent of the path. (See the corollary given on page 2-3)

EXERCISE - I

1. Evaluate

(a) $\int_0^{1+i} (x^2 - iy)\,dz$ along the path (i) $y = x$, (ii) $y = x^2$. (M.U. 2015)

$\left[\text{Ans. : (i) } \frac{5}{6} - \frac{i}{6},\ \text{(ii) } \frac{5}{6} + \frac{i}{6}\right]$

(b) $\int_{1-i}^{2+i} (2x + iy + 1)\,dz$ along the curve $x = t + 1$, $y = 2t^2 - 1$. $\left[\text{Ans. : } 4 + \frac{25}{3}i\right]$

2. Evaluate $\int_0^{3+i} z^2 dz$

(i) along the real axis from 0 to 3 and then vertically to $3 + i$.

(ii) along the imaginary axis from 0 to i and then horizontally to $3 + i$.

(iii) along the parabola $x = 3y^2$.

(M.U. 2006) [Ans. : $6 + \frac{26}{3}i$ in each case.]

3. Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along

(i) the line $x = 2y$. (M.U. 2001, 02, 03)

(ii) the real axis from 0 to 2 and then vertically to $2 + i$. (M.U. 2006)

(iii) the parabola $2y^2 = x$.

(**Hint :** $(\bar{z})^2 = (x - iy)^2$) [Ans. : (i) $\frac{10}{3} - \frac{5}{3}i$, (ii) $\frac{14}{3} + \frac{11}{3}i$, (iii) $\frac{8}{3} - \frac{41}{15}i$]

4. Evaluate $\int_C (y - x - 3x^2 i)\,dz$, where C is a straight line from $z = 0$ to $z = 1 + i$.

[Ans. : $1 - i$]

5. Evaluate $\int_C \frac{dz}{z}$, where C is the circle $|z| = r$ in the positive sense.

[Ans. : $2\pi i$. Put $z = re^{i\theta}$, $dz = r\,i\,e^{i\theta} d\theta$]

6. Evaluate $\int (z - z^2)\,dz$ along the upper half of the circle $|z| = 1$. (M.U. 1994)

[Ans. : $\frac{2}{3}$. Put $z = re^{i\theta}$]

7. Evaluate $\int_C z^2 dz$ from $P\,(1, 1)$ to $Q\,(2, 4)$ where

(i) C is the curve $y = x^2$,

(ii) C is the line $y = 3x - 2$,

(iii) C is the curve $x = t$, $y = t^2$. [Ans. : $-\frac{86}{3} - 6i$ in each case.]

8. Evaluate $\int_C |z|^2\,dz$ where C is the boundary of the square C with vertices (0, 0), (1, 0), (1, 1), (0, 1).

[Ans. : $-1 + i$]

9. Evaluate $\int_C (z + 1)\,dz$, where C is the boundary of the square whose vertices are $z = 0$, $z = 1$, $z = 1 + i$, $z = i$.

[Ans. : 0]

10. Evaluate $\int_C \frac{z + 2}{z}\,dz$, where C is the semi-circle $z = 2e^{i\theta}$, $0 \le \theta \le \pi$.

[Ans. : $-4 + 2\pi i$. Put $z = 2e^{i\theta}$]

11. Evaluate $\int_C (x - iy)\,dz$ from (0, 0) to (4, 2) where C is first, the line segment joining (0, 0) to (0, 2) and then the line segment joining (0, 2) to (4, 2). [Ans. : $10 - 8i$]

12. Evaluate $\int_C (2z^3 + 8z + 2)\,dz$, where C is the arc of the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$ between the points (0, 0) and ($2\pi a$, 0). (M.U. 1991, 93)

(**Hint :** Find $\int_0^{2\pi a} f(z)\,dz$.) [Ans. : $4\pi a\,[2\pi^3 a^3 + 4\pi a + 1]$]

13. Evaluate $\int_0^{1+2i} z^2 dz$ along the curve $2x^2 = y$. (M.U. 1998, 2003) $\left[\text{Ans. : } -\frac{11}{3} - \frac{2i}{3}\right]$

14. Evaluate $\int_C (3z^2 + 2z + 1)\,dz$, where C is the arc of the cycloid $x = a\,(\theta + \sin\theta)$, $y = a\,(1 - \cos\theta)$ between $\theta = 0$ to $\theta = 2\pi$. (M.U. 1997)

$\left[\text{Ans. : } 2\pi a\left[4\pi^2 a^2 + 2\pi a + 1\right]\right]$

15. Evaluate $\int_C z^2 dz$, where C is the arc of the circle $x = r\cos\theta$, $y = r\sin\theta$ from $\theta = 0$ to $\theta = \pi/3$.

[Ans. : $-2r^3/3$]

16. Evaluate $\int f(z)\,dz$ along the square whose vertices are (1, 1), (2, 1), (2, 2), (1, 2) in anti-clockwise direction where $f(z) = x - 2iy$. (M.U. 1990) [Ans. : $3i$]

17. Evaluate $\int_C (z^2 - 2\bar{z} + 1)\, dz$, where C is the circle $x^2 + y^2 = 2$.

[Ans. : $-8\pi i$]

(**Hint :** Put $z = \sqrt{2}\, e^{i\theta}$, $\bar{z} = \sqrt{2}\, e^{-i\theta}$.)

18. Evaluate $\int_C (z^2 + 3z^{-4})\, dz$, where C is the upper half of the unit circle from (1, 0) to (–1, 0).

(M.U. 2000) [Ans. : 4 / 3]

19. Evaluate $\int_0^{2+i} z^2 dz$ (M.U. 2005)

(i) along the line $x = 2y$.

(ii) along the real axis from $z = 0$ to $z = 2$ and then along the line parallel to the imaginary axis from $z = 2$ to $z = 2 + i$.

(iii) along the imaginary axis from $z = 0$ to $z = i$ and then along the line parallel to the real axis from $z = i$ to $z = 2 + i$.

(iv) along the parabola $2y^2 = x$. [Ans. : $\frac{1}{3}(2 + 11i)$ in each case.]

20. Evaluate $\int_C \bar{z}\, dz$ from $z = 0$ to $z = 4 + 2i$, where C is

(i) the curve given by $z = t^2 + i\,t$.

(ii) the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$.

[Ans. : (i) $10 - \frac{8}{3}i$, (ii) $10 - 4i$]

21. Evaluate $\int_C z\, dz$ from $z = 0$ to $z = 1 + i$ along the curve $z = t^2 + it$. [Ans. : i]

22. Evaluate $\int_{1-i}^{1+i} (ix + iy + 1)\, dx$ along straight line joining $(1 - i)$ to $(1 + i)$.

[Ans. : $2\,(i - 1)$]

EXERCISE - II

Theory

1. Define line integral of $f(z)$ where $f(z)$ is a function of complex variable z.
2. Explain how the line integral $\int_C f(z)\, dz$ is evaluated
3. Explain why the line integral $\int_C f(z)\, dz$ depends upon the path. When is it independent of the path ?

☆☆☆

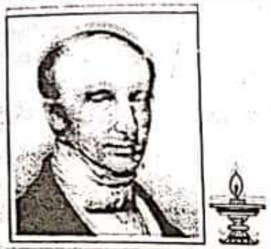
CHAPTER 2

Cauchy's Theorem

1. Introduction

In this chapter we shall study a very important theorem in the field of complex integration due to Cauchy. He proved a very beautiful and apparently very simple theorem about integration of an analytic function around a closed contour.

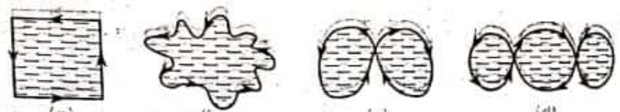
Augustin Louis (Baron de) Cauchy (Pronounced as Coshy) (1789 : 1857)



A French mathematician of great repute who contributed to various branches of mathematics. He wanted to be an engineer but because of poor health he was advised to pursue mathematics. His mathematical work began in 1811 when he gave briliant solutions to some difficult problems of that time. In the next 35 years he published 700 papers in various branches of mathematics. He is supposed to have initiated the era of modern analysis.

2. Simply and Multiply Connected Regions

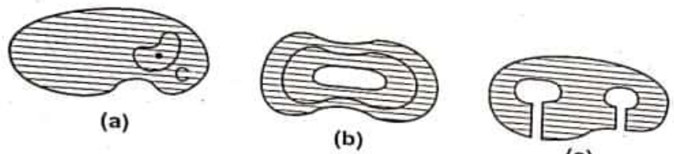
If a closed curve does not intersect itself, it is called a **simple closed curve** or a **Jordan Curve** [See Fig. 2.1 (a), (b)]. If a closed curve intersects itself it is called a **multiple curve** (See Fig. 2.1 (c), (d)].



(a) (b) (c) (d)

Fig. 2.1

A region R is called a **simply connected region** if every closed curve in the region encloses points of the region R only. In other words this means that every closed curve which lies in R can be contracted to a point without leaving R [See Fig. 2.2 (a)]. A region which is not simply connected is called multiply connected [See Fig. 2.2 (b)]. In simple terms a simply connected region is one which has no holes in it. A multiply connected region can be converted into simply connected region by giving it one or more cuts [See Fig. 2.2 (c)].



(a) (b) (c)

Fig. 2.2

3. Theorem

If $f(z)$ is continuous on a closed curve C of length l where, $|f(z)| \le M$, then

$$\left|\int_C f(z)\,dz\right| \le Ml.$$

Proof : We first note that the integral is the limit of infinite sums and that the modulus of a sum is less than or equal to sum of their modulie.

$$\therefore \quad \left|\int_C f(z)\,dz\right| \le \int_C |f(z)\,dz| = \int_C |f(z)|\,|dz|$$

$$\le \int_C M\,|dz| \qquad [\because |f(z)| \le M]$$

$$\le M\int_C \sqrt{dx^2+dy^2} \qquad [\because dz = dx + i\,dy]$$

$$\le M\int_C ds \qquad \left[\because ds = \sqrt{dx^2+dy^2}\right]$$

$$\le Ml \qquad \left[\because \int_C ds = l\right]$$

Hence, we get $\left|\int_C f(z)\,dz\right| \le Ml$. .... (1)

4. Cauchy's Integral Theorem

If $f(z)$ is an analytic function and if its derivative $f'(z)$ is continuous at each point within and on a simple closed curve C then the integral of $f(z)$ along the closed curve C is zero *i.e.*

$$\boxed{\oint f(z)\,dz = 0} \qquad \text{(M.U. 2005)}$$

Proof : Let the region enclosed by the curve C be denoted by R.

Let $f(z) = u(x, y) + iv(x, y) = u + iv$

$dz = dx + i\,dy$

$$\therefore \quad \oint_C f(z)\,dz = \oint_C (u+iv)(dx+i\,dy)$$

$$= \oint_C (udx - vdy) + i\oint_C (vdx + udy)$$

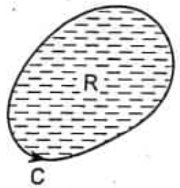


Fig. 2.3

Since, $f'(z)$ is continuous, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in R and on C.

Hence, we can apply Green's theorem

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \text{ to the above integrals.} \qquad \text{..................... (2)}$$

$$\therefore \quad \oint f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \qquad \text{..................... (3)}$$

But $f(z)$ is analytic at each point of the region R, hence, by Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{..................... (4)}$$

Putting these values from (4) in (3) the r.h.s. of (3) becomes zero.

Hence, $\oint_C f(z)\,dz = 0$.

Note ....

The French mathematician Goursat proved the theorem without assuming that $f'(z)$ is continuous. Hence, the theorem now becomes, **"if $f(z)$ is analytic in and on a closed counter C then $\oint_C f(z)\,dz = 0$."**

This theorem is known as Cauchy-Goursat Theorem.

(a) Corollary

If $f(z)$ is analytic in R then the line integral of $f(z)$ along any curve in R joining any two points of R is the same if the curve wholly lies in R *i.e.* the line integral is independent of the path joining the two points.

(b) Extension of Cauchy's Integral Theorem

Cauchy's theorem can be applied even if the region is multiply connected.

Theorem : If $f(z)$ is analytic in R between two simple closed curves C_1 and C_2 then,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz \qquad \text{(M.U. 2001)}$$

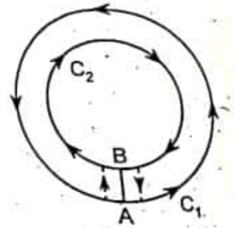


Fig. 2.4

Proof : To prove the theorem we introduce a cross-cut AB as shown in the figure. Then by Cauchy's Theorem

$$\int_C f(z)\,dz = 0$$

where, C is the curve indicated by arrows in the Fig. 2.4 *viz.* the path along $\overrightarrow{AB}$, then the curve C_2 in clockwise direction, then the path $\overrightarrow{BA}$ and then C_1 in anti-clockwise direction.

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$$\therefore \int_{AB} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{C_1} f(z)\,dz = 0$$

Now, the integrals along AB and BA cancel out

$$\therefore \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = 0$$

Reversing the direction of the integral along C_2, we get

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0.$$ And transposing, we get

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

where the curves C_1 and C_2 are taken in the same anti-clockwise direction.

Generalisation

The line integral of a single valued analytic function $f(z)$ over the outer contour C of a multiply connected region is equal to the sum of the integrals over the inner contours $C_1, C_2, C_3, \ldots\ldots C_n$ where $C_1, C_2, C_3, \ldots\ldots C_n$ are the boundaries of the multiply connected region.

Example 1 : Evaluate $\int_C \frac{z+3}{z^2-2z+5}\,dz$, where C is the circle $|z-1|=1$.

Sol. : C is a closed curve i.e. a circle with centre (1, 0) and radius 1.

Now, $z^2 - 2z + 5 = 0$ gives $z = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$.

These points are outside C.

Hence, $f(z) = \frac{z+3}{z^2-2z+5}$ is analytic in and on C.

Hence, by Cauchy's theorem

$$\int_C f(z)\,dz = 0$$

$$\therefore \int_C \frac{z+3}{z^2-2z+5}\,dz = 0.$$

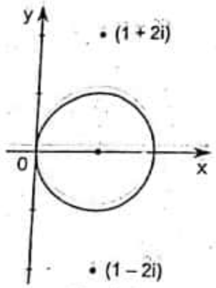

Fig. 2.5

Example 2 : Evaluate $\int_C \tan z\,dz$, where C is $|z| = 1/2$.

Sol. : The contour C is a circle with centre (0, 0) and radius 1/2.

Now $\tan z = \frac{\sin z}{\cos z}$

and $\cos z = 0$ for $z = \pm \pi/2$.

But $z = \pm \pi/2$ lies outside C. Hence, $f(z)$ is analytic in and on C.

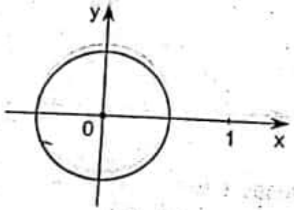

Fig. 2.6

Hence, by Cauchy's theorem

$$\int_C f(z)\,dz = 0 \quad \therefore \quad \int_C \tan z\,dz = 0.$$

Example 3 : Evaluate $\int_C (8\bar{z} + 3z)\,dz$ around the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

(M.U. 2004)

Sol. : Since C is a closed curve and the part $f(z) = 3z$ is analytic by Cauchy's Theorem

$$\int_C 3z\,dz = 0.$$

Hence, $I = \int 8\bar{z}\,dz = \int_C 8(x - iy)(dx + i\,dy)$

$$= 8 \int \{(x\,dx + y\,dy) + i(x\,dy - y\,dx)\}$$

Now, we put $x = a\cos^3\theta$, $y = a\sin^3\theta$

and $dx = -3a\cos^2\theta\sin\theta\,d\theta$,

$dy = 3a\sin^2\theta\cos\theta\,d\theta$.

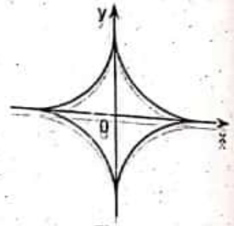

Fig. 2.7

$$\therefore I = 8 \cdot 4 \cdot 3a^2 \int_0^{\pi/2} [(-\cos^5\theta\sin\theta) + \sin^5\theta\cos\theta] + i[(\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta)]\,d\theta$$

[For reduction formulae. See App. Maths. - II.]

$$= 96a^2 \left\{ \left[\frac{\cos^6\theta}{6}\right]_0^{\pi/2} + \left[\frac{\sin^6\theta}{6}\right]_0^{\pi/2} + \int_0^{\pi/2} i[\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta]\,d\theta \right\}$$

$$\therefore I = 96a^2 \left\{ \left(0 - \frac{1}{6}\right) + \left(\frac{1}{6} - 0\right) + i\left[\frac{3\cdot1\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2} + \frac{3\cdot1\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2}\right] \right\}$$

$$= 96a^2 \left[0 + i\left(\frac{\pi}{32} + \frac{\pi}{32}\right)\right] = \frac{96a^2 i}{16}\pi = 6\pi a^2 i.$$

5. Cauchy's Integral Formula (Fundamental Formula)

If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R and if z_0 is any point within C, then

$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz}$$

i.e. $$\boxed{\int_C \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0)}$$ (M.U. 1999, 2002, 03, 05)

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Proof : Since $f(z)$ is analytic inside and on C, $\frac{f(z)}{z-z_0}$ is also analytic inside and on C except at $z = z_0$. We draw a small circle C_1 around z_0 with centre at $z = z_0$ and radius r lying wholly inside C.

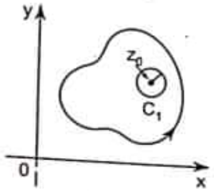

Fig. 2.8

Now, $\frac{f(z)}{z-z_0}$ is analytic in the region enclosed between the curves C and C_1.

Hence, by Cauchy's extended theorem,

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_1} \frac{f(z)}{z-z_0}\,dz \qquad \text{.................. (1)}$$

Putting $z - z_0 = r e^{i\theta}$, $dz = r i e^{i\theta} d\theta$, we get on C_1,

$$\int_{C_1} \frac{f(z)}{z-z_0}\,dz = \int_{C_1} \frac{f(z_0 + r e^{i\theta})}{r e^{i\theta}} \cdot r i e^{i\theta} d\theta$$

$$= \int_{C_1} f(z_0 + r e^{i\theta})\, i\, d\theta$$

As $r \to 0$, the circle tends to the point z_0. Hence, by taking the limit, we get

$$\therefore \int_{C_1} f(z_0 + r e^{i\theta})\, i\, d\theta = \int_{C_1} f(z_0)\, i\, d\theta = i f(z_0) \int_{C_1} d\theta$$

$$= i f(z_0) \int_0^{2\pi} d\theta = 2\pi i f(z_0)$$

Hence, from (1) we get $\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$.

Corollary : Cauchy's Integral Formula for derivatives

Differentiating under the integral sign with respect to the parameter z_0, we get

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f''(z_0) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3}\,dz$$

....

....

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$f^{n-1}(z_0) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^n}\,dz$$

The formula can be remembered as

$$\boxed{\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)}$$

Note ....

These formulae give us derivatives of a function at any interior point of R and also prove that an analytic function possesses derivatives of all orders and these derivatives are themselves analytic.

6. Theorem (Extension of Cauchy's Integral Formula To Multiply Connected Regions)

If $f(z)$ is analytic in a region R bounded by two closed curves C_1 and C_2 one within the other (as shown in the figure), if z_0 is any point in the region R then

$$f(z_0) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z-z_0}\,dz$$

Proof : We introduce a cross-cut AB. They by Cauchy's Integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\,dz$$

where C is the curve indicated by arrows in the figure viz. the path along $\vec{AB}$, then the curve C_2 in clockwise sense, then the path along $\vec{BA}$, then the curve C_1 in anti-clockwise direction.

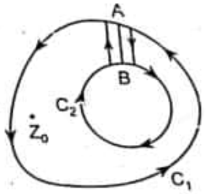

Fig. 2.9

$$\therefore f(z_0) = \frac{1}{2\pi i} \int_{AB} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i} \int_{BA} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z-z_0}\,dz$$

Now, since the paths along $\vec{AB}$ and $\vec{BA}$ are in opposite directions the integrals along AB and BA cancel out

$$\therefore f(z_0) = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z-z_0}\,dz$$

Reversing the direction of the integral along C_2, we get

$$f(z_0) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z-z_0}\,dz$$

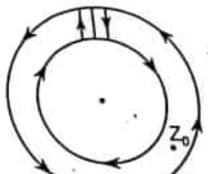

Fig. 2.10

Particular Case : If C_1 and C_2 are two concentric circles and z_0 is any point in the annulus region, then we have

$$f(z_0) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z-z_0}\,dz$$

7. Converse of Cauchy's Theorem (Morera's Theorem)

If $f(z)$ is continuous in a region R and if $\int_C f(z)\,dz = 0$ for every simple closed curve C which can be drawn in R then $f(z)$ is analytic in R. We accept this theorem without proof.

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8. Procedure to Find The Integral

$\int_C F(z)\,dz$ where C is a closed curve in a given region R.

(i) When $\int_C F(z)\,dz = \int_C \frac{\Phi(z)\,dz}{z-z_0}$ or $\int \frac{\Phi(z)}{(z-z_0)^n}\,dz$.

If z_0 is a point outside C, $F(z)$ is analytic in and on C. We put $F(z) = f(z)$ and then by Cauchy's theorem of § 4.

$$\int_C f(z)\,dz = 0 \qquad \text{[See Ex.1(a)]}$$

If z_0 is a point inside C, $F(z)$ is not analytic in C. We put $\Phi(z) = f(z)$ which may be analytic in C, then by Cauchy's formula of § 5.

$$\int_C F(z)\,dz = \int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

and $$\int_C F(z)\,dz = \int \frac{f(z)\,dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0) \qquad \text{[See Ex. 1(b), 2, 3 and Ex. 5]}$$

(ii) When $\int_C F(z)\,dz = \int \frac{\Phi(z)}{(z-z_1)(z-z_2)}\,dz$

If z_1 and z_2 are **both inside** C, the $F(z)$ is **not** analytic in C. We then express $\frac{1}{(z-z_1)(z-z_2)}$ as partial fraction $\frac{k_1}{z-z_1} + \frac{k_2}{z-z_2}$. We put $\Phi(z) = f(z)$ which may be analytic in C. Then by Cauchy's formula.

$$\therefore \int_C F(z)\,dz = k_1 \int_C \frac{f(z)}{z-z_1}\,dz + k_2 \int_C \frac{f(z)}{z-z_2}\,dz$$

$$= k_1 \cdot 2\pi i\, f(z_1) + k_2 \cdot 2\pi i\, f(z_2) \qquad \text{[See Ex. 1, 2, 3 pages 2-12, 2-13]}$$

(iii) When $\int_C F(z)\,dz = \int \frac{\Phi(z)}{(z-z_1)(z-z_2)}$

If z_1 and z_2 are **both outside** C, $F(z)$ is analytic in C. We put $F(z) = f(z)$. Then by Cauchy's theorem

$$\int_C F(z)\,dz = \int_C f(z)\,dz = 0 \qquad \text{[See Ex. 1, 2 pages 2-14]}$$

(iv) If z_1 **is outside** and z_2 **is inside** C, $F(z)$ is **not** analytic in C. We then write $\frac{\Phi(z)}{z-z_1} = f(z)$. Then $f(z)$ may be analytic in C. By Cauchy's integral formula

$$\int_C f(z)\,dz = \int_C \frac{f(z)}{z-z_2}\,dz = 2\pi i\, f(z_2). \qquad \text{[See Ex. 1, 2, 3, 4 pages 2-14, 2-15, 2-16]}$$

Type I : $\int_C F(z)\,dz = \int \frac{\Phi(z)\,dz}{z-z_0}$ or $\int \frac{\Phi(z)\,dz}{(z-z_0)^n}$ **and z_0 is a point inside or outside C**

Example 1 : Find (a) $\int_C \frac{dz}{z-z_0}$, (b) $\int \frac{dz}{(z-z_0)^n}$, $n \neq 1$ where C is a simple closed curve and $z = z_0$ is a point (a) outside C, (b) inside C. **(M.U. 2003, 04)**

Sol. : (a) Since $z = z_0$ is a point **outside** C, both $\frac{1}{z-z_0}$ and $\frac{1}{(z-z_0)^n}$ are analytic in and on C. Hence, by Cauchy's Theorem.

$$\int_C \frac{dz}{z-z_0} = 0 \quad \text{and} \quad \int_C \frac{dz}{(z-z_0)^n} = 0$$

(b) Since $z = z_0$ is **inside** C, both $\frac{1}{z-z_0}$ and $\frac{1}{(z-z_0)^n}$ are **not** analytic in C. We, therefore, put $f(z) = 1$ which is analytic in and on C. Hence, by Cauchy's formula.

$$\int_C \frac{dz}{z-z_0} = 2\pi i\, f(z_0) = 2\pi i \qquad [\because f(z) = 1,\ f(z_0) = 1]$$

and $$\int_C \frac{f(z)\,dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} \cdot f^{n-1}(z_0)$$

$$\int_C \frac{dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} \cdot f^{n-1}(z_0) = 0$$

$$[\because f(z) = 1,\ f'(z) = 0,\ f''(z) = 0, \ldots \text{ etc. } f^{n-1}(z_0) = 0]$$

Example 2 : Evaluate $\int_C \frac{1}{z} \cdot \cos z\,dz$ where C is the ellipse $9x^2 + 4y^2 = 1$. **(M.U. 2003)**

Sol. : The point $z = 0$ lies **inside** the ellipse $9x^2 + 4y^2 = 1$.

Hence, by Cauchy's integral formula,

$$\int_C \frac{1}{z} \cdot \cos z\,dz = 2\pi i \cos(0) = 2\pi i.$$

Example 3 : Evaluate $\int_C \frac{\cot z}{z}\,dz$ where C is the ellipse $9x^2 + 4y^2 = 1$.

Sol. : We have $\int_C \frac{\cot z}{z}\,dz = \int_C \frac{\cos z}{z \sin z}\,dz$

The point $z = 0$ lies **inside** the ellipse.

Hence, by Cauchy's integral formula

$$\int_C \frac{\cot z}{z}\,dz = \int_C \frac{\cos z}{z \sin z}\,dz = 2\pi i \cos(0) = 2\pi i.$$

Example 4 : Evaluate $\oint_C \frac{e^3 z}{z-\pi i}\,dz$ where C is the curve $|z-2|+|z+2|=6$. **(M.U. 2003)**

Sol. : As seen in Fig. 1-5, page 1-4, $|z-2|+|z+2|=6$ is an ellipse with semi-major axis $a=6/2=3$ and semi-minor axis given by $b^2=3^2-2^2=5$ as shown in the Fig. 2.11. (For alternative explanation see the next example.)

The point $z=\pi i$ *i.e.* (0, 3.14) lies **outside** the ellipse.

Hence, by Cauchy's Theorem,

$$\oint_C \frac{e^3 z}{z-\pi i}\,dz = 0.$$

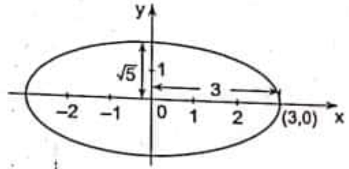

Fig. 2.11

Example 5 : Evaluate $\int_C \frac{e^{3z}}{z-i}\,dz$ where C is the curve $|z-2|+|z+2|=6$. **(M.U. 1994, 2003)**

Sol. : We can write $|z-2|+|z+2|=6$ as $|x-2+iy|+|x+2+iy|=6$

i.e. $\sqrt{(x-2)^2+y^2}+\sqrt{(x+2)^2+y^2}=6$

Putting $y=0$, $x-2+x+2=6 \quad \therefore x=3$.

Putting $x=0$, $\sqrt{y^2+4}+\sqrt{y^2+4}=6$

$\therefore 2\sqrt{y^2+4}=6 \quad \therefore y^2+4=9$

$\therefore y=\pm\sqrt{5}$

The curve $|z-2|+|z+2|=6$ is an ellipse with foci at (−2, 0), (2, 0) and intersecting the real axis in (−3, 0), (3, 0) and imaginary axis in $(0,\sqrt{5})$, $(0,-\sqrt{5})$.

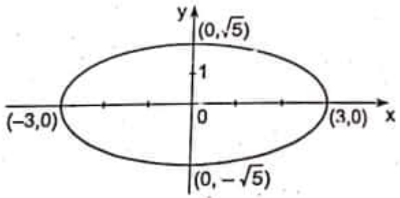

Fig. 2.12

The point $z=i$ *i.e.* (0, 1) lies **inside** C and $f(z)=e^{3z}$ is analytic in and on C. Hence, by Cauchy's integral formula.

$$\int_C \frac{e^{3z}}{z-i}\,dz = 2\pi i\, e^{3i} = 2\pi i(\cos 3 + i\sin 3)$$

Example 6 : Evaluate $\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$ where C is $|z|=1$. **(M.U. 1994, 95, 2000, 04)**

Sol. : The contour $|z|=1$ is a circle with centre at the origin and radius unity. Hence the point $z=\pi/6$ lies **inside** C. $f(z)=\sin^6 z$ is analytic in and on C. Hence, by corollary of Cauchy's Integral formula.

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

where $f(z)=\sin^6 6$ and $z_0=\pi/6$, $n=3$.

Now, $f(z)=\sin^6 z$

$\therefore \quad f'(z)=6\sin^5 z\cos z$

$f''(z)=6(5\sin^4 z\cos^2 z-\sin^6 z)$

$f''(\pi/6)=6(\sin^4 \pi/6\cos^2 \pi/6-\sin^6 \pi/6)$

$$=6\left(5\cdot\frac{1}{16}\cdot\frac{3}{4}-\frac{1}{64}\right)=\frac{6\cdot 14}{64}=\frac{21}{16}$$

$$\therefore \int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \frac{2\pi i}{2!}\cdot\frac{21}{16}=\frac{21\pi i}{16}$$

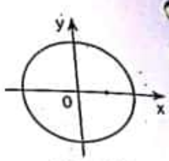

Fig. 2.13

Example 7 : Evaluate $\int_C \frac{\sin^6 z}{[z-(\pi/2)]^3}\,dz$ where C is the circle $|z|=2$. **(M.U. 1997, 2014)**

Sol. : The contour $|z|=2$ is a circle with centre at the origin and radius 2. Here, the point $z=\frac{\pi}{2}=\frac{3\cdot 14}{2}=1\cdot 57$ lies inside the circle. The root is repeated thrice. Hence, by the formula given on page 2-6.

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0) \quad \text{where, } f(z)=\sin^6 z \text{ and } z_0=\frac{\pi}{2},\ n=3.$$

As seen in the above example, $f'=6\sin^5 z\cos z$.

$f''(z)=6(5\sin^4 z\cos^2 z-\sin^6 z)$

$$f''(\pi/2)=6\left(5\sin^4\frac{\pi}{2}\cos^2\frac{\pi}{2}-\sin^6\frac{\pi}{2}\right)=-6$$

$$\therefore \int_C \frac{\sin^6 z}{[z-(\pi/2)]^3}\,dz = \frac{2\pi i}{2!}(-6) = -6\pi i.$$

Example 8 : Evaluate $\int_C \frac{e^{2z}}{(z-1)^4}\,dz$ where C is the circle $|z|=2$. **(M.U. 1996, 2016)**

Sol. : Since the point $z=1$ lies inside the circle $|z|=2$, $f(z)=e^{2z}$ is analytic in and on C. Hence, by corollary of Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^n} = \frac{2\pi i}{(n-1)!}\cdot f^{n-1}(a)$$

Now, $f(z)=e^{2z} \quad \therefore f'(z)=2e^{2z}, \quad f''(z)=4e^{2z}, \quad f'''(z)=8e^{2z}$.

$\therefore f'''(1)=8e^2$.

$$\therefore \int_C \frac{e^{2z}}{(z-1)^4} = \frac{2\pi i}{3!}\cdot 8e^2 = \frac{8\pi i}{3}\cdot e^2$$

Example 9 : Evaluate $\int_C \frac{ze^z}{(z-a)^3}\,dz$ where C is a circle with centre at the origin and radius $b > a$.

Sol. : Since radius b of the circle with centre at the origin is greater than a, the point $z = a$ lies **inside** C. $f(z) = ze^z$ is analytic in and on C. Hence, by corollary of Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(a)$$

Now, $f(z) = ze^z$

$$\therefore \quad f'(z) = ze^z + e^z$$

$$f''(z) = ze^z + e^z + e^z = ze^z + 2e^z$$

$$f''(a) = ae^a + 2e^a = (a+2)e^a$$

$$\therefore \quad \int_C \frac{ze^z}{(z-a)^3}\,dz = \frac{2\pi i}{2!} f''(a) = \frac{2\pi i}{2!}(a+2)e^a = (a+2)e^a \pi i$$

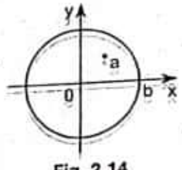

Fig. 2.14

Example 10 : Evaluate $\int_C \frac{\tan(z/2)}{(z-a)^2}\,dz$, $(-2 < a < 2)$ where C is the boundary of the square with centre at the origin and sides of length 2.

Sol. : Since $-2 < a < 2$ and the sides of the square are of length 2, $z = a$ lies **inside** the square.

If $f(z) = \tan\frac{z}{2}$, $f'(z) = \frac{1}{2}\sec^2\frac{z}{2}$ does not exist at $z = \pm\pi, \pm 3\pi, \ldots$ etc. But these points lie outside C. Hence, by corollary of Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(a)$$

$$\therefore \quad \int_C \frac{\tan(z/2)}{(z-a)^2}\,dz = \frac{2\pi i}{1!}\left(\frac{1}{2}\sec^2\frac{a}{2}\right) = \pi i \sec^2\frac{a}{2}$$

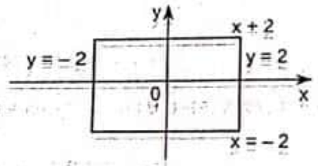

Fig. 2.15

Type II : $\int_C F(z)\,dz = \int_C \frac{\Phi(z)}{(z-z_1)(z-z_2)}\,dz$ **and z_1 and z_2 are both inside C.**

Example 1 : Evaluate $\int_C \frac{3z^2+z}{z^2-1}\,dz$, where C is the circle $|z| = 2$.

(M.U. 1998, 2005)

Sol. : The circle $|z| = 2$ has centre at (0, 0) and radius 2. Now, $z^2 - 1 = 0$ gives $(z-1)(z+1) = 0$. $\therefore z = 1, -1$. Both these points lie **Inside** the circle. Hence, we use partial fractions and write

$$\frac{1}{z^2-1} = \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]$$ and write $f(z) = 3z^2 + z$, which is analytic in and on C.

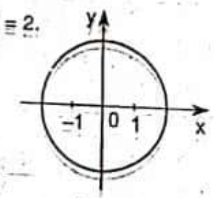

Fig. 2.16

$$\therefore \quad \int_C \frac{3z^2+z}{z^2-1}\,dz = \frac{1}{2}\int_C \frac{3z^2+z}{z-1}\,dz - \frac{1}{2}\int_C \frac{3z^2+z}{z+1}\,dz$$

$$= \frac{1}{2}\cdot 2\pi i f(1) - \frac{1}{2}\cdot 2\pi i f(-1) \quad \text{where } f(z) = 3z^2 + z$$

$$= \pi i(4) - \pi i(2) = 2\pi i$$

Example 2 : Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz$, where C is the circle $|z| = 3$. **(M.U. 1993)**

Sol. : The circle $|z| = 3$ has centre at (0, 0) and radius 3. The points $z = 1$ and $z = 2$ lie **inside** the circle. Hence, $f(z)$ is not analytic in C. Hence, we use the method of partial fractions and write

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

and write $f(z) = e^{2z}$ which is analytic in C.

$$\therefore \quad \int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = \int_C \frac{e^{2z}}{z-2}\,dz - \int_C \frac{e^{2z}}{z-1}\,dz$$

$$= 2\pi i f(2) - 2\pi i f(1) \quad \text{where } f(z) = e^{2z}$$

$$= 2\pi i e^4 - 2\pi i e^2 = 2\pi i e^2(e^2 - 1)$$

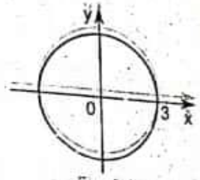

Fig. 2.17

Example 3 : Evaluate $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)(z-3)}\,dz$, where C is the circle $|z| = 4$.

Sol. : $|z| = 4$ is a circle with centre at the origin and radius 4. Hence $z = 2$, $z = 3$ lie inside the circle. Hence, the given function $f(z)$ is not analytic in C. Hence we use the method of partial fractions and write

$$\frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2}$$

and write $f(z) = \sin\pi z^2 + \cos\pi z^2$ which is analytic in C.

By Cauchy's integral formula,

$$\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)(z-3)}\,dz$$

$$= \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-3}\,dz - \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2}\,dz$$

$$= 2\pi i f(3) + 2\pi i f(2) \quad \text{where } f(z) = \sin\pi z^2 + \cos\pi z^2$$

$$= 2\pi i(\sin 9\pi + \cos 9\pi) - 2\pi i(\sin 4\pi + \cos 4\pi)$$

$$= -2\pi i - 2\pi i = -4\pi i$$

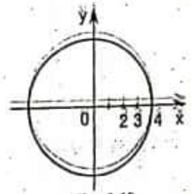

Fig. 2.18

Type III : $\int_C F(z)\,dz = \int \frac{\phi(z)}{(z-z_1)(z-z_2)}\,dz$ and z_1 and z_2 are both outside C.

Example 1 : Evaluate $\int_C \frac{z+2}{(z-3)(z-4)}\,dz$, where C is the circle $|z|=1$. (M.U. [illegible])

Sol. : $|z|=1$ is a circle with centre at the origin and radius 1. Hence, both points $z=3$ and $z=4$ lie outside the circle C and $f(z)$ is analytic in C.

By Cauchy's Theorem, $\int_C \frac{z+2}{(z-3)(z-4)}\,dz = 0$.

Example 2 : Evaluate $\int_C \frac{z^2+z+1}{z^2-7z+2}\,dz$, where C is the ellipse $25x^2+16y^2=1$.

Sol. : The equation of the ellipse can be written as (M.U. 1992, 93, 94)

$$\frac{x^2}{(1/25)}+\frac{y^2}{(1/16)}=1 \text{ i.e. } \frac{x^2}{(1/5)^2}+\frac{y^2}{(1/4)^2}=1$$

$$\therefore\ a=\frac{1}{5}=0{\cdot}2,\quad b=\frac{1}{4}=0{\cdot}25.$$

Further $z^2-7z+2=0$ gives $z=\frac{7\pm\sqrt{49-8}}{2}=\frac{7\pm\sqrt{41}}{2}$.

The root $\frac{7+\sqrt{41}}{2}=\frac{7+6{\cdot}4}{2}=\frac{13{\cdot}4}{2}=6{\cdot}7>0{\cdot}2$ and $0{\cdot}25$.

The root $\frac{7-\sqrt{41}}{2}=\frac{7-6{\cdot}4}{2}=\frac{0{\cdot}6}{2}=0{\cdot}3>0{\cdot}2$ and $0{\cdot}25$.

Since, both roots are outside the contour $f(z)$ is analytic in and on C. By Cauchy's Theorem

$$\int_C f(z)\,dz=0$$

Type IV : $\int_C F(z)\,dz = \int_C \frac{\Phi(z)}{(z-z_1)(z-z_2)}\,dz$ **and z_1 is inside and z_2 is outside C.**

Example 1 : Evaluate $\int_C \frac{z+3}{z^2+2z+5}\,dz$, where C is the circle

(i) $|z|=1$, (ii) $|z+1-i|=2$. (M.U. 1997, 2015)

Sol. : Now, $z^2+2z+5=0$ gives $(z+1)^2+2^2=0$

$\therefore\ (z+1+2i)(z+1-2i)=0 \qquad \therefore\ z=-1-2i,\ z=-1+2i.$

(i) Now, $|z|=1$ is a circle with centre at the origin and radius 1. Hence, both the points lie **outside** the circle C and $f(z)$ is analytic in C.

By Cauchy's Theorem $\int_C \frac{z+3}{z^2+2z+5}\,dz=0.$

(ii) Now, $|z+1-i|=2$ is a circle with centre at [illegible] and radius 2. [illegible] lies inside the circle. Hence, we put $f(z)=\frac{z+3}{z+1+2i}$ which is analytic in C. By Cauchy's formula

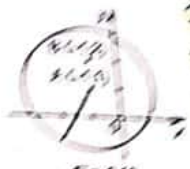

Fig. 2.19

$$\int_C \frac{z+3}{z^2+2z+5}\,dz=\int_C \frac{(z+3)/(z+1+2i)}{z+1-2i}\,dz$$

$$=\int_C \frac{f(z)}{z-z_0}\,dz=2\pi i\,f(z_0) \text{ where } z_0=-1+2i$$

$$=2\pi i\cdot\frac{-1+2i+3}{-1+2i+1+2i}=2\pi i\cdot\frac{(2+2i)}{4i}=\pi(1+i)$$

Example 2 : Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$ where C is the circle $|z+1-i|=2$.

(M.U. 2003, 05, 09)

Sol. : The circle $|z+1-i|=2$ has centre at $-1+i$ i.e., $(-1, 1)$ and radius 2.

Now, $z^2+2z+5=0$ gives $(z+1)^2+2^2=0$

$\therefore\ (z+1+2i)(z+1-2i)=0$

$\therefore\ z=-1-2i$ and $z=-1+2i$.

The root $z=-1+2i$ i.e., $(-1, 2)$ lies inside C and $z=-1-2i$ i.e., $(-1,-2)$ lies outside C.

Hence, we put $f(z)=\frac{z+4}{z+1+2i}$ which is analytic in C.

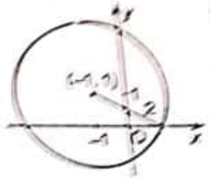

Fig. 2.20

By Cauchy's formula,

$$\int_C \frac{z+4}{z^2+2z+5}\,dz=\int_C \frac{(z+4)/(z+1+2i)}{(z+1-2i)}\,dz$$

$$=\int_C \frac{f(z)}{z+1-2i}\,dz=2\pi i\,f(z_0) \quad \text{where } z_0=-1+2i$$

$$\therefore\ \int_C \frac{z+4}{z^2+2z+5}\,dz=2\pi i\cdot\frac{-1+2i+4}{-1+2i+1+2i}$$

$$=2\pi i\cdot\frac{3+2i}{4i}=(3+2i)\frac{\pi}{2}$$

Example 3 : Evaluate $\int_C \frac{z+3}{2z^2+3z-2}\,dz$ where C is the circle $|z-i|=2$.

(M.U. 2001, 03, 05)

Sol. : The circle $|z-i|=2$ has centre at $(0, 1)$ and radius 2. Now, $2z^2+3z-2=(2z-1)(z+2)$. Hence, $z=1/2$, lies **inside** the circle and $z=-2$ lies **outside** the circle.

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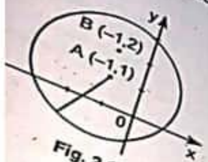

Fig. 2.19

$-i| = 2.$

U. 2003, 05, 16)

Hence, we write

$$\int_C \frac{z+3}{2z^2+3z-2}\,dz = \int_C \frac{(z+3)/(z+2)}{2z-1}\,dz$$

Now, $f(z) = \dfrac{z+3}{z+2}$ is analytic in and on C and the point $z = 1/2$ lies inside C.

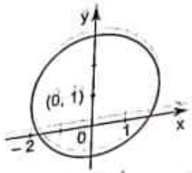

Fig. 2.21

$\therefore$ By Cauchy's integral formula

$$\int_C \frac{z+3}{2z^2+3z-2}\,dz = \int_C \frac{(z+3)/(z+2)}{2z-1}\,dz = 2\pi i\, f(z_0)$$

where, $f(z) = \dfrac{z+3}{z+2}$ and $z_0 = \dfrac{1}{2}$.

$\therefore \ f(z_0) = \dfrac{(1/2)+3}{(1/2)+2} = \dfrac{7}{5} \qquad \therefore \ \displaystyle\int_C \frac{z+3}{2z^2+3z-2}\,dz = 2\pi i \cdot \frac{7}{5} = \frac{14}{5}\pi i.$

Example 4 : Evaluate $\displaystyle\int_C \frac{z+6}{z^2-4}\,dz$ where C is the circle **(M.U. 2001; 10)**

(i) $|z| = 1$, (ii) $|z-2| = 1$, (iii) $|z+2| = 1$.

Sol. : Now $z^2 - 4 = 0$ gives $(z+2)(z-2) = 0 \quad \therefore \ z = -2, 2.$

(i) The circle $|z| = 1$ has centre at the origin and radius 1. The points $z = -2, 2$ lie outside the circle. Hence,

$f(z) = \dfrac{z+6}{z^2-4}$ is analytic inside C.

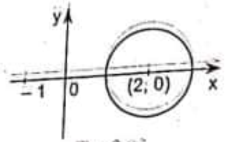

Fig. 2.22

$\therefore$ By Cauchy's integral formula

$$\int_C f(z)\,dz = 0 \qquad \therefore \int_C \frac{z+6}{z^2-4}\,dz = 0$$

(ii) The circle $|z-2| = 1$ has centre at (2, 0) and radius 1. The point (2, 0) lies **inside** C and the point (− 2, 0) lies **outside** C. Hence, we write

$$\int_C \frac{z+6}{(z-2)(z+2)}\,dz = \int_C \frac{(z+6)/(z+2)}{z-2}\,dz$$

Now, $f(z) = \dfrac{z+6}{z+2}$ is analytic in and on C and the point $z = 2$ lies **inside** C.

$\therefore$ By Cauchy's integral formula

$$\int_C \frac{z+6}{z^2-4}\,dz = \int_C \frac{(z+6)/(z+2)}{z-2}\,dz = 2\pi i\, f(z_0)$$

where, $f(z) = \dfrac{z+6}{z+2}$ and $z_0 = 2$.

Now, $f(z_0) = \dfrac{2+6}{2+4} = \dfrac{8}{4} = 2 \qquad \therefore \ \displaystyle\int_C \frac{z+5}{z^2-4}\,dz = 2\pi i(2) = 4\pi i.$

(iii) The circle $|z+2| = 1$ has centre at $z = -2$ and radius 1. The point (−2, 0) lies **inside** C and the point (2, 0) lies **outside** C. Hence, we write

$$\int_C \frac{z+6}{(z-2)(z+2)}\,dz = \int_C \frac{(z+6)/(z+2)}{z+2}\,dz$$

Now, $f(z) = \dfrac{z+6}{z-2}$ is analytic in and on C and the point $z = -2$ lies **inside** C.

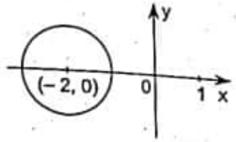

Fig. 2.23

$\therefore$ By Cauchy's integral formula

$$\int_C \frac{z+6}{z^2-4}\,dz = \int_C \frac{(z+6)/(z-2)}{z+2}\,dz = 2\pi i\, f(z_0)$$

where, $f(z) = \dfrac{z+6}{z-2}$ and $z_0 = -2$.

Now, $f(z_0) = \dfrac{-2+6}{-2+4} = \dfrac{4}{-4} = -1. \qquad \therefore \ \displaystyle\int_C \frac{z+6}{z^2-4}\,dz = 2\pi i(-1) = -2\pi i.$

Example 5 : Evaluate $\displaystyle\int_C \frac{4z-1}{z^2-3z-4}\,dz$, where C is the ellipse $x^2 + 4y^2 = 4$. **(M.U. 2003)**

Sol. : The ellipse $x^2 + 4y^2 = 4$ *i.e.* $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$ has centre at the origin and major axis 2 and minor axis 1.

Now, $z^2 - 3z - 4 = (z-4)(z+1)$. Hence, $z = -1$ lies **inside** C and $z = 4$ lies **outside** C. Hence, we write

$$\int_C \frac{4z-1}{z^2-3z-4}\,dz = \int \frac{(4z-1)/(z-4)}{z+1}\,dz$$

Now, $f(z) = \dfrac{4z-1}{z-4}$ is analytic in and on C and the point $z = -1$ lies **inside** C.

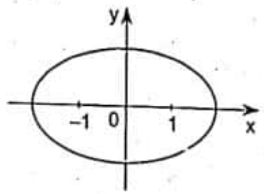

Fig. 2.24

$\therefore$ By Cauchy's integral formula

$$\int_C \frac{4z-1}{z^2-3z-4}\,dz = \int_C \frac{(4z-1)/(z-4)}{z+1}\,dz$$

$$= 2\pi i\, f(z_0) \qquad \text{where } f(z) = \frac{4z-1}{z-4} \text{ and } z_0 = -1$$

$\therefore \ f(z_0) = \dfrac{-4-1}{-1-4} = \dfrac{-5}{-5} = 1 \qquad \therefore \ \displaystyle\int_C \frac{4z-1}{z^2-3z-4}\,dz = 2\pi i(1) = 2\pi i$

Example 6 : Evaluate $\displaystyle\int_C \frac{z+2}{z^3-2z^2}\,dz$, where C is the circle $|z-2-i| = 2$. **(M.U. 2009)**

Sol. : The circle $|z-2-i| = 2$ has centre at $2+i$ *i.e.* (2, 1) and radius 2. The point $z = 0$ lies **outside** the circle but $z = 2$ *i.e.* (2, 0) lies **inside** the circle.

Hence, we write $\displaystyle\int \frac{z+2}{z^2(z-2)}\,dz = \int \frac{(z+2)/z^2}{z-2}\,dz.$

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Now, $f(z) = \frac{z+2}{z^2}$ is analytic in and on C and the point $z = 2$ lies inside C.

$\therefore$ By Cauchy's integral formula

$$\int_C \frac{z+2}{z^3 - 2z^2}\, dz = \int_C \frac{(z+2)/z^2}{(z-2)}\, dz = 2\pi i\, f(z_0)$$

where, $f(z) = \frac{z+2}{z^2}$ and $z_0 = 2$.

$$\therefore \quad f(z_0) = \frac{2+2}{4} = \frac{4}{4} = 1$$

$$\therefore \quad \int_C \frac{z+2}{z^2(z-2)}\, dz = 2\pi i (1) = 2\pi i.$$

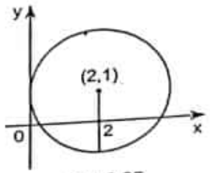

Fig. 2.25

Example 7 : Evaluate $\int_C \frac{z\, dz}{(z-1)^2 (z-2)}$ where C is the circle $|z - 2| = 0.5$. **(M.U. 2016)**

Sol. : Here C is the circle with centre at (2, 0) and radius 0·5.

Now, $(z-1)^2 (z-2) = 0$ gives $z = 1$ and $z = 0$ *i.e.*, z is (1, 0) and z is (2, 0).

From the figure we see that $z = (2, 0)$ lies **inside** C and $z = (1, 0)$ lies **outside** C.

Hence, we put $f(z) = \frac{z}{(z-1)^2}$ which is analytic in C.

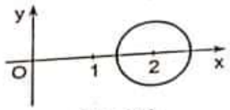

Fig. 2.26

By Cauchy's formula,

$$\int_C \frac{z}{(z-1)^2 (z-2)}\, dz = \int_C \frac{z/(z-1)^2}{z-2}\, dz$$

$$= \int_C \frac{f(z)}{z-2}\, dz = 2\pi i\, f(z_0) \quad \text{where } z_0 = 2$$

$$= 2\pi i \cdot \frac{2}{(2-1)^2} = 4\pi i.$$

Example 8 : Evaluate $\int_C \frac{z^2}{z^4 - 1}\, dz$, where C is the circle

(i) $|z| = 1/2$, (ii) $|z - 1| = 1$, (iii) $|z + i| = 1$. **(M.U. 1994, 2005)**

Sol. : We first express the denominator in terms of linear factors.

$$\text{Let} \quad \frac{1}{z^4 - 1} = \frac{a}{z-1} + \frac{b}{z+1} + \frac{c}{z-i} + \frac{d}{z+i}$$

$$\therefore \quad 1 = a(z+1)(z^2+1) + b(z-1)(z^2+1) + c(z+i)(z^2-1) + d(z-i)(z^2-1)$$

When $z = 1$, $1 = a \cdot 4$ $\therefore$ $a = 1/4$.

When $z = -1$, $1 = -4b$ $\therefore$ $b = -1/4$.

When $z = i$, $1 = -4ic$ $\therefore$ $c = -1/4i$.

When $z = -i$, $1 = 4id$ $\therefore$ $d = 1/4i$.

$$\therefore \quad \frac{z^2}{z^4 - 1} = \frac{1}{4} \cdot \frac{z^2}{z-1} - \frac{1}{4} \cdot \frac{z^2}{z+1} - \frac{1}{4i} \cdot \frac{z^2}{z-i} + \frac{1}{4i} \cdot \frac{z^2}{z+i}$$

(i) $|z| = 1/2$ is a circle with centre at the origin and radius 1/2. Hence, all the points $z = 1, -1, i, -i$ lie outside the circle $|z| = 1/2$. The function $f(z) = \frac{z^2}{z^4 - 1}$ is analytic in and on the C. Hence, by Cauchy's Theorem

$$\int_C \frac{z^2}{z^4 - 1}\, dz = 0.$$

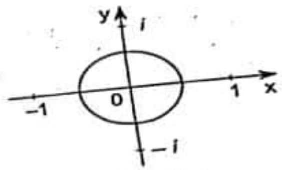

Fig. 2.27

(ii) $|z - 1| = 1$ is a circle with centre at (1, 0) and radius 1. Hence, the points $z = -1, i, -i$ lie **outside** the circle. The only point $z = 1$ lies inside the circle. The given function

$$\Phi(z) = \frac{z^2/(z^2+1)(z+1)}{(z-1)}.$$

Now, $f(z) = \frac{z^2}{(z^2+1)(z+1)}$ is analytic in C and on C.

Hence, by Cauchy's formula,

$$\int_C f(z)\, dz = 2\pi i\, f(z_0) \quad \text{where } z_0 = 1$$

$$= 2\pi i \cdot \frac{1}{4}(1) = \frac{\pi i}{2}$$

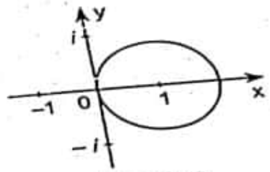

Fig. 2.28

(iii) $|z + i| = 1$ is a circle with centre at $(0, -i)$ and radius 1. Hence, the points $z = +1, -1, i$ lie **outside** the circle. The only point $z = -i$ lies inside the circle. The given function

$$\Phi(z) = \frac{z^2/(z^2-1)(z-i)}{(z+i)}.$$

$$\text{Now,} \quad f(z) = \frac{z^2}{(z^2-1)(z-i)}$$

is analytic in and on C. Hence, by Cauchy's formula,

$$\int_C f(z)\, dz = 2\pi i\, f(z_0) \quad \text{where } z_0 = -i$$

$$= 2\pi i \cdot \frac{1}{4i}(-1) = -\frac{\pi}{2}.$$

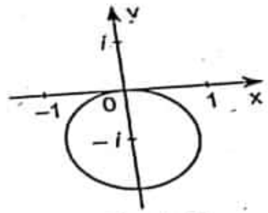

Fig. 2.29

Example 9 : Evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\, dz$, where C is the circle $|z - 1| = 3$. **(M.U. 1999)**

Sol. : The circle $|z - 1| = 3$ has centre at C (1, 0) and radius 3. Further, $z + 1 = 0$ gives A. $z = -1$. The point A lies **inside** the circle. Hence, $e^{2z}/(z+1)^4$ is not analytic in C. We take $f(z) = e^{2z}$ which is analytic in C.

By Corollary of Cauchy's Formula

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{3!} f^3(z_0)$$

$$= \frac{2\pi i}{3!}\cdot\frac{8}{e^2}$$

$$\left[\because f(z) = e^{2z},\ f^3(z) = 8e^{2z} \text{ and } z_0 = -1\right]$$

$$= \frac{8\pi i}{3e^2}.$$

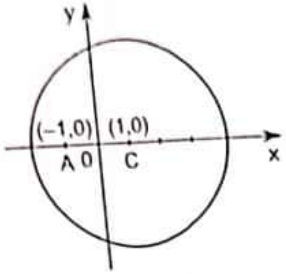

Fig. 2.30

Example 10 : Evaluate $\int_C \frac{dz}{z^3(z+4)}$, where C is the circle $|z| = 2$.

(M.U. 1993, 97, 2004)

Sol. : $|z| = 2$ is a circle with centre at the origin and radius 2. $z^3(z+4) = 0$ gives $z = 0$ and $z = -4$. The point $z = 0$ lies **inside** the circle and $z = -4$ lies **outside** the circle.

∴ We take $f(z) = \frac{1}{z+4}$ which is analytic in C.

Hence, by Cauchy's formula

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$\therefore \int_C \frac{dz/(z+4)}{z^3} = \frac{2\pi i}{2!} f^2(z_0) \quad \text{where } f(z) = \frac{1}{z+4}$$

$$= \pi i\left[\frac{2}{(z+4)^3}\right] \text{ at } z_0 = 0$$

$$= \frac{\pi i}{32}.$$

Example 11 : Evaluate $\oint_C \frac{1}{(z^3-1)^2}\,dz$ where C is $|z-1| = 1$. (M.U. 1998, 2016)

Sol. : $|z-1| = 1$ is a circle with centre at (1, 0) and radius 1.

Now, $z^3 - 1 = (z-1)(z^2+z+1) = 0$ gives $z = 1$ or $z = \frac{-1 \pm \sqrt{3}\cdot i}{2}$.

The point $z = 1$ lies **inside** the circle and the points $z = \frac{-1 \pm \sqrt{3}\cdot i}{2}$ lie **outside** the circle.

Hence, we write $\oint_C \frac{dz}{(z^3-1)^2} = \oint_C \frac{1/(z^2+z+1)^2}{(z-1)^2}\,dz$

But $z = 1$ is repeated twice. Hence, by Corollary of Cauchy's formula

$$\int_C \frac{f(z)}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$\therefore \int_C \frac{dz}{(z^3-1)^2} = \int_C \frac{1/(z^2+z+1)^2}{(z-1)^2}\,dz$$

$$= 2\pi i\left[\frac{d}{dz}\cdot\frac{1}{(z^2+z+1)^2}\right]_{z=1}$$

$$= 2\pi i\left[\frac{-2(2z+z)}{(z^2+z+1)^2}\right]_{z=1}$$

$$= 2\pi i\left[\frac{-2(3)}{3^2}\right] = -\frac{4\pi i}{9}.$$

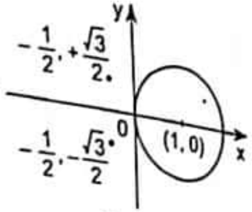

Fig. 2.31

Example 12 : Evaluate $\oint_C \frac{\sin^6 z}{[z-(\pi/2)]^3}\,dz$ where C is $|z| = 2$. (M.U. 1997)

Sol. : $z - \pi/2 = 0$ gives $z = \pi/2$ and $|z| = 2$ is a circle with center at the origin and radius 2. The point $z = \pi/2$ lies inside C. Hence, by corollary of Cauchy's formula

$$\int_C \frac{f(z)}{z-z_0}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$\int_C \frac{\sin^6 z}{[z-(\pi/2)]^3}\,dz = \frac{2\pi i}{2!} f''(z_0)$$

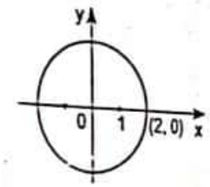

Fig. 2.32

Where, $f(z) = \sin^6 z$.

Now, $f'(z) = 6\sin^5 z\cos z$.

∴ $f''(z) = 30\sin^4 z\cos^2 z + 6\sin^5 z(-\sin z)$

∴ $f''(z = \pi/2) = 30(0) + 6(-1) = -6$.

$$\therefore \int_C \frac{\sin^6 z}{[z-(\pi/2)]^3}\,dz = \frac{2\pi i}{2}(-6) = -6\pi i$$

Type V : To find the value of the function $f(\xi) = \int_C \frac{\Phi(z)}{z-\xi}$ at $\xi = \xi_0$.

Example 1 : If $f(\xi) = \int_C \frac{3z^2+2z+1}{z-\xi}\,dz$, where C is the circle $x^2+y^2 = 4$, find the values of (i) $f(3)$, (ii) $f'(1-i)$, (iii) $f''(1-i)$. (M.U. 2014)

Sol. : The circle $x^2+y^2 = 4$ is $|z| = 2$.

(i) The point $z = 3$ lies **outside** the circle $|z| = 2$.

∴ By Cauchy's Integral Theorem

$$f(3) = \int_C \frac{3z^2+2z+1}{z-3}\,dz = 0$$

(ii) The point $z = 1 - i$ *i.e.* $(1, -1)$ lies **inside** the circle. Hence, we take $\Phi(z) = 3z^2 + 2z + 1$ which is analytic everywhere

$$\therefore \int_C \frac{\Phi(z)}{z - \xi}\, dz = 2\pi i\, \Phi(\xi) = 2\pi i (3\xi^2 + 2\xi + 1)$$

$$\therefore f(\xi) = \int_C \frac{3z^2 + 2z + 1}{z - \xi}\, dz = 2\pi i (3\xi^2 + 2\xi + 1)$$

$$\therefore f'(\xi) = 2\pi i (6\xi + 2) \quad \text{and} \quad f''(\xi) = 2\pi i (6)$$

$$\therefore f'(1 - i) = 2\pi i [6(1 - i) + 2] = 2\pi i (8 - 6i)$$

and $f''(1 - i) = 12\pi i.$

Example 2 : If $f(a) = \int_C \frac{4z^2 + z + 5}{z - a}\, dz$, where C is $|z| = 2$, find the values of $f(1)$, $f(i)$, $f'(-1)$, $f''(-i)$.

Sol. : The circle $|z| = 2$ has centre at the origin and radius 2.

The point $z = 1$ lies inside the circle. $f(z) = 4z^2 + z + 5$ is analytic in and on C and $z = 1$ lies **inside** it. Hence, by Cauchy's formula

$$f(1) = \int_C \frac{4z^2 + z + 5}{z - 1}\, dz = 2\pi i\, \Phi(z_0) \qquad \text{where, } \Phi(z) = 4z^2 + z + 5 \text{ and } z_0 = 1.$$

$$\therefore f(1) = 2\pi i (4 + 1 + 5) = 20\pi i$$

The point $z = i$ also lies **inside** the circle

$$\therefore f(i) = \int_C \frac{4z^2 + z + 5}{z - i}\, dz = 2\pi i\, \Phi(z_0) \qquad \text{where, } \Phi(z) = 4z^2 + z + 5 \text{ and } z_0 = i.$$

$$\therefore f(i) = 2\pi i (-4 + i + 5) = 2\pi i (1 + i) = 2\pi (i - 1)$$

$$\therefore f'(a) = 2\pi i\, \Phi'(a) = 2\pi i (8z + 1)$$

$$\therefore f'(-1) = -14\pi i$$

$$\therefore f''(a) = 2\pi i (8) = 16\pi i$$

$$\therefore f''(-1) = 16\pi i$$

Example 3 : If $f(\xi) = \int_C \frac{4z^2 + z + 5}{z - \xi}\, dz$ where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, find the values of $f(i)$, $f'(-1)$, $f''(-i)$ and $f(3)$. **(M.U. 2003, 06, 16)**

Sol. : (i) The point $z = i$ *i.e.*, $(0, 1)$ lies **inside** the ellipse,

$$f(z) = 4z^2 + z + 5$$

is analytic in and on C. Hence, by Cauchy's formula

$$f(i) = \int_C \frac{4z^2 + z + 5}{z - i}\, dz = 2\pi i\, \Phi(z_0) \qquad \text{where, } \Phi(z) = 4z^2 + z + 5 \text{ and } z_0 = i.$$

$$\therefore f(i) = 2\pi i (4i^2 + i + 5)$$

$$= 2\pi i (-4 + i + 5) = 2\pi i (1 + i)$$

(ii) The point $(-1, 0)$ also lies **inside** the ellipse. Hence, we take $\Phi(z) = 4z^2 + z + 5$, which is analytic in and on C.

$$\therefore f(\xi) = \int_C \frac{4z^2 + z + 5}{z - \xi}\, dz = 2\pi i (4\xi^2 + \xi + 5)$$

$$\therefore f'(\xi) = 2\pi i (8\xi + 1) \quad \text{and} \quad f''(\xi) = 2\pi i \cdot 8$$

$$\therefore f'(-1) = 2\pi i \cdot [8(-1) + 1] = -14\pi i$$

(iii) $f''(\xi) = 16\pi i \qquad \therefore f''(-i) = 16\pi i$

(iv) The point $(3, 0)$ lies **outside** the ellipse. Hence,

$$\int_C \frac{4z^2 + z + 5}{z - \xi}\, dz = 0.$$

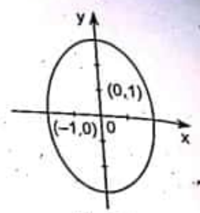

Fig. 2.33

Miscellaneous Examples

Example 1 : If C is the circle $|z| = 1$, using the integral $\int_C \frac{e^{kz}}{z}\, dz$, where k is real, show that $\int_0^{\pi} e^{k\cos\theta} \cos(k \sin\theta)\, d\theta = \pi.$ **(M.U. 1993, 2001, 03, 14)**

Sol. : We obtain the integral in two different ways.

(i) $|z| = 1$ is a circle with centre at the origin and radius = 1. The point $z = 0$ lies within the circle. Hence, we write $f(z) = e^{kz}$ which is analytic in and on C. Hence, by Cauchy's Integral Formula

$$\int_C \frac{e^{kz}}{z}\, dz = 2\pi i\, f(z_0) \quad \text{where } f(z) = e^{kz} \text{ and } z_0 = 0$$

$$= 2\pi i\, e^{kz_0} \text{ at } z_0 = 0$$

$$= 2\pi i\, e^0 = 2\pi i.$$

(ii) Now, if we put $z = e^{i\theta}$, $dz = i e^{i\theta} d\theta$

$$\int_C \frac{e^{kz}}{z}\, dz = \int_0^{2\pi} \frac{e^{ke^{i\theta}}}{e^{i\theta}}\, i e^{i\theta}\, d\theta = i \int_0^{2\pi} e^{ke^{i\theta}}\, d\theta$$

$$\therefore \int_C \frac{e^{kz}}{z}\, dz = i \int_0^{2\pi} e^{k(\cos\theta + i\sin\theta)}\, d\theta = i \int_0^{2\pi} e^{k\cos\theta} \cdot e^{ik\sin\theta}\, d\theta$$

$$= i \int_0^{2\pi} e^{k\cos\theta} \{\cos(k\sin\theta) + i \sin(k\sin\theta)\}\, d\theta$$

$$\therefore 2\pi i = i \int_0^{2\pi} e^{k\cos\theta} \{\cos(k\sin\theta) + i \sin(k\sin\theta)\}\, d\theta$$

Equating imaginary parts

$$2\pi = \int_0^{2\pi} e^{k\cos\theta} \cos(k\sin\theta)\, d\theta$$

$$2\pi = 2\int_0^{\pi} e^{k\cos\theta}\cos(k\sin\theta)\,d\theta \quad \left[\because \int_0^{a} f(x)\,dx = 2\int_0^{a/2} f(a-x)\,dx \text{ if } f(a-x) = f(x)\right]$$

$$\therefore \int_0^{\pi} e^{k\cos\theta}\cos(k\sin\theta)\,d\theta = \pi.$$

(For another method see Ex. 16 page 4-52.)

Example 2 : Show that $\int_C \frac{dz}{z+1} = 2\pi i$, where C is the circle $|z| = 2$. Hence, deduce that $\int_C \frac{(x+1)\,dx + y\,dy}{(x+1)^2+y^2} = 0$ and $\int_C \frac{(x+1)\,dy - y\,dx}{(x+1)^2+y^2} = 2\pi$. **(M.U. 1991)**

Sol. : $|z| = 2$ is a circle with centre at $z = 0$ and radius $= 2$. The point $z = -1$ lies **inside** the circle. We take $f(z) = 1$, which is analytic in and on C. Hence, by Cauchy's Integral Formula

$$\int_C \frac{dz}{z+1} = 2\pi i\, f(z_0) = 2\pi i \qquad \text{.................... (1)}$$

Now $$\int_C \frac{dz}{z+1} = \int_C \frac{dx + i\,dy}{(x+1)+iy} = \int_C \frac{dx+i\,dy}{(x+1)+iy}\cdot\frac{(x+1)-iy}{(x+1)-iy}$$

$$= \int_C \frac{(x+1)\,dx + y\,dy}{(x+1)^2+y^2} + i\int_C \frac{(x+1)\,dy - y\,dx}{(x+1)^2+y^2} \qquad \text{.................... (2)}$$

From (1) and (2) equating real and imaginary parts

$$\int_C \frac{(x+1)\,dx + y\,dy}{(x+1)^2+y^2} = 0 \quad \text{and} \quad \int_C \frac{(x+1)\,dy - y\,dx}{(x+1)^2+y^2} = 2\pi.$$

Example 3 : If $f(z)$ is analytic in and on a simple closed curve C, prove that

$$f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-a)^4}\,dz.$$

Hence, evaluate $\int_C \frac{e^{iz}}{z^4}\,dz$ where C is the circle $|z| = 2$. **(M.U. 1997)**

Sol. : Since $f(z)$ is analytic in and on C, by Cauchy's integral formula

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

Differentiating under the integral sign, w.r.t. a,

$$\int_C \frac{f(z)}{(z-a)^2}\,dz = 2\pi i\, f'(a)$$

Differentiating twice again w.r.t. a,

$$\int_C \frac{2 f(z)}{(z-a)^3}\,dz = 2\pi i\, f''(a) \quad \text{and} \quad \int_C 3!\frac{f(z)}{(z-a)^4}\,dz = 2\pi i\, f'''(a)$$

$$\therefore f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-a)^4}\,dz.$$

Now, let $f(z) = e^{iz}$, $a = 0$ and C be $|z| = 2$.

Since, $f(z)$ is analytic in and on C by the above result

$$f'''(0) = \frac{3!}{2\pi i}\int_C \frac{e^{iz}}{z^4}\,dz$$

Now $f(z) = e^{iz}$, $f'(z) = i\,e^{iz}$, $f''(z) = -e^{iz}$, $f'''(z) = -i\,e^{iz}$

$$\therefore f'''(0) = -i\,e^0 = -i \quad \therefore (-i) = \frac{3!}{2\pi i}\int_C \frac{e^{iz}}{z^4}\,dz$$

$$\therefore \int_C \frac{e^{iz}}{z^4}\,dz = -\frac{2\pi i^2}{3!} = \frac{\pi}{3}.$$

Example 4 : Evaluate $\int_C \frac{z+1}{z^3-2z^2}\,dz$ where C is (a) the circle $|z| = 1$, (b) the circle $|z-2-i| = 2$, (c) the circle $|z-1-2i| = 2$. **(M.U. 2002)**

Sol. : (a) $|z| = 1$ is a circle with centre at the origin and radius 1.

Now, $z^3 - 2z^2 = z^2(z-2)$. $\therefore$ $z = 0$ lies **inside** the circle and $z = 2$ lies **outside** it.

Hence, we write $\frac{z+1}{z^3-2z^2} = \frac{(z+1)/(z-2)}{z^2}$ and take $f(z) = \frac{z+1}{z-2}$ which is analytic in C.

Then by the corollary of Cauchy's formula,

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0) \quad \text{and} \quad z_0 = 0.$$

$$\therefore f'(z) = \frac{(z-2)\cdot 1 - (z+1)\cdot 1}{(z-2)^2} = -\frac{3}{(z-2)^2}$$

$$\therefore f'(0) = -\frac{3}{2^2} = -\frac{3}{4}$$

$$\therefore \int_C \frac{z+1}{z^3-2z^2}\,dz = \int_C \frac{(z+1)/(z+2)}{z^2}\,dz = \frac{2\pi i}{1!}\left(-\frac{3}{4}\right) = -\frac{3\pi}{2}.$$

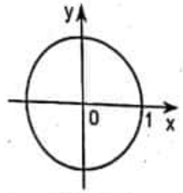

Fig. 2.34

(b) $|z-2-i| = 2$ is a circle with centre at (2, 1) and radius 2.

$\therefore$ The point $z = 0$ *i.e.* (0, 0) lies **outside** the circle and the point $z = 2$ *i.e.* (2, 0) lies **inside** the circle. Hence, we take $f(z) = (z+1)/z^2$ which is analytic in C.

$\therefore$ By Cauchy's formula

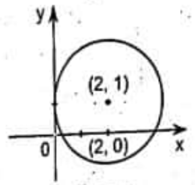

Fig. 2.35

$$\int_C \frac{z+1}{z^3-2z^2}\,dz = \int_C \frac{(z+1)/z^2}{z-2}\,dz$$

$$= \int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0) \quad \text{where } f(z) = \frac{z+1}{z^2} \text{ and } z_0 = 2$$

$$= 2\pi i\cdot\frac{2+1}{2^2} = 2\pi i\cdot\frac{3}{4} = \frac{3\pi i}{2}.$$

(c) $|z-1-2i| = 2$ is a circle with centre at (1, 2) and radius 2.

$\therefore$ The points $z_0 = 0$ and $z = 2$ both lie **outside** the circle.

$\therefore$ $f(z) = \dfrac{z+1}{z^3 - 2z^2}$ is analytic in C.

$\therefore$ By Cauchy's Theorem

$$\int_C f(z)\,dz = 0.$$

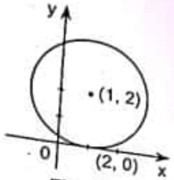

Fig. 2.36

Example 5 : Evaluate $\displaystyle\int_C \frac{1}{(z^3-1)^2}\,dz$ where C is the circle $|z-1| = 1$. **(M.U. 1998)**

Sol. : The circle $|z-1| = 1$ has centre at (1, 0) and radius 1.

$\therefore\ z^3 - 1 = 0 \qquad \therefore\ (z-1)(z^2+z+1) = 0$

$\therefore\ z = 1$ or $z = \dfrac{-1 \pm \sqrt{3}\,i}{2}$

The point $z = 1$ *i.e.* (1, 0) lies **inside** the circle and the points $z = \dfrac{-1 \pm \sqrt{3}\,i}{2}$ *i.e.* $\left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$ and $\left(-\dfrac{1}{2}, \dfrac{-\sqrt{3}}{2}\right)$ lie in the second and third quadrant and hence, **outside** the circle. Further note that the roots are **repeated twice**.

$\therefore$ We take $f(z) = \dfrac{1}{(z^2+z+1)^2}$ which is analytic in C.

Then by the corollary of Cauchy's formula,

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0) \quad \text{and} \quad z_0 = 1.$$

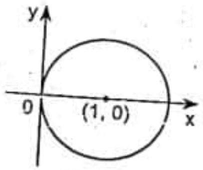

Fig. 2.37

$$\therefore\ f'(z) = -\frac{2}{(z^2+z+1)^3} \cdot (2z+1)$$

$$\therefore\ f'(z_0) = f'(1) = -\frac{2}{(1+1+1)^3}(2 \cdot 1 + 1) = -\frac{6}{27} = -\frac{2}{9}.$$

$$\therefore\ \int_C \frac{1}{(z^3+1)^2}\,dz = \frac{2\pi i}{1!}\left(-\frac{2}{9}\right) = -\frac{4\pi i}{9}.$$

EXERCISE - I

(A) On Cauchy's Theorem

1. Evaluate $\displaystyle\int_C (x^2 - y^2 + 2ixy)\,dz$, where C is the circle $|z| = 2$. **[Ans. : $I = 0$]**

2. $\displaystyle\int_C \cot z \cdot dz$ where C is $\left|z + \dfrac{1}{2}\right| = \dfrac{1}{3}$. **(M.U. 2004) [Ans. : $I = 0$]**

(B) On Cauchy's Formula

Type I : The point *a* lies inside *C*

1. $\displaystyle\int_C \operatorname{cosec} z \cdot dz$ where C is $|z| = 1$. **(M.U. 2003) [Ans. : $2\pi i$]**

2. $\displaystyle\int_C \frac{dz}{\sin hz}$ where C is the circle $x^2 + y^2 = 16$. **(M.U. 2001, 03) [Ans. : $2\pi i$]**

3. $\displaystyle\int_C \frac{\sin 3z}{z + (\pi/2)}\,dz$ where C is the circle $|z| = 5$. **(M.U. 2003) [Ans. : $2\pi i$]**

4. $\displaystyle\int_C \frac{dz}{z-2}$, where C is the circle (i) $|z-2| = 1$, (ii) $|z-1| = 1/2$. **(M.U. 1991, 93)**
   [Ans. : (i) $2\pi i$, (ii) 0]

5. $\displaystyle\int_C \frac{e^{2z}}{z-1}\,dz$, where C is the circle, find (i) $|z| = 2$, (ii) $|z| = 1/2$. **(M.U. 1998)**
   [Ans. : (i) $2\pi i e^2$, (ii) 0]

6. $\displaystyle\int_C \frac{\sin^6 z}{(z - \pi/6)^n}\,dz$ where C is the circle $|z| = 1$ for $n = 1, n = 3$. **(M.U. 2004)**
   [Ans. : (i) $\dfrac{\pi i}{32}$, (ii) $\dfrac{21\pi i}{16}$.]

7. $\displaystyle\int_C \frac{dz}{z^4 e^z}$, where C is the circle $|z| = 1$. **[Ans. : $-\dfrac{\pi i}{3}$]**

8. $\displaystyle\int_C \frac{ze^{2z}}{(z-1)^3}\,dz$, where C is $|z+i| = 2$. **(M.U. 1995) [Ans. : $8\pi i e^2$]**

Type II : Both points lie inside *C*

1. $\displaystyle\int_C \frac{\cos(\pi z^2)}{(z^2 - 3z + 2)}\,dz$, where C is the circle $|z| = 3$. **[Ans. : $4\pi i$]**

2. $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$, where C is the circle $|z| = 3$. **(M.U. 1995, 2003, 06, 15)**
   [Ans. : $4\pi i$]

3. If $f(z) = z^3 + iz^2 - 4z - 4i$, evaluate $\displaystyle\int_C \frac{f'(z)}{f(z)}\,dz$ where C is a simple closed curve enclosing zeros of $f(z)$. **(M.U. 2004) [Ans. : $6\pi i$]**

Type III : One point lies inside and the other lies outside *C*

1. $\displaystyle\int_C \frac{z+6}{z^2-4}\,dz$, where (i) C is the circle $|z| = 1$, (ii) C is the circle $|z-2| = 1$, (iii) C is the circle $|z+2| = 1$. **(M.U. 1994,2001)**
   [Ans. : (i) 0, (ii) $4\pi i$, (iii) $-2\pi i$]

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2. $\int_C \frac{z^2+1}{z^2-1} dz$, where the contour C is a circle with radius 1 and centre at

(i) $z = i$, (ii) $z = 1$, (iii) $z = -1$. [Ans. : (i) 0, (ii) $2\pi i$, (iii) $-2\pi i$]

3. $\int_C \frac{z-1}{(z+1)^2(z-2)} dz$, where C is $|z-i| = 2$. (M.U. 1998, 2005) [Ans. : $-2\pi i/9$]

4. $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z^2+3z+2} dz$, where C is (i) $|z| = \frac{1}{2}$, (ii) $|z| = \frac{3}{2}$. (M.U. 1998, 2011)

[Ans. : (i) 0, (ii) $-2\pi i$]

5. $\int_C \frac{z \cos \pi z}{z^2-z-2} dz$, where C is $|z-i| = 2$. (M.U. 1995) [Ans. : $\frac{-2\pi i}{3}$]

6. $\int_C \frac{(z-1)(z-2)}{(z-3)(z-4)} dz$, where C is (i) $|z| = 3\cdot5$, (ii) $|z| = 4\cdot5$. (M.U. 1996, 2006)

[Ans. : (i) $-4\pi i$, (ii) $8\pi i$]

7. $\int_C \frac{z+1}{z^3-2z} dz$, where C is the circle $|z| = 1$. (M.U. 1992, 2002) [Ans. : $-3\pi i/2$]

8. $\int_C \frac{z^2+4}{(z-2)(z+3i)} dz$, where C is (i) $|z+1| = 2$, (ii) $|z-2| = 2$.

(M.U. 1991, 94, 95, 2003, 06) [Ans. : (i) 0, (ii) $16\pi i/(2+3i)$]

9. $\int_C \frac{dz}{z^2+9}$, when (i) the point $3i$ lies inside C and the point $-3i$ lies outside C. (ii) the point $3i$ lies outside C and the point $-3i$ lies inside C. (iii) both the points $3i$ and $-3i$ lie outside C.

(M.U. 1991, 92, 97) [Ans. : (i) $\pi/3$, (ii) $-\pi/3$, (iii) 0]

10. $\int_C \frac{\cos \pi z}{z^2-1} dz$, where C is the rectangle whose vertices are

(i) $2 \pm i$, $-2 \pm i$, (ii) $-i, 2-i, 2+i, i$. (M.U. 1998) [Ans. : (i) 0, (ii) $-\pi i$]

11. $\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2} dz$ where C is $|z| = 1$. (M.U. 2003, 04) [Ans. : $-\frac{16\pi i}{9}$]

12. $\int_C \frac{z\,dz}{(z-1)(z-3)}$, where C is the circle (i) $|z| = 3$, (ii) $|z| = 1\cdot5$.

[Ans. : (i) $2\pi i$, (ii) $-\pi i$]

13. $\int_C \frac{e^z}{(z-1)(z-4)} dz$, where C is the circle $|z| = 2$. [Ans. : $\frac{-2\pi i e}{3}$]

14. $\int_C \frac{dz}{z^3(z+4)}$, where C is the circle $|z| = 2$. (M.U. 2000) [Ans. : $\frac{2\pi i}{27}$]

15. $\int_C \frac{z^2+2z+3}{z^2+3z-4} dz$, where C is the ellipse $4x^2+9y^2 = 36$. [Ans. : $\frac{12\pi i}{5}$]

16. $\int_C \frac{z+1}{z^3-2z^2} dz$, where C is (a) the circle $|z| = 1$, (b) the circle $|z-2-i| = 2$, (c) the circle $|z-1-2i| = 2$.

(M.U. 2002)

[Ans. : (a) $-\frac{3\pi i}{2}$, (b) $\frac{3\pi i}{2}$, (c) 0]

Type IV : To find the value of the function $\Phi(\alpha)$

1. If $\Phi(\alpha) = \int_C \frac{ze^z}{z-\alpha} dz$, where C is $|z-2i| = 3$, find the values of (i) $\Phi(1)$, (ii) $\Phi'(2)$, (iii) $\Phi(3)$. (M.U. 1996, 2014) [Ans. : (i) $2\pi i e$, (ii) $6\pi i e^2$, (iii) 0]

2. If $\Phi(\alpha) = \int_C \frac{4z^2+z+5}{z-\alpha} dz$, where C is the contour of the ellipse $\frac{x^2}{4}+\frac{y^2}{9} = 1$, find the values of (i) $\Phi(3\cdot5)$, (ii) $\Phi(i)$, (iii) $\Phi'(-1)$, (iv) $\Phi''(-i)$.

[Ans. : (i) 0, (ii) $2\pi(i-1)$, (iii) $-14\pi i$, (iv) $16\pi i$]

3. If $f(\xi) = \int_C \frac{4z^2+z+4}{z-\xi} dz$, where C is the ellipse $4x^2+9y^2 = 36$. find the values of (i) $f(4)$, (ii) $f(1)$, (iii) $f(i)$, (iv) $f'(-1)$, (v) $f''(-i)$. (M.U. 1997, 2006)

[Ans. : (i) 0, (ii) $18\pi i$, (iii) -2π, (iv) $-14\pi i$, (v) $16\pi i$]

4. If $f(\xi) = \int_C \frac{3z^2+7z+1}{z-\xi} dz$, where C is a circle $|z| = 2$, find the values of (i) $f(-3)$, (ii) $f(i)$, (iii) $f'(1-i)$, (iv) $f''(1-i)$. (M.U. 1993, 99, 2006, 15)

[Ans. : (i) 0, (ii) $-2\pi(2i+7)$, (iii) $2\pi(6+13i)$, (iv) $12\pi i$]

5. Verify Cauchy's Theorem for $\int_C f(z)\,dz = 3z^2$ where C is the circle $|z| = 2$.

(M.U. 2004)

EXERCISE - II

Theory

1. Define a simple curve and also simply and multiply connected region with suitable diagrams. (M.U. 2003)
2. State Cauchy's Integral Theorem.
3. State and prove Cauchy's Theorem. (M.U. 2003)
4. State and prove Cauchy's integral formula. (M.U. 1999, 2003, 09)
5. State Morera's Theorem.
6. State Cauchy's extended integral formula for a doubly connected region.
7. State Cauchy's integral formula for the n-th derivative of an analytic function.
8. If $f(z)$ is continuous on a closed curve C of length l, where $|f(z)| < M$, then prove that

$$\left| \int_C f(z)\,dz \right| \le Ml.$$

9. Find the following integrals.

(i) $\int_C \cot z\, dz$, where C is $|z| = 1$.

(ii) $\int_C \tan hz\, dz$, where C is $|z| = 3$.

(iii) $\int_C \frac{(z^2 + 2z + 1)}{z - 4} dz$, where C is $|z| = 1$. (M.U. 2000)

(iv) $\int_C (z - a)^n dz$, $n \neq -1$ where C is $|z - a| = r$. (M.U. 2002)

[Ans. : Each is zero by Cauchy's Theorem.]

10. State true or false with proper justification.

(i) $\oint_C z^3 dz = \oint_C z^{-3} dz$, where C is $|z - 3| = 4$.

(ii) If $\oint_C f(z)\, dz = 0$ for a any closed curve C then $f(z)$ is an analytic function. (M.U. 1996)

(iii) $\oint_C z^3\, dz = \oint_C \frac{1}{z^3} dz$ where C is $|z - 2i| = 1$. (M.U. 2004)

(iv) $\int_C \frac{dz}{z} = 2\pi i$ where C is the circle $|z| = r$. (M.U. 2001)

(v) $\int_C (z^2 + 2z)\, dz = 0$ where C is a triangle whose vertices are (0, 0), (0, 1), (1, 1). (M.U. 2001)

(vi) If $f(z) = (x^2 - 2x - y^2) + i(2xy - 2y)$, then $\int_A^B f(z)\, dz$ is the same along

(i) $y = x^2$ and (ii) $y = x$ where A is (0, 0) and B is (1, 1). (M.U. 2002)

(vii) $\int_C \frac{dz}{(z - a)^4} = 0$ where C is $|z - a| = a$. (M.U. 2002)

(viii) $\int_0^{1+i} [(x^2 - y^2 + 2x) + 2iy(x + 1)]\, dz$ is same along $y = x^2$ and $x = y^2$. (M.U. 2002)

(ix) If $\Phi(a) = \oint_C \frac{z^2 + 2}{z - a} dz$ where C is $|z - 1 + i| = 2$, then $\Phi'(1) = 2\pi i$. (M.U. 2002)

(x) $\oint_C \frac{dz}{z - 4} = 0$ where C is $|z - 4| = 2$. (M.U. 2002)

(xi) $\int_C (x^2 - y^2 + 2ixy)\, dz = 0$ where C is $|z| = 2$. (M.U. 2004)

[Ans. : (i) z^3 is analytic in and on C.

$\therefore \oint z^3 dz = 0$. But z^{-3} is not analytic in C. Hence, $\oint z^{-3} dz \neq 0$. Hence, false.

(ii) If $\oint_C f(z)\, dz = 0$ for any closed curve C, then $f(z)$ is analytic.

Morera's Theorem. Hence, True.

(iii) True. z^3 and $1/z^3$ are both analytic on C. (iv) True. (v) True.

(vi) True $f(z)$ is analytic. (vii) False. a is inside C.

(viii) True. (ix) False. (x) False. (xi) True.

☆☆☆

CHAPTER

3

Taylor's and Laurent's Series

1. Introduction

In this chapter we shall study how to expand an analytic function as a power series. We shall also learn in this chapter some important concepts in complex analysis *viz.* zeros, poles and residues and applications of residues to find the integrals in the next chapter.

2. Complex Power Series

A series of the type $(a_1 + ib_1) + (a_2 + ib_2) + \ldots.. + (a_n + ib_n) + \ldots..$ where $a_1, a_2, \ldots.. a_n, \ldots..$ and $b_1, b_2, \ldots.. b_n, \ldots..$ are real numbers is called a series of **complex numbers**. It can be written as

$$\sum_{n=1}^{\infty} (a_n + ib_n)$$

The above series is convergent if both the series Σa_n and Σb_n are convergent.

A series of the type $c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots.. + c_n(z-a)^n + \ldots..$ is called **a power series in powers of** $(z-a)$. It can be written as

$$\sum_{n=1}^{\infty} c_n(z-a)^n$$

where z is a complex number. The constants $c_0, c_1, c_2, \ldots..$ are called the **coefficients** and the constant a is called **the centre** of the series.

It can be proved that there exists a real number R such that the power series $\Sigma c_n(z-a)^n$ is convergent for $|z-a| < R$, is divergent for $|z-a| > R$ and may or may not be convergent for $|z-a| = R$. This means that there is a circle with centre a and radius R, such that the power series $\Sigma c_n(z-a)^n$ is convergent at all points inside the circle $|z-a| = R$, is divergent at all points outside the circle and may and may not be convergent at points on the circle. For this reason the circle $|z-a| = R$ is called **the circle of convergence** for the power series $\Sigma C_n(z-a)^n$ and R is called the **radius of convergence.**

Power series are important in complex analysis because every power series is analytic and every analytic function can be represented as power series. Such series are known as **Taylor's Series.**

Analytic functions can also be represented by another type of series containing positive as well as negative powers of $(z-a)$. Such series are called **Laurent's Series.** They are useful for evaluating integrals real and complex.

We shall accept the validity of the expansions of functions as Taylor's Series and Laurent's Series without proof.

Brook Taylor (1685 - 1731)

He was an English mathematician best known for Taylor's Theorem and Taylor's Series. Initially he had interest in law and got doctorate in law in 1714. But he had also keen interest in mathematics. His publication **'Methodus Incrementorum Directa et Inversa'** is considered as the beginning of new branch of mathematics called **"Calculus of finite differences".** The famous Taylor's theorem remained unrecognised untill 1712 when Lagrange realised its powers. He was elected to Royal Society in the same year. From 1715 he took interest in the studies of religion and philosophy.

Pierre Alphonse Laurent (1813 - 1854)

He studied in Ecole polytechnic and got his degree with the highest rank in his class. He was then immediately given the rank of second lieutenant in the engineering corps. After few years in Algeria he returned to France. He spent six years in the project of enlargement of the port of Le Harve. In the midst of these technical operations he submitted a "Memoire . Sur le calcul des variations" to the Academy des Sciences. In 1843 he discovered the well-known theorem for the expansion of a function in the form of series now known as **Laurent Series**. Though he had keen interest in Mathematics, by profession he was a military engineer.

To find Radius of Convergence

We accept the following two formulae for finding the radius of convergence of power series $\Sigma C_n(z-a)^n$.

(i) D'Alembert's Ratio Formula : $\quad R = \lim_{n\to\infty} \left|\dfrac{C_n}{C_{n+1}}\right|$

(ii) *N*-th Root Formula : $\quad R = \lim_{n\to\infty} \dfrac{1}{\sqrt[n]{C_n}}$

Example 1 : Find the radius of convergence of each of the following power series.

(i) $\Sigma \dfrac{z^n}{3^n+1}$ (ii) $\Sigma \dfrac{(n!)^2}{(2n)!} z^n$ (iii) $\Sigma \dfrac{n+1}{(n+2)(n+3)} z^n$

(iv) $\Sigma \left(1+\dfrac{1}{n}\right)^{n^2} \cdot z^n$ (v) $\Sigma a^n \cdot z^n \; (a > 0)$

Sol. : (i) We have $C_n = \dfrac{1}{3^n+1}$ $\quad \therefore \; C_{n+1} = \dfrac{1}{3^{n+1}+1}$

$$\therefore \left|\frac{C_n}{C_{n+1}}\right| = \frac{1}{3^n+1} \cdot \frac{3^{n+1}+1}{1} = \frac{3+(1/3^n)}{1+(1/3^n)}$$

$$\lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right| = \lim_{n\to\infty}\frac{3+(1/3^n)}{1+(1/3^n)} = 3$$

∴ Radius of convergence = 3.

(ii) We have $C_n = \dfrac{(n!)^2}{(2n)!}$. ∴ $C_{n+1} = \dfrac{[(n+1)!]^2}{[2(n+1)]!}$

$$\therefore \left|\frac{C_n}{C_{n+1}}\right| = \frac{n!\cdot n!}{(2n)!}\cdot\frac{(2n+2)!}{(n+1)!\,(n+1)!} = \frac{n!\,n!\,(2n+2)(2n+1)(2n)!}{(2n)!\,(n+1)\,n!\,(n+1)\,n!}$$

$$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$$

$$\therefore \lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim_{n\to\infty}\frac{[2+(2/n)]\,[2+(1/n)]}{[1+(1/n)]\,[1+(1/n)]}$$

$$= 2\cdot 2 = 4$$

∴ Radius of convergence = 4.

(iii) Here $C_n = \dfrac{n+1}{(n+2)(n+3)}$. ∴ $C_{n+1} = \dfrac{n+2}{(n+3)(n+4)}$

$$\therefore \left|\frac{C_n}{C_{n+1}}\right| = \frac{n+1}{(n+2)(n+3)}\cdot\frac{(n+3)(n+4)}{(n+2)} = \frac{(n+1)(n+4)}{(n+2)(n+2)}$$

$$\therefore \lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)(n+4)}{(n+2)(n+2)}$$

$$= \lim_{n\to\infty}\frac{[1+(1/n)]\,[1+(4/n)]}{[1+(2/n)]\,[1+(2/n)]} = 1$$

∴ Radius of convergence = 1.

(iv) Here $C_n = \left(1+\dfrac{1}{n}\right)^{n^2}$ ∴ $\sqrt[n]{C_n} = \left[\left(1+\dfrac{1}{n}\right)^{n^2}\right]^{1/n}$

$$\therefore \lim_{n\to\infty}\sqrt[n]{C_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

∴ Radius of convergence $= \lim_{n\to\infty}\dfrac{1}{\sqrt[n]{C_n}} = \dfrac{1}{e}$.

(v) Here $C_n = a^n$.

$$\therefore \lim_{n\to\infty}\sqrt[n]{C_n} = \lim_{n\to\infty}\sqrt[n]{a^n}\ \lim_{n\to\infty}(a^n)^{1/n}$$

$$= \lim_{n\to\infty} a = a.$$

∴ Radius of convergence $= \lim_{n\to\infty}\dfrac{1}{\sqrt[n]{C_n}} = \dfrac{1}{a}$.

EXERCISE - I

Find the radius of convergence of the following series.

(i) $\Sigma\dfrac{1}{n^p}z^n$ (ii) $\Sigma\left(1-\dfrac{1}{n}\right)^{n^2}z^n$ (iii) $\Sigma\dfrac{n+3}{3^n}z^n$ (iv) $\Sigma\dfrac{z^n}{2^n+1}$

[Ans. : (i) 1, (ii) 1 / e, (iii) 1 / 3, (iv) 2]

3. Expansion of A Complex Function $f(z)$ as Taylor's Series

If $f(z)$ is analytic inside a circle C with centre at z_0, then for all z inside C, $f(z)$ can be expanded as

$$\boxed{f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \ldots\ldots\infty} \quad \text{.............. (A)}$$

The series is convergent at every point inside C and is known as Taylor's Series.

Cor. 1 : If we put $z = z_0 + h$, then

$$f(z_0+h) = f(z_0) + h\,f'(z_0) + \frac{h^2}{2!}f''(z_0) + \ldots\ldots\infty$$

Cor. 2 : If we put $z_0 = 0$, then

$$f(z) = f(0) + z\,f'(0) + \frac{z^2}{2!}f''(0) + \ldots\ldots\infty$$

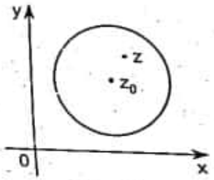


Fig. 3.1

This is known as **Maclaurin's Series.**

For every point z_0 we can always find a circle C such that Taylor's series expansion of $f(z)$ is possible. The largest circle with centre z_0 such that $f(z)$ is analytic inside it is called the **circle of convergence.** Its radius is called the **radius of convergence. Clearly the radius of convergence is the distance between z_0 and the nearest singularity of $f(z)$.**

From (A), we see that Taylor's series consists of positive integral powers of $(z-z_0)$.

4. Laurent's Series Expansion

If C_1 and C_2 are two concentric circles of radii r_1 and r_2 with centre at z_0 and if $f(z)$ is analytic on C_1 and C_2 and in the annular region R between the two circles, then for any point z in R,

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}} \quad \text{.............. (B)}$$

where $\boxed{a_n = \dfrac{1}{2\pi i}\oint_{C_1}\dfrac{f(\omega)}{(\omega-z_0)^{n+1}}\,d\omega}$ .... (1)

and $\boxed{b_n = \dfrac{1}{2\pi i}\oint_{C}\dfrac{f(\omega)}{(\omega-z_0)^{-n+1}}\,d\omega}$ .... (2)

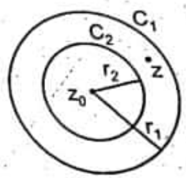


Fig. 3.2

We accept this without proof.

From (B), we see that Laurent's series consists of positive as well as negative integral powers of $(z - z_0)$.

It is clear that if $f(z)$ is analytic inside C_2, then Laurent's series reduces to Taylor's series. In this case the coefficients b_n of negative powers are zero.

The Taylor's expansion and Laurent's expansion are **unique** in the given regions and are usually found by using Binomial Series as illustrated below.

The part $\sum_{n=0}^{\infty} a_n(z-a)^n$ consisting of positive integral powers of $(z-a)$ is called the **analytic part or regular part** and the part $\sum_{n=0}^{\infty} b_n(z-a)^{-n}$ consisting of negative integral powers of $(z-a)$ is called **principal part of Laurent's Series.**

Notes ....

1. The Maclaurin's Series of some elementary functions are given below for ready reference.

   1. $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$ $\quad |z| < \infty$
   2. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...$ $\quad |z| < \infty$
   3. $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + ...$ $\quad |z| < \infty$
   4. $\sin hz = z + \frac{z^3}{3!} + \frac{z^5}{5!} + ...$ $\quad |z| < \infty$
   5. $\cos hz = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + ...$ $\quad |z| < \infty$
   6. $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + ...$ $\quad |z| < 1$

2. We often need the following expansions.

   $(1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - .....$ where $|z| < 1$

   $(1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + ...$ where $|z| < 1$

   $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + ...$ where $|z| < 1$

   $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + ...$ where $|z| < 1$

(A) Taylor's Series Expansion

Example 1 : Expand $f(z) = \sin z$ as a Taylor's series around $z = \pi/4$. **(M.U. 1993)**

Sol. : We first note that $f(z) = \sin z$ is analytic everywhere. By Taylor's series,

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + ...$$

Now $f(z) = \sin z$ $\quad \therefore f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z$ $\quad \therefore f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z$ $\quad \therefore f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$f'''(z) = -\cos z$ $\quad \therefore f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$$\therefore \sin z = \frac{1}{\sqrt{2}}\left[1 + \frac{[z-(\pi/4)]}{1!} - \frac{[z-(\pi/4)]^2}{2!} - \frac{[z-(\pi/4)]^3}{3!} + ...\infty\right]$$

Example 2 : Expand $\cos z$ as Taylor's series at $z = \pi/2$. **(M.U. 1995)**

Sol. : By Taylor's series,

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + ...$$

Here $f(z) = \cos z$ and $a = \pi/2$ $\quad \therefore f(\pi/2) = \cos(\pi/2) = 0$

$\therefore f'(z) = -\sin z$ $\quad \therefore f'(\pi/2) = -\sin(\pi/2) = -1$

$f''(z) = -\cos z$ $\quad \therefore f''(\pi/2) = -\cos(\pi/2) = 0$

$f'''(z) = \sin z$ $\quad \therefore f'''(\pi/2) = \sin(\pi/2) = 1$

$$\therefore f(z) = 0 + [z-(\pi/2)](-1) + \frac{[z-(\pi/2)]^3}{3!}(1) + \frac{[z-(\pi/2)]^5}{5!}(-1) + \frac{[z-(\pi/2)]^7}{7!}(1) + ...$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \cdot \frac{[z-(\pi/2)]^{2n+1}}{(2n+1)!}$$

Aliter : Put $z = u + (\pi/2)$

$$\therefore \cos z = \cos[u + (\pi/2)] = \cos u \cos(\pi/2) - \sin u \sin(\pi/2) = -\sin u$$

$$= -\left[u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + ...\right] = -u + \frac{u^3}{3!} - \frac{u^5}{5!} + \frac{u^7}{7!} + ...$$

$$= -[z-(\pi/2)] + \frac{[z-(\pi/2)]^3}{3!} - \frac{[z-(\pi/2)]^5}{5!} + ...$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{[z-(\pi/2)]^{2n+1}}{(2n+1)!}$$

Example 3 : Show that $\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty} \frac{(n+1)!}{n!} \cdot (z+1)^n$, where $|z+1| < 1$. **(M.U. 1991)**

Sol. : By Taylor's series,

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + ...$$

Here $f(z) = \frac{1}{z^2}$ and $a = -1$ $\quad \therefore f(-1) = 1$

$$\therefore \quad f'(z) = -\frac{2}{z^3} \quad \therefore f'(-1) = 2! \; ; \quad f''(z) = \frac{2 \cdot 3}{z^4} \quad \therefore f''(-1) = 3!$$

$$f'''(z) = -\frac{2 \cdot 3 \cdot 4}{z^5} \quad \therefore f'''(-1) = 4! \text{ and so on.}$$

$$\therefore \quad \frac{1}{z^2} = 1 + (z+1) \cdot 2! + \frac{(z+1)^2}{2!} \cdot 3! + \frac{(z+1)^3}{3!} \cdot 4! + ...$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(n+1)!}{n!} \cdot (z+1)^n$$

Aliter : $\frac{1}{z^2} = \frac{1}{[1-(z+1)]^2} = [1-(z+1)]^{-2}$ where $|z+1| < 1$

$$= 1 + 2(z+1) + 3(z+1)^2 + 4(z+1)^3 + ...$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(n+1)!}{n!} (z+1)^n$$

Example 4 : Show that for every finite value of z, $e^z = e + e\,\Sigma \frac{(z-1)^n}{n!}$.

(M.U. 1993, 94)

Sol. : By Taylor's series

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + ...$$

Here $f(z) = e^z$ and the r.h.s. suggests that $a = 1$.

Also $f'(z) = f''(z) = f'''(z) = ... = f^n(z) = e^z$

$$\therefore \quad f'(1) = f''(1) = f'''(1) = ... = f^n(1) = e^1$$

$$\therefore \quad e^z = e + (z-1) \cdot e + \frac{(z-1)^2}{2!} \cdot e + \frac{(z-1)^3}{3!} \cdot e + ...$$

$$= e + e\,\Sigma \frac{(z-1)^n}{n!}.$$

Example 5 : Obtain Taylor's expansion of $f(z) = \frac{1-z}{z^2}$ in powers of $(z-1)$.

Sol. : $f(z)$ is not analytic at $z = 0$. However if we consider the region $0 < |z-1| < 1$, the function is analytic in the region.

Now $f(z) = \frac{1-z}{z^2} = \frac{1}{z^2} - \frac{1}{z} \quad \therefore f(1) = 0$

$$\therefore \quad f^n(z) = (-1)^n \cdot \frac{(n+1)!}{z^{n+2}} - (-1)^n \cdot \frac{n!}{z^{n+1}}$$

$$\therefore \quad f^n(1) = (-1)^n \cdot (n+1)! - (-1)^n \cdot n!$$

$$= (-1)^n \cdot n! \, [(n+1) - 1] = (-1)^n \cdot n!\, n$$

By Taylor's series,

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + ...$$

$$\therefore \quad f(z) = \sum_{n=1}^{\infty} (z-1)^n (-1)^n \cdot \frac{n!\, n}{n!} = \sum_{n=1}^{\infty} (-1)^n \cdot n \cdot (z-1)^n$$

Alternatively we can obtain the above expansion by using Binomial theorem as follows.

$$f(z) = -\frac{(z-1)}{(z-1+1)^2}$$

Now put $z - 1 = u$

$$\therefore \quad f(z) = -\frac{u}{(u+1)^2} = -u[1+u]^{-2}$$

$$= -u\left[1 + (-2)u + \frac{(-2)(-3)}{2!} u^2 + \frac{(-2)(-3)(-4)}{3!} u^3 + ...\right]$$

$$= -u\left[1 - 2u + 3u^2 - 4u^3 + ...\right]$$

$$= -u + 2u^2 + 3u^3 - 4u^4 + ...$$

$$= \sum_{n=1}^{\infty} (-1)^n \cdot n \cdot u^n = \sum_{n=1}^{\infty} (-1)^n \cdot n \cdot (z-1)^n.$$

Example 6 : Obtain Taylor's expansion of $f(z) = \frac{z+2}{(z-1)(z-4)}$ at $z = 2$.

Sol. : Let us put $z - 2 = u$.

$$\therefore \quad f(z) = \frac{u+4}{(u+1)(u-2)} = -\frac{1}{u+1} + \frac{2}{u-2}$$

$$= -\frac{1}{(1+u)} - \frac{1}{2[1-(u/2)]} = -(1+u)^{-1} - \frac{1}{2}\left(1 - \frac{u}{2}\right)^{-1}$$

$$= -(1 - u + u^2 - u^3 + ...) - \frac{1}{2}\left(1 + \frac{u}{2} + \frac{u^2}{2^2} + \frac{u^3}{2^3} + ...\right)$$

$$= -\Sigma(-1)^n \cdot u^n - \frac{1}{2}\Sigma \frac{1}{2^n} \cdot u^n = -\Sigma\left[(-1)^n + \frac{1}{2^{n+1}}\right] u^n$$

$$= -\Sigma\left[(-1)^n + \frac{1}{2^{n+1}}\right](z-2)^n$$

The region of convergence is $|z-2| < 1$ and the singularities $z = 1$, $z = 4$ lie outside the region of convergence.

Example 7 : Expand the function $f(z) = \frac{\sin z}{z - \pi}$ about $z = \pi$. (M.U. 2003, 05)

Sol. : $f(z)$ is not analytic at $z = \pi$. Hence, we put $z - \pi = u$ i.e. $z = \pi + u$.

$$\therefore \quad f(z) = \frac{\sin(\pi + u)}{u} = -\frac{\sin u}{u} = -\frac{1}{u}\left[u - \frac{u^3}{3!} + \frac{u^4}{5!} - .....\right]$$

$$\therefore \quad f(z) = -1 + \frac{u^2}{3!} - \frac{u^4}{5!} + \ldots\ldots = -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!} + \ldots\ldots$$

Example 8 : Expand e^{3z} about $z = 3i$.

Sol. :

$$e^{3z} = e^{3(z-3i)+9i} = e^{9i} \cdot e^{3(z-3i)}$$

$$= e^{9i}\left[1 + \frac{3(z-3i)}{1!} + \frac{3^2(z-3i)^2}{2!} + \frac{3^3(z-3i)^3}{3!} + \ldots\ldots\right]$$

Example 9 : Obtain Taylor's series for $f(z) = \dfrac{2z^3+1}{z(z+1)}$ about $z = i$.

Sol. : By actual division and partial fractions.

$$f(z) = 2z - 2 + \frac{1}{z} + \frac{1}{z+1}$$

Since, we want Taylor's series about $z = i$, we have to express $f(z)$ in positive powers of $(z - i)$.

$$\therefore \quad f(z) = 2(z-i) + (2i-2) + \frac{1}{(z-i)+i} + \frac{1}{(z-i)+(1+i)}$$

$$= 2(z-i) + 2(i-1) + \frac{1}{i\left[1+\frac{(z-i)}{i}\right]} + \frac{1}{(1+i)\left[1+\frac{(z-i)}{(1+i)}\right]}$$

$$= 2(z-i) + 2(i-1) + \frac{1}{i}\left[1+\frac{(z-i)}{i}\right]^{-1} + \frac{1}{(1+i)}\left[1+\frac{(z-i)}{(1+i)}\right]^{-1}$$

$$= 2(z-i) + 2(i-1) + \frac{1}{i}\left[1-\frac{(z-i)}{i}+\frac{(z-i)^2}{i^2}-\ldots\ldots\right]$$

$$+ \frac{1}{(1+i)}\left[1-\frac{(z-i)}{(1+i)}+\frac{(z-i)^2}{(1+i)^2}-\ldots\ldots\right]$$

$$= 2(z-i) + 2(i-1) + (-1)^n \sum\left(\frac{1}{i^{n+1}}+\frac{1}{(1+i)^{n+1}}\right)(z-i)^n$$

(B) Laurent's Series Expansion

Example 1 : Find the Laurent's series for $f(z) = z^3 e^{1/z}$ about $z = 0$. (M.U. 2006)

Sol. : $f(z)$ is not analytic at $z = 0$. Hence, for $|z| > 0$.

$$f(z) = z^3\left[1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{3!}\cdot\frac{1}{z^3} + \ldots\ldots\right]$$

$$= z^3 + z^2 + \frac{1}{2!}\cdot z + \frac{1}{3!} + \frac{1}{4!}\cdot\frac{1}{z} + \frac{1}{5!}\cdot\frac{1}{z^2} + \ldots\ldots$$

Example 2 : Find Laurent's series for $f(z) = \dfrac{e^{3z}}{(z-1)^3}$ about $z = 1$. (M.U. 2004, 05)

Sol. : $f(z)$ is not analytic at $z = 1$. Hence, for $|z - 1| > 0$.

$$f(z) = \frac{e^{3(z-1)}\cdot e^3}{(z-1)^3} = \frac{e^3}{(z-1)^3}\left[e^{3(z-1)}\right]$$

$$= \frac{e^3}{(z-1)^3}\left[1 + 3(z-1) + \frac{3^2(z-1)^2}{2!} + \frac{3^3(z-1)^3}{3!} + \frac{3^4(z-1)^4}{4!} + \ldots\ldots\right]$$

$$= e^3\left[\frac{1}{(z-1)^3} + \frac{3}{(z-1)^2} + \frac{3^2}{2!(z-1)} + \frac{3^3}{3!} + \frac{3^4}{4!}\cdot\frac{1}{(z-1)} + \ldots\ldots\right]$$

Example 3 : Find Laurent's series for $f(z) = (z-3)\sin\left(\dfrac{1}{z+2}\right)$ about $z = -2$. (M.U. 2000)

Sol. : $f(z)$ is not analytic at $z = -2$. Hence, put $z + 2 = u$.

$$\therefore \quad f(z) = (u-5)\sin\frac{1}{u}$$

But $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \ldots\ldots$

$$\therefore \quad f(z) = (u-5)\left[\frac{1}{u} - \frac{1}{3!}\cdot\frac{1}{u^3} + \frac{1}{5!}\cdot\frac{1}{u^5} - \ldots\ldots\right]$$

$$= \left(1 - \frac{5}{u}\right) - \frac{1}{3!}\cdot\frac{1}{u^2} + \frac{5}{3!}\cdot\frac{1}{u^3} + \ldots\ldots$$

$$= 1 - \frac{5}{(z+2)} - \frac{1}{3!}\cdot\frac{1}{(z+2)^2} + \frac{5}{3!}\cdot\frac{1}{(z+2)^3} + \ldots\ldots$$

Example 4 : Expand the unction $f(z) = \dfrac{1}{z^2 \sin hz}$ about $z = 0$. (M.U. 2000)

Sol. : We have

$$f(z) = \frac{1}{z^2 \sin hz} = \frac{1}{z^2\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\ldots\right]} = \frac{1}{z^3\left[1 + \frac{z^2}{6} + \frac{z^4}{120} + \ldots\ldots\right]}$$

$$= \frac{1}{z^3}\left[1 + \left(\frac{z^2}{6} + \frac{z^4}{120} + \ldots\ldots\right)\right]^{-1}$$

$$= \frac{1}{z^3}\left[1 - \left(\frac{z^2}{6} + \frac{z^4}{120} + \ldots\ldots\right) + \left(\frac{z^2}{6} + \frac{z^4}{120} + \ldots\ldots\right)^2 + \ldots\ldots\right]$$

$$= \frac{1}{z^3}\left[1 - \frac{z^2}{6} - \frac{z^4}{120} + \frac{z^4}{36} + \ldots\ldots\right] = \frac{1}{z^3} - \frac{1}{6z} + \frac{7}{360}z + \ldots\ldots$$

Example 5 : Find all possible Laurent's expansions of the function

$$f(z) = \frac{2 - z^2}{z(1-z)(2-z)}$$

about $z = 0$ indicating the region of convergence in each case. **(M.U. 2006)**

Sol. : Let $f(z) = \frac{a}{z} + \frac{b}{1-z} + \frac{c}{2-z}$

$$\therefore \quad 2 - z^2 = a(1-z)(2-z) + bz(2-z) + cz(1-z)$$

When $z = 0$, $\quad 2 = 2a \qquad \therefore\ a = 1$

When $z = 1$, $\quad 1 = b \qquad \therefore\ b = 1$

When $z = 2$, $\quad -2 = -2c \qquad \therefore\ c = 1$

$$\therefore \quad \frac{2-z^2}{z(1-z)(2-z)} = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

Clearly $f(z)$ is not analytic at $z = 0$, $z = 1$ and $z = 2$.

Hence, we consider the regions (i) $0 < |z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$.

In these regions $f(z)$ is analytic and can be expanded by Laurent's series.

Hence, we consider the following three cases.

Note ....

We note that the series $\frac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + ...$ .... (A)

is convergent only if $|z| < 1$. Hence, in $\frac{1}{a-z}$, we take a common if $a > z$ and write

$$\frac{1}{a-z} = \frac{1}{a[1-(z/a)]} = \frac{1}{a}\cdot\left(1 - \frac{z}{a}\right)^{-1}$$

In this case now $\left|\frac{z}{a}\right| < 1$ and the expansion (A) can be used.

In the same way in $\frac{1}{a-z}$, we take z common if $z > a$ and write

$$\frac{1}{a-z} = \frac{1}{z[(a/z)-1]} = -\frac{1}{z[1-(a/z)]} = -\frac{1}{z}\left(1 - \frac{a}{z}\right)^{-1}$$

In this case now $\left|\frac{a}{z}\right| < 1$ and the expansion (A) can be used.

Case (i) : $f(z) = \frac{1}{z} + (1-z)^{-1} + \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$

$$= \frac{1}{z} + (1 + z + z^2 + z^3 + ...) + \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + ...\right)$$

$$= \frac{1}{z} + \left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{2^2}\right)z + \left(1 + \frac{1}{2^3}\right)z^2 + \left(1 + \frac{1}{2^4}\right)z^3 + ...$$

$$\therefore \quad f(z) = \frac{1}{z} + \left(1 + \frac{1}{2^{n+1}}\right)z^n$$

The series is convergent when $|z| < 1$. But this includes the point $z = 0$, which is a singularity of $f(z)$. Hence, the region of convergence of the series is $0 < |z| < 1$.

This is the unit circle without the centre $z = 0$.

Case (ii) : $f(z) = \frac{1}{z} - \frac{1}{z[1-(1/z)]} + \frac{1}{2[1-(z/2)]}$

$$= \frac{1}{z} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$

$$= \frac{1}{z} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...\right) + \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + ...\right)$$

$$= \frac{1}{z} - \Sigma\frac{1}{z^{n+1}} + \Sigma\frac{z}{2^{n+1}}$$

The series is convergent if $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$ *i.e.* $1 < |z|$ and $|z| < 2$ *i.e.* $1 < |z| < 2$. Hence, the region of convergence of the series is $1 < |z| < 2$.

2 and this excludes $z = 0$. Hence, the region of convergence of the series is $1 < |z| < 2$.

This is the annular region between two circles of radii $r = 1$ and $r = 2$ with centre at the origin.

Case (iii) : $f(z) = \frac{1}{z} - \frac{1}{z[1-(1/z)]} - \frac{1}{z[1-(2/z)]} = \frac{1}{z} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1}$

$$= \frac{1}{z} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...\right) - \frac{1}{z}\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + ...\right)$$

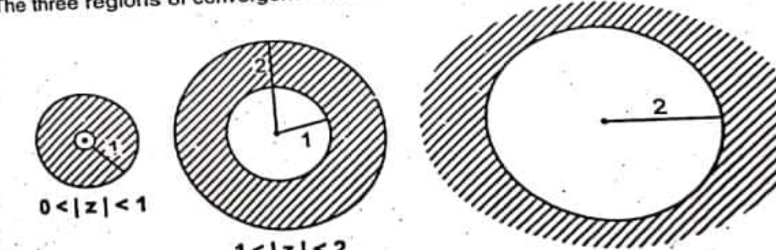
$$= \frac{1}{z} - \Sigma\frac{1}{z^{n+1}} - \Sigma\frac{2^n}{z^{n+1}}.$$

The series is convergent if $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$ *i.e.* $1 < |z|$ and $2 < |z|$ *i.e.* if $|z| > 2$.

Hence, the region of convergence of the series is $2 < |z| < \infty$.

This is the exterior of the circle $|z| = 2$.

The three regions of convergence are shown below.



$0 < |z| < 1$ $\qquad$ $1 < |z| < 2$ $\qquad$ $2 < |z| < \infty$

Fig. 3.3

Example 6 : Expand $f(z) = \dfrac{1}{z(z+1)(z-2)}$

(i) within the unit circle about the origin, (ii) within the annulus region between the concentric circles about the origin having radii 1 and 2 respectively, (iii) in the exterior of the circle with centre at the origin and radius 2. **(M.U. 1998, 2000, 01, 17)**

Sol. : Let $f(z) = \dfrac{1}{z(z+1)(z-2)} = \dfrac{a}{z} + \dfrac{b}{z+1} + \dfrac{c}{z-2}$

$$\therefore \quad 1 = a(z+1)(z-2) + bz(z-2) + cz(z+1)$$

When $z = 0$, $\quad 1 = -2a \quad \therefore \; a = -1/2$

When $z = -1$, $\quad 1 = 3b \quad \therefore \; b = 1/3$

When $z = 2$, $\quad 1 = 6c \quad \therefore \; c = 1/6.$

$$\therefore \quad f(z) = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)}$$

Case (i) : When $0 < |z| < 1$, we write

$$f(z) = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)} \quad \text{as}$$

$$f(z) = -\frac{1}{2z} + \frac{1}{3(1+z)} - \frac{1}{12[1-(z/2)]}$$

When $|z| < 1$, clearly $|z| < 2$

$$\therefore \quad f(z) = -\frac{1}{2z} + \frac{1}{3}(1+z)^{-1} - \frac{1}{12}[1-(z/2)]^{-1}$$

$$= -\frac{1}{2z} + \frac{1}{3}[1 - z + z^2 - z^3 + z^4 - .....] - \frac{1}{12}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + ...\right]$$

Case (ii) : When $1 < |z| < 2$, we write

$$f(z) = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)} \quad \text{as}$$

$$= -\frac{1}{2z} + \frac{1}{3z[1+(1/z)]} - \frac{1}{12[1-(z/2)]}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left(1 + \frac{1}{z}\right)^{-1} - \frac{1}{12}\left(1 - \frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left[1 - \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + ...\right]$$

$$- \frac{1}{12}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + ...\right]$$

Case (iii) : When $|z| > 2$, we write

$$f(z) = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)} \quad \text{as}$$

$$f(z) = -\frac{1}{2z} + \frac{1}{3z[1+(1/z)]} + \frac{1}{6z[1-(2/z)]}$$

When $|z| > 2$, clearly $|z| > 1$

$$\therefore \quad f(z) = -\frac{1}{2z} + \frac{1}{3z}\left[1 + \left(\frac{1}{z}\right)\right]^{-1} + \frac{1}{6z}\left[1 - \left(\frac{2}{z}\right)\right]^{-1}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left[1 - \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + ...\right]$$

$$+ \frac{1}{6z}\left[1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + ...\right]$$

The regions of convergence are shown on the page 3-12.

Example 7 : Find the Laurent's series for

$$f(z) = \frac{4z+3}{z(z-3)(z+2)} \quad \text{valid for } 2 < |z| < 3.$$

(M.U. 1990, 2011)

Sol. : Let $f(z) = \dfrac{a}{z} + \dfrac{b}{z-3} + \dfrac{c}{z+2}$

$$\therefore \quad 4z + 3 = a(z-3)(z+2) + bz(z+2) + cz(z-3)$$

When $z = 0$, $\quad 3 = -6a \quad \therefore \; a = -1/2$

When $z = -2$, $\quad -5 = 10c \quad \therefore \; c = -1/2$

When $z = 3$, $\quad 15 = 15b \quad \therefore \; b = 1$

$$\therefore \quad \frac{4z+3}{z(z-3)(z+2)} = -\frac{1}{2z} + \frac{1}{z-3} - \frac{1}{2(z+2)}$$

When $2 < |z| < 3$, we write

$$f(z) = -\frac{1}{2z} + \frac{1}{z-3} - \frac{1}{2(z+2)} \quad \text{as}$$

$$= -\frac{1}{2z} - \frac{1}{3[1-(z/3)]} - \frac{1}{2z[1+(2/z)]}$$

$$= -\frac{1}{2z} - \frac{1}{3}\left[1 - \left(\frac{z}{3}\right)\right]^{-1} - \frac{1}{2z}\left[1 + \left(\frac{2}{z}\right)\right]^{-1}$$

$$= -\frac{1}{2z} - \frac{1}{3}\left[1 + \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + ...\right]$$

$$- \frac{1}{2z}\left[1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + ...\right]$$

Example 8 : Find Laurent's series which represents the function $f(z) = \dfrac{2}{(z-1)(z-2)}$

when (i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$. **(M.U. 1990, 2003, 09, 10, 14)**

Sol. : Let $\frac{2}{(z-1)(z-2)} = \frac{a}{z-1} + \frac{b}{z-2}$ $\quad \therefore\ 2 = a(z-2) + b(z-1)$

When $z = 1$, $\ 2 = -a$ $\quad \therefore\ a = -2$

When $z = 2$, $\ 2 = b$

$$\therefore \frac{2}{(z-1)(z-2)} = \frac{-2}{z-1} + \frac{2}{z-2}$$

Case (i) : When $|z| < 1$, clearly $|z| < 2$

$$\therefore f(z) = \frac{2}{1-z} - \frac{2}{2[1-(z/2)]} = 2[1-z]^{-1} - [1-(z/2)]^{-1}$$

$$\therefore f(z) = 2[1+z+z^2+z^3+.....] - \left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+.....\right]$$

Case (ii) : When $1 < |z| < 2$, we write

$$\frac{2}{(z-1)(z-2)} = -\frac{2}{(z-1)} + \frac{2}{(z-2)} \text{ as}$$

$$= -\frac{2}{z[1-(1/z)]} - \frac{2}{2[1-(z/2)]}$$

$$= -\frac{2}{z}[1-(1/z)]^{-1} - [1-(z/2)]^{-1}$$

$$= -\frac{2}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+.....\right] - \left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+.....\right]$$

Case (iii) : When $|z| > 2$, $\frac{|z|}{2} > 1$ *i.e.* $\frac{2}{|z|} < 1$.

Also when $|z| > 2$, $|z| > 1$ $\therefore$ $\frac{1}{|z|} < 1$. We write

$$f(z) = -\frac{2}{z}\cdot\frac{1}{[1-(1/z)]} + \frac{2}{z}\cdot\frac{1}{[1-(2/z)]}$$

$$= -\frac{2}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{2}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+.....\right) + \frac{2}{z}\left(\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+.....\right)$$

$$= -2\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+.....\right) + 4\left(\frac{1}{z^2}+\frac{2}{z^3}+\frac{4}{z^4}+.....\right)$$

Example 9 : Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in the regions

(i) $1 < |z-1| < 1$, (ii) $1 < |z-3| < 2$, (iii) $|z| < 1$. **(M.U. 2002, 03)**

Sol. : Let $f(z) = \frac{a}{z-1} + \frac{b}{z-2}$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{a(z-2)+b(z-1)}{(z-1)(z-2)}$$

$\therefore\ 1 = a(z-2) + b(z-1)$

Putting $z = 1$, we get $1 = a(1-2)$ $\quad \therefore\ a = -1$.

Putting $z = 2$, we get $1 = b(2-1)$ $\quad \therefore\ b = 1$.

$$\therefore f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

Case (i) : When $1 < |z-1| < 2$, we write

$$f(z) = -\frac{1}{z-1} + \frac{1}{(z-1)-1} = -\frac{1}{z-1} - \frac{1}{1-(z-1)} = -\frac{1}{z-1} - [1-(z-1)]^{-1}$$

$$= -\frac{1}{z-1} \cdot [1+(z-1)+(z-1)^2+(z-1)^3+.....]$$

Case (ii) : When $1 < |z-3| < 2$, we write

$$f(z) = -\frac{1}{(z-3)+2} + \frac{1}{(z-3)+1}$$

$$= -\frac{1}{2}\cdot\frac{1}{1+[(z-3)/2]} + \frac{1}{(z-3)}\cdot\frac{1}{1+[1/(z-3)]}$$

$$= -\frac{1}{2}\left[1+\left(\frac{z-3}{2}\right)\right]^{-1} + \frac{1}{(z-3)}\left[1+\frac{1}{(z-3)}\right]^{-1}$$

$$= -\frac{1}{2}\left[1-\left(\frac{z-3}{2}\right)+\left(\frac{z-3}{2}\right)^2-\left(\frac{z-3}{2}\right)^3+.....\right]$$

$$+ \frac{1}{(z-3)}\left[1-\frac{1}{z-3}+\frac{1}{(z-3)^2}-\frac{1}{(z-3)^3}+.....\right]$$

Case (iii) : When $|z| < 1$, clearly $|z| < 2$. Hence, we write

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2} = \frac{1}{1-z} - \frac{1}{2-z}$$

$$= \frac{1}{1-z} - \frac{1}{2}\cdot\frac{1}{[1-(z/2)]} = [1-z]^{-1} - \frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$$

$$= 1+z+\frac{z^2}{2!}+.....-\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2!\cdot 4}+.....\right]$$

Example 10 : Obtain Taylor's or Laurent's series for the function

$f(z) = \frac{1}{(1+z^2)(z+2)}$ for (i) $1 < |z| < 2$ and (ii) $|z| > 2$. **(M.U. 2003, 14)**

Sol. : Let $f(z) = \frac{a}{z+2} + \frac{bz+c}{z^2+1}$ $\quad \therefore \frac{1}{(z^2+1)(z+2)} = \frac{a(z^2+1)+(bz+c)(z+2)}{(z^2+1)(z+2)}$

$\therefore\ 1 = (a+b)z^2 + (2b+c)z + (a+2c)$

Equating the coefficients of like powers of z, on both sides

$$a+b=0,\ 2b+c=0 \ \therefore\ a+2c=1.$$

$$\therefore\ 2c-b=1 \text{ i.e. } 4c-2b=2 \text{ and } 2b+c=0.$$

$$\therefore\ 5c=2 \text{ i.e. } c=2/5,\ \therefore\ a=1-2c=1-(4/5)=1/5$$

$$\therefore\ b=-a=-1/5$$

$$\therefore\ f(z)=\frac{1}{5}\cdot\frac{1}{z+2}-\frac{z-2}{5(z^2+1)}=\frac{1}{5}\left[\frac{1}{z+2}-\frac{z-2}{z^2+1}\right]$$

Case (i) : When $1<|z|<2$, $|z|<2$ i.e. $\frac{|z|}{2}<1$ and $|z^2|>1$ i.e. $\frac{1}{|z^2|}<1$.

Hence, we write,

$$f(z)=\frac{1}{5}\cdot\frac{1}{z+2}-\frac{1}{5}\cdot\frac{z-2}{z^2+1}$$

$$=\frac{1}{5}\cdot\frac{1}{2}\cdot\frac{1}{1+(z/2)}-\frac{z-2}{5}\cdot\frac{1}{z^2}\cdot\frac{1}{1+(1/z^2)}$$

$$=\frac{1}{10}\left(1+\frac{z}{2}\right)^{-1}-\left(\frac{z-2}{5}\right)\cdot\left(1+\frac{1}{z^2}\right)^{-1}$$

$$=\frac{1}{10}\left[1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\ldots..\right]-\left(\frac{z-2}{5}\right)\left[1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\ldots..\right]$$

Case (ii) : When $|z|>2$, $\frac{|z|}{2}>1$ i.e. $\frac{2}{|z|}<1$.

Also when $|z|>2$, $|z|>1$ i.e. $|z^2|>1$ $\therefore\ \frac{1}{|z^2|}<1$.

Hence, we write,

$$f(z)=\frac{1}{5}\cdot\frac{1}{z}\cdot\frac{1}{1+(2/z)}-\frac{(z-2)}{5}\cdot\frac{1}{z^2}\cdot\frac{1}{1+(1/z^2)}$$

$$=\frac{1}{5z}\left[1+\frac{2}{z}\right]^{-1}-\frac{(z-2)}{5\cdot z^2}\cdot\left[1+\frac{1}{z^2}\right]^{-1}$$

$$=\frac{1}{5z}\left[1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+\ldots..\right]-\frac{(z-2)}{5\cdot z^2}\left[1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\ldots..\right]$$

Example 11 : Obtain Taylor's and Laurent's expansions of $f(z)=\frac{z-1}{z^2-2z-3}$ indicating regions of convergence. **(M.U. 1996, 2004, 10, 14, 16)**

Sol : Let $f(z)=\frac{z-1}{(z+1)(z-3)}=\frac{a}{z+1}+\frac{b}{z-3}$ $\therefore\ z-1=a(z-3)+b(z+1)$

Putting $z=-1$, $\quad -2=-4a \quad \therefore\ a=1/2$

Putting $z=3$, $\quad 2=b\cdot 4 \quad \therefore\ b=1/2$

$$\therefore\ \frac{z-1}{(z+1)(z-3)}=\frac{1/2}{z+1}+\frac{1/2}{z-3}$$

Hence, $f(z)$ is not analytic at $z=-1$ and $z=3$.

$\therefore$ $f(z)$ is analytic in (i) $|z|<1$, (ii) $1<|z|<3$, (iii) $|z|>3$.

Case (i) : When $|z|<1$, we also get $|z|<3$.

$$\therefore\ f(z)=\frac{1/2}{z+1}+\frac{1/2}{z-3}=\frac{1}{2}\cdot\frac{1}{1+z}+\frac{1}{2}\cdot\frac{1}{(-3)}\cdot\frac{1}{1-(z/3)}$$

$$=\frac{1}{2}\cdot[1+z]^{-1}-\frac{1}{6}\left(1-\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{2}\left[1-z+z^2-z^3+\ldots..\right]-\frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\ldots..\right]$$

$$=\frac{1}{3}-\frac{5}{9}z+\frac{13}{27}z^2+\ldots..$$

This is the required Taylor's series.

Case (ii) : When $1<|z|<3$, we get $\left|\frac{1}{z}\right|<1$ and $\left|\frac{z}{3}\right|<1$.

$$\therefore\ f(z)=\frac{1}{2}\cdot\frac{1}{1+z}+\frac{1}{2}\cdot\frac{1}{z-3}$$

$$=\frac{1}{2\cdot z}\cdot\frac{1}{1+(1/z)}+\frac{1}{2}\cdot\frac{1}{(-3)}\cdot\frac{1}{1-(z/3)}$$

$$=\frac{1}{2z}\cdot\left[1+\frac{1}{z}\right]^{-1}-\frac{1}{6}\left[1-\frac{z}{3}\right]^{-1}$$

$$=\frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\ldots..\right]-\frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\ldots..\right]$$

$$=\frac{1}{2}\left[\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\ldots..\right]-\frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\ldots..\right]$$

This is the required Laurent's series.

Case (iii) : When $|z|>3$, clearly, $|z|>1$.

$$\therefore\ \frac{|z|}{3}>1 \text{ and } \frac{|z|}{1}>1 \quad \therefore\ \frac{3}{|z|}<1 \text{ and } \frac{1}{|z|}<1$$

$$\therefore\ f(z)=\frac{1}{2}\cdot\frac{1}{z+1}+\frac{1}{2}\cdot\frac{1}{z-3}$$

$$=\frac{1}{2z}\cdot\frac{1}{1+(1/z)}+\frac{1}{2\cdot z}\cdot\frac{1}{1-(3/z)}$$

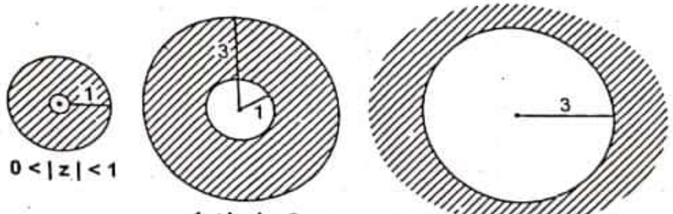
$$=\frac{1}{2z}\cdot\left[1+\frac{1}{z}\right]^{-1}+\frac{1}{2z}\cdot\left[1-\frac{3}{z}\right]^{-1}$$

$$=\frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\ldots..\right]+\frac{1}{2z}\left[1+\frac{3}{z}+\frac{9}{z^2}+\frac{27}{z^3}+\ldots..\right]$$

$$=\frac{1}{2z}\left[2+\frac{2}{z}+\frac{10}{z^2}+\frac{26}{z^3}+\ldots..\right]=\frac{1}{z}+\frac{1}{z^2}+\frac{5}{z^3}+\frac{13}{z^4}+\ldots..$$

This is the required Laurent's series.

The regions of convergence are as below.



$0 < |z| < 1$ $\quad$ $1 < |z| < 3$ $\quad$ $|z| > 3$

Fig. 3.4

Example 12 : Expand $f(z) = \dfrac{1}{z^2(z-1)(z+2)}$ about $z = 0$ for

(i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$. **(M.U. 1990, 91, 97, 2015)**

Sol. : Let $f(z) = \dfrac{a}{z} + \dfrac{b}{z^2} + \dfrac{c}{z-1} + \dfrac{d}{z+2}$

$\therefore\ 1 = az(z-1)(z+2) + b(z-1)(z+2) + cz^2(z+2) + dz^2(z-1)$

When $z = 0$, $\quad 1 = -2b \quad \therefore\ b = -1/2$

When $z = 1$, $\quad 1 = 3c \quad \therefore\ c = 1/3$

When $z = -2$, $\quad 1 = -12c \quad \therefore\ d = -1/12$

Equating powers of z^3,

$$0 = a + c + d \quad \therefore\ a = -\frac{1}{3} + \frac{1}{12} = -\frac{3}{12} = -\frac{1}{4}$$

$$\therefore\ f(z) = -\frac{1}{4}\cdot\frac{1}{z} - \frac{1}{2}\cdot\frac{1}{z^2} + \frac{1}{3}\cdot\frac{1}{z-1} - \frac{1}{12}\cdot\frac{1}{z+2} \qquad \text{.................... (1)}$$

Case (i) : When $0 < |z| < 1$,

$$\frac{1}{z-1} = -\frac{1}{1-z} = -(1-z)^{-1} = -(1 + z + z^2 + z^3 + ...)$$

$$\frac{1}{z+2} = \frac{1}{2[1+(z/2)]} = \frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} = \frac{1}{2}\left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + ...\right)$$

Hence, from (1), we get,

$$f(z) = -\frac{1}{4}\cdot\frac{1}{z} - \frac{1}{2}\cdot\frac{1}{z^2} - \frac{1}{3}(1 + z + z^2 + z^3 + ...) - \frac{1}{24}\left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + ...\right)$$

Case (ii) : When $1 < |z| < 2$

$$\frac{1}{z-1} = \frac{1}{z[1-(1/z)]} = \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...\right)$$

$$\frac{1}{z+2} = \frac{1}{2[1+(z/2)]} = \frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} = \frac{1}{2}\left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + ...\right)$$

Hence, from (1), we get

$$f(z) = -\frac{1}{4}\cdot\frac{1}{z} - \frac{1}{2}\cdot\frac{1}{z^2} + \frac{1}{3z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...\right) - \frac{1}{24}\left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + ...\right)$$

Case (iii) : When $|z| > 2$

When $|z| > 2$ clearly $|z| > 1$, and we get

$$\frac{1}{z-1} = \frac{1}{z[1-(1/z)]} = \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...\right)$$

And $$\frac{1}{z+2} = \frac{1}{z[1+(2/z)]} = \frac{1}{z}\left(1+\frac{2}{z}\right)^{-1} = \frac{1}{z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + ...\right)$$

Hence, from (1), we get

$$f(z) = -\frac{1}{4}\cdot\frac{1}{z} - \frac{1}{2}\cdot\frac{1}{z^2} + \frac{1}{3z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...\right) - \frac{1}{12z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + ...\right)$$

(See figures given on page 3-12.)

Example 13 : Expand $\dfrac{z^2-1}{z^2+5z+6}$ around $z = 0$. **(M.U. 1991, 93, 2001)**

Sol. : Since the degree of the numerator is equal to the degree of the denominator we first **divide the numerator by the denominator.**

$$\therefore\ f(z) = \frac{z^2-1}{z^2+5z+6} = 1 - \frac{5z+7}{z^2+5z+6}$$

Let $\dfrac{-5z-7}{z^2+5z+6} = \dfrac{a}{z+3} + \dfrac{b}{z+2} \qquad \therefore\ -5z-7 = a(z+2) + b(z+3)$

When $z = -2$, $\quad b = 3$; $\quad$ When $z = -3$, $\quad a = -8$.

$$\therefore\ f(z) = \frac{z^2-1}{z^2+5z+6} = 1 - \frac{8}{z+3} + \frac{3}{z+2} \qquad \text{.................... (1)}$$

Case (i) : When $|z| < 2$, we write

$$f(z) = 1 - \frac{8}{3[1+(z/3)]} + \frac{3}{2[1+(z/2)]}$$

When $|z| < 2$, clearly $|z| < 3$

$$\therefore\ f(z) = 1 - \frac{8}{3}[1+(z/3)]^{-1} + \frac{3}{2}[1+(z/2)]^{-1}$$

$$= 1 - \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - .....\right] + \frac{3}{2}\left[1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + ...\right]$$

Case (ii) : When $2 < |z| < 3$, we write

$$f(z) = 1 - \frac{8}{3[1+(z/3)]} + \frac{3}{z[1+(2/z)]} = 1 - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1} + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= 1 - \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - .....\right] + \frac{3}{z}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - .....\right]$$

Case (iii) : When $|z| > 3$, we write

$$f(z) = 1 - \frac{8}{z[1+(3/z)]} + \frac{3}{z[1+(2/z)]}$$

When $|z| > 3$, clearly $|z| > 2$

$$\therefore \quad f(z) = 1 - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1} + \frac{3}{2}\left(1+\frac{2}{z}\right)^{-1}$$

$$= 1 - \frac{8}{z}\left[1 - \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \ldots\right] + \frac{3}{z}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \ldots\right]$$

Example 14 : Expand $f(z) = \dfrac{z^2-1}{z^2+5z+6}$ around $z = 1$. **(M.U. 1993, 2016)**

Sol. : As in the previous example dividing numerator by denominator

$$f(z) = 1 - \frac{8}{z+3} + \frac{3}{z+2}$$

Since, we want expansion around $z = 1$, we have to obtain Laurent's series in powers of $(z-1)$.

$$\therefore \quad \frac{z^2-1}{z^2+5z+6} = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3}$$

There now arise three cases.

Case (i) : When $|z-1| < 3$, we write

$$f(z) = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3} \quad \text{as}$$

$$f(z) = 1 - \frac{8}{4[1+(z-1)/4]} + \frac{3}{3[1+(z-1)/3]}$$

(When $|z-1| < 3$ clearly $|z-1| < 4$.)

$$\therefore \quad f(z) = 1 - \frac{8}{4}\left[1+\left(\frac{z-1}{4}\right)\right]^{-1} + \frac{3}{3}\left[1+\left(\frac{z-1}{3}\right)\right]^{-1}$$

$$\therefore \quad f(z) = 1 - 2\left[1 - \left(\frac{z-1}{4}\right) + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \ldots\right]$$
$$+ \left[1 - \left(\frac{z-1}{3}\right) + \left(\frac{z-1}{3}\right)^2 - \left(\frac{z-1}{3}\right)^3 + \ldots\right]$$

Case (ii) : When $3 < |z-1| < 4$, we write

$$f(z) = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3} \quad \text{as}$$

$$= 1 - \frac{8}{4[1+(z-1)/4]} + \frac{3}{(z-1)[1+3/(z-1)]}$$

$$= 1 - 2\left[1+\left(\frac{z-1}{4}\right)\right]^{-1} + \frac{3}{(z-1)}\left[1+\left(\frac{3}{z-1}\right)\right]^{-1}$$

$$\therefore \quad f(z) = 1 - 2\left[1 - \left(\frac{z-1}{4}\right) + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \ldots\right]$$
$$+ \frac{3}{(z-1)}\left[1 - \left(\frac{3}{z-1}\right) + \left(\frac{3}{z-1}\right)^2 - \left(\frac{3}{z-1}\right)^3 + \ldots\right]$$

Case (iii) : When $|z-1| > 4$, we write

$$f(z) = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3} \quad \text{as}$$

$$= 1 - \frac{8}{(z-1)[1+4/(z-1)]} + \frac{3}{(z-1)[1+3/(z-1)]}$$

When $|z-1| > 4$, clearly $|z-1| > 3$

$$\therefore \quad f(z) = 1 - \frac{8}{z-1}\left[1+\left(\frac{4}{z-1}\right)\right]^{-1} + \frac{3}{z-1}\left[1+\left(\frac{3}{z-1}\right)\right]^{-1}$$

$$= 1 - \frac{1}{z-1}\left[1 - \left(\frac{4}{z-1}\right) + \left(\frac{4}{z-1}\right)^2 - \ldots\right]$$
$$+ \frac{3}{z-1}\left[1 - \left(\frac{3}{z-1}\right) + \left(\frac{3}{z-1}\right)^2 + \ldots\right]$$

The regions of convergence of the three series are shown below.

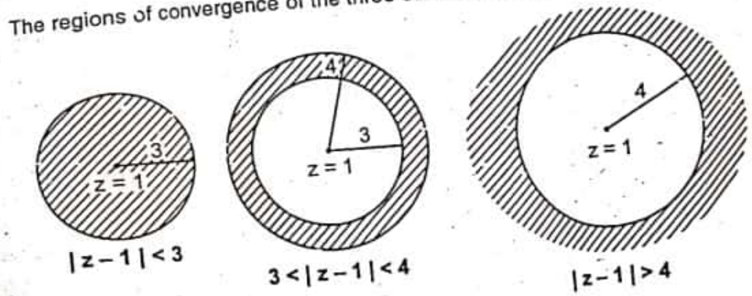


Fig. 3.5

Example 15 : Obtain two distinct Laurent's series expansions of

$$f(z) = \frac{1}{z^2(2-z)}.$$ **(M.U. 1997, 2005)**

Sol. : $f(z)$ is not analytic at $z = 0$ and $z = 2$. However, if we consider the regions $0 < |z| < 2$ and $|z| > 2$, the function is analytic in the regions.

Case (i) : When $0 < |z| < 2$. We write

$$f(z) = \frac{1}{z^2(2-z)} = \frac{1}{2z^2[1-(z/2)]} = \frac{1}{2z^2}\left(1-\frac{z}{2}\right)^{-1}$$

$$\therefore \quad f(z) = \frac{1}{2z^2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right) = \frac{1}{2z^2} + \frac{1}{4z} + \frac{1}{8} + \frac{z}{16} + \ldots$$

Case (ii) : When $|z| < 2$. We write

$$f(z) = -\frac{1}{z^2(z-2)} = -\frac{1}{z^3[1-(2/z)]}$$

$$= -\frac{1}{z^3}\left(1 - \frac{2}{z}\right)^{-1} = -\frac{1}{z^3}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \ldots\right)$$

$$= -\frac{1}{z^3} - \frac{2}{z^4} - \frac{4}{z^5} - \frac{8}{z^6} - \ldots$$

These are the two required expansions of $f(z)$.

Example 16 : Obtain Taylor's and Laurent's series for $f(z) = \dfrac{2z-3}{z^2-4z-3}$ in powers of $(z-4)$ indicating the regions of convergence. **(M.U. 1996, 2004, 13, 15)**

Sol. : Let $f(z) = \dfrac{2z-3}{z^2-4z-3} = \dfrac{a}{(z-1)} + \dfrac{b}{(z-3)}$

$\therefore \quad 2z - 3 = a(z-3) + b(z-1)$

When $z = 1$, $\quad -1 = a(-2) \qquad \therefore \ a = 1/2$

When $z = 3$, $\quad 3 = 2b \qquad \therefore \ b = 3/2$

$$\therefore \quad f(z) = \frac{1}{2}\cdot\frac{1}{(z-1)} + \frac{3}{2}\cdot\frac{1}{(z-3)}$$

$$= \frac{1}{2}\cdot\frac{1}{[(z-4)+3]} + \frac{3}{2[(z-4)+1]}$$

Case (i) : When $|z-4| < 1$, we write

$$f(z) = \frac{1}{2[(z-4)+3]} + \frac{3}{2[(z-4)+1]} \quad \text{as}$$

$$= \frac{1}{2\cdot 3[1+(z-4)/3]} + \frac{3}{2[1+(z-4)]}$$

When $|z-4| < 1$, clearly $|z-4| < 3$.

$$\therefore \quad f(z) = \frac{1}{6}\left[1 + \left(\frac{z-4}{3}\right)\right]^{-1} + \frac{3}{2}[1+(z-4)]^{-1}$$

$$= \frac{1}{6}\left[1 - \left(\frac{z-4}{3}\right) + \left(\frac{z-4}{3}\right)^2 - \left(\frac{z-4}{3}\right)^3 + \ldots\right]$$

$$+ \frac{3}{2}\left[1 - (z-4) + (z-4)^2 - (z-4)^3 + \ldots\right]$$

This is the required Taylor's series Expansion.

Case (ii) : When $|z-4| > 3$, we write

(When $|z-4| > 3$, clearly $|z-4| > 1$)

$$f(z) = \frac{1}{2[(z-4)+3]} + \frac{3}{2[(z-4)+1]} \quad \text{as}$$

$$= \frac{1}{2(z-4)[1+3/(z-4)]} + \frac{3}{2(z-4)[1+1/(z-4)]}$$

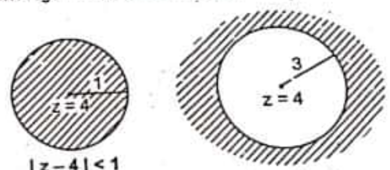
$$= \frac{1}{2(z-4)}\left[1 + \left(\frac{3}{z-4}\right)\right]^{-1} + \frac{3}{2(z-4)}\left[1 + \left(\frac{1}{z-4}\right)\right]^{-1}$$

$$= \frac{1}{2(z-4)}\left[1 - \left(\frac{3}{z-4}\right) + \left(\frac{3}{z-4}\right)^2 - \ldots\right]$$

$$+ \frac{3}{2(z-4)}\left[1 - \left(\frac{1}{z-4}\right) + \left(\frac{1}{z-4}\right)^2 - \ldots\right]$$

This is the required Laurent's series expansion.

The regions of convergence are shown below.



$|z-4| < 1$ $\qquad$ $|z-4| > 3$

Fig. 3.6

Example 17 : Expand $f(z) = \dfrac{3z-3}{(2z-1)(z-2)}$ in a Laurent's series about $z = 1$ convergent in $\dfrac{1}{2} < |z-1| < 1$. **(M.U. 2004)**

Sol. : Let $\dfrac{3z-3}{(2z-1)(z-2)} = \dfrac{a}{2z-1} + \dfrac{b}{z-2}$

$\therefore \quad 3z - 3 = a(z-2) + b(2z-1)$

When $z = 2$, $\quad 3 = 3b \qquad \therefore \ b = 1$.

When $z = 1/2$, $\quad -3/2 = -(3/2)\cdot a \qquad \therefore \ a = 1$.

$$\therefore \quad f(z) = \frac{1}{2z-1} + \frac{1}{z-2}$$

When $\dfrac{1}{2} < |z-1| < 1$, we get

$$f(z) = \frac{1}{2(z-1)+1} + \frac{1}{(z-1)-1}$$

$$= \frac{1}{2(z-1)\left[1 + \dfrac{1}{2(z-1)}\right]} - \frac{1}{1-(z-1)}$$

$$\therefore \quad f(z) = \frac{1}{2(z-1)}\left[1+\frac{1}{2(z-1)}\right]^{-1} - [1-(z-1)]^{-1}$$

[Since $\frac{1}{2} < |z-1| < 1 \quad \therefore 1 < 2|z-1| < 2 \quad \therefore 1 > \frac{1}{2|z-1|} > \frac{1}{2} \quad \therefore \frac{1}{2|z-1|} < 1.$]

$$\therefore \quad f(z) = \frac{1}{2(z-1)}\left[1-\frac{1}{2(z-1)}+\frac{1}{4(z-1)^2}-\frac{1}{8(z-1)^3}+\ldots\right]$$
$$-\left[1+(z-1)+(z-1)^2+(z-1)^3+\ldots\right]$$

Example 18 : Find all possible Laurent's expansions of the function $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ about $z = -1$. **(M.U. 1999, 2003, 09, 14, 16)**

Sol. : Let $\frac{7z-2}{z(z-2)(z+1)} = \frac{a}{z} + \frac{b}{z-2} + \frac{c}{z+1}$

$\therefore \quad 7z-2 = a(z-2)(z+1) + bz(z+1) + cz(z-2)$

When $z = 0$, $\quad -2 = -2a \quad \therefore a = 1$

When $z = -1$, $\quad -9 = 3c \quad \therefore c = -3$

When $z = 2$, $\quad 12 = 6b \quad \therefore b = 2.$

$$\therefore \quad \frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$\therefore \quad f(z) = \frac{1}{(z+1)-1} + \frac{2}{(z+1)-3} - \frac{3}{z+1}$$

Case (i) : When $|z+1| < 1$, we write

$$f(z) = -\frac{3}{z+1} - \frac{1}{1-(z+1)} - \frac{2}{3-(z+1)}$$

When $|z+1| < 1$, clearly $|z+1| < 3$.

$$\therefore \quad f(z) = -\frac{3}{z+1} - [1-(z+1)]^{-1} - \frac{2}{3}\left[1-\left(\frac{z+1}{3}\right)\right]^{-1}$$
$$= -\frac{3}{z+1} - \left[1+(z+1)+(z+1)^2+(z+1)^3+\ldots\right]$$
$$-\frac{2}{3}\left[1+\frac{(z+1)}{3}+\frac{(z+1)^2}{3^2}+\frac{(z+1)^3}{3^3}+\ldots\right]$$
$$= -\frac{3}{z+1} - \sum\left(1+\frac{2}{3^{n+1}}\right)(z+1)^n, \quad 0 < |z+1| < 1$$

Case (ii) : When $1 < |z+1| < 3$

$$f(z) = -\frac{3}{z+1} + \frac{1}{(z+1)\left(1-\frac{1}{z+1}\right)} - \frac{2}{3\left[1-\frac{(z+1)}{3}\right]}$$

$$\therefore \quad f(z) = -\frac{3}{z+1} + \frac{1}{z+1}\left(1-\frac{1}{z+1}\right)^{-1} - \frac{2}{3}\left(1-\frac{z+1}{3}\right)^{-1}$$
$$= -\frac{3}{z+1} + \frac{1}{z+1}\left[1+\frac{1}{(z+1)^1}+\frac{1}{(z+1)^2}+\ldots\right]$$
$$-\frac{2}{3}\left[1+\frac{(z+1)}{3}+\frac{(z+1)^2}{3^2}+\frac{(z+1)^3}{3^3}+\ldots\right]$$
$$= -\frac{3}{z+1} + \sum\frac{1}{(z+1)^{n+1}} - 2\sum\frac{(z+1)^n}{3^{n+1}} \; ; \; 1 < |z+1| < 3$$

Case (iii) : When $|z+1| > 3$, we write

$$f(z) = -\frac{3}{z+1} + \frac{1}{(z+1)\left(1-\frac{1}{z+1}\right)} + \frac{2}{(z+1)\left(1-\frac{3}{z+1}\right)}$$

When $|z+1| > 3$, $|z+1| > 1$.

$$\therefore \quad f(z) = -\frac{3}{z+1} + \frac{1}{z+1}\left(1-\frac{1}{z+1}\right)^{-1} + \frac{2}{z+1}\left(1-\frac{3}{z+1}\right)^{-1}$$
$$= -\frac{3}{z+1} + \frac{1}{z+1}\left[1+\frac{1}{(z+1)}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\ldots\right]$$
$$+\frac{2}{z+1}\left[1+\frac{3}{z+1}+\frac{3^2}{(z+1)^2}+\frac{3^3}{(z+1)^3}+\ldots\right]$$
$$= -\frac{3}{z+1} + \sum\frac{1+2\cdot 3^{n-1}}{(z+1)^n} \; ; \; |z+1| > 3.$$

Example 19 : Find all possible Laurent's expansions of $\frac{z}{(z-1)(z-2)}$ about $z = -2$. **(M.U. 2016)**

Sol. : Let $\frac{z}{(z-1)(z-2)} = \frac{a}{z-1} + \frac{b}{z-2} \quad \therefore z = a(z-2) + b(z-1)$

Putting $z = 1$, $\quad 1 = -a \quad \therefore a = -1$

Putting $z = 2$, $\quad 2 = b \quad \therefore b = 2.$

$$\therefore \quad f(z) = -\frac{1}{z-1} + \frac{2}{z-2} = -\frac{1}{(z+2)-3} + \frac{2}{(z+2)-4}$$

Case (i) : When $[z-(-2)] < 3$, $|z+2| < 3$. When $|z+2| < 3$, clearly $|z+2| < 4$.

We write,

$$\therefore \quad f(z) = -\frac{1}{(z+2)-3} + \frac{2}{(z+2)-4}$$
$$= \frac{1}{3}\cdot\frac{1}{\left[1-\frac{z+2}{3}\right]} - \frac{2}{4\left[1-\frac{z+2}{4}\right]} = \frac{1}{3}\left[1-\frac{z+2}{3}\right]^{-1} - \frac{1}{2}\left[1-\frac{z+2}{4}\right]^{-1}$$

$$\therefore \quad f(z) = \frac{1}{3}\left[1+\left(\frac{z+2}{3}\right)+\left(\frac{z+2}{3}\right)^2+\ldots\right]-\frac{1}{2}\left[1+\left(\frac{z+2}{4}\right)+\left(\frac{z+2}{4}\right)^2+\ldots\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{3^{n+1}}-\frac{1}{2\cdot 4^n}\right)(z+2)^n$$

Case (ii) : When $3 < |z+2| < 4$, we write,

$$f(z) = -\frac{1}{(z+2)-3}+\frac{2}{(z+2)-4}$$

$$= -\frac{1}{(z+2)\left[1-\dfrac{3}{z+2}\right]}-\frac{2}{4\left[1-\dfrac{z+2}{4}\right]}$$

$$= -\frac{1}{(z+2)}\left[1-\frac{3}{z+2}\right]^{-1}-\frac{1}{2}\left[1-\frac{z+2}{4}\right]^{-1}$$

$$= -\frac{1}{(z+2)}\left[1+\left(\frac{3}{z+2}\right)+\left(\frac{3}{z+2}\right)^2+\ldots\right]-\frac{1}{2}\left[1+\left(\frac{z+2}{4}\right)+\left(\frac{z+2}{4}\right)^2+\ldots\right]$$

$$= -\sum_{n=0}^{\infty}\frac{3^n}{(z+2)^{n+1}}-\frac{1}{2}\sum\left(\frac{z+2}{4}\right)^n$$

Case (iii) : When $|z+2| > 4$, clearly $|z+2| > 3$, we write,

$$f(z) = -\frac{1}{(z+2)\left[1-\dfrac{3}{z+2}\right]}+\frac{2}{(z+2)\left[1-\dfrac{4}{z+2}\right]}$$

$$= -\frac{1}{(z+2)}\left(1-\frac{3}{z+2}\right)^{-1}+\frac{2}{(z+2)}\left(1-\frac{4}{z+2}\right)^{-1}$$

$$= -\frac{1}{(z+2)}\left[1+\left(\frac{3}{z+2}\right)+\left(\frac{3}{z+2}\right)^2+\ldots\right]$$

$$+\frac{2}{(z+2)}\left[1+\left(\frac{4}{z+2}\right)+\left(\frac{4}{z+2}\right)^2+\ldots\right]$$

$$= \sum(2\cdot 4^n-3^n)\cdot\frac{1}{(z+2)^{n+1}}$$

EXERCISE - II

1. Expand the following in Taylor's series

(i) $\cos z$ about $z = \pi/4$. **(M.U. 2003)**
(ii) $\cos z$ about $z = 0$.
(iii) $\sin z$ about $z = 0$.
(iv) e^z about $z = 0$.
(v) $\cos z$ about $z = \pi/2$.
(vi) e^{2z} about $z = 2i$.
(vii) $\cos z$ about $z = \pi/3$. **(M.U. 2003)**
(viii) e^z about $z = -i$.
(ix) e^z about $z = 1$.
(x) $\log\left(\dfrac{1+z}{1-z}\right)$ about $z = 0$.

[Ans. : (i) $\dfrac{1}{\sqrt{2}}\left[1-\left(z-\frac{\pi}{4}\right)+\frac{1}{2!}\left(z-\frac{\pi}{4}\right)-\ldots\right]$ (ii) $1-\dfrac{z^2}{2!}+\dfrac{z^4}{4!}-\ldots$

(iii) $z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\ldots$ (iv) $1+z+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\dfrac{z^4}{4!}+\ldots$

(v) $\dfrac{[z+(\pi/2)]}{1!}-\dfrac{[z+(\pi/2)]^3}{3!}+\dfrac{[z+(\pi/2)]^5}{5!}-\dfrac{[z+(\pi/2)]^7}{7!}+\ldots$

(vi) $e^{4i}\left[1+\dfrac{2(z-2i)}{1!}-\dfrac{2^2(z-2i)^2}{2!}+\dfrac{2^3(z-2i)^3}{3!}+\ldots\right]$

(vii) $\dfrac{1}{2}\left[1-\sqrt{3}\left(z-\frac{\pi}{3}\right)-\frac{1}{2!}\left(z-\frac{\pi}{3}\right)^2+\frac{\sqrt{3}}{3!}\left(z-\frac{\pi}{3}\right)^3+\ldots\right]$

(viii) $e^{-i}\left[1+\dfrac{(z+i)}{1}+\dfrac{(z+i)^2}{2!}+\ldots\right]$

(ix) $e^{-1}\left[1-\dfrac{(z-1)}{1!}+\dfrac{(z-1)^2}{2!}-\dfrac{(z-1)^3}{3!}+\ldots\right]$

(x) $2\left(z+\dfrac{z^3}{3}+\dfrac{z^5}{5}+\ldots\right)$; $|z|<1$]

2. Find the radius of convergence in each of the following cases if the given function is expanded as a Taylor's series about the indicated point without actually expanding.

(i) $\dfrac{z+1}{z-1}$ about $z = 0$.
(ii) $\dfrac{z}{z^2+9}$ about $z = 0$.
(iii) $\dfrac{z+2}{(z-3)(z-4)}$ about $z = 2$.
(iv) $\sec \pi z$ about $z = 1$.

(**Hint :** The radius of convergence is the distance between the centre of Taylor's series and the nearest singularity of the function.)

[Ans. : (i) $|z| < 1$, **(ii)** $|z| < 3$, **(iii)** $|z-2| < 1$, **(iv)** $|z-1| < 1/2$ **]**

3. Find the Taylor's expansion of $f(z) = \dfrac{1}{z(1-z)}$ about $z = -1$, indicating the region of convergence. **[Ans. :** $f(z) = \sum\left(\dfrac{1}{2^{n+1}}-1\right)(z+1)^n$, $|z+1|<1$ **]**

4. Find the Taylor's expansion of $f(z) = \dfrac{z}{(z+1)(z+2)}$ about $z = i$, indicating the region of convergence. **[Ans. :** $f(z) = \sum(-1)^n\left\{\dfrac{2}{(2+i)^{n+1}}-\dfrac{1}{(1+i)^{n+1}}\right\}(z-i)^n$; $|z-i|<1$ **]**

5. Obtain Taylor's expansion of $f(z) = \dfrac{z-1}{z+1}$ indicating the region of convergence.

(M.U. 1998) [Ans. : $f(z) = 1-2(1+z+z^2+z^3+\ldots)$; $|z|<1$ **]**

6. Find Taylor's expansion of $f(z) = \frac{1}{z^2+4}$ about $z = -i$. (M.U. 2003)

[Ans. : $\frac{1}{3} + \frac{5}{18}(z+i) \cdot i + \frac{7}{27}(z+i)^2 - \frac{20}{81}(z+i)^3 i + \ldots$]

7. Find the Taylor's series expansion of $f(z) = \frac{1}{(z-1)(z-3)}$ about the point $z = 4$. Find the region of convergence. (M.U. 2000, 05)

[Ans. : $\frac{1}{2}\left[1-(z-4)+(z-4)^2-(z-4)^3+\ldots\right] - \frac{1}{6}\left[1-\left(\frac{z-4}{3}\right)+\left(\frac{z-4}{3}\right)^2-\ldots\right]$]

8. Obtain Laurent's series for $f(z) = \frac{1}{z(z+2)(z+1)}$ about $z = -2$. (M.U. 2001)

[Ans. : $\frac{1}{2u} - \frac{1}{4}\left(1+\frac{u}{2}+\frac{u^2}{4}+\ldots\right) + (1+u+u^2+u^3+\ldots)$ where, $u = z+2$.]

9. Obtain the expansion of $f(z) = \frac{z+1}{(z-3)(z-4)}$ about $z = 2$. (M.U. 2000)

[Ans. : $4\left[1+u+u^2+u^3+\ldots\right] - \frac{5}{2}\left[1+\frac{u}{2}+\frac{u^2}{4}+\frac{u^3}{8}+\ldots\right]$ where $u = z-2$.]

10. Obtain Laurent's series expansion of $f(z) = \frac{1}{z^2+4z+3}$ when (M.U. 2000)

(i) $1 < |z| < 3$, (ii) $|z| > 3$.

[Ans. : (i) $\frac{1}{2}\left[\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\ldots\right] - \frac{1}{3}\left[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\ldots\right]$

(ii) $\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \ldots$]

11. Expand $f(z) = \frac{1}{z^3-3z^2+2z}$ as Laurent's series about $z = 0$ for

(i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$. (M.U. 1997, 98, 2000)

[Ans. : (i) $\frac{1}{2z} + \frac{3}{4} + \frac{7}{8}z + \frac{15}{16}z^2 + \ldots$

(ii) $-\frac{1}{4} - \frac{1}{8} \cdot z - \frac{1}{16} \cdot z^2 - \ldots - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \ldots$

(iii) $\frac{1}{z^3} + \frac{3}{z^4} + \frac{7}{z^5} + \frac{15}{z^6} + \ldots$]

12. Obtain Taylor's and Laurent's expansions of $f(z) = \frac{z-1}{z^2-2z-3}$ indicating regions of convergence. (M.U. 1996)

[Ans. : (i) $\frac{1}{2}\left[\frac{2}{3} - \frac{10}{9}z + \frac{29}{27}z^2 + \ldots\right]$; $|z| < 1$

(ii) $\frac{1}{2}\left[-\frac{1}{3} - \frac{z}{9} - \frac{z^2}{27} \ldots - \frac{1}{z^2} + \frac{1}{z^3} - \ldots\right]$; $1 < |z| < 3$

(iii) $\frac{1}{z} + \frac{1}{z^2} + \frac{5}{z^3} + \ldots$, $|z| > 3$.]

13. Obtain the Laurent's series valid in the indicated region.

(i) $\frac{1}{z^2(z-2)}$; $0 < |z| < 2$, (M.U. 1997) (ii) $\frac{1+2z}{z+z^2}$; $0 < |z| < 1$

(iii) $\frac{1}{z-z^2}$; $1 < |z+1| < 2$ (iv) $\frac{7z-2}{z(z-2)(z+1)}$; $1 < |z+1| < 3$

(v) $\frac{1}{z^2-z}$; $0 < |z-1| < 1$ (vi) $\frac{1}{z^3(1-z)}$; $|z| > 1$

(vii) $\frac{1}{z(4-z)}$; $0 < |z| < 4$ (viii) $\frac{z}{(z+1)(z+2)}$; $0 < |z+2| < 1$

(ix) $\frac{1}{1-z^2}$; $0 < |z-1| < 2$ (x) $\frac{z-1}{z^2}$; $|z-1| > 1$ (M.U. 2003)

(xi) $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ (a) $1 < |z| < 4$, (b) $|z| > 4$. (M.U. 2004, 05)

[Ans. : (i) $-\frac{1}{2z^2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\ldots\right)$ (ii) $\frac{1}{z}(1+z+z^2+z^3-\ldots)$

(iii) $\frac{1}{z+1}\left(1+\frac{1}{(z+1)}+\frac{1}{(z+1)^2}+\ldots\right)+\frac{1}{2}\left[1+\left(\frac{z+1}{2}\right)+\left(\frac{z+1}{2}\right)^2+\ldots\right]$

(iv) $-\frac{2}{(z+1)} + \sum_{n=2}^{\infty} \frac{1}{(z+1)^n} - \frac{2}{3}\Sigma\left(\frac{z+1}{3}\right)^n$

(v) $\frac{1}{z-1}\left[1-(z-1)+(z-1)^2-(z-1)^3+\ldots\right]$

(vi) $-\frac{1}{z^4}\left[1+\frac{1}{z}+\frac{1}{z^2}+\ldots\right]$ (vii) $\sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$

(viii) $\frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + (z+2)^3 + \ldots$

(ix) $\Sigma \frac{(-1)^{n+1}}{2^{n+1}}(z-1)^{n-1}$

(x) $(z-1)^{-1} - 2(z-1)^{-2} + 3(z-1)^{-3} - 4(z-1)^{-4} + \ldots$

(xi) (a) $1-\left[\frac{1}{z}-\frac{1}{z^2}+\ldots\right]-\left[1-\frac{z}{4}+\left(\frac{z}{4}\right)^2-\ldots\right]$

(b) $1-\left[\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right]-4\left[\frac{1}{z}-\frac{4}{z^2}+\frac{16}{z^2}+\ldots\right]$

14. Find all possible Laurent's expansions of the following functions indicating the regions of convergence.

(i) $\dfrac{4-3z}{z(1-z)(2-z)}$ about $z=0$.
(ii) $\dfrac{z^2-1}{z^2+5z+6}$ about $z=0$.
(iii) $\dfrac{z}{(z-1)(z-2)}$ about $z=-2$.
(iv) $\dfrac{z^3-6z-1}{(z-1)(z-3)(z+2)}$ about $z=3$.
(v) $\dfrac{1}{z^2(1-z)}$ about $z=0$.
(vi) $\dfrac{z^2-1}{(z+2)(z+3)}$ about $z=0$.

[**Ans. :** (i) $f(z)=\dfrac{2}{z}+\sum_{n=0}^{\infty}\left(1+\dfrac{1}{2^{n+1}}\right)z^n\ ;\ |z|<1$

$f(z)=\dfrac{2}{z}-\sum_{n=0}^{\infty}\dfrac{1}{z^{n+1}}+\sum_{n=0}^{\infty}\dfrac{z^n}{2^{n+1}}\ ;\ 1<|z|<2$

$f(z)=\dfrac{2}{z}-\sum_{n=0}^{\infty}(1-2^n)\cdot\dfrac{1}{z^{n+1}}\ ;\ 2<|z|<\infty$

(ii) $f(z)=1+\sum(-1)^n\left\{\dfrac{3}{2^{n+1}}-\dfrac{8}{3^{n+1}}\right\}z^n\ ;\ |z|<2$

$f(z)=1+3\sum(-1)^n\cdot\dfrac{2^n}{z^{n+1}}-8\sum(-1)^n\cdot\dfrac{z^n}{3^{n+1}}\ ;\ 2<|z|<3$

$f(z)=1+\sum(-1)\cdot\{3\cdot 2^n-8\cdot 3^n\}\dfrac{1}{z^{n+1}}\ ;\ |z|>3$

(iii) $f(z)=\sum\left(-\dfrac{1}{2\cdot 4^n}+\dfrac{1}{3^{n+1}}\right)(z+2)^n\ ;\ |z+2|<3$

$f(z)=-\dfrac{1}{2}\sum\dfrac{(z+2)^n}{4^n}-\sum\dfrac{3^n}{(z+2)^{n+1}}\ ;\ 3<|z+2|<4$

$f(z)=\sum(2\cdot 4^n-3^n)\dfrac{1}{(z+2)^{n+1}}\ ;\ |z+2|>4$

(iv) $f(z)=1+\dfrac{4/5}{z-3}+\sum(-1)^n\cdot\left\{\dfrac{1}{2^{n+1}}+\dfrac{1}{5^{n+2}}\right\}(z-3)^n\ ;\ 0<|z-3|<2$

$f(z)=1+\dfrac{4/5}{z-3}+\sum(-1)^n\cdot\left\{\dfrac{2^n}{(z-3)^{n+1}}+\dfrac{1}{5^{n+2}}\right\}(z-3)^n\ ;\ 2<|z-3|<5$

$f(z)=1+\dfrac{4/5}{3}+\sum(-1)^n\cdot\{2^n+5^{n-1}\}\cdot\dfrac{1}{(z-3)^n}\ ;\ |z-3|>5$

(v) $f(z)=\sum z^{n+2}\ ;\ 0<|z|<1\ ;\quad f(z)=-\sum\dfrac{1}{z^{n+3}}\ ;\ |z|>1$

(vi) $f(z)=1+\dfrac{3}{2}\sum(-1)^n\left(\dfrac{z}{2}\right)^n-\dfrac{8}{3}\sum(-1)^n\left(\dfrac{z}{3}\right)^3\ ;\ |z|<2$

$f(z)=1+\dfrac{3}{z}\sum(-1)^n\left(\dfrac{2}{z}\right)^n-\dfrac{8}{z}\sum(-1)^n\left(\dfrac{3}{z}\right)^n\ ;\ |z|>3$

$f(z)=1+\dfrac{3}{2}\sum(-1)^n\left(\dfrac{2}{z}\right)^n-\dfrac{8}{3}\sum(-1)^n\left(\dfrac{z}{3}\right)^n\ ;\ 2<|z|<3$]

EXERCISE - III

Theory

1. Define radius and circle of convergence of a power series.
2. Define Taylor's series.
3. Define Laurent's series.
4. Define a Laurent's series and state its analytic part and the principal part.
5. State whether the following statements are true or false.
   (i) A function analytic at z_0 may have two Taylor series expansions centred at z_0.
   (ii) A function $f(z)$ may have two Laurent's series centre at z_0 depending upon the annulus of convergence. (M.U. 1999) [**Ans. :** Both are false.]

☆☆☆

CHAPTER 4

Residues

1. Introduction

In this chapter we shall learn some new concepts viz. a zero of an analytic function, poles, residues etc. We shall then study a very important theorem viz. residue theorem. We shall then see how to use this theorem to find some integrals.

2. Zero of an Analytic Function

If a function which is analytic in a region R is equal to zero at a point $z = z_0$ in R then z_0 is called a **zero of** $f(z)$ in R.

If $f(z_0) = 0$ but $f'(z_0) \neq 0$ then z_0 is called a **simple zero or a zero of first order.**

If $f(z_0) = 0$ and also $f'(z_0) = f''(z_0) = \ldots = f^{n-1}(z_0) = 0$ but $f^n(z_0) \neq 0$ then z_0 is called **a zero of order** n.

Since $f(z)$ is analytic at $z = z_0$ there exists a neighbourhood of z_0 in which $f(z)$ can be expanded as a Taylor's series.

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \ldots + \frac{(z-z_0)^n}{n!} f^n(z_0) + \ldots \qquad (|z - z_0| < r)$$

(a) If $z = z_0$ is a simple zero then $f(z_0) = 0$ and $f'(z_0) \neq 0$.

(b) If $z = z_0$ is a zero of order n then, since

$$f'(z_0) = f''(z_0) = \ldots = f^{n-1}(z_0) = 0,$$

we get from the above series

$$f(z) = \frac{(z-z_0)^n}{n!} f^n(z_0) + \ldots$$

Example 1 : Find the zeros of $f(z) = (z - 1)\, e^z$.

Sol. : Clearly $f(z) = 0$ when $z = 1$ and $f'(z) = (z - 1)\, e^z + e^z = e$ at $z = 1$. Thus, $f'(z) \neq 0$. Hence, $f(z)$ has simple zero at $z = 1$.

Example 2 : Find the zero of $f(z) = \sin z$.

Sol. : Clearly, $\sin z = 0$ when $z = 0, \pm\pi, \pm 2\pi, \ldots$ and $f'(z) = \cos z$ is not equal to zero for these values.

Hence, $f(z)$ has simple zeros at $z = 0, \pm\pi, \pm 2\pi, \ldots$

Example 3 : Find the zeros of $f(z) = z^2 \sin z$.

Sol. : $f(z) = z^2 \sin z = 0$ when $z = 0$, and $z = \pm\pi, \pm 2\pi, \ldots$

Now, $f(z) = z^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right) = z^3 - \frac{z^5}{3!} + \frac{z^7}{5!} - \ldots$

Now, $f'(z) = 3z^2 - \frac{5z^4}{3!} + \ldots,\qquad f''(z) = 6z - \frac{20z^3}{3!} + \ldots,$

$f'''(z) = 6 - \frac{60z^2}{3!} + \ldots$

Clearly for $z = 0$, $f'(z)$, $f''(z)$ are zero but $f'''(z) = 6 \neq 0$.

$\therefore$ $z = 0$ is a zero order 3.

But $f'(z) \neq 0$ for $z = \pm\pi, \pm 2\pi, \ldots$

Hence, $z = \pm\pi, \pm 2\pi, \ldots$ each is a simple zero.

Example 4 : Find the zeros of $f(z) = \left(\frac{z-1}{z^2+2}\right)^3$.

Sol. : $f(z) = 0 \quad \therefore \quad (z-1)^3 = 0$

$\therefore$ $z = 1$ is a zero of order 3.

EXERCISE - I

Find the zeros of the following functions.

1. $\cos z$ 2. $\frac{(z+1)^3}{z^2+3}$ 3. $(z-1)^2 (z-2)^3$ 4. $\frac{1}{(z-2)^2}$

[**Ans. : (1)** $z = \pm\pi/2, \pm 3\pi/2, \ldots$ are simple zeros. **(2)** $z = -1$ is a zero of order 3. **(3)** $z = 1$ is a zero of order 2 and $z = 2$ is a zero of order 3. **(4)** a zero of order 2 at infinity.]

2. Find the zeros of the following functions and determine their order.

(i) $z \tan z$, **(ii)** $(z^2 - 1)(z^2 + 3z + 2)$, **(iii)** $z^3 \sin z$, **(iv)** $(z-1)^3 (z+2)$.

[**Ans. : (i)** $z = 0$ is a zero of order 2 and $z = \pm\pi, \pm 2\pi, \pm 3\pi \ldots$ each is a zero of order 1. **(ii)** $z = 1, 2$ is a zero of order 1, $z = -1$ is a zero of order 2. **(iii)** $z = 0$ is a zero of order 4 and $z = \pm\pi, \pm 2\pi, \pm 3\pi \ldots$ each is a zero of order 1. **(iv)** $z = 1$ is a zero of order 3 and $z = -2$ is a zero of order 1.]

3. Singular Points

Definition : If a function $f(z)$ is analytic at every point in the neighbourhood of a point z_0 except at z_0 itself then $z = z_0$ is called **a singular point** or **a singularity** of $f(z)$.

For example, if $f(z) = \frac{z^2}{z-2}$, $z = 2$ is a singularity of $f(z)$.

If $f(z) = \frac{z}{z(z+1)}$, $z = 0$ and $z = -1$ are two singularities of $f(z)$.

There are different types of singularities. They are :-

(a) Isolated Singularity

If the singular point z_0 of $f(z)$ is such that there is no other singular point in the neighbourhood of z_0 then such a singularity is called an **isolated singularity** ; the point z_0 is called the **isolated singular point**. In other words z_0 is an isolated singularity or an isolated singular point, if we can find a circle $|z - z_0| = \delta$ with centre at z_0 and radius δ, such that there is no singular point of $f(z)$ within the circle other than z_0.

If we cannot find any such δ *i.e.* we cannot find a circle $|z - z_0| = \delta$ which does not contain a singularity other than z_0, z_0 is called **non-isolated singularity**.

(i) Isolated Singularity : There is no other singular point inside the neighbourhood of z_0 other than z_0 whch is a singular point.

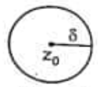


Fig. 4.1

For example, if $f(z) = \dfrac{z^2}{z-2}$, then $z = 2$ is an isolated singularity.

If $f(z) = \dfrac{z-2}{z(z+1)(z-1)}$, then $z = 0, z = 1, z = -1$ are the finite isolated singularities of $f(z)$.

If $f(z) = \dfrac{1}{z(z^2+1)(z^2-1)}$ then $z = 0, i, -i, +1, -1$ are the finite isolated singularities of $f(z)$.

If $f(z) = \dfrac{1}{\sin \pi z}$, then $z = 0, \pm 1, \pm 2, \ldots, \pm n, \ldots$ are infinite singularities of $f(z)$.

Note ....

If a function $f(z)$ has only a **finite** number of singularities then they are necessarily **isolated.**

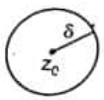


Fig. 4.2

If z_0 is **not** a singularity and the circle $|z - z_0| = \delta$ **does not contain** any singular point of $f(z)$, z_0 is called a **regular point** or **ordinary point** of $f(z)$.

(ii) Ordinary Point or **Regular Point :** No singular point inside the neigh-bourhood and z_0 is **not** a singular point.

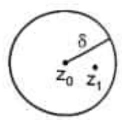


Fig. 4.3

(iii) Non-isolated Singularity : Every neighbourhood of z_0 **contains atleast one more singular point z_1** other than z_0.

1. Pole : If $z = z_0$ is an isolated singularity of $f(z)$ then we can find a region $0 < |z - z_0| < \delta$ in which $f(z)$ is analytic. In such a region $f(z)$ can be expanded by a Laurent's series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n} \quad \ldots\ldots (1)$$

The part $\Sigma a_n (z - z_0)^n$ of the above expansion is called the **analytic part, regular part** and the part $\Sigma b_n (z - z_0)^{-n}$ is called the **principal part** of $f(z)$ in the neighbourhood of z_0.

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots + \frac{b_n}{(z - z_0)^n} + \ldots \quad \ldots\ldots (2)$$

(i) If in the above series the coefficients, $b_{n+1} = b_{n+2} = \ldots = 0$

i.e., if $\quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \dfrac{b_1}{z - z_0} + \dfrac{b_2}{(z - z_0)^2} + \ldots + \dfrac{b_n}{(z - z_0)^n}$

then $z = z_0$ is called a **pole of order** ***n***.

(ii) A pole of order 1 is called a **simple pole.** If $z = z_0$ is a simple pole then $f(z)$ can be written as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0}.$$

2. Isolated Essential Singularity : In the above series (i) if we have infinite terms of negative powers of $(z - z_0)$ *i.e.* if the principal part of $f(z)$ does not terminate as in (i) *i.e.*, if $b_1, b_2, \ldots, b_n, \ldots, \infty$ are **not** zero then $z = z_0$ is called an **isolated essential singularity.**

We note here that the behaviour of a function at an essential singularity is very complicated.

Example 1 : Show that $f(z) = \dfrac{z^2 - 3z + 4}{z - 3}$ has a simple pole at $z = 3$.

Sol. : By actual division $f(z) = z + \dfrac{4}{z-3} = 3 + (z - 3) + \dfrac{4}{z-3}$

Hence, $f(z)$ has a simple pole at $z = 3$.

Example 2 : Show that $f(z) = \dfrac{1}{z(z-1)}$ has a simple pole at $z = 0$ and at $z = 1$.

Sol. : We can expand $f(z)$ around the singularities $z = 0$ and $z = 1$ as Laurent's series as follows.

$$f(z) = -\frac{1}{z(1-z)} = -\frac{1}{z}(1-z)^{-1}$$

$$= -\frac{1}{z}(1 + z + z^2 + z^3 + z^4 + \ldots \infty)$$

$$= -(1 + z + z^2 + \ldots) - \frac{1}{z} \text{ in } 0 < |z| < 1$$

It has only one term in the principal part, $b_1 \neq 0$, $b_2 = b_3 = \ldots = 0$. Hence, $z = 0$ is a simple pole.

Again $f(z) = \dfrac{1}{(z-1)[1 + (z-1)]} = \dfrac{1}{z-1}[1 + (z-1)]^{-1}$

$$= \frac{1}{z-1}\left[1 - (z-1) + (z-1)^2 - (z-1)^3 + \ldots\right]$$

$$= -1 + (z-1) - (z-1)^2 + \ldots + \frac{1}{(z-1)} \qquad 0 < |z-1| < 1$$

As before it has only one term in the principal part, $b_1 \neq 0$, $b_2 = b_3 = \ldots = 0$.

Hence, $z = 1$, is a simple pole.

Example 3 : Show that $f(z)=\dfrac{1}{(z-1)^2(z-2)^3}$ has a pole of order 2 at $z=1$ and a pole of order 3 at $z=2$.

Sol. : We can expand $f(z)$ around the singularities $z=1$, and $z=2$ as Laurent's series as follows.

$$f(z)=\frac{1}{(z-1)^2(z-1-1)^3}=-\frac{1}{(z-1)^2\left[1-(z-1)\right]^3}$$

$$=-\frac{1}{(z-1)^2}\left[1-(z-1)\right]^{-3}$$

$$=-\frac{1}{(z-1)^2}\left[1+3(z-1)+6(z-1)^2+10(z-1)^3+....\right]$$

$$=-6-10(z-1)-..........-\frac{3}{z-1}-\frac{1}{(z-1)^2}$$

It has two terms in the principal part, $b_1\neq 0$, $b_2\neq 0$, $b_3=b_4=.....=0$.

Hence, $z=1$ is a pole order 2.

$$\text{Again } f(z)=\frac{1}{(z-1)^2(z-2)^3}=\frac{1}{(z-2+1)^2(z-2)^3}$$

$$=\frac{1}{(z-2)^3\left[1+(z-2)\right]^2}=\frac{1}{(z-2)^3}\left[1+(z-2)\right]^{-2}$$

$$=\frac{1}{(z-2)^3}\left[1-2(z-2)+3(z-2)^2-4(z-2)^3+5(z-2)^4-.....\right]$$

$$=-4+5(z-2)+.....+.....+\frac{3}{z-2}-\frac{2}{(z-2)^2}+\frac{1}{(z-2)^3}$$

It has 3 terms in the principal part, $b_1\neq 0$, $b_2\neq 0$, $b_3\neq 0$, $b_4=b_5=.....=0$.

Hence, $z=2$ is a pole of order 3.

Example 4 : Find the order of the pole of $f(z)=\dfrac{\sin hz}{z^7}$.

Sol. : We have $f(z)=\dfrac{1}{z^7}\left(z+\dfrac{z^3}{3!}+\dfrac{z^5}{5!}+\dfrac{z^7}{7!}+\dfrac{z^9}{9!}+.....\right)$

$$=\frac{1}{z^6}+\frac{1}{3!\,z^4}+\frac{1}{5!\,z^2}+\frac{1}{7!}+\frac{1}{9!}z^2+.....$$

$\therefore$ $f(z)$ has a pole at $z=0$ of order 6.

Example 5 : Find the poles of cosec $h\,2z$ within $|z|=4$. **(M.U. 2004)**

Sol. : The poles of cosec $h\,2z=\dfrac{1}{\sin h\,2z}$ are given by $\sin h\,2z=0$.

$$\therefore\ \frac{e^{2z}-e^{-2z}}{2}=0\quad \therefore\ e^{2z}=e^{-2z}\quad \therefore\ e^{4z}=1$$

$$\therefore\ e^{4z}=e^{2\pi i}=e^{2n\pi i}\quad \therefore\ 4z=\pm 2n\pi i\quad \therefore\ z=\pm\frac{n\pi}{2}i$$

The poles within $|z|=4$ are $\pm\dfrac{\pi}{2}i$ and $\pm\pi i$.

(For $n=3, 4, ...;\ |z|>4$.)

Example 6 : Find the singular points of $f(z)=\dfrac{1}{z^4+1}$. **(M.U. 2001)**

Sol. : The singular points of $f(z)$ are given by

$$z^4+1=0\quad \therefore\ z^4=-1,\quad \therefore\ z=(-1)^{1/4}$$

$$z=(\cos\pi+i\sin\pi)^{1/4}$$

$$=\cos\frac{(2n+1)\pi}{4}+i\sin\frac{(2n+1)\pi}{4},\quad n=0,1,2,3.$$

Example 7 : Show that $f(z)=e^{1/z}$ has an isolated essential singularity at $z=0$.

Sol. : We have $e^{1/z}=1+\dfrac{1}{z}+\dfrac{1}{z^2}+.....+\dfrac{1}{z^n}+.....\infty$

The principal part of $f(z)$ has in infinite number of terms. Hence, $f(z)$ has an isolated essential singularity at $z=0$.

Example 8 : Show that $f(z)=(z+2)\sin\left(\dfrac{1}{z-1}\right)$ has isolated essential singularity at $z=1$.

Sol. : We have $f(z)=(z-1+3)\left[\dfrac{1}{z-1}-\dfrac{1}{3!(z-1)^3}+\dfrac{1}{5!(z-1)^5}-.....\right]$

$$=(z-1)\left[\frac{1}{z-1}-\frac{1}{3!(z-1)^3}+\frac{1}{5!(z-1)^5}-.....\right]$$

$$+3\left[\frac{1}{z-1}-\frac{1}{3!(z-1)^3}+\frac{1}{5!(z-1)^5}-.....\right]$$

$$=1-\frac{1}{3!(z-1)^2}+\frac{1}{5!(z-1)^4}-.....+\frac{3}{(z-1)}-\frac{3}{3!(z-1)^3}+.....$$

The principal part has infinite number of terms. Hence, $f(z)$ has **essential** isolated singularity at $z=1$.

Limit point of sequence : Loosely speaking if a small neighbourhood of a point '*a*'. contains a large number of elements of a sequence, then it is called the **limit point** (or a **cluster point**) of the sequence.

Example (i) : The sequence $\left\{1,\dfrac{1}{2},\dfrac{1}{3},.....\dfrac{1}{n}.....\right\}$ has limit point zero because in a small neighbourhood of zero, there lie a large number of members of the elements of the sequence.

(ii) The sequence $\left\{1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ldots\right\}$ has two limit points 1 and 0.

(iii) The sequence $(-1)^n\left(1+\frac{1}{n}\right)$ has two limit points -1 and 1.

(b) Non-isolated Essential Singularity

If we have a sequence of poles of $f(z)$, $z_1, z_2, \ldots, z_n, \ldots$ such that z_0 is the limit point of these poles then z_0 is called **non-isolated essential** singularity.

Example 1 : If $f(z) = \tan\left(\frac{1}{z}\right)$ then $f(z)$ has singularities at $\cos\frac{1}{z} = 0$ *i.e.* at $\frac{1}{z} = \frac{n\pi}{2}$, where $n = \pm 1, \pm 3, \pm 5, \ldots$ *i.e.* when $z = \frac{2}{n\pi}$, $n = \pm 1, \pm 3, \pm 5, \ldots$ The limit point of the sequence of the poles $\frac{2}{\pm\pi}, \frac{2}{\pm 3\pi}, \frac{2}{\pm 5\pi}, \ldots$ is $z = 0$. Hence, $z = 0$ is non-isolated essential singularity.

Example 2 : State the nature of singularity of $f(z) = \left[\sin\left(\frac{1}{z}\right)\right]^{-1}$. **(M.U. 2001)**

Sol. : We have $f(z) = \frac{1}{\sin(1/z)}$

$$\sin\left(\frac{1}{z}\right) = 0 \text{ when } \frac{1}{z} = n\pi \text{ where, } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The limit point of the sequence of the poles

$$\left\{z = \frac{1}{n\pi}\right\} \; i.e. \; \left\{\frac{1}{\pm\pi}, \frac{1}{\pm 2\pi}, \frac{1}{\pm 3\pi}, \ldots\right\} \text{ is } z = 0.$$

Hence, $z = 0$ is non-isolated essential singularity.

Example 3 : State the nature of the singularity of $f(z) = \left[\sin\frac{\pi}{z}\right]^{-1}$. **(M.U. 2000)**

Sol. : We have $f(z) = \frac{1}{\sin(\pi/z)}$

$$\sin\left(\frac{\pi}{z}\right) = 0 \text{ when } \frac{\pi}{z} = n\pi \text{ where, } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The limit point of the sequence of the pole

$$\left\{z = \frac{1}{n}\right\} \; i.e. \; \left\{\frac{1}{\pm\pi}, \frac{1}{\pm 2\pi}, \frac{1}{\pm 3\pi}, \ldots\right\} \text{ is } z = 0$$

Hence, $z = 0$ is non-isolated essential singularity.

Example 4 : Determine the nature of singularities of

(i) $\frac{z-2}{z^2}\sin\left(\frac{1}{z-1}\right)$ **(M.U. 2005)** (ii) $\frac{z^2+4}{e^z}$ **(M.U. 1998)**

Sol. : (i) Clearly $z = 0$ is a pole of order 2.

$$\text{Now, } \frac{z-2}{z^2}\sin\left(\frac{1}{z-1}\right) = \left(\frac{z-1-1}{z^2}\right)\left[\frac{1}{(z-1)} - \frac{1}{6(z-1)^3} + \ldots\right]$$

$$= \frac{(z-1)}{z^2}\cdot\left[\frac{1}{(z-1)} - \frac{1}{6(z-1)^3} + \ldots\right] - \frac{1}{z^2}\left[\frac{1}{(z-1)} - \frac{1}{6(z-1)^3} - \ldots\right]$$

$$= \frac{1}{z^2}\left[1 - \frac{1}{6(z-1)^2} + \ldots\right] - \frac{1}{z^2}\left[\frac{1}{(z-1)} - \frac{1}{6(z-1)^3} + \ldots\right]$$

$\therefore$ $z = 1$ is an essential singularity.

(ii) $f(z) = \frac{z^2+4}{e^z}$ is analytic everywhere. $z = -\infty$ is its singularity.

(c) Removable Singularity

If $z = z_0$ is a singularity of $f(z)$ such that $\lim_{z \to z_0} f(z)$ exists then $z = z_0$ is called a **removable singularity.**

Alternatively : If the expansion of $f(z)$ about $z = z_0$ [See (2), page 4-3] does not contain negative powers of $(z - z_0)$ *i.e.* $b_n = 0$ for all n then $z = z_0$ is called a **removable singularity** of $f(z)$. (See Ex. 1 below)

Note ....

From the above discussion it is clear that

(i) if $\lim_{z \to z_0} f(z)$ exists then $z = z_0$ is a removable singularity.

This type of singularity can be removed by suitably defining $f(z)$ at z_0.

(ii) $\lim_{z \to z_0} f(z) = \infty$ if z is a pole.

Example 1 : Show that $f(z) = \frac{\sin z}{z}$ has a removable singularity at $z = 0$. **(M.U. 2006)**

Sol. : Now $f(z) = \frac{\sin z}{z}$ is not defined at $z = 0$ but $\lim_{z \to 0}\frac{\sin z}{z} = 1$. Hence, $z = 0$ is a removable singularity of $f(z)$.

Laurent's expansion of $f(z)$ is

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right] = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$$

It does not contain negative powers of z.

The singularity is removed if we ***define*** $f(z) = \frac{\sin z}{z} = 0$ at $z = 0$.

Example 2 : Show that $f(z) = \dfrac{1-\cos z}{z}$ has a removable singularity at $z = 0$.

Sol. : Now $f(z) = \dfrac{1-\cos z}{z}$ is not defined at $z = 0$, but

$$\lim_{z\to 0} \frac{1-\cos z}{z} = \lim_{z\to 0} \frac{2\sin^2(z/2)}{z} = \lim_{z\to 0} \frac{2\sin^2(z/2)}{(z/2)^2}\cdot\frac{z}{2} = 0$$

Hence, $z = 0$ is a removable singularity.

Laurent's expansion of $f(z)$ is

$$f(z) = \frac{1}{z}\left[1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots\right)\right] = \frac{z}{2!}-\frac{z^3}{4!}+\frac{z^5}{6!}-\ldots$$

It does not contain negative powers of z.

The singularity is removed if we *define* $f(z) = \dfrac{1-\cos z}{z} = 0$ at $z = 0$.

(d) Singularity At ∞

The singularity of $f(z)$ at $z = \infty$ is the same as the singularity of $f(1/\omega)$ at at $\omega = 0$.

Example : $f(z) = z^4$ has a pole of order 4 at $z = \infty$, since $f(\omega) = \dfrac{1}{\omega^4}$ has a pole of order 4 at $\omega = 0$.

Notes ....

1. A function which is analytic everywhere in the finite z-plane is called an **entire function** or an **integral function.** An entire function can be expressed as a Taylor's series where radius of convergence is ∞ and conversely a power series whose radius of convergence is ∞ is an **entire** or **integral function.**

   For example, functions e^z, $\sin z$, $\cos nz$ are entire functions.

2. A function which is analytic everywhere in the finite z-plane except at a finite number of poles is called a **mermorphic function.**

   For example, $f(z) = \dfrac{1}{z(z-1)^2}$, $f(z) = \dfrac{3z}{(z-1)^2(z-2)^3}$ are mermorphic functions, since they have finite number of poles.

Example 3 : Determine the nature of singularities of the following functions.

(i) $\dfrac{e^z}{(z-1)^4}$, **(ii)** $(z+1)\cdot\sin\left(\dfrac{1}{z+2}\right)$, **(iii)** $\dfrac{1}{z^2(e^z-1)}$, **(iv)** $e^{1/z}$,

(v) $\dfrac{\cot \pi z}{(z-a)^3}$, **(vi)** $\sin\left(\dfrac{1}{z-1}\right)$, **(vii)** $\sec\left(\dfrac{1}{z}\right)$, **(viii)** $\left[\sin\dfrac{1}{z}\right]^{-1}$ **(M.U. 2001)**

Sol. : (i) $z = 1$ is a pole of order 4.

(ii) We have $f(z) = (z+2-1)\cdot\sin\left(\dfrac{1}{z+2}\right)$

$$\therefore\quad f(z) = [(z+2)-1]\cdot\left[\frac{1}{z+2}-\frac{1}{3!(z+2)^3}+\frac{1}{5!(z+2)^5}-\ldots\right]$$

$$= (z+2)\left[\frac{1}{z+2}-\frac{1}{3!(z+2)^3}+\frac{1}{5!(z+2)^5}-\ldots\right] - 1\left[\frac{1}{z+2}-\frac{1}{3!(z+2)^3}+\frac{1}{5!(z+2)^5}-\ldots\right]$$

$$= 1-\frac{1}{(z+2)}-\frac{1}{3!(z+2)^2}+\frac{1}{3!(z+2)^3}-\ldots$$

Since the principal part of $f(z)$ contains infinite number of terms in $(z+2)$, $z = -2$ is an isolated essential singularity.

(iii) We have $f(z) = \dfrac{1}{z^2(e^z-1)}$

The singularities are given by $z^2(e^z-1) = 0 \quad \therefore z = 0$

or $e^z = 1 = e^{2n\pi i}$, $n = 0, \pm 1, \pm 2, \ldots$

Hence, $z = 0$ is a pole of order **three**. The other singularities $\pm 2\pi, \pm 4\pi, \ldots$ are simple poles.

(iv) $f(z) = e^{1/z} = 1+\dfrac{1}{z}+\dfrac{1}{z^2\cdot 2!}+\dfrac{1}{z^3\cdot 3!}+\ldots$

Since, the principle part contains an infinite number of terms, $z = 0$ is an isolated essential singularity.

(v) Singularities of $f(z) = \dfrac{\cot \pi z}{(z-a)^3}$ *i.e.*, of $f(z) = \dfrac{\cos \pi z}{(z-a)^3 \sin \pi z}$ are given by $z - a = 0$ and $\sin \pi z = 0$.

$\therefore$ $z = a$ is a pole of order three.

Further, $\sin n\pi = 0$, $n = 0, \pm 1, \pm 2, \ldots$ *i.e.* $z = n$.

Hence, $z = 0, \pm 1, \pm 2, \ldots$ are simple poles.

(vi) $f(z) = 0$ when $\dfrac{1}{z-1} = n\pi$ *i.e.* $z - 1 = \dfrac{1}{n\pi}$ *i.e.* $z = 1+\dfrac{1}{n\pi}$ when $n = 0, \pm 1, \pm 2, \ldots$

The limit point of the sequence of poles is 1. Hence, $z = 1$ is non-isolated essential singularity.

(vii) $f(z) = \sec\dfrac{1}{z} = \dfrac{1}{\cos(1/z)} = 0$ when $\dfrac{1}{z} = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots$

i.e., $z = \pm\dfrac{2}{\pi}, \pm\dfrac{2}{3\pi}, \pm\dfrac{2}{5\pi}, \ldots$

The limit point of the sequence of poles is $z = 0$. Hence, $z = 0$ is non-isolated essential singularity.

(viii) $f(z) = \dfrac{1}{\sin(1/z)}$ is singular when $\sin\dfrac{1}{z} = 0$ *i.e.* when $\dfrac{1}{z} = 0$ or $n\pi$.

$\therefore$ The singularities are $z = 0, \pm\dfrac{1}{\pi}, \pm\dfrac{1}{2\pi}, \pm\dfrac{1}{3\pi}, \ldots$

These are all isolated singularities.

EXERCISE - II

1. Determine the nature of singularities, if any, for the following functions.

(i) $f(z) = \frac{z-2}{4} \sin\left(\frac{1}{z-1}\right)$ (M.U. 1997) (ii) $f(z) = \frac{z^2+1}{e^z}$ (M.U. 1997) (iii) $f(z) = z^2 e^{1/z}$

(iv) $f(z) = \frac{\sin 3z}{z}$ (v) $f(z) = (z+1)\, e^{1/(z-3)}$ (vi) $\frac{\sin h(z-z_0)}{z-z_0}$

(vii) e^{-1/z^2} (M.U. 2005)

[Ans. : (i) $z = 1$ is an essential singularity. (ii) $z = 0$ is an ordinary point. (iii) $z = 0$ is an essential singularity. (iv) $z = 0$ is a removable singularity. (v) $z = 3$ is an essential singularity. (vi) $z = z_0$ is a removable singularity. (vii) No singularity.]

2. Determine the nature of singularities of the following functions.

(i) $\frac{\tan z}{z}$, (ii) $z^3 \cdot e^{1/(z-1)}$, (iii) $\frac{z^2+1}{(z-1)^2(z+1)}$, (iv) $z^2 e^{-z}$,

(v) $\frac{2-e^z}{z^3}$, (vi) $\frac{z}{\cos z}$, (vii) $\frac{z}{\sin z}$.

[Ans. : (i) Removable singularity at $z = 0$. (ii) $z = 1$ is an essential singularity. (iii) $z = +1$ is an isolated singularity of order 2 and $z = -1$ is an isolated singularity of order 1. (iv) $z = 0$ is an ordinary point. (v) $z = 0$ is a pole of order 3. (vi) $z = (2n+1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots..$ each is a simple pole. (vii) $z = n\pi$, $n = \pm 1, \pm z, \ldots..$ each is a simple pole.]

3. Expand each of the following functions in Laurent's series about $z = 0$ and identify the singularity.

(i) $z^2 e^{-z^2}$, (ii) $\frac{1-\cos 2z}{z}$, (iii) $z^{-1} e^{-z}$, (iv) $z^{-3} e^z$,

(v) $(z+1)\sin\frac{1}{z}$, (vi) $\frac{1-e^z}{z}$, (vii) $\frac{1-e^{2z}}{z^3}$.

[Ans. : (i) $z^2 e^{-z^2} = z^2\left[1 - z^2 + \frac{z^4}{2!} - \frac{z^6}{3!} + \ldots..\right] = z^2 - 1 + \frac{z^6}{2!} - \frac{z^8}{3!} + \ldots..$

$\therefore z = 0$ is an ordinary point.

(ii) $$\frac{1-\cos 2z}{z} = \frac{1}{z}\left[1 - \left(1 - \frac{4z^2}{2!} + \frac{16z^4}{4!} - \ldots..\right)\right]$$

$$= 2z - \frac{2}{3}z^3 + \ldots.. ; \quad \lim_{z\to 0} \frac{1-\cos 2z}{z} = 0$$

$\therefore z = 0$ is a removable singularity.

(iii) $$z^{-1} e^{-z} = \frac{1}{z}\left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \ldots..\right) = \frac{1}{z} - 1 + \frac{z}{2!} - \frac{z^2}{3!} + \ldots..$$

$\therefore z = 0$ is a pole of order 1 *i.e.*, a simple pole.

(iv) $$z^{-3} e^z = \frac{1}{z^3}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots..\right) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z \cdot 2!} + \frac{1}{3!} + \frac{z}{4!} + \ldots..$$

$\therefore z = 0$ is a pole of order 3.

(v) $$(z+1)\sin\frac{1}{z} = (z+1)\left[1 - \frac{1}{z^3 \cdot 3!} + \frac{1}{z^5 \cdot 5!} - \ldots..\right]$$

$$= z - \frac{1}{z^2 \cdot 3!} + \frac{1}{z^4 \cdot 5!} - \ldots.. + 1 - \frac{1}{z^3 \cdot 3!} + \ldots..$$

$\therefore z = 0$ is an essential singularity.

(vi) $$\frac{1-e^z}{z} = \frac{1}{z}\left[1 - \left(1 + z + \frac{z^2}{2!} + \ldots..\right)\right]$$

$$= -1 - \frac{z}{2!} - \frac{z^2}{3!} - \ldots.. ; \quad \lim_{z\to 0} \frac{1-e^z}{z} = -1.$$

$\therefore z = 0$ is a removable singularity.

(vii) $$\frac{1-e^{2z}}{z^3} = \frac{1}{z^3}\left[1 - \left(1 + 2z + \frac{4z^2}{2!} + \frac{8z^3}{3!} + \ldots..\right)\right] = \frac{2}{z^2} + \frac{2}{z} + \frac{4}{3} + \frac{2}{3}z + \ldots..$$

$\therefore z = 0$ is a pole of order 2.]

4. Expand each of the following functions in Laurent's series about $z = 0$ and identify the singularity

(i) $z e^{1/z^2}$ (ii) $\frac{\sin^2 z}{z}$ (iii) $\frac{1}{z(4-z)}$ (M.U. 2004)

[Ans. : (i) $z e^{1/z^2} = z + \frac{1}{z} + \frac{1}{2!\, z^3} + \ldots..$ $\therefore z = 0$ is an essential singularity.

(ii) $\frac{\sin^2 z}{z} = z - \frac{1}{3} \cdot z^3 + \frac{2}{45} z^5 - \ldots..$ $\therefore z = 0$ is an ordinary point.

(iii) $$\frac{1}{z(4-z)} = \frac{1}{4z(z-(z/4))} = \frac{1}{4z}\left(1 - \frac{z}{4}\right)^{-1}$$

$$\frac{1}{z(4-z)} = \frac{1}{4z}\left(1 + \frac{z}{4z} + \frac{z^2}{64} + \frac{z^3}{256} + \ldots..\right) = \frac{1}{4z} + \frac{1}{16} + \frac{z}{64} + \ldots..$$

$\therefore z = 0$ is a simple pole.]

4. Residues

If $z = z_0$ is an isolated singularity then the constant b_1 *i.e.* **the coefficient of** $\frac{1}{z - z_0}$ in the Laurent's expansion of $f(z)$ at $z = z_0$ is called the **residue** of $f(z)$ at $z = z_0$.

$\therefore$ Residue of $f(z)$ (at $z = z_0$) $= b_1 =$ coefficient of $\frac{1}{z - z_0}$

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$$= \frac{1}{2\pi i} \oint_C f(z)\, dz$$ [See formula (2) on page 3-4 and put $n = 1$]

Or $\oint_C f(z)\, dz = 2\pi i$ (Residue at $z = z_0$)

Calculation of Residues at Poles

(i) If $z = z_0$ is a simple pole of $f(z)$ then

$$\boxed{\textbf{Residue of } f(z) \textbf{ (at } z = z_0) = \lim_{z \to z_0} (z - z_0) f(z)}$$

Proof : Since $z = z_0$ is a simple pole, the Laurent's series of

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z - z_0)}$$ becomes

$$\therefore \quad (z - z_0) f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+1} + b_1$$

$$\therefore \quad \lim_{n \to z_0} (z - z_0) f(z) = b_1 = \text{Residue of } f(z) \text{ (at } z = z_0)$$

(ii) If $z = z_0$ is a simple pole of $f(z) = \dfrac{P(z)}{Q(z)}$ then

$$\boxed{\textbf{Residue of } f(z) \textbf{ (at } z = z_0) = \lim_{z \to z_0} \frac{P(z)}{Q'(z)}}$$

Proof : Since $z = z_0$ is a simple pole by the above result,

$$\text{Residue (at } z = a_0) = \lim_{z \to z_0} \left[(z - z_0) \frac{P(z)}{Q(z)} \right]$$

Since at $z = z_0$, $Q(z) = 0$, the limit in the r.h.s. is of the form $\frac{0}{0}$. Hence, by Hospital's rule.

$$\text{Residue (at } z = a_0) = \lim_{z \to z_0} \left[\frac{(z - z_0) P'(z) + P(z)}{Q'(z)} \right] = \lim_{z \to z_0} \frac{P(z)}{Q'(z)}$$

(iii) If $z = z_0$ is a pole of order m then

$$\boxed{\textbf{Residue of } f(z) \textbf{ (at } z = z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right]}$$

Proof : Since $z = z_0$ is a pole of order m, the Laurent's series of $f(z)$ becomes

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots\ldots + \frac{b_m}{(z - z_0)^m}$$

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{m+n} + b_1 (z - z_0)^{m-1} + b_2 (z - z_0)^{m-2} + \ldots\ldots + b_m$$

Differentiating this $(m - 1)$ times and taking the limit as $z \to z_0$, we get

$$\frac{1}{m - 1!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) = b_1 = \text{Residue of } f(z) \text{ (at } z = z_0).$$

Note ....

1. To find the residues and to evaluate an integral using residues you are advised to draw a rough sketch of the contour or find the distance of centre z_0 of the given circle from the pole z_1 by using

$$| z_1 - z_0 | = | (x_1 - x_0) + i (y_1 - y_0) | = [(x_1 - x_0)^2 + (y_1 - y_0)^2]$$

and also use L' Hospitals rule wherever necessary.

2. Residue at $z = z_0$ can be obtained either by limit method (Ex. 1) or by expansion method [Ex. 2 (ii), Ex. 3].

Type I : To find the residues by taking limits

Example 1 : Determine the pole of the function $f(z) = \dfrac{z^2}{(z-1)^2 (z+2)}$ and also find the residue at each pole. **(M.U. 1998, 2003, 05, 11)**

Sol. : $\because$ $(z - 1)^2 (z + 2) = 0$ gives $z = -2$, 1 and 1. Hence, $f(z)$ has a simple pole at $z = -2$ and a pole of order 2 at $z = 1$.

(i) Residue of $f(z)$ (at $z = -2$) $= \lim_{z \to -2} (z + 2) f(z) = \lim_{z \to -2} \dfrac{z^2}{(z-1)^2} = \dfrac{4}{9}$

(ii) Residue of $f(z)$ (at $z = 1$) $= \lim_{z \to 1} \dfrac{d}{dz} \left[(z-1)^2 f(z) \right] = \lim_{z \to 1} \dfrac{d}{dz} \left[\dfrac{z^2}{z+2} \right]$

$$= \lim_{z \to 1} \frac{(z+2) 2z - z^2}{(z+2)^2} = \lim_{z \to 1} \frac{z^2 + 4z}{(z+2)^2} = \frac{5}{9}.$$

Example 2 : Determine the nature of poles of the following functions and find the residue at each pole.

(i) $\dfrac{z e^z}{(z-a)^3}$ **(M.U. 1998)** **(ii)** $\dfrac{1 - e^{2z}}{z^3}$ **(M.U. 1996, 2005)**

Sol. : (i) $z = a$ is a pole of order 3.

$\therefore$ Residue of $f(z)$ (at $z = a$)

$$= \lim_{z \to a} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z-a)^3 f(z) \right]$$

$$= \lim_{z \to a} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z-a)^3 \frac{z e^z}{(z-a)^3} \right]$$

$\therefore$ Residue of $f(z)$ (at $z = a$)

$$= \frac{1}{2!} \lim_{z \to a} \frac{d^2}{dz^2} (z e^z) = \frac{1}{2!} \lim_{z \to a} \frac{d}{dz} (z e^z + e^z)$$

$$= \frac{1}{2!} \lim_{z \to a} \left[z e^z + e^z + e^z \right] = \frac{1}{2} (a e^a + 2 e^a)$$

$$= \frac{1}{2} (a + 2) e^a.$$

(ii) $z = 0$ is a pole of order 3.

$\therefore$ Residue of $f(z)$ (at $z = 0$)

$$= \lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}\left[z^3 f(z)\right]$$

$$= \lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{z^3 \cdot (1-e^{2z})}{z^3}\right]$$

$$= \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}(1-e^{2z}) = \frac{1}{2!}\lim_{z\to 0}\frac{d}{dz}(-2e^{2z})$$

$$= \frac{1}{2!}\lim_{z\to 0}(-4e^{2z}) = \frac{1}{2}\cdot(-4) = -2$$

Alternatively

$$f(z) = \frac{1-e^{2z}}{z^3} = \frac{1-\left\{1+2z+\frac{4z^2}{2!}+\frac{8z^3}{3!}+\ldots\right\}}{z^3} = -\frac{2}{z^2}-\frac{2}{z}-\frac{4}{3}-\ldots$$

$\therefore$ Residue (at $z = 0$) $= b_1 =$ coefficient of $\frac{1}{z} = -2$.

Example 3 : Find the residue of $\frac{1-e^{2z}}{z^4}$ at its pole. **(M.U. 1992, 96)**

Sol. : Clearly $z = 0$ is a pole of order 4.

$\therefore$ Residue of $f(z)$ (at $z = 0$)

$$= \lim_{z\to 0}\frac{1}{3!}\frac{d^3}{dz^3}\left[z^4 f(z)\right]$$

$$= \lim_{z\to 0}\frac{1}{3!}\frac{d^3}{dz^3}\left[\frac{z^4\cdot(1-e^{2z})}{z^4}\right] = \frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(1-e^{2z})$$

$$= \frac{1}{3!}\lim_{z\to 0}\frac{d^2}{dz^2}(-2e^{2z}) = \frac{1}{3!}\lim_{z\to 0}\frac{d}{dz}(-4e^{2z})$$

$$= \frac{1}{3!}\lim_{z\to 0}(-8e^{2z}) = -\frac{8}{3!} = -\frac{4}{3}.$$

Alternatively

$$f(z) = \frac{1-\left[1+2z+\frac{4z^2}{2!}+\frac{8z^3}{3!}+\frac{16z^4}{4!}+\frac{32z^5}{5!}+\ldots\right]}{z^4}$$

$$= -\frac{2}{z^3}-\frac{2}{z^2}-\frac{4}{3}\cdot\frac{1}{z}-\frac{2}{3}-\frac{4}{15}z-\ldots$$

$\therefore$ Residue (at $z = 0$) $= b_1 =$ coefficient of $\frac{1}{z} = -\frac{4}{3}$.

Example 4 : Determine the nature of poles and find the residue thereat for $f(z) = z^2 \sec \pi z$. **(M.U. 1996)**

Sol. : We have $f(z) = \frac{z^2}{\cos \pi z}$

The singularities are given by $\cos \pi z = 0$.

$\therefore \quad \pi z = (2n+1)\frac{\pi}{2} \quad \therefore \quad z = \frac{(2n+1)}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$

and each is a simple pole.

$$\therefore \text{ Residue}\left[\text{at } z = \frac{(2n+1)}{2}\right] = \lim_{z\to(2n+1)/2}\frac{[z-(2n+1)/2]\,z^2}{\cos \pi z}$$

$$= \lim_{z\to(2n+1)/2}\left[\frac{2n+1}{2}\right]^2\cdot\frac{z-(2n+1)/2}{\cos\pi z}$$

$$= \left(\frac{2n+1}{2}\right)^2\cdot\lim_{z\to(2n+1)/2}\cdot\frac{1}{-\sin\pi z\cdot\pi} \quad \text{[By L'Hospital's Rule]}$$

$$= \left(\frac{2n+1}{2}\right)^2\left[-\frac{1}{\pi\sin[(2n+1)\pi/2]}\right]$$

$$= (-1)^{n+1}\left(\frac{2n+1}{2}\right)^2\cdot\frac{1}{\pi}$$

Example 5 : Find the residues of $f(z) = \frac{\sin \pi z}{(z-1)^2(z-2)}$ at its poles. **(M.U. 2003)**

Sol.: Clearly $z = 1$ is a pole of order 2 and $z = 2$ is a simple pole.

$$\therefore \text{ Residue (at } z = 1) = \lim_{z\to 1}\frac{1}{1!}\frac{d}{dz}\left[(z-1)^2\cdot\frac{\sin\pi z}{(z-1)^2(z-2)}\right]$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{\sin\pi z}{z-2}\right] = \lim_{z\to 1}\frac{(z-2)(\cos\pi z)\cdot\pi-\sin\pi z}{(z-2)^2}$$

$$= \frac{(1-2)(-1)\pi-0}{(1-2)^2} = \pi$$

$$\therefore \text{ Residue (at } z = 2) = \lim_{z\to 2}(z-2)\cdot\frac{\sin\pi z}{(z-1)^2(z-2)} = \lim_{z\to 2}\frac{\sin\pi z}{(z-1)^2} = 0$$

Example 6 : Find the residues of $\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)^2}$ at its poles. **(M.U. 1998, 2000, 15)**

Sol. : $f(z)$ has a simple pole at $z = 1$ and a pole of order 2 at $z = 2$.

$$\therefore \text{ Residue (at } z = 1) = \lim_{z\to 1}(z-1)\left\{\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)^2}\right\}$$

$$= \lim_{z\to 1}\frac{\sin\pi z^2+\cos\pi z^2}{(z-2)^2} = \frac{\sin\pi+\cos\pi}{(1-2)^2} = -1$$

$$\text{Residue (at } z=2) = \lim_{z\to 2}\frac{1}{1!}\frac{d}{dz}\left[\frac{(z-2)^2(\sin\pi z^2+\cos\pi z^2)}{(z-1)(z-2)^2}\right]$$

$$=\lim_{z\to 2}\frac{d}{dz}\left[\frac{\sin\pi z^2+\cos\pi z^2}{z-1}\right]$$

$$=\lim_{z\to 2}\frac{\left[(z-1)\{(\cos\pi z^2)(2\pi z)-(\sin\pi z^2)(2\pi z)\}-(\sin\pi z^2+\cos\pi z^2)\right]}{(z-1)^2}$$

$$=\frac{(2-1)[4\pi-0-0-1]}{(2-1)^2}=4\pi-1.$$

Type II : To find the residues using Laurent's Series

Example 1 : Find the residues at each pole of the following by Laurent's Expansion.

(i) $z^2e^{1/z}$ (M.U. 2000) (ii) $e^{-1/(z-1)^2}$ (M.U. 2000)

Sol. : (i) Since $f(z)=z^2e^{1/z}=z^2\left[1+\frac{1}{z}+\frac{1}{z^2\cdot 2!}+\frac{1}{z^3\cdot 3!}+.....\right]$

$$=z^2+z+\frac{1}{2!}+\frac{1}{6z}+\frac{1}{4!z^2}+\frac{1}{5!z^3}+.....$$

$z=0$ is an essential singularity.

$\therefore$ Residue (at $z=0$) $=b_1=$ coefficient of $\frac{1}{z}=\frac{1}{6}$.

(ii) $$f(z)=e^{-1/(z-1)^2}=1-\frac{1}{(z-1)^2}+\frac{1}{(z-1)^4\cdot 2!}-.....$$

$z=1$ is an essential singularity.

$\therefore$ Residue (at $z=1$) $=b_1=$ coefficient of $\frac{1}{z-1}=0$.

Example 2 : Find the residue of $f(z)=\frac{1}{z-\sin z}$ at its singularity, using Laurent's series expansion. (M.U. 1997)

Sol. : We have

$$f(z)=\frac{1}{z-\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-......\right]}=\frac{1}{\frac{z^3}{3!}-\frac{z^5}{5!}+.....}=\frac{1}{\frac{z^3}{3!}\left[1-3!\frac{z^2}{5!}+.....\right]}$$

Hence, $z=0$ is a pole of order 3.

Now, $$f(z)=\frac{1}{z^3/3!}\left[1-\frac{1}{20}z^2+.....\right]^{-1}=\frac{6}{z^3}\left[1+\frac{z^2}{20}+.....\right]=\frac{6}{23}+\frac{3}{10}\cdot\frac{1}{z}+.....$$

$\therefore$ Residue (at $z=0$) $=b_1=$ coefficient of $\frac{1}{z}=\frac{3}{10}$.

Example 3 : Find the residue of $f(z)=\frac{z}{\cos z-\cos hz}$ at its singularity, using Laurent's series expansion.

Sol. : We have

$$f(z)=\frac{z}{\left[1-\frac{z^2}{2!}+\frac{z^4}{4!}-.....\right]-\left[1+\frac{z^2}{2!}+\frac{z^4}{4!}+.....\right]}$$

$$=\frac{z}{-z^2-\frac{2}{6!}z^6-.....}=-\frac{1}{z+\frac{2}{6!}z^6+.....}=-\frac{1}{z\left[1+\frac{2}{6!}z^5+.....\right]}$$

Hence, $z=0$ is a simple pole.

Now, $$f(z)=-\frac{1}{z}\left[1+\frac{2}{6!}z^5+.....\right]^{-1}=-\frac{1}{z}\left[1+\frac{2}{6!}z^5+.....\right]=-\frac{1}{z}+\frac{2}{6!}z^4-.....$$

$\therefore$ Residue ($z=0$) $=b_1=$ coefficient of $=\frac{1}{z}=-1$.

Example 4 : Find the residue $f(z)=\frac{1-z}{1-\cos z}$ at its singularity, using Laurent's series expansion.

Sol. : We have

$$f(z)=\frac{1-z}{1-\left[1-\frac{z^2}{2!}+\frac{z^4}{4!}-.....\right]}=\frac{(1-z)}{\frac{z^2}{21}\left[1-\frac{2}{4!}z^2+\frac{2}{6!}z^4-.....\right]}$$

$$=2\left(\frac{1-z}{z^2}\right)\left[1-\frac{2}{4!}z^2+\frac{2}{6!}z^4-.....\right]^{-1}$$

$$=2\left(\frac{1}{z^2}-\frac{1}{z}\right)\left[1+\left(\frac{2}{4!}z^2-.....\right)+\left(\frac{2z^2}{4!}-.....\right)^2+.....\right]$$

$$\therefore\ f(z)=\frac{2}{z^2}-\frac{2}{z}+2\cdot\frac{2}{4!}-2\cdot\frac{2}{4!}z+.....$$

$\therefore$ Residue (at $z=0$) $=b_1=$ coefficient of $\frac{1}{z}=-2$.

Example 5 : Find the residues of at $f(z)=\frac{z}{(z-1)(z+2)^2}$ at its isolated singularities using Laurent's series expansion. (M.U. 2003)

Sol. : Now $\frac{z}{(z-1)(z+2)^2}=\frac{1/9}{z-1}-\frac{1/9}{z+2}+\frac{2/3}{(z+2)^2}$

Clearly $z=1$ and $z=-2$ are isolated singularities.

To find the residue at $z=1$, we have to expand $f(z)$ in powers of $(z-1)$, in $0<|z-1|<r$ and find the coefficient of $\frac{1}{z-1}$ in it.

$$f(z) = \frac{1/9}{(z-1)} - \frac{1/9}{3+(z-1)} + \frac{2/3}{[3+(z-1)]^2}$$

$$= \frac{1}{9} \cdot \frac{1}{(z-1)} - \frac{1}{9 \cdot 3}\left[1+\left(\frac{z-1}{3}\right)\right]^{-1} + \frac{2}{3 \cdot 9}\left[1+\left(\frac{z-1}{3}\right)\right]^{-2}$$

$$= \frac{1}{9} \cdot \frac{1}{(z-1)} - \frac{1}{27}\left[1 - \frac{(z-1)}{3} + \frac{(z-1)^2}{3^2} - \frac{(z-1)^3}{3^3} + \ldots\right]$$

$$+ \frac{2}{27}\left[1 - \frac{2(z-1)}{3} + \frac{2 \cdot 3}{2!} \cdot \frac{(z-1)^2}{3^2} - \frac{2 \cdot 3 \cdot 4}{3!} \cdot \frac{(z-1)^3}{3^3} - \ldots\right]$$

$$= \frac{1}{9} \cdot \frac{1}{(z-1)} - \frac{1}{27}\Sigma(-1)^n \frac{(z-1)^n}{3^n} + \frac{2}{27}\Sigma(-1)^n(n+1)\frac{(z-1)^n}{3^n}$$

The expansion is valid in $\left|\frac{z-1}{3}\right| < 1$ *i.e.* $0 < |z-1| < 3$.

$\therefore$ Residue (at $z = 1$) $= b_1 =$ coefficient of $\frac{1}{z-1} = \frac{1}{9}$.

To find the residue at $z = -2$, we have to expand $f(z)$ in powers of $(z+2)$ in $0 < |z+2| < r$ and find the coefficient of $\frac{1}{z+2}$.

$$f(z) = \frac{1}{9[(z+2)-3]} - \frac{1}{9(z+2)} + \frac{2}{3} \cdot \frac{1}{(z+2)^2}$$

$$= -\frac{1}{27}\left[1-\left(\frac{z+2}{3}\right)\right]^{-1} - \frac{1}{9(z+2)} + \frac{2}{3} \cdot \frac{1}{(z+2)^2}$$

$$= -\frac{1}{27}\left[1 - \frac{(z+2)}{3} + \frac{(z+2)^2}{3^2} - \ldots\right] - \frac{1}{9(z+2)} + \frac{2}{z(z+2)^2}$$

The expansion is valid in $\left|\frac{z+2}{3}\right| < 1$ *i.e.*, $0 < |z+2| < 3$.

$\therefore$ Residue (at $z = -2$) $= b_1 =$ coefficient of $\frac{1}{z+2} = -\frac{1}{9}$.

Type III : To find the sum of the residues

Example 1 : Prove that the sum of the residues of $f(z) = \frac{e^z}{z^2 - a^2}$ is $\frac{\sin ha}{a}$.

Sol. : Clearly the poles of $f(z)$ are given by $z^2 - a^2 = 0$.

$\therefore$ $(z-a)(z+a) = 0$ $\quad \therefore$ $z = a, -a$.

$$r_1 = \text{Residue (at } z = a) = \lim_{z \to a}(z-a) \cdot \frac{e^z}{(z^2-a^2)} = \lim_{z \to a} \frac{e^z}{z+a} = \frac{e^a}{2a}.$$

$$r_2 = \text{Residue (at } z = -a) = \lim_{z \to -a}(z+a) \cdot \frac{e^a}{z^2-a^2} = \lim_{z \to -a} \frac{e^z}{z-a} = \frac{e^{-a}}{-2a}$$

$\therefore$ Sum of the residues $= r_1 + r_2 = \frac{e^a}{2a} - \frac{e^{-a}}{2a} = \frac{1}{a}\left[\frac{e^a - e^{-a}}{2}\right] = \frac{\sin ha}{a}$.

Example 2 : Find the sum of the residues at singular points of **(M.U. 2001, 2016)**

$$f(z) = \frac{z}{(z-1)^2(z^2-1)}.$$

Sol. : We have $f(z) = \frac{z}{(z-1)^2(z-1)(z+1)} = \frac{z}{(z-1)^3(z+1)}$.

$\therefore$ $z = 1$ is a pole of order 3 and $z = -1$ is a simple pole.

$$\therefore \text{ Residue (at } z = 1) = \frac{1}{2!}\lim_{z \to 1}\frac{d^2}{dz^2}\left[(z-1)^3 \cdot \frac{z}{(z-1)^3(z+1)}\right]$$

$$= \frac{1}{2}\lim_{z \to 1}\frac{d^2}{dz^2}\left[\frac{z}{z+1}\right] = \frac{1}{2}\lim_{z \to 1}\frac{d}{dz}\left[\frac{(z+1)-(z)(1)}{(z+1)^2}\right]$$

$$= \frac{1}{2}\lim_{z \to 1}\frac{d}{dz}\left[\frac{1}{(z+1)^2}\right]$$

$$= \frac{1}{2}\lim_{z \to 1}\left[-\frac{2}{(z+1)^3}\right] = \frac{1}{2}\left[-\frac{2}{2^3}\right] = -\frac{1}{8}$$

$$\text{Residue (at } z = -1) = \lim_{z \to -1}\left[(z+1) \cdot \frac{z}{(z+1)(z-1)^3}\right]$$

$$= \lim_{z \to -1}\left[\frac{z}{(z-1)^3}\right] = \frac{-1}{(-1-1)^3} = \frac{1}{8}$$

$\therefore$ Sum of the residues $= -\frac{1}{8} + \frac{1}{8} = 0$.

Example 3 : Find the sum of the residues at singular points of

$$f(z) = \frac{z}{az^2 + bz + c}$$ **(M.U. 2001)**

Sol. : $f(z)$ is singular at $z = \alpha, \beta$ where α, β are the roots of $az^2 + bz + c = 0$.

$\therefore$ Further $z = \alpha$, $z = \beta$ are simple poles.

Also it should be noted that $az^2 + bz + c = a(z-\alpha)(z-\beta)$.

$$\therefore \text{ Residue (at } z = \alpha) = \lim_{z \to \alpha}(z-\alpha) \cdot \frac{z}{a(z-\alpha)(z-\beta)}$$

$$= \lim_{z \to \alpha}\frac{z}{a(z-\beta)} = \frac{\alpha}{a(\alpha-\beta)}$$

$$\text{Residue (at } z = \beta) = \lim_{z \to \beta}(z-\beta) \cdot \frac{z}{a(z-\alpha)(z-\beta)}$$

$\therefore$ Residue (at $z = \beta$) $= \lim_{z \to \beta} \dfrac{z}{a(z-\alpha)} = \dfrac{\beta}{a(\beta - \alpha)}$

$\therefore$ Sum of residues $= \dfrac{\alpha}{a(\alpha-\beta)} + \dfrac{\beta}{a(\beta-\alpha)}$

$= \dfrac{1}{a}\left[\dfrac{\alpha}{\alpha-\beta} - \dfrac{\beta}{\alpha-\beta}\right] = \dfrac{1}{a}\left(\dfrac{\alpha-\beta}{\alpha-\beta}\right) = \dfrac{1}{a}.$

EXERCISE - III

1. Determine the poles of the following and find the residue at each pole.

1. $\dfrac{3z+1}{z(z-2)}$, 2. $\dfrac{z+3}{z(z-1)(z+2)}$, 3. $\dfrac{z+2}{z^2(z-1)}$, 4. $\dfrac{1-e^{2z}}{z^4}$,

5. $\dfrac{e^{2z}}{z^2+\pi^2}$, 6. $\dfrac{z^2-z}{(z+1)^2(z^2+4)}$, (M.U. 2006) 7. $\dfrac{1}{(z^2+1)^3}$, 8. $\dfrac{z}{\cos z}$,

9. $\dfrac{1}{(z^2+a^2)^2}$, 10. $\dfrac{e^z}{(z-1)^3}$, (M.U. 2004) 11. $\dfrac{z^3}{(z-1)(z-2)(z-3)}$,

12. $\dfrac{z+2}{(z-2)(z+1)^2}$, 13. $\dfrac{1}{(z^2+1)^2}$, 14. $\dfrac{1}{z^3+z^5}$ (M.U. 2005)

15. e^{1/z^2} 16. $\dfrac{\sin^2 z}{z^3}$ (M.U. 2002)

[Ans. : (1) $z = 0, z = 2; -1/2, 7/2$. (2) $z = 0, 1, -2; -3/2, 4/3, 1/6$.
(3) $z = 0, z = 1; -3, 3$. (4) $z = 0; -4/3$.
(5) $z = \pi i, z = -\pi i; 1/2\pi i, -1/2\pi i$.
(6) $z = -1, z = \pm 2i; -11/25, (11 \pm 2i)/50$. (7) $z = i, z = -i; 3/16, 3/16$.
(8) $z = (2n+1)\pi/2; (-1)^{n+1}(2n+1)\pi/2$ where n is any integer.
(9) $z = ai, -ai, 1/(4a^3 i), -1/(4a^3 i)$. (10) $z = 1; e/2$.
(11) $z = 1, z = 2, z = 3; 1/2, -8, 27/2$. (12) $z = 2, -1; 4/9, -4/9$.
(13) $z = +i, -i, -i/4, i/4$. (14) $(1/2, -1/2, -1)$,
(15) $(0, 0)$, (16) $(0, 1/4)$]

2. Find the sum of the residues at singular points of the following functions.

(i) $\dfrac{z}{2z^2+3z+1}$ (ii) $\dfrac{z}{3z^2+2z+1}$ (iii) $\dfrac{z-4}{z(z-1)(z-2)}$

(iv) $\dfrac{z}{(z-1)(z-2)^2}$ (v) $\dfrac{z}{z^3+1}$ (M.U. 2002)

[Ans. : (i) 1/2, (ii) 1/3, (iii) 0, (iv) 0, (v) 0]

3. Find the residue of the following.

(i) $\dfrac{e^z}{\sin z}$ at $z = 0$, (ii) $\dfrac{1+e^z}{z\cos z + \sin z}$ at $z = 0$, (iii) $\dfrac{e^z}{z^2+a^2}$ at $z = ai$.

[Ans. : (i) 1, (ii) 1, use Hospital's rule for (i) and (ii). (iii) $e^{ai}/2ai$]

4. Find the residues of the following functions at their singularities, using Laurent's series expansion.

(i) $\dfrac{z}{\cos hz - \cos z}$ (ii) $\dfrac{1+z}{1-\cos z}$ (iii) $z^3 \cdot e^{1/z}$ (M.U. 2006)

(iv) $\dfrac{1}{z - \sin hz}$ (v) $\dfrac{z^2}{\sin hz - \sin z}$ (vi) $\operatorname{cosec}^2 z$ (M.U. 2004)

[Ans. : (i) 1, (ii) 2, (iii) 1/24, (iv) 3/10, (v) 3, (vi) 0]

5. Determine the nature of poles and find the residue at each pole of

$\dfrac{z^2+1}{(z^2-1)(z^2+4)}$ (M.U. 1997) [Ans. : $z = 1, -1, 2i, -2i; 1, -1, -\dfrac{3i}{20}, \dfrac{3i}{20}$.]

6. Prove that the sum of the residues of

(i) $f(z) = \dfrac{e^z}{z^2+a^2}$ is $\dfrac{\sin a}{a}$, (ii) $f(z) = \dfrac{e^{-z}}{z+a^2}$ is $-\dfrac{\sin a}{a}$,

(iii) $f(z) = \dfrac{e^{-z}}{z^2-a^2}$ is $-\dfrac{\sin ha}{a}$.

7. Find the residue of $f(z) = \dfrac{\sin z}{z\cos z}$ at its pole inside the circle $|z| = 2$. (M.U. 2005)

[Ans. : 0]

8. Find the residue of $f(z) = z^2 \sin\dfrac{1}{z}$ at $z = 0$. (M.U. 2000) [Ans. : $-\dfrac{1}{6}$]

5. Cauchy's Residue Theorem

If $f(z)$ is analytic inside and on a simple closed curve C, except at a finite number of isolated singular points $z_1, z_2 \ldots z_n$ inside C then (M.U. 2001)

$$\boxed{\oint_C f(z)\, dz = 2\pi i \text{ [sum of residues at } z_1, z_2 \ldots z_n \text{]}}$$

Note ....

When integration of $f(z)$ is carried out along a simple closed curve C, it is denoted by $\oint$. But when integration is carried out putting a small circle on the integration sign $\int$ like this $\oint$. But when integration is carried out along a specified circle or an ellipse or a rectangle which obviously is a simple closed curve we may denote the integral by $\int$ or by $\oint$.

Proof : We draw small circles $C_1, C_2, \ldots, C_n$ with centres at $z_1, z_2, \ldots, z_n$ which lie wholly inside C and which do not intersect mutually. Since, $f(z)$ is now analytic in multiply connected region bounded by $C, C_1, C_2, \ldots, C_n$, we have by Cauchy's Integral Theorem

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \ldots + \oint_{C_n} f(z)\,dz$$

$$= 2\pi i[\text{Res. at } z_1] + 2\pi i[\text{Res. at } z_2] + \ldots + 2\pi i[\text{Res. at } z_n]$$

$$= 2\pi i \text{ [Sum of residues at } z_1, z_2, \ldots, z_n]$$

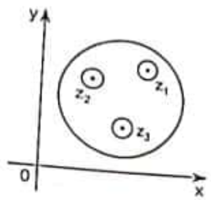

Fig. 4.4

Note ....

Cauchy's integral formula for $f(z_0)$ that $\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$ and for $f^n(z)$ that $\oint_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$ can be deduced from Cauchy's residue theorem.

Proof : Let $f(z)$ be an analytic function on and inside a closed curve C of a simple connected region R and let z_0 be any point within C and consider $\oint_C \frac{f(z)}{z-z_0}\,dz$.

Now $\frac{f(z)}{z-z_0}$ has a simple pole at $z = z_0$.

Further residue (at $z = z_0$) $= \lim_{z\to z_0} (z-z_0)\cdot\frac{f(z)}{(z-z_0)} = \lim_{z\to z_0} f(z) = f(z_0)$

$\therefore$ By Cauchy's residue theorem

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\,(\text{Residue}) = 2\pi i f(z_0)$$

No consider $\int_C \frac{f(z)}{(z-z_0)^n}\,dz$. It has a pole of order n at $z = z_0$.

And residue (at $z = z_0$) $= \frac{1}{(n-1)!} \lim_{z\to 0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n \cdot \frac{f(z)}{(z-z_0)^n}$

$$= \frac{1}{(n-1)!}\cdot f^{n-1}(z_0)$$

$\therefore$ By Cauchy's residue theorem

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = 2\pi i\,(\text{Residue}) = 2\pi i \frac{1}{(n-1)!} f^{n-1}(z_0).$$

Example 1 : Evaluate $\int_C \frac{dz}{\sin hz}$ where C is $x^2 + y^2 = 16$. (M.U. 2003)

Sol. : Here $f(z) = \frac{1}{\sin hz}$ and C is $|z| = 4$. $\therefore$ $z = 0$ is a simple pole.

$\therefore$ Residue (at $z = 0$) $= \lim_{z\to 0} z\cdot\frac{1}{\sin hz} = \lim_{z\to 0}\frac{z}{\sin hz} = 1.$

$\therefore$ $\int_C \frac{dz}{\sin hz} = 2\pi i\,(1) = 2\pi i.$

Example 2 : Evaluate $\int_C \frac{dz}{\cos z}$ where C is $|z| = 2$. (M.U. 2000)

Sol. : Singularities of $f(z) = \frac{1}{\cos z}$ are given $\cos z = 0$.

$\therefore$ $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

Now, the poles lying within $|z| = 2$ are $\pm\pi/2$ which are simple poles.

$\therefore$ Residue (at $z = \pi/2$) $= \lim_{z\to\pi/2}\left(z - \frac{\pi}{2}\right)\cdot\frac{1}{\cos z}$ $\left[\text{Form } \frac{0}{0}\right]$

$= \lim_{z\to\pi/2}\frac{1}{-\sin z} = -1$ [By Hospital's rule]

Similarly, residue (at $z = -\pi/2$) $= -(-1) = 1$.

$\therefore$ $\int_C \frac{dz}{\sin z} = 2\pi i$ (Sum of the residues) $= 2\pi(-1+1) = 0$.

Example 3 : Evaluate $\int_C \frac{dz}{z\sin z}$ where C is $x^2 + y^2 = 1$.

Sol. : Poles are given by $z\sin z = 0$ *i.e.* $z = 0$ and $\sin z = 0$.

$\therefore$ $z = 0$ and $z = n\pi$, $n = 0, \pm 1, \pm 2$.

Thus, $z = 0$ is a pole of order 2 in C.

$\therefore$ Residue $f(z)$ (at $z = 0$) $= \lim_{z\to 0}\frac{d}{dz}\left[z^2\cdot\frac{1}{z\sin z}\right] = \lim_{z\to 0}\frac{d}{dz}\left(\frac{z}{\sin z}\right)$

$= \lim_{z\to 0}\frac{\sin z - z\cos z}{\sin^2 z}$ $\left[\text{Form } \frac{0}{0}\right]$

$= \lim_{z\to 0}\frac{\cos z - \cos z + z\sin z}{2\sin z\cos z}$ [By Hospital's Rule]

$= \lim_{z\to 0}\frac{z}{2\cos z} = 0$

$\therefore$ $\int_C \frac{dz}{z\sin z} = 0.$

Example 4 : Evaluate $\int_C \frac{z^2}{(z-1)^2(z-2)}\,dz$ where C is the circle $|z| = 2{\cdot}5$.

(M.U. 1993, 2001, 03, 05)

Sol. : $f(z)$ has a simple pole at $z = 2$ and a pole of order 2 at $z = 1$.
Both poles lie inside C.

$$\therefore \text{ Residue (at } z = 2) = \lim_{z \to 2} (z-2) \cdot \frac{z^2}{(z-1)^2 (z-2)} = \lim_{z \to 2} \frac{z^2}{(z-1)^2} = 4.$$

$$\text{Residue (at } z = 1) = \lim_{z \to 1} \frac{d}{dz} (z-1)^2 \cdot \frac{z^2}{(z-1)^2 (z-2)} = \lim_{z \to 1} \frac{d}{dz} \cdot \frac{z^2}{z-2}$$

$$= \lim_{z \to 1} \frac{(z-2)\,2z - z^2}{(z-2)^2} = -3$$

$$\therefore \int_C f(z)\,dz = 2\pi i \text{ (Sum of residues)} = 2\pi i (4-3) = 2\pi i$$

Note ....

This example and also the other examples which follow can also be solved by using Cauchy's Integral Theorem.

Example 5 : Using Cauchy's residue theorem evaluate $\oint_C \frac{z^2+3}{z^2-1}\,dz$ where C is the circle (i) $|z-1| = 1$, (ii) $|z+1| = 1$. **(M.U. 2016)**

Sol. : The poles are given by $z^2 - 1 = 0$.

$\therefore\ (z-1)(z+1) = 0 \qquad \therefore\ z = 1, -1$ are simple poles.

(i) Now $|z-1| = 1$ is a circle with centre at $(1, 0)$ and radius 1.

$\therefore\ z = 1$ lies inside C and $z = -1$ lies outside C.

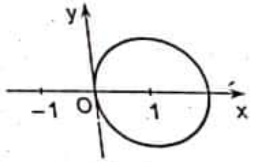

Fig. 4.5

$$\therefore \text{ Residue at } (z = 1) = \lim_{z \to 1} (z-1) \cdot \frac{z^2+3}{(z-1)(z+1)}$$

$$= \lim_{z \to 1} \frac{z^2+3}{z+1} = \frac{4}{2} = 2$$

$$\therefore \oint_C \frac{z^2+3}{z^2-1}\,dz = 2\pi i (2) = 4\pi i$$

(ii) $|z+1| = 1$ is a circle with centre at $(-1, 0)$ and radius 1.

$\therefore\ z = -1$ lies inside C and $z = 1$ lies outside C.

$$\therefore \text{ Residue at } (z = -1) = \lim_{z \to -1} (z+1) \cdot \frac{z^2+3}{(z+1)(z-1)}$$

$$= \lim_{z \to -1} \frac{z^2+3}{z-1} = \frac{4}{-2} = -2$$

$$\therefore \oint_C \frac{z^2+3}{z^2-1}\,dz = 2\pi i(-2) = -4\pi i$$

Example 6 : Evaluate $\int_C \frac{z^2}{(z-1)^2 (z+1)}\,dz$ where C is (i) $|z| = 1/2$, (ii) $|z| = 2$.

Sol. : $f(z)$ has a simple pole at $z = -1$ and a pole of order two at $z = 1$.

(i) If C is $|z| = 1/2$, both poles lie outside C and hence by Cauchy's Theorem

$$\int_C f(z)\,dz = \int_C \frac{z^2}{(z-1)^2 (z+1)}\,dz = 0$$

(ii) If C is $|z| = 2$, both poles lie inside C.

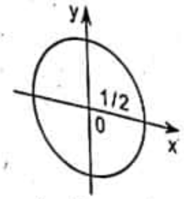

Fig. 4.6

Now, $$\text{Residue (at } z = -1) = \lim_{z \to -1} (z+1)\,f(z)$$

$$= \lim_{z \to -1} \frac{z^2}{(z-1)^2} = \frac{1}{4}.$$

$$\text{Residue (at } z = 1) = \lim_{z \to 1} \frac{d}{dz}\left[(z-1)^2 f(z)\right] = \lim_{z \to 1} \frac{d}{dz}\left[\frac{z^2}{z+1}\right]$$

$$= \lim_{z \to 1} \frac{(z+1)\,2z - z^2}{(z+1)^2} = \lim_{z \to 1} \frac{z^2+2z}{z+1} = \frac{3}{2}$$

$$\therefore \int_C f(z)\,dz = 2\pi i \text{ (sum of residues)} = 2\pi i \left[\frac{1}{4} + \frac{3}{2}\right] = \frac{7\pi i}{2}.$$

Example 7 : Evaluate $\oint_C \frac{e^z}{\cos \pi z}\,dz$ where C is the circle $|z| = 1$. **(M.U. 1998, 2002)**

Sol. : $f(z)$ has simple poles when $\cos \pi z = 0$ *i.e.* when $z = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}$..... Of these $z = \pm\frac{1}{2}$ lie within the circle $|z| = 1$.

$$\therefore \text{ Residue}\left(\text{at } z = \frac{1}{2}\right) = \lim_{z \to 1/2} \frac{[z-(1/2)]\,e^z}{\cos \pi z} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z \to 1/2} \frac{[z-(1/2)] \cdot e^z + e^z}{-\pi \cdot \sin \pi z} \qquad \text{[L' Hospital]}$$

$$= -\frac{e^{1/2}}{\pi}$$

Similarly, $$\text{Residue}\left(\text{at } z = -\frac{1}{2}\right) = \frac{e^{-1/2}}{\pi}$$

$$\therefore \oint_C \frac{e^z}{\cos \pi z}\,dz = 2\pi i \left[-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi}\right] = -4i\left(\frac{e^{1/2} - e^{-1/2}}{2}\right) = -4i \sinh\left(\frac{1}{2}\right)$$

Example 8 : Evaluate $\int \frac{z \sec z}{(1-z^2)}\,dz$ where C is the (i) ellipse $4x^2 + \frac{9}{4}y^2 = 9$. (ii) circle $|z| = 2$.

Sol. : (i) $f(z) = \frac{z}{(1-z^2)\cos z}$. It has simple poles at $z = \pm 1$ and $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$...... The ellipse $\frac{x^2}{9/4}+\frac{y^2}{4}=1$ intersects the x-axis at $x=\pm\frac{3}{2}$ and the y-axis at $y=\pm 2$. Hence, the only poles which lie within the ellipse are ± 1.

$$\therefore \text{ Residue (at } z=1) = \lim_{z\to 1}(z-1)\frac{z}{(1-z^2)\cos z}$$

$$= \lim_{z\to 1} -\frac{z}{(1+z)\cos z}$$

$$= -\frac{1}{2\cos(1)}$$

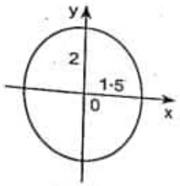

Fig. 4.7

Similarly, Residue (at $z=-1$) $= -\frac{1}{2\cos(1)}$

$$\therefore \oint_C \frac{z\sec z}{(1-z^2)}dz = 2\pi i\left[-\frac{1}{2\cos(1)}-\frac{1}{2\cos(1)}\right] = -\frac{2\pi i}{\cos(1)}.$$

(ii) There are four poles within the circle $|z|=2$. They are ± 1, $\pm\frac{\pi}{2}$.

As before the residues at $z=+1$ and $z=-1$ are each $-\frac{1}{2\cos 1}$.

$$\text{Residue}\left(\text{at } z=\frac{\pi}{2}\right) = \lim_{z\to\pi/2}\left(z-\frac{\pi}{2}\right)\cdot\frac{z}{(1-z^2)\cos z}$$

$$= \lim_{t\to 0} t\cdot\frac{(\pi/2+t)}{[1-(\pi/2+t)^2]}\cdot\frac{1}{(-\sin t)} \quad \text{where } z-\frac{\pi}{2}=t$$

$$= -\frac{\pi/2}{1-(\pi/2)^2} = \frac{2}{\pi(\pi^2-2^2)} \qquad \left[\because \lim_{t\to 0}\left(\frac{t}{\sin t}\right)=1\right]$$

Similarly, Residue $\left(\text{at } z=-\frac{\pi}{2}\right) = \frac{2}{\pi(\pi^2-2^2)}$

$$\therefore \int_C \frac{z\sec z}{(1-z)^2}dz = 2\pi i \text{ (Sum of the residues)}$$

$$= 2\pi i\left[-\frac{1}{\cos 1}+\frac{4}{\pi(\pi^2-2^2)}\right] = \frac{8i}{\pi^2-2^2}-\frac{2\pi i}{\cos(1)}.$$

Example 9 : Evaluate $\int_C \tan z\,dz$ where C (i) is the circle $|z|=2$, (ii) is the circle $|z|=1$.

(M.U. 1995, 98, 2009)

Sol. : The poles of $\tan z = \frac{\sin z}{\cos z}$ are given by $\cos z = 0$ i.e. $z=\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$......

(i) Of these $z=+\pi/2$ and $z=-\pi/2$ lie inside the circle

$$\therefore \text{ Residue}\left(\text{at } z=\frac{\pi}{2}\right) = \lim_{z\to\pi/2}\frac{[z-(\pi/2)]\cdot\sin z}{\cos z}$$

$$= \lim_{z\to\pi/2}\frac{[z-(\pi/2)]\cos z+\sin z}{-\sin z} = -1 \qquad \text{[By L' Hospital's Rule]}$$

$$\text{Residue}\left(\text{at } z=-\frac{\pi}{2}\right) = \lim_{z\to-\pi/2}\frac{[z+(\pi/2)]\cdot\sin z}{\cos z}$$

$$= \lim_{z\to-\pi/2}\frac{[z+(\pi/2)]\cos z+\sin z}{-\sin z} \qquad \text{[By L' Hospital's Rule]}$$

$$= -1$$

$$\therefore \oint_C \tan z\,dz = 2\pi i(-1-1) = -4\pi i.$$

(ii) Of these poles, none lies within the circle $|z|=1$.

Hence, by Cauchy's Integral Theorem $\oint_C \tan z\,dz = 0$.

Example 10 : Evaluate $\oint_C \frac{z-1}{z^2+2z+5}dz$ where C is the circle

(i) $|z|=1$, (ii) $|z+1+i|=2$, (iii) $|z+1-i|=2$. **(M.U. 2010)**

Sol. : $(z^2+2z+1)+4=0 \qquad \therefore z+1+2i=0,\ z+1-2i=0.$

$\therefore z=-1-2i,\ z=-1+2i$ are simple poles.

(i) Since both pole lie outside the circle $|z|=1$.

By Cauchy's Integral Theorem

$$\oint_C \frac{z-1}{z^2+2z+5}dz = 0 \text{ where } C \text{ is } |z|=1.$$

(ii) The centre C of the circle $|z+1+i|=2$ is $-1-i$ and radius is 2. If A is $z=-1-2i$ then

$CA = |(-1-2i)-(-1-i)| = |i| = 1 < 2$

Hence, A lies inside C the circle.

$$\therefore \text{ Residue (at } z=-1-2i) = \lim_{z\to(-1-2i)}\frac{z-1}{z+1-2i}$$

$$= \frac{-1-2i-1}{-1-2i+1-2i}$$

$$= \frac{-2-2i}{-4i} = \frac{1+i}{2i}$$

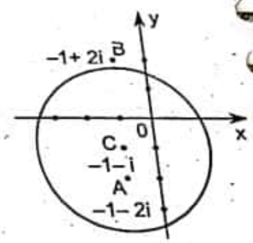

Fig. 4.8

$$\therefore \oint_C \frac{z-1}{z^2+2z+5}dz = 2\pi i\left(\frac{1+i}{2i}\right) = \pi(1+i)$$

(iii) The centre C of the circle $|z+1-i|=2$ is $-1+i$ and radius is 2. If B is $z=-1+2i$ then

$CB = |-1+2i+1-i| = |i| = 1 < 2$

Hence, B lies inside the circle.

$$\therefore \text{ Residue (at } z=-1+2i) = \lim_{z\to(-1+2i)}\frac{z-1}{z+1+2i}$$

$$= \frac{-1+2i-1}{-1+2i+1+2i}$$

$$= \frac{-2+2i}{4i} = \frac{-1+i}{2i}$$

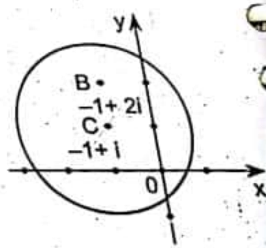

Fig. 4.9

$$\therefore \oint_C \frac{z-1}{z^2+2z+5}\,dz = 2\pi i\left(\frac{-1+i}{2i}\right) = \pi(-1+i)$$

Note ....

Whether a point at which the function is not analytic lies inside or outside the circle can be determined either (i) by calculating the distances or (ii) by drawing a sketch.

Example 11 : Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$ where C is (i) $|z+1-i|=2$, (ii) $|z|=1$.

(M.U. 1995, 2001)

Sol. : $z^2+2z+5=0 \quad \therefore z = \frac{-2\pm\sqrt{4-20}}{2}$

$\therefore z = -1 \pm 2i$. Both are simple poles.

(i) The centre C of the circle $|z+1-i|=2$ is $-1+i$ and radius is 2. Hence, as in Example 9, the point $z=-1+2i$ lies inside the circle

$$\therefore \text{Residue (at } z=-1+2i) = \lim_{z\to(-1+2i)} \frac{z+4}{z+1+2i}$$

$$\therefore \text{Residue (at } z=-1+2i) = \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$$

$$\therefore \oint_C \frac{z+4}{z^2+2z+5}\,dz = 2\pi i\left(\frac{3+2i}{4i}\right) = \frac{\pi(3+2i)}{2}$$

(ii) If C is $|z|=1$ both the poles lie outside the circle and the function is analytic every where inside C. Hence by Cauchy's Integral Theorem.

$$\int_C f(z)\,dz = \int_C \frac{z-4}{z^2+2z+5}\,dz = 0.$$

Example 12 : Evaluate (i) $\int_C \frac{3z^2+z}{z^2-1}\,dz$ where C is the circle $|z|=2$.

(ii) $\int_C \frac{\cos\pi z}{z^2-1}\,dz$ where C is the rectangle whose vertices are $2\pm i, -2\pm i$.

(M.U. 1998)

Sol. : (i) $z^2-1=0$ gives $z=+1,-1$. Hence, both the poles lie within the circle $|z|=2$.

$$\text{Residue of } f(z) \text{ (at } z=1) = \lim_{z\to1}(z-1)\cdot\frac{3z^2+z}{(z-1)(z+1)}$$

$$= \lim_{z\to1}\frac{3z^2+z}{z+1} = \frac{4}{2} = 2$$

$$\text{Residue of } f(z) \text{ (at } z=-1) = \lim_{z\to-1}(z+1)\cdot\frac{3z^2+z}{(z-1)(z+1)}$$

$$= \lim_{z\to-1}\frac{3z^2+z}{z-1} = \frac{2}{-2} = -1.$$

$$\therefore \int_C f(z)\,dz = 2\pi i(\text{Sum of the residues}) = 2\pi i(2-1) = 2\pi i$$

(ii) As before the poles are $z=\pm1$.

Both the poles lie within the rectangle.

Residue of $f(z)$ (at $z=1$)

$$= \lim_{z\to1}(z-1)\cdot\frac{\cos\pi z}{(z-1)(z+1)}$$

$$= \lim_{z\to1}\frac{\cos\pi z}{z+1} = \frac{-1}{2}$$

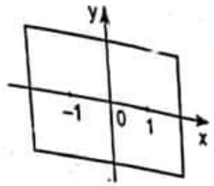

Fig. 4.4

$$\text{Residue of } f(z) \text{ at } (z=-1) = \lim_{z\to-1}(z+1)\cdot\frac{\cos\pi z}{(z+1)(z-1)}$$

$$= \lim_{z\to-1}\frac{\cos\pi z}{z-1} = \frac{-1}{-2} = \frac{1}{2}$$

$$\therefore \int_C f(z)\,dz = 2\pi i(\text{Sum of the residues}) = 2\pi\left(-\frac{1}{2}+\frac{1}{2}\right) = 0.$$

Example 13 : Evaluate $\int_C \frac{2z-1}{z(2z+1)(z+2)}\,dz$ using residue theorem, where C is the circle $|z|=1$.

(M.U. 2016)

Sol. : The poles of $f(z)$ are given by $z(2z+1)(z+2)=0$.

$\therefore$ The poles are $z=0, z=-1/2, z=-2$. Of these poles $z=0, z=-1/2$ lie inside C.

$$\text{Residue (at } z=0) = \lim_{z\to0} z\cdot\frac{2z-1}{z(2z+1)(z+2)} = \lim_{z\to0}\frac{2z-1}{(2z+1)(z+2)}$$

$$= \frac{-1}{1\cdot2} = -\frac{1}{2}$$

$$\text{Residue (at } z=-1/2) = \lim_{z\to(-1/2)}\left(z+\frac{1}{2}\right)\cdot\frac{2z-1}{z(2z+1)(z+2)}$$

$$= \lim_{z\to(-1/2)}\left(z+\frac{1}{2}\right)\cdot\frac{2z-1}{z\cdot2[z+(1/2)](z+2)}$$

$$= \frac{-2}{(-1)(2)(3/2)} = \frac{2}{3}.$$

$$\therefore \int_C f(z)\,dz = 2\pi i(\text{Sum of the residues})$$

$$= 2\pi i\left(-\frac{1}{2}+\frac{2}{3}\right) = 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3}$$

Example 14 : Evaluate $\int_C \frac{dz}{z^3(z+4)}$ where C is $|z|=2$.

(M.U. 1994, 95)

Sol. : The poles are given by $z^3(z+4)=0$.

$\therefore z=0$ is a pole of order 3 and $z=-4$ is a simple pole.

$|z| = 2$ is a circle with centre at the origin and radius 2. Hence, $z = 0$ lies inside C and $z = -4$ lies outside.

$$\text{Residue (at } z=0) = \lim_{z\to 0} \frac{1}{2!}\cdot\frac{d^2}{dz^2}\left[z^3\cdot\frac{1}{z^3(z+4)}\right]$$

$$= \lim_{z\to 0}\frac{1}{2}\cdot\frac{d^2}{dz^2}\left(\frac{1}{z+4}\right) = \lim_{z\to 0}\frac{1}{2}\cdot\frac{d}{dz}\left(-\frac{1}{z+4}\right)$$

$$= \lim_{z\to 0}\frac{1}{2}\cdot\frac{2}{(z+4)^3} = \frac{1}{64}$$

$$\therefore \int_C \frac{dz}{z^3(z+4)} = 2\pi i\left(\frac{1}{64}\right) = \frac{\pi i}{32}$$

Example 15 : Evaluate the following using residue theorem.

(i) $\int_C \frac{dz}{4z^2+1}$ where C is $|z| = 1$. (M.U. 1993)

(ii) $\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2}\,dz$ where C is $|z| = 1$. (M.U. 1993, 2003)

(iii) $\int_C \operatorname{cosec} z\,dz$ where C is $|z| = 1$. (M.U. 1992, 2003)

Sol. : (i) We have $f(z) = \frac{1}{4z^2+1}$ $\therefore$ $f(z) = \frac{1}{(2z+i)(2z-i)}$

$\therefore$ The poles of $f(z)$ are $z = -i/2,\ i/2$

$$\therefore \text{Residue of } f(z) \text{ (at } z = -i/2) = \lim_{z\to -i/2}[z+(i/2)]\cdot\frac{1}{(2z+i)(2z-i)}$$

$$= \lim_{z\to -i/2}\frac{(2z+i)}{2}\cdot\frac{1}{(2z+i)(2z-i)}$$

$$= \frac{1}{2\cdot(-i-i)} = -\frac{1}{4i}$$

$$\therefore \text{Residue of } f(z) \text{ (at } z = i/2) = \lim_{z\to i/2}[z-(i/2)]\cdot\frac{1}{(2z+i)(2z-i)}$$

$$= \lim_{z\to i/2}\frac{(2z-i)}{2}\cdot\frac{1}{(2z+i)(2z-i)}$$

$$= \frac{1}{(i+i)\cdot 2} = \frac{1}{4i}$$

$$\therefore \oint_C f(z)\,dz = 2\pi i\ \text{(Sum of the residues)} = 2\pi i\left[-\frac{1}{4i}+\frac{1}{4i}\right] = 0$$

(ii) The poles of $f(z)$ are given by $z^2(z^2+5z+6) = 0$

i.e. $z^2(z+2)(z+3) = 0$

$\therefore$ The poles are $z = 0,\ z = -2,\ z = -3$.

The last two poles lie outside the circle $|z| = 1$ and $z = 0$ is a pole of order 2.

$$\therefore \text{Residue of } f(z) \text{ (at } z=0) = \lim_{z\to 0}\frac{d}{dz}z^2\cdot f(z)$$

$$= \lim_{z\to 0}\frac{d}{dz}z^2\cdot\frac{(z+4)^2}{z^2(z+2)(z+3)}$$

$$= \lim_{z\to 0}\frac{d}{dz}\cdot\frac{(z+4)^2}{z^2+5z+6}$$

$$= \lim_{z\to 0}\frac{(z^2+5z+6)2\cdot(z+4)-(z+4)^2(2z+5)}{(z^2+5z+6)^2}$$

$$= \frac{6\cdot 2\cdot 4-16\cdot 5}{36} = -\frac{32}{36} = -\frac{8}{9}$$

$$\therefore \oint f(z)\,dz = 2\pi i\ \text{(Sum of the residues)} = 2\pi i\left(-\frac{8}{9}\right) = -\frac{16\pi i}{9}$$

(iii) $\operatorname{cosec} z = \frac{1}{\sin z}$

$\sin z = 0$ when $z = 0, \pm\pi, \pm 2\pi, \ldots$

Of these poles $z = 0$ lies within the circle $|z| = 1$.

$\therefore$ $z = 0$ is a simple pole.

$$\therefore \text{Residue of } f(z) \text{ (at } z=0) = \lim_{z\to 0} z f(z) = \lim_{z\to 0} z\cdot\frac{1}{\sin z}$$

$$\therefore \oint_C f(z)\,dz = 2\pi i\ \text{(Sum of the residues)} = 2\pi i(1) = 2\pi i.$$

Example 16 : Using residue theorem evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$ where C is $|z| = 4$. (M.U. 2009, 16)

Sol. : The poles of $f(z)$ are given by

$(z^2+\pi^2) = 0$ $\therefore$ $(z-\pi i)(z+\pi i) = 0$

$\therefore$ $z = \pi i, -\pi i$ are the poles of order 2.

$$\text{Residue (at } z=\pi i) = \frac{1}{1!}\lim_{z\to\pi i}\frac{d}{dz}\left[(z-\pi i)^2\cdot\frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right]$$

$$= \lim_{z\to\pi i}\frac{d}{dz}\left[\frac{e^z}{(z-\pi i)^2}\right] = \lim_{z\to\pi i}\left[\frac{(z+\pi i)^2\cdot e^z - e^z\cdot(z+\pi i)\cdot 2}{(z+\pi i)^4}\right]$$

$$= \lim_{z\to\pi i}\frac{e^z(z+\pi i-2)}{(z+\pi i)^3} = e^{\pi i}\cdot\frac{2\cdot(\pi i-1)}{(2\pi i)^3}$$

$$= e^{\pi i}\cdot\frac{2i(\pi+i)}{-8\pi^3 i} = -\frac{e^{\pi i}(\pi+i)}{4\pi^3}$$

$$= \frac{\pi+i}{4\pi^3} \qquad \left[\because e^{\pi i} = \cos\pi + i\sin\pi = -1\right]$$

$$\text{Residue (at } z=-\pi i) = \frac{1}{1!}\lim_{z\to-\pi i}\frac{d}{dz}\left[(z+\pi i)^2\cdot\frac{e^z}{(z+\pi i)^2(z-\pi i)^2}\right]$$

$$=\lim_{z\to-\pi i}\frac{d}{dz}\left[\frac{e^z}{(z-\pi i)^2}\right]=\lim_{z\to-\pi i}\left[\frac{(z-\pi i)^3e^z-e^z(z-\pi i)^2}{(z-\pi i)^4}\right]$$

$$=\lim_{z\to-\pi i}\left[\frac{e^z(z-\pi i-2)}{(z-\pi i)^3}\right]=e^{-\pi i}\frac{(-2\pi i-2)}{(-2\pi i)^3}=e^{-\pi i}\frac{(-2i)(\pi-i)}{8\pi^3 i}$$

$$=e^{-\pi i}\cdot\frac{(\pi-i)}{4\pi^3}=\frac{\pi-i}{4\pi^3}$$

$$\therefore\ \oint_C\frac{e^z}{(z^2+\pi^2)^2}dz=2\pi i\,(\text{Sum of the residues})$$

$$=2\pi i\left[\frac{\pi+i}{4\pi^3}+\frac{\pi-i}{4\pi^3}\right]$$

$$=\frac{4\pi^2 i}{4\pi^3}=\frac{i}{\pi}.$$

Example 17 : *Using the residue theorem evaluate*

$$\oint_C\frac{\cos\pi z^2+\sin\pi z^2}{z-z^2}dz$$

where C is $|z-2|=4$. **(M.U. 2003)**

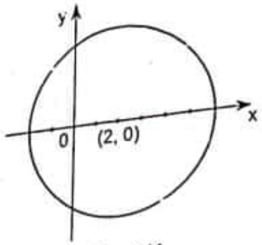

Fig. 4.11

Sol. : *The poles are given by* $z-z^2=0$ *i.e.* $z(1-z)=0$

$\therefore\ z=0,1.$

$|z-2|=4$ *is a circle with center at (2, 0) and radius 4. Hence, both the poles lie inside C.*

$$\text{Residue (at } z=0)=\lim_{z\to0}(z-0)\cdot\frac{\cos\pi z^2+\sin\pi z^2}{(z-0)(1-z)}$$

$$=\lim_{z\to0}\frac{\cos\pi z^2+\sin\pi z^2}{1-z}=1$$

$$\text{Residue (at } z=1)=\lim_{z\to1}(z-1)\cdot\frac{\cos\pi z^2+\sin\pi z^2}{z(1-z)}$$

$$=\lim_{z\to1}\frac{\cos\pi z^2+\sin\pi z^2}{(-1)z}=1$$

$$\therefore\ \oint_C f(z)\,dz=2\pi i\,(\text{Sum of the residues})=2\pi i(1+1)=4\pi i.$$

Example 18 : *Using residue theorem evaluate* $\oint_C\frac{e^{2z}}{(z-\pi i)^3}dz$ *where C is* $|z-2i|=2.$

(M.U. 2003)

Sol. : *Clearly* $z=\pi i$ *is a pole of order 3.*

$|z-2i|=4$ is a circle with centre at (0, 2) and radius 4. Hence, the pole $(0,\pi)$ lies inside C.

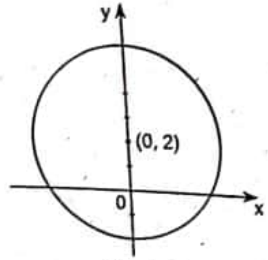

Fig. 4.12

$$\text{Residue (at } z=\pi i)$$

$$=\lim_{z\to\pi i}\frac{1}{2!}\frac{d^2}{dz^2}\left[(z-\pi i)^3\cdot\frac{e^{2z}}{(z-\pi i)^3}\right]$$

$$=\lim_{z\to\pi i}\frac{1}{2!}\frac{d^2}{dz^2}(e^{2z})$$

$$=\lim_{z\to\pi i}\frac{1}{2}\cdot\frac{d}{dz}(2e^{2z})$$

$$=\lim_{z\to\pi i}\frac{1}{2}\cdot4e^{2z}=2e^{2\pi i}.$$

$$\therefore\ \oint_C f(z)\,dz=2\pi i(2e^{2\pi i})=4\pi i(\cos2\pi+i\sin2\pi)=4\pi i.$$

Example 19 : By using residue theorem evaluate $\oint_C\frac{\sin^6 z}{(z-\pi/6)^3}dz$ where C is $|z|=1$.

(M.U. 2000, 10)

Sol : Clearly $z=\pi/6$ is a pole of order 3.

$|z|=1$ is a circle with center at (0, 0) and radius 1.

Then pole $(\pi/6, 0)$ *i.e.* $(3.14/6, 0)$ lies inside C.

$$\text{Residue (at } z=\pi/6)=\frac{1}{2!}\lim_{z\to\pi/6}\frac{d^2}{dz^2}\left[(z-\pi/6)^3\cdot\frac{\sin^6 z}{(z-\pi/6)^3}\right]$$

$$=\frac{1}{2!}\lim_{z\to\pi/6}\frac{d^2}{dz^2}(\sin^6 z)=\frac{1}{2!}\lim_{z\to\pi/6}\frac{d}{dz}(6\sin^5 z\cdot\cos z)$$

$$=\frac{1}{2!}\lim_{z\to\pi/6}\left[6\cdot5\sin^4 z\cos^2 z-6\sin^5 z\sin z\right]$$

$$\text{Residue (at } z=\pi/6)=\frac{1}{2}\left[30\cdot\left(\frac{1}{2}\right)^4\left(\frac{\sqrt3}{2}\right)^2-6\left(\frac{1}{2}\right)^6\right]$$

$$=\frac{1}{2}\left[30\cdot\frac{1}{16}\cdot\frac{3}{4}-6\cdot\frac{1}{64}\right]=\frac{1}{2}\cdot\frac{84}{64}=\frac{21}{32}$$

$$\therefore\ \oint_C f(z)\,dz=2\pi i\left(\frac{21}{32}\right)=\frac{21}{16}\pi i.$$

Example 20 : Using residue theorem evaluate $\oint_C e^{-1/z}\sin\left(\frac{1}{z}\right)dz$ where C is $|z|=1$.

(M.U. 1997)

Sol. : We expand $f(z)$ as Laurent's series.

$$e^{-1/z}\cdot\sin\left(\frac{1}{z}\right)=\left[1-\frac{1}{z}+\frac{1}{2z^2}-\ldots\right]\left[\frac{1}{z}-\frac{1}{6z^3}+\ldots\right]$$

$$\therefore \quad e^{-1/z} \cdot \sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{3z^3} - \dots$$

$z = 0$ is an isolated essential singularity and

Residue (at $z = 0$) $= b_1$ = coefficient of $1/z = 1$

$$\therefore \quad \int_C e^{-1/z} \sin\left(\frac{1}{z}\right) dz = 2\pi i\,(\text{Residue}) = 2\pi i(1) = 2\pi i$$

Example 21 : Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}\, dz$ where C encloses both poles of $f(z)$. **(M.U. 2006)**

Sol. : Clearly $z = -1$ is a pole of order 2 and $z = 2$ is a simple pole.

$$\therefore \quad \text{Residue (at } z = 2) = \lim_{z\to 2}\left[\frac{(z-2)\cdot(z-1)}{(z+1)^2(z-2)}\right] = \lim_{z\to 2} \frac{z-1}{(z+1)^2} = \frac{1}{9}.$$

$$\text{Residue (at } z = -1) = \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2 \cdot \frac{z-1}{(z+1)^2(z-2)}\right]$$

$$= \lim_{z\to -1} \frac{d}{dz}\left(\frac{z-1}{z-2}\right) = \lim_{z\to -1}\left[\frac{(z-2)\cdot 1 - (z-1)\cdot 1}{(z-2)^2}\right]$$

$$= \lim_{z\to -1}\left[-\frac{1}{(z-2)^2}\right] = \lim_{z\to -1} -\frac{1}{9}.$$

$$\therefore \quad \int_C f(z)\, dz = 2\pi i \text{ (Sum of the residues)}$$

$$= 2\pi i\left(\frac{1}{9} - \frac{1}{9}\right) = 0.$$

Example 22 : If $f(z) = \frac{\Phi(z)}{\Psi(z)}$, where $\Phi(z)$ and $\Psi(z)$ are complex polynomials of degree 2 has (i) pole of order 2 at $z = 1$. (ii) residue at $z = 2$ is -1. (iii) $f(0) = f(-1) = 0$, find $f(z)$.

Sol. : Since $z = 1$ is a single pole of order 2, we assume the required function $f(z)$ as

$$f(z) = \frac{az^2 + bz + c}{(z-1)^2}$$

Since $f(0) = 0$, $c = 0$. Since $f(-1) = 0$, $a - b = 0 \quad \therefore \; a = b$.

$$\therefore \quad f(z) = \frac{az^2 + az}{(z-1)^2}$$

$$\text{Now, residue (at } z = 1) = \frac{1}{1!} \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 \cdot \frac{az^2 + az}{(z-1)^2}\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}(az^2 + az)$$

$$= \lim_{z\to 1} 2az + a = 2a + a = 3a$$

But this is $-1 \quad \therefore \; 3a = -1 \quad \therefore \; a = -1/3$

$\therefore \; b = a = -1/3$ and $c = 0$

$$\therefore \quad f(z) = -\frac{1}{3} \cdot \frac{z^2 + z}{(z-1)^2}$$

Example 23 : Construct a function $f(z) = \frac{\Phi(z)}{\Psi(z)}$ which is analytic except at four distinct points z_1, z_2, z_3 and z_4 where it has (i) a simple pole at z_1, (ii) simple zeroes at z_2, z_3, z_4, (iii) simple pole at $z = 0$, (iv) $\lim_{|z|\to\infty} \frac{f(z)}{z} = 2$. **(M.U. 1999)**

Sol. : (i) Since $z = z_1$ is a simple pole $\Psi(z)$ must contain the factor $z - z_1$.

(ii) Since z_2, z_3, z_4 are simple zeros, $\Phi(z)$ must contain the factor $(z - z_2)(z - z_3)(z - z_4)$.

(iii) Since $f(z)$ has a simple pole at zero, $\Psi(z)$ must contain a factor z.

(iv) Since $\lim_{|z|\to\infty} \frac{f(z)}{z} = 2$ we introduce a constant k (to be determined) in $f(z)$ and consider

$$f(z) = k\frac{(z-z_2)(z-z_3)(z-z_4)}{z(z-z_1)}$$

$$\text{Since } \lim_{|z|\to\infty} \frac{f(z)}{z} = 2, \text{ we have } \lim_{|z|\to\infty} \frac{k(z-z_2)(z-z_3)(z-z_4)}{z^2(z-z_1)} = 2.$$

$$\therefore \quad \lim_{|z|\to\infty} \frac{k(1-z_2/z)(1-z_3/z)(1-z_4/z)}{(1-z_1/z)} = 2 \quad \therefore \; k = 2$$

$$\therefore \quad f(z) = \frac{2(z-z_2)(z-z_3)(z-z_4)}{z(z-z_1)}$$

Example 24 : If $f(z) = \frac{\Phi(z)}{\Psi(z)}$, where $\Phi(z)$ and $\Psi(z)$ are complex polynomials of degree 3 has

(i) poles of order 1 and 2 at $z = 2$, $z = 1$ respectively.

(ii) residues at 2 and 1 equal to 3 and 1 respectively.

(iii) $f(0) = 3/2$, $f(-1) = 1$, find $f(z)$.

Sol. : Since $z = 1$ and $z = 2$ are the poles of order 2 and 1, we assume the required function $f(z)$ as

$$f(z) = \frac{az^3 + bz^2 + cz + d}{(z-2)(z-1)^2}$$

$$\text{Now, Residue (at } z = 2) = \lim_{z\to 2} \frac{(z-2)(az^3 + bz^2 + cz + d)}{(z-2)(z-1)^2}$$

$$= \lim_{z\to 2} \frac{az^3 + bz^2 + cz + d}{(z-1)^2}$$

$$3 = 8a + 4b + 2c + d$$

$$\text{Residue (at } z=1) = \lim_{z\to 1} \frac{d}{dz}(z-1)^2 \cdot \frac{(az^3+bz^2+cz+d)}{(z-1)^2(z-2)}$$

$$1 = \lim_{z\to 1} \frac{d}{dz}\left\{\frac{az^3+bz^2+cz+d}{z-2}\right\}$$

$$= \lim_{z\to 1} \frac{(z-2)(3az^2+2bz+c)-(az^3+bz^2+cz+d)}{(z-2)^2}$$

$$1 = -4a-3b-2c-d$$

$$f(0) = \frac{d}{-2} = \frac{3}{2} \quad \therefore d = -3$$

$$f(-1) = \frac{-a+b-c+d}{-12} = 1 \quad \therefore \frac{a-b+c-d}{12} = 1$$

$\therefore$ $8a+4b+2c=6$ and $4a+3b+2c=2$ and $a-b+c=9$.

Solving these simultaneous equations, we get $a=2,\ b=-4,\ c=3$.

$$\therefore f(z) = \frac{2z^3-4z^2+3z-3}{(z-2)(z-1)^2}.$$

EXERCISE - IV

Using Cauchy's residue theorem evaluate the following.

1. $\oint_C e^{1/z^2} dz$ where C is the circle $|z|=1$. [Ans. : 0]

2. $\oint_C \frac{\sin z}{z^6} dz$ where C is the circle $|z|=1$. [Ans. : $\frac{\pi i}{60}$]

3. $\oint_C z e^{1/z} dz$ where C is the circle $|z|=1$. [Ans. : πi]

4. $\oint_C z^4 e^{1/z} dz$ where C is $|z|=1$. (M.U. 2000) [Ans. : $\frac{\pi i}{60}$]

   (**Hint :** In Ex. 2, 3, 4, expand $\sin z$, $e^{1/z}$ to find the residue *i.e.* b_1.)

5. $\oint_C \frac{z}{(z-1)^2(z+1)} dz$ where C is the circle (i) $|z| = \frac{5}{4}$, (ii) $|z| = \frac{3}{2}$. [Ans. : (i) 0, (ii) 0]

6. $\oint_C \frac{z-2}{z^2-z} dz$ where C is the ellipse $4x^2+9y^2=36$. [Ans. : $2\pi i$]

7. $\oint_C \frac{e^z}{(z+1)^2} dz$ where C is the circle $(x-1)^2+y^2=3^2$. [Ans. : $\frac{2\pi i}{e}$]

8. $\oint_C \frac{z}{\cos z}$ where C is $\left|z+\frac{\pi}{2}\right| = \frac{\pi}{2}$. [Ans. : $-\pi^2 i$]

9. $\oint_C \frac{1}{(z-a)^m}$ where C is the circle $|z-a|=a$ and m is any positive integer. [Ans. : If $m=1$, $I=2\pi i$; if $m\neq 1$, $I=0$]

10. $\oint_C \frac{z}{(z+1)^2(z-2)} dz$ where C is the circle $|z-i|=2$. [Ans. : $-\frac{4\pi i}{9}$]

11. $\oint_C \frac{dz}{(z^2+1)^2}$ where C is the circle $|z-i|=1$. [Ans. : $\frac{\pi}{2}$]

12. $\oint_C \frac{1-2z}{z(z-1)(z-2)} dz$ where C is the circle $|z|=1{\cdot}5$. [Ans. : $3\pi i$]

13. $\oint_C \frac{dz}{4z^2+1}$ where C is the circle $|z|=1$. (M.U. 1993) [Ans. : 0]

14. $\oint_C \frac{(z+4)^2}{z^4+5z^3+6z^2} dz$ where C is the circle $|z|=1$. (M.U. 1993) [Ans. : $\frac{-16\pi i}{9}$]

15. $\oint_C \frac{3z^2+2z-4}{z^3-4z} dz$ where C is the circle $|z-i|=3$. (M.U. 1995) [Ans. : $6\pi i$]

16. $\oint_C \frac{4z^2+1}{(2z-3)(z+1)^2}$ where C is the circle $|z|=4$. (M.U. 2003) [Ans. : $4\pi i$]

17. $\oint_C \frac{z^2+4}{(z-2)(z+3i)} dz$ where C is (i) $|z+1|=2$, (ii) $|z-2|=2$. (M.U. 2006) [Ans. : (i) 0, (ii) $16\pi i/(2+3i)$]

18. $\oint_C \frac{4z-1}{z^2-3z-4}$ where C is the ellipse $x^2+4y^2=4$. (M.U. 2003) [Ans. : $2\pi i$]

19. $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z^2+3z+2}$ where C is (i) $|z|=0{\cdot}5$, (ii) $|z|=1{\cdot}5$. (M.U. 1998, 2003) [Ans. : (i) 0, (ii) $-2\pi i$]

20. $\oint_C \frac{\sin 3z}{z+\pi/2} dz$ where C is (i) $|z|=5$. (M.U. 1998) [Ans. : $2\pi i$]

21. $\oint_C \frac{z+3}{2z^2+3z-2} dz$ where C is the circle with centre (0, 1) and radius 2. (M.U. 1998) [Ans. : $7\pi i/5$]

    (**Hint :** C is $|z-i|=2$)

22. $\oint_C \operatorname{cosec} z\, dz$ where C is the circle $|z|=1$. (M.U. 1992) [Ans. : $2\pi i$]

23. $\oint_C \frac{z-1}{(z+1)^2(z-2)} dz$ where C is the circle $|z-i|=2$. (M.U. 1994, 98, 2005) [Ans. : $-2\pi i/9$]

24. $\oint_C \frac{ze^{2z}}{(z-1)^3}dz$ where C is $|z+i|=2$. (M.U. 1996) [Ans. : $8\pi i e^2$]

25. $\oint_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz$ where C is the circle $|z|=3$. (M.U. 1991, 95) [Ans. : $4\pi i$]

26. $\oint_C \frac{dz}{z^2+1}$ where C is the circle $x^2+y^2=4$. [Ans. : 0]

27. $\oint_C \frac{12z-7}{(z-1)^2(2z+3)}dz$ where C is the circle (i) $|z|=1/2$, (ii) $|z|=2$, (iii) $|z+i|=\sqrt{3}$. (M.U. 2009) [Ans. : (i) 0, (ii) 0, (iii) $4\pi i$]

28. $\oint_C \frac{e^z}{\cos\pi z}dz$ where C is the circle $|z|=1$. (M.U. 1998) [Ans. : $-4\pi\sin h\left(\frac{1}{2}\right)$]

29. $\oint_C \frac{2z-1}{z(z+1)(z-3)}dz$ where C is the circle $x^2+y^2=4$. [Ans. : $-5\pi i/6$]

30. $\oint_C \frac{z-3}{z^2+2z+5}dz$ where C is the circle
(i) $x^2+y^2=1$, (ii) $|z+1+i|=2$, (iii) $|z+1-i|=2$. (M.U. 1993)
[Ans. : (i) 0, (ii) $\pi(2+i)$, (iii) $\pi(-2+i)$]

31. $\oint_C \frac{z^2+3}{z^2-1}$ where C is the circle (i) $|z-1|=1$, (ii) $|z+1|=1$. (M.U. 2016)
[Ans. : (i) $4\pi i$, (ii) $-4\pi i$]

32. $\oint_C \frac{15z+9}{z^3-9z}dz$ where C is $|z-1|=3$. (M.U. 2004) [Ans. : $4\pi i$]

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6. Applications of Residues

Cauchy's residue theorem can be used to evaluate certain types of definite integrals of real variables by using suitable contours - a circle or a semi-circle and its diameter.

Type I : Integral of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$

If f is a rational function of $\sin\theta$, $\cos\theta$ then we put $z=e^{i\theta}$. Then, we get,

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{z+z^{-1}}{2}=\frac{z^2+1}{2z}$$

$$\sin\theta=\frac{e^{i\theta}-e^{i\theta}}{2i}=\frac{z-z^{-1}}{2i}=\frac{z^2-1}{2iz}$$

$$dz=ie^{i\theta}d\theta=iz\,d\theta \qquad \therefore\ d\theta=\frac{dz}{iz}.$$

Now, as θ changes from 0 to 2π it is clear that $z=e^{i\theta}=\cos\theta+i\sin\theta$ moves around the circle $|z|=1$.

The new integral z can be evaluated by using Cauchy's residue theorem. **Because this integral is equal to $2\pi i$ (sum of the residues) at poles in $|z|=1$.**

Example 1 : Evaluate $\int_0^{2\pi}\frac{d\theta}{5+3\sin\theta}$. **(M.U. 2004, 09, 15)**

Sol. : Let $e^{i\theta}=z \quad \therefore e^{i\theta}\cdot i\,d\theta=dz$; $d\theta=\frac{dz}{iz}$ and $\sin\theta=\frac{z^2-1}{2iz}$

$$I=\int_C\frac{1}{5+3\left(\frac{z^2-1}{2iz}\right)}\cdot\frac{dz}{iz}=\int_C\frac{2}{3z^2+10iz-3}dz$$

$$=\int_C\frac{2}{(3z+i)(z+3i)}dz \text{ where } C \text{ is the circle } |z|=1.$$

Now, the poles of $f(z)$ are given by $(3z+i)(z+3i)=0 \ \therefore\ z=-(i/3)$ and $z=-3i$ are simple poles. But $z=-(i/3)$ lies inside and $z=-3i$ lies outside the circle $|z|=1$.

$$\text{Residue}\left(\text{at } z=-\frac{i}{3}\right)=\lim_{z\to-\frac{i}{3}}[z+(i/3)]\cdot\frac{2}{(3z+i)(z+3i)}$$

$$=\lim_{z\to-\frac{i}{3}}\frac{2}{3(z+3i)}=\frac{1}{4i}$$

$$\therefore\ I=2\pi i\left(\frac{1}{4i}\right)=\frac{\pi}{2}.$$

Example 2 : Evaluate $\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}$, $a>b$. **(M.U. 2005)**

Sol. : Let $z=e^{i\theta} \quad \therefore\ dz=ie^{i\theta}d\theta$

$$\therefore\ dz=i\cdot z\cdot d\theta \quad \therefore\ d\theta=\frac{dz}{iz} \quad \therefore\ \sin\theta=\frac{z-(1/z)}{2i}$$

$$\therefore\ I=\int_C\frac{1}{a+b\left[\frac{z-(1/z)}{2i}\right]}\cdot\frac{dz}{iz}=\int_C\frac{1}{a+b\left[\frac{z^2-1}{2iz}\right]}\cdot\frac{dz}{iz}$$

$$=\int_C\frac{2}{bz^2+2aiz-b}dz$$

The roots of $bz^2+2aiz-b=0$ are

$$z=\frac{-2ai\pm\sqrt{-4a^2+4b^2}}{2b}=-\frac{-ai\pm\sqrt{a^2-b^2}\cdot i}{b} \quad [\because a>b]$$

...oves around

...cause this

...09, 15)

$$\therefore \quad z = \left[\frac{-a \pm \sqrt{a^2-b^2}}{b}\right] i$$

Let $\alpha = \dfrac{-a+\sqrt{a^2-b^2}}{b} \cdot i, \quad \beta = \dfrac{-a-\sqrt{a^2-b^2}}{b} \cdot i$

Now, $\alpha = \dfrac{-a+\sqrt{a^2-b^2}}{b} \cdot i \cdot \dfrac{-a-\sqrt{a^2-b^2}}{-a-\sqrt{a^2-b^2}}$

$$= \frac{a^2-(a^2-b^2)}{-b(a+\sqrt{a^2-b^2})} \cdot i = \frac{-b}{a+\sqrt{a^2-b^2}} \cdot i$$

$$\therefore \quad |\alpha| = \left|\frac{-b \cdot i}{a+\sqrt{a^2-b^2}}\right| = \frac{b}{a+\sqrt{a^2-b^2}}$$

Since $a > b$, $|\alpha| < 1$. $\quad \therefore$ α lies inside $|z| = 1$.

By the same reasing $\beta > 1$ and hence, β lies outside C.

$$\therefore \text{ Residue of } f(z) \text{ (at } z=\alpha) = \lim_{z\to\alpha} (z-\alpha) \cdot \frac{2}{(z-\alpha)(z-\beta)}$$

$$= \lim_{z\to\alpha} \frac{2}{z-\beta} = \frac{2}{\alpha-\beta}$$

Now, $\alpha - \beta = \dfrac{-a+\sqrt{a^2-b^2}}{b} \cdot i - \dfrac{-a-\sqrt{a^2-b^2}}{b} \cdot i = \dfrac{2\sqrt{a^2-b^2}}{b} \cdot i$

$$\therefore \text{ Residue of } f(z) \text{ (at } z=\alpha) = \frac{2b}{2\sqrt{a^2-b^2} \cdot i} = \frac{b}{\sqrt{a^2-b^2}} \cdot i$$

$$\therefore \quad I = 2\pi i (\text{Residue}) = 2\pi i \cdot \frac{1}{\sqrt{a^2-b^2} \cdot i} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Example 3 : *Using residues evaluate* $\displaystyle\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$. **(M.U. 1997, 2004, 15)**

Sol. : *Let* $e^{i\theta} = z$, $e^{i\theta} \cdot i\, d\theta = dz \quad \therefore \; d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \dfrac{z+z^{-1}}{2}$.

$$\therefore \quad I = \int_C \frac{1}{\left(2+\dfrac{z^2+1}{2z}\right)^2} \cdot \frac{dz}{iz} = \frac{4}{i} \cdot \int_C \frac{z\,dz}{(z^2+4z+1)^2} \quad \text{where } C \text{ is the circle } |z| = 1.$$

Now the poles of $f(z)$ *are given by* $z^2+4z+1 = 0$.

$$\therefore \quad z = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

Let $\alpha = -2-\sqrt{3}$ *and* $\beta = -2+\sqrt{3}$.

Both are the poles of order 2. But the pole α *lies outside the circle and the pole* β *lies inside the circle.* $(\because \sqrt{3} = 1{\cdot}73)$

Since, $f(z) = \dfrac{z}{(z^2+4z+1)^2} = \dfrac{z}{(z-\alpha)^2(z-\beta)^2}$

$$\therefore \text{ Residue (at } z=\beta) = \lim_{z\to\beta} \frac{1}{1!} \frac{d}{dz}\left[(z-\beta)^2 \cdot \frac{z}{(z-\alpha)^2(z-\beta)^2}\right]$$

$$= \lim_{z\to\beta} \frac{d}{dz}\left[\frac{z}{(z-\alpha)^2}\right] = \lim_{z\to\beta}\left[\frac{(z-\alpha)^2 \cdot 1 - z \cdot 2(z-\alpha)}{(z-\alpha)^4}\right]$$

$$= \lim_{z\to\beta} \frac{-z-\alpha}{(z-\alpha)^3} = -\frac{\beta+\alpha}{(\beta-\alpha)^3}$$

But $\beta+\alpha = -4$ and $\beta-\alpha = 2\sqrt{3}$

$$\therefore \text{ Residue (at } z=\beta) = \frac{4}{24\sqrt{3}} = \frac{1}{6\sqrt{3}}$$

$$\therefore \quad I = 2\pi i \cdot \frac{4}{i} \cdot \frac{1}{6\sqrt{3}} = \frac{4\pi}{3\sqrt{3}}$$

Example 4 : From the integral $\displaystyle\int_C \frac{dz}{z+2}$ where C denotes the circle $|z| = 1$, deduce that

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0. \qquad \textbf{(M.U. 1999)}$$

Sol. : Let $z = e^{i\theta} = \cos\theta + i\sin\theta \quad \therefore \; dz = (-\sin\theta + i\cos\theta)\, d\theta$.

$$\therefore \quad \int_C \frac{dz}{z+2} = \int_0^{2\pi} \frac{(-\sin\theta+i\cos\theta)}{(\cos\theta+i\sin\theta)+2}\, d\theta$$

$$= \int_0^{2\pi} \frac{(-\sin\theta+i\cos\theta)}{[(\cos\theta+2)+i\sin\theta]} \cdot \frac{[(\cos\theta+2)-i\sin\theta]}{[(\cos\theta+2)-i\sin\theta]}\, d\theta$$

$$= \int_0^{2\pi} \frac{(-\sin\theta\cos\theta + i\cos^2\theta - 2\sin\theta + 2i\cos\theta + i\sin^2\theta + \sin\theta\cos\theta)}{\cos^2\theta + 4\cos\theta + 4 + \sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \frac{-2\sin\theta + i(1+2\cos\theta)}{5+4\cos\theta}\, d\theta$$

Now, by Cauchy's Integral Theorem $\displaystyle\int_C \frac{dz}{z+2} = 0$ because $f(z) = \dfrac{1}{z+2}$ is analytic inside and on the circle $|z| = 1$.

$$\therefore \quad \int_0^{2\pi} \frac{-2\sin\theta + i(1+2\cos\theta)}{5+4\cos\theta}\, d\theta = 0$$

Equating to zero (real and) imaginary parts,

$$\therefore \quad \int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0$$

Now, $\int_0^a f(x)\,dx = 2\int_0^{\pi/2} f(x)$ if $f(a-x) = f(x)$.

And since $\cos(2\pi - \theta) = \cos\theta$, we get $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$.

Example 5 : Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ where $a > b > 0$. **(M.U. 1996, 2006)**

Sol. : Let $z = e^{i\theta}$ $\therefore dz = i\,e^{i\theta}\cdot d\theta = i\,z\,d\theta$ $\therefore d\theta = \frac{dz}{iz}$; $\cos\theta = \frac{z^2+1}{2z}$

$$\therefore I = \int_C \frac{1}{a+b\cdot\frac{(z^2+1)}{2z}}\cdot\frac{dz}{iz} = \int_C \frac{2\,dz}{(bz^2+2az+b)\,i}$$ where C is the circle $|z| = 1$.

Now, the poles of $f(z)$ are given by $z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b}$ i.e. say $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ and $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$ which are simple poles. Since $a > b > 0$, α lies inside and β lies outside the circle $|z| = 1$.

$$\therefore \text{Residue of } f(z) \text{ (at } z=\alpha) = \lim_{z\to\alpha}(z-\alpha)\cdot\frac{2}{b(z-\alpha)(z-\beta)\,i} = \frac{2}{bi(\alpha-\beta)}$$

But $\alpha - \beta = \frac{2\sqrt{a^2-b^2}}{b}$

$$\therefore \text{Residue of } f(z) \text{ (at } z=\alpha) = \frac{2}{bi\cdot\frac{2\sqrt{a^2-b^2}}{b}} = \frac{1}{i\sqrt{a^2-b^2}}$$

$$\therefore I = 2\pi i\left(\frac{1}{i\sqrt{a^2-b^2}}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Example 6 : Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$. **(M.U. 1994, 2003, 05, 09, 14)**

Sol. : Consider $\int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta}\,d\theta$.

Now, put $z = e^{i\theta}$ $\therefore dz = i\,e^{i\theta}\cdot d\theta$ $\therefore dz = i\,z\,d\theta$ $\therefore d\theta = \frac{dz}{iz}$

And $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+(1/z)}{2}$

$$\therefore \int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta}\,d\theta = \int_C \frac{z^2}{5+4\left(\frac{z+(1/z)}{2}\right)}\cdot\frac{dz}{iz}$$

$$\therefore \int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta}\,d\theta = \int_C \frac{z^2}{i(2z^2+5z+2)}\,dz$$ where C is the circle $|z| = 1$.

Now, the poles are given by $2z^2+5z+2 = 0$ $\therefore (2z+1)(z+2) = 0$

$\therefore z = -1/2$ and $z = -2$.

The pole $z = -1/2$ lies inside the unit circle and the pole $z = -2$ lies outside.

$$\text{Now, Residue of } f(z) \text{ (at } z=-1/2) = \lim_{z\to-\frac{1}{2}}\left(z+\frac{1}{2}\right)\cdot\frac{z^2}{2[z+(1/2)](z+2)\,i}$$

$$= \frac{(-1/2)^2}{2[-(1/2)+2]\,i} = \frac{1}{12i}$$

$$\therefore \int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta}\,d\theta = 2\pi i\left(\frac{1}{12i}\right) = \frac{\pi}{6}$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \text{Real part of } \int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta}\,d\theta = \frac{\pi}{6}.$$

Example 7 : Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$. **(M.U. 1995, 96, 2013, 14)**

Sol. : Consider $\int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta}\,d\theta$.

Now, put $z = e^{i\theta}$ $\therefore dz = i\,e^{i\theta}\cdot d\theta$ $\therefore dz = i\,z\,d\theta$ $\therefore d\theta = \frac{dz}{iz}$

And $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+(1/z)}{2} = \frac{z^2+1}{2z}$

$$\therefore \int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta}\,d\theta = \int_C \frac{z^3}{5-4\left(\frac{z^2+1}{2z}\right)}\cdot\frac{dz}{iz}$$

$$\therefore \int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta}\,d\theta = \int_C \frac{z^3}{-2z^2+5z-2}\cdot\frac{dz}{i} = -\int_C \frac{z^3\,dz}{i(2z^2-5z+2)}$$

where C is the circle $|z| = 1$.

Now, the poles are given by $2z^2-5z+2 = 0$ $\therefore (2z-1)(z-2) = 0$

$\therefore z = 1/2$ and $z = 2$.

The pole $z = 1/2$ lies inside the unit circle and $z = 2$ lies outside it.

$$\text{Now, Residue of } f(z) \text{ (at } z=1/2) = \lim_{z\to(1/2)}\left(z-\frac{1}{2}\right)\cdot\frac{(-1)\cdot z^3}{i\cdot 2[z-(1/2)](z-2)}$$

$$= \lim_{z\to(1/2)} \frac{(-1)\cdot(1/2)^3}{i\cdot 2[(1/2)-2]} = \frac{1}{24i}$$

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$$\therefore \int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta}\, d\theta = 2\pi i\left(\frac{1}{24i}\right) = \frac{\pi}{12}$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \text{Real part of } \int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta}\, d\theta = \frac{\pi}{12}.$$

Example 8 : Evaluate $\int_0^{2\pi} \frac{d\theta}{25-16\cos^2\theta}$ **(M.U. 2013)**

Sol : Put $z = e^{i\theta}$ $\therefore dz = i e^{i\theta} \cdot d\theta$ $\therefore dz = i z\, d\theta$ $\therefore d\theta = \frac{dz}{iz}$

And $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+(1/z)}{2}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{25-16\cos^2\theta} = \int_C \frac{1}{25-16\left[\frac{z+(1/z)}{2}\right]^2}\cdot\frac{dz}{iz}$$

$$= \int_C \frac{1}{25-\left[4z^2+8+(4/z^2)\right]}\cdot\frac{dz}{iz} = -\frac{1}{i}\int_C \frac{z}{4z^4-17z^2+4}\, dz$$

where C is the circle $|z| = 1$.

Now, the poles of $4z^4 - 17z^2 + 4 = 0$ *i.e.* of $(4z^2-1)(z^2-4) = 0$ are $z = \frac{1}{2}, -\frac{1}{2}, 2, -2$.

The poles $z = \frac{1}{2}, -\frac{1}{2}$ lie inside the unit circle and $z = 2, -2$ lie outside it.

Residue of $f(z)$ (at $z = 1/2$) $= \lim_{z\to 1/2}\left(z-\frac{1}{2}\right)\cdot\frac{-z}{i(2z-1)(2z+1)(z^2-4)}$

$$= \lim_{z\to(1/2)}\left(\frac{2z-1}{2}\right)\cdot\frac{-z}{i(2z-1)(2z+1)(z^2-4)}$$

$$= \lim_{z\to(1/2)} \frac{1}{2}\cdot\frac{-z}{i(2z+1)(z^2-4)}$$

$$= \lim_{z\to(1/2)} \frac{1}{2}\cdot\frac{-1/2}{i(2)[(1/4)-4]} = \frac{-1/2}{-15i} = \frac{1}{30i}$$

Residue of $f(z)$ (at $z = -\frac{1}{2}$) $= \lim_{z\to-(1/2)}\left(\frac{2z+1}{2}\right)\cdot\frac{-z}{i(2z-1)(2z+1)(z^2-4)}$

$$= \lim_{z\to(-1/2)} \frac{1}{2}\cdot\frac{-z}{i(2z-1)(z^2-4)}$$

$$= \frac{1}{2}\cdot\frac{1/2}{i(-2)[(1/4)-4]} = \frac{1/2}{15i} = \frac{1}{30i}$$

$$\therefore \oint_C f(z)\, dz = 2\pi i\,(\text{Sum of the residues}) = 2\pi i\left(\frac{2}{30i}\right) = \frac{2\pi}{15}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{25-16\cos^2\theta} = \frac{2\pi}{15}.$$

Example 9 : Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta}\, d\theta.$ **(M.U. 2003, 04, 16)**

Sol. : Consider $\int_0^{2\pi} \frac{e^{3i\theta}}{5+4\cos\theta}\, d\theta$

Now, put $z = e^{i\theta}$ $\therefore dz = i e^{i\theta} \cdot d\theta$ $\therefore dz = i z\, d\theta$ $\therefore d\theta = \frac{dz}{iz}$

And $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+(1/z)}{2}$

$$\therefore \int_0^{2\pi} \frac{e^{3i\theta}}{5+4\cos\theta}\, d\theta = \int_C \frac{z^3}{5+4\left[\frac{z+(1/z)}{2}\right]}\cdot\frac{dz}{iz}$$

$$= \int_C \frac{z^3}{i(2z^2+5z+2)}\, dz \quad \text{where, } C \text{ is the cirlce } |z| = 1.$$

Now, the roots of $2z^2 + 5z + 2 = 0$ i.e. of $(2z+1)(z+2) = 0$ are $z = -1/2$ and $z = -2$.

$\therefore$ The pole $z = -1/2$ lies inside the unit circle and $z = -2$ lies outside it.

Now, residue of $f(z)$ (at $z = -1/2$) $= \lim_{z\to-1/2}\left(z+\frac{1}{2}\right)\cdot\frac{z^3}{i(2z+1)(z+2)}$

$$= \lim_{z\to-1/2}\left(\frac{2z+1}{2}\right)\cdot\frac{z^3}{i(2z+1)(z+2)}$$

$$= \lim_{z\to-1/2} \frac{1}{2}\cdot\frac{z^3}{i(z+2)} = \frac{1}{2}\cdot\frac{(-1/2)^3}{i[(-1/2)+2]}$$

$$= -\frac{1}{8\cdot 3i} = -\frac{1}{24i}$$

$$\therefore \int_0^{2\pi} \frac{e^{3i\theta}}{5+4\cos\theta}\, d\theta = 2\pi i\left(-\frac{1}{24i}\right) = -\frac{\pi}{12}$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta}\, d\theta = \text{Real part of } \int_0^{2\pi} \frac{e^{3i\theta}}{5+4\cos\theta}\, d\theta = -\frac{\pi}{12}.$$

Example 10 : Evaluate $\int_0^{\pi} \frac{d\theta}{3+2\cos\theta}.$ **(M.U. 2001, 03, 10, 16, 17)**

Sol. : Let $z = e^{i\theta}$ $\therefore dz = i e^{i\theta} \cdot d\theta$ $\therefore dz = i z\, d\theta$

$\therefore d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{(z^2+1)}{2z}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \int_C \frac{1}{3+2\left(\frac{z^2+1}{2z}\right)}\cdot\frac{dz}{iz} \quad \text{where, } C \text{ is the unit circle } |z| = 1.$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{i}\int_C \frac{1}{z^2+3z+1}\,dz$$

The roots of $z^2+3z+1=0$ are $z = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3\pm\sqrt{5}}{2}$.

Let $\alpha = \frac{-3+\sqrt{5}}{2}$ and $\beta = \frac{-3-\sqrt{5}}{2}$

Clearly α lies within the unit circle and β lies outside it.

$$\therefore \text{ Residue of } f(z) \text{ (at } z=\alpha) = \lim_{z\to\alpha} (z-\alpha)\cdot\frac{1}{(z-\alpha)(z-\beta)}\cdot\frac{1}{i}$$

$$= \lim_{z\to\alpha} \frac{1}{z-\beta}\cdot\frac{1}{i} = \frac{1}{(\alpha-\beta)\cdot i}$$

Now, $\alpha - \beta = \frac{-3+\sqrt{5}}{2} - \frac{-3-\sqrt{5}}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}$.

$$\therefore \text{ Residue of } f(z) \text{ (at } z=\alpha) = \frac{1}{\sqrt{5}\cdot i}.$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = 2\pi i(\text{Residue}) = 2\pi i\cdot\frac{1}{\sqrt{5}\,i} = \frac{2\pi}{\sqrt{5}}$$

$$\therefore \int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{3+2\cos\theta} = \frac{1}{2}\cdot\frac{2\pi}{\sqrt{5}} = \frac{\pi}{\sqrt{5}}$$

$$\left[\int_0^{2\pi}\frac{dx}{3+\cos 2x} = \int_0^{\pi}\frac{dx}{3+2\cos x} + \int_\pi^{2\pi}\frac{dx}{3+2\cos x}\right.$$

In the second integral put $x = 2\pi - t$ $\quad\therefore\ dx = -dt$

When $x=\pi$, $t=\pi$, when $x=2\pi$, $t=0$.

$$\therefore \int_\pi^{2\pi}\frac{dx}{3+2\cos x} = \int_\pi^0 \frac{-dt}{3+2\cos t} = \int_0^\pi \frac{dt}{3+2\cos t} = \int_0^\pi\frac{dx}{3+2\cos x}$$

$$\left.\therefore \int_0^{2\pi}\frac{dx}{3+2\cos x} = 2\int_0^\pi \frac{dx}{3+2\cos x}\right]$$

Example 11 : Evaluate $\int_0^{2\pi}\frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta$ where $-1<a<1$.

(M.U. 1993, 95, 99, 2003)

Sol. : Let $z = e^{i\theta}$ $\therefore$ $dz = i e^{i\theta}\cdot d\theta = iz\,d\theta$ and $\cos\theta = \frac{z^2+1}{2z}$.

Now consider

$$\int_0^{2\pi}\frac{e^{2i\theta}}{1-2a\cos\theta+a^2}\,d\theta = \int_C \frac{z^2}{1-2a\left(\frac{z^2+1}{2z}\right)+a^2}\cdot\frac{dz}{iz}$$

$$= \int_C \frac{z^2}{a^2z-az^2-a+z}\cdot\frac{dz}{iz} = \frac{1}{i}\int_C\frac{z^2}{(az-1)(a-z)}\,dz$$

where C is the unit circle $|z|=1$.

Now, $f(z)$ has simple poles at $z = 1/a$ and $z=a$. But as $-1<a<1$, the pole $z=a$ lies within the unit circle and $z = 1/a$ lies outside it.

$$\text{Residue of } f(z) \text{ (at } z=a) = \lim_{z\to a}(z-a)\cdot\frac{z^2}{i(az-1)(a-z)}$$

$$= \lim_{z\to a}\frac{-z^2}{i(az-1)} = \frac{-a^2}{i(a^2-1)}$$

$$\therefore \int_0^{2\pi}\frac{e^{2i\theta}}{1-2a\cos\theta+a^2}\,d\theta = \frac{1}{i}\cdot 2\pi i\cdot\left(\frac{-a^2}{a^2-1}\right) = \frac{2\pi a^2}{1-a^2}$$

$$\therefore \int_0^{2\pi}\frac{\cos 2\theta + i\sin 2\theta}{1-2a\cos\theta+a^2} = \frac{2\pi a^2}{1-a^2}$$

Equating real parts,

$$\int_0^{2\pi}\frac{\cos 2\theta}{1-2a\cos\theta+a^2} = \frac{2\pi a^2}{1-a^2}.$$

Example 12 : Show that $\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$. $(0<b<a)$

Sol. : Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$, $\sin\theta = \frac{z^2-1}{2iz}$

$$\therefore\ I = \int_C \frac{[(z^2-1)/2iz]^2}{a+b\left(\frac{z^2+1}{2z}\right)}\cdot\frac{dz}{iz} = -\frac{1}{2i}\int_C\frac{(z^2-1)^2}{z^2(bz^2+2az+b)}\,dz$$

where C is the circle $|z|=1$.

Now, the poles of $f(z)$ are given by $z=0$, which is a pole of order 2 and

$$z = \frac{-2a\pm\sqrt{4a^2-4b^2}}{2b} \quad \text{i.e., say}$$

$$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b} \quad\text{and}\quad \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

which are simple poles. Since $a>b>0$, α lies inside and β lies outside the circle $|z|=1$.

Residue of $f(z)$ (at $z=0$)

$$= \lim_{z\to 0}\frac{d}{dz}\left[z^2\cdot\frac{(z^2-1)^2}{z^2(bz^2+2az+b)}\right] = \lim_{z\to 0}\frac{d}{dz}\left[\frac{(z^2-1)^2}{bz^2+2az+b}\right]$$

$$= \lim_{z\to 0}\frac{(bz^2+2az+b)\cdot 2(z^2-1)\cdot 2z-(z^2-1)^2(2bz+2a)}{(bz^2+2az+b)^2}$$

$$= -\frac{2a}{b^2}$$

Note that $bz^2+2az+b=b(z-\alpha)(z-\beta)$ (since the roots of $ax^2+bx+c=0$ are obtained after dividing by a, the function must be multiplied by a). Also note that in this case $\alpha\beta=1$.

$\therefore\ \beta = 1/\alpha.$

Now, Residue of $f(z)$ (at $z=\alpha$) $= \lim_{z\to\alpha}(z-\alpha)\dfrac{(z^2-1)^2}{z^2b(z-\alpha)(z-\beta)}$

$$= \lim_{z\to\alpha}\frac{(z^2-1)^2}{z^2b(z-\beta)} = \frac{(\alpha^2-1)^2}{b\alpha^2(\alpha-\beta)} = \frac{[\alpha-(1/\alpha)]^2}{b(\alpha-\beta)} \qquad \left[\because\ \beta=\frac{1}{\alpha}\right]$$

$$= \frac{(\alpha-\beta)^2}{b(\alpha-\beta)} = \frac{1}{b}(\alpha-\beta) = \frac{1}{b}\cdot\frac{2\sqrt{a^2-b^2}}{b}$$

$$\therefore\ I = 2\pi i\left(-\frac{1}{2i}\right)\left[-\frac{2a}{b^2}+\frac{2\sqrt{a^2-b^2}}{b^2}\right] = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right]$$

Alternatively $I = \displaystyle\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1-\cos 2\theta}{a+b\cos\theta}\,d\theta$

$$= \text{Real part of } \frac{1}{2}\int_0^{2\pi}\frac{1-e^{2i\theta}}{a+b\cos\theta}\,d\theta$$

Then putting as before $z=e^{i\theta}$

$$I = \text{Real part of } \frac{1}{2}\int_0^{2\pi}\frac{1-z^2}{bz^2+2az+b}\,\frac{2\,dz}{i}$$

Then proceed as above.

Example 13 : Evaluate $\displaystyle\int_0^{2\pi}\frac{\sin^2\theta}{5-4\cos\theta}\,d\theta.$

Sol. : *Putting* $a=5,\ b=-4$, *we get from the above result*

$$\int_0^{2\pi}\frac{\sin^2\theta}{5-4\cos\theta}\,d\theta = \frac{2\pi}{16}\left[5-\sqrt{25-16}\right] = \frac{\pi}{4}$$

Or proceed independently by putting $z=e^{i\theta}$.

Example 14 : Evaluate $\displaystyle\int_{-\pi}^{\pi}\frac{a\cos\theta}{a+\cos\theta}\,d\theta,\ a>1.$

Sol. : Let $z=e^{i\theta}$ $\therefore\ d\theta=\dfrac{dz}{iz},\ \cos\theta=\dfrac{z^2+1}{2z}$

Consider $\displaystyle\int_0^{2\pi}\frac{a\cos\theta}{a+\cos\theta}\,d\theta = \int_C\frac{a(z^2+1)/2z}{a+(z^2+1)/2z}\cdot\frac{dz}{iz} = \int_C\frac{a(z^2+1)}{z(z^2+2az+1)}\cdot\frac{dz}{i}$

$\therefore$ The roots of $z(z^2+2az+1)=0$ are $z=0,\ z=\dfrac{-2a\pm\sqrt{4a^2-4}}{2} = -a\pm\sqrt{a^2-1}.$

$\therefore$ The pole $z=0,\ z=\alpha=-a+\sqrt{a^2-1}$ lie inside the unit circle and $z=\beta=-a-\sqrt{a^2-1}$ lies outside it because $a>1$.

$\therefore$ Residue of $f(z)$ (at $z=0$) $= \lim_{z\to 0} z\cdot\dfrac{a(z^2+1)}{z(z^2+2az+1)i} = \dfrac{a}{i}$

$\therefore$ Residue of $f(z)$ (at $z=\alpha$) $= \lim_{z\to\alpha}(z-\alpha)\cdot\dfrac{a(z^2+1)}{z(z-\alpha)(z-\beta)}\cdot\dfrac{1}{i}$

$$= \lim_{z\to\alpha}\frac{a(z^2+1)}{z(z-\beta)}\cdot\frac{1}{i} = \frac{a(\alpha^2+1)}{\alpha(\alpha-\beta)}\cdot\frac{1}{i} \qquad \text{.................... (1)}$$

Now, $\dfrac{a(\alpha^2+1)}{\alpha(\alpha-\beta)\,i} = \dfrac{a[\alpha+(1/\alpha)]}{(\alpha-\beta)}\cdot\dfrac{1}{i}$ .... (2)

Now, $\alpha+\dfrac{1}{\alpha} = \left(-a+\sqrt{a^2-1}\right)+\dfrac{1}{-a+\sqrt{a^2-1}}$

$$= \frac{\left(-a+\sqrt{a^2-1}\right)^2+1}{-a+\sqrt{a^2-1}} = \frac{a^2+a^2-1-2a\sqrt{a^2-1}+1}{-a+\sqrt{a^2-1}}$$

$$= \frac{2a^2-2a\sqrt{a^2-1}}{-a+\sqrt{a^2-1}} = \frac{2a\left(a-\sqrt{a^2-1}\right)}{-a+\sqrt{a^2-1}} = -2a \qquad \text{.................... (3)}$$

And $\alpha-\beta = \left(-a+\sqrt{a^2-1}\right)-\left(-a+\sqrt{a^2-1}\right) = 2\sqrt{a^2-1}$ .... (4)

Hence, from (1), (2), (3) and (4)

$\therefore$ Residue of $f(z)$ (at $z=\alpha$) $= a\cdot(-2a)\dfrac{1}{2\sqrt{a^2-1}}\cdot\dfrac{1}{i} = \dfrac{-a^2}{\sqrt{a^2-1}\cdot i}$

$\therefore\ \displaystyle\int_C\frac{a(z^2+1)}{z(z^2+2az+1)}\,dz = 2\pi i$(Sum of the residues)

$$= 2\pi i\left[\frac{a}{i}-\frac{a^2}{\sqrt{a^2-1}\cdot i}\right] = 2\pi a\left[1-\frac{a}{\sqrt{a^2-1}}\right].$$

Now, $\int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta}\, d\theta = 2\int_{0}^{\pi} \frac{a\cos\theta}{a+\cos\theta}\, d\theta = 2\cdot\frac{1}{2}\int_{0}^{2\pi} \frac{a\cos\theta}{a+\cos\theta}\, d\theta$

$$= \int_{0}^{2\pi} \frac{a\cos\theta}{a+\cos\theta}\, d\theta = 2\pi a\left[1-\frac{a}{\sqrt{a^2-1}}\right]$$

Example 15 : By using Cauchy's residue theorem evaluate $\int_0^{2\pi} \frac{\cos^2\theta}{5+4\cos\theta}\, d\theta$. **(M.U. 2016)**

Sol. : Let $z = e^{i\theta}$ $\therefore\ dz = e^{i\theta}\cdot i\, d\theta$ $\therefore\ dz = iz\, d\theta$ $\therefore\ d\theta = \frac{dz}{iz}$.

Now, $\cos\theta = \frac{z^2+1}{2z}$

$$\therefore\ \frac{\cos^2\theta}{5+4\cos\theta} = \left(\frac{z^2+1}{2z}\right)^2 \cdot \frac{1}{5+4\left(\frac{z^2+1}{2z}\right)}$$

$$= \frac{z^4+2z^2+1}{4z^2}\cdot\frac{z}{2z^2+5z+2} = \frac{z^4+2z^2+1}{4z(2z^2+5z+2)}$$

$$\therefore\ I = \int_C \frac{z^4+2z^2+1}{4z(2z^2+5z+2)}\cdot\frac{dz}{iz} = \frac{1}{4i}\int_C \frac{z^4+4z^2+1}{z^2(2z^2+5z+2)}\, dz$$

$$= \frac{1}{4i}\int_C \frac{z^4+4z^2+1}{z^2(2z+1)(z+2)}\, dz \text{ where } C \text{ is the circle } |z| = 1.$$

Now, the poles are given by $z^2 = 0$, $2z+1 = 0$ and $z+2 = 0$.

$\therefore$ The poles are $z = 0$, $z = -1/2$, $z = 2$.

Of these poles $z = 0$ and $z = -1/2$ lie within the unit circle and $z = 2$ lies outside it.

Further, $z = -1/2$ is a simple pole and $z = 0$ is a pole of order 2.

Now, residue at $\left(z=-\frac{1}{2}\right) = \lim_{z\to(-1/2)}\left(z+\frac{1}{2}\right)\cdot\frac{z^4+2z^2+1}{z^2\cdot 2[z+(1/2)]\cdot(z+2)}$

$$= \lim_{z\to(-1/2)}\frac{z^4+2z^2+1}{z^2\cdot 2(z+2)} = \frac{(1/16)+2(1/4)+1}{(1/4)\cdot 2[(-1/2)+2]}$$

$$= \frac{25}{16}\cdot\frac{4}{3} = \frac{25}{12}$$

Residue at $(z=0)$ $= \lim_{z\to 0}\frac{d}{dz}\left[\frac{z^4+2z^2+1}{2z^2+5z+2}\right]$

$$= \lim_{z\to 0}\frac{(2z^2+5z+2)(4z^3+4z)-(z^4+2z^2+1)(4z+5)}{(2z^2+5z+2)^2}$$

$$= \frac{0-5}{4} = -\frac{5}{4}$$

$\therefore$ Sum of the residues $= \frac{25}{12} - \frac{5}{4} = \frac{10}{12} = \frac{5}{6}$

$\therefore$ $I = 2\pi i\cdot\frac{1}{4i}\cdot\frac{5}{6} = \frac{5\pi}{12}$.

Example 16 : Evaluate $\oint_C \frac{e^{kz}}{z}\, dz$ where C is $|z| = 1$. Hence, deduce that

$$\int_0^{\pi} e^{k\sin\theta}\cos(k\sin\theta)\, d\theta = \pi$$ **(M.U. 2003, 14)**

Sol. : Let $f(z)\frac{e^{kz}}{z}$. Clearly $z = 0$ is a simple pole.

Residue (at $z = 0$) $= \lim_{z\to 0} z\cdot\frac{e^{kz}}{z} = \lim_{z\to 0} e^{kz} = e^0 = 1$.

$$\therefore\ \int_C \frac{e^{kz}}{z}\, dz = 2\pi i(1) = 2\pi i$$

Now, put $z = e^{i\theta}$ and proceed as in Ex. 1 page 2-23.

Example 17 : If $f(z) = z^3 + iz^2 - 4z - 4i$, evaluate $\oint_C \frac{f'(z)}{f(z)}\, dz$ where C encloses zeros of $f(z)$. **(M.U. 2004)**

Sol. : $\because\ f(z) = z^3 + iz^2 - 4z - i$ $\therefore\ f'(z) = 3z^2 + 2iz - 4$

$$\therefore\ \frac{f'(z)}{f(z)} = \frac{3z^2+2iz-4}{z^2(z+i)-4(z+i)} = \frac{3z^2+2iz-4}{(z^2-4)(z+i)}$$

Now, zeros of $f(z)$ are given by $(z^2-4)(z+i) = 0$.

$\therefore\ z = 2, -2, -i$.

Since, C encloses all zeros of $f(z)$ we calculate residues at $z = 2, -2, -i$ which are simple poles.

Residue (at $z = 2$) $= \lim_{z\to 2}(z-2)\cdot\frac{3z^2+2iz-4}{(z-2)(z+2)(z+i)}$

$$= \lim_{z\to 2}\frac{3z^2+2iz-4}{(z+2)(z+i)} = \frac{12+4i-4}{4(2+i)} = \frac{8+4i}{8+4i} = 1$$

Residue (at $z = -2$) $= \lim_{z\to -2}(z+2)\cdot\frac{3z^2+2iz-4}{(z-2)(z+2)(z+i)}$

$$= \lim_{z\to -2}\frac{3z^2+2iz-4}{(z-2)(z+i)} = \frac{12-4i-4}{-4(-2+i)} = \frac{8-4i}{8-4i} = 1$$

Residue (at $z = -i$) $= \lim_{z\to -i}(z+i)\cdot\frac{3z^2+2iz-4}{(z+i)(z^2-4)}$

Residue (at $z=-i$) $= \lim_{z \to -i} \frac{3z^2+2iz-4}{z^2-4} = \frac{-3+2-4}{-5} = \frac{-5}{-5} = 1$

$\therefore \int_C \frac{f'(z)}{f(z)}\,dz = 2\pi i$ (Sum of the residues) $= 2\pi i(3) = 6\pi i$.

EXERCISE - V

Using the residue theorem evaluate

1. $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ (M.U. 2004)
2. $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ (M.U. 2005)
3. $\int_0^{2\pi} \frac{d\theta}{5-4\sin\theta}$
4. $\int_0^{2\pi} \frac{d\theta}{17-8\cos\theta}$
5. $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$ (M.U. 2001, 03, 10, 11)
6. $\int_0^{\pi/2} \frac{d\theta}{97-72\cos\theta}$
7. $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ $(a>b>0)$
8. $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$
9. $\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$ $a^2<1$
10. $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta}$ $|a|<1$
11. $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$, $|a|<1$
12. $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$ $(a>b>0)$
13. $\int_0^{2\pi} \frac{d\theta}{(5-4\cos\theta)^2}$
14. $\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2}$ $0<a<1$ (M.U. 1998, 2006)
15. $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$ (M.U. 1993, 2002, 04)
16. $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta}$ $|a|<1$ (M.U. 2000)
17. $\int_0^{2\pi} \frac{d\theta}{13+5\cos\theta}$ (M.U. 2002)
18. $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ (M.U. 2015)

[*Ans.:* (1) $\frac{\pi}{6}$, (2) $\frac{2\pi}{\sqrt{3}}$, (3) $\frac{2\pi}{3}$, (4) $\frac{2\pi}{15}$, (5) $\frac{\pi}{2}$,

(6) Factors are $(4z-9)(9z-4)$; $\frac{2\pi}{65}$,

(7) $\frac{2\pi}{\sqrt{a^2-b^2}}$, (8) $\frac{2\pi}{\sqrt{a^2-b^2}}$, (9) $\frac{2\pi}{\sqrt{1-a^2}}$, (10) $\frac{2\pi}{\sqrt{1-a^2}}$,

(11) $\frac{2\pi}{(1-a^2)}$, (12) $\frac{2\pi a}{(a^2-b^2)^{3/2}}$, (13) $\frac{10\pi}{27}$, (14) $\frac{2\pi}{(1-a^2)}$,

(15) $\frac{2\pi}{3}$, (16) $\frac{2\pi}{\sqrt{1-a^2}}$, (17) $\frac{\pi}{6}$, (18) $\frac{\pi}{6}$.]

Type II : Integral of the form $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$ where $P(x)$ and $Q(x)$ are polynomials and the degree of $P(x)$ is greater than the degree of $Q(x)$ at least by 2.

Consider the integral $\int_C \frac{P(z)}{Q(z)}\,dz$ where C is the closed contour consisting of a semi-circle C_1 of radius R in the upper half with centre at the origin and the part of the real axis from $-R$ to $+R$.

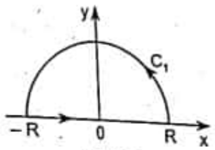

Fig. 4.13

We assume that **there is no singularity on the real axis** and we take R sufficiently large so that all the singularities of $f(z)$ lying in the upper half of the z-plane lie within the semi-circle.

Now, by Cauchy's Residue Theorem

$$\int_C f(z)\,dz = \int_C \frac{P(z)}{Q(z)}\,dz = \int_{C_1} \frac{P(z)}{Q(z)}\,dz + \int_{-R}^{R} \frac{P(z)}{Q(z)}\,dz$$

$= 2\pi i$ [sum of the residue at poles in the semi-circle] .... (1)

Now, as $R \to \infty$ i.e. $z \to \infty$ if $z\,f(z) \to 0$ i.e. $z\frac{P(z)}{Q(z)} \to 0$ then we have by Cauchy's Lemma

$$\int_{C_1} \frac{P(z)}{Q(z)}\,dz = \int_{C_1} f(z)\,dz = 0 \text{ in the limit.}$$

Hence, from (1) we get, as $R \to \infty$,

$$\int_C f(z)\,dz = \int_{-\infty}^{\infty} \frac{P(z)}{Q(z)}\,dz = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i$ [Sum of the residues at poles of $f(x)$ lying in the upper half]

Procedure to evaluate $\int_{-\infty}^{\infty} f(x)\,dx$

(i) Consider the contour consisting of a large semi-circle with centre at the origin, in the upper half of the plane and its diameter on the real axis.

(ii) See whether the degree polynomial in the numerator is greater than the degree of the denominator at least by two so that $z\,f(z) \to 0$ as $|z| \to \infty$.

(iii) Find the poles of $f(z)$ **lying in the upper half of the plane.**

(iv) Find the residues at the poles.

(v) Then $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i$ **[sum of the residues]**

Example 1 : Evaluate $\int_{-\infty}^{\infty} \frac{x^2 + x + 2}{x^4 + 10x^2 + 9}\, dx$ using contour integration. **(M.U. 2004, 05, 16)**

Sol. : (i) Consider the contour consisting of a large semi-circle with centre at the origin, in the upper half of the plane and its diameter on the real axis.

(ii) Now, $z f(z) = \frac{z^3 + z^2 + 2z}{z^4 + 10z^2 + 9} \to 0$ as $|z| \to \infty$

(iii) Further $z^4 + 10z^2 + 9 = 0$ *i.e.* $(z^2 + 1)(z^2 + 9) = 0$ gives $z = +i, -i, +3i, -3i$.

$\therefore$ The poles lying in the upper half are $z = i$, $z = 3i$.

(iv) Residue (at $z = i$) $= \lim_{z \to i} \frac{(z - i) \cdot (z^2 + z + 2)}{(z - i)(z + i)(z^2 + 9)} = \frac{1 + i}{16 i}$

Residue at $(z = 3i) = \lim_{z \to 3i} \frac{(z - 3i) \cdot (z^2 + z + 2)}{(z^2 + 1)(z - 3i)(z + 3i)} = \frac{7 - 3i}{48 i}$

(v) $\therefore \int_{-\infty}^{\infty} \frac{x^2 + x + 2}{x^4 + 10x^2 + 9}\, dx = 2\pi i \left[\frac{1 + i}{16 i} + \frac{7 - 3i}{48 i}\right] = \frac{5\pi}{12}$

Example 2 : Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)^2 (x^2 + 2x + 2)}\, dx$.

Sol. : (i) Consider the contour consisting of a large semi-circle with centre at the origin with diameter on the real axis in the upper half of the plane.

(ii) Now, $z f(z) = \frac{z \cdot z^2}{(z^2 + 1)^2 (z^2 + 2z + z)} \to 0$ as $z \to \infty$.

(iii) Further $(z^2 + 1)^2 (z^2 + 2z + 2) = 0$ gives $z = i, -i$, and $z = \frac{-2 \pm \sqrt{4 - 8}}{2}$ *i.e.* $z = -1 \pm i$.

The poles lying in the upper half are $z = i$ and $-1 + i$.

(iv) Residue (at $z = i$ which is a pole of order 2).

$$= \lim_{z \to i} \frac{d}{dz}\left[(z - i)^2 \cdot \frac{z^2}{(z - i)^2 (z + i)^2 (z^2 + 2z + 2)}\right]$$

$$= \lim_{z \to i} \frac{d}{dz}\left[\frac{z^2}{(z + i)^2 (z^2 + 2z + 2)}\right]$$

$$= \lim_{z \to i} \left[(z + i)^2 (z^2 + 2z + 2) \cdot 2z - z^2 \{2(z + i)(z^2 + 2z + 2) + (z + i)^2 \cdot (2z + 2)\}\right] + \left[(z + i)^4 (z^2 + 2z + 2)^2\right]$$

$$= \left[(2i)^2 (-1 + 2i + 2)(2i) - (-1)\{2(2i)(-1 + 2i + 2) + (2i)^2 (2i + 2)\}\right] + \left[(2i)^4 (-1 + 2i + 2)^2\right]$$

$$= \frac{[(-4)(1 + 2i)(2i) + \{4i(1 + 2i) - 4 \cdot 2(1 + i)\}]}{16(1 + 2i)^2}$$

$$= \frac{-8i + 16 + 4i - 8 - 8 - 8i}{16(-3 + 4i)} = \frac{-12i}{16(4i - 3)}$$

$$= -\frac{3i}{4(4i - 3)} \cdot \frac{4i + 3}{(4i + 3)} = -\frac{-12 + 9i}{4(-16 - 9)} = \frac{9i - 12}{100}$$

Residue at $(z = -1 + i)$

$$= \lim_{z \to -1+i} (z + 1 - i) \cdot \frac{z^2}{(z^2 + 1)^2 (z + 1 - i)(z + 1 + i)}$$

$$= \lim_{z \to -1+i} \frac{z^2}{(z^2 + 1)^2 (z + 1 + i)}$$

$$= \frac{(-1 + i)^2}{[(-1 + i)^2 + 1]^2 [-1 + i + 1 + i]} = \frac{1 - 2i - 1}{[1 - 2i - 1 + 1]^2 (2i)}$$

$$= \frac{-2i}{(1 - 2i)^2 \cdot 2i} = -\frac{1}{(1 - 2i)^2} = -\frac{1}{-3 - 4i} = \frac{1}{3 + 4i}$$

$$= \frac{1}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{3 - 4i}{25}$$

(v) $\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)^2 (x^2 + 2x + 2)}\, dx = 2\pi i$ (Sum of residues)

$$= 2\pi i \left[\frac{9i - 12}{100} + \frac{3 - 4i}{25}\right] = 2\pi i \frac{[9i - 12 + 12 - 16i]}{100}$$

$$= 2\pi i \cdot \frac{(-7i)}{100} = \frac{7\pi}{50}$$

Example 3 : Evaluate $\int_{0}^{\infty} \frac{dx}{x^4 + a^4}$ using contour integration. **(M.U. 2003, 11)**

Sol. : (i) Consider the contour consisting of a large semi-circle with centre at the origin, in the upper half of the plane and its diameter along the real axis.

(ii) Now, $z f(z) = \frac{z}{z^4 + a^4} \to 0$ as $|z| \to \infty$

(iii) Further $z^4 + a^4 = 0$ *i.e.* $z^4 = -a^4 = a^4 (\cos \pi + i \sin \pi)$

$$\therefore z = a(\cos \pi + i \sin \pi)^{1/4} = a\left(\cos \frac{(2n + 1)\pi}{4} + i \sin \frac{(2n + 1)\pi}{4}\right)$$

when $n = 0$, $z = a\left(\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)$, when $n = 1$, $z = a\left(-\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)$,

when $n = 2$, $z = a\left(-\frac{1}{\sqrt{2}} - i \cdot \frac{1}{\sqrt{2}}\right)$, when $n = 3$, $z = a\left(\frac{1}{\sqrt{2}} - i \cdot \frac{1}{\sqrt{2}}\right)$.

Of these first two poles lie in the upper half of the plane.

(iv) Residue at $z = a\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$ i.e. at $z = a e^{i\pi/4}$

$$= \lim_{z\to ae^{i\pi/4}} \frac{z-ae^{i\pi/4}}{z^4+a^4} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z\to ae^{i\pi/4}} \frac{1}{4z^3} \qquad [\text{By L'Hospital's Rule}]$$

$$= \frac{1}{4a^3}\cdot e^{-3i\pi/4}$$

Residue at $z = a\left(-\frac{1}{\sqrt{2}}+i\cdot\frac{1}{\sqrt{2}}\right)$ i.e. at $z = a e^{3i\pi/4}$

$$= \lim_{z\to ae^{3i\pi/4}} \frac{1}{4z^3} \qquad [\text{As before}]$$

$$= \frac{1}{4a^3}\cdot e^{-9i\pi/4}$$

(v) $$\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = \frac{1}{2}\cdot 2\pi i\left[\frac{1}{4a^3}e^{-3i\pi/4}+\frac{1}{4a^3}e^{-9i\pi/4}\right]$$

$$= \frac{\pi i}{4a^3}\left[\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{9\pi}{4}\right) - i\sin\left(\frac{9\pi}{4}\right)\right]$$

$$= \frac{\pi i}{4a^3}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] = \frac{\pi i}{4a^3}\left(-\frac{2i}{\sqrt{2}}\right)$$

$$= \frac{\pi}{2a^3\sqrt{2}} = \frac{\pi\sqrt{2}}{4a^3}$$

Example 4 : Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx,\ a>0,\ b>0.$

(M.U. 1995, 97, 2000, 02, 04, 06, 14)

Sol. : (i) Consider as before the contour consisting of a semi-circle and diameter on the real axis with centre at the origin.

(ii) Now, $z\,f(z) = \frac{z^3}{(z^2+a^2)(z^2+b^2)} \to 0$ as $|z| \to \infty$

(iii) Now, $(z^2+a^2)(z^2+b^2) = 0$ i.e. $z = +ai, -ai, +bi, -bi$. Of these $z = ai$, $z = bi$ lie in the upper of the z-plane.

(iv) Residue (at $z = ai$) $$= \lim_{z\to ai} (z-ai)\cdot\frac{z^2}{(z-ai)(z+ai)(z^2+b^2)}$$

$$= \frac{-a^2}{2ai(-a^2+b^2)} = \frac{a}{2i(a^2-b^2)}$$

Similarly, Residue (at $z = bi$) $= \frac{-b^2}{2bi(a^2-b^2)} = \frac{-b}{2i(a^2-b^2)}$

(v) $$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx = 2\pi i\left[\frac{a}{2i(a^2-b^2)} + \frac{-b}{2i(a^2-b^2)}\right] = \frac{\pi}{a+b}.$$

Example 5 : Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx.$ **(M.U. 2001, 03, 04, 05, 14)**

Sol. : Putting $a = 1$ and $b = 2$ in the above result, we get

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} = \frac{\pi}{1+2} = \frac{\pi}{3}$$

Or we obtain the result independently as follows,

(i) Consider the contour consisting of a semi-circle and diameter on the real axis with centre at the origin.

(ii) Now. $z\cdot f(z) = \frac{z^3}{(z^2+1)(z^2+4)} \to 0$ as $|z| \to \infty$

(iii) $(z^2+1)(z^2+4) = 0\ \therefore\ z = i, -i, 2i - 2i$ of there $z = i, 2i$ lie in the upper half of the z-plane.

(iv) Residue (at $z = i$) $$= \lim_{z\to i} (z-i)\cdot\frac{z^2}{(z+i)(z-i)(z^2+4)}$$

$$= \lim_{z\to i} \frac{z^2}{(z+i)(z^2+4)} = \frac{-1}{(2i)(3)} = -\frac{1}{6i}$$

Residue (at $z = 2i$) $$= \lim_{z\to 2i} (z-2i)\cdot\frac{z^2}{(z^2+1)(z-2i)(z+2i)}$$

$$= \lim_{z\to 2i} \frac{z^2}{(z^2+1)(z+2i)} = \frac{-4}{(-3)(4i)} = \frac{1}{3i}$$

(v) $$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx = 2\pi i\left[-\frac{1}{6i}+\frac{1}{3i}\right] = 2\pi i\cdot\frac{1}{6i} = \frac{\pi}{3}.$$

Example 6 : Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^3},\ a>0.$ **(M.U. 2003, 17)**

Sol. : (i) Consider the contour consisting of a semi-circle and diameter on the real axis with centre at the origin.

(ii) Now, $z\cdot f(z) = z\cdot\frac{1}{(z^2+a^2)^3} \to 0$ as $|z| \to \infty$

(iii) The poles are given by, $z^2+a^2 = 0\ \therefore\ z = ai, -ai$. Of these $z = ai$ lies in the upper half of the z-plane. It is a pole of order 3.

(iv) Residue (at $z = ai$) $$= \lim_{z\to ai} \frac{1}{2!}\cdot\frac{d^2}{dz^2}\left[(z-ai)^3\cdot\frac{1}{(z-ai)^3(z+ai)^3}\right]$$

$$= \lim_{z\to ai} \frac{1}{2!}\cdot\frac{d^2}{dz^2}\left[\frac{1}{(z+ai)^3}\right] = \lim_{z\to ai} \frac{1}{2}\cdot\frac{d}{dz}\left[\frac{-3}{(z+ai)^4}\right]$$

$\therefore$ Residue (at $z = ai$) $= \lim_{z \to ai} \frac{1}{2} \cdot \frac{(-3)(-4)}{(z+ai)^5} = \frac{6}{(2ai)^5} = \frac{3}{16a^5 i}$

(v) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} = 2\pi i \left[\frac{3}{16a^5 i}\right] = \frac{3\pi}{8a^5}$

$\therefore \int_0^{\infty} \frac{dx}{(x^3+a^2)^3} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)} = \frac{3\pi}{16a^5}.$

Example 7 : Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$. (M.U. 1990, 2001, 04)

Sol. : (i) Consider as before the contour consisting of a semi-circle and diameter on the real axis with centre at the origin.

(ii) Now, $z f(z) = z \cdot \frac{1}{(z^2+a^2)(z^2+b^2)} \to 0$ as $|z| \to \infty$

(iii) The poles are given by $z + a^2 = 0$, $z^2 + b^2 = 0$ $\therefore$ $z = \pm ai$, $z = \pm bi$. Of these $z = ai$, $z = bi$ lie in the upper half of the z-plane.

(iv) Residue (at $z = ai$) $= \lim_{z \to ai} (z - ai) \cdot \frac{1}{(z-ai)(z+ai)(z^2+b^2)} = \frac{1}{2ai(b^2-a^2)}$

Similarly, Residue (at $z = bi$) $= \lim_{z \to bi} (z - bi) \cdot \frac{1}{(z^2+a^2)(z-bi)(z+bi)}$

$= \frac{1}{2bi} \cdot \frac{1}{(a^2-b^2)}$

(v) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i \left[-\frac{1}{2ai(a^2-b^2)} + \frac{1}{2bi(a^2-b^2)}\right]$

$= \frac{\pi(a-b)}{(a^2-b^2)ab} = \frac{\pi}{ab(a+b)}.$

Example 8 : Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx.$ (M.U. 1995)

Sol. : (i) Considering the contour as above and nothing that $z f(z) \to 0$ as $|z| \to \infty$, we find the poles.

(ii) $z^6 + 1 = 0$ gives $z^6 = e^{(2n+1)\pi i}$

$\therefore z = e^{(2n+1)i\pi/6}$; $n = 0, 1, 2, 3, 4, 5.$

$\therefore z = e^{(i\pi)/6}, e^{(i\pi)/2}, e^{(5i\pi)/6}, e^{(7i\pi)/6}, e^{(9i\pi)/2}, e^{(11i\pi)/6}$

(iii) Of these first three poles $\alpha_1 = e^{i\pi/6}$, $\alpha_2 = e^{i\pi/2}$, $\alpha_3 = e^{5i\pi/6}$ lie in the upper half plane. Let α be one of the these poles.

(iv) Residue (at $z = \alpha$) $= \lim_{z \to \alpha} \frac{(z-\alpha)(z^2)}{z^6+1}$ $\left[\text{Form } \frac{0}{0}\right]$

Residue (at $z = \alpha$) $= \lim_{z \to \alpha} \frac{(z-\alpha)2z+z^2}{6z^5}$ [L' Hospital's Rule]

$= \frac{\alpha^2}{6\alpha^5} = \frac{\alpha^3}{6\alpha^6} = -\frac{\alpha^3}{6}$ $[\because \alpha^6 = -1]$

$\therefore$ Sum of the residues $= -\frac{1}{6}[\alpha_1^3 + \alpha_2^3 + \alpha_3^3]$

$= -\frac{1}{6}\left[e^{(i\pi)/2} + e^{(3i\pi)/2} + e^{(5i\pi)/2}\right]$

$= -\frac{1}{6}\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) + \cos\left(\frac{5\pi}{2}\right) + i\sin\left(\frac{5\pi}{2}\right)\right]$

$= -\frac{1}{6}[i - i + i] = -\frac{i}{6}$

(v) $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx = 2\pi i\left(-\frac{i}{6}\right) = \frac{\pi}{3}.$

EXERCISE - VI

Evaluate the following using contour integration.

1. $\int_0^{\infty} \frac{dx}{x^4+1}$
2. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)} dx$ (M.U. 2004)
3. $\int_0^{\infty} \frac{dx}{x^2+1}$ (M.U. 2005, 09)
4. $\int_0^{\infty} \frac{dx}{(x^2+1)^2}$
5. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$ (M.U. 2000, 03, 09)
6. $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$, $a > 0$
7. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+25)} dx$
8. $\int_{-\infty}^{\infty} \frac{x^2+x+3}{x^4+5x^2+4} dx$ (M.U. 2004)
9. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2}$
10. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2} dx$
11. $\int_0^{\infty} \frac{x^2}{(x^2+1)^3} dx$
12. $\int_0^{\infty} \frac{dx}{x^4+16}$ (M.U. 1998, 2011)
13. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2} dx$, $a > 0$ (M.U. 2000)
14. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+2x+2)} dx$ (M.U. 1997)

[**Ans. :** (1) $\frac{\pi\sqrt{2}}{4}$, (2) $\frac{\pi}{5}$, (3) $\frac{\pi}{2}$, (4) $\frac{\pi}{4}$, (5) $\frac{3\pi}{8}$, (6) $\frac{\pi}{4a^3}$, (7) $\frac{\pi}{8}$, (8) $\frac{5\pi}{6}$, (9) Use logarithmic differentiation ; $\frac{5\pi}{144}$, (10) $\frac{\pi}{100}$, (11) $\frac{\pi}{16}$, (12) $\frac{\pi\sqrt{2}}{32}$, (13) $\frac{3\pi}{2a}$, (14) $\frac{3\pi}{5}$.]

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Type III : Integral of the type $\int_{-\infty}^{\infty} \frac{\cos mx}{\Phi(x)}\,dx$ or $\int_{-\infty}^{\infty} \frac{\sin mx}{\Phi(x)}\,dx$ where $\Phi(x)$ is a polynomial in x.

We start with $\int_C F(z)\,dz = \int_C e^{imz} \cdot f(z)\,dz$ where $f(z) = \frac{1}{\Phi(z)}$. As in the previous case we consider a contour C consisting of a semi-circle C_1 of radius R with centre at the origin lying in the upper half of the z-plane and the part of the real axis from $-R$ to R. We assume **that there is no singularity on the real axis** and we take R sufficiently large, so that all the singularities of $f(z)$ lying in the upper half of the z-plane lie within the semi-circle.

Now, by Cauchy's Residue Theorem

$$\int_C F(z)\,dz = \int_{C_1} F(z)\,dz + \int_{-R}^{R} F(z)\,dz$$

$$= 2\pi i \text{ [sum of the residue at poles in the semi-circle]}$$

Now, as $R \to \infty$ i.e. $z \to \infty$, if $F(z) \to 0$, we have by Jordan's Lemma $\int_{C_1} F(z)\,dz = 0$ in the limit.

$$\therefore \int_C F(z)\,dz = \int_{-\infty}^{\infty} F(z)\,dz = \int_{-\infty}^{\infty} F(x)\,dx = \int_{-\infty}^{\infty} e^{imx} f(x)\,dx = 2\pi i \text{ [sum of residues]}$$

$$\therefore \int_{-\infty}^{\infty} e^{imx} f(x)\,dx = 2\pi i \text{ [sum of residues]}$$

Example 1 : Evaluate $\int_0^{\infty} \frac{\cos mx}{a^2 + x^2}\,dx$ and deduce that $\int_0^{\infty} \frac{x \sin mx}{a^2 + x^2}\,dx = \frac{\pi}{2} e^{-mx}$.

(M.U. 2005)

Sol. : Consider $\int_C F(z)\,dz = \int_C e^{imz} f(z)\,dz$ where $f(z) = \frac{1}{a^2 + z^2}$.

(i) Consider the contour C consisting of a large semi-circle with centre at the origin, in the upper half of the z-plane and its diameter on the real axis.

(ii) Now, $f(z) = \frac{1}{a^2 + z^2} \to 0$ as $z \to \infty$

(iii) Further, $a^2 + z^2 = 0 \quad \therefore z = + ai, - ai.$

The pole lying in the upper half of the z-plane is $z = ai$

(iv) Residue (at $z = ai$) $= \lim_{z \to ai} \frac{(z - ai)\, e^{imz}}{(z - ai)(z + ai)} = \frac{e^{-ma}}{2ai}$

(v) $$\int_{-\infty}^{\infty} \frac{e^{imx}}{a^2 + x^2}\,dx = 2\pi i \left(\frac{e^{-ma}}{2ai} \right) = \frac{\pi}{a} e^{-ma}$$

Equating real parts on both sides,

$$\int_{-\infty}^{\infty} \frac{\cos mx}{a^2 + x^2}\,dx = \frac{\pi}{a} e^{-ma} \qquad \therefore \int_0^{\infty} \frac{\cos mx}{a^2 + x^2}\,dx = \frac{\pi}{2a} e^{-ma}$$

Differentiating w.r.t. m,

$$\int_0^{\infty} \frac{-x \sin mx}{a^2 + x^2}\,dx = -a \cdot \frac{\pi}{2a} \cdot e^{-ma} \qquad \therefore \int_0^{\infty} \frac{x \sin mx}{a^2 + x^2}\,dx = \frac{\pi}{2} e^{-ma}$$

Example 2 : Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)}\,dx,\ a > 0,\ b > 0.$

(M.U. 1998)

Sol. : Consider $\int_C F(z)\,dz = \int_C e^{iz} f(z)\,dz$ where $f(z) = \frac{1}{(z^2 + a^2)(z^2 + b^2)}$.

(i) Consider the contour C consisting of a large semi-circle with centre at the origin, in the upper half of the z-plane and its diameter on the real axis.

(ii) Now, $f(z) = \frac{1}{(z^2 + a^2)(z^2 + b^2)} \to 0$ as $z \to \infty$

(iii) Further, $(z^2 + a^2)(z^2 + b^2) = 0.$

$\therefore z = \pm ai, \pm bi$ are simple poles.

Poles lying in the upper half of the z-plane are $z = ai, bi$.

(iv) Residue of $f(z)$ (at $z = ai$) $= \lim_{z \to ai} \frac{(z - ai) \cdot e^{iz}}{(z + ai)(z - ai)(z^2 + b^2)}$

$$= \lim_{z \to ai} \frac{e^{iz}}{(z + ai)(z^2 + b^2)} = \frac{e^{-a}}{-2ai(a^2 - b^2)}$$

Residue of $f(z)$ (at $z = bi$) $= \lim_{z \to bi} \frac{(z - bi) \cdot e^{iz}}{(z + bi)(z - bi)(z^2 + b^2)}$

$$= \lim_{z \to bi} \frac{e^{iz}}{(z + bi)(z^2 + a^2)} = \frac{e^{-b}}{2bi(a^2 - b^2)}$$

(v) $$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + a^2)(x^2 + b^2)}\,dx = \frac{2\pi i}{a^2 - b^2} \left(\frac{e^{-b}}{2bi} - \frac{e^{-a}}{2ai} \right) = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$$

Equating real parts,

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)}\,dx = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$$

EXERCISE - VII

Evaluate the following using residues.

1. $\int_0^{\infty} \frac{\cos x}{a^2 + x^2}\,dx \quad (a > 0)$ **(M.U. 1990)**

2. $\int_0^{\infty} \frac{\cos ax}{1 + x^2}\,dx \quad (a > 0)$

3. $\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 5x^2 + 4}\,dx$ **(M.U. 1996)**

4. $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 - 2x + 5} dx$ 5. $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} dx$ 6. $\int_{-\infty}^{\infty} \frac{\cos ax}{x^4 + 10x^2 + 9} dx$

[Ans. : (1) $\frac{\pi e^{-a}}{2a}$, (2) $\frac{\pi e^{-a}}{2}$, (3) $\pi\left(\frac{1}{3e} - \frac{1}{6e^2}\right)$, (4) $\frac{\pi}{2e^2} \sin 1$,

(5) $-\frac{\pi}{e} \sin 2$, (6) $\frac{\pi}{8}\left[\frac{e^{-a}}{1} - \frac{e^{-3a}}{3}\right]$.]

EXERCISE - VIII

Theory

1. State and prove Cauchy's Integral Theorem. (M.U. 1993, 96, 99)
2. State and prove Cauchy's Integral Formula. (M.U. 1995, 96, 98)
3. If $f(z)$ is analytic within and on a closed curve C and if 'a' is any point within C, then prove that

$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a).$$ (M.U. 1997)

4. Prove that $f'''(a) = \frac{3!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^4} dz$, if C is a closed curve enclosing $z = a$ and $f(z)$ is analytic inside and on C. Hence, evaluate $\oint_C \frac{e^{iz}}{z^4} dz$, where C is the circle $|z| = 2$.

(M.U. 1997) [Ans. : $\frac{2\pi}{3!} e^{ia}$]

5. Define : (i) Essential Singularity, (ii) Residue of a function, (iii) Pole of order m. (M.U. 1994, 97, 2003)
6. State Cauchy's residue theorem. (M.U. 1994)
7. Define (i) Pole, (ii) Isolated Singular Point. (M.U. 1995, 97, 2003)
8. State and prove Cauchy's residue theorem. (M.U. 1995, 97, 2003)
9. "Cauchy's integral formula can be deduced from Cauchy's residue theorem". Explain. (M.U. 1999)
10. If $f(z)$ has a pole of order m at $z = a$, then prove that

$$\text{Residue (at } z = a) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m \cdot f(z)\right]$$

11. Define : (i) Singular Point, (ii) Essential Singularity, (iii) Removable Singularity, (iv) Residue of a function. (M.U. 2006)

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CHAPTER 5

Eigenvalues and Eigenvectors

1. Introduction

In this chapter, we we goging to sudy two new concepts *viz.* eigenvalues and eigenvectors. Then we shall study a very important theorem *viz.* Cayley-Hamilton Theorem.

2. Definition

You are already familiar with matrices and with some operations on it. To repeat, a **matrix** is a system of mn numbers arranged in m rows and n columns. It is called an $m \times n$ matrix. Thus, A is an $m \times n$ matrix where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

3. Homogeneous Linear Equations

We have studied homogeneous linear equations in semester I, Applied Mathematics - I.

An equation of the form $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is called a homogeneous equation.

The important result we need is that, if the rank r of the matrix is less than the number unknowns n *i.e.*, if $r < n$ then the system has **non-trivial** (non-zero) solution.

The number of independent solutions *i.e.*, the **number of parameters is equal** to $n - r$.

In particular, in this chapter, we shall be concerned with the set of 3 equations in 3 unknowns.

If the rank of the matrix is 2, then there will be $3 - 2 = 1$ independent solution or one parameter. We shall denote this parameter by t.

If the rank of the matrix is 1, then there will be $3 - 1 = 2$ independent solutions or two parameters. We shall denote these parameters by s and t.

Since, the parameters, s and t are independent, the system then has **infinite number of solutions.**

4. Eigenvalues

Let A be any n-rowed square matrix, λ a scalar and I the unit matrix of the same order.

The matrix $A - \lambda I$ is called the **characteristic matrix.**

The determinant $| A - \lambda I |$ is called the **characteristic polynomial** of A.

The equation obtained by equating to zero this determinant *i.e.* the equation $| A - \lambda I | = 0$ is called the **characteristic equation** of the matrix A.

For example, consider a two-rowed square matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

Then, $A - \lambda I = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$ is the characteristic matrix of A.

Further, $| A - \lambda I |$ *i.e.* $\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix}$ *i.e.* $\lambda^2 - 4\lambda - 5$ is the characteristic polynomial of A.

And the equation $| A - \lambda I | = 0$ *i.e.* $\lambda^2 - 4\lambda - 5 = 0$ is the characteristic equation of A.

If we consider an n-th order square matrix A, then its characteristic equation will be

$$| A - \lambda I | = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \qquad \text{.................... (1)}$$

On expanding this determinant, we get an equation of the form,

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + ... + a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

This is the characteristic equation of A.

The roots of this equation are called the **characteristic roots** or **latent roots** or **characteristic values** or the **eigenvalues** or **proper values** of the matrix A. (Eigen is a German word meaning proper.)

For example if $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ then as seen before the characteristic equation of A is

$\lambda^2 - 4\lambda^2 - 5 = 0.$

By solving it we get $(\lambda - 5)(\lambda + 1) = 0$

$\therefore \ \lambda = -1, 5.$

$\therefore \ -1, 5$ are the eigenvalues of the matrix A.

5. Eigenvectors

Suppose λ_1 is a root of $| A - \lambda I | = 0$. Then, $| A - \lambda_1 I | = 0$. Suppose further we find a non-zero column matrix X such that

$$[A - \lambda_1 I] X = 0. \qquad \text{.................... (1)}$$

This system of equations has non-trivial solutions.

The vector X is called the **eigenvector** or **latent vector** corresponding to the root λ_1.

For example, if $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, we have seen that $\lambda = 5$ is a root. Corresponding to this root, suppose we have a column matrix X such that

$[A - \lambda_1 I] X = 0$ *i.e.* $[A - 5I] X = 0$

i.e., $\left\{ \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \ \begin{bmatrix} 1-5 & 2 \\ 4 & 3-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *i.e.* $\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $\quad R_2 + R_1 \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \ -4x_1 + 2x_2 = 0 \qquad \therefore \ x_2 = 2x_1.$

Putting $x_1 = t$, we get $x_2 = 2t$, we get, the vector $\begin{bmatrix} t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \therefore \ X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

or any non-zero multiple k of this vector is the *eigen-vector* corresponding to $\lambda = 5$.

Note ....

From (1) we get, $\quad AX = \lambda_1 X.$

Example 1 : If one eigenvalue of a matrix A is $a + ib$ then another eigenvalue must be $a - ib$. **(M.U. 2001)**

Sol. : Since eigenvalues are the roots of (its characteristic) equation, the complex roots occur in pairs. Hence, if $a + ib$ is one eigenvalue then another eigenvalue must $a - ib$.

Example 2 : Find the sum and the product of the eigenvalues of A where

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad \textbf{(M.U. 2004)}$$

Sol. : The characteristic equation is

$$\begin{bmatrix} a_1 - \lambda & a_2 & a_3 \\ b_1 & b_2 - \lambda & b_3 \\ c_1 & c_2 & c_3 - \lambda \end{bmatrix} = 0$$

If we expand the determinant on the l.h.s. then we get,

$$\lambda^3 - (a_1 + b_2 + c_3)\lambda^2 + (a_1 b_2 + b_2 c_3 + c_3 a_1 - a_2 b_1 - a_3 c_1 - b_3 c_2)\lambda - |A| = 0$$

By the theory of roots of an equation, (If $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n = 0$ is the given equation then (i) sum of the roots $= -a_1/a_0$ and (ii) product of the roots $= (-1)^n \dfrac{a_n}{a_0}$)

Sum of the eigenvalues $= a_1 + b_2 + c_3 =$ Sum of the diagonal elements.

Product of the eigenvalues = $|A|$.

Note ....

The result can be generalised for a square matrix of order n.

Example 3 : If $A = \begin{bmatrix} x & 4x \\ 2 & y \end{bmatrix}$ has eigenvalues 5 and -1, find the values of x and y. (M.U. 2014)

Sol. : By Ex. 2 above, we have

Sum of the eigenvalues = Sum of the diagonal elements

And Product of the eigenvalues = $|A|$.

$\therefore$ $x + y = 5 + (-1) = 4$ and $xy - 8x = 5(-1) = -5$

But $y = 4 - x$,

$\therefore$ $x(4-x) - 8x = -5$ $\quad\therefore -x^2 - 4x + 5 = 0$

$\therefore$ $x^2 + 4x - 5 = 0$ $\quad\therefore (x+5)(x-1) = 0$ $\quad\therefore x = -5$ or 1.

When $x = -5$, $y = 9$. When $x = 1$, $y = 3$.

Example 4 : If A is a non-singular square matrix of order n having eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$, prove that (i) $\lambda_1 + \lambda_2 + \ldots + \lambda_n = (a_{11} + a_{22} + \ldots + a_{nn})$ and (ii) $|A| = \lambda_1, \lambda_2, \ldots \lambda_n$. (M.U. 2000)

Sol. : As above.

Example 5 : If a matrix A is singular then at least one of the eigenvalues is zero and vice versa. (M.U. 2001)

Sol. : As proved above if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues then $|A| = \lambda_1 \lambda_2 \ldots \lambda_n$

But $|A| = 0$ $\therefore$ At least one of $\lambda_1, \lambda_2, \ldots \lambda_n$ must be zero and conversely. [See Ex. 1, page 5-14. One eigenvalue is zero and $|A| = 0$.]

Example 6 : Find the sum and the product of the eigenvalues of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Sol. : As proved above,

Sum of the eigenvalues = Sum of the diagonal element

$= 8 + 7 + 3 = 18$

Product of the eigenvalues $= |A|$

$= 8(21 - 16) + 6(-18 + 8) + 2(24 - 14)$

$= 40 - 60 + 20 = 0$

Note ....

Note that one of the eigenvalues of A is 0. See Ex. 1, page 5-14. This verifies Ex. 5 above and $|A| = 0$.

Remarks ....

1. The actual eigenvalues are 0, 3, 15.
2. Ex. 6 varifies the result of Ex. 5.

Example 7 : If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the matrix $\begin{bmatrix} -2 & -9 & 5 \\ -5 & -10 & 7 \\ -9 & -21 & 14 \end{bmatrix}$ then find $\lambda_1 + \lambda_2 + \lambda_3$ and $\lambda_1 \lambda_2 \lambda_3$. (M.U. 2000)

Sol. : As proved above,

Sum of the eigenvalues $= \lambda_1 + \lambda_2 + \lambda_3$

= Sum of the diagonal elements

$= -2 - 10 + 14 = 2$

Product of the eigenvalues $= \lambda_1 \lambda_2 \lambda_3$

$= |A|$

$= -2(-140 + 144) + 9(-70 + 63) + 5(105 - 90)$

$= -8 - 63 + 75 = 4$

Example 8 : If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$,

where a, b, c are positive integers, then prove that (i) $a + b + c$ is an eigenvalue of A and (ii) if A is non-singular, one of the eigenvalues is negative.

(See Ex. 24, page 5-40)

Sol. : The characteristic equation is

$$\begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0$$

By $C_1 + C_2 + C_3$

$$\begin{vmatrix} a+b+c-\lambda & b & c \\ a+b+c-\lambda & c-\lambda & a \\ a+b+c-\lambda & a & b-\lambda \end{vmatrix} = 0 \quad \therefore (a+b+c-\lambda)\begin{vmatrix} 1 & b & c \\ 1 & c-\lambda & a \\ 1 & a & b-\lambda \end{vmatrix} = 0$$

$\therefore$ $a + b + c - \lambda = 0$ $\quad\therefore \lambda = a + b + c$.

Further if $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of A, then

$\lambda_1 + \lambda_2 + \lambda_3 = a + b + c$ [By Ex. 2]

But one value, say $\lambda_1 = a + b + c$

$\therefore$ $\lambda_2 + \lambda_3 = 0$ .... (1)

Since A is non-singular

$\lambda_1 \lambda_2 \lambda_3 = |A| \neq 0$ .... (

$\therefore$ From (1) and (2) it is clear that one of λ_2 and λ_3 is negative.

Example 9 : Show that the following matrices have the same characteristic equation

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}, \begin{bmatrix} b & c & a \\ c & a & b \\ a & b & c \end{bmatrix}, \begin{bmatrix} c & a & b \\ a & b & c \\ b & c & a \end{bmatrix}$$ **(M.U. 2010)**

Sol. : The second matrix can be obtained from the first by interchanging R_1 and R_2 then R_2 and R_3.

The third matrix canbe obtained from the first by interchanging R_1 and R_3 and then R_2 and R_3.

Since the interchange of two rows (or two columns) changes the sign of the determinant, the resulting determinant in the above two cases remains the same as there are two changes.

Hence, the resulting equation remains the same (and the roots of the equation *i.e.,* eigenvalues are the same).

Note ....

The eigenvalues of the above three matrices are the same.

Example 10 : Two of the eigenvalues of a 3×3 matrix are $-1, 2$. If the determinant of the matrix is 4, find its third eigenvalue. **(M.U. 2000)**

Sol. : If the third eigenvalue is x then their product is equal to 4.

$\therefore\ (-1)(2)(x) = 4 \ \therefore\ x = -2$

Hence, the third eigenvalue is -2.

Example 11 : If A and B are two square matrices of the same order then AB and BA have the same characteristic roots.

Sol. : We accept this result without giving proof but we shall verify it by an example.

Example 12 : If A and B are as given below, find the eigenvalues of AB and BA.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 5 & 6 \end{bmatrix}$$

Sol. : $AB = \begin{bmatrix} 3 & 0 \\ 26 & 24 \end{bmatrix}, \ BA = \begin{bmatrix} 3 & 0 \\ 17 & 24 \end{bmatrix}$

The characteristic equations are

$$\begin{vmatrix} 3-\lambda & 0 \\ 26 & 24-\lambda \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} 3-\lambda & 0 \\ 17 & 24-\lambda \end{vmatrix} = 0$$

$\therefore\ (3-\lambda)(24-\lambda) = 0$ and $(3-\lambda)(24-\lambda) = 0$.

The eigenvalues of both AB and BA are 3, 24. .

6. Procedure to Find Eigenvalues and Eigenvectors of a Given Matrix A

1. First write the characteristic equation $|A - \lambda I| = 0$.
2. Solve the above equation and find the roots, say, $\lambda_1, \lambda_2, \lambda_3, \ldots$
3. For $\lambda = \lambda_1$, consider the matrix equality, $[A - \lambda_1 I] X = 0$.

4. Apply **elementary row transformations** and **reduce the above matrix $[A - \lambda_1 I]$ to echelon form.**
5. Now write the equations. If there are n unknowns and the rank of the matrix is r, there will be $(n - r)$ independent parameters.
6. Sometimes you will require to use, Cramer's rule.
7. Solve the equations and get the eigenvectors $X_1 = \ldots, X_2 = \ldots$

The following examples will make the procedure clear.

Type I : A is a non-symmetric matrix and eigenvalues are distinct

Example 1 : Find the eigenvalues and eigenvectors of the following matrix. [Verify that the eigenvectors are linearly independent.]

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$ **(M.U. 2004, 05, 09)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\therefore\ (2-\lambda)[(2-\lambda)^2 - 1] + 1[1(2-\lambda) + 1] + 1[-1-(2-\lambda)] = 0$

$\therefore\ \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \qquad \therefore\ \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 6\lambda - 6 = 0$

$\therefore\ (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0 \qquad \therefore\ (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

$\therefore\ \lambda = 1, 2, 3.$

(i) For $\lambda = 1$, $[A - \lambda_1 I] X = O$ gives

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We may use matrix method to obtain the roots of the above equation

$$\text{By } \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ x_1 - x_2 + x_3 = 0 \qquad \therefore\ 2x_2 - 2x_3 = 0$ *i.e.* $x_2 - x_3 = 0$

We note that the rank of the matrix is 2 and the number of variables is 3. Hence, there is $3 - 2 = 1$ linearly independent solution.

Putting $x_3 = t$, we get $x_2 = t$ and $x_1 = x_2 - x_3 = t - t = 0$.

$\therefore\ X = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Hence, corresponding to $\lambda = 1$, the eigenvector is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

[The eigenvector can be denoted as a column vector $\begin{bmatrix}0\\1\\1\end{bmatrix}$ or as the transpose of the row vector [0, 1, 1]'.]

(ii) For $\lambda = 2$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix}0 & -1 & 1\\1 & 0 & -1\\1 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore\ -x_2 + x_3 = 0,\quad x_1 - x_3 = 0,\quad x_1 - x_2 = 0$

Putting $x_2 = t$, we get $x_1 = t$, $x_3 = t$.

$\therefore\ X = \begin{bmatrix}t\\t\\t\end{bmatrix} = t\begin{bmatrix}1\\1\\1\end{bmatrix}$. Hence, corresponding to $\lambda = 2$, the eigenvector is $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

(iii) For $\lambda = 3$, $[A - \lambda_3 I] = O$ gives

$$\begin{bmatrix}-1 & -1 & 1\\1 & -1 & -1\\1 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{By } \begin{matrix}R_2 + R_1\\R_2 + R_3\end{matrix} \begin{bmatrix}-1 & -1 & 1\\0 & -2 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore\ -x_1 - x_2 + x_3 = 0 \quad \therefore\ -2x_2 = 0 \quad \therefore\ x_2 = 0.$

Putting $x_3 = t$, we get $x_1 = t$.

$\therefore\ X = \begin{bmatrix}t\\0\\t\end{bmatrix} = t\begin{bmatrix}1\\0\\1\end{bmatrix}$. Hence, corresponding to $\lambda = 3$, the eigenvector is $\begin{bmatrix}1\\0\\1\end{bmatrix}$.

[Definition : Vectors X_1, X_2, X_3 are said to be independent if $k_1X_1 + k_2X_2 + k_3X_3 = 0$ implies that $k_1 = 0, k_2 = 0, k_3 = 0$ for all values of k_1, k_2, k_3.

Here, $k_1X_1 + k_2X_2 + k_3X_3 = 0$ gives

$k_1[0, 1, 1] + k_2[1, 1, 1] + k_3[1, 0, 1] = 0.$

$\therefore\ 0\,k_1 + k_2 + k_3 = 0$ .... (i)

$k_1 + k_2 + 0\,k_3 = 0$ .... (ii)

and $k_1 + 0\,k_2 + k_3 = 0$ .... (iii)

From (i), $k_2 + k_3 = 0$ and from (ii) $k_1 + k_2 = 0$ and from (iii) $k_1 + k_3 = 0$.

Subtracting (iii) from (i), we get, $k_2 - k_1 = 0$.

Adding this to (ii), we get,

$k_2 = 0 \quad \therefore\ k_1 = 0 \quad \therefore\ k_3 = 0.$

$\therefore$ The vectors are linearly independent.]

Example 2 : Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{bmatrix}8 & -8 & -2\\4 & -3 & -2\\3 & -4 & 1\end{bmatrix} \qquad \textbf{(M.U. 2010, 16)}$$

Sol. : The characteristic equation is

$$\begin{vmatrix}8-\lambda & -8 & -2\\4 & -3-\lambda & -2\\3 & -4 & 1-\lambda\end{vmatrix} = 0$$

$\therefore\ (8-\lambda)[(3+\lambda)(\lambda-1) - 8] + 8[4 - 4\lambda + 6] - 2[-16 + 9 + 3\lambda] = 0$

$\therefore\ \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \qquad \therefore\ \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 6\lambda - 6 = 0$

$\therefore\ (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0 \qquad \therefore\ (\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\therefore\ \lambda = 1, 2, 3.$

(i) For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix}7 & -8 & -2\\4 & -4 & -2\\3 & -4 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

We may obtain the eigenvector by solving simultaneous equations obtained from the above matrix equation.

$\therefore$ From the first two rows, we get

$7x_1 - 8x_2 - 2x_3 = 0; \qquad 4x_1 - 4x_2 - 2x_3 = 0$

Solving by Cramer's rule,

$$\frac{x_1}{\begin{vmatrix}-8 & -2\\-4 & -2\end{vmatrix}} = \frac{-x_2}{\begin{vmatrix}7 & -2\\4 & -2\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}7 & -8\\4 & -4\end{vmatrix}}$$

$\therefore\ \dfrac{x_1}{8} = \dfrac{x_2}{6} = \dfrac{x_3}{4} \qquad \therefore\ \dfrac{x_1}{4} = \dfrac{x_2}{3} = \dfrac{x_3}{2} = t$, say

$\therefore\ x_1 = 4t,\ x_2 = 3t,\ x_3 = 2t$

$\therefore\ X = \begin{bmatrix}4t\\3t\\2t\end{bmatrix} = t\begin{bmatrix}4\\3\\2\end{bmatrix}$. Hence, corresponding to $\lambda = 1$, the eigenvector is $\begin{bmatrix}4\\3\\2\end{bmatrix}$.

[If k is a non-zero scalar then kX also is an eigenvector. The eigenvector can be denoted as a column matrix or as the transpose of a row matrix.]

(ii) For $\lambda = 2$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix}6 & -8 & -2\\4 & -5 & -2\\3 & -4 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

From the first two rows, we get,

$6x_1 - 8x_2 - 2x_3 = 0; \qquad 4x_1 - 5x_2 - 2x_3 = 0$

Solving by Cramer's rule,

$$\frac{x_1}{\begin{vmatrix}-8 & -2\\-5 & -2\end{vmatrix}} = \frac{-x_2}{\begin{vmatrix}6 & -2\\4 & -2\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}6 & -8\\4 & -5\end{vmatrix}}$$

$$\therefore \quad \frac{x_1}{6}=\frac{x_2}{4}=\frac{x_3}{2} \qquad \therefore \quad \frac{x_1}{3}=\frac{x_2}{2}=\frac{x_3}{1}=t, \text{ say}$$

$\therefore \quad x_1 = 3t,\ x_2 = 2t,\ x_3 = t.$

$$\therefore \quad X = \begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 2$, the eigenvector is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

(If k is a non-zero scalar then kX also is an eigenvector.)

(iii) For $\lambda = 3$, $[A - \lambda_3 I]X = O$ gives

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the first two rows, we get,

$$5x_1 - 8x_2 - 2x_3 = 0\ ; \qquad 4x_1 - 6x_2 - 2x_3 = 0$$

Solving by Cramer's rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$$\therefore \quad \frac{x_1}{4}=\frac{x_2}{2}=\frac{x_3}{2} \qquad \therefore \quad \frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{1}=t, \text{ say}$$

$\therefore \quad x_1 = 2t,\ x_2 = t,\ x_3 = t.$

$$\therefore \quad X = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 3$, the eigenvector is $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

(If k is a non-zero scalar then kX also is an eigenvector.)

Remark ....

Since there is no 1 in the first column Cramer's rule is more convenient than elementary operations.

Type II : *A* is a non-symmetric matrix and eigenvalues are repeated

Example 1 : Find the eigenvalues and eigenvectors of the following matrix.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$ **(M.U. 2001, 09)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

After simplification, we get,

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0 \quad \therefore \quad (\lambda - 1)(\lambda - 1)(\lambda + 5) = 0 \quad \therefore \quad \lambda = 1, 1, 5$$

Hence, 1, 1, 5 are the eigenvalues.

(i) For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 + 2x_2 + x_3 = 0.$

We see that the rank of the matrix is 1 and number of variables is 3. Hence, there are $3 - 1 = 2$ linearly independent solutions *i.e.*, there are two parameters. We shall denote these parameters by s and t.

Putting $x_2 = -s$, $x_3 = -t$, we get $x_1 = -2x_2 - x_3 = 2s + t$

$$\therefore \quad X = \begin{bmatrix} 2s+t \\ -s+0 \\ 0-t \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \therefore \quad X_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix},\quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The vectors $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are linearly independent.

[Let us verify that the vectors X_1 and X_2 are independent.

Consider $K_1X_1 + K_2X_2 = 0$.

$\therefore \quad K_1(2, -1, 0) + K_2(1, 0, -1) = 0$

$\therefore \quad 2K_1 + K_2 = 0,\ -K_1 = 0,\ -K_2 = 0$

Hence, the vectors are linearly independent.]

Hence, corresponding to $\lambda = 1$, the eigenvectors are $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(ii) For $\lambda = 5$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_{13} \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_2 - R_1 \\ R_3 + 3R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 8 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_3 + 2R_2 \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 + 2x_2 - 3x_3 = 0$ and $-4x_2 + 4x_3 = 0.$

Putting $x_3 = t$, we get $x_2 = t$ and $x_1 = -2x_2 + 3x_3 = -2t + 3t = t.$

$$\therefore \quad X = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 5$, the eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Note ....

- When the matrix A is non-symmetric and the eigenvalues are repeated, the eigenvectors corresponding to **repeated root** may or may not be linearly independent. Verify that the vectors X_1 and X_2 are independent. See Examples 1, 2, 3. In Ex. 1, page 5-22, we see that the eigen vectors corresponding to the repeated root $\lambda = 2$ are **not** independent.

Example 2 : Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

(M.U. 2016)

Sol. : The characteristic equation is

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\therefore\ (-2-\lambda)\,[\,(1-\lambda)(-\lambda)-(-2)(-6)\,] - 2\,[-2\lambda-6] - 3\,[-4+1(1-\lambda)] = 0$$

$$\therefore\ (-2-\lambda)\,[-\lambda+\lambda^2-12\,] + 2\,[2\lambda+6] + 3\,[3+\lambda] = 0$$

$$\therefore\ (2+\lambda)\,[\,12+\lambda-\lambda^2\,] + 4\lambda+12+9+3\lambda = 0$$

$$\therefore\ 24+2\lambda-2\lambda^2+12\lambda+\lambda^2-\lambda^3+7\lambda+21 = 0$$

$$\therefore\ -\lambda^3-\lambda^2+21\lambda+45 = 0$$

$$\therefore\ \lambda^3+\lambda^2-21\lambda-45 = 0$$

$$\therefore\ \lambda^3-5\lambda^2+6\lambda^2-30\lambda+9\lambda-45 = 0$$

$$\therefore\ \lambda^2(\lambda-5)+6(\lambda-5)+9(\lambda-5) = 0 \qquad \therefore\ (\lambda-5)(\lambda^2+6\lambda+9) = 0$$

$$\therefore\ (\lambda-5)(\lambda+3)^2 = 0 \qquad \therefore\ \lambda = 5,\ -3,\ -3.$$

Hence, the eigenvalues are $5, -3, -3$.

(i) For $\lambda = 5$, $[\,A-\lambda_1 I\,]\,X = O$ gives

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_{13} \\ -R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_2-2R_1 \\ R_3+7R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_3+2R_2 \\ -\frac{1}{8}R_2 \end{matrix} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ x_1 + 2x_2 + 5x_3 = 0$ and $x_2 + 2x_3 = 0$.

Putting $x_3 = t$, we get $x_2 = -2t$.

Then $x_1 - 4t + 5t = 0 \quad \therefore\ x_1 = -t$.

Changing the sign of t, $x_1 = t,\ x_2 = 2t,\ x_3 = -t$.

$\therefore\ X = \begin{bmatrix} t \\ 2t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. Hence, corresponding to $\lambda = 5$, the eigenvector is $X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

(ii) For $\lambda = -3$, $[\,A-\lambda_2 I\,]\,X = O$ gives

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2-2R_1 \\ R_3+R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ x_1 + 2x_2 - 3x_3 = 0$

We see that the rank of the matrix is 1 and the number of variables is 3. Hence, there are $3 - 1 = 2$ linearly independent solutions.

Putting $x_3 = s,\ x_2 = -t$, we get $x_1 - 2t - 3s = 0 \quad \therefore\ x_1 = 2t + 3s$.

$$\therefore\ X = \begin{bmatrix} 2t+3s \\ -t+0s \\ 0t+s \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \qquad \therefore\ X_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix},\ X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

[The vectors are $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ linearly independent. (Verify it.)]

Example 3 : Find the eigenvalues and eigenvectors of the following matrix.

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

(M.U. 1996, 2005, 14, 16)

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix} = 0$$

$$\therefore\ (2-\lambda)[(3-\lambda)(4-\lambda)-6] - 1[2(4-\lambda)-6] + 1[6-3(3-\lambda)] = 0$$

$$\therefore\ \lambda^3-9\lambda^2+15\lambda-7 = 0 \qquad \therefore\ \lambda^3-\lambda^2-8\lambda^2-8\lambda+7\lambda-7 = 0$$

$$\therefore\ (\lambda-1)(\lambda^2-8\lambda+7) = 0 \qquad \therefore\ (\lambda-1)(\lambda-1)(\lambda-7) = 0$$

$$\therefore\ \lambda = 1,\ 1,\ 7.$$

(i) For $\lambda = 1$, $[\,A-\lambda_1 I\,]\,X = O$ gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2-2R_1 \\ R_3-3R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ x_1 + x_2 + x_3 = 0$

We see that the rank of the matrix is 1 and the number of variables is 3. Hence, there are $3 - 1 = 2$ linearly independent solutions.

Putting $x_2 = -s,\ x_3 = -t$, we get $x_1 = -x_2 - x_3 = s + t$.

$$\therefore\ X = \begin{bmatrix} s+t \\ -s+0 \\ 0-t \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \therefore\ X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Now, vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are linearly independent. (Verify it)

Hence, corresponding to $\lambda = 1$, the eigenvectors are $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Further $kX_1 + kX_2$ is also an eigenvector.

(ii) For $\lambda = 7$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 + 3R_1 \end{matrix} \begin{bmatrix} -5 & 1 & 1 \\ 12 & -6 & 0 \\ -12 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 + R_3 \\ (1/6) R_2 \end{matrix} \begin{bmatrix} -5 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \ -5x_1 + x_2 + x_3 = 0$ and $2x_1 - x_2 = 0$.

Putting $x_2 = 2t$, we get $2x_1 = x_2 = 2t \ \therefore \ x_1 = t$.

$\therefore \ x_3 = 5x_1 - x_2 = 5t - 2t = 3t$

$\therefore \ X = \begin{bmatrix} t \\ 2t \\ 3t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Hence, corresponding to $\lambda = 7$, the eiven vector is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Type III : A is symmetric and eigenvalues are distinct

Example 1 : Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

(M.U. 2003)

Sol. : (a) The characteristic equation is

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

On simplification, we get,

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0 \quad \therefore \ \lambda(\lambda - 3)(\lambda - 15) = 0 \quad \therefore \ \lambda = 0, 3, 15.$$

Hence, 0, 3, 15 are the eigenvalues.

(i) For $\lambda = 0$. $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad R_{31} \begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 + 3R_1 \\ R_3 - 4R_1 \end{matrix} \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad R_3 + 2R_1 \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \ 2x_1 - 4x_2 + 3x_3 = 0$

$-5x_2 + 5x_3 = 0$ i.e. $x_2 = x_3$

We note that the rank of the matrix is 2 and the number of variables is 3. Hence, there is $3 - 2 = 1$ linearly independent solution.

Putting $x_3 = 2t$, we get $x_2 = 2t$. $\therefore \ 2x_1 = 4x_3 - 3x_3 = 8t - 6t = 2t \quad \therefore \ x_1 = t$.

$\therefore \ X = \begin{bmatrix} t \\ 2t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Hence, corresponding to $\lambda = 0$, the eiven vector is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

(If k is a non-zero scalar, then kX is also an eigenvector.)

(ii) For $\lambda = 3$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad R_1 + R_2 \begin{bmatrix} -1 & -2 & -2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 6R_1 \\ R_3 + 2R_1 \end{matrix} \begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & -8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad \begin{matrix} R_3 + (1/2) R_2 \\ -R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \ x_1 + 2x_2 + 2x_3 = 0$ and $16x_2 + 8x_3 = 0$ i.e. $2x_2 + x_3 = 0$

Putting $x_2 = t$, we get $x_3 = -2x_2 = -2t$. $\quad \therefore \ x_1 = -2x_2 - 2x_3 = -2t + 4t = 2t$.

$\therefore \ X = \begin{bmatrix} 2t \\ t \\ -2t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Hence, corresponding to $\lambda = 3$, the eigenvector is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

(If k is a non-zero scalar then kX is also an eigenvector.)

(iii) For $\lambda = 15$, $[A - \lambda_3 I]X = O$ gives

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad R_1 - R_2 \begin{bmatrix} -1 & 2 & 6 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 6R_1 \\ R_3 + 2R_1 \end{matrix} \begin{bmatrix} -1 & 2 & 6 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \ -x_1 + 2x_2 + 6x_3 = 0 \quad \therefore \ x_1 - 2x_2 - 6x_3 = 0$

$-20x_2 - 40x_3 = 0 \quad x_2 = -2x_3$

Putting $x_3 = t$, $x_2 = -2t$, $\therefore\ x_1 = 2x_2 + 6x_3 = 2t$.

$$\therefore\ X = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 15$, the eigenvector is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

(If k is a non-zero scalar then kX is also an eigenvector.)

Note ....

The eigen vectors $X_1 = [1, 2, 2]'$, $X_2 = [2, 1, -2]'$ and $X_3 = [2, -2, 1]$ are orthogonal because $X_1 \cdot X_2 = (2 + 2 - 4) = 0$, $X_1 \cdot X_3 = (2 + 2 - 4) = 0$ and $X_2 \cdot X_3 = (4 - 2 - 2) = 0$.

Example 2 : Find the eigenvalues and eigenvectors of the following matrix.

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$ **(M.U. 1993, 2005)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\therefore\ (3-\lambda)[(5-\lambda)(3-\lambda) - 1] + 1[-1(3-\lambda) + 1] + 1[1 - (5-\lambda)] = 0$

$\therefore\ \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$ $\quad\therefore\ \lambda^3 - 2\lambda^2 - 9\lambda^2 + 18\lambda + 18\lambda - 36 = 0$

$\therefore\ (\lambda - 2)(\lambda^2 - 9\lambda + 18) = 0$ $\quad\therefore\ (\lambda - 2)(\lambda - 3)(\lambda - 6) = 0$

$\therefore\ \lambda = 2, 3, 6.$

(i) For $\lambda = 2$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2 + R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ x_1 - x_2 + x_3 = 0$, $x_2 = 0$.

Putting $x_1 = t$, we get $x_3 = -t$.

$$\therefore\ X = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 2$, the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(If k is a non-zero scalar kX also is an eigenvector.)

(ii) For $\lambda = 3$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_2 + R_1 \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } R_3 + R_2 \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ $\therefore\ -x_2 + x_3 = 0$, $-x_1 + x_2 = 0$.

Putting $x_2 = t$, we got $x_1 = t$, $x_3 = t$.

$$\therefore\ X = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 3$, the eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(If k is a non-zero scalar kX is also an eigenvector.)

(iii) For $\lambda = 6$, $[A - \lambda_3 I]X = O$ gives

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2 + R_1 \\ R_3 + 3R_1 \end{matrix} \begin{bmatrix} -3 & -1 & 1 \\ -4 & -2 & 0 \\ -8 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_3 - (1/2)R_2 \\ (1/2)R_2 \end{matrix} \begin{bmatrix} -3 & -1 & 1 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ -3x_1 - x_2 + x_3 = 0;$ $\quad -2x_1 - x_2 = 0$

Putting $x_2 = -2t$, we get $2x_1 = -x_2 = 2t$ $\quad\therefore\ x_1 = t$

$\therefore\ x_3 = 3x_1 + x_2 = 3t - 2t = t.$

$$\therefore\ X = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 6$, the eigenvector is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

(If k is a non-zero scalar kX also is an eigenvector.)

(Verify that the vectors are orthogonal.)

Example 3 : Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$$ **(M.U. 2014)**

Sol. : The characteristic equation is

$$\begin{vmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{vmatrix} = 0$$

On simplification, we get $(\lambda + 3)(\lambda - 12)(\lambda + 6) = 0$ $\quad\therefore\ \lambda = -3, 12, -6$

$\therefore\ -3, 12, -6$ are the eigenvalues.

(i) For $\lambda = -3$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1 & 5 & 4 \\ 5 & 10 & 5 \\ 4 & 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_2 - R_3 \begin{bmatrix} 1 & 5 & 4 \\ 1 & 5 & 4 \\ 4 & 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } R_2 - R_1 \begin{bmatrix} 1 & 5 & 4 \\ 0 & 0 & 0 \\ 4 & 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_{32} \begin{bmatrix} 1 & 5 & 4 \\ 4 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

...or is $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

$= \begin{bmatrix}0\\0\\0\end{bmatrix}$

By $R_2 - R_1 \begin{bmatrix}1 & 5 & 4\\3 & 0 & -3\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore \quad x_1 + 5x_2 + 4x_3 = 0; \quad 3x_1 - 3x_3 = 0$

Putting $x_3 = t,\ x_1 = t$ and $5x_2 + 5t = 0 \quad \therefore x_2 = -t.$

$\therefore \quad X = \begin{bmatrix}t\\-t\\t\end{bmatrix} = t\begin{bmatrix}1\\-1\\1\end{bmatrix}$. Hence, corresponding to $\lambda = -3$, the given vector is $\begin{bmatrix}1\\-1\\1\end{bmatrix}$.

(ii) For $\lambda = 12$, $[A - \lambda_2 I]\,X = O$ gives

$$\begin{bmatrix}-14 & 5 & 4\\5 & -5 & 5\\4 & 5 & -14\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore \quad -14x_1 + 5x_2 + 4x_3 = 0; \quad 5x_1 - 5x_2 + 5x_3 = 0; \quad 4x_1 + 5x_2 - 14x_3 = 0$

Solving the first two equations by Crammer's rule, we get

$$\frac{x_1}{45} = \frac{-x_2}{-90} = \frac{x_3}{45} \qquad \therefore \quad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1} = t$$

$\therefore \quad x_1 = t,\ x_2 = 2t,\ x_3 = t.$

$\therefore \quad X = \begin{bmatrix}t\\2t\\t\end{bmatrix} = t\begin{bmatrix}1\\2\\1\end{bmatrix}$. Hence, corresponding to $\lambda = 12$, the eigenvector is $\begin{bmatrix}1\\2\\1\end{bmatrix}$.

(iii) For $\lambda = -6$, $[A - \lambda_3 I]\,X = O$ gives

$$\begin{bmatrix}4 & 5 & 4\\5 & 13 & 5\\4 & 5 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore \quad 4x_1 + 5x_2 + 4x_3 = 0; \quad 5x_1 + 13x_2 + 5x_3 = 0; \quad 4x_1 + 5x_2 + 4x_3 = 0$

Solving the first two equations by Cramer's rule, we get

$$\frac{x_1}{-27} = \frac{-x_2}{0} = \frac{x_3}{27} \qquad \therefore \quad \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} = t$$

$\therefore \quad X = \begin{bmatrix}t\\0t\\-t\end{bmatrix} = t\begin{bmatrix}1\\0\\-1\end{bmatrix}$. Hence, corresponding to $\lambda = -6$, the eigenvector is $\begin{bmatrix}1\\0\\-1\end{bmatrix}$.

(*Verify that the vectors are orthogonal.*)

Type IV : *A* is symmetric and eigenvalues are repeated

Example 1 : Find the eigenvalues and eigenvectors of the following matrix.

$$\begin{bmatrix}6 & -2 & 2\\-2 & 3 & -1\\2 & -1 & 3\end{bmatrix}$$

(M.U. 1993, 96, 2003)

Sol. : The characteristic equation is

$$\begin{vmatrix}6-\lambda & -2 & 2\\-2 & 3-\lambda & -1\\2 & -1 & 3-\lambda\end{vmatrix} = 0$$

On simplification, we get, $\quad (2-\lambda)(\lambda-2)(\lambda-8) = 0 \qquad \therefore \lambda = 2, 2, 8.$

Hence, 2, 2, 8 are the eigenvalues.

(i) For $\lambda = 2$, $[A - \lambda_1 I]\,X = O$ gives

$$\begin{bmatrix}4 & -2 & 2\\-2 & 1 & -1\\2 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{By } R_{12} \begin{bmatrix}-2 & 1 & -1\\4 & -2 & 2\\2 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

By $\begin{matrix}R_2 + 2R_1\\R_3 + R_1\end{matrix} \begin{bmatrix}-2 & 1 & -1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore \quad -2x_1 + x_2 - x_3 = 0 \quad i.e. \quad 2x_1 - x_2 + x_3 = 0$

We again see that the rank of the matrix is 1 and number of variables is 3. Hence, there are $3 - 1 = 2$ linearly independent solutions.

Putting $x_2 = 2s$ and $x_3 = -2t$, we get

$2x_1 = x_2 - x_3 = 2s + 2t \quad \therefore \quad x_1 = s + t$

$$\therefore \quad X = \begin{bmatrix}s+t\\2s+0\\0-2t\end{bmatrix} = s\begin{bmatrix}1\\2\\0\end{bmatrix} + t\begin{bmatrix}1\\0\\-2\end{bmatrix}$$

The vectors $\begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\-2\end{bmatrix}$ are linearly independent. (Verifity it.)

Hence, corresponding to $\lambda = 2$, the eigenvectors are $\begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\-2\end{bmatrix}$.

Further $k_1X + k_2X_2$ is also an eigenvector.

(ii) For $\lambda = 8$, $[A - \lambda_2 I]\,X = O$ gives,

$$\begin{bmatrix}-2 & -2 & 2\\-2 & -5 & -1\\2 & -1 & -5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{By } \begin{matrix}R_2 - R_1\\R_3 + R_1\end{matrix} \begin{bmatrix}-2 & -2 & 2\\0 & -3 & -3\\0 & -3 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

By $R_3 - R_2 \begin{bmatrix}-2 & -2 & 2\\0 & -3 & -3\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore \quad -2x_1 - 2x_2 + 2x_3 = 0 \quad i.e., \quad x_1 + x_2 - x_3 = 0$

and $\quad -3x_2 - 3x_3 = 0 \quad i.e., \quad x_2 = -x_3$

Putting $x_3 = t$, we get $x_2 = -t$ $\quad \therefore \; x_1 = -x_2 + x_3 = t + t = 2t.$

$$\therefore \quad X = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 8$, the eigenvectors is $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

(If k is a non-zero scalar then kX also is an eigenvector.)

Example 2 : Find the eigenvalues and eigenvectors of the following matrix.

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$ (M.U. 2004)

Sol. : The characteristic equation is

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (3-\lambda)[(3-\lambda)^2 - 1] + 1[-1(3-\lambda) + 1] + 1[1 - (3-\lambda)] = 0$$

$$\therefore \quad \lambda^3 - 9\lambda^2 + 24\lambda - 20 = 0 \qquad \therefore \quad \lambda^3 - 2\lambda^2 - 7\lambda^2 + 14\lambda + 10\lambda - 20 = 0$$

$$\therefore \quad (\lambda - 2)(\lambda^2 - 7\lambda + 10) = 0 \qquad \therefore \quad (\lambda - 2)(\lambda - 2)(\lambda - 5)$$

$$\therefore \quad \lambda = 2, 2, 5$$

(i) For $\lambda = 2$, $[A - \lambda_1 I] X = O$ gives

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{By} \quad \begin{matrix} R_2 + R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad x_1 - x_2 + x_3 = 0$$

The rank of the matrix is 1 and the number of variables is 3. Hence, there are $3 - 1 = 2$ linearly independent solutions.

Putting $x_2 = s$, $x_3 = -t$, we get $\quad x_1 = x_2 - x_3 = s + t.$

$$\therefore \quad X = \begin{bmatrix} s+t \\ s+0 \\ 0-t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Now, the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are linearly independent. (Verify it.)

Hence, corresponding to $\lambda = 2$, the eiven vectors are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(ii) For $\lambda = 5$, $[A - \lambda_2 I] X = O$ gives

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad \begin{matrix} R_2 + R_1 \\ R_3 + 2R_1 \end{matrix} \begin{bmatrix} -2 & -1 & 1 \\ -3 & -3 & 0 \\ -3 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_3 - R_2 \\ -\frac{1}{3} R_2 \end{matrix} \begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad -2x_1 - x_2 + x_3 = 0 ; \quad x_1 + x_2 = 0$$

Putting $x_2 = -t$, we get $\quad x_1 = t$ and $x_3 = 2x_1 + x_2 = 2t - t = t.$

$$\therefore \quad X = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 5$, the eigenvectors is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

[**Definition :** Two vectors $X = [x_1, x_2, x_3]$ and $Y = [y_1, y_2, y_3]$ are said to be orthogonal if $X \cdot Y = (x_1 y_1 + x_2 y_2 + x_3 y_3) = 0$.

In the above example, $\quad X_1 \cdot X_3 = [1, 1, 0] \cdot [1, -1, 1] = 0$

and $\quad X_2 \cdot X_3 = [1, 0, -1] \cdot [1, -1, 1] = 0.$]

Example 3 : Find the eigenvalues and eigenvectors of the following matrix.

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Sol. : The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (1-\lambda)[(1-\lambda)^2 - 4] - 2[2(1-\lambda) - 4] + 2[4 - 2(1-\lambda)] = 0$$

$$\therefore \quad \lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0 \qquad \therefore \quad \lambda^3 + \lambda^2 - 4\lambda^2 - 4\lambda - 5\lambda - 5 = 0$$

$$\therefore \quad (\lambda + 1)(\lambda^2 - 4\lambda - 5) = 0 \qquad \therefore \quad (\lambda + 1)(\lambda + 1)(\lambda - 5) = 0$$

$$\therefore \quad \lambda = -1, -1, 5.$$

(i) For $\lambda = -1$, $[A - \lambda_1 I] X = O$ gives

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_2 \\ (1/2) R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad x_1 + x_2 + x_3 = 0$$

The rank of the matrix is 1 and the number of variables is 3. Hence, there are $3 - 1 = 2$ linearly independent solutions.

Putting $x_2 = -s$, $x_3 = -t$, we get $\quad x_1 = -x_2 - x_3 = s + t.$

$$\therefore \quad X = \begin{bmatrix} s+t \\ -s+0 \\ 0-t \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Now, the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are linearly independent. (Verify it.)

Hence, corresponding to $\lambda = 1$, the eigenvectors are $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

The dimension of the eigensapce is 2.

(ii) For $\lambda = 5$, $[A - \lambda_2 I]\, X = O$ gives

$$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first two rows give

$$-4x_1 + 2x_2 + 2x_3 = 0\,; \qquad 2x_1 - 4x_2 + 2x_3 = 0$$

i.e. $\quad -2x_1 + x_2 + x_3 = 0\,; \qquad x_1 - 2x_2 + x_3 = 0$

By Crammer's rule

$$\frac{x_1}{1+2} = \frac{-x_2}{-2-1} = \frac{x_3}{4-1}$$

$$\therefore \quad \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \quad \therefore \quad \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = t, \text{ say} \quad \therefore \quad x_1 = t,\ x_2 = t,\ x_3 = t.$$

$\therefore \quad X = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Hence, corresponding to $\lambda = 5$, the eigenvectors is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Remark ....

Verify that the vectors corresponding to **distinct** *eigenvalues of a symmetric matrix are orthogonal i.e.* $(X_1\ X_3) = 0$ *and* $(X_2\ X_3) = 0$. (See Theorem 1, page 5-36)

Type V : A has repeated Eigenvalues but same eigenvectors

Example 1 : Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$$ **(M.U. 2015)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$$

On simplification, we get

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0 \quad \therefore \quad (\lambda - 1)(\lambda - 2)(\lambda - 2) = 0 \quad \therefore \quad \lambda = 1, 2, 2.$$

(i) For $\lambda = 1$, $[A - \lambda_1 I]\, X = O$ gives

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \frac{1}{3}R_1 \quad \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } R_2 - R_1 \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2 - R_1 \\ R_3 + R_1 \\ R_{2,3} \end{matrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 + 2x_2 + 2x_3 = 0$ and $-3x_2 - x_3 = 0$.

Putting $x_3 = -3t$, we get, $3x_2 = 3t \quad \therefore \quad x_2 = t$.

Then $x_1 = -2x_2 - 2x_3 = -2t + 6t = 4t$.

$$\therefore \quad X = \begin{bmatrix} 4t \\ t \\ -3t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$$

Since, the rank is 2 and number of variables is 3 there is only 3 − 2 = 1. linearly independent solution.

Hence, corresponding to $\lambda = 1$, the eigenvector is $\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$.

(ii) For $\lambda = 2$, $[A - \lambda_2 I]\, X = O$ gives

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_{1,2} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 6 & 6 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } R_3 - R_1 + R_2 \begin{bmatrix} 1 & 1 & 2 \\ 2 & 6 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_2 - 3R_1 \begin{bmatrix} 1 & 1 & 2 \\ -1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 + x_2 + 2x_3 = 0$ and $-x_1 + 3x_2 = 0$.

Putting $x_1 = 3t$, we get,

$3x_2 = 3t \qquad \therefore \quad x_2 = t$.

Then from $x_1 + x_2 + 2x_3 = 0$, we get,

$3t + t + 2x_3 = 0 \qquad \therefore \quad x_3 = -2t$.

$$\therefore \quad X = \begin{bmatrix} 3t \\ t \\ -2t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \quad \therefore \quad X_3 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

Since, the number variables is 3 and the rank of the matrix is 2, there is 3 − 2 = 1 independent solution.

Hence, corresponding to $\lambda = 2$, the eigenvector is $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$.

Note ....

Corresponding to the repeated root 2, we get only one eigenvector. The other vector is dependent on this vector. It may be taken as any multiple of this vector. For $\lambda = 2$, the eigen vectors are **not independent**.

Example 2 : Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

On simplification, we get $(2-\lambda)^3 = 0$ $\quad \therefore \lambda = 2, 2, 2.$

For $\lambda = 2$ $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore\ 0x_1 + x_2 + 0x_3 = 0\ ; \qquad 0x_1 + 0x_2 + x_3 = 0$$

By Cramer's rule

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0} = t, \text{ say.} \quad \therefore\ x_1 = t,\ x_2 = 0t \text{ and } x_3 = 0t.$$

$$\therefore\ X = \begin{bmatrix} t \\ 0t \\ 0t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We note for the last time that the rank of the matrix is 2 and the number of variables is 3. Hence, there is 3 − 2 = 1 linearly independent solution.

Hence, corresponding to $\lambda = 2$, the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

7. Certain Relations Between Eigenvalues and Eigenvectors

We state below certain theorems without proof.

Theorem 1 : ***λ is an eigenvalue of the matrix A if and only if there exists a non-zero vector X such that $AX = \lambda X$.***

Theorem 2 : ***If X is an eigenvector of a matrix A corresponding to an eigenvalue λ, then kX (k is a non-zero scalar) is also an eigenvector of A corresponding to the same eigenvalue λ.***

Theorem 3 : **(Uniqueness of Eigenvector) :** ***If X is an eigenvector of a matrix A, then X cannot correspond to more than one eigenvalues of A.***

Theorem 4 : **(Linear Independence of Eigenvectors) :** ***Eigenvectors corresponding to distinct eigenvalues of a matrix are linearly independent.***

(Eigenvectors corresponding to a repeated eigenvalue may or may not be linearly independent. See note page 5-12.)

Theorem 5 : ***The eigenvectors of a real symmetric matrix whose eigenvalues are distinct are orthogonal.***

(For proof, see page 5-36) (See Ex. 1, page 5-14.)

8. Certain Results about the Nature of Eigenvalues

Theorem 1 : *Eigenvalues of a Hermitian matrix are real.*

(M.U. 1996, 2001, 03, 04, 06, 14)

Proof : Let A be a Hermitian matrix with λ as an eigenvalue and X as a corresponding eigenvector. Then,

$$AX = \lambda X.$$

Premultiplying by X^{θ} we get

$$X^{\theta} A X = X^{\theta} \lambda X = \lambda X^{\theta} X \qquad \text{.................... (1)}$$

Taking complex conjugate transpose of both sides

$$(X^{\theta} A X)^{\theta} = (\lambda X^{\theta} X)^{\theta}$$

$$X^{\theta} A^{\theta} (X^{\theta})^{\theta} = \bar{\lambda} X^{\theta} (X^{\theta})^{\theta}$$

Since, A is Hermitian $A^{\theta} = A$ and also $(X^{\theta})^{\theta} = X$

$$\therefore\ X^{\theta} A X = \bar{\lambda} X^{\theta} X \qquad \text{.................... (2)}$$

From (i) and (ii), we get

$$\lambda X^{\theta} X = \bar{\lambda} X^{\theta} X$$

$$\therefore\ (\lambda - \bar{\lambda}) X^{\theta} X = 0$$

Since, X is not a zero vector $X^{\theta} X \neq 0$.

$$\therefore\ \lambda - \bar{\lambda} = 0 \quad \therefore\ \lambda = \bar{\lambda}$$

Hence, λ is real.

Example 1 : Verify that the eigenvalues of the following Hermitian matrix are real.

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 1 \end{bmatrix}.$$

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1+i \\ 1-i & 1-\lambda \end{vmatrix} = 0$$

$\therefore\ (2-\lambda)(1-\lambda)-(1+1)=0 \qquad \therefore\ 2-3\lambda+\lambda^2-2=0$

$\therefore\ \lambda^2-3\lambda=0 \qquad \therefore\ \lambda(\lambda-3)=0 \qquad \therefore\ \lambda=0, 3.$

Thus, the eigenvalues of the given Hermitian matrix are real.

Corollary 1 : The determinant of a Hermitian matrix is real.

Proof : The determinant of a matrix is equal to the product of its eigenvalues. Since all eigenvalues of a Hermitian matrix are real, their product is real.

Example 2 : Verify corollary 1 for $A=\begin{bmatrix} 1 & 2i \\ -2i & -1 \end{bmatrix}$, *i.e.*, verify that the determinant of Hermitian matrix is real for A given above.

Sol. : The determinant of A is $\Delta = 2-(1+1)=0$

$\therefore$ Thus, Δ is real.

Example 3 : Verify corollery 1 for $A=\begin{bmatrix} 4 & 2i \\ -2i & 5 \end{bmatrix}$.

Sol. : Left to you.

Corollary 2 : *Eigenvalues of a real symmetric matrix are all real.* **(M.U. 1997, 99, 2003)**

If the elements of Hermitian matrix are all real, it becomes a symmetric matrix. Thus, a real symmetric matrix is Hermitian and hence, its all eigenvalues are real.

Example 4 : Verify that the eigenvalues of a real symmetric matrix are real for $A=\begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix}$.

Sol. : The matrix A is symmetric and the characteristic equation is

$$\begin{vmatrix} 12-\lambda & 6 \\ 6 & 3-\lambda \end{vmatrix} = 0$$

$\therefore\ (12-\lambda)(3-\lambda)-36=0 \qquad \therefore\ 36-15\lambda+\lambda^2-36=0$

$\therefore\ \lambda^2-15\lambda=0 \qquad \therefore\ \lambda(\lambda-15)=0 \qquad \therefore\ \lambda=0, 15.$

The eigenvalues of A are real. (See Ex. 1, Type - III, page 5-14)

Corollary 3 : *Eigenvalues of a Skew-Hermitian matrix are either purely imaginary or zero.* **(M.U. 2003, 04)**

Suppose A is a Skew-Hermitian matrix. Then iA is Hermitian. Let λ be an eigenvalue and X be the corresponding eigenvector

$\therefore\ AX=\lambda X$ [By Theorem 1]

$\therefore\ (iA)X=(i\lambda)X$

$\therefore\ i\lambda$ is an eigenvalue of iA which is Hermitian. Hence, $i\lambda$ is real. Therefore, λ must be either purely imaginary or zero.

Corollary 4 : *The eigenvalues of a real skew-symmetric matrix are purely imaginary or zero.* **(M.U. 2002)**

If the elements of a Skew-Hermitian matrix are all real, it becomes a skew-symmetric matrix. Thus, a real skew-symmetric matrix is Skew-Harmitian and hence, its all eigenvalues are either purely imaginary or zero.

Theorem 2 : *The Eigenvalues of a unitary matrix are of unit modulus. (have absolute value one).* **(M.U. 2001, 03, 15)**

Proof : Suppose A is a unitary matrix. Then, $A^{\theta}A=I$

Let λ be an eigenvalue of A and X be the corresponding eigenvector.

Then, $AX=\lambda X$ .... (1)

Taking complex conjugate transposes of both sides of (1), we get

$(AX)^{\theta}=(\lambda X)^{\theta} \qquad \therefore\ X^{\theta}A^{\theta}=\bar{\lambda}X^{\theta}$ .... (2)

From (1) and (2) by multiplication, we get,

$(X^{\theta}A^{\theta})(AX)=(\bar{\lambda}X^{\theta})(\lambda X) \qquad \therefore\ X^{\theta}(A^{\theta}A)X=\lambda\bar{\lambda}X^{\theta}X$

$\therefore\ X^{\theta}IX=\lambda\bar{\lambda}X^{\theta}X \qquad \therefore\ X^{\theta}X=\lambda\bar{\lambda}X^{\theta}X$

$\therefore\ X^{\theta}X(\lambda\bar{\lambda}-1)=0$

Since, X is not a zero vector,

$X^{\theta}X\neq 0 \qquad \therefore\ \lambda\bar{\lambda}-1=0 \qquad \therefore\ \lambda\bar{\lambda}=1 \qquad \therefore\ |\lambda|=1.$

Example 5 : Verify that the eigenvalues of a unitary matrix are of unit modulus for

$$A=\frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}.$$

Sol. : We leave it to you to verify that A is a unitary matrix.

Now, consider $\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$.

Its characteristic equation is

$$\begin{vmatrix} -4-\lambda & -2-4i \\ 2-4i & -4-\lambda \end{vmatrix} = 0$$

$\therefore\ (4+\lambda)^2+(4+16)=0 \qquad \therefore\ \lambda^2+8\lambda+36=0$

$$\therefore\ \lambda=\frac{-8\pm\sqrt{64-144}}{2}=\frac{-8\pm 4\sqrt{5}\,i}{2}=-4\pm 2\sqrt{5}\,i$$

$$\therefore\ \text{Eigenvalues of } A \text{ are } \frac{-4\pm 2\sqrt{5}\,i}{6} \quad i.e.\ -\frac{2}{3}\pm\frac{\sqrt{5}\,i}{3}.$$

$$\therefore\ |\lambda_1|=\left|-\frac{2}{3}+\frac{\sqrt{5}}{3}\right|=\sqrt{\frac{4}{9}+\frac{5}{9}}=\sqrt{\frac{9}{9}}=1$$

Similarly, $|\lambda_2|=1$.

Example 6 : Prove that the eigenvalues of $\begin{bmatrix} \frac{(1+i)}{2} & \frac{-(1-i)}{2} \\ \frac{(1+i)}{2} & \frac{(1-i)}{2} \end{bmatrix}$ are of unit modulus.

(M.U. 1999, 2006)

Sol. : The characteristic equation is

$$\begin{bmatrix} \frac{(1+i)}{2} - x & \frac{-(1-i)}{2} \\ \frac{(1+i)}{2} & \frac{(1-i)}{2} - x \end{bmatrix} = 0$$

$$\therefore \quad \left[\left(\frac{1+i}{2}\right) - x\right]\left[\frac{(1-i)}{2} - x\right] + \frac{(1+i)}{2}\frac{(1-i)}{2} = 0$$

$$\frac{(1+i)}{2}\frac{(1-i)}{2} - \frac{(1+i)}{2}x - \frac{(1-i)}{2}x + x^2 + \left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right) = 0$$

$$\therefore \quad \frac{(i-i^2)}{4} - \frac{1}{2}x - \frac{i}{2}x - \frac{1}{2}x + \frac{i}{2}x + x^2 + \frac{1-i^2}{4} = 0$$

$$\therefore \quad \frac{1}{2} - x + x^2 + \frac{1}{2} = 0 \qquad \therefore \quad x^2 - x + 1 = 0$$

The roots of this equation are $x = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1 \pm i\sqrt{3}}{2}$.

Hence, the eigenvalues are $\dfrac{1+i\sqrt{3}}{2}$ and $\dfrac{1-i\sqrt{3}}{2}$.

$$\therefore \quad |\lambda_1| = \left|\frac{1}{2} + \frac{i\sqrt{3}}{2}\right| = \left|\frac{1}{4} + \frac{3}{4}\right| = \sqrt{1} = 1. \text{ Similarly, } |\lambda_2| = 1.$$

Hence, eigenvalues are of unit modulus.

Corollary : ***Eigenvalues of an orthogonal matrix are of unit modulus.*** **(M.U. 2003)**

If the elements of a unitary matrix A are all real, then A becomes an orthogonal matrix. Thus, an orthogonal matrix is unitary and hence, its eigenvalues are of unit modulus.

(See also Ex. 12, page 5-32)

Example 7 : Verify that the eigenvalues of an orthogonal matrix are of unit modulus for

$$A = \begin{bmatrix} \sqrt{3/2} & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}.$$

Solution : It can be verified that A is orthogonal.

The characteristic equation is

$$\begin{vmatrix} (\sqrt{3}/2) - \lambda & 1/2 \\ -1/2 & (\sqrt{3/2}) - \lambda \end{vmatrix} = 0$$

$$\therefore \quad \left(\frac{\sqrt{3}}{2} - \lambda\right)^2 + \frac{1}{4} = 0 \qquad \therefore \quad \frac{3}{4} - \sqrt{3}\cdot\lambda + \lambda^2 + \frac{1}{4} = 0$$

$$\therefore \quad \lambda^2 - \sqrt{3}\,\lambda + 1 = 0 \qquad \therefore \quad \lambda = \frac{\sqrt{3} \pm \sqrt{3-4}}{2} = \frac{\sqrt{3} \pm i}{2}$$

$$\therefore \quad \lambda_1 = \frac{\sqrt{3}}{2} + \frac{i}{2}, \quad \lambda_2 = \frac{\sqrt{3}}{2} - \frac{i}{2}.$$

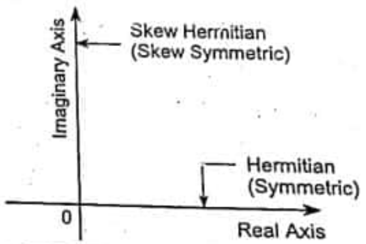
Now, $|\lambda_1| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1.$ Also $|\lambda_2| = 1.$

Example 8 : Verify that the matrix $A = \begin{bmatrix} 1/\sqrt{3} & \sqrt{2/3} \\ \sqrt{2/3} & -1/\sqrt{3} \end{bmatrix}$ is orthogonal and that the eigenvalues of A are of unit modulus.

Sol. : Left to you.

Note ....

The following diagram may help you to remember the relationship between different types of matrices and their eigenvalues.



9. Certain Relations between Matrix *A* and its Eigenvalues

Example 1 : Show that the matrices A and A' have the same eigenvalues.

(M.U. 2003)

Sol. : We see that $(A - \lambda I)' = A' - \lambda I'$

$$\therefore \quad (A - \lambda I)' = A' - \lambda I$$

$$\therefore \quad |(A - \lambda I)'| = |A' - \lambda I|$$

$$|A - \lambda I| = |A' - \lambda I| \qquad [\because |B'| = |B|]$$

$$\therefore \quad |A - \lambda I| = 0 \text{ if and only if } |A' - \lambda I| = 0$$

This means the roots of the two equations are the same. Hence, eigenvalues of A and A' are the same.

Aliter : The result is true because the value of a determinant is not changed if its rows and columns are interchanged, the diagonal elements remaining the same.

Example 2 : Find the eigenvalues of A' where A is given by Ex. 1, page 5-7.

Sol. : The eigenvalues of A and A' are the same. Hence, the eigenvalues of A' are 1, 2, 3.

Example 3 : If λ is an eigenvalue of A then $\bar{\lambda}$ is an eigenvalue of A^{θ}.

Sol. : We have,

$$|A^{\theta} - \bar{\lambda} I| = |(A - \lambda I)^{\theta}| = |\overline{A - \lambda I}|$$

(Because for any square matrix A, $|A^{\theta}| = |(\bar{A}')| = |\bar{A}'| = |\bar{A}|$)

$\therefore\ |A^{\theta} - \bar{\lambda} I| = 0$ if and only if $|\overline{A - \lambda I}| = 0$

$\therefore\ |A^{\theta} - \bar{\lambda} I| = 0$ if and only if $|A - \lambda I| = 0$

(Because for a complex number z, $\bar{z}$ is zero if and only if $z = 0$)

$\therefore\ \bar{\lambda}$ is an eigenvalue of A^{θ} if λ is an eigenvalue of A.

Example 4 : If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A then show that $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ are the eigenvalue kA.

Sol. : (i) If $k = 0$ then $kA = O$. Since, each eigenvalue of O (zero matrix) is 0, the eigenvalues of kA will be $0\lambda_1, 0\lambda_2, \ldots, 0\lambda_n$.

(ii) If $k \neq 0$, we have

$$|kA - \lambda k I| = |k(A - \lambda I)| = k^n |A - \lambda I|$$

$\therefore\ |kA - \lambda k I| = 0$ if and only if $|A - \lambda I| = 0$

$\therefore\ k\lambda$ is an eigenvalue of kA if and only if λ is an eigenvalue of A.

Thus, $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ are the eigenvalues of kA if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

Example 5 : Eigenvalues of unit matrix I_n are $1, 1, \ldots, 1$ and those of kI_n are $k, k, \ldots, k$.

Sol. : The characteristic equation of I_n is

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1-\lambda & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Expanding it, we get,

$$(1-\lambda)(1-\lambda)\ldots(1-\lambda) = 0 \qquad \therefore\ l = 1, 1, \ldots, 1.$$

Eigenvalues of a unit matrix are $1, 1, \ldots, 1$.

And by the above example 3, the eigenvalues of kI_n are $k \cdot 1, k \cdot 1, \ldots, k \cdot 1$, i.e., $k, k, \ldots, k$.

Example 6 : Find the eigenvalues of $4A$ where A is given by Ex. 2, page 5-8.

Sol. : The eigenvalues of A are 1, 2, 3. Hence, eigenvalues of $4A$ are $4 \times 1, 4 \times 2, 4 \times 3$ i.e., 4, 8, 12.

Example 7 : For the following matrix verify the above result of Ex. 4 for $k = 3$.

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Sol. : In Ex. 1, page 5-7, we have obtained the eigenvalues of the above matrix as 1, 2, 3.

Now, consider $B = 3A = \begin{bmatrix} 6 & -3 & 3 \\ 3 & 6 & -3 \\ 3 & -3 & 6 \end{bmatrix}$

The characteristic equation of B is

$$\begin{vmatrix} 6-\lambda & -3 & 3 \\ 3 & 6-\lambda & -3 \\ 3 & -3 & 6-\lambda \end{vmatrix} = 0$$

$\therefore\ (6-\lambda)[(6-\lambda)(6-\lambda) - 9] + 3[3(6-\lambda) + 9] + 3[-9 - 3(6-\lambda)] = 0$

$\therefore\ \lambda^3 - 3\lambda^2 - 15\lambda^2 + 45\lambda + 54\lambda - 162 = 0$

$\therefore\ \lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0 \qquad \therefore\ (\lambda - 3)(\lambda - 6)(\lambda - 9) = 0$

$\therefore\ (\lambda - 3)(\lambda^2 - 15\lambda + 54) = 0$

$\therefore\ \lambda = 3, 6, 9$

Thus, the eigenvalues of B are 3 times the eigenvalues of A.

Example 8 : If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A then show that $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ are the eigenvalues of A^{-1}. **(M.U. 1999, 2000, 01, 16)**

Sol. : If λ is an eigenvalue of A and X is the corresponding eigenvector then,

$$AX = \lambda X \qquad \therefore\ X = A^{-1}(\lambda X) = \lambda(A^{-1}X)$$

$$\therefore\ \frac{1}{\lambda} X = A^{-1} X \qquad \therefore\ \frac{1}{\lambda} \text{ is an eigen value of } A^{-1}.$$

Hence, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ are the eigenvalues of A^{-1}.

(And X is the eigenvector of A^{-1} corresponding to $1/\lambda$.)

Example 9 : Find the eigenvalues of A^{-1} where A is given by Ex. 2, page 5-16.

Sol. : The eigenvalues of A are 2, 3, 6.

Hence, eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ i.e., $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

Example 10 : If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A then show that $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$ are the eigenvalues of A^2. **(M.U. 1998)**

Sol. : If λ is an eigen-value of A and X is the corresponding eigenvector then,

$$AX = \lambda X \qquad \therefore\ A(AX) = A\lambda X$$

$$\therefore\ A^2 X = \lambda(AX) = \lambda(\lambda X) \qquad [\because\ AX = \lambda X]$$

$$\therefore\ A^2 X = \lambda^2 X \qquad \therefore\ \lambda^2 \text{ is an eigenvalue of } A^2.$$

$\therefore$ If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A then $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$ are the eigenvalues of A^2.

Similarly, we can prove that $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k where k is a positive integer. **(M.U. 2006)**

Example 11 : Find the eigenvalues of A^3 where A is given by Ex. 3, page 5-21.

Sol. : The eigenvalues of A are $-1, -1, 5$. Hence, the eigenvalues of A^3 are $(-1)^3, (-1)^3, (5)^3$ *i.e.*, $-1, -1, 125$.

Example 12 : The eigenvalues of an orthogonal matrix are +1 or −1.

Sol. : Let A be an orthogonal matrix and λ be an eigenvalue.

$\therefore \quad AA' = A'A = I$ [By definition]

If X is the eighen vector corresponding to λ then

$$AX = \lambda X$$

$$\therefore \quad (AX)' = (\lambda X)' \qquad \therefore \quad X'A' = \lambda X' \qquad \text{.....................} (1)$$

Multiply (2) by (1), $\qquad \text{.....................} (2)$

$$\therefore \quad (X'A')(AX) = (\lambda X)'(\lambda X) \qquad \therefore \quad X'(A'A)X = \lambda^2 X'X$$

$$\therefore \quad X'IX = \lambda^2 X'X \qquad \therefore \quad X'X = \lambda^2 X'X$$

$$\therefore \quad X'X(\lambda^2 - 1) = 0.$$

But $\quad X'X \neq 0 \quad \therefore \quad \lambda^2 - 1 = 0 \quad \therefore \quad \lambda = \pm 1.$

Note ....

Ex. 26 (xvi) on page 5-41 is an orthogonal matrix. Its eigenvalues are ±1.

Example 13 : Show that the eigenvalues of a triangular matrix are just the diagonal elements of the matrix.

Sol. : Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$ be a triangular matrix of order n.

It is characteristic equation is

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ 0 & a_{22} - \lambda & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

After expanding the determinant, we get

$$(a_{11} - \lambda)(a_{22} - \lambda) \,.....\, (a_{nn} - \lambda) = 0 \qquad \therefore \quad \lambda = a_{11}, a_{22}, \,.....\, a_{nn}.$$

Hence, the eigenvalues are the diagonal elements.

Example 14 : If λ is an eigenvalue of a non-singular matrix A, prove that $\dfrac{|A|}{\lambda}$ is an eigenvalue of adj. A. **(M.U. 1998, 2003, 05)**

Proof : By definition A adj. $A = |A| I$

Premultiplying by A^{-1},

$$(A^{-1}A) \text{ adj. } A = A^{-1}|A| \qquad \therefore \quad \text{adj. } A = A^{-1}|A| = |A|A^{-1}$$

Post multiplying by X where X is an eigenvector corresponding to the eigenvalue λ.

$$\text{adj. } AX = |A|A^{-1}X = |A|\frac{1}{\lambda}X \qquad \left[\because A^{-1}X = \frac{1}{\lambda}X \text{ by Example 5}\right]$$

$$\therefore \quad \text{adj. } AX = \frac{|A|}{\lambda}X \qquad \therefore \quad \frac{|A|}{\lambda} \text{ is an eigenvalue of adj. } A.$$

Cor. 1 : If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of a 3×3 matrix A then find Eigenvalues of adj. A. **(M.U. 2000)**

Sol. : As proved above the eigenvalues of adj. A are $\dfrac{|A|}{\lambda_1}, \dfrac{|A|}{\lambda_2}, \dfrac{|A|}{\lambda_3}$.

Example 15 : Find the eigenvalues of adj. A where A is given by Ex. 3, page 5-17.

Sol. : The eigenvalues of A are $-3, 12, -6$. Hence, eigenvalues of adj. A are $\dfrac{|A|}{\lambda_1}, \dfrac{|A|}{\lambda_2}, \dfrac{|A|}{\lambda_3}$.

But $|A| = 216$. Hence, the eigenvalues of adj. A are $\dfrac{216}{-3}, \dfrac{216}{12}, \dfrac{216}{-6}$ *i.e.*, $-72, 18, -36$.

Cor. 2 : If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of a 3×3 matrix then prove that the eigenvalues of adj. A are $\lambda_1\lambda_2, \lambda_2\lambda_3$ and $\lambda_3\lambda_1$. **(M.U. 1997, 2003)**

Sol. : As proved above the eigenvalues of adj. A are $\dfrac{|A|}{\lambda_1}, \dfrac{|A|}{\lambda_2}, \dfrac{|A|}{\lambda_3}$. But $|A| = \lambda_1\lambda_2\lambda_3$.

Hence, the eigenvalues are $\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1$.

Example 16 : If λ is an eigenvalue of a matrix A with corresponding eigenvector X, prove that λ^n is an eigenvalue of A^n with corresponding eigenvector X. **(M.U. 2015, 17)**

Hence, find the characteristic roots of A^4 where $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. **(M.U. 1996)**

Sol. : Since λ is an eigenvalue of A if X is the corresponding eigenvector.

$$AX = \lambda X.$$

Premultiply by A.

$$AAX = A\lambda X = \lambda AX \qquad \therefore \quad A^2X = \lambda AX = \lambda(\lambda X) = \lambda^2 X$$

Similarly, $\quad A^3X = \lambda^3 X$.

Continuing in this way $A^n X = \lambda^n X$.

$\therefore \quad \lambda^n$ is an eigenvalue of A^n and the corresponding eigenvector is X. Now, the characteristic equation of the given matrix A is

$$\begin{bmatrix} 1 - \lambda & 1 \\ 1 & 1 - \lambda \end{bmatrix} = 0$$

$$\therefore \quad (1 - \lambda)^2 - 1 = 0 \qquad \therefore \quad 1 - 2\lambda + \lambda^2 - 1 = 0$$

$$\therefore \quad \lambda^2 - 2\lambda = 0 \qquad \therefore \quad \lambda(\lambda - 2) = 0 \qquad \therefore \quad \lambda = 0, \ \lambda = 2.$$

$\therefore$ The characteristic roots of A are 0, 2.

By the above theorem the characteristic roots of A^4 are λ^4 *i.e.*, 0^4, 2^4 *i.e.*, 0 and 16.

Example 17 : *If A is a real symmetric matrix then the eigenvalues of* $A\bar{A}$ *are positive.* **(M.U. 2005)**

Sol. : Since A is symmetric its eigenvalues are real, positive or negative and also $\bar{A} = A$

$\therefore \; A\bar{A} = AA = A^2$

If $\lambda_1, \lambda_2, \ldots\ldots$ are the eigenvalues of A, then $\lambda_1^2, \lambda_2^2, \ldots\ldots$ are the eigenvalues of A^2 *i.e.* of $A\bar{A}$.

$\therefore$ The eigenvalues of $A\bar{A}$ are positive.

Example 18 : *If A is a square matrix of order n where n is an odd positive integer, defined over a field of real numbers then show that A has at least one real eigenvalue.* **(M.U. 2003)**

Sol. : We know that the complex roots of an algebraic polynomial appear in pairs such as $a + ib$ and $a - ib$. Since n is odd, there is atleast one root which is not complex and hence is real.

Example 19 : *If λ is an eigenvalue of the matrix A then $\lambda \pm k$ is an eigenvalue of $A \pm kI$.*

Sol. : Let X be the eigenvector corresponding to the eigenvalue λ. Then

$$AX = \lambda X$$

$$\therefore \; (A + kI)\, X = AX + kIX = \lambda X + kX = (\lambda + k)\, X$$

Hence, $\lambda + k$ is the characteristic root of $(A + kI)$.

Similarly, we can prove that $\lambda - k$ is an eigenvalue of $A - kI$.

Example 20 : *If $f(x)$ is an algebraic polynomial in x and λ is an eigenvalue and X is the corresponding eigenvector of a square matrix A then $f(\lambda)$ is an eigenvalue and X is the corresponding eigenvector of $f(A)$.* **(M.U. 1996, 2006, 16)**

Sol. : Let λ be an eigenvalue and X the corresponding eigenvector of A then,

$$AX = \lambda X \qquad \ldots\ldots\ldots\ldots (1)$$

$$\therefore \; A(AX) = A\lambda X$$

$$\therefore \; A^2X = \lambda AX$$

$$\therefore \; A^2X = \lambda(\lambda X) = \lambda^2 X \qquad \ldots\ldots\ldots\ldots (2)$$

The equation (2), shows that λ^2 is an eigenvalues of A^2 and X is the corresponding eigenvector.

Continuing in this way we can show that if m is a positive integer then

$$A^m X = \lambda^m X \qquad \ldots\ldots\ldots\ldots (3)$$

The equation (3) shows that λ^m is an eigenvalue of A^m and X is the corresponding eigenvector.

Let $\quad f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$

Then $\quad f(A) = a_0 I + a_1A + a_2A^2 + \ldots\ldots + a_nA^n$

$$\begin{aligned}\therefore \; f(A)\,X &= (a_0 I + a_1 A + \ldots\ldots + a_n A^n)\,X \\ &= a_0 X + a_1 AX + \ldots\ldots + a_n A^n X \\ &= a_0 X + a_1 \lambda X + \ldots\ldots + a_n \lambda^n X \qquad \text{[By 3]} \\ &= (a_0 + a_1 \lambda + \ldots\ldots + a_n \lambda^n)\,X \qquad \ldots\ldots\ldots\ldots (4) \\ \therefore \; f(A)\,X &= f(\lambda)\,X\end{aligned}$$

The equation (4) shows that $f(\lambda)$ is an eigenvalue of $f(A)$ and X is the corresponding eigenvector.

(See Ex. 20, 21 and 22 below.)

Example 21 : If $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ then find the Eigenvalues of $4A^{-1} + 3A + 2I$. **(M.U. 2000)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 \\ 2 & 4-\lambda \end{vmatrix} = 0 \qquad \therefore \; (1-\lambda)(4-\lambda) = 0 \qquad \therefore \; \lambda = 1, 4$$

$\therefore$ The eigenvalues of A are 1 and 4.

Now the eigenvalues of A^{-1} are (1 / 1), (1 / 4) and those of $4A^{-1}$ are 4 (1 / 1), 4 (1 / 4) *i.e.*, 4 and 1. The eigenvalues of $3A$ are 3 (1), 3 (4) *i.e.* 3 and 12. Hence, eigenvalues of $4A^{-1} + 3A$ are 4 + 3, 1 + 12 *i.e.* 7 and 13. Hence, the eigenvalues of $4A^{-1} + 3A + 2I$ by the above example are 7 + 2 and 13 + 2 *i.e.* 9 and 15.

Example 22 : If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$, find the eigenvalues of $A^3 + 5A + 8I$. **(M.U. 2014)**

Sol. : Since A is a triangular matrix, the eigenvalues are its diagonal elements *i.e.*, – 1, 3, – 2 by Ex. 13, page 5-32.

Now, the eigenvalues of A^3 are $(-1)^3$, $(3)^3$, $(-2)^3$. The eigenvalues of $5A$ are 5 (– 1), 5 (3), 5 (– 2). The eigenvalues of $8I$ are 8, 8, 8. Hence, the eigenvalues of $A^3 + 5A + 8I$ are

$$(-1)^3 + 5(-1) + 8, \qquad (3)^3 + 5(3) + 8$$

$$\text{and} \quad (-2)^3 + 5(-2) + 8 \qquad i.e., \quad 2, 50, -10.$$

Example 23 : If $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find the characteristic roots $A^3 + I$ and characteristic vectors of $3A^4 + 2A^3 + A^2 + A + 4I$. **(M.U. 2000, 01, 05, 09)**

Sol. : We have obtained the eigenvalues and the eigenvectors of A In Ex. 1, page 5-10.

Now, by the result proved in Ex. 10, page 5-31 and in Ex. 20, page 5-34, the eigenvalues of $A^3 + I$ are $1^3 + 1$, $1^3 + 1$ and $5^3 + 1$ *i.e.* 2, 2 and 126, with eigenvectors [2, – 1, 0] ', [1, 0, – 1] ' and [1, 1, 1] '.

By Ex. 20 above, the charactristic vectors of any function of A and hence of $3A^4 + 2A^3 + A^2 + A + 4I$ are the same as the eigenvectors of A *viz.* [2, –1, 0] ', [1, 0, –1] ' and [1, 1, 1] '.

Example 24 : Find the eigenvalues of $A^2 + 2I$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 5 & 3 \end{bmatrix}$ and I is the unit matrix of order 3. (M.U. 2016)

Sol. : Since A is a triangular matrix by Ex. 13, page 5-32, its eigenvalues are the diagonal elements $1, -2, 3$. Also by Ex. 10, page 5-31, the eigenvalues of A^2 are $(1)^2, (-2)^2, (3)^2$ *i.e.,* 1, 4, 9. The eigenvalues of I are 1, 1, 1 and of $2I$ are 2, 2, 2. By Ex. 19, page 5-34, the eigenvalues of $A^2 + 2I$ are $1 + 2, 4 + 2, 9 + 2$ *i.e.,* 3, 6, 11.

Example 25 : Find the characteristic roots of $A^{30} - 9A^{28}$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. (M.U. 2004)

Sol. : The eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ are given by $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$.

$$\therefore \quad (1-\lambda)^2 - 4 = 0 \qquad \therefore \quad 1 - 2\lambda + \lambda^2 - 4 = 0$$

$$\therefore \quad \lambda^2 - 2\lambda - 3 = 0 \qquad \therefore \quad (\lambda - 3)(\lambda + 1) = 0 \qquad \therefore \quad \lambda = -1, 3.$$

By Ex. 10 above, the eigenvalues of A^{30} are λ^{30} and of $9A^{28}$ are $9\lambda^{28}$. Hence, the eigenvalues of $A^{30} - 9A^{28}$ are $\lambda^{30} - 9\lambda^{28} = \lambda^{28}(\lambda^2 - 9)$.

When $\lambda = -1$, this gives -8 and when $\lambda = 3$, this gives 0. Hence, the eigenvalues $A^{30} - 9A^{28}$ of are 0 and -8.

10. Some Theorems on Eigenvectors

Theorem 1 : ***The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.***

Proof : Let X_1, X_2 be eigenvectors corresponding to two distinct eigenvalues λ_1, λ_2.

$$\therefore \quad A X_1 = \lambda_1 X_1 \qquad \text{.................... (1)}$$

$$\text{and} \quad A X_2 = \lambda_2 X_2 \qquad \text{.................... (2)}$$

Premultiplying (1) by X_2' and (2) by X_1', we get,

$$X_2' A X_1 = X_2' \lambda_1 X_1 = \lambda_1 X_2' X_1$$

$$\text{and} \quad X_1' A X_2 = X_1' \lambda_2 X_2 = \lambda_2 X_1' X_2 \qquad \text{.................... (3)}$$

$$\text{But} \quad (X_2' A X_1)' = (\lambda_1 X_2' X_1)' \qquad \therefore \quad X_1' A X_2 = \lambda_1 X_1' X_2 \qquad \text{.................... (4)}$$

From (3) and (4), we get

$$\lambda_1 X_1' X_2 = \lambda_2 X_1' X_2 \qquad \therefore \quad (\lambda_1 - \lambda_2) X_1' X_2 = 0$$

$$\text{But} \quad \lambda_1 - \lambda_2 \neq 0 \quad \therefore \quad X_1' X_2 = 0.$$

$\therefore$ X_1, X_2 are orthogonal.

Theorem 2 : ***Any two eigenvectors corresponding to two distinct eigenvalues of a unitary matrix are orthogonal.***

Proof : Let A be a unitary matrix and let X_1, X_2 be two eigenvectors of A corresponding to two distinct eigenvalues λ_1 and λ_2 of A.

$$\text{Then,} \quad A X_1 = \lambda_1 X_1 \qquad \text{.................... (1)}$$

$$A X_2 = \lambda_2 X_2 \qquad \text{.................... (2)}$$

Taking complex conjugate transpose of (2),

$$(A X_2)^{\theta} = (\lambda_2 X_2)^{\theta} \qquad \therefore \quad X_2^{\theta} A^{\theta} = \bar{\lambda}_2 X_2^{\theta} \qquad \text{.................... (3)}$$

Post multiplying (3) by AX_1.

$$X_2^{\theta} A^{\theta} A X_1 = \bar{\lambda}_2 X_2^{\theta} A X_1 \qquad \therefore \quad X_2^{\theta} X_1 = \bar{\lambda}_2 X_2^{\theta} \lambda_1 X_1$$

[$\because A^{\theta} A = I$, since A is unitary and $AX_1 = \lambda X_1$ by (1)]

$$\therefore \quad X_2^{\theta} X_1 = \lambda_1 \bar{\lambda}_2 X_2^{\theta} X_1 \qquad \therefore \quad (1 - \lambda_1 \bar{\lambda}_2) X_2^{\theta} X_1 = 0 \qquad \text{.................... (4)}$$

But eigenvalues of unitary matrix are of unit modulus. (See Theorem 2, page 5-27)

$$\therefore \quad \lambda_2 \bar{\lambda}_2 = 1 \quad \therefore \quad \bar{\lambda}_2 = \frac{1}{\lambda_2}.$$

Hence, from (4), we get $\left(1 - \frac{\lambda_1}{\lambda_2}\right) X_2^{\theta} X_1 = \underline{0}$.

Since, $\lambda_1 \neq \lambda_2$, $X_2^{\theta} X_1 = 0$. $\quad \therefore$ X_1, X_2 are orthogonal vectors.

The following table will help you to memorise the above results. If λ is a eigenvalue of A then

	Matrix	Eigenvalue
1.	A^2	λ^2
2.	A^n	λ^n
3.	A'	λ
4.	A^{θ}	$\bar{\lambda}$
5.	A^{-1}	$\frac{1}{\lambda}$
6.	kA	$k\lambda$
7.	I	1
8.	kI	k
9.	adj. A	$\frac{\lvert A \rvert}{\lambda}$
10.	$\lvert A \rvert = 0$	Atleast one zero
11.	Hermitian	All real
12.	Symmetric	All real
13.	Skew Hermitian	Zero or imaginary
14.	Skey symmetric	Zero of imaginary
15.	Unitary	$\lvert \lambda \rvert = 1$
16.	Orthogonal	± 1
17.	Triangular	diag. elements

EXERCISE - I

1. Find the sum and the product of the eigenvalues of the following matrix without solving the characteristic equation.

$$A=\begin{bmatrix}-17 & 18 & -6\\ -18 & 19 & -6\\ -9 & 9 & -2\end{bmatrix}$$

[Ans. : Sum $= -17+19-2=0$ and the product $= |A| = -2$]

2. Find the eigenvalues of the following matrices without solving the characteristic equation.

(i) $\begin{bmatrix}1 & 3 & -4 & 6\\ 0 & -1 & 5 & 3\\ 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 3\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 0 & 0 & 0\\ 3 & 2 & 0 & 0\\ 4 & -2 & -1 & 0\\ 5 & 2 & 3 & 3\end{bmatrix}$ [Ans. : (i) 1, – 1, 2, 3 ; (ii) 1, 2, – 1, 3]

3. Find the eigenvalues of A^2-2A+I if $A=\begin{bmatrix}2 & 1 & -2\\ 0 & 1 & 4\\ 0 & 0 & 3\end{bmatrix}$. [Ans. : 1, 0, 4]

4. If the product of two eigenvalues of $\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$ is 16, find the third eigenvalue.

(M.U. 2004) [Ans. : 2]

5. The sum of the eigenvalues of a 3×3 matrix is 6 and the product of the eigenvalue is also 6. If one of the eigenvalues is one, find the other two eigenvalues. [Ans. : 2, 3]

6. Find the eigenvalues if A^3-3A^2+A where $A=\begin{bmatrix}4 & 6 & 6\\ 1 & 3 & 2\\ -1 & -4 & -3\end{bmatrix}$

[Ans. : Eigenvalues of A are – 1, 1, 4. Required eigenvalues are – 5, – 1, 20]

7. If $A=\begin{bmatrix}3 & 1 & -1\\ 2 & 2 & -1\\ 2 & 2 & 0\end{bmatrix}$, find the characteristic roots and characteristic vectors of A^2.

(M.U. 2000)

[Ans. : The characteristic roots of A are 1, 2, 2 and hence, those of A^2 are 2, 4, 4. The characteristic vectors of both are [1, 0, 2]', [1, 1, 2]'.]

8. If $A=\begin{bmatrix}-1 & 0 & 0\\ 2 & -3 & 0\\ 1 & 4 & -2\end{bmatrix}$, find the eigenvalues of A^2. (M.U. 2000) [Ans. : 1, 9, 4]

9. If $A=\begin{bmatrix}2 & 4\\ 0 & 3\end{bmatrix}$, then find the eigenvalues of $6A^{-1}+A^3+2I$. (M.U. 2009)

[Ans. : 13, 31]

10. Two of the eigenvalues of a 3×3 matrix whose determinant is 6 are 1, 3. Find the third eigenvalue. [Ans. : 2]

11. If A is a square matrix of order 2 with $|A| = 1$ then prove that A and A^{-1} have the same eigenvalues. (M.U. 1996, 2003)

[Ans. : If α, β are the eigenvalues of A then $1/\alpha$, $1/\beta$ are eigenvalues of A^{-1}. But $\alpha\beta = 1$. Hence, the result.]

12. Find the characteristic roots of $A^2-2A+3I$ where $A=\begin{bmatrix}2 & 3\\ 0 & 1\end{bmatrix}$. (M.U. 1996)

[Ans. : Eigenvalues of A are 1, 2. Those of $A^2-2A+2I$ are 2, 3.]

13. Find the characteristic roots of $A'^2-3A'+4I$ where $A=\begin{bmatrix}3 & 0\\ 1 & 2\end{bmatrix}$.

[Ans. : Eigenvalues of A or A' are 3, 2. Those of $A'^2-3A'+4I$ are 4, 2]

14. Find the eigenvalues of adj. A if $A=\begin{bmatrix}1 & 2 & 3 & -2\\ 0 & 2 & 4 & 6\\ 0 & 0 & 4 & -5\\ 0 & 0 & 0 & 6\end{bmatrix}$. [Ans. : 48, 24, 12, 8]

15. Find the eigenvalues of the adjoint of $A=\begin{bmatrix}2 & 0 & -1\\ 0 & 2 & 0\\ -1 & 0 & 2\end{bmatrix}$. (M.U. 2004)

[Ans. : Eigenvalues of the adjoint are $\dfrac{|A|}{\lambda}$; 6, 3, 2]

16. Find the eigenvalues of adj. A and of A^2-2A+I, where $A=\begin{bmatrix}2 & 3 & 4\\ 0 & 4 & 2\\ 0 & 0 & 3\end{bmatrix}$.

(M.U. 2006) [Ans. : (i) 8, 12, 6 ; (ii) 1, 9, 4]

17. If λ_1, λ_2, λ_3 and λ_4 are eigenvalues of A where $A=\begin{bmatrix}2 & 4 & 6 & 8\\ 3 & -2 & 0 & 2\\ 1 & 3 & 2 & 6\\ 3 & 2 & 1 & 4\end{bmatrix}$,

find the eigenvalues of adj. A. [Ans. : $\lambda_1\lambda_2\lambda_3$, $\lambda_1\lambda_2\lambda_4$, $\lambda_1\lambda_3\lambda_4$, $\lambda_2\lambda_3\lambda_4$]

18. If $A=\begin{bmatrix}\sin\theta & \operatorname{cosec}\theta & 1\\ \sec\theta & \cos\theta & 1\\ \tan\theta & \cot\theta & 1\end{bmatrix}$ then prove that there does not exist a real value of θ for which characteristic roots of A are – 1, 1, 3. (M.U. 2000)

(Hint : Sum of the roots = $\sin\theta + \cos\theta + 1$. This cannot be equal to $-1+1+3=3$.)

19. Given $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, find the eigenvalues of A. Also find eigen vales of $4A^{-1}$ and eigenvector of $A^2 - 4I$.

(M.U. 2004)

[**Ans. :** 5, −3, −3 , 4/5, − 4/3, − 4/3 ; [1, 2, −1]', [−2, 1, 0]', [3, 0, 1]']

20. Verify that $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ is orthogonal.

Also verify that $1/\lambda$ is an eigen-value of A if λ is an eigenvalue and that eigenvalues of A are of unit modulus.

(M.U. 2004)

21. Prove that the characteristic roots of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are of unit modulus.

(M.U. 2004)

22. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$, find the eigenvalues of $A^3 + 5A + 8I$. **(M.U. 2001, 11)**

(**Hint :** Eigenvalues of A are −1, 3, 2 and of $A^3 + 5A + 8I$ are 2, 50, − 10.)

23. Verify that $X = [2, 3, -2, -3]'$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$ of the matrix.

$$A = \begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$$

(M.U. 2005) (**Hint :** Verify that $[A - \lambda I]X = 0$.)

24. If $A = \begin{bmatrix} 123 & 231 & 312 \\ 231 & 312 & 123 \\ 312 & 123 & 231 \end{bmatrix}$, prove that

(i) one of the characteristic root of A is 666. (ii) (given that A is non-singular) one of the characteristic roots of A is negative. **(M.U. 2002)**

*(**Hint :** Refer to Ex. 8, page 5-5.)*

***25.** Find the eigenvalues of the following matrices.*

(i) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & c & c \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 5 & 7 \\ 0 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

[**Ans. :** (i) 2, $-1 \pm \sqrt{2}$; (ii) a, b, c; (iii) 2, 3, 4 ; (iv) 1, 2, 5]

26. Find the eigenvalues and bases for eigenspaces of the following matrices.

(i) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

(M.U. 1993, 2016)

(iv) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ (v) $\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(vii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (viii) $\begin{bmatrix} -2 & 1 & 1 \\ -11 & 4 & 5 \\ -1 & 1 & 0 \end{bmatrix}$ (ix) $\begin{bmatrix} -3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

(M.U. 1996, 2004, 05, 06)

(x) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$ (xi) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ (xii) $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

(xiii) $\begin{bmatrix} 8 & -1 & 9 \\ 3 & -1 & 3 \\ -7 & 1 & -7 \end{bmatrix}$ (xiv) $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ (xv) $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

(M.U. 1999, 2002)

(xvi) $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

[**Ans. :** (i) $\lambda = 2, 2, 3$; (5, 2, − 5)', (1, 1, −2)'

(ii) $\lambda = 1, 2, 4$; (1, 0, 0)', (0, 1, − 1)'.

(iii) $\lambda = 2, 3, 6$; (1, 0, − 1)', (1, 1, 1)', (1, −2, 1)'.

(iv) $\lambda = 1, 2, 2$; (1, 1, − 1)', (2, 1, 0)'.

(v) $\lambda = 2, 0, 1$; (2, − 1, 0)', (4, − 1, 0)', (4, 0, − 1)'.

(vi) $\lambda = 1, 2, 3$; (1, 0, − 1)', (0, 1, 0)', (1, 0, 1)'.

(vii) $\lambda = 5, -3, -3$; (1, 2, − 1)', (2, − 1, 0)', (3, 0, 1)'.

(viii) $\lambda = 1, -1, 2$; (1, 2, 1)', (0, − 1, 1)', (1, 3, 1)'.

(ix) $\lambda = 0, 0, 1$; (3, − 1, 0)', (12, − 4, − 1)'.

(x) $\lambda = 1, 1, 1$; (1, 1, 1)'.

(xi) $\lambda = 1, 2, 3$; (1, − 1, 0)', (2, − 1, − 2)', (1, − 1− 2)'.

(xii) $\lambda = 0, 1, -2$; (10, 3, − 11)', (1, 0, − 1), (4, 3, − 7)'.

(xiii) $\lambda = 0, -1, 2$; (1, 0, − 1)', (1, 1, − 1)', (4, 1, − 3)'.

(xiv) $\lambda = 1, -1, 4$; (6, 2, − 7)', (0, 1, − 1)', (3, 1, − 1)'.

(xv) $\lambda = -2, 1, 3$; (11, 1, − 14)', (1, − 1, − 1)', (1, 1, 1)'.

(xvi) $\lambda = 1, 1, -1$; (1, −1, 0), (1, 0, − 1), (1, − 1, − 1).]

11. Cayley-Hamilton Theorem

We shall now see a very important theorem known as Cayley-Hamilton Theorem.

Theorem : ***Every square matrix satisfies its characteristic equation.***

Explanation : We shall not go into the proof of this theorem. We have seen that corresponding to every square matrix we get an equation called its characteristic equation *viz.* $|A-\lambda I|=0$ which can be written as,

$$a_n\lambda^n+a_{n-1}\lambda^{n-1}+a_{n-2}\lambda^{n-2}+\ldots..+a_2\lambda^2+a_1\lambda+a_0=0$$

Cayley - Hamilton Theorem states that this equation is satisfied by the matrix A itself.

i.e., $a_nA^n+a_{n-1}A^{n-1}+a_{n-2}A^{n-2}+\ldots..+a_2A^2+a_1A+a_0I=0$

Remark ....

Caylay-Hamilton theorem can be used to find A^{-1}, A^{-2} and A^4 etc. (See Ex. 1, 2).

Arthor Cayley (1821 - 1895)

A great British mathematician. He had shown his talent at the age of 17 when he was recognised by his teachers as "above the first". He had published his first paper in mathematics at the age of 20 and in the next five years he published 25 papers, when he was at Cambridge. In 1846 he left Cambridge to study law. He worked as a lawyer for the next 14 years but in the same period he published more than 200 papers. But in 1863 he left law and again joined the faculty at Cambridge University. He pursued his mathematical interest till his death. Cayley knew French, German, Italian, Greek and Latin besides English.

William Rown Hamilton (1805 - 1865)

Hamilton was a child prodigy. He red English at the age of 3, Greek and Hebrew at the age of 5, German, Frech, Italian and Spanish at the age of 12. He had also some command on Syriac, Persian, Arabic, Sanskrit and Hindustani. At the age of 17 he had mastered calculus and astronomy on his own. At the age of 22 he was appointed professor at Trinity College, Dublin. He was knighted at the age of 30. In 1833, he first developed the concept of complex numbers as ordered pairs. In 1843 he discovered quaternions and for the next 22 years he developed them further and wrote two monumental books. He had significant contributions to abstract algebra, dynamics, and optics.

Example 1 : Verify Cayley-Hamilton Theorem for the matrix A and hence, find A^{-1}, A^{-2} and A^4 where

$$A=\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}$$

(M.U. 2005, 06, 10, 15, 16, 17)

Prove that $A^{-1}=A^2-5A+9I$.

(M.U. 2017)

Sol. : The characteristic equation is

$$\begin{vmatrix}1-\lambda&2&-2\\-1&3-\lambda&0\\0&-2&1-\lambda\end{vmatrix}=0$$

$\therefore\ (1-\lambda)[(3-\lambda)(1-\lambda)-0]-2[-1(1-\lambda)-0]-2[2-0]=0$

$\therefore\ (1-\lambda)[3-4\lambda+\lambda^2]+2(1-\lambda)-4=0$

$\therefore\ 3-4\lambda+\lambda^2-3\lambda+4\lambda^2-\lambda^3+2-2\lambda-4=0 \qquad \therefore\ \lambda^3-5\lambda^2+9\lambda-1=0.$

Cayley-Hamilton theorem states that this equation is satisfied by the matrix A,

i.e. $A^3-5A^2+9A-I=0$ .... (1)

Now, $$A^2=\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}=\begin{bmatrix}-1&12&-4\\-4&7&2\\2&-8&1\end{bmatrix}$$

and $$A^3=\begin{bmatrix}-1&12&-4\\-4&7&2\\2&-8&1\end{bmatrix}\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}=\begin{bmatrix}-13&42&-2\\-11&9&10\\10&-22&-3\end{bmatrix}$$

It can now be easily seen that

$$A^3-5A^2+9A-I=\begin{bmatrix}-13&42&-2\\-11&9&10\\10&-22&-3\end{bmatrix}-5\begin{bmatrix}-1&12&-4\\-4&7&2\\2&-8&1\end{bmatrix}+9\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

(a) Now, multiply the above equation by A^{-1}.

$\therefore\ A^2-5A+9I-A^{-1}=0$

$\therefore\ A^{-1}=A^2-5A+9I$ .... (2)

$$=\begin{bmatrix}-1&12&-4\\-4&7&2\\2&-8&1\end{bmatrix}-5\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}+9\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}3&2&6\\1&1&2\\2&2&5\end{bmatrix}$$

(b) To find A^{-2} multiply (2) by A^{-1}.

$\therefore\ A^{-1}\cdot A^{-1}=A^{-1}\cdot A^2-5\cdot A^{-1}\cdot A+9A^{-1}I$

$\therefore\ A^{-2}=A-5I+9A^{-1}$

$$=\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}-5\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+9\begin{bmatrix}3&2&6\\1&1&2\\2&2&5\end{bmatrix}=\begin{bmatrix}23&20&52\\8&7&18\\18&-16&41\end{bmatrix}$$

(c) To find A^4 multiply (1) by

$$\therefore \quad A^4 - 5A^3 + 9A^2 - A = 0$$

$$\therefore \quad A^4 = 5A^3 - 9A^2 + A$$

$$\therefore \quad A^4 = 5\begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 9\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & 13 \end{bmatrix}$$

(You should verify that $AA^{-1} = I$)

Example 2 : Find the characteristic equation of the matrix A where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

Show that the matrix A satisfies the characteristic equation and hence, find (a) A^{-1}, (b) A^{-2}, (c) A^4. **(M.U. 1998, 2003)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (1-\lambda)[(1+\lambda)(1+\lambda) - 4] - 2[-2(1+\lambda) - 12] + 3[2 + 3(1+\lambda)] = 0$$

$$\therefore \quad (1-\lambda)(-3 + 2\lambda + \lambda^2) + 2(14 + 2\lambda) + 3(5 + 3\lambda) = 0$$

$$\therefore \quad -3 + 2\lambda + \lambda^2 + 3\lambda - 2\lambda^2 - \lambda^3 + 28 + 4\lambda + 15 + 9\lambda = 0$$

$$\therefore \quad \lambda^3 + \lambda^2 - 18\lambda - 40 = 0$$

Cayley-Hamilton Theorem states that this equation is satisfied by A *i.e.* .... (1)

$$A^3 + A^2 - 18A - 40I = 0$$

Now, $$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$

It can be seen that $A^3 + A^2 - 18A - 40I$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - \begin{bmatrix} 18 & 36 & 54 \\ 36 & -18 & 72 \\ 54 & 18 & -18 \end{bmatrix} - \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the theorem is verified.

Now, multiplying (1) by A^{-1}, we get $A^2 + A - 18I - 40A^{-1} = 0$. .... (2)

(a) $\therefore \quad 40A^{-1} = A^2 + A - 18I$

$$\therefore \quad 40A^{-1} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

(b) To find A^{-2}, multiply (2) by A^{-1} again.

$$\therefore \quad 40A^{-1} \cdot A^{-1} = A^{-1} \cdot A^2 + A^{-1} \cdot A - 18A^{-1}$$

$$= A + I - 18A^{-1}$$

$$\therefore \quad 40A^{-2} = A + I - 18A^{-1}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{18}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 1 & 0 \end{bmatrix} - \frac{9}{20}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 67 & -5 & 51 \\ -86 & 90 & 62 \\ 15 & -25 & 45 \end{bmatrix}$$

$$\therefore \quad A^{-2} = \frac{1}{800}\begin{bmatrix} 67 & -5 & 51 \\ -86 & 90 & 62 \\ 15 & -25 & 45 \end{bmatrix}$$

(c) Further, multiplying (1) by A, we get,

$$A^4 + A^3 - 18A^2 - 40A = 0$$

$$\therefore \quad A^4 = 40A + 18A^2 - A^3$$

$$\therefore \quad A^4 = \begin{bmatrix} 40 & 80 & 120 \\ 80 & -40 & 160 \\ 120 & 40 & -40 \end{bmatrix} + \begin{bmatrix} 252 & 54 & 144 \\ 216 & 162 & -36 \\ 36 & 72 & 252 \end{bmatrix} - \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} = \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$$

Example 3 : Find the characteristic equation of the matrix $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify that it is satisfied by A and hence, obtain A^{-1}. **(M.U. 1995, 98, 2005, 16)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (2-\lambda)[(2-\lambda)^2 - 1] + 1[-1(2-\lambda) + 1] + 1[1 - (2-\lambda)] = 0$$

$$\therefore \quad \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

Cayley-Hamilton Theorem states that this equation is satisfied by A.

i.e. $A^3 - 6A^2 + 9A - 4I = 0$ .... (1)

Now $$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

It can be seen that $A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Now multiply (1) by A^{-1}. $\quad \therefore A^2 - 6A + 9I - 4A^{-1} = 0$

$\therefore 4A^{-1} = (A^2 - 6A + 9I)$

$$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Example 4 : Show that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies Cayley-Hamilton Theorem and hence, find A^{-1} if it exists. **(M.U. 2000, 03)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 0-\lambda & c & -b \\ -c & 0-\lambda & a \\ b & -a & 0-\lambda \end{vmatrix} = 0$$

$$\therefore -\lambda(\lambda^2 + a^2) - c(c\lambda - ab) - b(ac + b\lambda) = 0$$

$$\therefore -\lambda^3 - \lambda a^2 - c^2\lambda + abc - abc - b^2\lambda = 0$$

$$\therefore \lambda^3 + (a^2 + b^2 + c^2)\lambda = 0 \quad \text{............ (1)}$$

Now $$A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}$$

$$\therefore A^3 = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c^3 - cb^2 - ca^2 & b^3 + bc^2 + ba^2 \\ c^3 + ca^2 + cb^2 & 0 & -ab^2 - ac^2 - a^3 \\ -bc^2 - b^3 - a^2b & ac^2 + ab^2 + a^3 & 0 \end{bmatrix}$$

$$\therefore A^3 = -(a^2 + b^2 + c^2)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = -(a^2 + b^2 + c^2)A$$

$\therefore A^3 + (a^2 + b^2 + c^2)A = 0$. $\quad \therefore$ A satisfies the equation (1).

Hence, A satisfies Cayley-Hamilton Theorem.

Now the determinant of the matrix A.

i.e. $$|A| = \begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix} = 0(0 + a^2) - c(0 - ab) - b(ac - 0) = abc - abc = 0.$$

Since the matrix A is singular A^{-1} does not exist.

Example 5 : Verify Cayley-Hamilton theorem and hence, find the matrix represented by $A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I$, where A is $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ **(M.U. 2016)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$$

$$\therefore (3-\lambda)[-(3+\lambda)(7-\lambda) + 20] - 10[-2(7-\lambda) + 12] + 5[-10 - 3(-3-\lambda)] = 0$$

$$\therefore (3-\lambda)(-1 - 4\lambda + \lambda^2) - 10(-14 + 2\lambda + 12) + 5(-10 + 9 + 3\lambda) = 0$$

$$\therefore -3 - 12\lambda + 3\lambda^2 + \lambda + 4\lambda^2 - \lambda^3 + 20 - 20\lambda - 5 + 15\lambda = 0$$

$$\therefore -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0$$

$$\therefore \lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

Cayley-Hamilton theorem states that this equation is satisfied by the matrix A.

i.e., $A^3 - 7A^2 + 16A - 12I = O$

Now, $$A^2 = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 25 & 10 \\ -2 & -31 & -26 \\ 20 & 50 & 44 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 4 & 25 & 10 \\ -2 & -31 & -26 \\ 20 & 50 & 44 \end{bmatrix}\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} -8 & 15 & -10 \\ -52 & -157 & -118 \\ 92 & 270 & 208 \end{bmatrix}$$

$$7A^2 = \begin{bmatrix} 28 & 175 & 70 \\ -84 & -217 & -182 \\ 140 & 350 & 308 \end{bmatrix}$$

$$16A = \begin{bmatrix} 48 & 160 & 80 \\ -32 & -48 & -64 \\ 48 & 80 & 112 \end{bmatrix}$$

$$12I = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

It can be easily seen that $A^3 - 7A^2 + 16A - 12I = O$. .... (1)

Thus, the Cayley Hamilton theorem is verified.

Now, by actual division, we see that

$$A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I = (A^3 - 7A^2 + 16A - 12I)(A^3 + A^2) + 2A - I$$

$$= O + 2A - I \qquad [\because A^3 - 7A^2 + 16A - 12I = O \text{ by } (1)]$$

$$= 2\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 20 & 10 \\ -4 & -7 & -8 \\ 6 & 10 & 13 \end{bmatrix}$$

Example 6 : Find the characteristic equation of the matrix A given below and hence, find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$ **(M.U. 2000, 03)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\therefore\ (2-\lambda)[(1-\lambda)(2-\lambda) - 0] - 1[0-0] + 1[0 - 1(1-\lambda)] = 0$

$\therefore\ (4 - 4\lambda + \lambda^2)(1-\lambda) - (1-\lambda) = 0$

$\therefore\ 4 - 4\lambda + \lambda^2 - 4\lambda + 4\lambda^2 - \lambda^3 - 1 + \lambda = 0$

$\therefore\ \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$

This equation is satisfied by A.

Now dividing $\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1$ by $\lambda^3 - 5\lambda^2 + 7\lambda - 3$, we get the quotient $\lambda^5 + \lambda$ and the remainder $\lambda^2 + \lambda + 1$.

In terms of the matrix A this means

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$
$$= (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I)$$

But $\quad (A^3 - 5A^2 + 7A - 3I) = 0$

$\therefore$ L.H.S. $= A^2 + A + I$

Now, $$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\therefore\ A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

$$\therefore\ \text{The given expression} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 7 : Use Cayley Hamilton theorem to find $2A^4 - 5A^3 - 7A + 6I$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ **(M.U. 2004, 15, 19)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$\therefore\ (1-\lambda)(2-\lambda) - 4 = 0 \qquad \therefore\ 2 - 3\lambda + \lambda^2 - 4 = 0 \qquad \therefore\ \lambda^2 - 3\lambda - 2 = 0$

By Cayley Hamilton theorem, this equation is satisfied by A.

$\therefore\ A^2 - 3A - 2I = 0$

Now, dividing $2\lambda^4 - 5\lambda^3 - 7\lambda + 6$ by $\lambda^2 - 3\lambda - 2$, we get

$$2\lambda^4 - 5\lambda^3 - 7\lambda + 6I = (\lambda^2 - 3\lambda - 2)(2\lambda^2 + \lambda + 7) + 16\lambda + 20$$

In terms of matrix A, this means

$$2A^4 - 5A^3 - 7A + 6 = (A^2 - 3A - 2I)(2A^2 + A + 7I) + 16A + 20I$$

But as seen above $\quad A^2 - 3A - 2I = 0$

$\therefore\ 2A^4 - 5A^3 - 7A + 6I = 16A + 20I$

$$= 16\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 36 & 32 \\ 32 & 52 \end{bmatrix}.$$

Example 8 : Apply Cayley Hamilton theorem to $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and deduce that

$A^8 = 625\,I.$ **(M.U. 2004, 14, 15)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$\therefore\ -(1-\lambda)(1+\lambda) - 4 = 0 \qquad \therefore\ 1 - \lambda^2 + 4 = 0 \qquad \therefore\ \lambda^2 = 5$

By Cayley-Hamilton theorem this equation is satisfied by A, i.e., $A^2 = 5\,I$.

Squaring, we get $A^4 = 25\,I$ and again squaring, we get $A^8 = 625\,I$.

Example 9 : If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$.

Hence, find A^{50}. **(M.U. 1997, 98, 2006)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$\therefore \quad (1-\lambda)[\lambda^2-1]-0+0=0$

$\therefore \quad \lambda^2-1-\lambda^3+\lambda=0 \quad \therefore \quad \lambda^3-\lambda^2-\lambda+1=0.$ .... (1)

By Cayley-Hamilton theorem, this equation is satisfied by A. .... (2)

$\therefore \quad A^3-A^2-A+I=0$

$\therefore \quad A^3=A+A^2-I$

We prove the required result by the method of mathematical induction.

Let the result be true for $n=k$ i.e. suppose $A^k = A^{k-2}+A^2-I$ be true.

Now, multiply the equation by A.

$\therefore \quad A^{k+1}=A^{k-1}+A^3-A$

But by (1), $A^3-A=A^2-I$

$\therefore \quad A^{k+1}=A^{k-1}+A^2-I=A^{(k+1)-2}+A^2-I.$

Hence, the result is true for $n=k+1$.

But by (2), the result is true for $n=3$.

Hence, by mathematical induction, it is true for $n=4, 5, \ldots$ for all $n \geq 3$. .... (3)

Hence, $\quad A^n=A^{n-2}+A^2-I$

To find A^{50}, we put successively, $n=2, 4, \ldots, 46, 48, 50$ in (3).

(1) $\quad A^2 = I+A^2-I$

(2) $\quad A^4 = A^2+A^2-I$

(3) $\quad A^6 = A^4+A^2-I$

....

....

(23) $\quad A^{46} = A^{44}+A^2-I$

(24) $\quad A^{48} = A^{46}+A^2-I$

(25) $\quad A^{50} = A^{48}+A^2-I$

Adding these results, columnwise. Since there are 25 equalities, we get

$$A^{50}=25A^2-24I$$

But $\quad A^2=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\therefore \quad A^{50}=25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}-24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

EXERCISE - II

1. Using Cayley-Hamilton Theorem,

(1) for the matrix $A=\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, prove that $A^{-1}=A^2-5A+9I$. **(M.U. 2006)**

(2) for the matrix $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$, prove that $A^{-1}=\dfrac{1}{40}\left[A^2+A-18I\right]$.

(3) for the matrix $A=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, prove that $A^{-1}=\dfrac{1}{4}\left[A^2-6A+9I\right]$.

2. Find the characteristic equation of each of the following matrices and obtain the inverse.

1. $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ **(M.U. 2017)**

2. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ **(M.U. 2004)**

3. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 1 & 2 \\ 2 & -3 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 5 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$ **(M.U. 2001)**

[**Ans. :** (1) $\dfrac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ (2) $\begin{bmatrix} -3 & 0 & 2 \\ -1 & 1/2 & 1/2 \\ 2 & 0 & -1 \end{bmatrix}$ (3) $\dfrac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$

(4) $-\dfrac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ -6 & 2 & 8 \\ 3 & -1 & -5 \end{bmatrix}$ (5) $\dfrac{1}{18}\begin{bmatrix} 3 & -3 & 6 \\ 7 & -1 & -4 \\ 1 & 5 & 2 \end{bmatrix}$ (6) $-\dfrac{1}{6}\begin{bmatrix} -4 & 4 & -2 \\ 2 & -2 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

(7) $\dfrac{1}{12}\begin{bmatrix} 3 & 3 & 6 \\ 2 & -2 & 4 \\ 5 & 1 & -4 \end{bmatrix}$ (8) $\dfrac{1}{2}\begin{bmatrix} 3 & 1 & -5 \\ 1 & 1 & -1 \\ 1 & 1 & -3 \end{bmatrix}$ (9) $\dfrac{1}{5}\begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{bmatrix}$

(10) $-\frac{1}{16}\begin{bmatrix}-12 & 5 & 1\\ 8 & 14 & -10\\ -4 & -13 & 7\end{bmatrix}$ (11) $\frac{1}{9}\begin{bmatrix}0 & 3 & 3\\ 3 & 2 & -7\\ 3 & -1 & -1\end{bmatrix}$ (12) $\frac{1}{9}\begin{bmatrix}0 & 3 & 3\\ 3 & 2 & -1\\ 3 & -7 & -1\end{bmatrix}$

(13) $\frac{1}{35}\begin{bmatrix}-4 & 11 & -5\\ -1 & -6 & 25\\ 6 & 1 & -10\end{bmatrix}$ (14) $-\frac{1}{37}\begin{bmatrix}-1 & -4 & -6\\ 4 & 16 & -13\\ -6 & 13 & 1\end{bmatrix}$ (15) $\frac{1}{7}\begin{bmatrix}-3 & 8 & 6\\ 7 & -14 & -7\\ -1 & 5 & 2\end{bmatrix}$

3. Find the eigenvalues of the matrix $\begin{bmatrix}10 & -2 & -5\\ -2 & 2 & 3\\ -5 & 3 & 5\end{bmatrix}$ and show that the corresponding eigenvectors are orthogonal. (M.U. 2003)

4. Find the eigenvalues and the corresponding eigenvectors of the matrix

$$\begin{bmatrix}-2 & 5 & 4\\ 5 & 7 & 5\\ 4 & 5 & -2\end{bmatrix}$$

Show that the eigenvectors are orthogonal. (M.U. 2001, 03, 04)

[Ans. : $-3, -6, 12$; $[1, -1, 1]'$, $[-1, 0, 1]'$, $[1, 2, 1]'$.]

5. Find the characteristic equation of the matrix given below and verify that it satisfies Cayley-Hamilton Theorem.

1. $\begin{bmatrix}2 & 1 & 3\\ 1 & -1 & 2\\ 1 & 2 & 1\end{bmatrix}$ 2. $\begin{bmatrix}1 & 1 & 3\\ 1 & 0 & -3\\ -2 & 1 & 0\end{bmatrix}$ 3. $\begin{bmatrix}2 & 3 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}$ 4. $\begin{bmatrix}1 & 0 & 2\\ 1 & 1 & 2\\ 2 & 1 & -1\end{bmatrix}$ 5. $\begin{bmatrix}7 & -1 & 3\\ 6 & 1 & 4\\ 2 & 4 & 8\end{bmatrix}$

(M.U. 2003, 04) (M.U. 2006)

6. Find the characteristic equation of matrix A and verify that it satisfies Cayley-Hamilton Theorem. Hence, find A^{-1} and A^4.

1. $\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 2 & 1 & -2\end{bmatrix}$ 2. $\begin{bmatrix}1 & 1 & 3\\ 1 & 3 & -3\\ -2 & -4 & -4\end{bmatrix}$ 3. $\begin{bmatrix}2 & 0 & -1\\ 0 & 2 & 0\\ -1 & 0 & 2\end{bmatrix}$

(M.U. 2003)

[Ans. : (1) $\begin{bmatrix}-1 & 2 & 1\\ 2 & -2 & -1\\ 0 & 1 & 0\end{bmatrix}$; $\begin{bmatrix}1 & 2 & 4\\ 8 & 3 & -10\\ -12 & -2 & 23\end{bmatrix}$

(2) $\begin{bmatrix}3 & 1 & 3/2\\ -5/4 & -1/4 & -3/4\\ -1/4 & -1/4 & -1/4\end{bmatrix}$; $\begin{bmatrix}-88 & -160 & -240\\ 200 & 432 & 120\\ 40 & 40 & 432\end{bmatrix}$

(3) $\frac{1}{6}\begin{bmatrix}4 & 0 & 2\\ 0 & 3 & 0\\ 2 & 0 & 4\end{bmatrix}$; $\begin{bmatrix}41 & 0 & -40\\ 0 & 16 & 0\\ -40 & 0 & 41\end{bmatrix}$

7. Verify that the matrix $A=\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$ satisfies the characteristic equation. Hence, find A^{-2}. (M.U. 1999, 2004, 09, 14)

[Ans. : $A^3 + A^2 - 5A - 5I = 0$; $A^{-2} = \frac{1}{5}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

8. Find the characteristic equation of the matrix A given below and hence, find the matrix represented by

1. $A^7 - 4A^6 - 20A^5 - 34A^4 - 4A^3 - 20A^2 - 33A + I$ where $A=\begin{bmatrix}1 & 3 & 7\\ 4 & 2 & 3\\ 1 & 2 & 1\end{bmatrix}$ (M.U. 2009)

2. $A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I$ where $A=\begin{bmatrix}3 & 10 & 5\\ -2 & -3 & -4\\ 3 & 5 & 7\end{bmatrix}$

[Ans. : (1) $\begin{bmatrix}3 & 6 & 14\\ 8 & 5 & 6\\ 2 & 4 & 3\end{bmatrix}$ (2) $\begin{bmatrix}5 & 20 & 10\\ -4 & -7 & -8\\ 6 & 10 & 13\end{bmatrix}$

$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$ $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

9. Verify Cayley-Hamilton theorem and find A^{-1} for $A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$.

Hence, find $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A. (M.U. 2005)

[Ans. : $A + 5I$]

10. If $A=\begin{bmatrix}1 & 8\\ 2 & 1\end{bmatrix}$, find (i) $2A^3 - A^2 - 35A - 44I$, (ii) Characteristic roots of $A^2 + 2A + I$.

(M.U. 2001) [Ans. : (i) $A + I$, (ii) 16, 36]

11. Verify Cayley-Hamilton Theorem for $A=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ and obtain A^{-1}.

(M.U. 2011) [Ans. : $A^{-1}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$]

12. Use Cayley-Hamilton theorem to find $2A^5 - 3A^4 + A^2 - 4I$ where $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$.

(M.U. 2000) [Ans. : $138A - 403I = \begin{bmatrix}11 & 138\\ -138 & -127\end{bmatrix}$]

13. If $A=\begin{bmatrix}1 & 4\\ 1 & 1\end{bmatrix}$, find $A^7 + 31A^2 + I$. (M.U. 1997, 2016) [Ans. : $609A + 640I$]

14. Verify Cayley-Hamilton Theorem for (i) $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, (ii) $A=\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}$ and hence find A^{-1} and A^3-5A^2. (M.U. 2004) [Ans. : (i) $\begin{bmatrix}-2 & 1\\ 1.5 & -0.5\end{bmatrix}$, $2A$ (ii) No A^{-1} ; A^2]

15. Compute $A^9-6A^8+10A^7-3A^6+A+I$, where $A=\begin{bmatrix}1 & 2 & 3\\ -1 & 3 & 1\\ 1 & 0 & 2\end{bmatrix}$. (M.U. 2003)

[Ans. : $\lambda^3-6\lambda^2+10\lambda-3=0$; $\begin{bmatrix}2 & 2 & 3\\ -1 & 4 & 1\\ 1 & 0 & 3\end{bmatrix}$]

12. Similarity of Matrices

(a) Definition

If A and B are two square matrices of order n then B is said to be **similar** *to A if there exists a non-singular matrix M such that*

$$B=M^{-1}AM$$

(b) Definition

A square matrix A is said to be **diagonalisable** *if it is similar to a diagonal matrix.*

Combining the two definitions we see that A is diagonalisable if there exists a matrix M such that $M^{-1}AM=D$ where D is a diagonal matrix. In this case M is said to **diagonalise A or transform A** *to diagonal form.*

Theorem 1 : *If A is similar to B and B is similar to C, then A is similar to C.*

Proof : Since A is similar to B, $A=M^{-1}BM$ and since B is similar to C, $B=N^{-1}CN$.

$$\therefore\ A=M^{-1}(B)M=M^{-1}(N^{-1}CN)M$$
$$=(M^{-1}N^{-1})C(NM)=(NM)^{-1}C(NM)$$

$\therefore$ A is similar to C.

Theorem 2 : *If A and B are similar matrices then $|A|=|B|$.*

Proof : Since A is similar to B, $A=M^{-1}BM$

$$\therefore\ \det A=\det(M^{-1}BM)=\det M^{-1}\cdot\det B\cdot\det M$$
$$=\det M^{-1}\cdot\det M\cdot\det B=\det(M^{-1}M)\det B$$
$$=\det I\cdot\det B=\det B.$$

Theorem 3 : *If A is similar to B, then A^2 is similar to B^2.* (M.U. 1999)

Proof : Since A is similar to B there exists P, such that

$$A=M^{-1}BM$$
$$\therefore\ A^2=(M^{-1}BM)(M^{-1}BM)=M^{-1}B(MM^{-1})BM$$

$$=M^{-1}BIBM=M^{-1}B^2M$$

$\therefore$ A^2 is similar to B^2.

(Continuing this argument, we can prove that A^n is similar to B^n.)

Cor. : If A is diagonable then A^2 is diagonable. (M.U. 2004)

Proof : If A is diagonable

$$M^{-1}AM=D$$
$$\therefore\ (M^{-1}AM)(M^{-1}AM)=D^2$$
$$\therefore\ M^{-1}AIAM=D^2$$
$$\therefore\ M^{-1}AMM^{-1}AM=D^2$$
$$\therefore\ M^{-1}A^2M=D^2.$$

$\therefore$ A^2 is diagonable.

Theorem 4 : If A and B are two similar matrices then they have the same eigenvalues. (M.U. 2002, 03)

Proof : Since A and B are similar matrices, there exists a non-singular matrix M such that

$$B=M^{-1}AM$$
$$\therefore\ B-\lambda I=M^{-1}AM-\lambda I$$

Because $M^{-1}(\lambda I)M=\lambda M^{-1}M=\lambda I$

$$\therefore\ B-\lambda I=M^{-1}AM-M^{-1}(\lambda I)M$$
$$=M^{-1}(A-\lambda I)M.$$
$$\therefore\ \det(B-\lambda I)=\det M^{-1}\cdot\det(A-\lambda I)\cdot\det M$$
$$=\det M^{-1}\cdot\det M\cdot\det(A-\lambda I)$$
$$=\det(M^{-1}M)\cdot\det(A-\lambda I)$$
$$=1\cdot\det(A-\lambda I)$$

Thus, the matrices A, B have the same characteristic polynomial and hence, the same eigenvalues.

We list below **the properties which two similar matrices A and B have in common.**

Property	Statement
1. Determinant	A, B have the same determinant.
2. Rank	A, B have the same rank.
3. Trace	A, B have the same trace. [Trace means the sum of the diagonal elements of a square matrix.]
4. Characteristic Polynomial	A, B have the same characteristic polynomial.
5. Eigenvalues	A, B have the same eigenvalues.
6. Dimension	Corresponding to an eigenvalue λ, the eigenspaces of A and B have the same dimension.

13. Algebraic and Geometric Multiplicity of an Eigenvalue

Definition (Algebraic Multiplicity) : If an eigenvalue λ_1 of matrix A is repeated t times then t is called the **albebraic multiplicity** of λ_1.

If λ_1 is an eigenvalue occurring once then the algebraic multiplicity of λ_1 is one.

If λ_1 is an eigenvalue occurring twice then the algebraic multiplicity of λ_1 is two, etc.

Definition (Geometric Multiplicity) : If corresponding to an eigenvalue λ_1, there are s linearly independent eigenvectors then s is called the **geometric multiplicity** of λ_1.

If there is only one eigenvector corresponding to an eigenvalue λ_1 then the geometric multiplicity of λ_1 is one.

If there are two linearly independent eigenvectors corresponding to an eigenvalue λ_1 then the geometric multiplicity of λ_1 is two.

Relation between the number of variables n in the matrix, the rank r of the matrix and the number s of independent eigenvectors of λ_1.

We note the relation that $\boxed{n - r = s}$

Example 1 : Determine the algebraic and geometric multiplicity of each eigenvalue of $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ if the eigenvalues of A are 7, 1, 1 and the eigenvectors are $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ corresponding to $\lambda = 7$ and $\lambda = 1$ in this order.

Sol. : Since the value $\lambda = 7$ is occurs only once, the algebraic multiplicity of $\lambda = 7$ is 1.

Since there is only one eigenvector corresponding to $\lambda = 7$, the geometric multiplicity of $\lambda = 7$ is 1.

Since the value $\lambda = 1$ is repeated twice, the algebraic multiplicity of $\lambda = 1$ is two.

Since there are two linearly independent eigenvectors corresponding to $\lambda = 1$, the geometric multiplicity of $\lambda = 1$ is 2.

Example 2 : Find the eigenvalues and eigenvectors of the matrix A and discuss algebraic and geometric multiplicity of each eigenvalue where

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Sol. : The characteristic equation of A is

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$\therefore \lambda^2(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1 + \lambda) = 0$ $\quad \therefore -\lambda^3 + \lambda + \lambda + 1 + \lambda + 1 = 0$

$\therefore \lambda^3 - 3\lambda - 2 = 0$ $\quad \therefore \lambda^3 + \lambda^2 - \lambda^2 - \lambda - 2\lambda - 2\lambda = 0$

$\therefore (\lambda^2 - \lambda - 2)(\lambda + 1) = 0$ $\quad \therefore (\lambda - 2)(\lambda + 1)(\lambda + 1) = 0$

$\therefore \lambda = -1, -1, 2.$

As seen above the algebraic multiplicity of $\lambda = -1$ is 2 and algebraic multiplicity of $\lambda = 2$ is 1.

(i) For $\lambda = -1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since there are three variables and rank is 1, there are $3 - 1 = 2$ independent eigenvectors.

Now, we have $x_1 + x_2 + x_3 = 0$.

Putting $x_2 = -s$, $x_3 = -t$, we get $x_1 = -x_2 - x_3 = s + t$

$$\therefore \quad X = \begin{bmatrix} s + t \\ -s + 0 \\ 0 - t \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\therefore \quad X_1 = [1, -1, 0]'$ and $X_2 = [1, 0, -1]'$.

Since there are two linearly independent eigenvectors corresponding to $\lambda = -1$, the geometric multiplicity of $\lambda = -1$ is 2.

(ii) For $\lambda = 2$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_{2,1} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_2 + 2R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_3 + R_2 \\ -(1/3)R_2 \end{matrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since there are three variables and the rank is two, there is $3 - 2 = 1$, eigenvector.

$\therefore \quad x_1 - 2x_2 + x_3 = 0, \quad x_2 - x_3 = 0$

Putting $x_3 = t$, we get $x_2 = t$, $x_1 = 2x_2 - x_3 = 2t - t = t$.

$$\therefore \quad X = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \therefore \quad X = [1, 1, 1]'.$$

As seen above the geometric multiplicity of $\lambda = 2$ is 1.

Example 3 : Find the algebraic multiplicity and geometric multiplicity of each eigenvalue of the matrix. (M.U. 2005)

$$\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$$

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$$

$\therefore \quad (4-\lambda)[-(3-\lambda)(2+\lambda)+10] - 6[-(2+\lambda)1+2] + 6[-5+(3-\lambda)] = 0$

$\therefore \quad \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0 \qquad \therefore (\lambda-2)(\lambda-2)(\lambda-1) = 0$

$\therefore \quad \lambda = 1, 2, 2.$

(i) For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \therefore \text{By } \frac{R_1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_2 - R_1 \\ R_3 + R_1 \end{matrix}\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & -3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \therefore x_1 + 2x_2 + 2x_3 = 0, \quad 3x_2 + x_3 = 0.$$

Putting $x_2 = t$, $x_3 = -3t$, $x_1 = 4t$.

$$\therefore \quad X_1 = \begin{bmatrix} 4t \\ t \\ -3t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} \quad \therefore \text{Eigenvector is } [4, 1, -3]'.$$

There are 3 variables and the rank is 2, hence, there is only $3 - 2 = 1$ independent eigenvector.

$\therefore$ For $\lambda = 1$, algebraic multiplicity = 1, geometric multiplicity = 1.

(ii) For $\lambda = 2$, $[A - \lambda_2 I] = O$ gives

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \frac{R_1}{2}\begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_2 - R_1 \\ R_3 + R_2 \end{matrix}\begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -1 \\ 0 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_3 - 2R_2 \\ -R_2 \end{matrix}\begin{bmatrix} 1 & 3 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 + 3x_2 + 3x_3 = 0, \quad 2x_2 + x_3 = 0$

Putting $x_2 = -t$, $x_3 = 2t$, $x_1 = -3t$.

$$\therefore \quad X_2 = \begin{bmatrix} -3t \\ -t \\ 2t \end{bmatrix} = t\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} \quad \therefore \text{Eigenvector is } [-3, -1, 2]'.$$

There are 3 variables and rank is 2, hence there is only $3 - 2 = 1$ independent eigenvector.

$\therefore$ For $\lambda = 2$, algebraic multiplicity = 2, geometric multiplicity = 1.

Theorem : ***The necessary and sufficient condition of a square matrix to be similar to a diagonal matrix is that the geometric multiplicity of each of its eigenvalues coincides with the algebraic multiplicity.***

We shall accept this theorem without proof. The theorem states that we can diagonalise a given square matrix if and only if algebraic multiplicity of each of its eigenvalues is equal to the geometric multiplicity. If corresponding to any eigenvalue, if algebraic multiplicity is **not equal** to geometric multiplicity then the matrix is **not diagonalisable.**

(See Ex. 6, page 5-69 and Ex. 10, page 5-73)

Cor. Every matrix whose eigenvalues are distinct is similar to a diagonal matrix.

14. Modal Matrix

We shall now define an important matrix *viz.* modal matrix.

Theorem : ***A square non singular matrix A whose eigenvalues are all distinct can be diagonalised by a similarity transformation $D = M^{-1} A M$ where M is the matrix whose columns are the eigenvectors of A and D is the diagonal matrix whose diagonal elements are the eigenvalues of A.***

Proof : Let the roots of the characteristic equation $|A - \lambda I| = 0$ be $\lambda_1, \lambda_2, \ldots, \lambda_n$ which are distinct. Let the corresponding eigenspaces be $X_1, X_2, X_3, \ldots, X_n$. Let M be the matrix whose columns are $X_1, X_2, \ldots, X_n$. The matrix M is called the **modal** matrix.

i.e. $\quad M = [X_1, X_2, \ldots, X_n] \qquad \therefore AM = [AX_1, AX_2, \ldots, AX_n]$

Since $X_1, X_2, \ldots, X_n$ are eigenspaces of A we have,

$$AX_1 = \lambda_1 X_1, \quad AX_2 = \lambda_2 X_2, \ldots, AX_n = \lambda_n X_n.$$

$$\therefore \quad AM = [\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n] = [X_1, X_2, \ldots, X_n]\begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} = MD$$

$\therefore \quad AM = MD$

Operating by M^{-1}, $\qquad M^{-1}AM = M^{-1}MD = D.$

Thus, if a non-singular square matrix A has all distinct eigenvalues then, $M^{-1}AM = D$.

From the two theorems we learn that :-

(i) If all eigenvalues of A are distinct then A can be diagonalised (See Ex. 1, 2, 3)

(ii) If an eigenvalue of A is repeated then A **may be** diagonalisable or A **may not be** diagonalisable.

(iii) If the algebraic multiplicity and geometric multiplicity of a repeated value **are equal** then A is diagonalisable. (See Ex. 5).

(iv) If the algebraic multiplicity and geometric multiplicity of a repeated value **are not equal** then A is not diagonalisable (See Ex. 6 and 10).

15. Diagonalising a Given Matrix

The process of finding a diagonal matrix similar to a given matrix is known as **diagonalising the given matrix**. The procedure is given below.

Procedure to diagonalise a given matrix

(a) Distinct Eigenvalues

1. First find the eigenvalues of the given matrix.
2. If all eigenvalues are distinct the matrix is **diagonalisable**. (See Theorem of § 14, page 5-59)
3. The diagonal matrix whose diagonal elements are the eigenvalues obtained above is the required diagonal matrix.
4. Now, find the eigenvectors corresponding to the eigenvalues obtained above.
5. The matrix of eigenvectors obtained above is the diagonalising or transforming matrix.
6. If $\lambda_1, \lambda_2, \lambda_3$ are the (distinct) eigenvalues of A, then the diagonal matrix D is

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

7. If $[u_1, u_2, u_3]'$, $[v_1, v_2, v_3]'$ and $[w_1, w_2, w_3]'$ are the eigenvectors **corresponding** to $\lambda_1, \lambda_2, \lambda_3$ **in this order**, then the diagonalising matrix or the modal matrix M is

$$M = \begin{bmatrix} u_1 & v_1 & w_3 \\ u_2 & v_2 & w_2 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

(b) Repeated Eigenvalues

1. If an eigenvalue λ is repeated 't' times then its **algebraic multiplicity** is 't'.
2. Now, find the eigenvectors corresponding to eignevalue λ. If there are 's' linearly independent eigenvectors then the **geometric multiplicity** of λ is 's'.
3. If the algebraic multiplicity 't' of λ **is equal** to the geometric multiplicity 's', *i.e.*, if $t = s$, then the matrix is **diagonalisable**.
4. The matrix of the eigenvalues are the diagonal matrix and the matrix of the corresponding eigenvectors are the diagonalising or modal matrix as explained above.
5. If the algebraic multiplicity of λ is **not equal** the geometric multiplicity, then the matrix A is **not diagonalisable.**

Example 1 : Find a matrix M which diagonalizes the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.

Verify that $M^{-1}AM = D$ where D is the diagonal matrix. **(M.U. 2011)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$\therefore\ (4-\lambda)(3-\lambda) - 2 = 0 \qquad \therefore\ \lambda^2 - 7\lambda + 10 = 0$

$\therefore\ (\lambda-5)(\lambda-2) = 0 \qquad \therefore\ \lambda = 2, 5.$

Since the eigenvalues are distinct, the matrix A is diagonalisable.

(i) For $\lambda = 2$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 - R_1 \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore\ 2x_1 + x_2 = 0$

Putting $x_2 = -2t$, we get $2x_1 = -x_2 = 2t \quad \therefore\ x_1 = t$.

$$\therefore\ X = \begin{bmatrix} t \\ -2t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{or} \quad X_1 = [1, -2]'.$$

(ii) For $\lambda = 5$, $[A_1 - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 + 2R_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore\ -x_1 + x_2 = 0$

Putting $x_2 = t$, we get $x_1 = x_2 = t$.

$$\therefore\ X = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{or} \quad X_2 = [1, 1]'.$$

Thus, the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ will be diagonalised to the diagonal matrix $D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ by the transforming matrix $M = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$. And $M^{-1}AM = D$.

Verification : We recall that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$\therefore\ M^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

Now, $$M^{-1}AM = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -4 & 5 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 6 & 0 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D$$

Example 2 : Show that the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalisable. Find the transforming matrix and the diagonal matrix. **(M.U. 2005, 15)**

Sol. : In Ex. 1, page 5-14, we have obtained the eigenvalues and eigenvectors of the above matrix A. The eigenvalues are 0, 3, 15.

Since all eigenvalues are distinct, the matrix A is diagonalisable.

Now, the eigenvectors are (See page 5-14)

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore \quad M = [X_1, \ X_2, \ X_3] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Since, $M^{-1}AM = D$, the given matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalised to diagonal matrix $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$ by transforming matrix $M = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

[Let us verify that $M^{-1}AM = D$.

Now, $|M| = 1(1-4) - 2(2+4) + 2(-4-2) = -27$

The cofactors of the first column are

$$\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 1 - 4 = -3, \qquad -\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = -(2+4) = -6,$$

$$\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} = -4 - 2 = -6$$

The cofactors of the second column are

$$-\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -(2+4) = -6, \qquad \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3$$

$$-\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = -(-2-4) = 6$$

The cofactors of the third column are

$$\begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} = -4 - 2 = -6, \qquad -\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = -(-2-4) = 6,$$

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3$$

$$\therefore \quad M^{-1} = -\frac{1}{27} \begin{bmatrix} -3 & -6 & -6 \\ -6 & -3 & 6 \\ -6 & 6 & -3 \end{bmatrix} = \frac{1}{27} \begin{bmatrix} 3 & 6 & 6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$$

Now, $M^{-1}AM = D$

$\therefore \quad M(M^{-1}AM)M^{-1} = MDM^{-1}$

$\therefore \quad A = MDM^{-1}$

Instead of verifying $M^{-1}AM = D$, we shall verify that $MDM^{-1} = A$.

$$\text{Now, } \quad MDM^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \times \frac{1}{27} \begin{bmatrix} 3 & 6 & 6 \\ 6 & 3 & -6 \\ 6 & -6 & -3 \end{bmatrix}$$

$$= \frac{1}{27} \begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix} \begin{bmatrix} 3 & 6 & 6 \\ 6 & 3 & -6 \\ 6 & -6 & -3 \end{bmatrix}$$

$$= \frac{1}{27} \begin{bmatrix} 216 & -162 & 54 \\ -162 & 189 & -108 \\ 54 & -108 & 81 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} = A$$

This shows that M diagonalises A to diagonal matrix D.]

Example 3 : Show that the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is diagonalisable. Find the transforming matrix and the diagonal matrix. **(M.U. 1991, 93, 95, 2003, 05, 16)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$\therefore \quad (1-\lambda)(\lambda-2)(\lambda-3) = 0 \qquad \therefore \quad \lambda = 1, 2, 3.$

Since, all eigenvalues are distinct the matrix A is diagonalisable.

(i) For $\lambda = 1$, $[A - \lambda I]X = O$ gives

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad 7x_1 - 8x_2 - 2x_3 = 0; \qquad 4x_1 - 4x_2 - 2x_3 = 0$

By Cramer's rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$$

$$\therefore \quad \frac{x_1}{8} = \frac{x_2}{6} = \frac{x_3}{4} \qquad \therefore \quad \frac{x_1}{4} = \frac{x_2}{3} = \frac{x_3}{2} = t$$

$\therefore\ x_1 = 4t,\ x_2 = 3t,\ x_3 = 2t.$

$$\therefore\ X_1 = \begin{bmatrix} 4t \\ 3t \\ 2t \end{bmatrix} = t\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \quad \therefore\ X_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$\therefore$ Corresponding to eigenvalue 1, the eigenvector is [4, 3, 2] '.

(ii) For $\lambda = 2$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ 6x_1 - 8x_2 - 2x_3 = 0\ ;\qquad 4x_1 - 5x_2 - 2x_3 = 0$

By Cramer's rule,

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 6 & -8 \\ 4 & -5 \end{vmatrix}}$$

$$\therefore\ \frac{x_1}{6} = \frac{x_2}{4} = \frac{x_3}{2} \quad \therefore\ \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1} = t$$

$\therefore\ x_1 = 3t,\ x_2 = 2t,\ x_3 = t.$

$$\therefore\ X_2 = \begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \therefore\ X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ Corresponding to eigenvalue, 2 the eigenvector is [3, 2, 1] '.

(iii) For $\lambda = 3$, $[A - \lambda_3 I]X = O$ gives

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ 5x_1 - 8x_2 - 2x_3 = 0\ ;\qquad 4x_1 - 6x_2 - 2x_3 = 0$

By Cramer's rule,

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$$\therefore\ \frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{2} \quad \therefore\ \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1} = t$$

$\therefore\ x_1 = 2t,\ x_2 = t,\ x_3 = t.$

$$\therefore\ X_3 = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \therefore\ X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Corresponding to eigenvalue 3, the eigenvector is [2, 1, 1] '.

$$\therefore\ M = [X_1, X_2, X_3] = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Since, $M^{-1}AM = D$, the matrix $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ will be diagonalised to the diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the transforming matrix $M = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

Note ....

It should be noted that the columns of matrix M are to be taken **in the order** in which we take the eigenvalues in D. In the above example, if we change the order of eigenvalues in D, then we have to take the columns in M in the new order.

e.g., If $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then $M = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

Example 4 : Is the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalisable? If so find the diagonal form and the transforming matrix. **(M.U. 2014, 17)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\therefore\ (2-\lambda)[(2-\lambda)(1-\lambda) - 0] - 1[1(1-\lambda) - 0] + 1[0 - 0] = 0$

$\therefore\ (2-\lambda)(2-\lambda)(1-\lambda) - (1-\lambda) = 0$

$\therefore\ (1-\lambda)[(2-\lambda)(2-\lambda) - 1] = 0$

$\therefore\ (1-\lambda)(4 - 4\lambda + \lambda^2 - 1) = 0 \qquad \therefore\ (1-\lambda)(\lambda^2 - 4\lambda + 3) = 0$

$\therefore\ (1-\lambda)(\lambda - 3)(\lambda - 1) = 0 \qquad \therefore\ \lambda = 1, 1, 3.$

Since the eigenvalues are repeated the matrix A may be or may not be diagonalisable. [See (ii) at the bottom, page 5-59]

We shall now find algebraic multiplicity and geometric multiplicity of each eigenvalue and apply (iii) and (iv) at the bottom of page 5-59.

(i) For $\lambda = 3$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_2 + R_1 \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_3 + R_2 \\ (1/2) R_2 \end{matrix} \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \; -x_1 + x_2 + x_3 = 0, \quad x_3 = 0$

Putting $x_2 = t$, we get $x_1 = t$.

$\therefore \; X_1 = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\therefore$ Eigenvector is $[1, 1, 0]'$.

There are three variables and the rank is 2, hence, there is only $3 - 2 = 1$ independent solution.

$\therefore$ For $\lambda = 3$, algebraic multiplicity = 1 and the geometric multiplicity = 1.

(ii) For $\lambda = 1$, $[A - \lambda_2 I] X = O$ gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \; x_1 + x_2 + x_3 = 0$

Let $x_2 = -s, \; x_3 = -t \quad \therefore \; x_1 = s + t$

$$\therefore \; X_2 = \begin{bmatrix} s+t \\ -s+0 \\ 0-t \end{bmatrix} = \begin{bmatrix} s \\ -s \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore \; X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

There are three variables and the rank of the matrix is one and hence there are $3 - 1 = 2$ independent vectors.

$\therefore$ For $\lambda = 1$, since the eigenvalue is repeated twice, the algebraic multiplicity = 2 and since X_2, X_3 are two independent vectors corresponding to $\lambda = 1$, the geometric multiplicity = 2.

Since the algebraic multiplicity and geometric multiplicity of each eigenvectors are equal, by Theorem of § 14, page 5-59, the matrix is diagonalisable.

The diagonalising matrix is

$$M = [X_1, \; X_2, \; X_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Thus, the given matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalised to the diagonal matrix $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by transforming matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

[We shall below verify that $M^{-1} AM = D$. We shall first obtain M^{-1} by elementary transformations. For this we write $A = I A$ where $A = M$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

By $\quad R_2 - R_1 \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

By $\quad R_1 + R_3 \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

By $\quad R_2 - R_3 \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} A$

By $\quad R_1 + \frac{1}{2} R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} A$

By $\quad \begin{matrix} -(1/2) R_2 \\ -R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix} A$

$$\therefore \quad M^{-1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix}$$

Now, $\quad M^{-1} A M = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 3 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D \quad]$$

Example 5 : Show that the matrix $A=\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}$ is diagonalisable. Find the diagonal form D and the diagonalising matrix M.

(M.U. 1992, 93, 94, 2004, 05, 14, 15, 17)

Sol. : The characteristic equation is

$$\begin{vmatrix}-9-\lambda & 4 & 4\\ -8 & 3-\lambda & 4\\ -16 & 8 & 7-\lambda\end{vmatrix}=0$$

$\therefore\ (1+\lambda)(1+\lambda)(3-\lambda)=0 \qquad \therefore\ \lambda=-1,-1,3.$

(i) For $\lambda=-1$, $[A-\lambda_1 I]X=O$ gives

$$\begin{bmatrix}-8 & 4 & 4\\ -8 & 4 & 4\\ -16 & 8 & 8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \text{By } \begin{matrix}R_2-R_1\\ R_3-2R_1\\ -(1/4)R_1\end{matrix}\begin{bmatrix}2 & -1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore\ 2x_1-x_2-x_3=0$

The rank of the coefficient matrix is 1. The number of unknowns is 3. Hence, there are $3-1=2$ linearly independent solutions.

Putting $x_2=2t$ and $x_3=2s$, we get $2x_1=x_2+x_3$.

$\therefore\ 2x_1=2t+2s \qquad \therefore\ x_1=t+s$

$$\therefore\ X_1=\begin{bmatrix}s+t\\ 0+2t\\ 2s+0\end{bmatrix}=s\begin{bmatrix}1\\0\\2\end{bmatrix}+t\begin{bmatrix}1\\2\\0\end{bmatrix}$$

$\therefore$ Corresponding to the eigenvalue -1, we get the following **two** linearly independent eigenvectors.

$$X_1=[1,0,2]' \text{ and } X_2=[1,2,0]'.$$

For $\lambda=-1$, the algebraic multiplicity is 2 and the geometric multiplicity is 2.

(ii) For $\lambda=3$, $[A-\lambda_2 I]X=O$ gives

$$\begin{bmatrix}-12 & 4 & 4\\ -8 & 0 & 4\\ -16 & 8 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \text{By } \begin{matrix}R_2-R_1\\ R_3-R_1\end{matrix}\begin{bmatrix}-12 & 4 & 4\\ 4 & -4 & 0\\ -4 & 4 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\text{By } R_3+R_2\begin{bmatrix}-12 & 4 & 4\\ 4 & -4 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \text{By } \begin{matrix}-(1/4)R_1\\ (1/4)R_2\end{matrix}\begin{bmatrix}3 & -1 & -1\\ 1 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore\ 3x_1-x_2-x_3=0$ and $x_1-x_2=0 \quad \therefore\ x_1=x_2$

Putting $x_2=t$, we get $x_1=t$ and $x_3=3x_1-x_2=3t-t=-2t$.

$$\therefore\ X_3=\begin{bmatrix}t\\ t\\ 2t\end{bmatrix}=t\begin{bmatrix}1\\1\\2\end{bmatrix}$$

$\therefore$ Corresponding to eigenvalue 3, we get the eigenvector $X_3=[1,1,2]'$.

For $\lambda=3$, the algebraic multiplicity is 1 and the geometric multiplicity is 1.

Although eigenvalues of A are not distinct the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity, A is diagonisable.

Now, $M=[X_1,\ X_2,\ X_3]=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 1\\ 2 & 0 & 2\end{bmatrix}$

Since, $M^{-1}AM=D$, the matrix $A=\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}$ will be diagonalised to the diagonal matrix $D=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 3\end{bmatrix}$ by the transforming matrix $M=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 1\\ 2 & 0 & 2\end{bmatrix}$

Example 6 : Show that the matrix $A=\begin{bmatrix}2 & 3 & 4\\ 0 & 2 & -1\\ 0 & 0 & 1\end{bmatrix}$ is **not** similar to a diagonal matrix.

(M.U. 1992)

Sol. : The characteristic equation is

$$\begin{vmatrix}2-\lambda & 3 & 4\\ 0 & 2-\lambda & -1\\ 0 & 0 & 1-\lambda\end{vmatrix}=0$$

$\therefore\ (2-\lambda)(2-\lambda)(1-\lambda)=0 \qquad \therefore\ \lambda=1,2,2.$

Since the eigenvalue 2 is repeated twice, its algebraic multiplicity is two.

For $\lambda=2$, $[A-\lambda I]X=0$ gives

$$\begin{bmatrix}0 & 3 & 4\\ 0 & 0 & -1\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \text{By } \quad R_3-R_2\begin{bmatrix}0 & 3 & 4\\ 0 & 0 & -1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Since, the rank of the coefficient matrix is 2 and since there are 3 variables, the number of independent solutions is $3-2=1$. Hence, the geometric multiplicity of $\lambda=2$ is one.

Since, the algebraic multiplicity of the eigenvalue 2 is 2 and the geometric multiplicity is 1, A cannot be similar to a diagonal matrix.

Example 7 : Show that the matrix $\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}$ is diagonalsable. Find the transforming matrix and diagonal form.

(M.U. 2016)

Sol. : The characteristic equation is

$$\begin{vmatrix}2-\lambda & 2 & 1\\ 1 & 3-\lambda & 1\\ 1 & 2 & 2-\lambda\end{vmatrix}=0$$

$\therefore\ (2-\lambda)[(3-\lambda)(2-\lambda)-2]-2[1(2-\lambda)-1]+1[2-(3-\lambda)]=0$

$\therefore\ (2-\lambda)[6-5\lambda+\lambda^2-2]-2[1-\lambda]+[-1+\lambda]=0$

$\therefore\ (2-\lambda)(4-5\lambda+\lambda^2)-2(1-\lambda)+(\lambda-1)=0$

$\therefore\ 8-10\lambda+2\lambda^2-4\lambda+5\lambda^2-\lambda^3-2+2\lambda+\lambda-1=0$

$\therefore\ -\lambda^3+7\lambda^2-11\lambda+5=0 \qquad \therefore\ \lambda^3-7\lambda^2+11\lambda-5=0$

$\therefore\ \lambda^3-\lambda^2-6\lambda^2+6\lambda+5\lambda-5=0 \qquad \therefore\ (\lambda-1)(\lambda^2-6\lambda+5)=0$

$\therefore\ (\lambda-1)(\lambda-1)(1-5)=0 \qquad \therefore\ \lambda=1,1,5.$

Since the eigenvalues are repeated the matrix may or may not be diagonisable. [See (ii), bottom of page 5-59].

We shall now find algebraic and geometric multiplicity of each eigenvalue and apply (iii) and (iv) given at the bottom of page 5-59.

(i) For $\lambda=5$, $[A-\lambda_1 I]X=O$ gives

$$\begin{bmatrix}-3 & 2 & 1\\ 1 & -2 & 1\\ 1 & 2 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{By } R_{13}\begin{bmatrix}1 & 2 & -3\\ 1 & -2 & 1\\ -3 & 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\text{By } \begin{matrix}R_2-R_1\\R_3+3R_1\end{matrix}\begin{bmatrix}1 & 2 & -3\\ 0 & -4 & 4\\ 0 & 8 & -8\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{By } \begin{matrix}R_3+2R_2\\-\frac{1}{4}R_2\end{matrix}\begin{bmatrix}1 & 2 & -3\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore\ x_1+2x_2-3x_3=0, \quad x_2-x_3=0$

Putting $x_3=t,\ x_2=t \qquad \therefore\ x_1+2t-3t=t.$

$$\therefore\ X_1=\begin{bmatrix}t\\t\\t\end{bmatrix}=t\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \therefore\ X_1=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

There are three variables and the rank is 2, hence there is only $3-2=1$ independent solution.

$\therefore$ For $\lambda=5$, occurring once, algebraic multiplicity = 1 and the geometric multiplicity = 1.

(ii) For $\lambda=1$, $[A-\lambda_2 I]X=O$ gives

$$\begin{bmatrix}1 & 2 & 1\\ 1 & 2 & 1\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{By } \begin{matrix}R_2-R_1\\R_3-R_1\end{matrix}\begin{bmatrix}1 & 2 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore\ x_1+2x_2+x_3=0$

Putting $x_2=-s,\ x_3=-t$, we get $x_1-2s-t=0 \qquad \therefore\ x_1=2s+t.$

$$\therefore\ X=\begin{bmatrix}2s+t\\-s+0\\0-t\end{bmatrix}=s\begin{bmatrix}2\\-1\\0\end{bmatrix}+t\begin{bmatrix}1\\0\\-1\end{bmatrix} \qquad \therefore\ X_2=\begin{bmatrix}2\\-1\\0\end{bmatrix},\ X_3=\begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

There are three variables and the rank of the matrix is 1, hence there are $3-1=2$ independent solutions.

$\therefore$ For $\lambda=1$, occurring twice, algebraic multiplicity = 2 and geometric multiplicity = 2.

Since the algebraic multiplicity and geometric multiplicity of each vector are equal by theorem explained in § 13, page 5-59, the matrix is diagonisable.

$$\text{The diagonalising matrix is} \quad M=[X_1\ \ X_2\ \ X_3]=\begin{bmatrix}1 & 2 & 1\\ 1 & -1 & 0\\ 1 & 0 & -1\end{bmatrix}.$$

$$\text{The diagonlised matrix is } \begin{bmatrix}5 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Example 8 : Prove that the matrix A given below is diagonalisable matrix. Also find the diagonal matrix and the transforming matrix. **(M.U. 2016)**

$$A=\begin{bmatrix}1 & -6 & -4\\ 0 & 4 & -2\\ 0 & -6 & -3\end{bmatrix}.$$

Sol. : The characteristic equation of A is

$$\begin{vmatrix}1-\lambda & -6 & -4\\ 0 & 4-\lambda & 2\\ 0 & -6 & -3-\lambda\end{vmatrix}=0$$

$\therefore\ (1-\lambda)[(4-\lambda)(-3-\lambda)+12]+6(0)-4(0)=0$

$\therefore\ -(1-\lambda)[(4-\lambda)(3+\lambda)-12]=0$

$\therefore\ -(1-\lambda)(12+\lambda-\lambda^2-12)=0$

$\therefore\ (1-\lambda)(\lambda^2-\lambda)=0 \qquad \therefore\ (1-\lambda)(\lambda)(\lambda-1)=0$

$\therefore\ \lambda=0,1,1.$

(i) For $\lambda=0$, $[A-\lambda_1 I]X=O$ gives

$$\begin{bmatrix}1 & -6 & -4\\ 0 & 4 & 2\\ 0 & -6 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\text{By } \begin{matrix}R_3+(3/2)R_2\\(1/2)R_2\end{matrix}\begin{bmatrix}1 & -6 & -4\\ 0 & 2 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore\ x_1-6x_2-4x_3=0$ and $2x_2+x_3=0.$

Putting $x_2=-t$, we get $x_3=2t$ and $x_1=6x_2+4x_3=-6t+8t=2t.$

$$\therefore\ X_1=\begin{bmatrix}2t\\-t\\2t\end{bmatrix}=t\begin{bmatrix}2\\-1\\2\end{bmatrix}$$

$\therefore$ Corresponding to $\lambda=0$, we get the eigenvector $X_1=[2,-1,2]'$.

There are three variables and the rank is two, hence there is only one independent solution.

$\therefore$ For $\lambda=0$, algebraic multiplicity is 1 and geometric multiplicity is 1.

(ii) For $\lambda = 1$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \begin{matrix} R_2 + (1/2)R_1 \\ R_3 - R_1 \\ -(1/2)R_1 \end{matrix} \begin{bmatrix} 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

There are three variables and the rank of the matrix is one. Hence, there are $3 - 1 = 2$ independent solutions. *i.e.*, two independent parameters.

Now, we have $3x_2 + 2x_3 = 0$.

Since x_1 can take any value, let us put $x_1 = t$, where t is a parameter.

Now, putting $x_3 = -3s$, we get

$$3x_2 = -2x_3 = 6s \qquad \therefore \; x_2 = 2s \quad \text{where } s \text{ is another parameter.}$$

$$\therefore \; X = \begin{bmatrix} t + 0 \\ 0 + 2s \\ 0 - 3s \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \qquad \therefore \; X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

Now, the algebraic multiplicity of $\lambda = 1$ is 2 and geometric multiplicity is also 2.

Hence, the matrix A is diagonalisable.

Thus, A is diagonalised to $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the diagonalising matrix is $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 0 & -3 \end{bmatrix}$.

[**Verification :** We can verify that $M^{-1}AM = D$.

For this, we shall first find M^{-1}.

$$\text{Now,} \quad |M| = \begin{vmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 0 & -3 \end{vmatrix} = 2(0) - 1(3 - 4) + 0 = 1$$

Now, cofactors are

$$\begin{vmatrix} 0 & 2 \\ 0 & -3 \end{vmatrix} = 0, \; -\begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} = 1, \; \begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} = 0, \; -\begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} = 3,$$

$$\begin{vmatrix} 2 & 0 \\ 2 & -3 \end{vmatrix} = -6, \; -\begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} = 2, \; \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2, \; -\begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix} = -4, \; \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1$$

$$\text{Hence,} \quad M^{-1} = \begin{bmatrix} 0 & 3 & 2 \\ 1 & -6 & -4 \\ 0 & 2 & 1 \end{bmatrix}.$$

$$\text{Now,} \; M^{-1}AM = \begin{bmatrix} 0 & 3 & 2 \\ 1 & -6 & -4 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & -6 & -4 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

Example 9 : Determine the diagonal matrix unitarily similar to the Hermitian matrix $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$. Also obtain the transformation matrix.

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \; (1-\lambda)^2 + i^2 = 0 \quad \therefore \; 1 - 2\lambda + \lambda^2 - 1 = 0$$

$$\therefore \; \lambda^2 - 2\lambda = 0 \qquad \therefore \; \lambda = 0, 2.$$

$$\therefore \; D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

(i) For $\lambda = 0$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \therefore \; x_1 + ix_2 = 0 \text{ and } -ix_1 + x_2 = 0$$

Putting $x_2 = it$, we get $-ix_1 = -x_2 = -it$ $\quad \therefore \; x_1 = t$

$$\therefore \; X_1 = \begin{bmatrix} t \\ it \end{bmatrix} = t\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \therefore \; X_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

(ii) For $\lambda = 2$, $[A - \lambda_2 I]X = 0$ gives

$$\begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \therefore \; -x_1 + ix_2 = 0 \text{ and } ix_1 + x_2 = 0$$

Putting $x_2 = it$, we get $ix_1 = -it$ $\quad \therefore \; x_1 = -t$

$$\therefore \; X_2 = \begin{bmatrix} -t \\ it \end{bmatrix} = t\begin{bmatrix} -1 \\ i \end{bmatrix} \qquad \therefore \; X_2 = \begin{bmatrix} -1 \\ i \end{bmatrix}$$

$$\therefore \; M = \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix}$$

(You can verify that $M^{-1}AM = D$.)

Example 10 : If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1/2 & 2 \end{bmatrix}$, prove that both A and B are **not** diagonalisable but AB is diagonalisable. **(M.U. 2000)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = 0 \qquad \therefore \; (1-\lambda)^2 = 0 \quad \therefore \; \lambda = 1, 1.$$

For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix 1 and the number of variables is 2. Hence, there is only one solution.

...ar to the Hermitian matrix

Now, the algebraic multiplicity of the eigenvalue 1 is 2 but its geometric multiplicity is 1. Hence, the matrix A is not diagonalisable.

The characteristic equation of B is

$$\begin{vmatrix} 2-\lambda & 0 \\ 1/2 & 2-\lambda \end{vmatrix} = 0 \quad \therefore (2-\lambda)^2 = 0 \quad \therefore \lambda = 2, 2$$

For $\lambda = 2$, $[A - \lambda_2 I]X = O$ gives $\begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The rank of the coefficient matrix is 1 and the number of variables is 2. Hence, there is only one solution.

Now, the algebraic multiplicity of the eigenvalue 2 is 2 and its geometric multiplicity is one. Hence, the matrix B is not diagonalisable.

Now, $C = AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1/2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1/2 & 2 \end{bmatrix}$

The characteristic equation of C is

$$\begin{vmatrix} 3-\lambda & 4 \\ 1/2 & 2-\lambda \end{vmatrix} = 0$$

$\therefore (3-\lambda)(2-\lambda) - 2 = 0 \quad \therefore 4 - 5\lambda + \lambda^2 = 0$

$\therefore (\lambda - 4)(\lambda - 1) = 0 \quad \therefore \lambda = 1, 4.$

(i) For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 2 & 4 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{By } R_2 - \frac{1}{4}R_1 \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ The rank of the coefficient matrix is 1. $\quad \therefore 2x_1 + 4x_2 = 0$

There is only one solution. Hence, the algebraic multiplicity and geometric multiplicity of eigenvalue 1 are equal.

(ii) For $\lambda = 4$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -1 & 4 \\ 1/2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{By } R_2 + \frac{1}{2}R_1 \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The rank of the matrix is 1. $\quad \therefore -x_1 + 4x_2 = 0.$

There is only one solution. Hence, the algebraic multiplicity and geometric multiplicity of eigenvalue 4 are equal.

Hence, the matrix $C = AB$ is diagonalisable.

Example 11 : Find the symmetric matrix $A_{3\times 3}$ having the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = 15$, with the corresponding eigenspaces $X_1 = [1, 2, 2]'$, $X_2 = [-2, -1, 2]'$ and X_3.

(M.U. 2004)

Sol. : Let $X_3 = [x_1, x_2, x_3]'$ be the third eigenvector corresponding to the eigenvalue 15.

Since the required matrix A is symmetric and all eigenvalues are distinct the three eigenvectors corresponding to the three eigenvalues are **orthogonal**.

(See Theorem 1, page 5-36)

$\therefore X_3$ is orthogonal to X_1 and X_2. $\quad$ (i.e., $u_1v_1 + u_2v_2 + u_3v_3 = 0$)

$\therefore x_1 + 2x_2 + 2x_3 = 0 \; ; \quad -2x_1 - x_2 + 2x_3 = 0$

$$\therefore \frac{x_1}{\begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix}}$$

$\therefore \dfrac{x_1}{6} = \dfrac{x_2}{-6} = \dfrac{x_3}{3} \quad \therefore \dfrac{x_1}{2} = \dfrac{x_2}{-2} = \dfrac{x_3}{1} = t$

$\therefore x_1 = 2t, \; x_2 = -2t, \; x_3 = t.$

$$\therefore X_3 = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad \text{or} \quad X_3 = [2, -2, 1]'$$

Since A is symmetric it is orthogonally similar to a diagonal matrix D. There exists an orthogonal matrix P such that

$P^{-1}AP = D \quad$ i.e. $\quad A = PDP^{-1} = PDP'$

Since P is an orthogonal matrix, we divide each vector by its norm. Now, the norm of each vector is $\sqrt{1+4+4} = 3$.

$$\text{Hence,} \quad P = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

(Since P is orthogonal $P^{-1} = P'$)

$$A = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 0 & -2 & 10 \\ 0 & -1 & -10 \\ 0 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

(See now Ex. 1, page 5-14)

Note ....

We can obtain A from $M = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$ using $M^{-1}AM = D$ i.e., $A = MAM^{-1}$ as in Ex. 2, page 5-62. But it is easier to obtain A by normalising vectors as above because P^{-1} is P' if P is an orthogonal matrix.

EXERCISE - III

1. Diagonalise the matrix $\begin{bmatrix} -3 & 2+2i \\ 2-2i & 4 \end{bmatrix}$. **(M.U. 2010) [Ans. :** $D = \begin{bmatrix} 5 & 0 \\ 0 & -4 \end{bmatrix}$

2. Diagonalise the matrix $\begin{bmatrix} 4 & 1+i \\ 1-i & 5 \end{bmatrix}$. [Ans. : $D = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$]

3. Diagonalise the Hermitian matrix $A = \begin{bmatrix} -3 & 2+2i \\ 2-2i & 4 \end{bmatrix}$. (M.U. 2003, 10)

[Ans. : Characteristic roots are 5, – 4. $\therefore$ $D = \begin{bmatrix} 5 & 0 \\ 0 & -4 \end{bmatrix}$]

4. Diagonalise the Hermitian matrix $A = \begin{bmatrix} 4 & 1+i \\ 1-i & 5 \end{bmatrix}$ [Ans. : $D = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$]

5. Determine the diagonal matrix unitarily similar to the Hermitian matrix A. Find also the transformation matrix.

$$A = \begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$$

[Ans. : Eigenvalues are 3, – 3. Corresponding eigenvectors are

$$X_1 = \begin{bmatrix} 5 \\ 1+2i \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1-2i \\ 5 \end{bmatrix}$$

Lengths of the vectors X_1 and X_2 are each $\sqrt{30}$.

$$M = \left[\frac{1}{\sqrt{30}} X_1, \ -\frac{1}{\sqrt{30}} X_2 \right]$$]

6. Reduce the following matrix to diagonal form

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

(M.U. 1998, 2003) [Ans. : $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$]

7. Reduce the following matrix to diagonal form

$$A = \begin{bmatrix} 9 & -1 & 9 \\ 3 & -1 & 3 \\ -7 & 1 & -7 \end{bmatrix}$$

(M.U. 2010) [Ans. : $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$]

8. Reduce the following matrix to diagonal form

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

(M.U. 2006) [Ans. : $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$]

9. Transform the following matrix into diagonal form

$$A = \begin{bmatrix} 8 & -12 & 5 \\ 15 & -25 & 11 \\ 24 & -42 & 19 \end{bmatrix}$$

(M.U. 2003) [Ans. : $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$]

10. Show that the following matrix is diagonalisable. Also find the diagonal form and a diagonalising matrix. (M.U. 2003, 15)

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

[Ans. : $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, $M = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}$]

11. Show that the following matrices are similar to diagonal matrices. Find the diagonal form and the diagonalising matrix.

(i) $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ (M.U. 2009, 17) (ii) $\begin{bmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & 2 \end{bmatrix}$ (M.U. 2009) (iii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (M.U. 2004, 05)

[Ans. : (i) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$; (ii) $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $M = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

(iii) $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, $M = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$]

12. Show that the following matrices are not similar to diagonal matrices.

(i) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$ (M.U. 2011) (iii) $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (M.U. 2003)

13. Is the following matrix diagonisable ? Justify your answer.

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$ (M.U. 2004)

[Ans. : Not diagonisable. The characteristic equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$.
The roots 1, 1, 4 are not distinct and geometric multiplicity of $\lambda = 1$ is 1.]

14. Find the symmetric matrix $A_{3\times3}$ having the eigenvalues $\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = 9$ with the corresponding eigenvectors $X_1 = [1, 2, 2]'$, $X_2 = [-2, 2, -1]'$ and X_3.

[Ans. : $\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$]

15. Find a square symmetric matrix of order 3 whose eigenvalues are 3, –3, 9 with corresponding eigenvectors [2, 2, –1]', [2, –1, 2]' and X_3.

[Ans. : $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$]

16. Functions of A Square Matrix

If A is a non-singular square matrix with distinct eigenvalues, we can find the modal matrix M as explained in § 14. We can also find the diagonal matrix D as seen earlier.

We know how to find A^2 from A by the process of matrix multiplication. From A^2, we can find A^3, and also A^4. In fact we can find any positive integral power A by this process. However, this method becomes obviously tedious if we want a large power A. For obtaining a large power we use the method explained below in (a).

The method is quite general and can also be used to find other functions of A such e^A, 4^A etc. as explained in the examples which follow.

(a) Calculation of the powers of a matrix

If A is a non-singular square matrix with distinct eigenvalues then we can find any power of A *i.e.* A^k (k is a positive integer) by the process explained below.

As seen in § 14 on page 5-59, we have $M^{-1}AM = D$.

Operating by M on the left and by M^{-1} on the right,

$$MM^{-1}AMM^{-1} = MDM^{-1}$$

$$\therefore \quad (MM^{-1})A(MM^{-1}) = MDM^{-1}$$

$$\therefore \quad A = MDM^{-1}$$

$$\therefore \quad A^n = (MDM^{-1})(MDM^{-1})\ldots(MDM^{-1})\ldots(n \text{ times})$$

$$\therefore \quad A^n = MD(M^{-1}M)D(M^{-1}M)\ldots(M^{-1}M)DM^{-1}$$

$$= MD\ldots DM^{-1}$$

$$= MD^nM^{-1}$$

$$= M\begin{bmatrix} \lambda_1^n & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2^n & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda^n \end{bmatrix}M^{-1}$$

Example 1 : If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, find A^{50}. **(M.U. 2000, 04, 15, 17)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (2-\lambda)^2 - 1 = 0 \qquad \therefore \quad 4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\therefore \quad \lambda^2 - 4\lambda + 3 = 0 \qquad \therefore \quad (\lambda - 3)(\lambda - 1) = 0 \qquad \therefore \quad \lambda = 1, 3.$$

(i) For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{By } R_2 - R_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad x_1 + x_2 = 0$$

Putting $x_2 = -t$, $x_1 = t$. $\qquad \therefore \quad X_1 = \begin{bmatrix} t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Hence, corresponding to $\lambda = 1$, the eigenvector is $[1, -1]'$.

(ii) For $\lambda = 3$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{By } R_2 + R_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\therefore \quad -x_1 + x_2 = 0 \text{ and } x_1 = x_2$$

Putting $x_2 = t$, $x_1 = t$. $\qquad \therefore \quad X_2 = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence, corresponding to $\lambda = 3$, the eigenvector is $[1, 1]'$.

$\therefore$ Modal matrix $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Now, $|M| = 2 \quad \therefore \quad M^{-1} = \dfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

[If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.]

Now, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad \therefore \quad D^{50} = \begin{bmatrix} 1^{50} & 0 \\ 0 & 3^{50} \end{bmatrix}$

$$\therefore \quad A^{50} = MD^{50}M^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1^{50} & 0 \\ 0 & 3^{50} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 3^{50} \\ -1 & 3^{50} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$$

Example 2 : Find e^A and 4^A if $A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$. **(M.U. 1997, 2001, 03, 05, 06, 15)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} (3/2-\lambda) & 1/2 \\ 1/2 & (3/2-\lambda) \end{vmatrix} = 0$$

$$\therefore \quad \left(\frac{3}{2} - \lambda\right)^2 - \frac{1}{4} = 0 \qquad \therefore \quad \frac{9}{4} - 3\lambda + \lambda^2 - \frac{1}{4} = 0$$

$$\therefore \quad \lambda^2 - 3\lambda + 2 = 0 \qquad \therefore \quad (\lambda - 1)(\lambda - 2) = 0 \qquad \therefore \quad \lambda = 1, 2.$$

(i) For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } \begin{matrix} 2R_1 \\ 2R_2 \end{matrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{By } R_2 - R_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \therefore \quad x_1 + x_2 = 0$$

Putting $x_2 = -t$, we get $x_1 = t$. $\therefore X_1 = \begin{bmatrix} t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Hence, the eigenvector is $[1, -1]'$.

(ii) For $\lambda = 2$, $[A - \lambda_2 I] X = O$ gives

$$\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } \begin{matrix} 2R_1 \\ 2R_2 \end{matrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $R_2 + R_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \therefore -x_1 + x_2 = 0 \quad \therefore x_1 = x_2$

Putting $x_2 = t$, $x_1 = t$, we get $\quad \therefore X_2 = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence, the eigenvector is $[1, 1]'$.

$\therefore M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \therefore |M| = 2 \qquad \therefore M^{-1} = \dfrac{\text{adj.} M}{|M|} = \dfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Now $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

If $f(A) = e^A$, $f(D) = e^D = \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}$. $\quad$ If $f(A) = 4^A$, $f(D) = 4^D = \begin{bmatrix} 4^1 & 0 \\ 0 & 4^2 \end{bmatrix}$

$\therefore e^A = M f(D) M^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e & e^2 \\ -e & e^2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e + e^2 & -e + e^2 \\ -e + e^2 & e + e^2 \end{bmatrix}$$

Similarly, replacing e by 4 we get,

$$4^A = \frac{1}{2}\begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

Example 3 : If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ then prove that $3 \tan A = A \tan 3$. **(M.U. 2000, 05, 06)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$\therefore -(1+\lambda)(1-\lambda) - 8 = 0 \qquad \therefore \lambda^2 - 9 = 0 \qquad \therefore \lambda = 3, -3.$

(i) For $\lambda = 3$, $[A - \lambda_1 I] X = O$ gives

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 + \frac{1}{2}R_1 \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore -4x_1 + 4x_2 = 0 \qquad \therefore x_1 - x_2 = 0$

Putting $x_2 = t$, we get $x_1 = t$. $\qquad \therefore X_1 = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ The eigenvector is $[1, 1]'$.

(i) For $\lambda = -3$, $[A - \lambda_2 I] X = O$ gives

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 - R_1 \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore 2x_1 + 4x_2 = 0$ i.e. $x_1 + 2x_2 = 0$

Putting $x_2 = -t$, we get $x_1 = -2x_2 = 2t$.

$\therefore X_2 = \begin{bmatrix} 2t \\ -t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\therefore$ The eigenvector is $[2, -1]'$.

$\therefore M = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ and $|M| = -3 \qquad \therefore M^{-1} = \dfrac{\text{adj. } M}{|M|} = -\dfrac{1}{3}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$

Now, $D = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$

$\because f(A) = \tan A, \quad f(D) = \begin{bmatrix} \tan 3 & 0 \\ 0 & \tan(-3) \end{bmatrix} = \begin{bmatrix} \tan 3 & 0 \\ 0 & -\tan 3 \end{bmatrix}$

$\therefore \tan A = M f(D) M^{-1}$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \tan 3 & 0 \\ 0 & \tan(-3) \end{bmatrix}\left(-\frac{1}{3}\right)\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} \tan 3 & -2\tan 3 \\ \tan 3 & -\tan 3 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} \tan 3 & -4\tan 3 \\ -2\tan 3 & -\tan 3 \end{bmatrix}$$

$$\therefore 3\tan A = \begin{bmatrix} -\tan 3 & 4\tan 3 \\ 2\tan 3 & \tan 3 \end{bmatrix} = \tan 3\begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} = \tan 3 \cdot A$$

$= A \tan 3.$

Example 4 : If $A = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix}$, prove that $e^A = e^\alpha\begin{bmatrix} \cos h\alpha & \sin h\alpha \\ \sin h\alpha & \cos h\alpha \end{bmatrix}$. **(M.U. 2006)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} \alpha - \lambda & \alpha \\ \alpha & \alpha - \lambda \end{vmatrix} = 0$$

$\therefore (\alpha - \lambda)^2 - \alpha^2 = 0 \qquad \therefore \lambda^2 - 2\alpha\lambda = 0$

$\therefore \lambda(\lambda - 2\alpha) = 0 \qquad \therefore \lambda = 0, \ \lambda = 2\alpha.$

(i) For $\lambda = 0$, $[A - \lambda_1 I] X = O$ gives

$$\begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 - R_1 \begin{bmatrix} \alpha & \alpha \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \alpha x_1 + \alpha x_2 = 0 \qquad \therefore x_1 + x_2 = 0$

Putting $x_2 = -t,\ x_1 = t.$ $\quad \therefore\ X_1 = \begin{bmatrix} t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore$ The eigenvector is $[1, -1]'$.

(ii) For $\lambda = 2\alpha$, $[A - \lambda_2 I]\,X = O$ gives

$$\begin{bmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 + R_1 \begin{bmatrix} -\alpha & \alpha \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore\ -\alpha x_1 + \alpha x_2 = 0 \qquad \therefore\ x_1 - x_2 = 0$

Putting $x_2 = t,\ x_1 = t.$ $\quad \therefore\ X_1 = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ The eigenvector is $[1, 1]'$.

$$\therefore\ M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \therefore\ |M| = 2 \quad \therefore\ M^{-1} = \frac{\text{adj. } M}{|M|} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Now, $D = \begin{bmatrix} 0 & 0 \\ 0 & 2\alpha \end{bmatrix}$

If $f(A) = e^A$, $f(D) = e^D = \begin{bmatrix} e^0 & 0 \\ 0 & e^{2\alpha} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\alpha} \end{bmatrix}$

$$\therefore\ e^A = M\,f(D)\,M^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{2\alpha} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ e^{2\alpha} & e^{2\alpha} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 + e^{2\alpha} & -1 + e^{2\alpha} \\ -1 + e^{2\alpha} & 1 + e^{2\alpha} \end{bmatrix}$$

Dividing the matrix by e^{α}.

$$e^A = e^{\alpha}\begin{bmatrix} \dfrac{e^{\alpha} + e^{-\alpha}}{2} & \dfrac{e^{\alpha} - e^{-\alpha}}{2} \\ \dfrac{e^{\alpha} - e^{-\alpha}}{2} & \dfrac{e^{\alpha} + e^{-\alpha}}{2} \end{bmatrix} = e^{\alpha}\begin{bmatrix} \cos h\alpha & \sin h\alpha \\ \sin h\alpha & \cos h\alpha \end{bmatrix}$$

Example 5 : If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, find e^{At}. **(M.U. 1999)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 0 - \lambda & 1 \\ -1 & 0 - \lambda \end{vmatrix} = 0 \quad \therefore\ \lambda^2 + 1 = 0 \quad \therefore\ \lambda = i, -i.$$

(i) For $\lambda = i$, $[A - \lambda_1 I]\,X = O$ gives

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore\ -i x_1 + x_2 = 0$ and $x_1 + i x_2 = 0$

Putting $x_2 = it$, we get $x_1 = -ix_2 = -i^2 t = t$.

$\therefore\ X_1 = \begin{bmatrix} t \\ it \end{bmatrix} = t\begin{bmatrix} 1 \\ i \end{bmatrix}$ $\quad \therefore$ The eigenvector is $[1, i]'$.

(ii) For $\lambda = -i$, $[A - \lambda_2 I]\,X = O$ gives

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \therefore\ i x_1 + x_2 = 0 \text{ and } -x_1 + i x_2 = 0$$

Putting $x_2 = -it$, we get $-x_1 = -ix_2 = i^2 t = -t$ $\quad \therefore\ x_1 = t$

$\therefore\ X_2 = \begin{bmatrix} t \\ -it \end{bmatrix} = t\begin{bmatrix} 1 \\ -i \end{bmatrix}$ $\quad \therefore$ The eigenvector is $[1, -i]'$.

$\therefore$ Modal matrix $M = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ $\quad \therefore\ |M| = -2i$

$$\therefore\ M^{-1} = -\frac{1}{2i}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix}$$

Now, $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$\therefore$ If $f(A) = e^{At}$, $f(D) = e^D = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$

$$e^{At} = M\,f(D)\,M^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}\frac{1}{2i}\begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{bmatrix}\begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix}$$

$$= \frac{1}{2i}\begin{bmatrix} i(e^{it} + e^{-it}) & e^{it} - e^{-it} \\ -e^{it} - e^{-it} & i(e^{it} + e^{-it}) \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

(b) Another method

The method used in the above examples is useful **if the matrix A is diagonisable *i.e.*, if the eigenvalues are distinct.** If the diagonal matrix of A cannot be found out, the above method cannot be used. In such cases we use the method stated below.

If the given matrix is of order 3, we assume that the required function $\Phi(A)$ can be written as

$$\Phi(A) = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I \quad \text{where, } \alpha_0, \alpha_1, \alpha_2 \text{ are constants to be determined.}$$

If the given matrix is of order 2, we assume that the required function

$$\Phi(A) = \alpha_1 A + \alpha_0 I \quad \text{where, } \alpha_0, \alpha_1 \text{ are constants to be determined.}$$

We then find the constants $\alpha_0, \alpha_1, \alpha_2$ as illustrated in Ex. 10 and 11.

Remark ....

The method is quite general when the matrix is diagonalisable or not *i.e.* when the eigenvalues are distinct or not distinct as is illustrated in Ex. 11 below.

Example 6 : If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, prove that $A^{50} - A^{49} = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$.

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$\therefore (1-\lambda)(3-\lambda) - 8 = 0 \quad \therefore 3 - 4\lambda + \lambda^2 - 8 = 0$

$\therefore \lambda^2 - 4\lambda - 5 = 0 \quad \therefore (\lambda - 5)(\lambda + 1) = 0 \quad \therefore \lambda = -1, 5.$

(i) For $\lambda = 5$, $[A - \lambda I] X = O$ gives

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_2 + (1/2)R_1 \\ -(1/4)R_1 \end{matrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore x_1 - x_2 = 0.$ Putting $x_2 = t$, we get $x_1 = t$.

$\therefore X = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \therefore X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(ii) For $\lambda = -1$, $[A - \lambda I] X = O$ gives

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_2 - R_1 \\ (1/2)R_1 \end{matrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore x_1 + 2x_2 = 0.$ Putting $x_2 = -t$, we get $x_1 = 2t$.

$\therefore X = \begin{bmatrix} 2t \\ -t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \therefore X_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Now, $D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ and $M = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

$\therefore |M| = -3 \quad \therefore M^{-1} = -\frac{1}{3}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

$\therefore D^{49} = \begin{bmatrix} 5^{49} & 0 \\ 0 & -1 \end{bmatrix} \quad \left[\because \text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]$

$\therefore A^{49} = M D^{49} M^{-1}$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5^{49} & 0 \\ 0 & -1 \end{bmatrix}\left(\frac{1}{3}\right)\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 5^{49} & -2 \\ 5^{49} & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5^{49} - 2 & 2\cdot 5^{49} + 2 \\ 5^{49} + 1 & 2 \cdot 5^{49} - 1 \end{bmatrix}$$

$\therefore A^{50} - 5A^{49} = A^{49}[A - 5I]$

$$= A^{49}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right\} = A^{49}\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$$

$$\therefore A^{50} - 5A^{49} = \frac{1}{3}\begin{bmatrix} 5^{49} - 2 & 2\cdot 5^{49} + 2 \\ 5^{49} + 1 & 2\cdot 5^{49} - 1 \end{bmatrix}\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 12 & -12 \\ -6 & 6 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$$

Example 7 : If $A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$, prove that $A^{50} = \begin{bmatrix} -149 & -150 \\ 150 & 151 \end{bmatrix}$.

(M.U. 1999, 2004, 16)

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 3 \\ -3 & -4-\lambda \end{vmatrix} = 0$$

$\therefore (2 - \lambda)(-4 - \lambda) + 9 = 0 \quad \therefore -8 - 2\lambda + 4\lambda + \lambda^2 + 9 = 0$

$\therefore \lambda^2 + 2\lambda + 1 = 0 \quad \therefore (\lambda + 1)^2 = 0 \quad \therefore \lambda = -1, -1.$

The eigenvalues are repeated. Hence, we use the second method.

Let $\Phi(A) = A^{50} = \alpha_1 A + \alpha_0 I$

We assume that the above equality is satisfied by the characteristic roots of A.

$\therefore \lambda^{50} = \alpha_1 \lambda + \alpha_0$ .... (1)

Putting $\lambda = -1$, we get $(-1)^{50} = \alpha_1(-1) + \alpha_0$

$\therefore 1 = -\alpha_1 + \alpha_0$ .... (2)

Since the characteristic roots are repeated to obtain another equation we differentiate (1) w.r.t. λ.

Now, differentiating (1), we get $\quad 50\lambda^{49} = \alpha_1$

Putting $\lambda = -1$, again, we get $\quad 50(-1)^{49} = \alpha_1 \quad \therefore \alpha_1 = -50.$

Putting $\alpha_1 = -50$ in (2), we get $\quad 1 = 50 + \alpha_0 \quad \therefore \alpha_0 = -49.$

From $A^{50} = \alpha_1 A + \alpha_0 I$, we get

$$A^{50} = -50\begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} - 49\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -149 & 150 \\ 150 & 151 \end{bmatrix}$$

Example 8 : If $A = \begin{bmatrix} \pi & \pi/4 \\ 0 & \pi/2 \end{bmatrix}$, find $\cos A$.

(M.U. 2003, 05, 10)

Sol. : The characteristic equation is

$$\begin{vmatrix} \pi - \lambda & \pi/4 \\ 0 & (\pi/2) - \lambda \end{vmatrix} = 0 \quad \therefore (\pi - \lambda)\left(\frac{\pi}{2} - \lambda\right) = 0 \quad \therefore \lambda = \frac{\pi}{2}, \pi.$$

Let $\Phi(A) = \cos A = \alpha_1 A + \alpha_0 I$ .... (1)

Since λ satisfies the above equation, we have

$\cos\lambda = \alpha_1\lambda + \alpha_0$ .... (2)

Putting $\lambda = \pi/2$, we get

$\cos\frac{\pi}{2} = \alpha_1\cdot\frac{\pi}{2} + \alpha_0 \quad \therefore 0 = \alpha_1\cdot\frac{\pi}{2} + \alpha_0$ .... (3)

$\cos\pi = \alpha_1\cdot\pi + \alpha_0 \quad \therefore -1 = \alpha_1\cdot\pi + \alpha_0$ .... (4)

From (3) and (4), we get

$$\alpha_1 \cdot \frac{\pi}{2} = -1 \quad \therefore \alpha_1 = -\frac{2}{\pi} \quad \therefore \alpha_0 = -1+2 = 1.$$

Putting these values in (1), we get

$$\cos A = -\frac{2}{\pi}\begin{bmatrix} \pi & \pi/4 \\ 0 & \pi/2 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

Example 9 : Show that $\cos O_{3\times3} = I_{3\times3}$. **(M.U. 2004)**

Sol. : We have $O_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Clearly eigenvalues of $O_{3\times3}$ are 0, 0 and 0.

Let $\cos O_{3\times3} = \alpha_2 O_{3\times3} + \alpha_1 O_{3\times3} + \alpha_0 I_3$ .... (1)

λ satisfies this equation. $\therefore \cos\lambda = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ .... (2)

Differentiating (2), w.r.t. λ, $-\sin\lambda = 2\alpha_2\lambda + \alpha_1$ .... (3)

Again differentiating w.r.t. λ, $-\cos\lambda = 2\alpha_2$ .... (4)

Putting $\lambda = 0$ in (2), (3) and (4), $1 = \alpha_0$, $0 = \alpha_1$, $-\frac{1}{2} = \alpha_2$

Hence, from (1), we get $\cos O_{3\times3} = -\frac{1}{2}O_{3\times3}^2 + I_3 = I_3$

Example 10 : If $A = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$, find A^n. **(M.U. 1998, 2003)**

Sol. : Since the eigenvalues are distinct, the matrix A is diagonalisable and hence, the method (b), page 5-83 can be used.

The characteristic roots of A are 4 and 9.

Since, the matrix is of order 2, we consider

$\Phi(A) = A^n = \alpha_1 A + \alpha_0 I$ .... (1)

We assume that the above equality is satisfied by the characteristic roots.

$\therefore 4^n = 4\alpha_1 + \alpha_0$ .... (2)

and $9^n = 9\alpha_1 + \alpha_0$ .... (3)

By subtraction, we get

$$9^n - \alpha^n = 5\alpha_1 \quad \therefore \alpha_1 = \frac{9^n - 4^n}{5}$$

Putting this value in (2) or (3) and after simplication, we get,

$$\frac{9\cdot4^n - 4\cdot9^n}{5} = \alpha_0$$

Hence, putting the value of α_1 and α_0 in (1), we get

$$A^n = \frac{9^n - 4^n}{5}\begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix} + \frac{9\cdot4^n - 4\cdot9^n}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Aliter : Now, for $\lambda = 4$, $[A - \lambda I] = O$ gives

$$\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \therefore x_1 + x_2 = 0.$$

Let $x_2 = -1 \therefore x_1 = 1$. $\therefore$ The eigenvector $= [1, -1]'$.

For $\lambda = 9$, $[A - \lambda I] = O$ gives

$$\begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore -2x_1 + 3x_2 = 0$. Let $x_2 = 2$ $\therefore x_1 = 3$

$\therefore$ The eigenvector $= [3, 2]'$.

$$\therefore M = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \quad \therefore |M| = 5 \quad \therefore M^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.$$

And $A^n = MD^nM^{-1}$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4^n & 0 \\ 0 & 9^n \end{bmatrix}\cdot\frac{1}{5}\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4^n & 3\cdot9^n \\ -4^n & 2\cdot9^n \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 4^n\cdot2 + 3\cdot9^n & -4^n\cdot3 + 3\cdot9^n \\ -4^n\cdot2 + 2\cdot9^n & 4^n\cdot3 + 2\cdot9^n \end{bmatrix}$$

Example 11 : If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find A^{50}. **(M.U. 2000, 09)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$\therefore (1-\lambda)[(-\lambda)(-\lambda) - 1] = 0 \quad \therefore (1-\lambda)(\lambda^2 - 1) = 0 \quad \therefore \lambda = 1, 1, -1.$

As the eigenvalue 1 is repeated, we use the second method.

Since, the matrix is of order 3, we consider

$\Phi(A) = A^{50} = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$ .... (1)

λ satisfies this equation. $\therefore \lambda^{50} = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ .... (2)

Putting $\lambda = 1$, $\lambda = -1$, we get $1 = \alpha_2 + \alpha_1 + \alpha_0$ .... (3)

$1 = \alpha_2 - \alpha_1 + \alpha_0$ .... (4)

Differentiating (2), w.r.t. λ, we get

$$50\lambda^{49} = 2\alpha_2\lambda + \alpha_1$$

Putting $\lambda = 1$, we get $50 = 2\alpha_2 + \alpha_1$ .... (5)

Solving (iii), (iv) and (v), we get

$\alpha_2 = 25$, $\alpha_1 = 0$ and $\alpha_0 = -24$.

Putting these values in (1), we get $A^{50} = 25A^2 - 24I$.

But $A^2 = \begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix}$

$\therefore\ A^{50} = 25\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix} - 24\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\25&1&0\\25&0&1\end{bmatrix}$ (See also Ex. 9, page 5-49)

Example 12 : If $A = \begin{bmatrix}1&2&-2\\0&2&1\\0&0&-1\end{bmatrix}$, find A^{100}. **(M.U. 2003)**

Sol. : The characteristic equation is

$$\begin{vmatrix}1-\lambda&2&-2\\0&2-\lambda&1\\0&0&-1-\lambda\end{vmatrix} = 0$$

$\therefore\ (1-\lambda)(2-\lambda)(-1-\lambda) = 0 \qquad \therefore\ \lambda = 1, -1, 2.$

Since the matrix is of order 3, we consider

$\Phi(A) = A^{100} = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$ .... (1)

λ satisfies this equation. $\therefore\ \lambda^{100} = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ .... (2)

Putting $\lambda = 1, -1$, $\quad 1 = \alpha_2 + \alpha_1 + \alpha_0$ .... (3)

$1 = \alpha_2 - \alpha_1 + \alpha_0$ .... (4)

Differentiating (2), w.r.t. λ, we get, $100\lambda^{99} = 2\alpha_2\lambda + \alpha_1$

Putting $\lambda = 1$, we get $100 = 2\alpha_2 + \alpha_1$ .... (5)

Solving (3), (4) and (5), we get $\alpha_2 = 50,\ \alpha_1 = 0,\ \alpha_0 = -49.$

Putting these values in (1), $A^{100} = 50A^2 - 49I$

But $A^2 = \begin{bmatrix}1&2&-2\\0&2&1\\0&0&-1\end{bmatrix}\begin{bmatrix}1&2&-2\\0&2&1\\0&0&-1\end{bmatrix} = \begin{bmatrix}1&6&2\\0&4&1\\0&0&1\end{bmatrix}$

$\therefore\ A^{100} = \begin{bmatrix}50&300&100\\0&200&50\\0&0&50\end{bmatrix} + \begin{bmatrix}-49&0&0\\0&-49&0\\0&0&-49\end{bmatrix} = \begin{bmatrix}1&300&100\\0&151&50\\0&0&1\end{bmatrix}$

EXERCISE - V

1. If $A = \begin{bmatrix}3&-4\\2&-3\end{bmatrix}$, prove that $A^{2n+1} = A$.

2. If $A = \begin{bmatrix}2&3\\-3&-4\end{bmatrix}$, find A^{100}. **(M.U. 2001, 03)** [**Ans. :** $\begin{bmatrix}-299&-300\\300&301\end{bmatrix}$]

3. If $A = \begin{bmatrix}\pi/2&\pi\\0&3\pi/2\end{bmatrix}$, find sin A. [**Ans. :** $\begin{bmatrix}1&-2\\0&-1\end{bmatrix}$]

4. If $A = \begin{bmatrix}3/2&1/2\\1/2&3/2\end{bmatrix}$, find A^{20}. [**Ans. :** $\frac{1}{2}\begin{bmatrix}1+2^{20}&-1+2^{20}\\-1+2^{20}&1+2^{20}\end{bmatrix}$]

5. If $A = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$, find $e^{A\pi/2}$. [**Ans. :** $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$]

6. If $A = \begin{bmatrix}3&-4\\1&-1\end{bmatrix}$, prove that $A^{100} = \begin{bmatrix}201&-400\\100&-199\end{bmatrix}$. **(M.U. 2003, 04)**

7. If $A = \begin{bmatrix}5&3\\1&3\end{bmatrix}$, find A^n in terms of A and also find A^4. **(M.U. 2006)**

[**Ans. :** $A^n = \frac{1}{4}\begin{bmatrix}2^n+3\cdot6^n&-3\cdot2^n+3\cdot6^n\\-2^n+6^n&3\cdot2^n+6^n\end{bmatrix}$

or $A^n = \frac{6^n-2^n}{4}\begin{bmatrix}5&3\\1&3\end{bmatrix} + \frac{3\cdot2^n-6^n}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix};\quad A^4 = \begin{bmatrix}976&960\\320&336\end{bmatrix}$]

8. If $A = \begin{bmatrix}4&3\\7&8\end{bmatrix}$, find A^{100}. [**Ans. :** $A^{100} = \frac{1}{10}\begin{bmatrix}7+3\cdot11^{100}&-3+3\cdot11^{100}\\-7+7\cdot11^{100}&3+7\cdot11^{100}\end{bmatrix}$]

(M.U. 1998)

9. If $A = \begin{bmatrix}3&1\\1&3\end{bmatrix}$, find e^A, 5^A. **(M.U. 2009, 17)**

[**Ans. :** $e^A = \frac{1}{2}\begin{bmatrix}e^2+e^4&-e^2+e^4\\-e^2+e^4&e^2+e^4\end{bmatrix};\quad 5^A = \begin{bmatrix}325&300\\300&325\end{bmatrix}$]

10. If $A = \begin{bmatrix}1&0\\0&2\end{bmatrix}$, find e^A. **(M.U. 2000)** [**Ans. :** $\begin{bmatrix}e&0\\0&e^2\end{bmatrix}$]

17. Minimal Polynomial and Minimal Equation of A Matrix

Let $f(x)$ be a polynomial in x and A be a square matrix of order n. If $f(A) = 0$ then we say that $f(x)$ **annihilates** the matrix A. We know that by Caley-Hamilton theorem every matrix satisfies its characteristic equation. Hence, the characteristic polynomial of the matrix A annihilates A.

For example, if $A = \begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}$, then the characteristic polynomial is $|A - xI|$

i.e., $\quad f(x) = x^3 - 5x^2 + 9x - 1$ [See Ex. 1 of Exercise - II, page 5-51]

But by Cayley-Hamilton theorem $\quad f(A) = A^3 - 5A^2 + 9A - I = 0.$

Hence, $f(x) = x^3 - 5x^2 + 9x - 1$ annihilates the matrix

$$A = \begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}$$

(a) Monic polynomial

A polynomial in x in which the coefficient of the highest power of x is unity is called a **monic polynomial**. Thus, $x^3 - 2x^2 + 3x - 7$ is a monic polynomial while $2x^3 - 3x^2 + 4x - 9$ is not a monic polynomial.

(b) Minimal polynomial of a matrix

The monic polynomial of lowest degree that annihilates a matrix A is called **minimal polynomial** of A. Further, if $f(x)$ is the minimal polynomial of A then the equation $f(x) = 0$ is called the **minimal equation** of the matrix A.

If a matrix is of order n then its characteristic polynomial is of degree n. We know that the characteristic polynomial of A annihilates A.

Hence, the degree of minimal polynomial of A cannot be greater than n.

18. Derogatory and Non-derogatory Matrices

An n-rowed square matrix is said to be **derogatory** or **non-derogatory** according as the degree of its minimal equation is less than or equal to n.

Alternatively we may define the terms derogatory and no-derogatory as follows.

Derogatory Matrix : If the degree of the minimal equation of a square matrix of order n **is less than n** then it is called **derogatory**.

Non-derogatory Matrix : If the degree of the minimal equation a square matrix of order n **is equal to n** then it is called **non-derogatory**.

Since by Cayley-Hamilton theorem the characteristic equation is satisfied by the matrix, every square matrix is either derogatory or non-derogatory.

Note ....

1. Each eigenvalue of a square matrix is a root of the minimal equation.
2. If all eigenvalues are distinct then the matrix is non-derogatory.
3. If all eigenvalues are distinct, say $\lambda_1, \lambda_2\, \lambda_3$ then the minimal equation is $(x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = 0$.

Procedure to determine whether A is Derogatory or Non-derogatory

1. First find the characteristic equation of A and solve it.
2. If the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are all distinct then A is non-derogatory (See Ex. 5, 6).
3. If $\lambda_1 = \lambda_2$, then find whether the equation $(x - \lambda_1)(x - \lambda_3) = 0$ is satisfied by A. If it is satisfied by A, then A is derogatory otherwise non-derogatory (See Ex. 1, 2).
4. If $\lambda_1 = \lambda_2 = \lambda_3$, then find whether the equation $x - \lambda_1 = 0$ is satisfied by A. If it is satisfied by A, then A is derogatory otherwise non-derogatory.

Example 1 : Show that the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is derogatory.

(M.U. 1993, 96, 2003, 04, 10, 14)

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)[-(16-\lambda^2)+12] + 6[4+\lambda-6] - 6[6-3(4-\lambda)] = 0$$

$$\therefore\ (5-\lambda)[-4+\lambda^2] + 6[-2+\lambda] - 6[-6+3\lambda] = 0$$

$$\therefore\ \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0 \qquad \therefore\ \lambda^3 - 2\lambda^2 - 3\lambda^2 + 6\lambda + 2\lambda - 4 = 0$$

$$\therefore\ (\lambda-2)(\lambda^2 - 3\lambda + 2) = 0 \qquad \therefore\ (\lambda-2)(\lambda-2)(\lambda-1) = 0$$

Hence, the roots of $|A - \lambda I| = 0$ are 2, 2, 1.

Let us now find the minimal polynomial of A. We know that each characteristic root of A is also a root of the minimal polynomial of A. So if $f(x)$ is the minimal polynomial of A then $x-1$ and $x-2$ are the factors of $f(x)$. Let us see whether the polynomial $(x-2)(x-1) = x^2 - 3x + 2$ annihilates A.

Now, $$A^2 = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}^2 = \begin{bmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix}$$

$$\therefore\ A^2 - 3A + 2I = \begin{bmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix} - 3\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ f(x) = x^2 - 3x + 2$ annihilates A.

Thus, $f(x)$ is the monic polynomial of lowest degree that annihilates A. Hence, $f(x)$ is the minimal polynomial of A. Since its degree **is less** than the order of A, A is derogatory.

Example 2 : Show that $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$ is derogatory.

(M.U. 1999, 2000, 03, 04, 05, 11, 14, 15)

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & -4 & 4-\lambda \end{vmatrix} = 0$$

$$\therefore\ (7-\lambda)[(7-\lambda)(4-\lambda)-4] - 4[4(4-\lambda)-4] - 1[-16+4(7-\lambda)] = 0$$

$$\therefore\ (7-\lambda)[24 - 11\lambda + \lambda^2] - 4[12 - 4\lambda] - [12 - 4\lambda] = 0$$

$$\therefore\ \lambda^3 - 18\lambda^2 + 81\lambda - 108 = 0 \qquad \therefore\ (\lambda-3)(\lambda^2 - 15\lambda + 36) = 0$$

$$\therefore\ (\lambda-3)(\lambda-12)(\lambda-3) = 0$$

Hence, the roots of $|A - \lambda I| = 0$ are 3, 3, 12.

Let us now find the minimal polynomial of A. We know that each characteristic root of A is also a root of the minimal polynomial of A. So if $f(x)$ is the minimal polynomial of A then

$x-3$ and $x-12$ are the factors of $f(x)$. Let us see whether $(x-3)(x-12) = x^2-15x+36$ annihilates A.

Now, $A^2 = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}^2 = \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix}$

$$\therefore\ A^2-15A+36I = \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix} - 15\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} + 36\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ f(x) = x^2-15x+36$ annihilates A.

Thus, $f(x)$ is the monic polynomial of lowest degree that annihilates A. Hence, $f(x)$ is the minimal polynomial of A. Since its degree is less than the order of A, A is derogatory.

Example 3 : Show that the matrix A is derogatory and find its minimal equation

$$A = \begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$ **(M.U. 2002, 16)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & -3 & 3 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\therefore\ (2-\lambda)[(3-\lambda)^2-1]+3[0]+3[0]=0$

$\therefore\ (2-\lambda)[9-6\lambda+\lambda^2-1]=0 \qquad \therefore\ (2-\lambda)(\lambda^2-6\lambda+8)=0$

$\therefore\ (2-\lambda)(\lambda-2)(\lambda-4)=0 \qquad \therefore\ \lambda = 2, 2, 4$

Hence, the roots of $|A-\lambda I| = 0$ are 2, 2, 4.

Let us now find the minimal polynomial of A. We know that each characteristic root of A is also a root of the minimal polynomial of A. So if $f(x)$ is the minimal polynomial of A, then $x-2$ and $x-4$ are the factors of $f(x)$.

Let us see whether $(x-2)(x-4) = x^2-6x+8$ annihilates A.

Now, $A^2-6A+8I$

$$= \begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} - 6\begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -18 & 18 \\ 0 & 10 & -6 \\ 0 & -6 & 10 \end{bmatrix} - 6\begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ f(x) = x^2-6x+8$ annihilates A.

Thus, $f(x)$ is the monic polynomial of lowest degree that annihilates A.

Hence, $f(x)$ is the minimal polynomial of A. Since the degree of $f(x)$ is less than the order of A, A is derogatory.

Example 4 : Show that $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is non-derogatory. **(M.U. 2000, 17)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 3-\lambda & 4 \\ 3 & 4 & 5-\lambda \end{vmatrix} = 0$$

$\therefore\ (1-\lambda)[(3-\lambda)(5-\lambda)-16]-2[2(5-\lambda)-12]+3[8-3(3-\lambda)]=0$

$\therefore\ (1-\lambda)[-1-8\lambda+\lambda^2]-2[-2-2\lambda]+3[-1+3\lambda]=0$

$\therefore\ \lambda^3-9\lambda^2-6\lambda=0 \qquad \therefore\ \lambda(\lambda^2-9\lambda-6)=0$

Since, all roots are distinct and since the characteristic equation is satisfied by A. The degree of minimal equation is equal to 3 and hence A is non-derogatory.

Example 5 : Show that the matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ is non-derogatory. **(M.U. 2005)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$\therefore\ (2-\lambda)[-(1-\lambda)(1+\lambda)-3]+2[-(1+\lambda)-1]+3[3-(1-\lambda)]=0$

$\therefore\ (2-\lambda)(-4+\lambda^2)-2(2+\lambda)+3(2+\lambda)=0$

$\therefore\ \lambda^3-2\lambda^2-5\lambda+6=0 \qquad \therefore\ \lambda^3-\lambda^2-\lambda^2+\lambda-6\lambda+6=0$

$\therefore\ (\lambda-1)(\lambda^2-\lambda-6)=0 \qquad \therefore\ (\lambda-1)(\lambda-3)(\lambda+2)=0$

$\therefore\ \lambda = 1, -2, 3.$

Since, all the roots of the characteristic equation are distinct,

$f(x) = (x-1)(x+2)(x-3)$

is the minimal polynomial. The degree of minimal equation is equal to 3. Hence, the matrix is non-derogatory.

Example 6 : Find eigenvalues and eigenvectors of A^3 where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Is A derogatory ? **(M.U. 2006)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$\therefore \quad (1-\lambda)[(2-\lambda)(3-\lambda)-2]-0[1(3-\lambda)-2]-1[2-2(2-\lambda)]=0$

$\therefore \quad (1-\lambda)[6-5\lambda+\lambda^2-2]-0[4-5\lambda+\lambda^2]+2[1-\lambda]=0$

$\therefore \quad (1-\lambda)(\lambda^2-5\lambda+6)=0 \qquad \therefore (1-\lambda)(\lambda-2)(\lambda-3)=0$

$\therefore \quad \lambda = 1, 2, 3.$

$\therefore$ Eigenvalues of A^3 are $1^3, 2^3, 3^3$.

(i) For $\lambda = 1$, $[A-\lambda_1 I]X = O$ gives

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_3 - 2R_2 \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_3 = 0, \quad x_1 + x_2 + x_3 = 0, \ \therefore \ x_1 + x_2 = 0$

Let $x_2 = -1$, $x_1 = 1 \ \therefore \ X_1 = [1, -1, 0]'$.

(ii) For $\lambda = 2$, $[A-\lambda_2 I]X = O$ gives

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2 + R_1 \\ R_2 + 2R_1 \end{matrix} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad -x_1 - x_3 = 0, \quad 2x_2 - x_3 = 0$

Let $x_2 = 1$, $x_3 = 2 \ \therefore \ x_1 = -x_3 = -2. \quad \therefore \ X_2 = [-2, 1, 2]'$.

(iii) For $\lambda = 3$, $[A-\lambda_3 I]X = o$ gives

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_1 + 2R_2 \\ R_3 + R_1 \end{matrix} \begin{bmatrix} 0 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad -2x_2 + x_3 = 0, \quad x_1 - x_2 + x_3 = 0$

Let $x_2 = 1 \quad \therefore \ x_3 = 2, \quad \therefore \ x_1 - 1 + 2 = 0 \quad \therefore \ x_1 = -1.$

$\therefore \quad X_3 = [-1, 1, 2]'$.

Now, if $AX = \lambda X$, then $A^n X = \lambda^n X$.

Hence, eigenvalues of A^3 are 1, 8, 27.

Eigenvectors of A^3 are

$X_1 = [1, -1, 0]', \qquad X_2 = [-2, 1, 2]', \qquad X_3 = [-1, 1, 2]'$.

Since, eigenvalues of A are all distinct, A is non-derogatory.

EXERCISE - VI

1. Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ is derogatory and find its minimal polynomial.

(M.U. 1996, 2001, 03, 05) [**Ans. :** $f(x) = x^2 - 1$]

2. Show that $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is derogatory and find its minimal polynomial.

(M.U. 2003, 06) [**Ans. :** $f(x) = x^2 - 10x + 16$]

3. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, find the minimal polynomial of A. (M.U. 2003)

[**Ans. :** $x - 1 = 0$]

4. Show that the matrix A is derogatory and find its minimal equation.

$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ (M.U. 2002, 09) [**Ans. :** $f(x) = x^2 - x$]

5. Given $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$,

find the eigenvalues of $4A^{-1}$ and the eigen-vectors of $A^2 - 4I$. Obtain the minimal polynomial of A. Determine if A is diagonisable and if so find the diagonalising matrix P and the diagonal matrix D. (M.U. 2004)

[**Ans. :** Eigenvalues of A are 5, −3, −3. Eigenvalues of $4A^{-1}$ are 4/5, − 4/3, − 4/3

Eigenvectors of A and also of $A^2 - 4I$ are [1, 2, −1]', [2, −1, 0]', [3, 0, 1]'.

Min. poly. $x^2 - 2x - 15 = 0$.]

6. Show that the following matrices are non-derogatory.

(i) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ (iii) $\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ (v) $\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$

[**Ans. :** (i) $f(x) = x^3 - 18x^2 + 45x$ (ii) $f(x) = x^3 - 4x^2 - x + 4$

(iii) $f(x) = x^3 - 3x^2 + 2x$ (v) $f(x) = x^3 - 2x^2 - x + 2$

(v) $f(x) = x^3 - 3x^2 - 90x - 216$.]

EXERCISE - VII

Theory

1. Define the following terms.

(a) Similarity of matrices.

(b) Diagonalisable matrix.

(c) Algebraic multiplicity of an eigenvalue.

(d) Geometric multiplicity of an eigenvalue.

(e) Minimal polynomial of a matrix.

(f) Derogatory and non-derogatory matrix. (M.U. 2003)

2. State Cayley-Hamilton theorem. (M.U. 2003)

3. Prove that a matrix satisfies its characteristic equation. (M.U. 1998)

4. Prove that the necessary and sufficient condition for a square matrix A (i) to be symmetric is that $A = A'$, (ii) to be skew-symmetric is that $A = -A'$.

5. Prove that if A is Hermitian then $|A|$ is real. (M.U. 2000)

6. Prove that the necessary and sufficient condition for a square matrix A to be (i) Hermitian is that $A = A^{\theta}$, (ii) Skew-Hermitian is that $A = -A^{\theta}$.

7. Prove that if A is Hermitian then iA is Skew-Hermitian.

8. Prove that if A is Skew-Hermitian then iA is Hermitian. (M.U. 1999)

9. If A and B are symmetric matrices then show that AB is symmetric if A and B commute. (M.U. 1995)

10. If A is a square matrix then show that (i) $A + A'$ is Symmetric and $A - A'$ is Skew-symmetric, (ii) $A + A^{\theta}$ is Hermitian and $A - A^{\theta}$ is Skew-Hermitian. (M.U. 1998)

11. Show that every square matrix can be uniquely expressed as the sum of (i) a Symmetric and a Skew-symmetric matrix. (ii) a Hermitian and a Skew-Hermitian matrix. (M.U. 1998, 99, 2005)

12. Prove that every square matrix can be expressed as $P + iQ$ when P and Q are Hermitian matrices. (M.U. 1997)

13. Prove that every Hermitian matrix can be written as $B + iC$ where B is real Symmetric and C is real Skew-symmetric matrices. (M.U. 1997, 98)

14. Prove that every Skew-Hermitian matrix A can be expressed as $B + iC$ where B is real Skew-symmetric and C is real Symmetric matrix.

15. Define : Unitary and Orthogonal matrices.

16. Prove that a real square matrix A is unitary if and only if A is orthogonal.

17. Prove that if A is unitary then $|A|$ is of unity modulus.

18. If two square matrices A, B are unitary then AB is also unitary. Prove.

19. If a square matrix A is unitary then A', $\overline{A}$, A^{θ}, A^{-1} are also unitary. Prove.

20. If a square matrix is orthogonal then prove that A' and A^{-1} are also orthogonal.

21. Prove that A, B are two square orthogonal matrices then AB is also orthogonal.

22. If a square matrix A is orthogonal then prove that $|A|$ is of unit modulus.

23. Define - (i) linear independence of a vector, (ii) linear combination of vector.

24. Define - (i) inner product of two vectors, (ii) norm of a vector, (iii) unit vector.

25. Define - (i) eigenvector, (ii) eigenvalue, (iii) characteristic equation.

26. If A is a non-singular matrix of order n and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A then prove that

(i) $\lambda_1 . \lambda_2 . \ldots . \lambda_n = |A|$

(ii) $\lambda_1 + \lambda_2 + \ldots + \lambda_n =$ Sum of the diagonal elements of A. (M.U. 2000)

27. Prove that eigenvalues of a Hermitian matrix are real. (M.U. 1996, 2003)

28. Eigenvalues of a real symmetric matrix are all real. Prove this. (M.U. 1997, 99, 2003)

29. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A then show that

(i) $\dfrac{1}{\lambda_1}, \dfrac{1}{\lambda_2}, \ldots, \dfrac{1}{\lambda_n}$ are the eigenvalues of A^{-1}. (M.U. 1999, 2000)

(ii) $\lambda_1^n, \lambda_2^n, \ldots, \lambda_n^n$ are the eigenvalues of A^n.

(iii) $\dfrac{|A|}{\lambda_1}, \dfrac{|A|}{\lambda_2}, \ldots, \dfrac{|A|}{\lambda_n}$ are the eigenvalues of adj. A.

30. State the Cayley-Hamilton theorem.

31. Define - (i) Similarity of matrices, (ii) Diagonalisable matrices.

32. Define the following terms.

(i) minimal polynomial

(ii) monic polynomial (M.U. 2011)

(iii) derogatory matrix. (M.U. 2011)

☆☆☆

CHAPTER 6

Probability

1. Introduction

You were introduced probability in the X-th standard and you have studied laws of probability, Bayes' theorem probability distributions, particularly Binomial distribution in XII-th standard. In this chapter we shall briefly review probability and then study once again Bays' Theorem in detail. Then we shall study probability distributions in general.

2. Terms used in Axiomatic Theory

The limitations of classical theory of probability are removed by putting the theory in axiomatic form. This approach was suggested by **A. N. Kolmogorov**, a Russian mathematician in 1933.

It should be noted that as in any other axiomatic theory (*e.g.* Geometry) we start with certain undefined terms, state certain axioms about them and then deduce theorems about these terms strictly from the axioms by rules of logic. Before developing the axiomatic probability theory we explain below certain peculiar terms used therein.

Andrey Nikolaevich Kolmogorov (1903 - 1987)

A well-known Russian mathema-tician known for his great contributions to the fields of probability theory, topology, intuitionistic logic, classical mechanics and others. In 1922 he constructed a Fourier Series that diverges almost everywhere. His pioneering work "About The Analytical Methods Of Probability Theory" was published in 1931, in which year he become professor at Moscow University. In 1933 he published "Foundations of The Theory of Probability" laying the foundation of modern axiomatic theory. In 1939 he become a member of U.S.S.R. Academy of Sciences. He along with British mathematician Chapman developed "The Chapman-Kolmogorov Equations" in Random processes. He was a founder of algorithm complexity theory known as "Kolmogorov Complexity Theory". A quotation of Kolmogorov. "Every mathematician believes that he is ahead of all others. The reason why they do not say this in public is they are intelligent people".

(1) Sample space

The aggregate or the set of all possible outcomes of an experiment is called *sample space*, the outcomes *i.e.* the members of the set themselves being called *sample points*. Sample space is denoted by S or Ω and sample points by $w_1, w_2, w_3, \ldots$. Thus, $\Omega = \{w_1, w_2, w_3, \ldots\}$.

Example 1 : In a random experiment of tossing a coin the sample points are its outcomes a head (H) and a tail (T) and the sample space is

$$\Omega = \{H, T\}.$$

Example 2 : In a random experiment of tossing a dice the sample points are 1, 2, 3, 4, 5 and 6 the sample space is

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

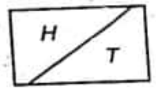
Example 3 : In a throw of two coins the sample points are $(H, H), (H, T), (T, H), (T, T)$, and the sample space is

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}.$$

(2) Event

A subset of a sample space is called an *event*.

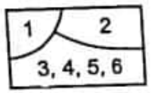
(i) In a throw of a coin $\Omega = \{H, T\}$. We have shown two events



$A_1 = \{H\}$,

$A_2 = \{T\}$.

(ii) In a throw of a die $\Omega = \{1, 2, 3, 4, 5, 6\}$, there are a number of events.



$A_1 = \{1\},\ A_2 = \{2\},\ A_3 = \{3, 4, 5, 6\}, \ldots$

$B_1 = \{1, 2\},\ B_2 = \{1, 3\} \ldots$

$C_1 = \{1, 2, 3\},\ C_2 = \{1, 2, 4\}, \ldots$ etc.

We have shown the first three.

(iii) In a throw of a coin and a die there are again a number of events.

$A_1 = \{(H, 1), (H, 2)\}$

$A_2 = \{(H, 1), (H, 3), (H, 5),\} \ldots$

$B_1 = \{(T, 1), (T, 2)\}$

$B_2 = \{(T, 1), (T, 3), (T, 5)\} \ldots$ etc.

3. Axiomatic Definition of Probability

Let E be a random experiment and S be the sample space. We define the probability $P(A)$ for every element A of S (*i.e.* for every subset A of the sample space S) satisfying the following axioms :

1. $P(A) \geq 0$ (Axiom 1)
2. $P(S) = 1$ (Axiom 2)
3. $P(A \cup B) = P(A) + P(B)$ (Axiom 3)

if A and B are any two exclusive events (*i.e.* they are disjoint sets).

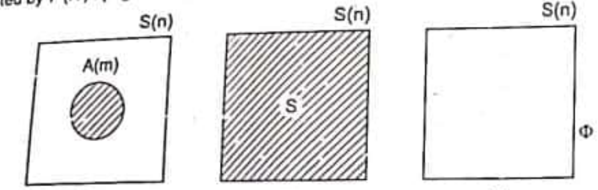
Explanation : The first axiom states that the probability of any event is greater than zero, which means the probability of any event cannot be negative.

The second axiom states that the sum of all the probabilities is equal to 1. This together with the first axiom states that the probability of an event must be less than or equal to 1 *i.e.* $P(A) \leq 1$.

The third axiom states that if two events are mutually exclusive, then the probability that either of them will occur is equal to the some of their probabilities.

For our study the classical definition is sufficient. We give that definition in terms of sets again for ready reference.

Definition : If a sample space S has n points which are equally likely and mutually exclusive and an event A has m points then the ratio m / n is called probability of A and is denoted by $P(A)$. [Fig. 6.1 (a)]



(a) (b) (c)

Fig. 6.1

Thus, $$P(A) = \frac{m}{n} = \frac{\text{number of points in } A}{\text{number of point in } S}$$

Note ....

We shall be concerned with only such random experiments in which all sample points are equally likely. And to find the probability of an event in such a random experiment we need to know the number of points in sample space S and the number of points in the event A.

4. Laws of Probability

There are two laws. The probability of **A or B** *i.e.* $P(A \cup B)$ is given by the *law of addition* and the probability of **A and B** *i.e.* $P(A \cap B)$ is given by the *law of multiplication.*

Theorem 1 : For any two events A and B the probability that exactly B will occur is given by

$$\boxed{P(B \cap \bar{A}) = P(B) - P(A \cap B)}$$

and that exactly A will occur is given by

$$\boxed{P(A \cap \bar{B}) = P(A) - P(A \cap B)}$$

where $\bar{A}$ denotes the complement of A and $\bar{B}$ denotes the complement of B.

Proof : To prove this result we first express the event B as the union of two exclusive events $A \cap B$ and $\bar{A} \cap B$.

$\therefore \quad B = (A \cap B) \cup (\bar{A} \cap B)$

Since the events on the r.h.s. are exclusive

$P(B) = P(A \cap B) + P(\bar{A} \cap B)$ [By Axiom 3]

$\therefore \quad P(B \cap \bar{A}) = P(B) - P(A \cap B)$

Similarly, we can prove that

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$

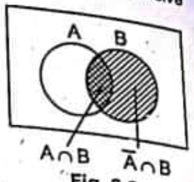


Fig. 6.2

Theorem 2 : (Additive Theorem) (Two Events)

Probability that atleast one of the events A and B will occur is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof : We first express the event $A \cup B$ is the union of two exclusive events A and $\bar{A} \cap B$.

$A \cup B = A \cup (\bar{A} \cap B)$

$\therefore \quad P(A \cup B) = P[A \cup (\bar{A} \cap B)]$

But the events on the r.h.s. are mutually exclusive.

$\therefore \quad P(A \cup B) = P(A) + P(\bar{A} \cap B)$ [By Axiom 3]

$= P(A) + P(B) - P(A \cap B)$ [By Theorem 2]

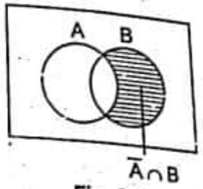


Fig. 6.3

Corollary 1 : If A and B are two mutually exclusive events then the probability that either A or B will happen is the sum of the probabilities of A and B *i.e.*,

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

Proof : Since events are exclusive $A \cap B = \Phi$.

$\therefore \quad P(A \cap B) = P(\Phi) = 0$ [By Theorem 1]

Hence, from the above theorem,

$P(A \cup B) = P(A) + P(B)$

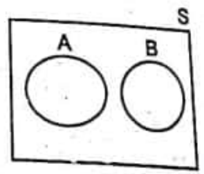


Fig. 6.4

Corollary 2 : If $A, B, C, \ldots, K$ are mutually exclusive events such that their union is the whole of sample space then

$$\boxed{P(A) + P(B) + P(C) + \ldots + P(K) = 1}$$

Proof : Since $A \cup B \cup C \cup \ldots \cup K = S$

$\therefore \quad P(A \cup B \cup C \cup \ldots \cup K) = P(S)$

Since, events are mutually exclusive, we get, from above,

$P(A) + P(B) + P(C) + \ldots + P(K) = P(S)$

But $P(S) = 1$ [By Axiom 2]

$\therefore \quad P(A) + P(B) + P(C) + \ldots + P(K) = 1$

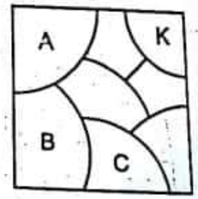


Fig. 6.5

Note ....

The group of all possible events $A, B, C, \ldots, K$ of the sample space S such that

$$P(A) + P(B) + P(C) + \ldots + P(K) = 1$$

is called **exhaustive.** e.g. in the toss of a coin H, T; in the toss of a dice 1, 2, 3, 4, 5, 6 are exhaustive events.

Theorem 3 : *Addition Theorem (Three Events)* : If A, B, C are any three events then the probability that *at least* one of them will occur is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

Proof : We prove this theorem by considering the union of A and $(B \cup C)$ and applying the above theorem.

$\therefore \quad P(A \cup (B \cup C)) = P(A) + P(B \cup C) - P(A \cap (B \cup C))$

But $\quad P(B \cup C) = P(B) + P(C) - P(B \cap C)$

And $\quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$\therefore \quad P(A \cap (B \cup C)) = P[(A \cap B) \cup (A \cap C)]$

$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$

$\therefore \quad P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$

Corollary : If A, D, C are three events which are pairwise exclusive then

$$\boxed{P(A \cup B \cup C) = P(A) + P(B) + P(C)}$$

Proof : Since, A, B are exclusive $A \cap B = \Phi$. Since, $P(\Phi) = 0$, $P(A \cap B) = 0$. Similarly, $P(B \cap C) = 0$, $P(C \cap A) = 0$, $P(A \cap B \cap C) = 0$, $P(C \cap A) = 0$, $P(A \cap B \cap C) = 0$.

Hence, from the above result,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C).$$

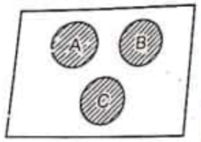


Fig. 6.6

Example : Three horses A, B and C are in a race. A is twice as likely to win as B and B is twice as likely to win as C. What are the probabilities of their winning ?

Sol. : Let $P(C) = x$ then $P(B) = 2x$ and $P(A) = 4x$.

But the sum of all probabilities is 1.

$\therefore \quad P(A) + P(B) + P(C) = 1 \qquad \therefore \quad 4x + 2x + x = 1 \qquad \therefore \quad x = \frac{1}{7}$

$\therefore \quad P(A) = \frac{4}{7}, \quad P(B) = \frac{2}{7}, \quad P(C) = \frac{1}{7}.$

Complementary Events

It is clear that $A \cup \overline{A} = S$ *i.e.* the union of the event and its complement is the whole of sample space S and $\overline{A} \cap A = \Phi$ *i.e.* the event A and its complement are exclusive.

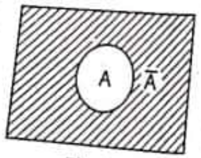


Fig. 6.7

Theorem 4 :

$$\boxed{P(\overline{A}) = 1 - P(A)}$$

Proof : Since $A \cup \overline{A} = S$, $P(A \cup \overline{A}) = P(S)$. But $P(S) = 1$ and A, $\overline{A}$ are exclusive.

By Axiom 2, $\quad P(A \cup \overline{A}) = P(A) + P(\overline{A})$

$\therefore \quad P(A) + P(\overline{A}) = 1 \qquad \therefore \quad P(\overline{A}) = 1 - P(A)$

Note .... Since A and $\overline{A}$ are exclusive, we have $A \cup \overline{A} = S$ and $A \cap \overline{A} = \Phi$.

Corollary : Probability of an event is always less than or equal to one.

i.e., $\quad \boxed{P(A) \leq 1}$

Proof : $P(A) = 1 - P(\overline{A})$. But $P(\overline{A}) \geq 0$ by axiom 1.

$\therefore \quad P(A) \leq 1.$

De Morgan's Laws

Since an event is a subset of sample space, De Morgan's are laws are applicable to events. Thus, we get

$$\boxed{\begin{aligned} P(\overline{A \cup B}) &= P(\overline{A} \cap \overline{B}) \\ P(\overline{A \cap B}) &= P(\overline{A} \cup \overline{B}) \end{aligned}}$$

Augustus De Morgan (1806 - 1871)

Augustus De Morgan was born in Madurai, Tamil Nadu. His father was a colonel in the Indian army. His family returned to England when he was 7 months old. When in schools he mastered Latin, Greek and Hebrew and developed strong interest in mathematics.

He was a fellow of the Astronomical Society and a founder of London Mathematical Society. De Morgan greatly influenced the development of mathematics in the 19th century. He was a prolific writer and wrote over 1000 articles in more than 15 journals, in addition to a number of books known for clarity, logical presentation and minute details. He made original contributions to analysis and logic. He coined the term mathematical induction and gave first precise definition of limit in his book "The Differential And Integral Calculus".

Example 1 : Find the probability that a card drawn will be black or picture ?

Sol. : Let A = the card is picture and B = the card is black.

Now A and B are not mutually exclusive events. There are 6 black pictures

$$P(A) = \frac{12}{52}, \quad P(B) = \frac{26}{52}, \quad P(A \cap B) = \frac{6}{52}$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{12}{52} + \frac{26}{52} - \frac{6}{52} = \frac{32}{32} = \frac{8}{13}.$$

Note ....

Many examples of this type can be more easily solved by counting the number of points in the desired event. In the above example $A \cup B$ = {Black or Picture Cards} has 26 + 12 − 6 = 32 points.

$$\therefore \quad P(A \cup B) = \frac{32}{52} = \frac{8}{13}.$$

Example 2 : Two cards are drawn from a pack of cards. Find the probability that they will be both red or both pictures.

Sol. : Let A = { both red } and B = { both pictures }

$$P(A) = \frac{^{26}C_2}{^{52}C_2}, \quad P(B) = \frac{^{12}C_2}{^{52}C_2}$$

But A and B are not exclusive. There are six red picture cards.

$$\therefore \quad P(A \cap B) = \frac{^{6}C_2}{^{52}C_2}.$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{^{26}C_2}{^{52}C_2} + \frac{^{12}C_2}{^{52}C_2} - \frac{^{6}C_2}{^{52}C_2} = \frac{203}{663}.$$

5. Conditional Probability

Consider general case of conditional probability (See Fig. 6.8 on page 6-9). Let n = the number of points in S, m_1 = number of points in A, m_2 = number of points in B, m_{12} = number of points in A and B both.

Then, $P(A) = \frac{m_1}{n}$, $P(B) = \frac{m_2}{n}$ and $P(A \cap B) = \frac{m_{12}}{n}$.

Now, suppose in a trial we know the result partially *i.e.* we know that A has occurred. What is the probability now that B has occurred along with A ?

Since we know that A has occurred the outcome of the trial is one of those points in A *i.e.* one of m_1 (and not n) And of these, m_{12} are in B. Hence, the probability that B will occur, when A has already occurred is (m_{12} / m_1). This is called conditional probability of B under the condition that A has occurred. It is denoted by $P(B / A)$.

Thus, $\quad P(B / A) = \frac{m_{12}}{m_1}$.

Definition : Let A and B be any two events in a sample space S. The probability that B will occur, given that A has already occurred is called the **conditional probability of B** and is denoted by $P(B / A)$.

Similarly the probability that A will occur given that B has already occurred is called the **conditional probability of A** and is denoted by $P(A / B)$.

Example : Suppose there are 100 students in a class and the results of an examination of the class are given in the following table.

	Passed	Failed	Total
Boys	28	32	60
Girls	26	14	40
Total	54	46	100

Let us define two events A, B as follows :

A = a student has passed, $\quad B$ = a student is a male student.

Suppose a student is selected at random and is known to be a male student. What is the probability that this student has passed ? In symbols, we want to find $P(A / B)$.

Since the student is a male student, the sample space of B has 60 points. Of these 28 have passed *i.e.* A has now 28 points. Hence,

$$\therefore \quad P(A / B) = \frac{28}{60} = \frac{7}{15}$$

But, from the table we see that 28 is the number of points in $A \cap B$ and 60 is the number of points in B

$$\boxed{P(A / B) = \frac{\text{No. of points in } A \cap B}{\text{No. of points in } B}} \quad \text{.................... (1)}$$

or

$$\boxed{P(A / B) = \frac{\text{No. of points in } A \text{ out of } B}{\text{No. of points in } B}} \quad \text{.................... (2)}$$

Similarly, we can find that

$$\boxed{P(B / A) = \frac{\text{No. of points in } B \cap A}{\text{No. of points in } A}} \quad \text{.................... (3)}$$

or

$$\boxed{P(B / A) = \frac{\text{No. of points in } B \text{ out of } A}{\text{No. of points in } A}} \quad \text{.................... (4)}$$

Further, we see that, (1) can be written as

$$P(A / B) = \frac{(\text{No. of points in } A \cap B) / \text{No. of points in } S}{(\text{No. of points in } B) / \text{No. of points in } S}$$

$$\therefore \quad P(A / B) = \frac{P(A \cap B)}{P(B)}$$

Similarly, from (3) we can get $\quad P(B / A) = \frac{P(A \cap B)}{P(A)}$.

From these we get $\quad P(A \cap B) = P(A / B) \times P(B)$ and $P(A \cap B) = P(B / A) \times P(A)$ which is called the law of multiplication of probability.

Theorem 5 : *Multiplication Theorem*

If A and B are two events and neither is null then the probability that both of them will occur is given by

$$\boxed{P(A \cap B) = P(A) \times P(B / A)} \quad \text{or} \quad \boxed{P(A \cap B) = P(B) \times P(A / B)}$$

where $P(A / B)$ and $P(B / A)$ denote the conditional probabilities which are greater than zero.

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Proof : Let the number of points in A be m_1 and those in B be m_2. Let the number of points in $(A \cap B)$ be m_{12} and let n be the total number of points in S.

$$P(A) = \frac{m_1}{n}, \quad P(B) = \frac{m_2}{n}$$

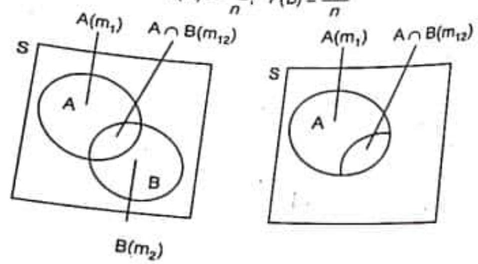

Fig. 6.8

To find the probability of B when A has happened, we have to consider the sample space of A which has m_1 points. In this sample space only B can occur (along with A) which has m_{12} points.

$$\therefore \quad P(B/A) = \frac{\text{No. of points in } A \cap B}{\text{No. of points in } A} = \frac{m_{12}}{m_1}$$

Now, $$P(A \cap B) = \frac{m_{12}}{n} = \frac{m_1}{n} \times \frac{m_{12}}{m_1}$$

$$\therefore \quad P(A \cap B) = P(A) \times P(B/A)$$

Similarly, we can prove that $P(A \cap B) = P(B) \times P(A/B)$.

Example 1 : There are 11 tickets in a box bearing numbers 1 to 11. Three tickets are drawn one after the other without replacement. Find the probability that they are drawn in the order bearing (i) even, odd, even number, (ii) odd, odd, even number. **(M.U. 1997)**

Sol. : This is an example on conditional probability. Out of 11 tickets, 5 are even and 6 are odd.

$$P(\text{even, odd, even}) = P(\text{even}) \cdot P(\text{odd}) \cdot P(\text{even})$$

$$= \frac{5}{11} \cdot \frac{6}{10} \cdot \frac{4}{9} = \frac{4}{33}$$

$$P(\text{odd, odd, even}) = P(\text{odd}) \cdot P(\text{odd}) \cdot P(\text{even})$$

$$= \frac{6}{11} \cdot \frac{5}{10} \cdot \frac{5}{9} = \frac{5}{33}$$

Example 2 : In a bag there are 4 white and 3 black balls. If four balls are drawn one by one at random without replacement, what is probability that the balls so drawn are alternately of different colours ?

Sol. : Try.

$$P(A) = \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{35}; \qquad P(B) = \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{35}$$

$$P(\text{Alternate colour}) = \frac{3}{35} + \frac{3}{35} = \frac{6}{35}.$$

Example 3 : In a certain college 4% of the boys and 1% of the girls are taller than 1·8 mts. Furthermore 60% of the students are girls. If a student selected at random is taller than 1·8 mts, what is the probability that the student was a boy ? **(M.U. 2002, 03)**

Sol. : For convenience suppose there are 1000 students in the college.

Then we can easily prepare the following table.

	Taller than 1·8	Less than 1·8	Total
Boys	16	384	400
Girsl	6	594	600
Total	22	978	1000

Since the student selected at random is found to be taller than 1·8, the student is one of 22. But out of these 22 students 16 are boys.

$$\therefore \quad \text{Required Probability} = \frac{16}{22} = \frac{8}{11}.$$

6. Partition of Sample Space

Definition : The events $A_1, A_2, \ldots, A_n$ are said to represent a partition of sample space S if

(i) $A_i \cap A_j = 0$ for all $i \neq j$

and (ii) $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S$.

In words : The events $A_1, A_2, \ldots, A_n$ are mutually exclusive and exhaustive *i.e.* when the experiment is performed *one and only one* of the events A_i occurs and must occur.

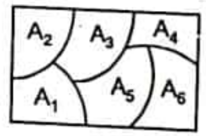

Fig. 6.9

For example, in tossing of a die the events $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$, $A_3 = \{6\}$ represent a partition of the sample space S. However, $B_1 = \{1, 2, 3\}$, $B_2 = \{3, 4\}$, $B_3 = \{4, 5, 6\}$ do not represent a partition (why ?). Also, $C_1 = \{1\}$, $C_2 = \{3, 4\}$, $C_3 = \{6\}$ do not represent a partition. (why ?)

Example 1 : A, B and C are bidding for a contract. It is believed that A has exactly half the chance than B has. B in turn is 4 / 5th as likely as C to get the contract. What is the probability for each to get the contract ? **(M.U. 2004)**

Sol. : Let $P(A)$, $P(B)$, $P(C)$ denote the probabilities that A, B, C will get the contract respectively. Further, let $P(C) = x$. Then,

$$P(B) = \frac{4}{5}x \text{ and } P(A) = \frac{1}{2} \cdot \frac{4}{5}x = \frac{2}{5}x$$

Since one of A, B, C must get the contract, the three events partition the sample space

$$x + \frac{4}{5}x + \frac{2}{5}x = 1 \quad \therefore \quad 11x = 5 \quad \therefore \quad x = \frac{5}{11}.$$

$$\therefore \quad P(A) = \frac{2}{11}, \; P(B) = \frac{4}{11}, \; P(C) = \frac{5}{11}$$

Example 2 : If the events A, B and C form a partition of the sample space and $P(A) = 2P(B) = 3P(C)$, find (i) $P(A \cup B)$, (ii) $P(B \cup C)$.

Sol. : Let $P(A) = x$, then $P(B) = \frac{x}{2}$ and $P(C) = \frac{x}{3}$.

Since A, B and C form partition S. These are exhaustive events.

$$\therefore \quad P(A) + P(B) + P(C) = 1$$

$$\therefore \quad x + \frac{x}{2} + \frac{x}{3} = 1 \qquad \therefore \quad \frac{11}{6}x = 1 \qquad \therefore \quad x = \frac{6}{11}$$

$$\therefore \quad P(A) = \frac{6}{11}, \quad P(B) = \frac{3}{11}, \quad P(C) = \frac{2}{11}.$$

Since, the events are mutually exclusive.

$$P(A \cup B) = P(A) + P(B) = \frac{6}{11} + \frac{3}{11} = \frac{9}{11}$$

and $$P(B \cup C) = P(B) + P(C) = \frac{3}{11} + \frac{2}{11} = \frac{5}{11}.$$

Example 4 : If the events A, B and C form a partition of the sample space S and $3P(A) = 2P(B) = 6P(C)$, find $P(A \cup B)$.

Please do it.

[Ans. : 5 / 6]

Example 5 : If the events A, B, C and D form a partition of the sample space S and the probabilities of A and C are equal, the event B is twice as likely as A and the event D is twice as likely as B, find $P(A \cup B \cup C)$.

Please do it.

[Ans. : 1 / 2]

Theorem on Total Probability

Let $A_1, A_2, A_3, \ldots, A_n$ be a partition of S and B be some event defined on S. Then,

$$P(B) = P(B / A_1) \times P(A_1) + P(B / A_2) \times P(A_2) + \ldots$$
$$\ldots + P(B / A_n) \times P(A_n)$$

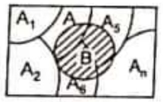

Fig. 6.10

We accept this theorem without proof.

If we denote the probabilities $P(A_i)$ by p_i and the conditional probabilities $P(B / A_i)$ by p'_i then, the above theorem can be stated as

$$\boxed{P(B) = p_1 p'_1 + p_2 p'_2 + p_3 p'_3 + \ldots p_n p'_n}$$

Remark ....

The above theorem is known as the theorem on **total probability**. In some problems it may be difficult to evaluate $P(B)$ directly. But when additional information that A_i has occurred is known, we may evaluate $P(B / A)$ and use the above theorem to find $P(B)$.

Example 1 : Four roads lead away from a jail. A prisoner trying to escape from the jail selects a road at random. If road A is selected, the probability of escaping is 1 / 8, for road B, it is 1 / 6, for road C it is 1 / 4 and for road D it is 9 / 10.

What is the probability that a prisoner will succeed in escaping from the jail ?

(M.U. 2005)

Sol. : Let E = Success in Escaping.

p_1 = selecting road A, $\quad p_2$ = selecting road B,
p_3 = selecting road C, $\quad p_4$ = selecting road D.
$p_1 = P(A) = 1/4$, $\quad p_2 = P(B) = 1/4$,
$p_3 = P(C) = 1/4$, $\quad p_4 = P(D) = 1/4$.
$p_1' = P(E / A) = 1/8$, $\quad p_2' = P(E / B) = 1/6$,
$p_3' = P(E / C) = 1/4$, $\quad p_4' = P(E / D) = 9/10$.

$$\therefore \quad P(E) = p_1 p_1' + p_2 p_2' + p_3 p_3' + p_4 p_4'$$

$$\therefore \quad P(E) = \frac{1}{4}\cdot\frac{1}{8} + \frac{1}{4}\cdot\frac{1}{6} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{9}{10}$$

$$= \frac{1}{4}\left(\frac{1}{8} + \frac{1}{6} + \frac{1}{4} + \frac{9}{10}\right) = \frac{1}{4}\cdot\frac{163}{120} = \frac{163}{480}.$$

Example 2 : In a box there are four tags numbered 1 and 6 tags numbered 2. There are two urns U_1 and U_2 containing 3 red and 7 black balls and 8 red and 2 black balls respectively. One tag is drawn from the box and one ball is drawn from the urn whose number is found on the tag drawn. Find the probability that a red ball is drawn.

Sol. : Try.

[Ans. : 0·6]

EXERCISE - I

1. The probabilities that three students A, B and C will pass the common entrance test for engineering are 4 / 9, 2 / 9 and 1 / 3 respectively. The probabilities that they will get admission in the same engineering college are 3 / 10, 1 / 2 and 4 / 5 respectively.

   Find the probability that they will get admission in the same engineering college.

   [Ans. : 23 / 45]

2. The chances that A, B and C will be the Education Minister of Government of India are in the ratio 4 : 1 : 2. The probabilities that they will introduce reservations in professional colleges for backward classes are 0·3, 0·8 and 0·5 respectively.

   Find the probability that the bill for reservation will be introduced. **[Ans. : 3 / 7]**

3. In a factory an article is produced on three machines. Their respective productions are 300 units by A, 250 units by B and 450 units by C. It is found that the percentages of defective articles for A, B, C are 1, 1·2 and 2 selected at random from a days production (which are mixed).

   Find the probability that the selected article is defective. **[Ans. : 0·015]**

We now state an important theorem known as Bayes' Theorem. It enables us to evaluate what may be called **reverse probabilities**. Suppose there are two boxes (I and II) which contain 2 white and 3 black balls; and 3 white and 4 black balls. If a box is chosen at random and a ball is drawn from it, what is the probability that the ball drawn is white ? We know how to calculate this probability. Now, consider the question : If the ball drawn is known to be white, what is the probability that it was drawn from the 1st box ? The question can be answered by Bayes' Theorem. If the 'result' is known, Bayes' Theorem enables us to find the probability of the 'cause'. For this reason it is also sometimes known as the formula for the "probability of causes".

7. Bayes' Theorem

Let the events $A_1, A_2, \ldots, A_n$ represent a partition of the sample space S. Let B be any other event defined on S. If $P(A_i) \neq 0$, $i = 1, 2, \ldots, n$ and $P(B) \neq 0$ then

$$P(A_i / B) = \frac{P(A_i) \times P(B/A_i)}{\Sigma P(A_i) \times P(B/A_i)}$$

If we write $p_1 = P(A_1),\ p_2 = P(A_2),\ p_3 = P(A_3) \ldots\ldots$ etc.

and $p'_1 = P(B/A_1),\ p'_2 = P(B/A_2),\ p'_3 = P(B/A_3) \ldots\ldots$ etc.

then Bayes' theorem can be stated as

$$\boxed{P(A_i / B) = \frac{p_i\, p'_i}{p_1 p'_1 + p_2 p'_2 + \ldots + p_n p'_n}}$$

Proof : We have by conditional probability

$$P(A_i / B) = \frac{P(A_i \cap B)}{P(B)} \qquad \ldots\ldots\ldots\ldots \text{(i)}$$

But $P(B/A_i) = \dfrac{P(B \cap A_i)}{P(A_i)}$ $\quad \therefore \ P(A_i \cap B) = P(A_i) \cdot P(B/A_i)$ $\quad \ldots\ldots\ldots\ldots$ (ii)

From (i) and (ii), we get

$$\therefore \ P(A_i / B) = \frac{P(A_i) \cdot P(B/A_i)}{P(B)} = \frac{p_i\, p'_i}{P(B)} \qquad \ldots\ldots\ldots\ldots \text{(iii)}$$

But $B = (B \cap A_1) \cup (B \cap A_2) \ldots\ldots (B \cap A_n)$

$\therefore \ P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots\ldots + P(B \cap A_n)$

But $P(B \cap A_i) = P(A_i) \cdot P(B/A_i)$ etc. $= p_1 p'_1$ etc.

Hence, from (iii) $\quad P(A_i / B) = \dfrac{p_i\, p'_i}{p_1 p'_1 + p_2 p'_2 + \ldots + p_n p'_n}$

Thomas Bayes (1701 - 1761)

Thomas Bayes was an English mathematician known for the theorem that bears his name. This theorem was published after his death by Richar Price. He published two works in his lifetime, one theological and one mathematical. He became a Fellow of the Royal Society in 1742. It is said that he learned mathematics and probability from a book by De Moivre. In his later years he took deep interest in probability.

Example 1 : There are in a bag three true coins and one false coin with head on both sides. A coin is chosen at random and tossed four times. If head occurs all the four times, what is the probability that the false coin was chosen and used ? **(M.U. 2003)**

Sol. : $P(\text{selecting true coin}) = p_1 = \dfrac{3}{4}$. $\quad P(\text{selecting false coin}) = p_2 = \dfrac{1}{4}$.

$$p'_1 = P(\text{getting all four heads with true coin}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}.$$

$$p'_2 = P(\text{getting all four heads with false coin}) = 1 \cdot 1 \cdot 1 \cdot 1 = 1.$$

$$\therefore \ \text{Required Probability} = \frac{p_2 p'_2}{p_1 p'_1 + p_2 p'_2} = \frac{(1/4) \cdot 1}{(3/4) \cdot (1/16) + (1/4) \cdot 1} = \frac{16}{19}.$$

Example 2 : A coin is tossed. If it turns up heads two balls are drawn from urn A otherwise two balls are drawn from urn B. Urn A contains 3 black and 5 white balls. Urn B contains 7 black and one white ball. What is the probability that urn A was used, given that both balls drawn are black ? **(M.U. 2001, 04)**

Sol. : We have $p_1 = P(H) = \dfrac{1}{2}$, $\ p_2 = P(T) = \dfrac{1}{2}$

$$p'_1 = P(\text{two black balls from } A) = \frac{{}^3C_2}{{}^8C_2}$$

$$p'_2 = P(\text{two black balls from } B) = \frac{{}^7C_2}{{}^8C_2}$$

$$\therefore \ \text{Required Probability} = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2} = \frac{\frac{1}{2} \cdot \frac{{}^3C_2}{{}^8C_2}}{\frac{1}{2} \cdot \frac{{}^3C_2}{{}^8C_2} + \frac{1}{2} \cdot \frac{{}^7C_2}{{}^8C_2}}$$

$$= \frac{{}^3C_2}{{}^3C_2 + {}^7C_2} = \frac{3 \cdot 2 / 2 \cdot 1}{(3 \cdot 2 / 2 \cdot 1) + (7 \cdot 6 / 2 \cdot 1)}$$

$$= \frac{6}{6 + 42} = \frac{6}{48} = \frac{1}{8}.$$

Example 3 : In a certain test there are multiple choice questions. There are four possible answers to each question and one of them is correct. An intelligent student can solve 90% questions correctly by reasoning and for the remaining 10% questions he gives answers by guessing. A weak student can solve 20% questions correctly by reasoning and for the remaining 80% questions he gives answers by guessing. An intelligent students gets the correct answer, what is the probability that he was guessing ? **(M.U. 2004)**

Sol. : Consider the intelligent student.

Let p_1 = answering by reasoning $= \dfrac{90}{100} = \dfrac{9}{10}$.

p_2 = answering by guessing $= \dfrac{10}{100} = \dfrac{1}{10}$.

p_1' = answer is correct (by reasoning) = 1

p_2' = answering is correct (by guessing) $= \frac{1}{4}$.

Required Probability $= \frac{p_2 p_2'}{p_1 p_1' + p_2 p_2'} = \frac{(1/10)\cdot(1/4)}{(9/10)\cdot 1 + (1/10)\cdot(1/4)}$

$= \frac{1/40}{37/40} = \frac{1}{37}$.

Example 4 : A certain test for a particular cancer is known to be 95% accurate. A person submits to the test and the result is positive. Suppose that a person comes from a population of 100,000 where 2000 people suffer from that disease. What can we conclude about the probability that the person under test has that particular cancer ? **(M.U. 2006)**

Sol. : We have

p_1 = probability a person has the cancer $= \frac{2000}{100{,}000} = \frac{2}{100} = 0{\cdot}02$

p_2 = probability that a person does not have the cancer $= 1 - 0{\cdot}02 = 0{\cdot}98$

p_1' = test is positive when a person has cancer $= \frac{95}{100} = 0{\cdot}95$

p_2' = test is positive when person does not have a cancer $= \frac{5}{100} = 0{\cdot}05$

$\therefore$ Required probability $= \frac{p_1 p_1'}{p_1 p_1' + p_2 p_2'}$

$\therefore$ Required probability $= \frac{(2/100)\cdot(95/100)}{(2/100)\cdot(95/100) + (98/100)\cdot(5/100)}$

$= \frac{190}{680} = 0{\cdot}279$

Example 5 : A bag contains 7 red and 3 black balls and another bag contains 4 red and 5 black balls. One ball is transferred from the first bag to the second bag and then a ball is drawn from the second bag. If this ball happens to be red, find the probability that a black ball was transferred. **(M.U. 2002)**

Sol. : We have

p_1 = Probability of transferring black ball $= \frac{3}{10}$

p'_1 = Probability of now drawing a red ball $= \frac{4}{10}$

p_2 = Probability of transferring red ball $= \frac{7}{10}$

p'_2 = Probability of now drawing red ball $= \frac{5}{10}$

$\therefore$ Required Probability $= \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2} = \frac{(3/10)\cdot(4/10)}{(3/10)(4/10) + (7/10)(5/10)}$

$= \frac{12}{47}$.

Example 6 : A man speaks truth 3 times out of 5. When a die is thrown, he states that it gave an ace. What is the probability that this event has actually happened ?

Sol. : We have

p_1 = Probability he speaks truth $= \frac{3}{5}$; p'_1 = Probability of an ace $= \frac{1}{6}$

p_2 = Probability he speaks a lie $= \frac{2}{5}$; p'_2 = Probability of not ace $= \frac{5}{6}$

P(he speaks truth when ace has occured) $= \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2}$

$= \frac{(3/5)(1/6)}{(3/5)(1/6) + (2/5)(5/6)} = \frac{3}{3+10} = \frac{3}{13}$.

Example 7 : A lot of IC chips is known to contain 3% defective chips. Each chip is tested before delivery but the tester is not completely reliable. It is known that :

P (Tester says the chip is good / The chip is actually good) $= 0{\cdot}95$ and P (Tester says the chip is defective / The chip is actually defective) $= 0{\cdot}96$.

If a tested chip is declared defective by the tester. What is the probability that it is actually defective ?

Sol. : Let p_1 = Chip is defective $= 0{\cdot}03$

p'_1 = Tester says the chip is defective / The chip is defective $= 0{\cdot}96$

p_2 = Chip is good $= 0{\cdot}97$

p'_2 = Tester says the chip is defective / The chip is good $= 0{\cdot}05$.

By Bayes' Theorem,

P (Chip is defective / Tester says it is defective)

$= \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2} = \frac{0{\cdot}03 \times 0{\cdot}96}{0{\cdot}03 \times 0{\cdot}96 + 0{\cdot}97 \times 0{\cdot}05} = 0{\cdot}37$.

Example 8 : A binary communication transmitter sends data as one of two types of signals denoted by 0 or 1. Due to noise, sometimes a transmitted 1 is received as 0 and vice versa.

If the probability that a transmitted 0 is correctly received as 0 is 0·9 and the probability that a transmitted 1 is correctly received as 1 is 0·8 and if the probability of transmitting 0 is 0·45, find the probability that (i) a 1 is received, (ii) a 0 is received, (iii) a 1 was transmitted given that 1 was received, (iv) a 0 was transmitted given that a 0 was received, (v) the error has occurred.

Sol. : We are given that

$P(T_0)$ = a 0 is transmitted $= 0{\cdot}45$

$P(T_1)$ = a 1 is transmitted $= 1 - P(T_0) = 1 - p_1 = 0{\cdot}55$

$P(R_0 / T_0)$ = a 0 is received when a 0 was transmitted $= 0{\cdot}9$

$P(R_1 / T_0)$ = a 1 received when a 0 was transmitted

$= 1 - 0{\cdot}9 = 0{\cdot}1$

$P(R_1 / T_1)$ = a 1 is received when a 1 was transmitted $= 0{\cdot}8$

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$P(R_0 / T_1)$ – a 0 is received when a 1 was transmitted
$= 1 - 0{\cdot}8 = 0{\cdot}2.$

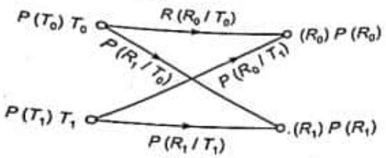

Fig. 6.11

Now, we calculate the required probabilities as follows :

(i) P (1 is received) = P (1 is received when 1 is transmitted)
+ P (1 is received when 0 is transmitted)

$$\therefore \; P(R_1) = P(R_1 / T_1) \cdot P(T_1) + P(R_1 / T_0) \cdot P(T_0)$$
$$= 0{\cdot}8 \times 0{\cdot}55 + 0{\cdot}1 \times 0{\cdot}45$$
$$= 0{\cdot}485$$

(ii) P (0 is received) = P (0 is received when 0 is transmitted)
+ P (0 is received when 1 is transmitted)

$$\therefore \; P(R_0) = P(R_0 / T_0) \cdot P(T_0) + P(R_0 / T_1) \cdot P(T_1)$$
$$= 0{\cdot}9 \times 0{\cdot}45 + 0{\cdot}2 \times 0{\cdot}55$$
$$= 0{\cdot}515$$

Now, by Bayes' Theorem

(iii) P (1 was transmitted given that 1 was received) *i.e.,*

$$P(T_1 / R_1) = \frac{P(R_1 / T_1)\, P(T_1)}{P(R_1)} = \frac{0{\cdot}8 \times 0{\cdot}55}{0{\cdot}485} = 0{\cdot}907$$

(iv) P (0 was transmitted given that 0 was received) *i.e.,*

$$P(T_0 / R_0) = \frac{P(R_0 / T_0) \cdot P(T_0)}{P(R_0)} = \frac{0{\cdot}9 \times 0{\cdot}45}{0{\cdot}515} = 0{\cdot}786$$

(v) P (Error) = P (0 was received when 1 is transmitted given that 1 was transmitted) + P (1 was received when 0 was transmitted given that 0 was transmitted)

$$= P(R_0 / T_1) \cdot P(T_1) + P(R_1 / T_0) \cdot P(T_0)$$
$$= 0{\cdot}2 \times 0{\cdot}55 + 0{\cdot}1 \times 0{\cdot}45 = 0{\cdot}155.$$

Example 9 : A box contains three biased coins A, B and C. The probability that a head will result when A is tossed is $1/3$, when B is tossed, it is $2/3$ and when C is tossed, it is $3/4$.

(a) *If one of the coins is chosen at random and is tossed 3 times, head resulted twice and tail once. What is the probability that the coin chosen was A ?*

(b) *What is the probability of getting head when a coin selected at random is tossed once?*

(c) *What is the probability that we would get two heads in the first three tosses and a head again in the fourth toss with the same coin ?*

Sol. : We have p_1 = Probability of choosing $A = \frac{1}{3}$.

p_2 = Probability of choosing $B = \frac{1}{3}$.

p_3 = Probability of choosing $C = \frac{1}{3}$.

p_1' = Probability of getting 2 heads in three tosses with the coin A

$$= {}^3C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right) = 3 \cdot \frac{1}{9} \cdot \frac{2}{3} = \frac{2}{9}.$$

p_2' = Probability of getting 2 heads in three tosses with the coin B

$$= {}^3C_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right) = 3 \cdot \frac{4}{9} \cdot \frac{1}{3} = \frac{4}{9}.$$

p_3' = Probability of getting 2 heads in three tosses with the coin C

$$= {}^3C_2 \left(\frac{3}{4}\right)^2 \cdot \left(\frac{1}{4}\right) = 3 \cdot \frac{9}{16} \cdot \frac{1}{4} = \frac{27}{64}.$$

$$\therefore \text{ Required Probability } = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2 + p_3 p_3'}$$
$$= \frac{(1/3)(2/9)}{(1/3)\cdot(2/9) + (1/3)\cdot(4/9) + (1/3)\cdot(27/64)}$$
$$= \frac{(2/9)}{(2/9) + (4/9) + (27/64)} = \frac{128}{627}.$$

(b) We do not know which coin was tossed.

If the coin was A, the probability that it will give head
= (Prob. of choosing A) × (Prob. of giving head)
$$= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}.$$

If the coin was B, probability of getting head.
= (Prob. of choosing B) × (Prob. of giving head)
$$= \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}.$$

If the coin was C, probability of getting head
= (Prob. of choosing C) × (Prob. of giving head)
$$= \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

$$\therefore \text{ The required probability} = \frac{1}{9} + \frac{2}{9} + \frac{1}{4} = \frac{7}{12}.$$

(c) Now, getting two heads in the first three tosses and a head in the fourth toss with A
= P (Choosing A) × P (Getting two heads in three tosses with A)
× P (Getting a head in the fourth toss with A)

$$= p_1 \cdot p_1' \cdot \frac{1}{3} = \frac{1}{3} \cdot \frac{2}{9} \cdot \frac{1}{3} = \frac{2}{81}.$$

Similarly, getting two heads in the first three tosses and a head in the fourth toss with B

$$= p_2 \cdot p_2' \cdot \frac{2}{3} = \frac{1}{3} \cdot \frac{4}{9} \cdot \frac{2}{3} = \frac{8}{81}$$

And getting two heads in the first three tosses and a head in the fourth toss with C

$$= p_3\, p_3' \cdot \frac{3}{4} = \frac{1}{3} \cdot \frac{27}{64} \cdot \frac{3}{4} = \frac{27}{256}$$

$\therefore$ Required Probability

$$= \frac{2}{81} + \frac{8}{81} + \frac{27}{256} = 0.23.$$

Example 10 : A bag contains five balls, the colours of which are not known. Two balls were drawn from the bag and they were found to be white. What is the probability that all balls are white ? **(M.U. 2005)**

Sol. : Since two balls drawn are white, the bag may contain 2 white or 3 white or 4 white or 5 white balls.

Let these events be denoted by A_1, A_2, A_3, A_4 respectively. We can assume that the probabilities of these events are equal.

Let $p_1 = P(A_1),\ p_2 = P(A_2),\ p_3 = P(A_3),\ p_4 = P(A_4)$.

$$\therefore\ p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$$

Now, two balls out of 5 can be drawn in ${}^5C_2$ ways.

$$\therefore\ p_1' = P\,(\text{drawing two balls when two balls are white}) = \frac{{}^2C_2}{{}^5C_2} = \frac{2 \cdot 1}{5 \cdot 4} = \frac{2}{20}.$$

$$p_2' = P\,(\text{drawing two white balls when 3 balls are white}) = \frac{{}^3C_2}{{}^5C_2} = \frac{3 \cdot 2}{5 \cdot 4} = \frac{6}{20}.$$

$$p_3' = P\,(\text{drawing 2 white balls when 4 balls are white}) = \frac{{}^4C_2}{{}^5C_2} = \frac{4 \times 3}{5 \times 4} = \frac{12}{20}.$$

$$p_4' = P\,(\text{drawing 2 white balls when 5 balls are white}) = \frac{{}^5C_2}{{}^5C_2} = \frac{5 \cdot 4}{5 \cdot 4} = \frac{20}{20}.$$

$\therefore$ By Baye's theorem Required Probability

$$= \frac{p_4\, p_4'}{p_1 p_1' + p_2 p_2' + p_3 p_3' + p_4 p_4'}$$

$$= \frac{(1/4) \cdot (20/20)}{(1/4) \cdot (2/20) + (1/4) \cdot (6/20) + (1/4) \cdot (12/20) + (1/4) \cdot (20/20)}$$

$$= \frac{20}{2 + 6 + 12 + 20} = \frac{20}{40} = \frac{1}{2}.$$

Example 11 : There are three boxes containing respectively 1 white 2 red, 3 black balls; 2 white, 3 red and 1 black ball; 3 white, 1 red and 2 black balls. A box is chosen at random and two balls are drawn from it. The two balls are found to be one red and one white. Find the probability that those have come from box 1, box 2 and box 3.

Sol. : Since there are three boxes, say, A_1, A_2, A_3, then

$$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3} \qquad \therefore\ p_1 = p_2 = p_3 = \frac{1}{3}.$$

Let B be the event that one ball is white and the other is red. Then,

$$p_1' = P(B/A_1) = \frac{{}^1C_1 \cdot {}^2C_1}{{}^6C_2} = \frac{1 \cdot 2 \cdot 2}{6 \cdot 5} = \frac{2}{15}$$

$$p_2' = P(B/A_2) = \frac{{}^2C_1 \cdot {}^3C_1}{{}^6C_2} = \frac{2 \cdot 3 \cdot 2}{6 \cdot 5} = \frac{6}{15}$$

$$p_3' = P(B/A_3) = \frac{{}^3C_1 \cdot {}^1C_1}{{}^6C_2} = \frac{3 \cdot 1 \cdot 2}{6 \cdot 5} = \frac{6}{15}$$

By Baye's Theorem,

$$P(A_1/B) = \frac{p_1\, p_1'}{p_1 p_1' + p_2 p_2' + p_3 p_3'}$$

$$\therefore\ P(A_1/B) = \frac{(1/3)(2/15)}{(1/3)(2/15) + (1/3)(6/15) + (1/3)(3/15)} = \frac{2/45}{11/45} = \frac{2}{11}$$

$$P(A_2/B) = \frac{p_2\, p_2'}{p_1 p_1' + p_2 p_2' + p_3 p_3'} = \frac{(1/3)(6/15)}{11/45} = \frac{6}{11}$$

$$P(A_3/B) = \frac{p_3\, p_3'}{p_1 p_1' + p_2 p_2' + p_3 p_3'} = \frac{(1/3)(3/15)}{11/45} = \frac{3}{11}.$$

Example 12 : Three factories A, B, C produce 30 %, 50 % and 20 % of the total production of an item. Out of their production 80 %, 50 % and 10 % are defective. An item is chosen at random and found to be defective. Find the probability that it was produced by the factory A.

Sol. :

$p_1 = P\,(\text{item is produced by } A) = 0{\cdot}3$

$p_2 = P\,(\text{item is produced by } B) = 0{\cdot}5$

$p_3 = P\,(\text{item is produced by } C) = 0{\cdot}2$

Let D be the event that the item is defective then

$$p_1' = P(D/A) = 0{\cdot}8, \quad p_2' = P(D/B) = 0{\cdot}5, \quad p_3' = P(D/C) = 0{\cdot}1$$

Now, the required event is A/D.

$$\therefore\ P(A/D) = \frac{p_1\, p_1'}{p_1 p_1' + p_2 p_2' + p_3 p_3'} = \frac{0{\cdot}3 \times 0{\cdot}8}{0{\cdot}3 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}5 + 0{\cdot}2 \times 0{\cdot}1}$$

$$= \frac{0{\cdot}24}{0{\cdot}24 + 0{\cdot}25 + 0{\cdot}02} = \frac{0{\cdot}24}{0{\cdot}51} = 0{\cdot}47.$$

EXERCISE - II

1. If $P(A) = 0{\cdot}7$, $P(B) = 0{\cdot}6$, $P(A/B) = 0{\cdot}4$, find $P(B/A)$. [Ans. : 12 / 35]

2. If A_1 and A_2 are two mutually exclusive and exhaustive events of sample space S and if B is another event defined on S, such that $P(A_1) = 0{\cdot}6$, $P(A_2) = 0{\cdot}4$ and $P(B/A_1) = 0{\cdot}4$, $P(B/A_2) = 0{\cdot}5$, find $P(B)$. [Ans. : 0·44]

3. A bag contains two coins one of which is a false coin with head on both sides and the other is a true coin. When a coin, taken at random from the bag, is tossed, it gave a head. What, is the probability that the true coin was taken and tossed ? [Ans. : 1 / 3]

4. A bag contains two dice, one of which is regular and fair and the other is false with number 6 on all its faces. A dice was drawn from the bag and tossed. It gave 6. What is the probability that the dice obtained was the false one ? [Ans. : 6 / 7]

5. In a bolt factory, machines A, B, C produce respectively 25 %, 35 % and 40 %. Of their output 5 %, 4 % and 2 % are defective. A bolt is drawn at random from a days production and is found defective. What is the probability that it was produced by machines A, B, C? (M.U. 1997) [Ans. : 25 / 69, 28 / 69, 16 / 69]

6. A manufacturing firm produces steel pipes in three plants with daily production of 500, 1000 and 2000 units. According to past experience it is known that the fraction of defective output produced by the three plants are respectively 0·005, 0·008 and 0·010.

If a pipe is selected at random from a day's output and was found to be defective, what is the probability that it came from the first plant ? [Ans. : 0·08]

7. A given lot of I.C. chips contains 3 % defective chips. Each chip is tested before delivery. The tester itself is not totally reliable, such that P (tester shows the chip is good / chip is actually good) = 0·96 and P (tester shows chip is defective / chip is actually defective) = 0·94. If a tested chip is shown to be defective, what is the probability that it is actually defective ? [Ans. : 0·42]

8. A binary communication system transmits and also receives data as '0' and '1'. Due to noise a transmitted '0' is sometimes received as '1' and a transmitted '1' is some times received as '0'. The probability that a transmitted '0' is correctly received as '0' is 0·95 and the probability that a transmitted '1' is correctly received as '1' is 0·90, probability of transmitting '0' is 0·45. If a signal is sent find :

(i) Probability that a '1' is received

(ii) Probability that a '0' is received.

(iii) Probability that a '1' is transmitted, given that a '1' is received.

(iv) Probability that a '0' is transmitted, given that a '0' is received.

[Ans. : (i) 0·5175, (ii) 0·4825, (iii) 0·9565, (iv) 0·8860, (v) 0·0775]

9. A newly constructed flyover is likely to collapse. The chance that the design is faulty is 0·5. The chance that the flyover will collapse if the design is faulty is 0·95 otherwise it is 0·30. The flyover collapsed. What is the probability that it collapsed because of faulty design? (M.U. 2004) [Ans. : 0·76]

10. Three identical urns have the following composition of black and white balls :

First urn : 2 black, 1 white

Second urn : 1 black, 2 white

Third urn : 2 black, 2 white

One of the urns is selected at random and one ball is drawn. What is the probability of drawing a white ball ? [Ans. : 1/2]

11. There are three urns having the following compositions of black and white balls :

Urn 1 : 7 white, 3 black balls,

Urn 2 : 4 white, 6 black balls,

Urn 3 : 2 white, 8 black balls.

One of the urns is chosen at random with probabilities 0·2, 0·6 and 0·2 respectively. From the chosen urn two balls are drawn at random without replacement. Calculate the probabilities that both these balls are white. [Ans. : 8 / 45]

12. Three boxes A, B, C contain 30, 50, 20 bulbs respectively. The percentages of defective bulbs in these boxes are 4, 3 and 5 respectively. A box is selected at random and a bulb is drawn. What is the probability that the bulb will be defective ? [Ans. : 0·037]

13. A man is equally likely to choose any one of three routes C_1, C_2, C_3 from his house to the railway station. The probabilities of missing the train by the routes C_1, C_2 and C_3 are 2 / 5, 3 / 10, 1 / 20. He sets out on a day and misses the train. What is the probability that the route C_3 was selected ? [Ans. : 1 / 15]

14. A factory manufacturing television sets gets its supply of components from 4 units A, B, C, D in the following proportion 15%, 20%, 30% and 35% respectively. It was found that of the supply received 1%, 2%, 2% and 3% articles are defective. A television set was chosen at random from the total output and was found to be defective. What is the probability that it came from unit D ? (M.U. 2002) [Ans. : 21 / 44]

15. A man has three coins A, B, C. A is unbiased. The probability that head will result when B is tossed is 2 / 3. The probability that head will result when C is tossed is 1/ 3. If one of the coins is chosen at random and is tossed three times it gave two heads and one tail. Find the probability that the coin A was chosen. (M.U. 1997, 2002) [Ans. : 9 / 25]

16. Bag I contains 6 blue and 4 red balls.

Bag II contains 2 blue and 6 red balls.

Bag III contains 1 blue and 8 red balls.

A bag is chosen at random and two balls are drawn from it. If both the balls were found to be blue, find the probability that bag II was chosen. (M.U. 2004) [Ans. : 3 / 31]

17. A coin is tossed. If it turns up head two balls are drawn from urn A , otherwise two balls are drawn from urn B. Urn A contains 3 black and 5 white balls. Urn B contains 7 black and one white ball. In both cases selection is made with replacement. What is the probability that A was chosen given that both the balls drawn are black. (M.U. 2002) [Ans. : 9 / 58]

18. An urn A contains ten red and three black balls. Another urn B contains three red and five black balls. Two balls are transferred from urn A to the urn B without noticing their colour. One ball now is drawn from the urn B and it is found to be red. What is the probability that one red and one black ball have been transferred ? [Ans. : 20 / 59]

19. There are three boxes A, B and C. The probability of getting a white ball from the box A is 1/ 3, from the box B is 2 / 3 and from the box C is 3 / 4.

A box is chosen at random and three balls are drawn from it (without replacement) and it was found that two of them were white.

(a) What is the probability that the box A was chosen ?

(b) If a ball is drawn from a box selected at random, what is the probability that it will be white ?

(c) What is the probability that we would get two white balls in the first three draws and a white ball again in the fourth draw ?

(Hint : See Solved Ex. 9) [Ans. : (a) 128 / 627, (b) 7 / 12, (c) 0·23]

20. For a certain binary communication channel, the probability that a transmitted '0' is received as a '0' is 0·95 while the probability that a transmitted '1' is received as '1' is 0.90. If the probability of transmitting a '0' is 0·4, find the probability that (i) a '1' is received, (ii) a '1' was transmitted given that '1' was received. [**Ans. :** (i) 0·56, (ii) 27 / 28]

8. Random Variable

In the previous chapter we learnt how to find the probability of an outcome and the laws of probability. We saw that the outcome of an experiment can be anything : it may be colour (black-white-red .....) of a ball, a gender (male-female) of a child, a suit (club-diamond-spade-heart) of a card or a number (1 – 2 – 3 – 4 – 5 – 6) of a die or a logical answer (yes - no) of a question or result of a toss (head - tail) of a coin. In most of the problems the outcome of an experiment is a number *e.g.* the salary of a person, the height of a student, the temperature at a place, the rainfall on a particular day. However, when the outcome is not a number we can express these outcomes in numbers by agreeing to denote,

(a) head by 1 and tail by 0 (b) boy by 1 and girl by 0

(c) yes by 1 and no by 0 (d) club by 1, diamond by 2, spade by 3 and heart by 4

(e) red by 1, white by 2 and black by 3, etc.

In probability problems it is found convenient to think of a variable and consider the values of the variable which describe the outcomes of the experiment. In the toss of a coin the variable takes values 1 and 0; in the selection of a child it takes values 1 and 0, in the answers of the question it takes the values 1 and 0, in drawing a card it takes values 1, 2, 3, 4, in drawing a ball it takes values 1, 2, 3 etc. This variable may take discrete values or may take any value in a range continuously *e.g.* the arrival time of a bus at a stop, say, between 9 a.m. and 9.10 a.m. may be any number between the range. Such a variable is called a **random variable**. Actually random variable is a misnomer. It is a function which assigns a real number to the outcome of an experiment. A random variable is denoted by X and a particular value of X is denoted by x. In the toss of a coin X assigns the value 1 to H and 0 to T, in the selection of a ball X assigns value 1 to red, 2 to white and 3 to black.

In these cases X takes discrete values and is called a discrete random variable. But in the case of arrival time of a bus X takes continuously any value between 9 am. and 9.10 am. and hence X is a continuous random variable. In this way, we consider X as a function from sample space S to the set of real numbers R. Thus we get the following definition.

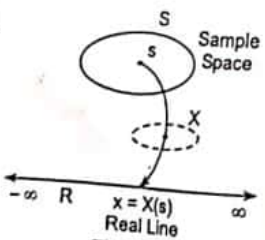

Fig. 6.12

(a) Definition

Let E be an experiment and S be the sample space associated with it. A function X assigning to every element s of S one and only one real number $x = X(s)$ of R is called a *random variable*.

Since X is a function whose domain is the set of outcomes of an experiment and whose range is a part or the whole of real line ($-\infty < x < \infty$), it can be shown pictorially as a mapping from the sample space to the real line.

The random variable X can be discrete or continuous depending upon the nature of its domain.

Remarks ....

1. In simple words a variable used to denote the numerical value of the outcome of an experiment is called the random variable, abbreviated as r.v.
2. X is a *function* and still we call it a variable.
3. We are not interested in the *functional nature* of X but in the *values of* X.
4. X must be single valued *i.e.* for every s of S there corresponds *exactly one value* of X. Different elements of S may lead to the same value of X (See Ex. 2). But two values of X cannot be assigned to the same sample point.
5. We shall denote *random variables* by capital letters X, Y, Z, ... and shall denote the *unknown values* of these random variables by small letters x, y, z, ..., x_1, x_2, ... y_1, y_2, ... z_1, z_2, ... etc. This is an important distinction and students should note it carefully. With this notation it is meaningless to write $P(x \geq 10)$ say since x being a value of X either is or is not ≥ 10. Instead we should write $P(X \geq 10)$.

Example 1 : Suppose the experiment E is to toss a fair coin.

Then $S = \{H, T\}$. If X is the random variable denoting the number of heads then we have $X(H) = 1$ and $X(T) = 0$.

[See Fig. 6.13]

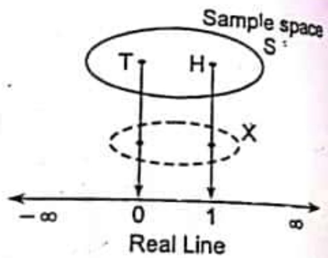

Fig. 6.13

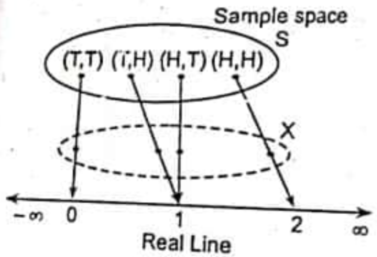

Fig. 6.14

Example 2 : Suppose the experiment E is to toss two fair coins.

Then $S = \{(H, H), (H, T), (T, H), (T, T)\}$. If X is the random variable denoting the number of heads then $X(H, H) = 2$, $X(H, T) = 1$, $X(T, H) = 1$, $X(T, T) = 0$.

[See Fig. 6.14]

Example 3 : Suppose the experiment is to record the temperature at a place.

If X denoted the temperature then X can take any value from $-273°$ F to $212°$ F say.

[See Fig. 6.15]

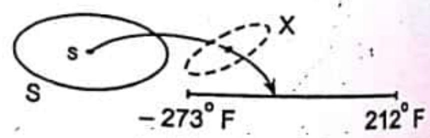

Fig. 6.15

Example 4 : Suppose the experiment is to record the time required to complete a software project.

If X denotes the time then X can take any value (theoretically) from 0 to ∞.

[See Fig. 6.16]

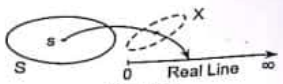

Fig. 6.16

In examples 1 and 2, X takes discrete values and in examples 3 and 4, X takes continuously all values between a specified interval. In the first case X is called a ***discrete random variable*** in the second case it is called ***continuous random variable***.

(b) Definition

Let X be a random variable. If X takes finite or countably infinite values $x_0, x_1, x_2, \ldots$ then X is called a ***discrete random variable.***

(c) Definition

Let X be a random variable. If X takes uncountably infinite values in a given interval then X is called a ***continuous random variable.***

9. Probability Distribution of a Discrete Random Variable

As we know already, with every possible outcome of an experiment there will be associated its probability. We shall be interested in the values of the random variable X along with their probabilities. If x_i is the value of X and $P(x_i)$ is the probability of x_i then set of pairs $\{x_i, P(x_i)\}$ is called the **probability distribution.**

Definition : Let X be a **discrete random variable.** Let $x_1, x_2, \ldots, x_n, \ldots$ be the possible values of X. With each possible outcome x_i we associate a number $p(x_i) = P(X = x_i)$ called the probability of x_i. The numbers $p(x_i)$, $i = 1, 2, \ldots, n \ldots$ must satisfy the following conditions :

1. $p(x_i) \geq 0$ for all i
2. $\sum_{i=1} p(x_i) = 1$

The function p is called the **probability function** or **probability mass function (p.m.f) or probability density function (p.d.f.)** of the random variable X and the set of pairs (x_i, p_i) is called the probability distribution of X.

The probability distribution of a discrete random variable X taking values $x_1, x_2, x_3, \ldots, x_n, \ldots$ with probabilities $p_1, p_2, p_3, \ldots, p_n, \ldots$ where $p_i \geq 0$ and $\Sigma p_i = 1$ can be given in tabular form as

X	x_1	x_2	x_3	.....	x_n	.....
$P(x_i)$	p_1	p_2	p_3	.....	p_n	.....

Example 1 : State true or false with justification :

(a) A random variable X takes values 0, 1, 2 and 3 then $p(X = x) = \frac{x-1}{2}$ can be its probability distribution.

(b) A random variable takes values 0, 1, 2 and $p(x) = \frac{x+1}{3}$ is its probability distribution.

Sol. : As seen above for a probability distribution, two conditions must be satisfied.

(i) Each probability must be equal to or greater than zero but less than one.

(ii) The sum of all probabilities must be equal to unity.

Putting $x = 0, 1, 2, 3$ in (a), we get

$$P(0) = -\frac{1}{2}, \quad P(1) = 0, \quad P(2) = \frac{1}{2}, \quad P(3) = 1$$

Since, the probability cannot be negative, $P(X = x) = \frac{x-1}{2}$ cannot be a probability distribution.

Putting $x = 0, 1, 2, 3$, in (b), we get $P(0) = \frac{1}{3}$, $P(1) = \frac{2}{3}$, $P(2) = 1$.

Although all probabilities are positive, the sum of all the probabilities is 2, (greater than 1). Hence, $P(X = x) = \frac{x+1}{3}$ also cannot be a probability distribution.

Example 2 : From the past experience it was found that the daily demand at an autogarage was as under.

Daily Demand	:	5	6	7
Probability	:	0·25	0·65	0·10

Check if this is a probability distribution. Find also the probability that over a period of two days the number of demands would be 11 or 12.

Sol. : Since the sum of all probabilities $= 0{\cdot}25 + 0{\cdot}65 + 0{\cdot}10 = 1$, it is a probability distribution.

P(11 requests over two days)

$= P$(5 requests on the first day and 6 on the second)

$\quad + P$(6 request on the first day and 5 on the second)

$= (0{\cdot}25 \times 0{\cdot}65) + (0{\cdot}65 \times 0{\cdot}25)$

$= 0{\cdot}1625 + 0{\cdot}1625 = 0{\cdot}325$

P(12 requests over two days)

$= P$(5 requests on the first day and 7 on the second)

$\quad + P$(6 requests on the first day and 6 on the second)

$\quad + P$(7 requests on the first day and 5 on the second)

$= (0{\cdot}25 \times 0{\cdot}10) + (0{\cdot}65 \times 0{\cdot}65) + (0{\cdot}10 \times 0{\cdot}25)$

$= 0{\cdot}025 + 0{\cdot}4224 + 0{\cdot}025 = 0{\cdot}7975.$

Example 3 : Find the probability distribution of number of heads (X) obtained when a fair coin is tossed 4 times.

Sol. : When a coin is tossed 4 times, there are $2^4 = 16$ outcomes which are listed below.

HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT,
THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT.

(Write first H, T, H, T alternately for 16 times. Then 2 H's and 2 T's alternately, then 4 H's and 4 T's alternatively and lastly 8 H's and 8 T's alternately). Hence, the probability distribution of X is

X	:	0	1	2	3	4
$P(X = x)$	:	1 / 16	4 / 16	6 / 16	4 / 16	1 / 16

Example 4 : Write down the probability distribution of the sum of numbers appearing on the toss of two unbiased dice. (M.U. 2004, 06)

Sol. : When two dice are thrown, we get the sum of numbers as shown below by slant dotted lines.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

It is easy to see that the sum 2 appears once, 3 twice, 4 thrice etc. The probability of each single event is 1/36.

The probability distribution obtained is shown below.

X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Example 5 : For the above distribution, (i) find the probability that X is an odd number, (ii) find the probability that X lies between 3 and 9. (M.U. 1997)

Sol. : $P(X \text{ is odd}) = P(X = 3, 5, 7, 9 \text{ or } 11)$

$= P(X=3) + P(X=5) + P(X=7) + P(X=9) + P(X=11)$

$$\therefore\ P(X \text{ is odd}) = \frac{2}{36}+\frac{4}{36}+\frac{6}{36}+\frac{4}{36}+\frac{2}{36}=\frac{18}{36}=\frac{1}{2}$$

$P(3 \le X \le 9) = P(X = 3, 4, 5, 6, 7, 8 \text{ or } 9)$

$= P(X=3) + P(X=4) + \ldots + P(X=9)$

$$= \frac{2}{36}+\frac{3}{36}+\ldots+\frac{4}{36}=\frac{29}{36}.$$

Example 6 : Two unbiased dice are thrown. Let x_1 and x_2 denote the scores obtained and y denote the maximum of them *i.e.* $y = \max.(x_1, x_2)$. Find the probability distribution of Y.

Sol. : Refer to the chart of Ex. 4 above. It is easy to see that y takes values, 2, 3, 4, 5, 6 with probabilities 1/36, 3/36, 5/36, 7/36, 9/36, 11/36 respectively as shown by thick arrows.

Y	1	2	3	4	5	6
$P(Y=y)$	1/36	3/36	5/36	7/36	9/36	11/36

Example 7 : For the above distribution, (i) find the probability that X is an odd number, (ii) find the probability that X lies between 3 and 9.

Sol. : $P(X \text{ is odd}) = P(X = 3, 5, 7, 9 \text{ or } 11)$

$\therefore\ P(X \text{ is odd}) = P(X=3) + P(X=5) + P(X=7) + P(X=9) + P(X=11)$

$$= \frac{2}{36}+\frac{4}{36}+\frac{6}{36}+\frac{4}{36}+\frac{2}{36}=\frac{18}{36}=\frac{1}{2}$$

$P(3 \le X \le 9) = P(X = 3, 4, 5, 6, 7, 8 \text{ or } 9)$

$= P(X=3) + P(X=4) + \ldots + P(X=9)$

$$= \frac{2}{36}+\frac{3}{36}+\ldots+\frac{4}{36}=\frac{29}{36}.$$

Example 8 : The probability mass function of a random variable X is zero except at the points $X = 0, 1, 2$. At these points it has the values $P(0) = 3c^3$, $P(1) = 4c - 10c^2$, $P(2) = 5c - 1$.

(i) Determine c, (ii) Find $P(X < 1)$, $P(1 < X \le 2)$, $P(0 < X \le 2)$. (M.U. 2001)

Sol. : Since $\Sigma p_i = 1$, we have, $P(0) + P(1) + P(2) = 1$.

$\therefore\ 3c^3 - 10c^2 + 4c + 5c - 1 = 1$ $\quad \therefore\ 3c^3 - 10c^2 + 9c - 2 = 0$

$(3c - 1)(c - 2)(c - 1)$ $\quad \therefore\ c = 1/3$

(The other values are not admissible. Why?)

$\therefore$ The probability distribution is

X	0	1	2
$P(X=x)$	1/9	2/9	2/3

$$\therefore\ P(X < 1) = P(X = 0) = \frac{1}{9};\quad P(1 < X \le 2) = P(X = 2) = \frac{2}{3};$$

$$P(0 < X \le 2) = P(X = 1) + P(X = 2) = \frac{2}{9}+\frac{2}{3}=\frac{8}{9}.$$

Example 9 : A random variable X has the following probability distribution

X	0	1	2	3	4	5	6	7
$P(X=x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i) Find k, (ii) $P\left(\dfrac{1.5 < X < 4.5}{X > 2}\right)$, (iii) The smallest value of λ for which $P(X \le \lambda) > \dfrac{1}{2}$.

Sol. : (i) Since $\Sigma p(x_i) = 1$, we get

$10k^2 + 9k = 1$ $\quad \therefore\ 10k^2 + 9k - 1 = 0$

$\therefore\ 10k^2 + 10k - k - 1 = 0$ $\quad \therefore\ 10k(k+1) - 1(k+1) = 0$

$\therefore\ (10k - 1)(k + 1) = 0$ $\quad \therefore\ k = 1/10.$ k cannot be -1.

$\therefore$ The probability distribution of X is

X	0	1	2	3	4	5	6	7
$P(X=x)$	0	1/10	2/10	2/10	3/10	1/100	2/100	17/100

(ii) Now, $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$$\therefore\ P\left(\frac{1.5 < X < 4.5}{X > 2}\right) = \frac{P[(1.5 < X < 4.5) \cap (X > 2)]}{P(X > 2)} = \frac{P(2 < X < 4.5)}{P(X > 2)}$$

$$= \frac{P(X = 3, 4)}{P(X = 3, 4, 5, 6, 7)} = \frac{5/10}{70/100} = \frac{5}{7}.$$

Now from the table we find that

$$P(X \le 3) = P(X=0)+P(X=1)+P(X=2)+P(X=3)$$

$$= 0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10} = \frac{5}{10} = \frac{1}{2}.$$

Hence, $P(X \le 4) = \frac{8}{10} > \frac{1}{2}$. Hence, $\lambda = 4$.

Example 10 : An urn contains 4 white and 3 red balls. Find the probability distribution of the number of red balls in three draws made successively with replacement from the urn. **(M.U. 2007)**

Sol. : We get the following probabilities.

No. of red balls	Probability
0	$\frac{4}{7}\cdot\frac{4}{7}\cdot\frac{4}{7}$
1	$\left\{\begin{array}{l}\frac{3}{7}\cdot\frac{4}{7}\cdot\frac{4}{7}\\ \frac{4}{7}\cdot\frac{3}{7}\cdot\frac{4}{7}\\ \frac{4}{7}\cdot\frac{4}{7}\cdot\frac{3}{7}\end{array}\right.$
2	$\left\{\begin{array}{l}\frac{3}{7}\cdot\frac{3}{7}\cdot\frac{4}{7}\\ \frac{3}{7}\cdot\frac{4}{7}\cdot\frac{3}{7}\\ \frac{4}{7}\cdot\frac{3}{7}\cdot\frac{3}{7}\end{array}\right.$
3	$\frac{3}{7}\cdot\frac{3}{7}\cdot\frac{3}{7}$

$\therefore$ Probability distribution is

x	0	1	2	3	Total
$p(x)$	$\frac{64}{343}$	$\frac{144}{343}$	$\frac{108}{343}$	$\frac{27}{343}$	1

Example 11 : A random variable X has the following probability function :

X	:	1	2	3	4	5	6	7
$P(X=x)$	:	k	$2k$	$3k$	k^2	k^2+k	$2k^2$	$4k^2$

Find (i) k, (ii) $P(X<5)$, (iii) $P(X>5)$, (iv) $P\left(\frac{X<5}{2<X\le 5}\right)$, (v) $P\left(\frac{X=4}{3\le X\le 5}\right)$.

Sol. : Since $\Sigma\, p(x_i) = 1$,

$$k+2k+3k+k^2+k^2+k+2k^2+4k^2 = 1$$

$\therefore\ 8k^2+7k-1=0 \quad \therefore\ (8k-1)(k+1)=0$

$\therefore\ k = 1/8$ or $k=-1$ which is impossible (why ?)

Thus, we have the following probability distribution.

X	:	1	2	3	4	5	6	7
$P(X=x)$	:	1/8	2/8	3/8	1/64	9/64	2/64	4/64

(i) $P(X<5) = P(X=1,2,3,4,)$

$$= P(X=1)+P(X=2)+P(X=3)+P(X=4)$$

$$= \frac{1}{8}+\frac{2}{8}+\frac{3}{8}+\frac{1}{64} = \frac{49}{64}.$$

(ii) $P(X>5) = P(X=6,7) = P(X=6)+P(X=7)$

$$= \frac{2}{64}+\frac{4}{64} = \frac{6}{64} = \frac{3}{32}.$$

(iii) $$P\left(\frac{X<5}{2<X\le 6}\right) = \frac{P(X<5 \cap 2<X\le 6)}{P(2<X\le 6)} = \frac{P(2<X<5)}{P(2<X\le 6)}$$

$$= \frac{P(X=3,4)}{P(X=3,4,5,6)} = \frac{25/64}{36/64} = \frac{25}{36}.$$

(iv) $$P\left(\frac{X=4}{3\le X\le 5}\right) = \frac{P(X=4 \cap 3\le X\le 5)}{P(3\le X\le 5)}$$

$$= \frac{P(X=4)}{P(X=3,4,5)} = \frac{1/64}{34/64} = \frac{1}{64}.$$

Example 12 : The probability of a man hitting the target is 1/4. How many times must he fire so that the probability of his hitting the target at least once is greater than 2/3? **(M.U. 2016)**

Sol. : We are given that the probability of hitting the target $p = 1/4$.

Probability of not hitting the target, $q = 1-\frac{1}{4} = \frac{3}{4}$.

$\therefore$ Probability of not hitting the target in n trials

$$= \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\ldots\ldots(n \text{ times}) = \left(\frac{3}{4}\right)^n$$

$\therefore$ Probability of hitting the target at least once in n trials

$$= 1-\left(\frac{3}{4}\right)^n$$

We want this probability to be greater 2/3.

$\therefore\ 1-\left(\frac{3}{4}\right)^n > \frac{2}{3} \qquad \therefore\ 1-\left(\frac{2}{3}\right) > \left(\frac{3}{4}\right)^n$

$\therefore\ \frac{1}{3} > \left(\frac{3}{4}\right)^n \qquad \therefore\ \left(\frac{3}{4}\right)^n < \frac{1}{3}$

Taking logarithms of both sides

$$n(\log 3 - \log 4) < -\log 3$$

$\therefore\ n(0{\cdot}4771 - 0{\cdot}6021) < -\log 3 \qquad \therefore\ -n(0{\cdot}1250) < -0{\cdot}4771$

$\therefore\ n > \dfrac{0.4771}{0.1250} = 3.81 \qquad \therefore\ n > 4$

$\therefore$ He must fire at least 4 times, so that he will hit the target at least once, with probability of success greater 2/3.

EXERCISE - II

1. If $P(X = x) = x/25$, $x = 1, 3, 5, 7, 9$, find $P(X = 1 \text{ or } 3)$ and $P(4 < X < 8)$.

[**Ans. :** (i) 4 / 25, (ii) 12 / 25]

2. Verify whether the following functions can be considered as p.m.f. and if so find $P(X = 1 \text{ or } 3)$. Give reasons.

(i) $P(X = x) = \dfrac{1}{5}$, $x = 0, 1, 2, 3, 4$. (ii) $P(X = x) = \dfrac{x^2+1}{18}$, $x = 0, 1, 2, 3$.

(iii) $P(X = x) = \dfrac{x^2-2}{8}$, $x = 1, 2, 3$. (iv) $P(X = x) = \dfrac{2x+1}{18}$, $x = 0, 1, 2, 3$.

[**Ans. :** (i) Yes, 2 / 5 ; (ii) Yes, 2 / 3 ; (iii) No ; (iv) No]

3. If the p.m.f. $P(X = x)$ of a discrete random variate which assumes values x_1, x_2, x_3 such that $P(x_1) = 2P(x_2) = 3P(x_3)$, obtain the probability distribution of X.

[**Ans. :** (i) $P(x_1) = 2/11$, (ii) $P(x_2) = 3/11$, (iii) $P(x_3) = 6/11$]

4. Presuming the daily demand to be independent, find the probability that over a two -days period the number of requests at a service station will be (i) 9, (ii) 10, if the past record show that the demand was 4, 5 or 6 with probabilities 0·50, 0·40 or 0·10 respectively.

(**Hint :** The event can occur as (i) (4, 5), (5, 4); (ii) (4, 6), (6, 4), (5, 5).)

[**Ans. :** (i) 0.40, (ii) 0·26]

5. Find the probability distribution and the probability mass function of the number of points obtained when a fair die is tossed. [**Ans. :** $P(X = x_i) = 1/6$, $i = 1, 2, 3, 4, 5, 6$]

6. Find the probability distribution and the p.m.f. of the number of heads obtained when an unbiased coin is tossed three times.

[**Ans. :** $P(X = x) = \dfrac{{}^3C_x}{2^3}$, $x = 0, 1, 2, 3$]

7. The probability density function of a random variable X is

X	0	1	2	3	4	5	6
$P(X = x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find $P(X < 4)$, $P(3 < X \le 6)$. (M.U. 2001, 05, 10, 15)

[**Ans. :** k = 1/49, 16/49, 33/49]

8. A random variable X has the following probability function

X	1	2	3	4	5	6	7
$P(X = x)$	k	$2k$	$3k$	k^2	k^2+k	$2k^2$	$4k^2$

Find (i) k, (ii) $P(X < 5)$, (iii) $P(X > 5)$, (iv) $P(0 \le X \le 5)$.

[**Ans. :** (i) $k = 1/8$, (ii) 49 / 64, (iii) 3 / 32, (iv) 29 / 32]

9. A discrete random variable X has the following probability distribution

X	−2	−1	0	1	2	3
$P(X = x)$	0·1	k	0·2	$2k$	0·3	$3k$

Find (i) k, (ii) $P(X \ge 2)$, (iii) $P(-2 < X < 2)$. (M.U. 2009)

[**Ans. :** (i) $k = 1/15$, (ii) 1 / 2, (iii) 2 / 5]

10. Given the following probability function of a discrete random variable X

X	0	1	2	3	4	5	6	7
$P(X = x)$	0	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2 + c$

(i) Find c, (ii) Find $P(X \ge 6)$, (iii) $P(X < 6)$, (iv) Find k if, $P(X \le k) > 1/2$, where k is a positive integer, (v) $P(1.5 < X < 4.5 / X > 2)$. (M.U. 1996, 2003, 05)

[**Ans. :** (i) $c = 0.1$, (ii) 0·19, (iii) 0·81, (iv) $k = 4$, (v) 5 / 7]

11. In the example 9 above, find

(i) $P(-1 \le X \le 1 / -2 \le X \le 3)$, (ii) $P(X \le 2 / 0 \le X \le 4)$, (iii) $P(X \le 1 / X \le 2)$.

[**Ans. :** (i) $\dfrac{1}{2}$, (ii) $\dfrac{19}{26}$, (iii) $\dfrac{15}{24}$]

12. A random variable X takes values −2, −1, 0, 1, 2 such that $P(X > 0) = P(X = 0) = P(X < 0)$; $P(X = -2) = P(X = -1)$, $P(X = 1) = P(X = 2)$. Obtain the probability distribution of X.

Also find (i) $P\left(\dfrac{-1 \le X \le 1}{-2 \le X \le 0}\right)$, (ii) $P\left(\dfrac{X = 1}{0 \le X \le 2}\right)$.

[**Ans. :**

X :	−2	−1	0	1	2
$P(X = x)$:	1/6	1/6	1/3	1/6	1/6

(i) 4/5, (ii) 1/5]

13. A random variable X takes values 0, 1, 2, ... n with probabilities proportional to ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ {}^nC_n$. Find the proportionality constant.

(**Hint :** $k[{}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n] = 1 \quad \therefore\ k \cdot 2^n = 1$) [**Ans. :** $k = 2^{-n}$]

14. A random variable X assumes four values with probabilities $(1 + 3x)/4$, $(1 - x)/4$, $(1 + 2x)/4$ and $(1 - 4x)/4$. For what value of x do these values represent the probability distribution of X? (M.U. 2004)

[**Ans. :** $\Sigma p_i = 1$. But $\dfrac{1+3x}{4} \ge 0$ if $x \ge -\dfrac{1}{3}$ and also $\dfrac{1-4x}{4} \ge 0$ if $x \le \dfrac{1}{4}$

$\therefore\ -\dfrac{1}{3} \le x \le \dfrac{1}{4}$]

15. The amount of bread X (in hundred kgs) that a certain bakery is able to sell in a day is a random variable with probability density function is given by

$$f_X(x) = \begin{cases} Ax, & 0 < x < 5 \\ A(10 - x), & 5 < x < 10 \\ 0, & \text{elsewhere} \end{cases}$$

Find (i) A, (ii) the probabilities of the events : B, the amount of bread sold in a day is more than 500 kgs, C : the amount of bread sold in a day is less than 500 kgs, D : the amount

of bread sold in a day is between 250 kgs and 750 kgs. (iii) Are the events B and C exclusive? **(M.U. 2005)**
(iv) Are the events B and D exclusive.

[Ans. : (i) $A = 1/25$, (ii) $1/2$, $1/2$, 0.75, (iii) Yes. (iv) No]

10. Distribution Function of a Discrete Random Variable X

Probability distribution of X gives us the probability $p(x)$ that X will take a particular value x. Sometimes we need to know the probability that X will take a value less than or equal to a given value x. This probability is obtained by adding the probabilities of all values less than or equal to x.

Suppose, X is a discrete random variable taking values $x_1, x_2 \ldots\ldots x_n$ with probabilities $p(x_i)$, $i = 1, 2, \ldots\ldots n \ldots\ldots$ such that

(i) $p(x_i) \geq 0$ for all i,

(ii) $\Sigma\, p(x_i) = 1$ and consider the following table.

X	x_1	x_2	x_3	.....	x_n	.....
$F(x_i) = F(X = x_i)$	$p(x_i)$	$\sum_1^2 p(x_i)$	$\sum_1^3 p(x_i)$	.....	$\sum_1^n p(x_i)$	.....

The table states that

$$F(x_1) = P(a \leq X < x) = p(x_1)$$
$$F(x_2) = P(X \leq x_2) = p(a \leq X \leq x_1) + p(x_1 \leq X \leq x_2)$$
$$= p(x_1) + p(x_2)$$
$$F(x_3) = P(X \leq x_3) = p(a \leq X \leq x_1) + p(x_1 \leq X \leq x_2) + p(x_2 \leq X \leq x_3)$$
$$= p(x_1) + p(x_2) + p(x_3).$$

And so on.

The graph shown in the Fig. 6.17 shows this diagrammatically.

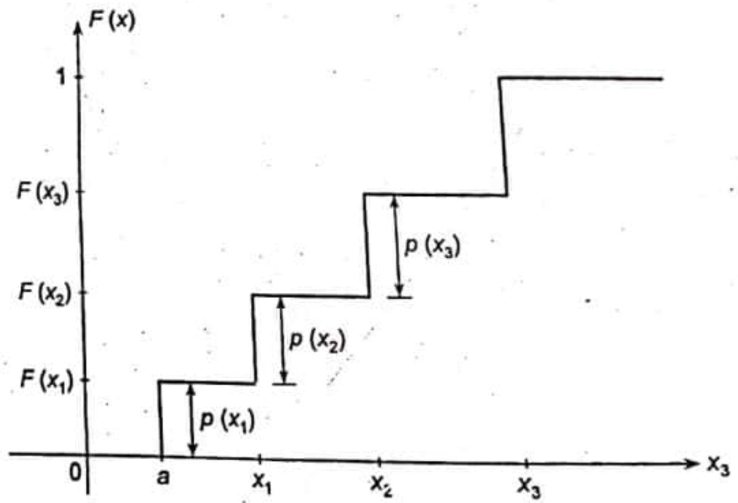

Fig. 6.17 : Graph of Distribution Function.

The function F is called the **distribution function.** We have more precise definition as follows.

(a) Definition

Let X be a discrete random variable taking values $x_1, x_2, \ldots\ldots$ such that $x_1 < x_2 < x_3 \ldots\ldots$ with probabilities $p(x_1), p(x_2), \ldots\ldots$ such that $p(x_i) \geq 0$ for all i and $\Sigma\, p(x_i) = 1$.

Consider F defined by

$$F(x_i) = p(X \leq x_i), \quad i = 1, 2, 3, \ldots\ldots$$

i.e. $$F(x_i) = p(x_1) + p(x_2) + \ldots\ldots + p(x_i)$$

then the function F is called the **cumulative distribution function** or simply **distribution function** and the set of pairs $\{x_i, F(x_i)\}$ is called the **cumulative probability distribution.**

Note ....

The distribution function is to the probability mass function as cumulative frequency distribution is to frequency distribution.

(b) Important Properties of Distribution function

The distribution function F of a random variable X has the following important properties.

1. $0 \leq F(x) \leq 1$

Proof : Since, $F(x_1) = p(x_1)$, $F(x_2) = p(x_1) + p(x_2)$

.... $F(x_n) = p(x_1) + p(x_2) + \ldots\ldots + p(x_n)$

and $0 \leq p(x_i) \leq 1$ for every i and $\Sigma\, p(x_i) = 1$, it is clear that $0 \leq F(x) \leq 1$.

2. $F(x) = 0$ for $x < a$ and $F(x) = 1$ for $x > b$ where $a < x_1 < x_2 < \ldots\ldots x_n < b$.

Proof : If $x < a$ where $a < x_1$, $p(x) = 0$ and hence, $F(x) = 0$ for $x < a$.

If $x > b$ then $F(x) = p(x_1) + p(x_2) + \ldots\ldots + p(x_n) = 1$.

3. $F(x)$ is a step function

Proof : By definition, $F(x_1) = F(X < x_1) = p(x_1)$

$F(x_2) = F(X < x_2) = p(x_1) + p(x_2)$

i.e. $F(X)$ has the same value $p(x_1)$ for $x_1 \leq x \leq x_2$,

and the same value $p(x_1) + p(x_2)$ for $x_2 \leq x \leq x_3 \ldots$

Hence, the graph of $F(x)$ is made up of horizontal line segments taking 'jumps' at the possible values x_i of X. The jump is of magnitude $p(x_i) = P(X = x_i)$.

Hence, $F(x)$ is a step functions.

Example 1 : A random variable X has the probability function given below :

$f(x) = k$ if $x = 0$; $f(x) = 2k$ if $x = 1$;

$f(x) = 3k$ if $x = 2$; $f(x) = 0$ otherwise.

(i) Determine the value of k, (ii) Evaluate $P(X < 2)$, $P(X \leq 2)$, $P(0 < X < 2)$, (iii) Obtain the distribution function.

Sol. : The probability distribution can be tabulated as

X	0	1	2
$p(x)$	k	$2k$	$3k$

(i) Since $\Sigma\, p_i = 1$, $k + 2k + 3k = 1$ $\therefore k = 1/6$.

(ii) $P(X<2) = P(X=0) + P(X=1) = k + 2k = 3/6 = 1/2$

$P(X \le 2) = P(X=0) + P(X=1) + P(X=2) = 6k = 1.$

$P(0 < X < 2) = P(X=1) = 2k = 1/3.$

(iii) Distribution function of X is

X	0	1	2
$F(x)$	1/6	1/2	1

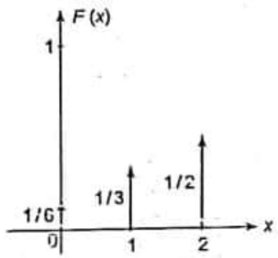

Fig. 6.18 (a)
Probability Density Function.

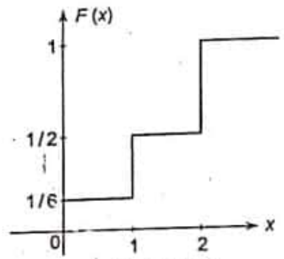

Fig. 6.18 (b)
Distribution Function.

EXERCISE - III

1. A random variable takes values 1, 2, 3, 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the probability distribution and the cumulative distribution function. (M.U. 2004)

[Ans. :

x	1	2	3	4
$P(x)$	15/61	10/61	30/61	6/61
$F(x)$	15/61	25/61	55/61	1

2. A shipment of 8 computers contains 3 that are defective. If a college makes a random purchase of 2 of these computers, find the probability distribution of the defective computers. Find also distribution function. (M.U. 2005)

[Ans. :

x	0	1	2
$P(x)$	10/28	15/28	3/28
$F(x)$	10/28	25/28	1

3. The probability density function of a random variable X is

X	0	1	2	3	4	5	6
$P(X=x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find $P(X<4)$, $P(3<X\le 6)$. [Ans. : $k = 1/49$, 16/49, 33/49]

11. Continuous Random Variable

Definition : A random variable is called a **continuous random variable** if it takes all values between an interval (a, b).

For example, age, height, weight are continuous random variables.

12. Probability Density Function of A Continuous Random Variable

Let $y = f(x)$ be a continuous function of x such that the area $f(x)\,\delta x$ represents the probability that X will lie in the interval $(x, x+\delta x)$. Symbolically,

$$P(x \le X \le x + \delta x) = f_x(x)\,\delta x$$

where, $f_x(x)$ denotes the value of $f(x)$ at x.

The adjoining figure denotes the curve $y = f(x)$ and the area under the curve in the interval $(x, x+\delta x)$. The function satisfying certain conditions giving the probability that x will lie between certain limits is called **probability density function**, or simply **density function** of a continuous random variable X and is abbreviated as p.d.f. The curve given by $y = f(x)$ is called the **probability density curve** or simply **probability curve**. The expression $f(x)\,dx$ is usually denoted by $df(x)$ and is known as **probability differential**.

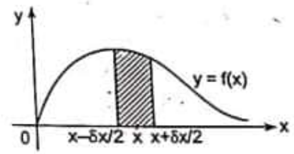

Fig. 6.19

Definition : A continuous function $y = f(x)$ such that

(i) $f(x)$ is integrable. (ii) $f(x) \ge 0$

(iii) $\int_a^b f(x)\,dx = 1$ if X lies in $[a, b]$ and

(iv) $\int_\alpha^\beta f(x)\,dx = P(\alpha \le X \le \beta)$ where $a < \alpha < \beta < b$

is called **probability desnity function** of a continuous random variable X.

Thus, for a continuous random variale X,

$$\boxed{P(\alpha \le X \le \beta) = \int_\alpha^\beta f(x)\,dx}$$

Clearly $\int_\alpha^\beta f(x)\,dx$ represents the area under the curve $y = f(x)$, the x-axis and the ordinates at $x = \alpha$ and $x = \beta$. Further since, the total probability is one, if X lies in the interval $[a, b]$ then $\int_a^b f(x)\,dx = 1$. For a continuous random variable X the range may be finite $[a, b]$ or infinite $[-\infty, \infty]$.

Properties of Probability Density Function

The probability density function $f(x)$ has the following properties.

(i) $f(x) \ge 0$, $-\infty < x < \infty$ (*i.e.* the curve $y = f(x)$ lies above the x-axis in the first and second quadrants only)

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (*i.e.* the total area under the curve and the x-axis is one.)

(iii) The probability that $\alpha \le X \le \beta$ is given by $P(\alpha \le X \le \beta) = \int_\alpha^\beta f(x)\,dx$.

Notes ....

1. The property (1) and the property (2) can be used to verify whether a given function $f(x)$ can be a probability density function.

2. You know that for A discrete random variable the probability at $X = c$ may not be zero. But, in a continuous random variable $P(X = c)$ is always zero because

$$P(X = c) = \int_c^c f_X(x)\,dx$$

and this definite integral is zero. Hence, for a continuous random variable X,

$$P(\alpha \le X \le \beta) = P(\alpha < X < \beta) = P(\alpha < X \le \beta) = P(\alpha < X < B)$$

In other words we may include or may not include the end-points in the interval.

3. Any function $f(x)$ of a real variable x can be a probability density function if it satisfies the first two properties given above *viz.* $f(x)$ is non-negative for all value of x and

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Sometimes $\int_{-\infty}^{\infty} f(x)\,dx$ is not equal to 1 but $\int_{-\infty}^{\infty} k f(x) = 1$ for some value of k.

In such cases k is called the ***normalising factor*** or ***normalisation constant***.

Example 1 : Find the normalising factor, k if the following function is a probability density function

$$f(x) = \begin{cases} k(1 - x^2) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Also find $P(0{\cdot}1 < X < 0{\cdot}2)$ and $P(X > 0{\cdot}5)$.

Sol. : Since $0 < x < 1$, $f(x) \ge 0$ for all x.

Now, $\int_0^1 f(x)\,dx = k\int_0^1 (1 - x^2)\,dx = k\left[x - \frac{x^3}{3}\right]_0^1 = k\frac{2}{3}$

But this must be equal to 1.

$$\therefore \quad \frac{2k}{3} = 1 \quad \therefore \quad k = \frac{3}{2}$$

(i) $P(0{\cdot}1 < X < 0{\cdot}2) = \int_{0\cdot1}^{0\cdot2} \frac{3}{2}(1 - x^2)\,dx = \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0\cdot1}^{0\cdot2}$

$$= \frac{3}{2}\left[\left\{0{\cdot}2 - \frac{(0{\cdot}2)^3}{3}\right\} - \left\{0{\cdot}1 - \frac{(0{\cdot}1)^3}{3}\right\}\right]$$

$$= \frac{3}{2}[0{\cdot}0977] = 0{\cdot}146$$

(ii) $P(X > 0{\cdot}5) = \int_{0\cdot5}^{1} \frac{3}{2}(1 - x^2)\,dx = \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0\cdot5}^{1}$

$$= \frac{3}{2}\left[\left\{1 - \frac{1}{3}\right\} - \left\{0{\cdot}5 - \frac{(0{\cdot}5)^3}{3}\right\}\right]$$

$$= \frac{3}{2}[0{\cdot}667 - 0{\cdot}458] = 0{\cdot}313$$

Example 2 : A continuous random variable X has the following probability law

$$f(x) = kx^2, \quad 0 \le x \le 2$$

Determine k and find the probabilities that (i) $0.2 \le X \le 0.5$, (ii) $X \ge 3/4$ given that $X \ge 1/2$. (M.U. 2005)

Sol. : Since the total probability *i.e.* the total area is unity

$$\int_a^b f(x)\,dx = \int_0^2 kx^2\,dx = 1$$

$$k\left[\frac{x^3}{3}\right]_0^2 = 1 \quad \therefore \quad k \cdot \frac{8}{3} = 1 \quad \therefore \quad k = \frac{3}{8}$$

(i) $P(0{\cdot}2 \le X \le 0{\cdot}5) = \frac{3}{8}\int_{0\cdot2}^{0\cdot5} x^2\,dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0\cdot2}^{0\cdot5} = \frac{1}{8}\left[0{\cdot}5^3 - 0{\cdot}3^3\right] = 0{\cdot}0123$

(ii) Let $A = (X \ge 1/2)$, $B = (X \ge 3/4)$

$$\therefore \quad P(A) = P(X \ge 1/2) = \frac{3}{8}\int_{0\cdot5}^{2} x^2\,dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0\cdot5}^{2}$$

$$= \frac{1}{8}\left[2^3 - 0{\cdot}5^3\right] = 0{\cdot}984$$

$$P(B) = P(X \ge 3/4) = \frac{3}{8}\int_{0\cdot75}^{2} x^2\,dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0\cdot75}^{2}$$

$$= \frac{1}{8}\left[2^3 - 0{\cdot}75^3\right] = 0{\cdot}947$$

$$P(A \cap B) = P(B) = 0{\cdot}947$$

$$\therefore \quad P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0{\cdot}947}{0{\cdot}984} = 0{\cdot}96$$

Example 3 : Let X be a continuous random variable with probability distribution

$$p(x) = \begin{cases} \dfrac{x}{6} + k & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

(M.U. 1998, 2004)

Evaluate k and find $P(1 \le x \le 2)$.

Sol. : Since the total probability is one

$$\int_{-\infty}^{\infty} p(x)\,dx = \int_0^3 \left(\frac{x}{6} + x\right)dx = \left[\frac{x^2}{12} + kx\right]_0^3 = \frac{3}{4} + 3k = 1$$

$$\therefore \quad 3\left(\frac{1}{4} + k\right) = 1 \quad \therefore \quad \frac{1}{4} + k = \frac{1}{3} \quad \therefore \quad k = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\therefore \quad p(x) = \begin{cases} \dfrac{x}{6} + \dfrac{1}{12} & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \quad P(1 \le x \le 2) = \int_1^2 \left(\frac{x}{6} + \frac{1}{12}\right) dx = \left[\frac{x^2}{12} + \frac{x}{12}\right]_1^2$$

$$= \frac{1}{12}[(4+2) - (1+1)] = \frac{1}{12}(4) = \frac{1}{3}.$$

Example 4 : Let X be a continuous random variable with p.d.f. $f(x) = kx(1-x)$, $0 \le x \le 1$. Find k and determine a number b such that $P(X \le b) = P(X \ge b)$.

(M.U. 2003, 11, 15)

Sol. : Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$, we have

$$k\int_0^1 (x - x^2)\,dx = 1 \quad \therefore k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$\therefore \quad k\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \quad \therefore k = 6.$$

Since, the total probability is 1 and $P(x \le b) = P(x \ge b)$, $P(x \le b) = 1/2$.

$$\therefore \quad \int_0^b f(x)\,dx = \frac{1}{2}$$

$$\therefore \quad 6\int_0^b (x - x^2)\,dx = \frac{1}{2} \quad \therefore \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b = \frac{1}{12}$$

$$\therefore \quad \frac{b^2}{2} - \frac{b^3}{3} = \frac{1}{12} \quad \therefore 6b^2 - 4b^3 = 1 \quad \therefore 4b^3 - 6b^2 - 1 = 0$$

$$\therefore \quad 4b^3 - 2b^2 - 4b^2 - 2b + 2b - 1 = 0$$

$$\therefore \quad (2b-1)(2b^2 - 2b + 1) = 0 \quad \therefore \quad b = 1/2.$$

Example 5 : The probability that a person will die in the time interval (t_1, t_2) is given by

$$P(t_1 \le t \le t_2) = \int_{t_1}^{t_2} f(t)\,dt$$

where, $$f(t) = \begin{cases} 3 \times 10^{-9}(100\,t - t^2)^2, & 0 < t < 100 \\ 0, & \text{elsewhere.} \end{cases}$$

Find (i) the probability that Mr. X will die between the ages 60 and 70. (ii) the probability that he will die between the ages 60 and 70, given that he has survived upto age 60.

(M.U. 2005)

Sol. : (i) $P(60 \le t \le 70) = \int_{60}^{70} 3 \times 10^{-9}(100\,t - t^2)^2\,dt$

$$= 3 \times 10^{-9} \int_{60}^{70} (100^2\,t^2 - 200\,t^3 + t^4)\,dt$$

$$= 3 \times 10^{-9}\left[100^2\frac{t^3}{3} - 200\frac{t^4}{4} + \frac{t^5}{5}\right]_{60}^{70}$$

$$\therefore \quad P(60 \le t \le 70) = 3 \times 10^{-9}\left[27.89 \times 10^8 - 22.75 \times 10^8\right]$$

$$= 0.1542.$$

(ii) $$P\left(\frac{60 \le T \le 70}{T \ge 60}\right) = \frac{P(60 \le t \le 70 \cap t \ge 60)}{P(t \ge 60)} = \frac{P(60 \le t \le 70)}{P(60 \le t \le 100)}$$

$$= \frac{\int_{60}^{70} f(t)\,dt}{\int_{60}^{100} f(t)\,dt} = \frac{0.1542}{0.3174} = 0.4858.$$

EXERCISE - IV

1. A function is defined as

$$f(x) = \begin{cases} 0 & \text{for } x < 2 \\ \dfrac{2x+3}{18} & \text{for } 2 \le x \le 4 \\ 0 & \text{for } x > 4 \end{cases}$$

Show that $f(x)$ is a probability density function and find the probability that $2 < x < 3$.

(M.U. 2005) [Ans. : 5 / 18]

2. A random variable X has the probability density function

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Find $P(1 \le X \le 3)$, $P(X \ge 0.5)$. **[Ans. : (i) 0.133, (ii) 0.368]**

3. A continuous random variable X has the following probability density function

$$f(x) = \begin{cases} kx & 0 \le x \le 1 \\ k & 1 \le x \le 2 \end{cases}$$

Find (i) the value of k, (ii) $P(x \le 1.5)$. **[Ans. : (i) $k = 2/3$, (ii) $2/3$]**

4. A continuous random variable has the following probability density function.

$$f(x) = \begin{cases} (x/4) + k & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Evaluate k and $P(1 \le X \le 2)$. **[Ans. : $k = 1/4$, $5/8$]**

5. Find the value of k such that the following will be the probability density function. Find also $P(x \le 1.5)$.

$$f(x) = \begin{cases} kx & 0 \le x \le 1 \\ k & 1 \le x \le 2 \\ k(3-x) & 2 \le k \le 3 \end{cases}$$

(M.U. 2003) [Ans. : $k = \frac{1}{2}, \frac{1}{2}$]

13. Continuous Distribution Function

Probability distribution of X or the probability density function of X helps us to find the probability that X will be within a given interval $[a, b]$ *i.e.* $P(a \le X \le b) = \int_a^b f(x)\,dx$, other conditions being satisfied.

However, sometimes we need to know that probability that X will be less than a given value x. For a continuous random variable X, this probability is obtained by integrating $f(x)$ from $-\infty$ (or the lower limit of the interval) to x. The function so obtained is called **distribution function.**

Definition : If X is a continuous random variable X, having the probability density function $f(x)$ then the function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt, \quad -\infty < x < \infty$$

is called **distribution function** or **cumulative distribution function** of the random variable X.

Some Important Properties of Distribution Function $F(x)$ of a Continuous Random Variable

1. The function $F(x)$ is defined for every real number x.
2. Since $F(x)$ denotes probability and probability of X lies between 0 and 1, $0 \le F(x) \le 1$.
3. $F(x)$ is a non-decreasing function which means if $x_1 \le x_2$, then $F(x_1) \le F(x_2)$.
4. **The derivative of $F(x)$** *i.e.* $F'(x)$ exists at all points (except perhaps at a finite number of points) and **is equal to the probability density function $f(x)$.**

$$F'(x) = \frac{d}{dx} F(x) = f(x) \ge 0 \text{ provided the derivative exists.}$$

5. If $F(x)$ is a distribution function of a continuous random variable then $P(a \le X \le b) = F(b) - F(a)$.

Example 1 : A continuous variable X has the following distribution function

$$F(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 \le x \le 1 \\ 1, & 1 \le x \end{cases} \qquad \text{.................... (1)}$$

Find the probability density function and draw the graphs of both p.d.f. and c.d.f.

Sol. : The p.d.f. is

$$f(x) = F'(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 0 & 1 < x \end{cases}$$

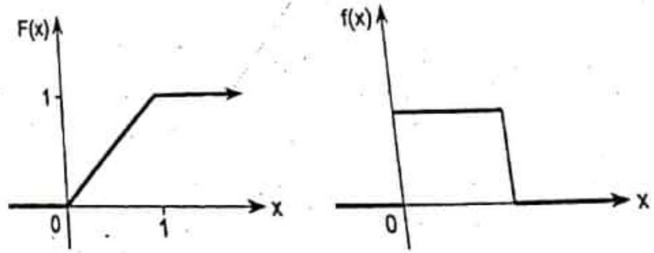

Fig. 6.20

From the graph of $F(x)$ we see that $F(x)$ is continuous at all points including $x = 0$ and $x = 1$. $f(x)$ is obtained by differentiating $F(x)$.

Example 2 : For the distribution function given below, find p.d.f.

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x/4}, & x \ge 0 \end{cases}$$

Also find the probabilities : $P(X \le 4)$, $P(X \ge 8)$, $P(4 \le X \le 8)$.

Sol. : $F(x)$ satisfies all the conditions of a distribution function. If $f(x)$ is the corresponding probability density function

$$f(x) = F'(x) = \begin{cases} \frac{1}{4} e^{-x/4}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

Now, we have to verify that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\therefore \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} \frac{1}{4} e^{-x/4}\,dx = 0 + \int_{0}^{\infty} \frac{1}{4} e^{-x/4}\,dx = \frac{1}{4}\left[\frac{e^{-x/4}}{-1/4}\right]_0^{\infty}$$

$$= -\left[e^{-x/4}\right]_0^{\infty} = -[0-1] = 1$$

Hence, $F(x)$ is a distribution function.

Now, $P(X \le 4) = F(4) = 1 - e^{-1} = 1 - \frac{1}{e} = \frac{e-1}{e}$

$$P(X \ge 8) = 1 - P(x \le 8) = 1 - F(8)$$

$$= 1 - \left[1 - e^{-2}\right] = e^{-2} = 1/e^2$$

$$P(4 \le X \le 8) = F(8) - F(4) = (1 - e^{-2}) - (1 - e^{-1})$$

$$= e^{-1} - e^{-2} = \frac{1}{e} - \frac{1}{e^2} = \frac{e-1}{e^2}$$

Example 3 : If $f(x) = \begin{cases} x e^{-x^2/2} & x \ge 0 \\ 0 & x < 0 \end{cases}$

(i) Show that $f(x)$ is a probability density function. (ii) Find its distribution function.

Sol. : If $f(x)$ is a p.d.f., we must have $f(x) \ge 0$ where $x \ge 0$ and $\int_0^{\infty} f(x)\,dx = 1$.

Clearly $f(x) = x e^{-x^2/2} \ge 0$ for $x \ge 0$.

Now, $\int_0^{\infty} x e^{-x^2/2}\,dx = \int_0^{\infty} e^{-t}\,dt \qquad [t = x^2/2]$

$$= \left[-e^{-t}\right]_0^{\infty} = -[0-1] = 1$$

$\therefore$ $f(x)$ is a probability density function.

Now, its distribution function is given by

$$F(x) = \int_0^x f(x)\,dx = \int_0^x x\,e^{-x^2/2}\,dx = -\left[e^{-x^2/2}\right]_0^x \qquad \text{[As above]}$$

$$= -\left[e^{-x^2/2} - 1\right] = 1 - e^{-x^2/2}, \quad x \ge 0.$$

EXERCISE - V

1. The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0 , & x < -1 \\ \dfrac{x+1}{4}, & -1 \le x \le 3 \\ 1 , & 3 < x \end{cases}$$

Find the probability density function. Draw the graphs of both p.d.f. and c.d.f.

2. The c.d.f. of a continuous random variable X is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

Find the p.d.f. Draw the graphs of both p.d.f. and c.d.f.

Also find $P\left(\frac{1}{2} \le X \le \frac{4}{5}\right)$. **[Ans. : 0·195]**

3. Find the distribution functions corresponding to the following probability density functions.

(i) $f(x) = \begin{cases} \frac{1}{2} x^2 e^{-x}, & 0 \le x < \infty \\ 0 , & \text{otherwise} \end{cases}$ (ii) $f(x) = \begin{cases} x , & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0 , & \text{otherwise} \end{cases}$

(iii) $f(x) = \begin{cases} \lambda (x-1)^4, & 1 \le x \le 3, \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$

[Ans. : (i) $F(x) = \begin{cases} 1 - e^{-x}\left(1 + x + \dfrac{x^2}{2}\right), & x \ge 0 \\ 0 , & \text{otherwise} \end{cases}$

(ii) $F(x) = \begin{cases} 0 , & x < 0 \\ x^2/2 , & 0 \le x \le 1 \\ 2x - 0{\cdot}5x^2 - 1, & 1 \le x \le 2 \\ 1 , & x > 2 \end{cases}$

(iii) $\lambda = \dfrac{5}{32}$; $F(x) = \begin{cases} 0 , & x \le 1 \\ \dfrac{(x-1)^5}{32}, & 1 \le x \le 3 \\ 1 , & x \ge 3 \end{cases}$ **]**

4. A continuous random variable X has the following probability density function

$$f(x) = \frac{a}{x^5}, \quad 2 \le x \le 10$$

Determine the constant a, distribution function of X and find the probability of the event $4 \le x \le 7$.

[Ans. : (i) $a = \dfrac{2500}{39}$, (ii) $\dfrac{625}{39}\left[\dfrac{1}{16} - \dfrac{1}{x^4}\right]$, (iii) 0·056 **]**

5. The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 1 - (1+x)e^{-x}, & x \ge 0 \\ 0 , & x < 0 \end{cases}$$

Find the probability density function and find $P(0 \le X \le 1)$.

[Ans. : (i) $f(x) = x\,e^{-x}$, (ii) $P(0 \le X \le 1) = F(1) - F(0) = \dfrac{e-2}{e}$ **]**

EXERCISE - VI

Theory

1. State and prove Bayes' Theorem.
2. Define the following terms giving suitable examples
   (a) Random variable
   (b) Discrete random variable
   (c) Continuous random variable
   (d) Probability density function
   (e) Distribution function
3. State the properties of probability density function.
4. State the properties of distribution function.

☆☆☆

CHAPTER 7

Mathematical Expectation

1. Introduction

Suppose two coins are tossed twenty times. Let X be the number of heads obtained in a toss. X then, takes values 0, 1 and 2. Suppose further that no heads, one head and two heads were obtained 4, 10, 6 times respectively. Then, the average number of heads per toss

$$= \frac{4(0) + 10(1) + 6(2)}{6 + 10 + 4} = 1.1$$

This is the average value and is not necessarily a possible outcome of the toss.

The ratios 4 / 20, 10 / 20, 6 / 20 of 0, 1, 2 heads to the total number of tosses are the relative frequencies of $X = 0, 1, 2$. If the experiment is repeated very large number of times, we know that, these relative frequencies tend to the probabilities 1 / 4, 1 / 2, 1 / 4 of 0, 1, 2 heads because in the toss of two coins we have the following.

Sample space	:	HH	$\underbrace{HT, TH}$	TT
Probability	:	1/4	1/2	1/4

The average calculated with probabilities in place of relative frequencies above is called **expected value** or **mathematical expectation** and is denoted by $E(X)$. Thus,

$$E(X) = \frac{1}{4}(0) + \frac{1}{2}(1) + \frac{1}{4}(2) = 1$$

$$E(X) = \text{sum of the products of the values and their probabilities}$$
$$= p_1 x_1 + p_2 x_2 + p_3 x_3 + ...$$

This means, a person who throws two coins over and over again will get one head per toss on the average. This suggests us that the expected value of X can be obtained by multiplying the values of X by their respective probabilities and taking the sum. This leads us to the following definition of expectation of a discrete random variable X.

2. Expectation of a Random Variable

(a) Definition : If a **discrete random variable** X assumes values $x_1, x_2, ..., x_n, ...$ with probabilities $p_1, p_2, ... p_n .....$ respectively then the **mathematical expectation of** X denoted by $E(X)$ (if it exists) is defined by

$$\boxed{E(X) = p_1 x_1 + p_2 x_2 + ... + p_n x_n + ...}$$

i.e. $\boxed{E(X) = \Sigma\, p_i x_i}$ where $\Sigma\, p_i = 1$.

if $\Sigma\, p_i x_i$ is absolutely convergent.

This value is also referred to as *mean value* of X. It is also denoted by μ_1'.

$\therefore \quad \mu_1' = E(X)$.

Notation : In this chapter we shall slightly deviate from our previous notation. Instead of denoting $P(X = x_i)$ by $p(x_i)$ we shall denote it simply by p_i. This will be found more convenient while dealing with expectations.

(b) Definition : Let X be a **continuous random variable** with probability density function $f(x)$. Then the **mathematical expectation of** X, denoted by $E(X)$ (if it exists), is defined by

$$\boxed{E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx} \qquad \text{where, } \int_{-\infty}^{\infty} f(x)\, dx = 1$$

if the integral is absolutely convergent.

Notes ....

1. If X assumes only a finite number of values then $E(X) = \Sigma\, p_i x_i$ and can be considered as "weighted average" of the values $x_1, x_2, ... x_n$ with weights $p_1, p_2, ..., p_n$.
2. If all values $x_1, x_2, ... x_n$ are equiprobable *i.e.* $p_1 = p_2 = ... = p_n = 1 / n$ then $E(X) = (1 / n) \Sigma\, x_i$ and can be seen to be simple arithmetic mean of the n values $x_1, x_2, ..., x_n$.
3. One should guard oneself from being mislead by the term 'expectation'. $E(X)$ does *not* give us the value of X, we can expect in a single trial. In the Example 4 below $E(X) = 7 / 2$ is not even a possible value of X when a die is tossed.

   $E(X)$ denotes *mathematical* expectation of X in the sense that if we toss a die for a fairly large number of times, observe the frequencies of the outcomes 1, 2, 3, 4, 5, 6 then the average of these values will be closer to 7 / 2 the more often the die were tossed.
4. $E(X)$ is expressed in the *same units* as X.
5. **Expectation of a constant is constant**

   (i) $E(c) = \Sigma\, p_i c = c \Sigma\, p_i = c \qquad [\because \Sigma\, p_i = 1]$

   (ii) $E(c) = \int_{-\infty}^{\infty} c\, f(x)\, dx = c \int_{-\infty}^{\infty} f(x)\, dx = c \qquad \left[\because \int_{-\infty}^{\infty} f(x)\, dx = 1\right]$

Example 1 : A fair coin is tossed 3 times. A person received ₹ X^2 if he gets X heads. Find his expectation. **(M.U. 2004)**

Sol. : When a coin is tossed three times, the sample space is

$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$

The probability distribution of X is

X	:	0	1	2	3
$P(X = x)$	:	1/8	3/8	3/8	1/8

Now, X^2 takes the following values

X^2	:	0	1	2	3
$P(X^2)$	:	1/8	3/8	3/8	1/8

$$\therefore \quad E(X) = \Sigma\, p_i x_i = \frac{1}{8} \times 0 + \frac{3}{8} \times 1 + \frac{3}{8} \times 4 + \frac{1}{8} \times 9$$

$$\therefore \quad E(X) = \frac{3+12+9}{8} = \frac{24}{8} = 3 \text{ ₹}$$

Example 2 : There are 10 counters in a bag, 6 of which are worth 5 rupees each while the remaining 4 are of equal but unknown value. If the expectation of drawing a single counter at random is 4 rupees, find the unknown value. **(M.U. 2015)**

Sol. : Let x be the value of the remaining 4 counters.

$$P(\text{of counter worth of ₹ } 5) = \frac{6}{10}$$

$$P(\text{of counter of unknown value}) = \frac{4}{10}$$

$$E(X) = \Sigma p_i x_i \qquad \therefore \quad 4 = \frac{6}{10} \cdot 5 + \frac{4}{10} \cdot x$$

$$\therefore \quad 40 = 30 + 4x \qquad \therefore \quad 4x = 10 \qquad \therefore \quad x = \text{₹ } 2{\cdot}5.$$

Example 3 : A fair coin is tossed till a head appears. What is the expectation of the number of tosses required ? **(M.U. 1996, 2010)**

Sol. : Let X denote the order of the toss at which we get the first head. We have

Event	H	TH	TTH	$TTTH$	......
X	1	2	3	4	......
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{16}$	......

$$\therefore \quad E(X) = \Sigma p_i x_i = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + \ldots..$$

Let

$$S = \frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 4\cdot\frac{1}{16} + \ldots..$$

$$\frac{1}{2}S = \qquad \frac{1}{4} + 2\cdot\frac{1}{8} + 3\cdot\frac{1}{16} + \ldots..$$

$$\therefore \quad S - \frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots..$$

$$\therefore \quad \frac{1}{2}S = \frac{1}{2}\cdot\frac{1}{1-(1/2)} = 1 \qquad \left[\text{G.P. } S_\infty = \frac{a}{1-r}\right]$$

$$\therefore \quad E(X) = 2.$$

Example 4 : Find the expectation of (i) the sum, (ii) the product of the number of points on the throw of n dice. **(M.U. 2004, 06)**

Sol. : Let X_i denote the number of points on the i th dice.

Then if S denotes the sum of the points of n dice then $S = \sum_{i=1}^{n} X_i$.

$$\text{Now,} \quad E(X_i) = \Sigma p_i x_i = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \ldots.. + \frac{1}{6}\cdot 6$$

$$\therefore \quad E(X_i) = \frac{1}{6}(1+2+3+3+4+5+6) = \frac{21}{6} = \frac{7}{2}$$

$$\therefore \quad S = \sum_{i=1}^{n} X_i = n\cdot\frac{7}{2} = \frac{7n}{2}$$

If π denotes the product of the points

$$E(\pi) = E(X_1)\cdot E(X_2) \ldots.. E(X_n) = \left(\frac{7}{2}\right)\cdot\left(\frac{7}{2}\right) \ldots.. \cdot \left(\frac{7}{2}\right) = \left(\frac{7}{2}\right)^n.$$

Example 5 : A box contains n tickets numbered 1, 2, ..., n. If m tickets are drawn at random from the box. What is the expectation of the sum of the numbers on the tickets drawn ? **(M.U. 2001)**

Sol. : Let X_i denote the number on the i th ticket drawn.

Then $S = X_1 + X_2 + \ldots.. + X_m$

$$\text{Now,} \quad E(X_i) = \Sigma p_i x_i = \frac{1}{n}\cdot 1 + \frac{1}{n}\cdot 2 + \ldots.. + \frac{1}{n}\cdot n$$

$$= \frac{1}{n}(1+2+\ldots..+n) = \frac{1}{n}\cdot\frac{n}{2}\cdot(n+1) = \frac{n+1}{2}$$

$$\therefore \quad E(S) = \sum_{i=1}^{m} E(X_i) = \frac{m(n+1)}{2}.$$

Example 6 : Three urns contain respectively 3 green and 2 white balls, 5 green and 6 white balls, 2 green and 4 white balls. One ball is drawn from each urn. Find the expected number of white ball drawn. **(M.U. 2007, 09)**

Sol. : If X denotes the number of white balls drawn from an urn then the expectation of X is as follows.

$$E(X) = p_i x_i = p_1 x_1 + p_2 x_2$$

x takes two values. $X = 1$ if white ball is drawn and $X = 0$ if greenball is drawn.

$$\therefore \text{ From first urn :} \qquad E(X_1) = 1\times\frac{2}{5} + 0\times\frac{3}{5} = \frac{2}{5}$$

$$\text{From second urn :} \qquad E(X_2) = 1\cdot\frac{6}{11} + 0\cdot\frac{5}{11} = \frac{6}{11}$$

$$E(X_3) = 1\cdot\frac{4}{6} + 0\cdot\frac{2}{6} = \frac{2}{3}$$

$$\therefore \text{ The required expectation } = E(X_1) + E(X_2) + E(X_3)$$

$$= \frac{2}{5} + \frac{6}{11} + \frac{2}{3} = \frac{266}{165} = 1{\cdot}61$$

Example 7 : A box contains 2^n tickets of which ${}^nC_r$ tickets bear the number r ($r = 0, 1, 2, \ldots.. n$). A group of m tickets is drawn. What is the expectation of the sum of their numbers ?

Sol. : Let $X_1, X_2, \ldots.., X_m$ be the variables denoting the number on the first, second, ..., m th ticket.

If S is the sum of the numbers on the tickets drawn then

$$S = \Sigma X_i \text{ and } E(S) = \Sigma E(X_i)$$

Now, X_i is a random variable which can take any one of the value 0, 1, 2, ..., n with probabilities ${}^nC_0/2^n, {}^nC_1/2^n, \ldots.., {}^nC_n/2^n$.

$$\therefore \quad E(X_i) = \Sigma p_i x_i$$

$$= \frac{1}{2^n}\left[0 \cdot {}^nC_0 + 1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 + 3 \cdot {}^nC_3 + \ldots\ldots + n \cdot {}^nC_n\right]$$

$$= \frac{1}{2^n}\left[1 \cdot n + 2 \cdot \frac{n(n-1)}{2!} + 3 \cdot \frac{n(n-1)(n-2)}{3!} + \ldots\ldots + n \cdot 1\right]$$

$$= \frac{n}{2^n}\left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \ldots\ldots + 1\right]$$

$$\therefore \quad E(X_i) = \frac{n}{2^n}[1+1]^{n-1} = \frac{n}{2^n} \cdot 2^{n-1} = \frac{n}{2}.$$

$$\therefore \quad E(S) = \Sigma E(X_i) = m \cdot \frac{n}{2} = \frac{mn}{2}.$$

Cor. 1 : If two tickets are drawn then putting $m = 2$, we get

$$E(\text{Sum}) = 2 \cdot \frac{n}{2} = n. \qquad \text{(M.U. 2003)}$$

Example 8 : A box contains 'a' white balls and 'b' black balls. 'c' balls are drawn from the box at random. Find the expected value of the number of white balls. (M.U. 2005)

Sol. : Let X_i be the variable denoting the result of the ith draw.

Let $X_i = 1$ if ith ball drawn is white and $X_i = 0$ if ith ball drawn is black.

Since, 'c' balls are drawn the sum of the white ball will be

$$S = X_1 + X_2 + \ldots\ldots + X_c = \sum_{i=1}^{c} X_i$$

Now, $P(X_i = 1) = P(\text{drawing a white ball}) = \dfrac{a}{a+b}$

$P(X_i = 0) = P(\text{drawing a black ball}) = \dfrac{b}{a+b}$

$$E(X_i) = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0)$$

$$= 1 \cdot \frac{a}{a+b} + 0 \cdot \frac{b}{a+b} = \frac{a}{a+b}$$

$$\therefore \quad E(S) = E(X_1) + E(X_2) + \ldots\ldots + E(X_c) = c \cdot \frac{a}{a+b} = \frac{ac}{a+b}.$$

Example 9 : A die is thrown until a five is obtained, find the expectation of the number of throws.

Sol. : Probability of getting 5 in the first toss = 1 / 6,

Probability of getting 5 in the second = (5 / 6) · (1 / 6),

Probability of getting 5 in third = (5 / 6) (5 / 6) (1 / 6) and so on,

and X takes values 1, 2, 3, ...

$$E(X) = 1(1/6) + 2(5/6)(1/6) + 3(5/6)^2(1/6) + \ldots\ldots$$

$$= (1/6)[1 + 2x + 3x^2 + \ldots\ldots] \qquad \text{where, } x = 5/6$$

$$= (1/6)(1-x)^{-2} = (1/6)[1-(5/6)]^{-2}$$

$$= 6.$$

Example 10 : A and B throw a fair die for a stake of ₹ 44, which is won by the player who throws 6 first. If A starts first, find their expectations.

Sol. : A can win the game, in the first throw or in the third throw or in the fifth throw and so on.

$$P(\text{A winning}) = \frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{5}{6}\right) \cdot \frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right) \cdot \frac{1}{6} + \ldots\ldots$$

$$\therefore \quad P(\text{A winning}) = \frac{1}{6}\left[1 + \left(\frac{25}{36}\right) + \left(\frac{25}{36}\right)^2 + \ldots\ldots\right] = \frac{1}{6} \cdot \frac{1}{1-(25/36)} = \frac{6}{11}.$$

$$P(\text{B winning}) = 1 - P(\text{A winning}) = \frac{5}{11}$$

$$\therefore \quad \text{Expectation of A} = p \cdot x = \frac{6}{11} \cdot 44 = ₹\ 24$$

$$\text{Expectation of B} = p \cdot x = \frac{5}{11} \cdot 44 = ₹\ 20.$$

Example 11 : A, B, C, D cut a pack of cards successfully in the order mentioned. The person who cuts a spade first wins ₹ 175. Find their expectations.

Sol. : Probability of cutting a spade $= \dfrac{13}{52} = \dfrac{1}{4}$.

Let A denote the success of A and $\bar{A}$ denote failure of A and so on.

Probability of A's success

$$= P(A) + P(\bar{A}\,\bar{B}\,\bar{C}\,\bar{D}\,A) + P(\bar{A}\,\bar{B}\,\bar{C}\,\bar{D} \cdot \bar{A}\,\bar{B}\,\bar{C}\,\bar{D}\,A) + \ldots\ldots$$

$$= \frac{1}{4} + \left(\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4}\right) + \left(\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4}\right) + \ldots\ldots$$

$$= \frac{1}{4} + \frac{81}{256} \cdot \frac{1}{4} + \left(\frac{81}{256}\right)^2 \cdot \frac{1}{4} + \ldots\ldots = \frac{1}{4} \cdot \frac{1}{1-(81/256)}$$

$$= \frac{1}{4} \cdot \frac{256}{256-81} = \frac{1}{4} \cdot \frac{256}{175} = \frac{64}{175}.$$

Probability of B's success

$$= P(\bar{A}\,B) + P(\bar{A}\,\bar{B}\,\bar{C}\,\bar{D} \cdot \bar{A}\,B) + \ldots\ldots$$

$$= \frac{3}{4} \cdot \frac{1}{4} + \left(\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}\right)\left(\frac{3}{4} \cdot \frac{1}{4}\right) + \ldots\ldots = \frac{3}{16}\left[1 + \frac{81}{256} + \ldots\ldots\right]$$

$$= \frac{3}{16}\left[\frac{1}{1-(81/256)}\right] = \frac{3}{16}\left[\frac{256}{175}\right] = \frac{48}{175}.$$

Probability of C's success

$$= \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} + \ldots\ldots = \frac{9}{64}\left[\frac{1}{1-(81/256)}\right] = \frac{9}{64} \cdot \frac{256}{256-81} = \frac{36}{175}.$$

Probability of D's success

$$= 1 - [P(A) + P(B) + P(C)] = 1 - \left[\frac{64}{175} + \frac{48}{175} + \frac{36}{175}\right] = \frac{27}{175}.$$

Now, the probabilities of A, B, C, D

$$\therefore \quad E(A) = p\,x = \frac{64}{175} \times 175 = ₹\ 64, \qquad E(B) = p\,x = \frac{48}{175} \times 175 = ₹\ 48,$$

$$E(C) = p\,x = \frac{36}{175} \times 175 = ₹\ 36, \qquad E(D) = p\,x = \frac{27}{175} \times 175 = ₹\ 27.$$

Example 12 : Find the expectation of number of failures preceeding the first success in an infinite series of independent trials with constant probabilities p and q of success and failure respectively. **(M.U. 1999, 2003, 17)**

Sol. : We have the following probability distribution

X	:	0	1	2	3	....
$P(X = x)$	:	p	qp	$q^2 p$	$q^3 p$	

Since, we may get success in the first trial where the number of failures $X = 0$ and the probability is p ; we may get success in the second trial when the number of failures $X = 1$ and the probability is $q\,p$ and so on.

$$\therefore \quad E(X) = \Sigma p_i x_i = p\,(0) + qp\,(1) + q^2 p\,(2) + q^3 p\,(3) + \ldots$$

$$= qp\left[1 + 2q + 3q^2 + \ldots\right] = qp\,(1-q)^{-2} = \frac{qp}{p^2} = \frac{q}{p}.$$

Example 13 : The daily consumption of electric power (in million kwh) is a random variable X with probability distribution function

$$f(x) = \begin{cases} kx\,e^{-x/3} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Find the value of k, the expectation of x and the probability that on a given day the electric consumption is more than expected value. **(M.U. 2003, 04, 16)**

Sol. : We must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad i.e. \quad k\int_0^{\infty} x\,e^{-x/3}\,dx = 1$$

$$\therefore \quad k\left[(x)\left(\frac{e^{-x/3}}{-1/3}\right) - (1)\left(\frac{e^{-x/3}}{1/9}\right)\right]_0^{\infty} = 1$$

$$\therefore \quad k\left[0 + 9\right] = 1 \quad \therefore \quad 9k = 1 \quad \therefore \quad k = 1/9$$

$$E(X) = \int_{-\infty}^{x} x\,f(x)\,dx = \frac{1}{9}\int_0^{\infty} x^2 \cdot e^{-x/3} \cdot dx$$

$$= \frac{1}{9}\left[x^2\left(\frac{e^{-x/3}}{-1/3}\right) - (2x)\left(\frac{e^{-x/3}}{1/9}\right) + 2\left(\frac{e^{-x/3}}{-1/27}\right)\right]_0^{\infty}$$

$$= \frac{1}{9}\left[0 - 0 + 0 + 54\right] = 6$$

$$\therefore \quad P(X > 6) = \frac{1}{9}\int_6^{\infty} x \cdot e^{-x/3} \cdot dx$$

$$\therefore \quad P(X > 6) = \frac{1}{9}\left[(x)\left(\frac{e^{-x/3}}{-1/3}\right) - (1)\left(\frac{e^{-x/3}}{1/9}\right)\right]_6^{\infty}$$

$$= \frac{1}{9}\left[(0 - 0) - (-18\,e^{-2} - 9e^{-2})\right]$$

$$= 3\,e^{-2} = 0.406$$

Example 14 : Find k and then $E(X)$ if X has the p.d.f.

$$f(x) = \begin{cases} k\,x(2-x), & 0 \le x \le 2,\ k > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Sol. : Now $\displaystyle\int_0^2 kx(2-x)\,dx = k\int_0^2 2x - x^2\,dx = k\left[x^2 - \frac{1}{3}x^3\right]_0^2 = k \cdot \frac{4}{3}$

$$\therefore \quad k \cdot \frac{4}{3} = 1 \quad \therefore \quad k = \frac{3}{4}$$

By definition

$$E(X) = \int_0^2 x\,f(x)\,dx = \int_0^2 x \cdot \frac{3}{4}x(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx$$

$$\therefore \quad E(X) = \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right] = \frac{3}{4} \cdot \frac{16}{12} = 1.$$

Example 15 : Find k and then $E(X)$ for the p.d.f.

$$f(x) = \begin{cases} k(x - x^2), & 0 \le x \le 1,\ k > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Sol. : Now $\displaystyle k\int_0^1 (x - x^2)\,dx = k \cdot \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = k\left[\frac{1}{2} - \frac{1}{3}\right] = k \cdot \frac{1}{6}$.

But $\quad k \cdot \dfrac{1}{6} = 1 \quad \therefore \quad k = 6$

By definition

$$E(X) = \int_0^1 x\,f(x)\,dx = \int_0^1 x \cdot 6(x - x^2)\,dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{6}{12} = \frac{1}{2}.$$

EXERCISE - I

1. From an urn containing 3 red balls and 2 white balls, a man is to draw 2 balls at random without replacement. He gets ₹ 20 for each red ball and ₹ 10 for each white ball he draws. Find his expectation. **[Ans. : ₹ 32]**
2. Two urns contain respectively 5 white and 3 black balls; 2 white and 3 black balls. One ball is drawn from each urn. Find the expected number of white balls drawn. **[Ans. : 41 / 40]**

3. A and B toss a fair coin alternately. One who gets a head first wins ₹ 12. A starts. Find their mathematical expectations. [Ans. : ₹ 8, 4]

4. A, B, and C toss a fair coin. The first one to throw a head wins the game and gets ₹ 28. If A starts, find their mathematical expectations. [Ans. : ₹ 16, 8, 4]

5. A, B and C draw a card in that order from a well shuffled pack of 52 cards. The first to draw a diamond wins ₹ 740. If A starts, find their expectations. [Ans. : ₹ 320, 240, 180]

6. Two unbiased dice are thrown. Find the expectation of the sum. (M.U. 2007) [Ans. : 7]

7. A man with n keys in his pocket wants to open the door of his case by trying the keys independently and randomly one by one. Find the mean and the variance of the number of trials required to open the door if unsuccessful keys are kept aside. (M.U. 1998) [Ans. : (i) $(n+1)/2$, (ii) $(n^2-1)/12$]

8. A player throwing an ordinary die is to receive ₹ $1/2^n$ where n denotes the number of throws required to get first 3. Find his expectation. (M.U. 2001, 04) [ans. : ₹ 1 / 7]

9. Three fair coins are tossed. Find the expectation and the variance of number of heads. (M.U. 2004) [Ans. : $\bar{x} = 3/2$, $\text{Var}(X) = 3/4$]

10. From a box containing n tickets bearing numbers 1, 2, 3, ..., n a ticket is drawn. If X denotes the number on the ticket drawn, find the mean and variance of X. [Ans. : $\bar{x} = (n+1)/2$, $\text{Var}(X) = (n^2-1)/2$]

11. In a game of chance a man is allowed to throw a fair coin indefinitely. He receives rupees 1, 2, 3, ... if he throws a head at the 1st, 2nd, 3rd, ... trial respectively. If the entry fee to participate in the game is ₹ 2, find the expected value of his net gain. [Ans. : Zero]

12. A person draws 3 balls from a bag containing 3 white, 4 red and 5 black balls. He is offered ₹ 10, ₹ 5 and ₹ 2 if he draws 3 balls of the same colour, 2 balls of the same colour and 1 ball of each colour respectively. Find his expectation. (M.U. 2004) [Ans. : $p_1 = \frac{3}{44}$, $p_2 = \frac{29}{44}$, $p_3 = \frac{12}{44}$; ₹ 4·52]

13. A continuous random variable X has the density function $f(x) = k(1+x)$ where $2 \le x \le 5$. Find k, $P(x \le 4)$ and $E(X)$. (M.U. 2005) [Ans. : $k = \frac{2}{27}$; $\frac{16}{27}$; $\frac{11}{3}$]

3. Expectation of a Function of a Random Variable X

We can now extend the concept of expectation of a random variable to the function of a random variable.

1. Definition : Let X be a discrete random variable taking values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ and $g(X)$ be a function of X then **mathematical expectation of $g(X)$** (if it exists) is defined by

$$\boxed{E[g(X)] = \Sigma p_i g(x_i)}$$

2. Definition : Let X be a continuous random variable with p.d.f. $f(x)$, let $g(X)$ be a function such that $g(X)$ is a random variable then $E[g(X)]$ (if it exists) is defined by,

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx}$$

Notes ....

1. If $g(X) = aX^n$, then $E[g(X)] = E[aX^n] = \Sigma p_i a x_i^n = a\Sigma p_i x_i^n = aE(X^n)$

And $E[g(X)] = E[aX^n] = \int_{-\infty}^{\infty} ax^n f(x)\,dx = a\int_{-\infty}^{\infty} x^n f(x)\,dx = aE(X^n)$

e.g. $E(aX) = aE(X)$

e.g. $E(aX^2) = aE(X^2)$, $E(aX^3) = aE(X^3)$

2. If $g(X) = aX + b$, then $E[g(X)] = E[aX+b] = aE(X) + b$.

3. It should be noted that, $E(X^2) \ne [E(X)]^2$ and $E(1/X) \ne 1/E(X)$

4. Putting $a = 1$, $E(X^n) = \Sigma p_i x_i^n$ and $E(X^n) = \int_{-\infty}^{\infty} x^n f(x)\,dx$

In particular $E(X^2) = \Sigma p_i x_i^2$ and $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$E(X^2)$ is denoted by μ_2'

$\therefore \quad \mu_2' = \Sigma p_i x_i^2 \quad$ or $\quad \mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

4. Mean and Variance

If we know the probability density function, discrete or continuous, we can find the mean and variance of the random variable as follows.

$\mu_1' = \text{Mean} = E(X) = \Sigma p_i x_i \quad$ or $\quad \mu_1' = \int_{-\infty}^{\infty} x f(x)\,dx$

We then find $\mu_2' = E(X^2) = \Sigma p_i x_i^2 \quad$ or $\quad \mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

Now,

$$\begin{aligned}\text{Var}(X) &= E(X - \bar{X})^2 = E[X - E(X)]^2 \\ &= E\left[X^2 - 2X\,E(X) + \{E(X)\}^2\right] \\ &= E(X^2) - 2E(X)\cdot E(X) + [E(X)]^2\end{aligned}$$

$$\boxed{\text{Var.}(X) = E(X^2) - [E(X)]^2}$$

But $E(X^2) = \mu_2'$ and $E(X) = \mu_1'$

$$\therefore \quad \boxed{\text{Var.}(X) = \mu_2' - \mu_1'^2}$$

Type I : Examples on Mean and Variance of a Discrete Probability Distribution

Example 1 : If X denotes the smaller of the two numbers that appear when a pair of dice is thrown, find the probability distribution of X, and also the mean and variance of X. (M.U. 2004)

Sol. : Refer to the table of Ex. 4 page 6-27.

We see that the number 1 appears as smaller (including equality) of the two numbers in 11 cases out of 36, the number 2 appears in 9 cases, 3 in 7 cases and so on.

The probability distribution of X is as given below.

X :	1	2	3	4	5	6
$P(X=x)$:	1/36	9/36	7/36	5/36	3/36	1/36

$$E(X) = \Sigma p_i x_i = \frac{11}{36}(1) + \frac{9}{36}(2) + \frac{7}{36}(3) + \frac{5}{36}(4) + \frac{3}{36}(5) + \frac{1}{36}(6) = 2{\cdot}5278$$

$$E(X^2) = \frac{11}{36}(1^2) + \frac{9}{36}(2^2) + \frac{7}{36}(3^2) + \frac{5}{36}(4^2) + \frac{3}{36}(5^2) + \frac{1}{36}(6^2)$$

$$= \frac{301}{36} = 8{\cdot}3611$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = 8{\cdot}3611 - (2{\cdot}5278)^2 = 1{\cdot}97$$

Example 2 : A discrete random variable has the probability density function given below.

X :	-2	-1	0	1	2	3
$P(X=x)$:	0·2	k	0·1	$2k$	0·1	$2k$

(M.U. 1997, 2001)

Find k, the mean and variance.

Solution : We must have $\Sigma p_i = 1$. Probability final value is 1 or below always.

$$\therefore \quad 5k + 0{\cdot}4 = 1 \quad \therefore \quad 5k = 0{\cdot}6 \quad \therefore \quad k = \frac{0{\cdot}6}{5} = \frac{3}{25}$$

Hence, the probability distribution is

X :	-2	-1	0	1	2	3
$P(X=x)$:	2/10	3/25	1/10	6/25	1/10	6/25

Now, Mean $= E(X) = \Sigma p_i x_i = -\frac{4}{10} - \frac{3}{25} + 0 + \frac{6}{25} + \frac{2}{10} + \frac{18}{25} = \frac{60}{250} = \frac{6}{25}$

$p_1x_1^2 + p_2x_2^2 + \ldots$ or $p_1x_1 + p_2x_2 + p_3x_3$

$$E(X^2) = \Sigma p_i x_i^2 = \frac{2}{10}(4) + \frac{3}{25}(1) + 0 + \frac{6}{25}(1) + \frac{1}{10}(4) + \frac{6}{25}(9) = \frac{73}{250}$$

$$\therefore \quad \text{Variance} = \sigma^2 = E(X^2) - [E(X)]^2 = \frac{73}{250} - \frac{36}{625} = \frac{293}{625}.$$

Example 3 : Find the value of k from the following data.

X :	0	10	15
$P(x)$:	$\frac{k-6}{5}$	$\frac{2}{k}$	$\frac{14}{5k}$

Also find the distribution function and the expectation of the distribution.

Sol. : Since $\Sigma p_i = 1$.

$$\frac{k-6}{5} + \frac{2}{k} + \frac{14}{5k} = 1 \quad \therefore \quad k^2 - 11k + 14 = 0$$

$$\therefore \quad (k-8)(k-3) = 0 \quad \therefore \quad k = 8 \text{ or } 3$$

But when $k = 3$ $P(X=0) = \frac{3-6}{5} = -\frac{3}{5}$ which is impossible. $\therefore$ $k = 8$.

$\therefore$ The p.d.f. and distribution function are

X :	0	10	15
$P(x)$:	2/5	1/4	7/20
$F(x)$:	2/5	13/20	1

$$\therefore \quad E(X) = \Sigma p_i x_i = \frac{2}{5}(0) + \frac{1}{4}(10) + \frac{7}{20}(15) = \frac{5}{2} + \frac{21}{4} = \frac{31}{4}.$$

Example 4 : If the mean of the following distribution is 16 find m, n and variance

X :	8	12	16	20	24
$P(X=x)$:	1/8	m	n	1/4	1/12

(M.U. 2006, 16)

Sol. : Since $\Sigma p_i = 1$,

$$\frac{1}{8} + m + n + \frac{1}{4} + \frac{1}{12} = 1 \quad \therefore \quad m + n = \frac{13}{24} \quad \ldots\ldots\ldots\ldots (1)$$

Since mean $= 16$, $\Sigma p_i x_i = 16$

$$\therefore \quad 1 + 12m + 16n + 5 + 2 = 16$$

$$\therefore \quad 12m + 16n = 8 \quad \therefore \quad 3m + 4n = 2 \quad \ldots\ldots\ldots\ldots (2)$$

Multiply (1) by 3 and subtract from (2),

$$\therefore \quad 3m + 4n = 2 \;;\; 3m + 3n = \frac{13}{8} \quad \therefore \quad n = \frac{3}{8}$$

$$\therefore \quad m + n = \frac{13}{24} \text{ gives } \quad m + \frac{3}{8} = \frac{13}{24} \quad \therefore \quad m = \frac{4}{24} = \frac{1}{6}$$

To find variance, consider

$$E(X^2) = \Sigma p_i x_i^2 = \frac{1}{8}(64) + \frac{1}{6}(144) + \frac{3}{8}(256) + \frac{1}{4}(400) + \frac{1}{12}(576) = 276$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 276 - 16^2 = 20.$$

Example 5 : A woman with n keys with her, wants to open the door of her house by trying keys independently and randomly one by one. Find the mean and the variance of the number of trials required to open the door, if unsuccessful keys are kept aside. **(M.U. 2016)**

Sol. : If unsuccessful keys are kept aside, she will get success in the first trial, or second trial or third trial and so on, the random variable X of the successful trial will take values 1, 2, 3, ..., n.

$$\therefore \quad P(\text{Success in the first trial}) = \frac{1}{n}$$

$$P(\text{Failure in the first trial}) = 1 - \frac{1}{n}$$

If there is failure in the first trial, the key is eliminated. There are now $(n-1)$ keys.

$$\therefore \quad P(\text{Success in the second trial}) = \frac{1}{n-1}$$

$\therefore$ P(Failure in the first trial and success in the second trial)

$$= \left(1 - \frac{1}{n}\right)\left(\frac{1}{n-1}\right) = \frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n}$$

$\therefore$ P(Failure in the first trial, failure in the second trial and success in the third trial)

$$= \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n-1}\right)\left(\frac{1}{n-2}\right)$$

$$= \frac{(n-1)}{n}\cdot\frac{(n-2)}{(n-1)}\cdot\frac{1}{(n-2)} = \frac{1}{n}$$

Thus, the probability of success at any trial remains constant $= \frac{1}{n}$.

Thus, the probability distribution of X is

X	:	1	2	3	4	....	n
$P(X=x)$	:	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	....	$\frac{1}{n}$

$$\therefore \quad E(X) = \Sigma\, x\, p(x) \cdot x = \frac{1}{n}\cdot 1 + \frac{1}{n}\cdot 2 + \frac{1}{n}\cdot 3 + \ldots + \frac{1}{n}\cdot n$$

$$= \frac{1}{n}(1+2+3+\ldots+n) = \frac{1}{n}\cdot\frac{n}{2}\cdot(n+1)$$

$$\therefore \quad E(X) = \frac{n+1}{2} \qquad \left[\because 1+2+\ldots+n = \frac{n+1}{2}\right]$$

$$E(X^2) = \Sigma\, p(x)\, x^2 = \frac{1}{n}\cdot 1^2 + \frac{1}{n}\cdot 2^2 + \frac{1}{n}\cdot 3^2 + \ldots + \frac{1}{n}\cdot n^2$$

$$= \frac{1}{n}(1^2+2^2+3^2+\ldots+n^2) = \frac{1}{n}\cdot\frac{n}{6}\cdot(n+1)(2n+1)$$

$$= \frac{(n+1)(2n+1)}{6} \qquad \left[\because 1^2+2^2+\ldots+n^2 = \frac{n}{6}(n+1)(2n+1)\right]$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{2(2n^2+3n+1) - 3(n^2+2n+1)}{12} = \frac{n^2-1}{12}.$$

Type II : Examples on Mean and Variance of a Continuous Probability Distribution

Example 1 : A continuous random variable X has the p.d.f. defined by $f(x) = A + Bx$, $0 \le x \le 1$. If the mean of the distribution is 1/3, find A and B. **(M.U. 2004, 14)**

Sol. : Since $f(x)$ is a probability distribution

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

By data $\int_0^1 (A+Bx)\,dx = 1$

$$\therefore \left[Ax + \frac{Bx^2}{2}\right]_0^1 = 1 \quad \therefore A + \frac{B}{2} = 1 \qquad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Since the mean is $\frac{1}{3}$. $\int_0^1 x\,f(x)\,dx = \frac{1}{3}$ $\quad\therefore \int_0^1 (A+Bx)\,x\,dx = \frac{1}{3}$

$$\therefore \int_0^1 (Ax+Bx^2)\,dx = \frac{1}{3} \quad \therefore \left[A\frac{x^2}{2} + \frac{Bx^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\therefore \frac{A}{2} + \frac{B}{3} = \frac{1}{3} \quad \therefore 3A + 2B = 2 \qquad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Solving the equations (i) and (ii), we get $A = 2$, $B = -2$.

$\therefore$ The p.d.f. is $f(x) = 2 - 2x$, $0 \le x \le 1$

Example 2 : The distribution function of a r.v. X is given by $F_X(x) = 1 - (1+x)\,e^{-x}$, $x \ge 0$. Find the mean and variance. **(M.U. 2011)**

Sol. : We have

$$f_X(x) = \frac{d\,F_X(x)}{dx} = (1+x)\,e^{-x} - e^{-x} = x\,e^{-x}, \quad x \ge 0$$

$$\therefore \quad \text{Mean } \bar{X} = \int_0^\infty x\cdot f_X(x)\,dx = \int_0^\infty x^2 e^{-x}\,dx$$

$$= \left[x^2(-e^{-x}) - 2x(e^{-x}) + 2(1)\cdot(-e^{-x})\right]_0^\infty = 2$$

$$E(X^2) = \int_0^\infty x^2 f_X(x)\,dx = \int_0^\infty x^3\cdot e^{-x}\,dx$$

$$E(X^2) = \left[x^3(-e^{x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x})\right]_0^\infty = 6$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2.$$

Example 3 : A continuous random variable X has the p.d.f. $f(x) = kx^2 e^{-x}$, $x \ge 0$. Find k, mean and variance. **(M.U. 2004)**

Solution : We must have $\int_0^\infty k\,x^2 e^{-x}\cdot dx = 1$

$$\therefore \quad k\left[x^2(-e^{-x}) - \int -e^{-x}2x\,dx\right]_0^\infty = 1$$

$$\therefore \quad k\left[-x^2e^{-x} + 2x(-e^{-x}) - \int -2e^{-x}dx\right]_0^\infty = 1 \qquad \text{[Integrating by parts]}$$

$$k\left[-x^2e^{-x} - 2x\,e^{-x} - 2e^{-x}\right]_0^\infty = 1$$

$$k[0 - (-0-0-2)] = 1 \quad \therefore 2k = 1 \quad \therefore k = \frac{1}{2}.$$

Now, mean $\bar{X} = \int_0^\infty x\,f(x)\,dx = \int_0^\infty \frac{1}{2}x^3 e^{-x}dx$

$$= \frac{1}{2}\left[x^3(-e^{-x}) - (3x^2)(e^{-x}) + (6x)(-e^{-x}) - (6)(e^{-x})\right]_0^\infty$$

(By the generalised rule of integration by parts.

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$$

where dashes denote the derivatives and suffixes denote the integrals.)

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$$\therefore \quad \overline{X} = \frac{1}{2}[0-(-6)] = \frac{1}{2}\cdot 6 = 3$$

Now, $\mu_2' = \frac{1}{2}\int_0^{\infty} x^2 f(x)\,dx = \frac{1}{2}\int_0^{\infty} x^2 \cdot x^2 e^{-x}\,dx$

$$= \frac{1}{4}\int_0^{\infty} x^4 \cdot e^{-x}dx$$

$$= \frac{1}{2}\left[x^4(-e^{-x}) - (4x^3)(e^{-x}) + (12x^2)(-e^{-x}) - (24x)(e^{-x}) + 24(-e^{-x})\right]_0^{\infty}$$

$$\therefore \quad \mu_2' = \frac{1}{2}[0-(-24)] = \frac{24}{2} = 12$$

$\therefore$ Variance $= \mu_2' - \mu_1'^2 = 12 - 9 = 3.$

Example 4 : If X is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} k(x-x^3), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) k, (ii) mean, (iii) variance. (M.U. 2016)

Sol. : (i) We have $\int_0^1 k(x-x^3)\,dx = 1$

$$\therefore \quad k\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 1 \qquad \therefore \quad k\left[\frac{1}{2} - \frac{1}{4}\right] = 1 \qquad \therefore \quad k\cdot\frac{1}{4} = 1 \qquad \therefore \quad k = 4$$

(ii) Mean, $\overline{X} = \mu_1' = \int_0^1 x\,f(x)\,dx = \int_0^1 x\cdot 4(x-x^3)\,dx$

$$= 4\int_0^1 (x^2-x^4)\,dx = 4\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 4\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{8}{15}$$

$$\mu_2' = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2\cdot 4(x-x^3)\,dx = 4\int_0^1 (x^3-x^5)\,dx$$

$$= 4\cdot\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 = 4\cdot\left[\frac{1}{4} - \frac{1}{6}\right] = 4\cdot\frac{1}{12} = \frac{1}{3}$$

(iii) Variance $(X) = \mu_2' - \mu_1'^2 = \frac{1}{3} - \frac{64}{225} = \frac{33}{225} = 0{\cdot}1467$.

Example 5 : Find the value of k, if the function

$$f(x) = kx^2(1-x^3), \quad 0 \le x \le 1$$
$$= 0 \quad \text{otherwise}$$

is a probability density function. Also find $P(0 \le x \le 1/2)$ and the mean and variance. (M.U. 2017)

Sol. : We have $\int_0^1 kx^2(1-x^3)\,dx = 1 \quad \therefore \quad \int_0^1 k(x^2-x^5)\,dx = 1$

$$\therefore \quad k\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = 1 \qquad \therefore \quad k\left[\frac{1}{3} - \frac{1}{6}\right] = 1 \qquad \therefore \quad k\cdot\frac{1}{6} = 1 \qquad \therefore \quad k = 6.$$

Now, $P(0 \le x \le 2) = 6\int_0^{1/2}(x^2-x^5)\,dx = 6\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^{1/2} = 6\left[\frac{1}{8\cdot 3} - \frac{1}{64\cdot 6}\right] = \frac{15}{64}.$

Mean $\overline{x} = \mu_1' = \int_0^1 x\,f(x)\,dx = 6\int_0^1 x\cdot x^2(1-x^3)\,dx = 6\int_0^1 (x^3-x^6)\,dx$

$$= 6\left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1 = 6\left[\frac{1}{4} - \frac{1}{7}\right] = \frac{18}{28} = \frac{9}{14}$$

$$\mu_2' = \int_0^1 x^2 f(x)\,dx = 6\int_0^1 x^2\left[x^2(1-x^3)\right]dx = 6\int_0^1 (x^4-x^7)\,dx$$

$$= 6\left[\frac{x^5}{5} - \frac{x^8}{8}\right]_0^1 = 6\left[\frac{1}{5} - \frac{1}{8}\right] = \frac{18}{40} = \frac{9}{20}$$

Variance $= \mu_2' - \mu_1' = \frac{9}{20} - \frac{91}{196} = \frac{441-405}{980} = \frac{36}{980} = \frac{9}{245}.$

Example 6 : A continuous random variable X has the following probability density function

$$f(x) = \begin{cases} kx & 0 \le x \le 2 \\ 2k & 2 \le x \le 4 \\ 6k - kx & 4 \le x \le 6 \end{cases}$$

Find k, $P(1 \le x \le 3)$ and the mean.

Sol. : For p.d.f. we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

$$\therefore \quad \int_0^2 kx\,dx + \int_2^4 2k\,dx + \int_4^6 (6k-kx)\,dx = 1$$

$$\therefore \quad k\left[\frac{x^2}{2}\right]_0^2 + 2k[x]_2^4 + k\left[6x - \frac{x^2}{2}\right]_4^6 = 1$$

$$\frac{k}{2}[4-0] + 2k[4-2] + k[(36-18)-(24-8)] = 1$$

$$2k + 4k + 6k = 1 \quad \therefore \quad 12k = 1 \quad \therefore \quad k = \frac{1}{2}$$

$$\therefore \quad P(1 \le x \le 3) = \int_1^2 \frac{x}{12}\,dx + \int_2^3 \frac{1}{6}\,dx = \frac{1}{12}\left[\frac{x^2}{2}\right]_1^2 + \frac{1}{6}[x]_2^3 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

Mean $\overline{x} = \mu_1' = \int_{-\infty}^{\infty} x\,f(x)\,dx$

$$= \frac{1}{12}\int_0^2 x\cdot x\,dx + \frac{1}{6}\int_2^4 x\,dx + \frac{1}{12}\int_4^6 x(6-x)\,dx$$

$$\therefore \text{ Mean } \bar{x} = \frac{1}{12}\left[\frac{x^3}{3}\right]_0^2 + \frac{1}{6}\left[\frac{x^2}{2}\right]_2^4 + \frac{1}{12}\left[3x^2 - \frac{x^3}{3}\right]_4^6$$

$$= \frac{8}{36} + \frac{1}{12}[16-4] + \frac{1}{12}\left[\left(108 - \frac{216}{3}\right) - \left(48 - \frac{64}{3}\right)\right]$$

$$= \frac{2}{9} + 1 + \frac{1}{12} + \frac{28}{3} = \frac{383}{36}.$$

Example 7 : If the distribution function of a random variable is given by

$$F(x) = \begin{cases} 1 - 4/x^2, & x > 2 \\ 0, & x \le 2 \end{cases}$$

find (i) $P(x < 3)$, (ii) $P(4 < x < 5)$, (iii) mean and the variance.

Sol. : The probability density function $f(x)$ is given by

$$f(x) = F'(x) = \begin{cases} 8/x^3, & x > 2 \\ 0, & x \le 2 \end{cases}$$

(i) $P(x < 3) = \int_2^3 \frac{8}{x^3}\,dx = \left[-\frac{8}{2x^2}\right]_2^3 = \left[-\frac{4}{x^2}\right]_2^3 = -4\left[\frac{1}{9} - \frac{1}{4}\right] = \frac{5}{9}.$

(ii) $P(4 < x < 5) = \int_4^5 \frac{8}{x^3}\,dx = \left[-\frac{4}{x^2}\right]_4^5 = -4\left[\frac{1}{25} - \frac{1}{16}\right] = 0.09$

(iii) $\mu_1' = \int_2^\infty x \cdot \frac{8}{x^3}\,dx = \int_2^\infty \frac{8}{x^2}\,dx = 8\left[-\frac{1}{x}\right]_0^\infty = -8\left[0 - \frac{1}{2}\right] = 4$

$\mu_2' = \int_2^\infty x^2 \cdot \frac{8}{x^3}\,dx = 8\int_2^\infty \frac{1}{x}\,dx = 8[\log x]_2^\infty = \infty$

$\therefore$ Var. $(x) = \mu_2' - \mu_1'^2$ $\qquad \therefore$ Var. (x) does not exist.

Example 8 : A continuous random variable has probability density function

$$f(x) = k(x - x^2), \quad 0 \le x \le 1$$

Find (i) k, (ii) mean, (iii) variance. **(M.U. 1997, 2001, 03, 15, 16)**

Sol. : (i) We have

$$\int_0^1 k(x - x^2)\,dx = 1 \qquad \therefore\ k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$\therefore\ k\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \qquad \therefore\ k \cdot \frac{1}{6} = 1 \qquad \therefore\ k = 6.$$

(ii) $\mu_1' = \int_0^1 x \cdot 6(x - x^2)\,dx = 6\int_0^1 (x^2 - x^3)\,dx$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}.$$

(iii) $\mu_2' = \int_0^1 x^2 \cdot 6(x - x^2)\,dx = 6\int_0^1 (x^3 - x^4)\,dx$

$$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 6\left[\frac{1}{4} - \frac{1}{5}\right] = \frac{6}{20} = \frac{3}{10}$$

$\therefore$ Var. $(x) = \mu_2' - \mu_1'^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$

EXERCISE - II

Type - I

1. The probability distribution of a random variable X is given by

X	:	-2	-1	0	1	2	3
$P(X = x)$	:	0·1	k	0·2	$2k$	0·3	k

Find k, the mean and variance. **[Ans. : (i)** $k = 0{\cdot}1$, **(ii)** $\bar{X} = 0{\cdot}8$, **(iii)** Var. 2·16]

2. If X denotes the larger of the two numbers that appear when a pair of dice is thrown, find the probability distribution of X, and also the mean and variance of X.

[Ans. :

X	:	1	2	3	4	5	6
$P(X = x)$	:	1/36	3/36	5/36	7/36	9/36	11/36

Mean = 4·47, Var. = 1·97]

3. Given the following distribution

X	:	-3	-2	-1	0	1	2
$P(X = x)$	:	0·01	0·1	0·2	0·3	0·2	0·15

Find (i) $P(X \ge 1)$, (ii) $P(X < 0)$, (iii) $E(X)$, (iv) $V(X)$. **(M.U. 1998)**

[Ans. : (i) 0·35, **(ii)** 0·35, **(iii)** 0·05, **(iv)** 1·8475]

4. Find the value of k from the following data.

X	:	0	10	15
$P(X = x)$	:	$(k-6)/5$	$2/k$	$14/5k$

Also find the distribution function and expectation of X. **(M.U. 2003)**

[Ans. : $k = 8$ or 3, 3 is impossible since $P(X = 0) = -\frac{3}{5}$ for $k = 3$

X	:	0	10	15
$F(x)$	:	2/5	13/20	1

$E(X) = 31/4$]

5. Find the mean and variance of X in Ex. 1 of Exercise II, page 6-31. **[Ans. :** 33/5, 136/25]

6. Find the mean and variance of X in Ex. 2 (ii) of Exercise II, page 6-31. **[Ans. :** 7/3, 14/9]

7. A random variable X has the probability law $P(X = x) = 1/n$, $x = 1, 2, \ldots, n$. Find $E(X)$ and $V(X)$. **[Ans. : (i)** $(n+1)/2$, **(ii)** $(n^2 - 1)/12$]

8. A random variable X has the probability distribution

X :	-2	-1	0	1	2	3
$p(x)$:	0·1	k	0·2	$2k$	0·3	k

Find k and then mean and variance of X. [Ans. : $k = 0{\cdot}1$, 0·8, 2·16]

9. Find the mean and the variance of the following distribution

X :	1	3	4	5
$P(X = x)$:	0·4	0·1	0·2	0·3

[Ans. : (i) $\overline{X} = 3$, (ii) Var. = 3]

10. A random variable X has the probability distribution $P(X=0) = P(X=2) = p$, $P(X=1) = 1-2p$ and $0 \le p \le 2/3$.
For what value of p is the Var. (X) maximum ? [Ans. : $p = 2/3$]

Type - II

1. Find the mean and the variance of the following distribution
$$f(x) = \begin{cases} 1-x, & 0 < x < 1 \\ x-1, & 1 < x < 2 \end{cases}$$
(M.U. 2006) [Ans. : (i) $\overline{X} = 1$, (ii) Var. = 1 / 2]

2. If the probability density function is given by
$$f(x) = \begin{cases} kx^2(1-x^3), & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$
(i) Find k, (ii) $P(0 < X < 1/2)$, (iii) $\overline{X}$, (iv) σ^2. (M.U. 1998, 2003, 04)
[Ans. : (i) $k = 6$, (ii) 15 / 64, (iii) 9 / 14, (iv) 9 / 245]

3. If the probability density of a random variable is given by
$$f(x) = \begin{cases} kx, & 0 \le x \le 2 \\ 2k, & 2 \le x \le 4 \\ 6k - kx, & 4 \le x \le 6 \end{cases}$$
Find (i) k, (ii) $P(1 \le X \le 3)$, (iii) $\overline{X}$. (M.U. 2002, 03, 05)
[Ans. : (i) 1 / 8, (ii) 7 / 16, (iii) 7 / 4]

4. Find the mean and the variance of
$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \end{cases}$$
(M.U. 2009) [Ans. : 1, 5 / 2]

5. If the probability density of a random variable is given by

(a) $f(x) = \begin{cases} kx\,e^{-x/3}, & x > 0 \\ 0, & x \le 0 \end{cases}$ (M.U. 2005)

(b) $f(x) = \begin{cases} kx^2\,e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$ (M.U. 2007)

(i) Find k, (ii) $\overline{X}$, (iii) σ^2. [Ans. : (a) (i) 1 / 9, (ii) 6, (iii) 18 ; (b) (i) 1 / 2, (ii) 3, (iii) 3]

6. A continuous random variable X has p.d.f. $f(x) = k\,x^2\,e^{-x}$, $x \ge 0$. Find k, mean and variance.
(M.U. 2004) [Ans. : (i) 1 / 2, (ii) 3, (iii) 3]

7. If the probability density of a random variable is given by
$$f(x) = \begin{cases} kx\,e^{-x/3}, & x > 0 \\ 0, & x \le 0 \end{cases}$$
(i) Find k, (ii) $\overline{X}$, (iii) σ^2. (M.U. 2005) [Ans. : (i) 1 / 9, (ii) 6, (iii) 18]

8. A continuous random variable has probability density function
$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$
Find (i) $E(X)$, (ii) $E(X^2)$, (iii) Var (X), (iv) S.D. of X. (M.U. 2005)
[Ans. : (i) $\frac{1}{2}$, (ii) $\frac{1}{2}$, (iii) $\frac{1}{4}$, (iv) $\frac{1}{2}$]

9. The length of time (in minutes) a lady speaks on telephone is found to be a random variable with probability density function
$$f(x) = \begin{cases} Ae^{-x/5} & \text{for } x \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$
Find A and the probability that she will speak for (i) more than 10 minutes, (ii) less than 5 minutes, (iii) between 5 and 10 minutes. (M.U. 2004)
[Ans. : (i) $A = \frac{1}{5}$, (ii) $\frac{1}{e^2}$, (iii) $\frac{(e-1)}{e}$, (iv) $\frac{(e-1)}{e^2}$]

10. The probability density function of a random variable is given by
$$f(x) = ke^{-x/\sigma}, \quad 0 < x < \infty$$
Find the mean and standard deviation of X. (M.U. 2002) [Ans. : (i) σ, (ii) σ]

11. A random variable X has the p.d.f. $f(x) = \dfrac{k}{1+x^2}$, $-\infty < x < \infty$. (M.U. 2004)
Determine k and the distribution function. Evaluate (i) $P(x \ge 0)$, (ii) Mean, (iii) Variance.
[Ans. : $k = \frac{1}{\pi}$, $F(x) = \frac{1}{\pi}\left[\tan^{-1} x + \frac{\pi}{2}\right]$, $P(x \ge 0) = \frac{1}{2}$, $\overline{X} = 0$, variance does not exist.]

12. If $f(x) = \begin{cases} x\,e^{-x^2/2}, & x \ge 0 \\ 0, & x < 0 \end{cases}$
prove that (i) $f(x)$ is a probability density function and (ii) obtain distribution function $F(x)$. (M.U. 1996)
[Ans. : (i) $\int_0^\infty x\,e^{-x^2/2}dx = \int_0^\infty e^{-t}dt = 1$ where $t = \frac{x^2}{2}$; (ii) $F(x) = 1 - e^{-x^2/2}$]

13. A continuous random variable X takes values between 2 and 5. Its density function is $f(x) = k(1+x)$. Find k and $P(x < 4)$. (M.U. 1996) [Ans. : $k = 2/27$, $16/27$]

14. The distribution function of a continuous random variable X is given by
$$F(x) = 1 - (1+x)\,e^{-x}, \quad x \ge 0.$$
Find the density function, mean and variance. [Ans. : $f(x) = xe^{-x}$, $x \ge 0$; 2 ; 2]

15. A continuous random variable X has a probability density function $f(x) = 3x^2, 0 \le x \le 1$. Find a and b such that (i) $P(X \le a) = P(X \ge a)$, (ii) $P(X > b) = 0.005$.

[Ans. : (i) $a = \sqrt[3]{1/2}$, (ii) $b = \sqrt[3]{95}$]

16. If $f(x)$ is probability density function of a continuous random variate, find k, mean and variance.

$$f(x) = \begin{cases} kx^2 & 0 \le x \le 1 \\ (2-x)^2 & 1 \le x \le 2 \end{cases}$$ (M.U. 2002)

[Ans. : $k = 2$, $\bar{x} = \frac{11}{12}$, Var. = 0·626]

17. A continuous random variable X has the probability density function given by

$$f(x) = \begin{cases} 2ax + b & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

If the mean of the distribution is 3, find the constants a and b. (M.U. 1996)

[Ans. : $a = \frac{3}{2}$, $b = -\frac{5}{2}$]

18. The probability density function of a random variable is given by

$$f(x) = \begin{cases} 0 & x < 2 \\ \frac{2x+3}{18} & 2 \le x \le 4 \\ 0 & x > 4 \end{cases}$$

Find the mean and variance. (M.U. 2001, 02) [Ans. : $\bar{X} = \frac{83}{27}$, Var. = 0·33]

19. Prove that $f(x) = \begin{cases} 1 - |1 - x| & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$

is a probability density function. Find its mean and variance. (M.U. 1998)

[Ans. : $\bar{X} = 12$, Var. = $\frac{1}{2}$]

20. A continuous random variable X has the following probability density function

$$f(x) = \begin{cases} x^3 & 0 \le x \le 1 \\ (2-x)^3 & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(0{\cdot}5 \le x \le 1{\cdot}5)$ and the mean of the distribution. (M.U. 1998) [Ans. : $\frac{15}{32}$, $\bar{X} = \frac{1}{2}$]

21. The probability density function of a continuous random variable X is given by $f(x) = kx(2-x),\ 0 \le x \le 2$.

Find k, mean and variance. (M.U. 1997, 99) [Ans. : $k = \frac{3}{4}$, $\bar{X} = 1$, $V(X) = \frac{1}{5}$]

5. Laws of Expectation

We shall now develop some laws or theorems that will simplify the calculations mathematical expectation of functions of random variables. These rules help us to calculate expectations in terms of known or easily computable expectations.

Theorem 1 : If X is a discrete random variable such that $x_i \ge 0$ for all i, then

$$\boxed{E(X) \ge 0}$$

Proof : Since $x_1, x_2, \ldots, x_n$ are all non-negative and since, $p_1, p_2, \ldots, p_n$ are also non-negative

$\therefore\ \Sigma p_i x_i \ge 0 \qquad \therefore\ E(X) \ge 0$

Theorem 2 : If X is a discrete (or continuous) random variate, a and b are constants then

$$\boxed{E(aX + b) = aE(X) + b}$$

Proof : We shall prove the result for discrete random variable and leave the continuous case to you. Let X take values $x_1, x_2, \ldots, x_i$, with probabilities $p_1, p_2, \ldots, p_n$. Then by definition,

$$\begin{aligned} E(aX + b) &= \Sigma p_i (a x_i + b) \\ &= p_1(ax_1 + b) + p_2(ax_2 + b) + \ldots + p_n(ax_n + b) \\ &= a(p_1x_1 + p_2x_2 + \ldots + p_nx_n) + b(p_1 + p_2 + \ldots + p_n) \\ &= aE(X) + b \qquad [\because \Sigma p_i = 1] \end{aligned}$$

Cor. 1 : Putting $a = 0$.

$E(b) = b$ i.e. **expectation of a constant is the constant itself.**

Cor. 2 : Putting $b = 0$.

$E(ax) = a\,E(X)$ i.e. for calculations **the constant can be taken out.**

Cor. 3 : Putting $a = 1$, $b = -\bar{X}$, $E(X - \bar{X}) = 0$.

Theorem 3 : Theorem of Addition (a) Discrete Variates

The expectation of the sum (or difference) of two (discrete) variates is equal to the sum (or difference) of their expectations.

In symbols, $\boxed{E(X \pm Y) = E(X) \pm (Y)}$

We shall accept the result without proof.

(a) **Theorem of Addition (Continuous Variates)**

The expectation of the sum (or difference) of two continuous variates is equal to the sum (or difference) of their expectations.

In symbols, $\boxed{E(X \pm Y) = E(X) \pm (Y)}$

We shall accept the result without proof.

Generalisation : The mathematical expectation of the sum (or difference) of n random variables is equal to **sum (or difference) of their expectations provided all the expectations exist.**

$$E(X_1 \pm X_2 \pm \ldots \pm X_n) = E(X_1) \pm E(X_2) \pm \ldots \pm E(X_n).$$

Theorem 4 : Theorem of Multiplication (a) Discrete variables : The expectation of the product of two **independent** variates is equal to the product of their expectations if the expectations exist.

In symbols, $\boxed{E(XY) = E(X) \cdot E(Y)}$

We shall accept the theorem without proof.

(b) Theorem of Multiplication (continuous variable)

The expectation of the product of two **independent variates** (discrete or continuous) is equal to the product of their expectations if the expectations exist.

In symbols, $\boxed{E(XY) = E(X) \cdot E(Y)}$

We shall accept the theorem without proof.

Generalisation : The mathematical expectation of the product of a number of independent continuous random variates is equal to the product of their expectations, provided all the expectations exist.

$$E(X, Y, Z \ldots\ldots W) = E(X) \cdot E(Y) \cdot E(Z) \ldots\ldots E(W).$$

5. Properties of Variance

We now prove some properties of variance which are useful in calculating variance.

1. Variance of a constant is zero, $V(C) = 0$.

Proof : $V(C) = E(C^2) - [E(C)]^2 = C^2 - C^2 = 0.$

2. If X **is a random variate and** a, b **are constants then**

$$\boxed{V(aX + b) = a^2\, V(X)}$$ (M.U. 2006)

Proof : By definition,

$$\begin{aligned} V(aX+b) &= E[(aX+b) - E(aX+b)]^2 = E[(aX+b) - a\,E(X) - b)]^2 \\ &= E[a\{X - E(X)\}]^2 = E[a^2\{X - E(X)\}^2] \\ &= a^2\, E[X - E(X)]^2 = a^2\, V(X). \end{aligned}$$

Cor. 1 : $\boxed{V(aX) = a^2\, V(X)}$

Putting $b = 0$, we get the result.

Cor. 2 : $\boxed{V(X+b) = V(X)}$

Putting $a = 1$, we get the result.

Note ....

Note that although $E(aX+b) = aE(X) + b$ we do not have $V(aX+b) = aV(X) + b$. Instead, we have $V(aX+b) = a^2 V(X)$.

3. $\boxed{V(a_1X_1 + a_2X_2) = a_1^2\, V(X_1) + a_2^2\, V(X_2)}$

where X_1 and X_2 are independent random variates.

Proof : Let $Y = a_1X_1 + a_2X_2$ where X_1 and X_2 are independent.

$$\begin{aligned} \therefore \quad \bar{Y} &= E(a_1X_1 + a_2X_2) \\ &= E(a_1X_1) + E(a_2X_2) \qquad \text{[By Theorem 3, page 7-22]} \\ &= a_1\, E(X_1) + a_2\, E(X_2) \end{aligned}$$

(This result is true even if X_1, X_2 are not independent.)

$$\begin{aligned} \therefore \quad V(Y) &= E[(a_1X_1 + a_2X_2) - \{a_1\, E(X_1) + a_2 E(X_2)\}]^2 \\ &= E[a_1\{X_1 - E(X_1)\} + a_2\{X_2 - E(X_2)\}]^2 \\ &= E\Big[a_1^2\{X_1 - E(X_1)\}^2 + a_2^2\{X_2 - E(X_2)\}^2 \\ &\qquad + 2a_1 a_2\{X_1 - E(X_1)\}\{X_2 - E(X_2)\}\Big] \\ &= a_1^2\, E\{X_1 - E(X_1)\}^2 + a_2^2\, E\{X_2 - E(X_2)\}^2 \\ &\qquad + 2a_1 a_2\, E\big[\{X_1 - E(X_1)\}\{X_2 - E(X_2)\}\big] \\ &= a_1^2\, E[X_1 - E(X_1)]^2 + a_2\, E[X_2 - E(X_2)]^2 \end{aligned}$$

$$+ 2a_1 a_2 [E(X_1)E(X_2) - E(X_1)E(X_2) - E(X_1)E(X_2) + E(X_1)E(X_2)]$$

[$\because$ $E(X_1X_2) = E(X_1) \cdot E(X_2)$ as X_1, X_2 aro independent.]

$$\therefore \quad V(a_1X_1 + a_2X_2) = a_1^2\, V(X_1) + a_2^2\, V(X_2)$$

Cor. 1 : If $a_1 = 1$, $a_2 = 1$, we get

$$\boxed{V(X_1 + X_2) = V(X_1) + V(X_2)}$$

If $a_1 = 1$, $a_2 = -1$, we get

$$\boxed{V(X_1 - X_2) = V(X_1) + V(X_2)}$$

Example 1 : If $E(X) = 2$ and $V(X) = 5$, find $E(3X+2)$ and $V(3X+2)$.

Sol. :
$$E(3X+2) = 3E(X) + 2 = 8$$
$$V(3X+2) = 9V(X) = 45.$$

Example 2 : If $E(X) = 1$ and $E(X^2) = 4$, find the mean and variance of $Y = 2X - 3$.

Sol. :
$$\begin{aligned} E(Y) &= E(2X-3) = E(2X) - E(3) \\ &= 2\,E(X) - 3 = 2 \cdot 1 - 3 = -1. \\ E(Y^2) &= E[(2X-3)^2] = E[\,4X^2 - 12X + 9\,] \\ &= E(4X^2) - E(12X) + E(9) \\ &= 4\,E(X^2) - 12E(X) + 9 \\ &= 4 \cdot 4 - 12 \cdot 1 + 9 = 13 \\ V(Y) &= E(Y^2) - [E(Y)]^2 = 13 - (-1)^2 = 12. \end{aligned}$$

Alternatively : $V(X) = E(X^2) - [E(X)]^2 = 4 - 1 = 3$

$\therefore$ $V(2X-3) = 4V(X) = 4 \cdot 3 = 12$ [By Cor. 1, 2, page 7-23]

Example 3 : If $E(X) = 10$, $\sigma_x^2 = 1$, and $Y = 2X(X+20)$, find $E(Y)$.

Sol. :
$$E(Y) = E[2X^2 + 20X] = E(2X^2) + E(20X) = 2E(X^2) + 20E(X)$$
Now, $E(X) = 10$
and $V(X) = E(X^2) - [E(X)]^2 = 1$
$\therefore$ $E(X^2) - 100 = 1$ $\quad\therefore E(X^2) = 101$
$\therefore$ $E(Y) = 2(101) + 20(10) = 202 + 200 = 402.$

Example 4 : Suppose that X is a r.v. with $E(X) = 10$ and $V(X) = 25$. Find the positive values of a and b such that $Y = aX - b$ has expectation 0 and variance 1.

Sol. : We have, $E(Y) = E[aX - b] = aE(X) - b$
$\therefore$ $0 = a10 - b$ $\quad\therefore b = 10a$
And $V(Y) = V(aX - b) = a^2 V(X)$
$\therefore$ $1 = a^2 \cdot 25$ $\quad\therefore a = 1/5.$
Thus, $a = 1/5$ and $b = 2$.

Example 5 : If X_1 has mean 4 and variance 9 and X_2 has mean -2 variance 4, and the two are independent, find $E(2X_1 + X_2 - 3)$ and $V(2X_1 + X_2 - 3)$.

Sol. : We have $E(X_1) = 4$, $V(X_1) = 9$, $E(X_2) = -2$ and $V(X_2) = 4$.
$$\therefore \quad E(2X_1 + X_2 - 3) = E(2X_1 + X_2) - 3 = 2E(X_1) + E(X_2) - 3 = 2(4) + (-2) - 3 = 3.$$
$$V(2X_1 + X_2 - 3) = V(2X_1 + X_2) \quad \text{[By above Cor. 2]}$$
$$= 2^2 V(X_1) + V(X_2) = 4(9) + 5 = 41.$$

EXERCISE - III

1. If $E(X) = 3$, $V(X) = 5$ and $Y = 3X + 4$, find $E(Y)$ and $V(Y)$. [Ans. : 13 ; 45]
2. Find $V(X + 2)$ and $V(3X + 2)$. [Ans. : $V(X)$, $9\,V(X)$]
3. Let X be a random variate with $E(X) = 12$ and $V(X) = 16$. Find the positive values of a and b such that $Y = aX - b$ has expectations 0 and variance 1.
[Ans. : $a = 1/4$, $b = 3$]
4. If X and Y are independent random variate, prove that
$$V(X + Y) = V(X) + V(Y)$$
(M.U. 2003)
5. Let X be a random variate with $E(X) = 15$ and $V(X) = 25$. Find the positive values of a and b such that $Y = aX - b$ has expectation 0 and variance 1. [Ans. : 1/5, 3]
6. If X_1 has mean 5 and variance 5, X_2 has mean -2 and variance 3, and if X_1, X_2 are independent, find :
(i) $E(X_1 + X_2)$, $V(X_1 + X_2)$
(ii) $E(X_1 - X_2)$, $V(X_1 - X_2)$
(iii) $E(2X_1 + 3X_2 - 5)$, $V(2X_1 + 3X_2 - 5)$
[Ans. : (i) 3, 8 ; (ii) 7, 8 ; (iii) -1, 52]

7. Covariance and Correlation

Definition : If X and Y are discrete random variates then the covariance between them denoted by cov. (X, Y) is defined by
$$\boxed{\text{cov.}(X, Y) = E[\{X - E(X)\} \cdot \{Y - E(Y)\}]} \quad \text{.................... (1)}$$
or
$$\text{cov.}(X, Y) = \frac{1}{n}\Sigma(x_i - \overline{X})(y_i - \overline{Y})$$

On simplification, we get another expression for cov. (X, Y)

Now, $\text{cov.}(X, Y) = E[XY - XE(Y) - YE(X) + E(X)E(Y)]$
$= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X) \cdot E(Y)$
$$\boxed{\text{cov.}(X, Y) = E(XY) - E(X)E(Y)} \quad \text{.................... (2)}$$

Example 1 : Prove that cov. $(aX, bY) = ab$ cov. (X, Y)

Sol. : Cov. $(aX, bY) = E(ab\,XY) - E(aX)E(bY) = ab\,E(XY) - ab\,E(X)E(Y)$
$= ab[E(XY) - E(X)E(Y)] = ab$ cov. (X, Y)

Definition of Correlation Coefficient r

Definition : If X, Y are two discrete random variates, then the correlation coefficient between them denoted by r is defined by
$$\boxed{r = \frac{\text{cov.}(X, Y)}{\sigma_x \sigma_y}}$$
where σ_x, σ_y are the standard deviations of x and y.

Theorem : If X, Y independent random variates, then they are not correlated.

Proof : By definition, coefficient of correlation $r = \dfrac{\text{cov.}(X, Y)}{\sigma_x \sigma_y}$.

But cov. $(x, y) = E(X, Y) - E(X) \cdot E(Y)$. [By (2)]
Since x, y are independent $E(X, Y) = E(X) \cdot E(Y)$
$\therefore$ cov. $(X, Y) = E(X) \cdot E(Y) - E(X) \cdot E(Y) = 0$
$\therefore$ $r = 0$

Note ....

However, the converse is not true *i.e.* if $r = 0$, it does not necessarily mean that the variables are not independent.

Consider the following series.

X	Y	$x_i - \overline{X}$	$y_i - \overline{Y}$	$(x_i - \overline{X})(y_i - \overline{Y})$
4	8	4	3	12
2	2	2	-3	-6
-2	2	-2	-3	6
-4	8	-4	3	-12
$\overline{X} = 0$	$\overline{Y} = 5$			Total 0

Since $\Sigma(x_i - \overline{X})(y_i - \overline{y}) = 0$, cov.$(X, Y) = 0$ $\therefore r = 0$

But it is clear that the above values of x and y satisfy the relation $y = \frac{1}{2}x^2$. This means X and Y are not independent.

Example : Let X and Y be two variates taking values 0 and 1. If $r(X, Y) = 0$, show that $X = 1$ and $Y = 1$ are independent events. (M.U. 2004)

Sol. : Let X take values 0, 1 with probability p and q.

Let Y take values 0, 1 with probabilities p' and q'.

Now, $r(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_x \sigma_y}$ $\because r = 0$, cov$(X, Y) = 0$

But cov $(X, Y) = E(X, Y) - E(X) \cdot E(Y)$

$\therefore E(X, Y) - E(X)\, E(Y) = 0$

$\therefore E(X, Y) = E(X) \cdot E(Y)$ .... (A)

Now, $E(X) = \Sigma p_i x_i = p \times 0 + q \times 1 = q$

$E(Y) = \Sigma p_i y_i = p' \times 0 + q' \times 1 = q'$

Now, $E(XY)\, \Sigma p_{ij} \cdot x_i \cdot y_j = p_{00}(0 \times 0) + p_{01}(0 \times 1) + p_{10}(1 \times 0) + p_{11}(1 \times 1) = p_{11}$

$\therefore$ By (A) $p_{11} = qq'$

$\therefore P(X = 1, Y = 1) = p(X = 1) \cdot p(Y = 1)$

Hence, the events $X = 1$ and $Y = 1$ are independent.

8. Raw and Central Moments

The following mathematical expectations have special significance in the study of probability distributions and hence, they are known by special names. They are denoted by special symbols.

(1) r-th moment about the origin (μ_r')

If we put $g(X) = X^r$, in § 3 (page 7-9), then the expectation is denoted by μ_r'. Thus,

$$\boxed{\mu_r' = E(X^r) = \Sigma p_i x_i^r} \quad \text{or} \quad \boxed{\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx}$$

This is called the r-th moment of the probability distribution of X about the origin, denoted by μ_r'.

Particular cases : (i) If $r = 0$, we get $\boxed{\mu_0' = 1}$

because $\mu_0' = E(X^0) = \Sigma p_i (x_i)^0 = \Sigma p_i = 1$

or $\mu_0' = E(X^0) = \int_{-\infty}^{\infty} x^0 f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx = 1$

(ii) If $r = 1$, we get $\boxed{\mu_1' = \overline{X}}$

because $\mu_1' = E(X) = \Sigma p_i x_i = \overline{X}$

or $\mu_1' = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \overline{X}$

(2) r-th moment about the value a (μ_r')

If we put $g(X) = (X - a)^r$, in § 3 (page 7-9), then the expectation is denoted by μ_r'. Thus,

$$\boxed{\mu_r' = E(X - a)^r = \Sigma p_i (x_i - a)^r}$$

or

$$\boxed{\mu_r' = E(X - a)^r = \int_{-\infty}^{\infty} (x - a)^r f(x)\, dx}$$

This is called the r-th raw moment of the probability distribution of X about a denoted by μ_r'.

Particular cases : (i) If $r = 0$, we get $\boxed{\mu_0' = 0}$

because $\mu_0' = E(X - a)^0 = \Sigma p_i (x_i - a)^0 = \Sigma p_i = 1.$

or $\mu_0' = E(X - a)^0 = \int_{-\infty}^{\infty} (x - a)^0 f(x)\, dx$

$= \int_{-\infty}^{\infty} f(x)\, dx = 1$

(ii) If $r = 1$, we get $\boxed{\mu_1' = \overline{X} - a}$

because $\mu_1' = E(X - a) = \Sigma p_i (x_i - a) = \Sigma p_i x_i - a \Sigma p_i = \overline{X} - a$

or $\mu_1' = E(X - a) = \int_{-\infty}^{\infty} (x - a) f(x)\, dx$

$= \int_{-\infty}^{\infty} x f(x)\, dx - a \int_{-\infty}^{\infty} f(x)\, dx = \overline{X} - a$

(3) r-th moment about the mean (μ_r)

If we put $g(X) = (X - \overline{X})^r$, in § 3 (page 7-9), then the expectation is denoted by μ_r'. Thus,

$$\boxed{\mu_r = E[(X - \overline{X})^r] = \Sigma p_i (x_i - \overline{X})^r}$$

or

$$\boxed{\mu_r = E(X - \overline{X})^r = \int_{-\infty}^{\infty} (x - \overline{X}) f(x)\, dx}$$

This is called the r-th moment of the probability distribution of X about the mean $\overline{X}$, denoted by μ_r.

Particular cases : (i) If $r = 0$, we get $\boxed{\mu_0 = 1}$

because $\mu_0 = \Sigma p_i (x_i - \overline{X})^0 = \Sigma p_i = 1$

or $\mu_0 = \int_{-\infty}^{\infty} (x - \overline{X})^0 f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx = 1$

(ii) If $r = 1$, we get $\boxed{\mu_1 = 0}$

because $\mu_1 = \Sigma p_i (x_i - \overline{X}) = \Sigma p_i x_i - \overline{X} \Sigma p_i = \overline{X} - \overline{X} = 0.$

or $\mu_1 = \int_{-\infty}^{\infty} (x - \overline{X}) f(x)\, dx = \int_{-\infty}^{\infty} x f(x)\, dx - \overline{X} \int_{-\infty}^{\infty} f(x)\, dx$

$= \overline{X} - \overline{X} = 0$

9. Moment Generating Function

(a) Definition (Discrete Random Variable) : The moment generating function (m.g.f.) of a discrete random variate X about a denoted by $M_a(t)$ is defined by

$$M_a(t) = E[e^{t(x-a)}] \quad \therefore \quad \boxed{M_a(t) = \Sigma\, p_i\, e^{t(x_i - a)}} \quad \text{.................. (1)}$$

The m.g.f. is a function of the real parameter t. The subscript a shows the point about which the m.g.f. is taken.

Expanding the exponential in (1), we get

$$M_a(t) = \Sigma p_i \left[1 + \frac{t}{1!}(x_i - a) + \frac{t^2}{2!}(x_i - a)^2 + \frac{t^3}{3!}(x_i - a)^3 + ...\right]$$

$$= \Sigma p_i + \frac{t}{1!}\Sigma p_i (x_i - a) + \frac{t^2}{2!}\Sigma p_i (x_i - a)^2 + \frac{t^3}{3!}\Sigma p_i (x_i - a)^3 + ... \quad \text{........ (1a)}$$

But $\Sigma\, p_i (x_i - a)^r$ is the r-th moment of X about a i.e. μ'_r. Hence, we have

$$M_a(t) = 1 + \mu'_1 \cdot \frac{t}{1!} + \mu'_2 \cdot \frac{t^2}{2!} + \mu'_3 \cdot \frac{t^3}{3!} + ... + \mu'_r \cdot \frac{t^r}{r!} + ... \quad \text{.................. (2)}$$

Thus, the coefficient of $(t^r / r!)$ is the r-th moment of X about a i.e. μ'_r. In this way $M_a(t)$ **generates moments. This is the** reason why the function $M_a(t)$ is called the **moment generating function.**

Thus,

$$\boxed{\mu'_r = \text{coefficient of } \frac{t^r}{r!}}$$

(b) Definition (Continuous Random Variable) : The moment generating function (m.g.f.) of a continuous random variate X about a denoted $M_a(t)$ is defined by

$$M_a(t) = E[e^{t(x-a)}] \quad \therefore \quad \boxed{M_a(t) = \int_{-\infty}^{\infty} e^{t(x-a)} \cdot f(x)\, dx} \quad \text{.................. (3)}$$

where $f(x)$ is the p.d.f. of X.

Expanding the exponential in (3), we get,

$$M_a(t) = \int_{-\infty}^{\infty} f(x) \cdot \left[1 + \frac{t}{1}(x-a) + \frac{t^2}{2!}(x-a)^2 + \frac{t^3}{3!}(x-a)^3 + ...\right] dx$$

$$M_a(t) = \int_{-\infty}^{\infty} f(x)\, dx + \frac{t}{1!}\int_{-\infty}^{\infty} (x-a) f(x)\, dx$$

$$+ \frac{t^2}{2!}\int_{-\infty}^{\infty} (x-a)^2 f(x)\, dx + \frac{t^3}{3!}\int_{-\infty}^{\infty} (x-a)^3 f(x)\, dx + ... \quad \text{........ (3a)}$$

But $\int_{-\infty}^{\infty} (x-a)^r f(x)\, dx$ is the r-th moment μ'_r of X about a. Hence,

$$M_a(t) = 1 + \mu_1' \cdot \frac{t}{1!} + \mu_2' \cdot \frac{t^2}{2!} + \mu_3' \cdot \frac{t^3}{3!} + ... + \mu_r' \cdot \frac{t^r}{r!} + ... \quad \text{.................. (4)}$$

Thus, the coefficient of $(t^r / r!)$ is the r-th moment of X about a.

(c) To find moments of various orders from m.g.f. : It is clear from (2) and (4) that the moment of order r is the coefficient of $(t^r / r!)$ in the expansion of m.g.f. Hence, one way of obtaining various moments is to obtain expansion of the m.g.f. of X and find the coefficient of $(t^r / r!)$ in the expansion.

However, in practice many a time obtaining the expansion of m.g.f. is not convenient. In such cases we differentiate m.g.f. w.r.t. t for r times and equate it to zero to get μ_r.

Thus, differentiating $M_a(t)$ from (2) (or from 4), successively, we get,

$$\frac{d}{dt}[M_a(t)] = \mu_1' + \mu_2' t + \mu_3' \frac{t^2}{2!} + ...$$

Putting $t = 0$, $\quad \frac{d}{dt}[M_a(t)]_{t=0} = \mu_1'$

$$\frac{d^2}{dt^2}[M_a(t)] = \mu_2' + \mu_3' t + \mu_4' \frac{t^2}{2!} + ...$$

Putting $t = 0$, $\quad \frac{d^2}{dt^2}[M_a(t)]_{t=0} = \mu_2'$

In general, $\quad \boxed{\frac{d^r}{dt^r}[M_a(t)]_{t=0} = \mu_r'}$

(d) Moment generating function about origin : Putting $a = 0$ in (1), we get,

$$\boxed{M_0 = \Sigma\, p_i\, e^{t x_i}} \quad \text{.................. (5)}$$

Putting $a = 0$ in (3), we get,

$$\boxed{M_0(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx} \quad \text{.................. (6)}$$

Putting $a = 0$ in (1a), we get since $\Sigma\, p_i x_i^r = \mu_r'$ about the origin.

$$M_0(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + ... + \mu_r' \frac{t^r}{r!} + ... \quad \therefore \quad M_0(t) = \Sigma\, \mu_r' \frac{t^r}{r!}$$

Note ....

It is assumed that the r.h.s. of (5) and (6) is absolutely convergent.

(e) If $L(t) = \log M(t)$ where $M(t)$ is the moment generating function of a random variable, prove that the mean = $L'(0)$ and variance = $L''(0)$. **(M.U. 2006)**

Proof : We have $\quad M(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2} + ...$

$$\therefore \quad L(t) = \log M(t) = \log\left[1 + \left(\mu_1' t + \mu_2' \frac{t^2}{2!} + ...\right)\right]$$

$$= \left(\mu_1' t + \mu_2' \frac{t^2}{2} + ...\right) - \frac{1}{2}\left(\mu_1' t + \mu_2' \frac{t^2}{2!} + ...\right)^2 + ...$$

$$\therefore \quad L(t) = \left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots\right) - \frac{1}{2}\left(\mu_1'^2 t^2 + \mu_1'\mu_2' t^3 + \ldots\right) + \ldots$$

$$\therefore \quad L'(t) = \mu_1' + \mu_2' t - \mu_1'^2 \cdot t - \frac{3}{2}\mu_1'\mu_2' t^2 + \text{terms in higher powers of } t$$

Putting $t = 0$, $\therefore L'(0) = \mu_1'$ .... (A)

Now, $L''(t) = \mu_2' - \mu_1'^2 - 3\mu_1'\mu_1'^2 t +$ terms in higher powers of t

Putting $t = 0$, $L''(0) = \mu_2' - \mu_1'^2$

Hence, Mean, $\mu = L'(0)$, Variance, $\mu_2 = L''(0)$. .... (B)

(f) Moment generating function of the sum of two independent random variates : ***"The moment generating function of the sum of two independent random variates is equal to the product of the m.g.f.s of the two variates."***

Proof : Let X, Y be two independent random variates then the m.g.f. of their sum $X + Y$ about the origin is given by

$$M_{(X+Y)}(t) = E[e^{t(X+Y)}] = E(e^{tX} \cdot e^{tY})$$

$$= E(e^{tX}) \cdot E(e^{tY}) \qquad [\because X \text{ and } Y \text{ are independent}]$$

$$\therefore \quad \boxed{M_{X+Y}(t) = M_X(t) \cdot M_Y(t)}$$

Generalisation : If $X_1, X_2, \ldots, X_n$ are n independent random variates, then the m.g.f. of their sum is equal to the product of their m.g.f.s

$$M_{X_1+X_2+\ldots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \ldots M_{X_n}(t)$$

(g) Uniqueness of Moment Generating Function : This is a very important property of m.g.f. It states that the m.g.f. of a distribution, if it exists, uniquely determines the distribution. In other words it means that for a given probability distribution there is one and only one m.g.f. and corresponding a given m.g.f. there is one and only one probability distribution. Thus, if m.g.f. of X and m.g.f. of Y are equal then X and Y must be identical.

Example 1 : Find the M.G.F. of the following distribution

X	:	-2	3	1
$P(X = x)$	:	1/3	1/2	1/6

Hence, find first four central moments. (M.U. 2010, 15)

Sol. : Since we want central moments we shall find m.g.f. about the mean.

Now, $\bar{X} = \Sigma p_i x_i = -\frac{2}{3} + \frac{3}{2} + \frac{1}{6} = \frac{-4+9+1}{6} = \frac{6}{6} = 1$

$\therefore$ M.G.F. about the mean

$$M_{\bar{x}}(t) = \Sigma p_i e^{t(x_i - \bar{x})} = \frac{1}{3} \cdot e^{t(-2-1)} + \frac{1}{2} e^{t(3-1)} + \frac{1}{6} e^{t(1-1)}$$

$$= \frac{1}{3} e^{-3t} + \frac{1}{2} e^{2t} + \frac{1}{6}$$

$$M_{\bar{x}}(t) = \frac{1}{3}\left[1 - 3t + \frac{9t^2}{2!} - \frac{27t^3}{3!} + \frac{81t^4}{4!} - \ldots\right] + \frac{1}{2}\left[1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \frac{16t^4}{4!} + \ldots\right] + \frac{1}{6}$$

Now, $\mu_r = \text{Coeff. of } \frac{t^r}{r!}$

$$\therefore \quad \mu_0 = \text{Coeff. of } t^0 = \text{constant term} = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1$$

$$\therefore \quad \mu_1 = \text{Coeff. of } \frac{t}{1!} = \frac{1}{3}(-3) + \frac{1}{2}(2) = 0$$

$$\mu_2 = \text{Coeff. of } \frac{t^2}{2!} = \frac{9}{3} + \frac{4}{2} = 5$$

$$\mu_3 = \text{Coeff. of } \frac{t^3}{3!} = \frac{1}{3}(-27) + \frac{1}{2}(8) = -5$$

$$\mu_4 = \text{Coeff. of } \frac{t^4}{4!} = \frac{1}{3}(81) + \frac{1}{2}(16) = 35.$$

Example 2 : Find the m.g.f. of the random variable X about the origin whose p.m.f. is given above in Ex. 1. Also find the first two moments about the ogirion. (M.U. 2004)

Sol. : By definition

$$\mu_0(t) = E(e^{tx}) = \Sigma p_i e^{tx_i} = \frac{1}{3} \cdot e^{-2t} + \frac{1}{2} e^{3t} + \frac{1}{6} e^t$$

Now, $$\mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = \left[-\frac{2}{3} e^{-2t} + \frac{3}{2} e^{3t} + \frac{1}{6} e^t\right]_{t=0}$$

$$= -\frac{2}{3} + \frac{3}{2} + \frac{1}{6} = \frac{6}{6} = 1$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = \left[\frac{4}{3} e^{-2t} + \frac{9}{2} e^{3t} + \frac{1}{6} e^t\right]_{t=0}$$

$$= \frac{4}{3} + \frac{9}{2} + \frac{1}{6} = \frac{36}{6} = 6.$$

Example 3 : A random variable X has probability density function $1/(2^x)$, $x = 1, 2, 3 \ldots$ Find the m.g.f. and hence, find the mean and variance.

Sol. : Since, $P(X = x) = \frac{1}{2^x}$, $x = 1, 2, 3 \ldots$

$$M_0(t) = E(e^{tx}) = \Sigma p e^{tx}$$

$$= \Sigma \frac{1}{2^x} e^{tx} = \sum_{x=1} \left(\frac{e^t}{2}\right)^x = \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \ldots$$

$$= \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \ldots\right] = \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1}$$

$$\therefore \quad M_0(t)=\frac{e^t}{2}\cdot\frac{1}{1-(e^t/2)}=\frac{e^t}{2}\cdot\frac{2}{2-e^t}=\frac{e^t}{2-e^t}$$

$$\therefore \quad \mu_1'=\left[\frac{d}{dt}M_0(t)\right]_{t=0}=\left[\frac{(2-e^t)e^t-e^t(-e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$=2\left[\frac{e^t}{(2-e^t)^2}\right]_{t=0}=2$$

$$\therefore \quad \mu_2'=\left[\frac{d^2}{dt^2}M_0(t)\right]_{t=0}=2\cdot\left[\frac{(2-e^t)^2\cdot e^t-e^t\cdot 2(2-e^t)\cdot(-e^t)}{(2-e^t)^4}\right]_{t=0}$$

$$=2\cdot\left[\frac{(2-e^t)\cdot e^t+2e^{2t}}{(2-e^t)^3}\right]_{t=0}=\frac{2(1+2)}{1}=6$$

$$\therefore \quad \mu_2=\mu_2'-\mu_1'^2=6-4=2$$

$\therefore$ Mean = Variance = 2.

Example 4 : If X denotes the outcome when a fair die is tossed, find M.G.F. of X and hence, find the mean and variance of X. **(M.U. 2005, 09)**

Sol. : We have, here X taking values 1, 2, 3, 4, 5, 6 each with probability 1/ 6.

$$M_0(t)=E(e^{tx_i})=\Sigma p_i e^{tx_i}$$

$$=\frac{1}{6}e^t+\frac{1}{6}e^{2t}+\frac{1}{6}e^{3t}+\ldots..+\frac{1}{6}e^{6t}$$

$$=\frac{1}{6}(e^t+e^{2t}+\ldots..+e^{6t})$$

$$\therefore \quad \mu_1'=\left[\frac{d}{dt}M_0(t)\right]_{t=0}=\frac{1}{6}\left[e^t+2e^{2t}+\ldots..+6e^{6t}\right]_{t=0}$$

$$=\frac{1}{6}[1+2+\ldots..+6]=\frac{21}{6}=\frac{7}{2}$$

$$\mu_2'=\left[\frac{d^2}{dt^2}M_0(t)\right]_{t=0}=\frac{1}{6}\left[e^t+4e^{2t}+9e^{3t}+\ldots..+36e^{6t}\right]_{t=0}$$

$$=\frac{1}{6}[1+4+9+\ldots..+36]=\frac{91}{6}$$

$$\therefore \quad \mu_2=\mu_2'-\mu_1'^2=\frac{91}{6}-\frac{49}{4}=\frac{35}{12}.$$

$$\therefore \quad \text{Mean}=\frac{7}{2} \text{ and Variance }=\frac{35}{12}.$$

Example 5 : Find the m.g.f. of a random variable X if the r-th moment about the origin is given by $\mu_r'=r!$.

Sol. : Be definition, $\mu_r'=E(x^r)=r!$

$\therefore \quad E(x)=1, \quad E(x^2)=2!\qquad E(x^3)=3!\ldots..$ .... (1)

Now, $M_0(t)=E(e^{tx})$

$$=E\left[1+tx+\frac{t^2x^2}{2!}+\frac{t^3x^3}{3!}+\ldots..\right]$$

$$=E(1)+t\,E(x)+\frac{t^2}{2!}E(x^2)+\frac{t^3}{3!}E(x^3)+\ldots..$$

Putting the values of $E(X)$, $E(X)^2$, ... from (1)

$$M_0(t)=1+t+\frac{t^2}{2!}\cdot 2!+\frac{t^3}{3!}\cdot 3!+\ldots..$$

$$=1+t+t^2+t^3+\ldots..=(1-t)^{-1}=\frac{1}{1-t}.$$

Example 6 : Find the m.g.f. of a random variable whose p.m.f. is

$$P(X=x)=\left(\frac{1}{2}\right)^x,\quad x=1,2,3\ldots..$$
$$=0 \qquad \text{else where}$$

Hence, find the mean and variance of X.

Sol. : By definition

$$M_0(t)=E(e^{tx})=\Sigma p_i e^{tx_i}=\Sigma\frac{1}{2^x}e^{tx}=\Sigma\left(\frac{e^t}{2}\right)^x$$

$$M_0(t)=\frac{e^t}{2}+\left(\frac{e^t}{2}\right)^2+\left(\frac{e^t}{2}\right)^3+\ldots..=\frac{e^t}{2}\left[1+\left(\frac{e^t}{2}\right)+\left(\frac{e^t}{2}\right)^2+\ldots.\right]$$

$$=\frac{e^t}{2}\,\frac{1}{1-(e^t/2)}=\frac{e^t}{2-e^t}.$$

To find the mean and variance

$$\mu_1'=\left[\frac{d}{dt}M_0(t)\right]_{t=0}=\left[\frac{(2-e^t)\cdot e^t-e^t\cdot(-e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$=\left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0}=2$$

$$\mu_2'=\left[\frac{d^2}{dt^2}M_0(t)\right]_{t=0}=\left[\frac{(2-e^t)^2\cdot e^t-e^t\cdot 2(2-e^t)(-e^t)}{(2-e^t)^4}\right]_{t=0}$$

$$=\frac{2[1+2]}{1}=6$$

$$\therefore \quad \text{Variance}=\mu_2'-\mu_1'^2=6-4=2.$$

Remark ....

Mean and variance of the above distribution can be obtained using $E(X)$ and $E(X)^2$. Find them.

Example 7 : A random variable X has the following probability density function

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0. & \text{otherwise} \end{cases}$$

Find m.g.f., μ_r', mean, and variance.

Sol. : We have

$$M_0(t) = E(e^{tx})$$

$$= \int_0^1 e^{tx}(1)\,dx = \left[\frac{e^{tx}}{t}\right]_0^1 = \frac{1}{t}\left[e^t - 1\right]$$

$$= \frac{1}{t}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\ldots - 1\right] = \frac{1}{t}\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\ldots\right]$$

$$= 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\ldots + \frac{t^r}{(r+1)!}$$

$$\therefore \quad \mu_r' = \text{Coefficient of } \frac{t^r}{r!} = \frac{1}{r+1}, \quad r = 1, 2, \ldots\ldots$$

Putting $r = 1, 2$

$$\therefore \quad \text{Mean} = \mu_1' = \frac{1}{2} \,;\; \mu_2' = \frac{1}{3}$$

$$\therefore \quad \text{Var. } (X) = \mu_2' - {\mu_1'}^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Example 8 : A random variable X has the following probability density function

$$f(x) = \begin{cases} ke^{-kx}, & x > 0,\ k > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the m.g.f. and hence, the mean and variance.

Sol. : We have

$$M_0(t) = E(e^{tx}) = \int_0^\infty e^{tx} \cdot k e^{-kx}\,dx = k\int_0^\infty e^{(t-k)x}dx$$

$$= \frac{k}{t-k}\left[e^{(t-k)x}\right]_0^\infty = \frac{k}{t-k}[0-1] = \frac{k}{k-t} \qquad [t \neq k]$$

$$\text{Now,} \quad M_0(t) = \frac{k}{k[1-(t/k)]} = \left[1 - \frac{t}{k}\right]^{-1} = 1 + \frac{t}{k} + \frac{t^2}{k^2} + \frac{t^3}{k^3} + \ldots\ldots$$

$$\therefore \quad \mu_1' = \text{Coefficient of } t = \frac{1}{k}; \qquad \mu_2' = \text{Coefficient of } \frac{t^2}{2!} = \frac{2}{k^2}$$

$$\therefore \quad \text{Mean} = \mu_1' = \frac{1}{k};$$

$$\text{Var. } (X) = \mu_2' - {\mu_1'}^2 = \frac{2}{k^2} - \frac{1}{k^2} = \frac{1}{k^2}.$$

Example 9 : If a random variable has the moment generating function $M_t = \frac{3}{3-t}$, obtain the mean and the standard deviation. **(M.U. 2010)**

Sol. : We have

$$M_0(t) = \frac{3}{3-t} = \frac{3}{3[1-(t/3)]} = \left(1 - \frac{t}{3}\right)^{-1} = 1 + \frac{t}{3} + \frac{t^2}{9} + \frac{t^3}{27} + \ldots\ldots\infty.$$

$$\text{Mean} = E(X) = \text{Coefficient of } \frac{t}{1!} = \frac{1}{3}$$

$$\mu_2' = E(X^2) = \text{Coefficient of } \frac{t^2}{2!} = \frac{2}{9}.$$

$$\therefore \quad \text{Var}(X) = \mu_2' - \mu_1^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9} \qquad \therefore \text{ S.D.} = \frac{1}{3}.$$

EXERCISE - IV

1. A random variable takes values $X = 0, 1$ with probabilities q and p respectively, such that $q + p = 1$. Find the moment generating function of X and show that all moments about the origin are equal to p.

   [**Ans. :** $M_0(t) = qe^0 + pe^t = 1 + (e^t - 1)\,p = 1 + \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right) \quad \therefore \mu_r' = p$]

2. A random variable X has the m.g.f. given by $M_0(t) = \frac{2}{2-t}$. Find the standard deviation of X. [**Ans. :** $\sigma = 1/2$]

3. A random variable X has the probability distribution

   $$P(X = x) = \frac{1}{8}\,{}^3C_x, \quad x = 0, 1, 2, 3.$$

   Find the moment generating function of X and then find mean and variance.

   (M.U. 2003) [**Ans. : (i)** $\frac{1}{8}(1 + e^t)$, **(ii)** $\frac{3}{2}, \frac{3}{4}$.]

4. A random variable X has the following probability distribution.

X	:	0	1	2	3
$P(X = x)$	:	1/6	1/3	1/3	1/6

   Compute (i) Moment generating function about the origin, (ii) first two raw moments.

   [**Ans. : (i)** $\frac{1}{6} + \frac{1}{3}e^t + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$, **(ii)** $\frac{3}{2}, \frac{19}{6}$.]

5. A random variable X has the following probability distribution.

X	:	0	1	2
$P(X = x)$	:	1/3	1/3	1/3

   Find (i) the moment generating function, (ii) find mean and variance.

   [**Ans. : (i)** $\frac{1}{3}(1 + e^t + e^{2t})$, **(ii)** $1, \frac{2}{3}$.]

6. Find the m.g.f. of the random variable having the following probability density function. Also find the mean and variance.

(i) $f(x)=\begin{cases}1/2, & -1\le x<1\\ 0, & \text{elsewhere}\end{cases}$ (ii) $f(x)=\begin{cases}e^{-(x-5)}, & x\ge 5\\ 0, & \text{elsewhere}\end{cases}$

$\left[\textbf{Ans. : } \text{(i) } \dfrac{(e^{t}-e^{-t})}{2t},\ 0,\ \dfrac{1}{3}\ ;\ \text{(ii) } \dfrac{e^{5t}}{1-t},\ 6,\ 1\right]$

7. A random variable X has the probability density function

$$f(x)=\begin{cases}1/3, & -1<x<2\\ 0, & \text{elsewhere}\end{cases}$$

Find the m.g.f. of X. $\left[\textbf{Ans. : } M_0(t)=\dfrac{e^{2t}-e^{-t}}{3t},\ t\ne 0\right]$

8. A random variable X has the following density function

$$f(x)=\begin{cases}2e^{-2x}, & x>0\\ 0, & x\le 0\end{cases}$$

Find the m.g.f. and hence its mean and variance.

$\left[\textbf{Ans. : } M_0(t)=\dfrac{2}{2-t}\ ;\ t\ne 2,\ \mu_1=\dfrac{1}{2},\ \mu_2=\dfrac{1}{4}\right]$

9. Find the m.g.f. of the random variable having probability density function

$$f(x)=\begin{cases}x, & 0\le x<1\\ 2-x, & 1\le x<2\\ 0, & \text{elsewhere}\end{cases}$$

$\left[\textbf{Ans. : } M_0(t)=\left(\dfrac{e^{t}-1}{t}\right)^2 ;\ t\ne 0\right]$

EXERCISE - V

Theory

1. Define mathematical expectation.
2. Define expectation of a function of a random variable X.
   (a) when X is a discrete r.v., (b) when X is a continuous r.v.
3. Explain the following terms :
   (i) Expectation of a random variable.
   (ii) Variance of a random variable. **(M.U. 1998)**
4. Explain the following terms
   (i) Moments about origin.
   (ii) Moment Generating Function. **(M.U. 2002)**
5. Prove that $V(X)=E(X^2)-\{E(X)\}^2$.
6. If X is a random variable and $Y=aX+b$ show that
   (i) $\mu_Y=a\,\mu_X+b$, (ii) $\sigma_Y^2=a^2\,\sigma_X^2$
   where μ denotes the mean and σ denotes the standard deviation.

7. Prove that $V(aX+b)=a^2\,V(x)$.
8. Define Moment Generating Function of a random variable X about a and explain how you will get moments from it.
9. Derive the formulae for rth moment about any point and rth moment about the mean. **(M.U. 1999)**
10. If $L(t)=\log M(t)$ where $M(t)$ is the m.g.f. of a discrete random variable, then prove that mean $=L'(0)$ and variance $=L''(0)$. **(M.U. 2006)**

☆☆☆

CHAPTER 8

Some Standard Distributions

1. Introduction

In this chapter we shall study some standard probability distributions. Study of such theoretical distributions is the foundation for the development of further topics. The first two distributions are discrete while the third is continuous.

2. Binomial Distribution

This was discovered by James Bernoulli's in 1700 and it expresses probabilities of events of dichotomous (dicho + tomy = two parts) nature *i.e.*, which results in only two ways, success or failure.

(a) To Derive Binomial Distribution

Consider an experiment which results in either success or failure. Let it be repeated n times, the probability p of success remaining constant every time and let $q = 1 - p$, the probability of failure.

The probability of x successes and hence $(n-x)$ failures in a trial in a particular order say, SSS ..... (x times) and then FFF ..... $(n-x)$ times, as given by the multiplication theorem on probability, is

$$= ppp \text{ ..... } (x \text{ times}) \times qqq \text{ ..... } (n-x) \text{ times}$$
$$= p^x q^{n-x}.$$

But x successes can occur in ${}^nC_x$ ways and the probability of each of these ways is the same *viz.* $p^x q^{n-x}$. Hence, the probability of x successes in any order by the addition theorem is ${}^nC_x p^x q^{n-x}$.

Hence, $p(x) = {}^nC_x p^x q^{n-x}, x = 0, 1, 2 \text{ ..... } n.$

This is Binomial distribution.

(b) Definition

A random variable is said to follow **Binomial distribution** if probability of x is given by

$$\boxed{P(X = x) = {}^nC_x p^x q^{n-x}, \quad x = 0, 1, 2, 3 \text{ ..... } n \text{ and } q = 1 - p}$$

The two constants n and p are called the **parameters** of the distribution.

(The meanings of 'success' and 'failure' are quite arbitrary. We may call 'a fatal accident' or 'getting infected by a disease' or 'premature delivery' a success !).

Remarks ....

1. The sum of the probabilities is 1.

$$\sum_{x=0} p(x) = \sum_{x=0} {}^nC_x p^x q^{n-x} = {}^nC_0\, q^n + {}^nC_1\, q^{n-1} p + {}^nC_2\, q^{n-2} p^2 + \text{.....} + p^n$$
$$= (q+p)^n = 1.$$

2. Let the experiment of n trials be repeated N times. Then we expect x successes to occur $N \cdot {}^nC_x\, p^x q^{n-x}$ times. This is called frequency function. The expected frequencies of 0, 1, 2, ...n successes are the successive terms of the binomial expansion $N(q+p)^n$. **(M.U. 1998)**

3. If x is a binomial variate with parameters n and p, it is denoted as $b(x, n, p)$.

4. The distribution is called **"Binomial Distribution"** because the probabilities ${}^nC_x\, p^x q^{n-x}$, $x = 0, 1, 2$, ..., n are the successive terms of the expansion of the binomial expression $(q+p)^n$.

(c) When do we get binomial distribution ?

As is clear from the previous discussion, we get a binomial distribution when the following conditions are satisfied :

(i) A trial is repeated n times where n is a finite number.

(ii) Each trial results only in two ways-success or failure.3

(iii) These possibilities are mutually exclusive, exhaustive but not necessarily equally likely.

(iv) If p and q are the probabilities of success and failure then $p + q = 1$.

(v) The events are independent, *i.e.*, the probability p of success in each trial remains constant in all trials.

(d) Uses

Naturally, we can use binomial distribution when these conditions are satisfied. Thus, in problems involving (i) the tossing of a coin-heads or tails, (ii) the result of an examination-success or failure, (ii) the result of an election-success or failure, (iii) the result of inspection of an article-defective or non-defective, (iv) habit of a person-smoker or non-smoker etc. binomial distribution can be used if other conditions are also satisfied.

(e) Mean and Variance **(M.U. 2003, 05)**

The first two moments about the origin are obtained as follows.

$$\mu_1' = E(X) = \Sigma p_i x_i = \sum_{x=0} {}^nC_x\, p^x q^{n-x} \cdot x$$
$$= {}^nC_0\, p^0 q^n \cdot 0 + {}^nC_1\, pq^{n-1} \cdot 1 + {}^nC_2\, p^2 q^{n-2} \cdot 2 + \text{.....} + p^n \cdot n$$
$$= npq^{n-1} + \frac{n(n-1)}{2!} \cdot p^2 q^{n-2} \cdot 2 + \text{.....} + p^n \cdot n$$
$$= np\left[q^{n-1} + (n-1)\, q^{n-2} \cdot p + \frac{(n-1)(n-2)}{2!} q^{n-3} p^3 + \text{.....} + p^{n-1}\right]$$
$$= np[q+p]^{n-1} = np. \qquad [\because p + q = 1]$$
$$\mu_2' = E(X^2) = \Sigma p \cdot x^2 = \Sigma\, {}^nC_x\, p^x q^{n-x} \cdot x^2$$

But x^2 can be written as $x^2 = x + x(x-1)$

$$\therefore \quad \mu_2' = \Sigma[x + x(x-1)]\,{}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0} x\,{}^nC_x\, p^x q^{n-x} + \sum_{x=0} x(x-1)\,{}^nC_x\, p^x q^{n-x}$$

But the first term on the r.h.s. is np as shown above.

$$\therefore \quad \mu_2' = np + [0\cdot{}^nC_0\, p^0 q^n + 0\cdot{}^nC_1\, p^1 q^{n-1} + 2\cdot 1\cdot{}^nC_2\, p^2 q^{n-2} + 3\cdot 2\,{}^nC_3 p^3 q^{n-3} + \ldots]$$

$$= np + 2\cdot\frac{n(n-1)}{2!}p^2q^{n-2} + 3\cdot 2\frac{n(n-1)(n-2)}{3!}p^3q^{n-3} + 4\cdot 3\frac{n(n-1)(n-2)(n-3)}{4!}p^4q^{n-4} + \ldots$$

$$= np + n(n-1)\,p^2q^{n-2} + n(n-1)(n-2)\,p^3q^{n-3} + \frac{n(n-1)(n-2)(n-3)}{2!}p^4q^{n-4} + \ldots$$

$$= np + n(n-1)p^2\left[q^{n-2} + (n-2)\,pq^{n-3} + \frac{(n-2)(n-3)}{2!}p^2q^{n-4} + \ldots\right]$$

$$= np + n(n-1)p^2(q+p)^{n-2} = np + n(n-1)p^2$$

$$= np[1 + (n-1)p] = np[1 - p + np]$$

$$= np[q + np] = npq + n^2p^2$$

$$\therefore \quad \mu_2 = \mu_2' - \mu_1'^2 = npq$$

$$\therefore \quad \boxed{\textbf{Mean} = np \quad \textbf{and} \quad \textbf{Variance} = npq}$$

(f) Moment Generating Function about origin (M.U. 2015)

By definition m.g.f. about origin is

$$M_0(t) = E(e^{tX}) = \Sigma\, p_i\, e^{tx_i}$$

$$= \Sigma\,{}^nC_x\, p^x q^{n-x} e^{tx} = \Sigma\,{}^nC_x\, q^{n-x}\cdot(pe^t)^x$$

$$= (q + pe^t)^n \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

[Note that $\Sigma\,{}^nC_r\, a^{n-r}b^r = {}^nC_0\, a^n + {}^nC_1 a^{n-1}\, b + {}^nC_2\, a^{n-2}\, b^2 + \ldots = (a+b)^n$.]

Differentiating $M_0(t)$ and putting $t = 0$, we get the required moments.

Now, $$\frac{d}{dt}[M_0(t)] = n(q + pe^t)^{n-1}pe^t = np\left[e^t(q + pe^t)^{n-1}\right]$$

$$\therefore \quad \left[\frac{d}{dt}M_0(t)\right]_{t=0} = np(q+p) = np$$

$$\frac{d^2}{dt^2}[M_0(t)] = np\left[e^t(n-1)(q + pe^t)^{n-2}pe^t + e^t(q + pe^t)^{n-1}\right]$$

$$= npe^t(q + pe^t)^{n-2}\left[e^t(n-1)\,p + (q + pe^t)\right]$$

$$\therefore \quad \left[\frac{d^2}{dt^2}M_0(t)\right]_{t=0} = np(q+p)[(n-1)\,p + (q+p)]$$

$$= np(np - p + 1) = np(np + q) = npq + n^2p^2$$

$$\therefore \quad \mu_2 = \mu_2' - \mu_1'^2 = npq.$$

(g) Additive property of Binomial Distribution

(1) If X_1 is a Binomial variate with parameter n_1 and p_1 and X_2 is another Binomial variate with parameter n_2 and p_2 then $X_1 + X_2$ in general is not a Binomial variate.

Proof : Since X_1, X_2 are Binomial with parameters n_1, p_1 and n_2, p_2

$$M_{X_1}(t) = (q_1 + p_1e^t)^{n_1}; \qquad M_{X_2}(t) = (q_2 + p_2e^t)^{n_2}$$

Since, X and Y are independent

$$M_{X_1+X_2}(t) = M_{X_1}(t)\cdot M_{X_2}(t) = (q_1 + p_1e^t)^{n_1}\cdot(q_2 + p_2e^t)^{n_2}$$

But this cannot be expressed in the form $(q + pe^t)^n$. Hence, $X_1 + X_2$ is not a Binomial variate.

Thus, the sum of two independent Binomial variable is **not** a Binomial variate. In other words, Binomial distribution does **not** possess additive property.

(However if $p_1 = p_2$, we get the additive property).

(2) If X_1 and X_2 are two Binomial variates with parameters n_1, p and n_2, p then $X_1 + X_2$ is a Binomial variate with parameters $(n_1 + n_2)$, p.

We have, $$M_{X_1}(t) = (q + pe^t)^{n_1}, \quad M_{X_2}(t) = (q + pe^t)^{n_2}$$

$$\therefore \quad M_{X_1+X_2}(t) = M_{X_1}(t)\cdot M_{X_2}(t)$$

$$= (q + pe^t)^{n_1}\cdot(q + pe^t)^{n_2}$$

$$= (q + pe^t)^{n_1+n_2}$$

$\therefore$ $X_1 + X_2$ is a Binomial variate with parameters $(n_1 + n_2)$, p.

Example 1 : Find the mean of the probability distribution of the number of heads obtained in three flips of a balanced coin.

Sol. : We have $p = 1/2$, $n = 3$

$$\therefore \quad \text{Mean} = E(X) = np = 3\times\frac{1}{2} = 1{\cdot}5$$

Example 2 : What is the expectation of heads if an unbiased coin is tossed 12 times ?

Sol. : Since the expectation of x in a binomial distribution is given by $E(X) = np$ and $n = 12$, $p = 1/2$. We could expect $12\times(1/2)$ *i.e.* 6 heads.

Example 3 : If X is Binomially distributed with $E(X) = 2$ and Var $(X) = 4/3$, find the probability distribution of X. **(M.U. 2004, 16, 17)**

Sol. : We have $E(X) = np = 2$ and Var $(X) = npq = 4/3$.

$$\therefore \quad \frac{npq}{np} = \frac{4/3}{2} \qquad \therefore \quad q = \frac{2}{3} \qquad \therefore \quad p = 1 - q = \frac{1}{3}$$

But $np = 2$. $\therefore\ n \cdot \frac{1}{3} = 2 \quad \therefore\ n = 6$

Hence, the distribution is

$$P(X = x) = {}^nC_x\, p^x\, q^{n-x} = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

Putting $x = 0, 1, 2, \ldots\ldots 6$, we get the following probability distribution of X.

x	0	1	2	3	4	5	6
$P(X = x)$	$\frac{64}{729}$	$\frac{192}{729}$	$\frac{240}{729}$	$\frac{160}{729}$	$\frac{60}{729}$	$\frac{12}{729}$	$\frac{1}{729}$

Example 4 : Prove that for all Binomial distributions with the same parameter n, the variance is maximum when $p = 1/2$.

Sol. : We have for Binomial distribution

$$p(x) = {}^nC_x\, p^x\, q^{n-x}$$

And the variance is given by

$$V = n\,p\,q \qquad \text{where, } q = 1 - p$$

$$\therefore\ V = np\,(1 - p) = np - np^2$$

For maxima, $\frac{dV}{dp} = 0$ and $\frac{d^2V}{dp^2} = -\text{ve}$

Now, $\frac{dV}{dp} = n - 2np$ and $\frac{d^2V}{dp^2} = -2n$, always $-$ ve.

$\therefore\ \frac{dV}{dp} = 0$ gives $\quad n - 2np = 0 \quad \therefore\ n(1 - 2p) = 0$

But $\quad n \neq 0 \quad \therefore\ 1 - 2p = 0 \quad \therefore\ p = \frac{1}{2}$.

Hence, the variance is maximum when $p = \frac{1}{2}$.

Example 5 : The mean and variance of a Binomial distribution are 4 and 4/3. Find the distribution and the probability of atleast one success. **(M.U. 2010)**

Sol. : We have $\quad E(X) = np = 4, \quad V(X) = npq = 4/3$

$\therefore\ \frac{npq}{np} = \frac{4/3}{4} = \frac{1}{3} \quad \therefore\ q = \frac{1}{3} \quad \therefore\ p = \frac{2}{3}$

$\because\ np = 4 \quad \therefore\ n \cdot \frac{2}{3} = 4 \quad \therefore\ n = 6.$

$\therefore$ The distribution is

$$P(X = x) = {}^nC_x\, p^x\, q^{n-x} = {}^6C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{6-x}$$

x	0	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{729}$	$\frac{12}{729}$	$\frac{60}{729}$	$\frac{160}{729}$	$\frac{240}{729}$	$\frac{192}{729}$	$\frac{64}{729}$

$$P(\text{atleast one success}) = 1 - P(0 \text{ success}) = 1 - \frac{1}{729} = \frac{728}{729}.$$

(Compare Ex. 3 with Ex. 5.)

Example 6 : Find the Binomial distribution if the mean is 4 and variance is 3. **(M.U. 2005)**

Sol. : We have mean $= np = 4$ and variance $= npq = 4$.

$\therefore\ \frac{np}{npq} = \frac{4}{3} \quad \therefore\ \frac{1}{q} = \frac{4}{3} \quad \therefore\ q = \frac{3}{4} \quad \therefore\ p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$

$\therefore\ n = 4 \quad \therefore\ n \cdot \frac{1}{4} = 4 \quad \therefore\ n = 16$

$$\therefore\ P(X = x) = {}^nC_x\, p^x q^{n-x} = {}^{16}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{16-x}$$

Example 7 : The probability that a man aged 60 will live upto 70 is 0·65. What is the probability that out of 10 such men now at 60 at least 7 will live upoto 70 ? **(M.U. 2003, 07)**

Sol. : P(man will live upto 70) = 0·65

P(man will **not** live upto 70) = 0·35.

We have $n = 10$, $p = 0{\cdot}65$, $q = 0{\cdot}35$.

$$\therefore\ P(X = x) = {}^{10}C_x\,(0{\cdot}65)^x\,(0{\cdot}35)^{10-x}$$

$$P(\text{at least 7 will survive}) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

$$= \sum_{x=7}^{10} {}^{10}C_x (0{\cdot}65)^x (0{\cdot}35)^{10-x}$$

Example 8 : "In a Binomial distribution the mean is 5 and the standard deviation is 3". Find the fallacy if any in this statement. **(M.U. 1996)**

Sol. : For a Binomial distribution the mean $= np = 5$ and variance $= npq = 9$.

$\therefore\ \frac{np}{npq} = \frac{5}{9} \quad \therefore\ \frac{1}{q} = \frac{5}{9} \quad \therefore\ q = \frac{9}{5} = 1{\cdot}8$

This is the probability of failure and probability of an event cannot be greater than 1. Hence, the statement is wrong.

Example 9 : With usual notation find p of Binomial distribution if $n = 6$, $9P(X = 4) = P(X = 2)$. **(M.U. 2001, 04)**

Sol. : We have $P(X = x) = {}^nC_x\, p^x\, q^{n-x} = {}^6C_x\, p^x\, q^{6-x}$

Since, $9\,P(X = 4) = P(X = 2)$

$\therefore\ 9\,{}^6C_4\, p^4\, q^{6-4} = {}^6C_2\, p^2\, q^{6-2}$

$\because\ {}^6C_4 = {}^6C_2, \quad 9p^2 = q^2 = (1-p)^2$

$\therefore\ 9p^2 = 1 - 2p + p^2 \quad \therefore\ 8p^2 + 2p - 1 = 0$

$\therefore\ (4p - 1)(2p + 1) = 0 \quad \therefore\ p = \frac{1}{4}$ or $p = -\frac{1}{2}$. $\quad \therefore\ p = \frac{1}{4}$ and $q = \frac{3}{4}$.

Example 10 : If X is Binomially distributed with parameters n and p, show that

$$E\left(\frac{X}{n}-p\right)^2 = \frac{pq}{n}.$$ (M.U. 2009)

Sol. : We have

$$E\left(\frac{X}{n}-p\right)^2 = E\left(\frac{X^2}{n^2}-\frac{2X}{n}p+p^2\right)$$

$$= E\left(\frac{X^2}{n^2}\right) - E\left(\frac{2X}{n}p\right) + E(p^2)$$

$$= \frac{1}{n^2}E(X^2) - \frac{2p}{n}E(X) + p^2 E(1) \qquad \text{.................... (1)}$$

For Binomial distribution $E(X) = np$ and $\text{Var}(X) = npq$

$\therefore\ E(X^2) = \text{Var}(X) + [E(X)]^2 = npq + n^2p^2$ and $E(X) = np$

$\because$ From (1),

$$E\left(\frac{X}{n}-p\right)^2 = \frac{1}{n^2}(npq+n^2p^2) - \frac{2p}{n}\cdot np + p^2$$

$$= \frac{pq}{n} + p^2 - 2p^2 + p^2 = \frac{pq}{n}.$$

Example 11 : What is the mean and variance of the Binomial Distribution $(0{\cdot}3+0{\cdot}7)^{10}$, $q = 0{\cdot}3$?

Sol. : Here, $p = 0{\cdot}7$, $q = 0{\cdot}3$, $n = 10$.

$\therefore$ Mean $= np = 10\times 0{\cdot}7 = 7$

$\therefore$ Variance $= npq = 10 \times 0{\cdot}3\times 0{\cdot}7 = 2{\cdot}1$.

Example 12 : A factory turns out an article by mass production methods. From the past experience it is found that 20 articles on an average are rejected out of every batch of 100. Find the mean and the variance of the number of rejected articles. (M.U. 1997)

Sol. : The number of rejected articles in a batch is a Binomial variate with $n = 100$ and $p = \frac{20}{100} = 0{\cdot}2$.

Hence, the mean of the distribution $= np = 100 \times \frac{2}{10} = 20$.

Variance $= \sqrt{npq} = \sqrt{100\times 0{\cdot}2\times 0{\cdot}8} = \sqrt{16} = 4$.

Example 13 : The ratio of the probability of 3 successes in 5 independent trials to the probability of 2 successes in 5 independent trials is 1/4. What is the probability of 4 successes in 6 independent trials ? (M.U. 2005, 10)

Sol. : For a Binomial distribution $P(X=x) = {}^nC_x p^x q^{n-x}$

When $n = 5$, $x = 3$, $P(X=3) = {}^5C_3 p^3q^2$

When $n = 5$, $x = 2$, $P(X=2) = {}^5C_2 p^2q^3$

The ratio of these probabilities is 1/4.

$$\therefore\ \frac{P(X=3)}{P(X=2)} = \frac{{}^5C_3\,p^3q^2}{{}^5C_2\,p^2q^3} = \frac{1}{4}.$$

Since, ${}^5C_3 = {}^5C_2$, we get

$$\frac{p}{q} = \frac{1}{4} \quad \therefore\ \frac{p}{1-p} = \frac{1}{4} \quad \therefore\ 4p = 1-p$$

$$\therefore\ 5p = 1 \quad \therefore\ p = \frac{1}{5} \quad \therefore\ q = 1-p = \frac{4}{5}$$

$$\therefore\ P(X=x) = {}^nC_x\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{n-x}$$

When $n = 6$ and $x = 4$, $P(X=4) = {}^6C_4\left(\frac{1}{5}\right)^4\cdot\left(\frac{4}{5}\right)^2$.

Example 14 : A biased coin is tossed n times. Prove that the probability of getting even number of heads is $0.5\,[1+(q-p)^n]$. (M.U. 2004)

Sol. : Let $P(\text{head}) = p$ $\therefore$ $q = 1-p$.

$\therefore$ $P(\text{even head}) = P(0, 2, 4, 6, \ldots\text{ heads})$

$$= {}^nC_0\,p^0q^n + {}^nC_2\,p^2q^{n-2} + {}^nC_4\,p^4q^{n-4} + \ldots..$$

Now, $(p+q)^n = {}^nC_0\,p^0q^n + {}^nC_1\,p^1q^{n-2} + \ldots.. = 1^n = 1$

and $(q-p)^n = {}^nC_0\,q^np^0 - {}^nC_1\,q^{n-1}p + {}^nC_2\,q^{n-2}p^2 - \ldots..$

By addition $2[{}^nC_0\,p^0q^n + {}^nC_2\,p^2q^{n-2} + \ldots.] = 1 + (q-p)^n$

$\therefore$ Required Probability $= \frac{1}{2}[1+(q-p)^n]$.

Example 15 : If m things are distributed among 'a' men and 'b' women, show that the probability that the number of things received by men is odd is

$$\frac{1}{2}\left[\frac{(b+a)^m-(b-a)^m}{(b+a)^m}\right]$$ (M.U. 2004, 05)

Sol. : There are 'a' men and 'b' women. Hence, the probability that a man will be selected for giving that thing is $p = \frac{a}{a+b}$.

The probability that a woman will be selected for giving that thing is $q = \frac{b}{a+b}$ where $p+q = 1$.

Now, the probability of giving r things to men $= {}^mC_r\,p^r q^{m-r}$, $r = 1, 2, 3, \ldots..$

Since, men are to get odd number of things, $r = 1, 3, 5, \ldots..$

$\therefore$ $P(\text{men receiving odd number of things}) = P(1) + P(3) + P(5) + \ldots..$

$$= {}^mC_1\,p^1q^{m-1} + {}^mC_3\,p^3q^{m-3} + \ldots..$$

$$= \frac{1}{2}\left[(q+p)^m - (q-p)^m\right]$$

(Even powers in $(q+p)^m$ are cancelled by $(q-p)^m$ and odd powers are added.)

But $q + p = 1$ and $q - p = \frac{b}{a+b} - \frac{a}{a+b} = \frac{b-a}{b+a}$.

$\therefore$ Required Probability $= \frac{1}{2}\left[1 - \frac{(b-a)^m}{(b+a)^m}\right] = \frac{1}{2}\left[\frac{(b+a)^m - (b-a)^m}{(b+a)^m}\right]$.

Example 16 : A communication system consists of n components, each of which functions independently with probability p. The total system will be able to function effectively if atleast one-half of its components are functioning. For what value of p is a 5-component system more likely to operate effectively than a 3-component system ? **(M.U. 2004, 09)**

Sol. : Here, we have a Binomial distribution with parameters n and p.

$\therefore \quad P(X = x) = {}^nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2, \ldots\ldots n$

P (5 component system will work effectively) $= P(X = 3$, or 4 or 5)

$= P(X = 3) + P(X = 4) + P(X = 5)$

$= \sum_{x=3}^{5} {}^3C_x\, p^x q^{5-x} \qquad [\because n = 5]$

P (3-component system will work effectively) $= P(X = 2$ or 3)

$= \sum_{x=2}^{3} {}^3C_x\, p^x q^{3-x} \qquad [\because n = 3]$

5-component system will work more effectively than 3-component system if

$$\sum_{x=3}^{5} {}^5C_x\, p^x q^{5-x} \geq \sum_{x=2}^{3} {}^3C_x\, p^x q^{3-x}$$

$\therefore \quad (5C_3\, p^3 q^2 + {}^5C_4\, p^4 q + 5\,C_5\, p^5) \geq (3\,C_2\, p^2 q + {}^3C_3\, p^3)$

$\therefore \quad (10p^3(1-p)^2 + 5p^4(1-p) + p^5) \geq 1(3p^2(1-p) + p^3)$

$10p^3 - 20p^4 + 10p^5 + 5p^4 - 5p^5 + p^5 - 3p^2 + 3p^3 - p^3 \geq 0$

$6p^5 - 15p^4 + 12p^3 - 3p^2 \geq 0$

$\therefore \quad 3p^2(2p^3 - 5p^2 + 4p - 1) \geq 0 \qquad \therefore \quad 3p^2(p-1)^2(2p-1) \geq 0$

$\therefore \quad 2p - 1 \geq 0 \quad [\because p^2 \geq 0, (p-1)^2 \geq 0]$

$\therefore \quad p \geq \frac{1}{2}$ is the required value.

Example 17 : It has been claimed that in 60% of all solar heat installations, the utility bill is reduced by atleast one third. Accordingly what are the probabilities that the utility bill will be reduced by atleast one third in (i) four of five installations, (ii) atleast four of five installations.

Sol. : For a Binomial distribution $\quad P(X = x) = {}^nC_x p^x q^{n-x}$

We have $n = 5$, $x = 4$, $p = 0{\cdot}6$, $q = 0{\cdot}4$.

(i) $\therefore \quad P(X = 4) = {}^5C_4\,(0{\cdot}6)^4\,(0{\cdot}4)^1 = 0{\cdot}259$

(ii) $\quad P$(atleast 4 of 5 installations) $= P(X \geq 4) = P(X = 4$ or $5)$

$= P(X = 4) + P(X = 5)$

$= {}^5C_4\,(0{\cdot}6)^4\,(0{\cdot}4)^1 + {}^5C_5\,(0{\cdot}6)^5\,(0{\cdot}4)^0$

$= 0{\cdot}259 + 0{\cdot}078 = 0{\cdot}337.$

Example 18 : The incidence of an occupational disease in an industry is such that the workers have 20% chance of suffering from it. What is the probability that out of 6 workers chosen at random 4 or more will be suffering from the disease ? **(M.U. 2005)**

Sol. : We have $p = 20\,\% = \frac{20}{100} = 0{\cdot}2$, $q = 1 - p = 0{\cdot}8$, $n = 6$

$\therefore \quad P(X = x) = {}^nC_x\, p^x q^{n-x} = {}^6C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{6-x}$

$\therefore \quad P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6)$

$$= {}^6C_4 \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^2 + {}^6C_5 \left(\frac{1}{5}\right)^5 \left(\frac{4}{5}\right)^1 + {}^6C_6 \left(\frac{1}{5}\right)^6 \left(\frac{4}{5}\right)^0$$

$\therefore \quad P(X \geq 4) = \frac{1}{5^6}\left[15{\cdot}4^2 + 6{\cdot}4 + 1\right] = \frac{205}{5^6} = \frac{41}{3125}.$

Example 19 : The probability that a bomb dropped from a plane will strike the target is 1/5. If six such bombs are dropped, find the probability that (i) exactly two bombs hit the target, (ii) at least two will hit the target.

Sol. : We have $p = \frac{1}{5}$, $q = \frac{4}{5}$, $\quad n = 6.$

By Binomial distribution, $P(x) = {}^nC_x\, p^x q^{n-x} = {}^6C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{6-x}$

$\therefore \quad P(2) = {}^6C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^4 = \frac{6 \cdot 5}{2} \cdot \frac{1}{25} \cdot \frac{256}{625} = \frac{1536}{6250} = 0{\cdot}24576.$

P(at least two) $= 1 - [\,P(1)\,]$

Now, $\quad P(1) = {}^6C_1 \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^5 = 6 \cdot \frac{1}{5} \cdot \frac{1024}{3125} = \frac{6144}{15625} = 0{\cdot}3932$

$\therefore \quad P$(at least two) $= 1 - 0.3932 = 0{\cdot}6068.$

Example 20 : The probability that at any moment one telephone line out of 10 will be busy is 0·2.

(i) What is the probability that 5 lines are busy ?

(ii) Find the expected number of busy lines and also find the probability of this number.

(iii) What is the probability that all lines are busy ?

Sol. : We have, $p = 0{\cdot}2$, $\quad q = 0{\cdot}8$, $\quad n = 10.$

By Binomial distribution, $P(X = x) = {}^nC_x\, p^x q^{n-x}$

$\therefore \quad P(X = x) = {}^{10}C_x\,(0{\cdot}2)^x\,(0{\cdot}8)^{10-x}$

(i) $\therefore\ P(X=5) = {}^{10}C_5\,(0.2)^5\,(0.8)^{0.5}$

(ii) Expected number of busy lines = mean = $np = 10 \times \frac{2}{10} = 2$.

$$P(X=2) = {}^{10}C_2\,(0.2)^2\,(0.8)^{0.8}$$

(iii) Probability of all lines busy = $P(X=10) = {}^{10}C_{10}\,(0.2)^{10}\,(0.8)^0 = 0.2^{10}$.

Example 21 : In a precision bombing attack there is a 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give at least 99% chance of destroying the target ? **(M.U. 2003, 04)**

Sol. : We have $p = 1/2$ and $q = 1/2$.

The probability distribution of X, the number of bombs hitting the target is

$$P(X=x) = {}^nC_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{n-x}$$

For completely destroying the target X must be greater than or equal to 2.

$$\therefore\ P(X \geq 2) \geq 0.99$$

$$[1 - P(X \leq 1)] \geq 0.99$$

$$[1 - P(X=0) - P(X=1)] \geq 0.99$$

But $P(X=0) = {}^nC_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n$; $\quad P(X=1) = {}^nC_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{n-1} = n\left(\frac{1}{2}\right)^n$

$$\therefore\ \left[1 - \left(\frac{1}{2}\right)^n - n\cdot\left(\frac{1}{2}\right)^n\right] \geq \frac{99}{100} \qquad \therefore\ -\left(\frac{1}{2}\right)^n - n\left(\frac{1}{2}\right)^n \geq -0.01$$

$$\therefore\ \left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^n \leq \frac{1}{100} \qquad \therefore\ 1 + n \leq \frac{2^n}{100} \qquad \therefore\ 2^n \geq 100 + 100\cdot n$$

We know that $2^1, 2^2, \ldots, 2^{10}, 2^{11}$ are 2, 4, ..., 1024, 2048.

Thus, by trial and error, were find that $n \geq 11$.

Hence, minimum 11 bombs are required to destroy the target.

Example 22 : Seven dice are thrown 729 times. How many times do you expect at least four dice to show three or five ? **(M.U. 2004, 15, 16)**

Sol. : Probability of getting (3 or 5) in a single toss $= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

This is a binomial distribution with $n = 7$, $p = 1/3$, $q = 2/3$.

$$\therefore\ P(X=x) = {}^nC_x\,p^x\,q^{n-x} = {}^7C_2\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{7-x}$$

P(at least 4 successes) $= P(x = 4, 5, 6, 7)$

$$= {}^7C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^3 + {}^7C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^2 + {}^7C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^1 + {}^7C_7\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right)^0$$

$$= \frac{379}{3^7}$$

$\therefore$ The expected number of times of getting (3 or 5) at least 4 times

$$= Np = 729 \times \frac{379}{3^7} = 126.3.$$

Example 23 : Out of 1000 families of 3 Children each, how many would you expect to have 2 boys and 1 girl?

Sol. : Here $P(\text{Boy}) = p = \frac{1}{2}$, $P(\text{Girl}) = q = \frac{1}{2}$, $n = 3$, $r = 2$.

$$\therefore\ P(\text{2 boys and 1 girl}) = {}^3C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{3}{8}$$

$$\therefore\ \text{Expected number of families} = Np = 1000 \times \frac{3}{8} = 375$$

Example 24 : If hens of a certain breed lay eggs on 5 days a week on an average; find on how many days during a season of 100 days, a poultry keeper with 5 hens of this breed, will expect to receive at least 4 eggs?

Sol. : Probability of an hen laying an egg, $p = 5/7$ and probability of not laying an egg, $q = 1 - p = 2/7$.

$$P(X=x) = {}^nC_x\,p^x q^{n-x} \quad \text{and } n = 5,\ p = \frac{5}{7},\ q = \frac{2}{7}$$

$$\therefore\ P(X \geq 4) = P(X=4) + P(X=5)$$

$$= {}^5C_4\left(\frac{5}{7}\right)^4\left(\frac{2}{7}\right)^1 + {}^5C_5\left(\frac{5}{7}\right)^5\left(\frac{2}{7}\right)^0 = 0.5578.$$

Expectation = $Np = 100 \times 0.5578 = 55.78 = 56$.

Example 25 : Let X, Y be two independent binomial variates with parameters ($n_1 = 6$, $p = 1/2$) and ($n_2 = 4$, $p = 1/2$) respectively. Evaluate $P(X+Y) = 3$. **(M.U. 2004)**

Sol. : By the additive property of Binomial variates $Z = X + Y$ is a Binomial variate with parameters $n = n_1 + n_2 = 6 + 4 = 10$ and $p = 1/2$.

$$\therefore\ P(Z) = {}^nC_z\,p^z \cdot q^{n-z}$$

$$\therefore\ P(Z=3) = {}^{10}C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7 = \frac{15}{128} = 0.1172.$$

Example 26 : In the above example, find $P(X+Y) \geq 3$. **(M.U. 2004)**

Sol. : We want,

$$P(Z \geq 3) = 1 - [P(Z=0) + P(Z=1) + P(Z=2)]$$

$$= 1 - \left[{}^{10}C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^9 + {}^{10}C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^8\right]$$

$$= 1 - \left[\left({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2\right)\left(\frac{1}{2}\right)^{10}\right]$$

$$= 0.945.$$

Example 27 : If X and Y are two independent Binomial variates with parameters (3, 1 / 3) and (5, 1 / 3) respectively. Find $P(X+Y \geq 1)$.

Sol. : As above $Z = X + Y$ is a Binomial variate with parameters $n = n_1 + n_2 = 3 + 5 = 8$ and $p = 1/3$.

$$P(Z) = {}^nC_z\, p^z q^{n-z} = {}^8C_z (1/3)^z (2/3)^{8-z}$$

$$P(Z=0) = {}^8C_0 (1/3)^0 (2/3)^8 = (2/3)^8$$

$$P(Z \geq 1) = 1 - P(Z=0) = 1 - (2/3)^8.$$

Example 28 : Three fair coins are tossed 3000 times. Find the frequencies of the distribution of heads and tails and tabulate the result. Also calculate the mean and standard deviation of the distribution.

Sol. : We have $p = \frac{1}{2},\ q = \frac{1}{2},\ n = 3.$

Let X be the number of heads obtained when the three coins are tossed.

$$\therefore\ P(X=x) = {}^nC_x\, p^x q^{n-x} = {}^3C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x}$$

Putting $X = 0, 1, 2, 3$ we get

$$P(X=0) = {}^3C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^3 = \frac{1}{8}; \qquad P(X=1) = {}^3C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^2 = \frac{3}{8};$$

$$P(X=2) = {}^3C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1 = \frac{3}{8}; \qquad P(X=3) = {}^3C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^0 = \frac{1}{8}.$$

$\therefore$ Expected frequency $= Np$

To obtain frequencies, we multiply these probabilities by 3000.

X :	0	1	2	3	Total
f :	375	1125	1125	375	3000

Also Mean $= np = 3000 \times \frac{1}{2} = 1500$

$$\text{S.D.} = \sqrt{npq} = \sqrt{3700 \times \frac{1}{2} \times \frac{1}{2}} = 27{\cdot}39$$

Example 29 : Seven coins are tossed and the number of heads obtained is noted. The experiment is repeated 128 times and the following distribution is obtained.

No. of heads :	0,	1,	2,	3,	4,	5,	6,	7	Total
Frequency :	7,	6,	19,	35,	30,	23,	7,	1	128

(M.U. 1998, 2005, 06)

Fit a Binomial distribution if (i) the coins are unbiased, (ii) if the nature of the coins is not known.

Sol. : To fit a distribution to given data means to find the constants of the distribution which will adequately describe the given situation.

(i) ***When the coins are unbiased***

$$p = \frac{1}{2},\ q = \frac{1}{2} \text{ and by data } n = 7 \qquad \therefore\ P(X=x) = {}^7C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{7-x}$$

Putting $x = 0, 1, 2, 3, \ldots, 7$, we get

$$P(0) = \frac{1}{2^7},\quad P(1) = \frac{7}{2^7},\quad P(2) = \frac{21}{2^7},\ \ldots$$

Expected frequency $= Np$ and $N = 128$.

Multiplying the above probabilities by 128 *i.e.* by 2^7 we get the expected frequencies as 1, 7, 21, 35, 35, 21, 7, 1.

(ii) ***When the nature of the coins is not known.***

We have $\bar{X} = \frac{\Sigma f_i x_i}{N} = \frac{0 \times 7 + 1 \times 6 + 2 \times 19 + \ldots + 7 \times 1}{128} = \frac{433}{128} = 3{\cdot}38$

But $\bar{X} = np$. $\quad \therefore\ p = \frac{\bar{X}}{n} = \frac{3{\cdot}38}{7} = 0{\cdot}48 \quad \therefore\ q = 1 - p = 0{\cdot}52$

$$\therefore\ P(X=x) = {}^7C_x (0{\cdot}48)^x (0{\cdot}52)^{7-x}$$

Putting $x = 0, 1, 2, 3, \ldots, 7$ we get

$$P(0) = 0{\cdot}01,\ P(1) = 0{\cdot}066,\ P(2) = 0{\cdot}184,\ \ldots$$

Multiply these probabilities by 128 we get the expected frequencies as

1, 8, 23, 36, 33, 18, 6, 3.

(Last term = 128 − sum of other terms).

Example 29 : Fit a Binomial distribution to the following data.

X :	0	1	2	3	4	5	6
Y :	5	18	28	12	7	6	4

(M.U. 2004, 05, 16)

Sol. : We have $n = 6$

$$N = 5 + 18 + 28 + 12 + 7 + 6 + 4 = 80$$

$$\Sigma f_i x_i = 5 \times 0 + 18 \times 1 + 28 \times 2 + 12 \times 3 + 7 \times 4 + 6 \times 5 + 4 \times 6$$
$$= 0 + 18 + 56 + 36 + 28 + 30 + 24 = 192$$

$$\therefore\ \bar{X} = \frac{\Sigma f_i x_i}{N} = \frac{192}{80} = 2{\cdot}4$$

But $\bar{X} = np \quad \therefore\ p = \frac{\bar{X}}{n} = \frac{2{\cdot}4}{6} = 0{\cdot}4$

$$\therefore\ q = 1 - p = 1 - 0{\cdot}4 = 0{\cdot}6$$

$$\therefore\ p(x) = {}^nC_x\, p^n q^{n-x} = {}^6C_x (0{\cdot}4)^x (0{\cdot}6)^{6-x}$$

Putting $x = 0, 1, 2, 3, 4, 5, 6$, we get

$p(0) = {}^6C_0 (0{\cdot}4)^0 (0{\cdot}6)^6 = 0{\cdot}0466;\quad p(1) = {}^6C_1 (0{\cdot}4)^1 (0{\cdot}6)^5 = 0{\cdot}1866;$

$p(2) = {}^6C_2 (0{\cdot}4)^2 (0{\cdot}6)^4 = 0{\cdot}3110;\quad p(3) = {}^6C_3 (0{\cdot}4)^3 (0{\cdot}6)^3 = 0{\cdot}2765;$

$p(4) = {}^6C_4 (0{\cdot}4)^4 (0{\cdot}6)^2 = 0{\cdot}1382;\quad p(5) = {}^6C_5 (0{\cdot}4)^5 (0{\cdot}6)^1 = 0{\cdot}0369;$

$p(6) = {}^6C_6 (0{\cdot}4)^6 (0{\cdot}6)^0 = 0{\cdot}0041.$

Multiplying these probabilities by $N = 80$, we get the expected frequencies as

x :	0	1	2	3	4	5	6
y :	4	15	25	22	11	3	0

(The frequencies are to be rounded).

Example 30 : Twelve dice were thrown 4096 times and the number of appearance of 6 each time was noted.

No. of successes	0	1	2	3	4	5	6 and above
Frequency	447	1145	1181	786	380	115	32

Fit a Binomial distribution when the dice are unbiased. (M.U. 2016)

Sol. : We have the probability of getting 6 in a throw of one die $p = \frac{1}{6}$.

The probability of not getting 6 is $q = 1 - \frac{1}{6} = \frac{5}{6}$.

The number of trials, $n = 12$.

The number of repetitions, $N = 4096$.

Now, $P(X = x) = {}^nC_x p^x q^{n-x} = {}^{12}C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{12-x}$

Putting $X = 0, 1, 2, 3, 4, 5, ...$

$P(0) = {}^{12}C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{12} = 0{\cdot}1122$; $\quad P(1) = {}^{12}C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^{11} = 0{\cdot}2692$;

$P(2) = {}^{12}C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{10} = 0{\cdot}2961$; $\quad P(3) = {}^{12}C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^9 = 0{\cdot}1974$;

$P(4) = {}^{12}C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^8 = 0{\cdot}0888$; $\quad P(5) = {}^{12}C_5 \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^7 = 0{\cdot}0284$;

P(6 and above) $= 1 - (0{\cdot}1122 + 0{\cdot}2692 + ... + 0{\cdot}0284) = 0{\cdot}0079$

Expected frequency $= Np = 4096p$

$\therefore$

No. of successes	0	1	2	3	4	5	6 and above
Frequency	460	1103	1213	808	364	116	32

EXERCISE - I

(A) 1. Find the fallacy if any in the following statements.

(a) "The mean of a Binomial distribution is 6 and standard deviation is 4." (M.U. 1998)

(b) "The mean of a Binomial distribution is 9 and its standard deviation is 4."

[**Ans. :** (a) False, $q = 8/3$ is impossible, (b) False, $q = 16/9$, is impossible]

2. The mean and variance of a Binomial variate are 3 and 1·2. Find 'n', 'p' and $P(X < 4)$. (M.U. 2002) [**Ans. :** $n = 5$, $p = 0{\cdot}6$, 2068 / 3125]

3. Find the Binomial distribution if the mean is 5 and the variance is 10 / 3. Find $P(X = 2)$. (M.U. 2003, 05) [**Ans. :** $P(X = x) = {}^{15}C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{15-x}$; 0·06]

4. Find the mean, mode and standard deviation of the Binomial distribution whose parameters are $n = 8$, $p = 1/4$. (M.U. 1998) [**Ans. :** $\overline{X} = 2$, $\sigma = \sqrt{1{\cdot}5}$]

5. In a Binomial distribution consisting of 5 independent trials, probabilities of 1 and 2 successes are 0·4096 and 0·2048 respectively. Find the parameter 'p' of the distribution. (M.U. 1999) [**Ans. :** $p = 0{\cdot}2$]

6. Find $P(X \geq 1)$, where X is a Binomial variate with mean 4 and variance 3. (M.U. 2004) [**Ans. :** 0·09]

7. A Binomial variate X satisfies the relation $9P(X = 4) = P(X = 2)$ when $n = 6$. Find the value of the parameter p. [**Ans. :** $p = 3/4$]

8. Let X be Binomial distributed with parameter n, p. For what value of p variance is maximum, if you assume n fixed ? (M.U. 2003) [**Ans. :** $p = 1/2$]

(B) 1. The odds in favour of X's winning a game against Y are 4:3. Find the probability of Y's winning 3 games out of 7 played. [**Ans. :** ${}^7C_3 \left(\frac{3}{7}\right)^3 \left(\frac{3}{7}\right)^4$]

2. If 10% of bolts produced by a machine are defective. Find the probability that out of 5 bolts selected at random at most one will be defective. [**Ans. :** $(1{\cdot}4)(0{\cdot}9)^4$]

3. On an average 3 out of ten students fail in an examination. What is the probability that out of 10 students that appear for the examination none will fail ? [**Ans. :** $(0{\cdot}7)^{10}$]

4. If on the average rain falls on 10 days in every thirty, find the probability, (i) that the first three days of a week will be fine and the remaining wet, (ii) that the rain will fall on just three days of a week. (M.U. 1999)

[**Ans. :** (i) $\left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^4$, (ii) ${}^7C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^4$]

5. 12% of the items produced by a machine are defective. What is the probability that out of a random sample of 20 items produced by the machine, 5 are defective? (Simplification is not necessary). [**Ans. :** ${}^{20}C_5 (0{\cdot}12)^5 (0{\cdot}88)^{15}$]

5. If 10% of the rivets produced are defective, what is the chance that a random sample of 5 rivets will contain (i) exactly two defectives, (ii) fewer than two defectives ? [**Ans. :** 0·0729, 0·918]

7. An unbiased die is rolled five times. What is the probability of its showing 5 twice ? What is the probability of its showing 5 at least once ? [**Ans. :** 0·13, 0·598]

8. Find the chance of getting exactly 5 heads in 6 throws of an unbiased coin. [**Ans. :** 3 / 32]

9. On an average 20% of population in an area suffer from T.B. What is the probability that out of 5 persons chosen at random from this area at least two suffer from T.B.? [**Ans. :** 821 / 3125]

10. If X is the random variable showing the number of boys in a family with 4 children, construct a table showing the probability distribution of X. [**Ans. :** 1 / 16, 4 / 16, 6 / 16, 4 / 16, 1 / 16]

11. Three balls are drawn from a box containing 4 red and 6 black balls. If X denotes the total number of red balls drawn construct probability distribution table. [**Ans. :** 27 / 125, 54 / 125, 36 / 125, 8 / 125]

12. Two unbiased dice are thrown three times. Find the probability that the sum nine would be obtained (i) once, (ii) twice. (M.U. 1997) [Ans. : (i) 0·26, (ii) 0·03]

13. If X is the random variable showing the number of boys in a family with 4 children, construct a table showing the probability distribution of X.

[Ans. : 1 / 16, 4 / 16, 6 / 16, 4 / 16, 1 / 16]

14. 5 defective bulbs are accidently mixed up with 20 good ones. It is not possible to just look at a bulb and tell whether or not it is defective. Find the probability distribution of defective bulbs if 4 bulbs are drawn from this lot. (M.U. 2001, 02, 14)

[Ans. :

X :	0	1	2	3	4
$P(X=x)$:	0·41	0·41	0·15	0·02	0·01]

15. For special security in a certain protected area it was decided to put three lighting bulbs on each pole. If each bulb has probability 'p' of burning out in the first 100 hours of service, calculate the probability that at least one of them is still good after 100 hours. If $p = 0{\cdot}3$ how many bulbs would be needed on each pole to ensure 99% safety that at least one is good after 100 hours ? Find also the probability that at least one of the bulbs is still working after 100 hours. (M.U. 2001, 04, 07, 09) [Ans : (i) $1-p^3$, (ii) 4, (iii) 0·973]

16. If 10% of the rivets produced by a machine are defective, find the probability that out of 5 randomly chosen rivets (i) none will be defective, (ii) at the most two will be defective.

(M.U. 1997) [Ans. : (i) 0·59, (ii) 0·99]

(C) 1. Assuming that half the population is female and assuming that 100 samples each of 10 individuals are taken, how many samples would you except to have 3 or less females ?

(M.U. 1998) [Ans. : 17]

2. Take 100 sets of 10 tosses of an unbiased coin. In how many cases do you expect to get (a) 7 heads and 3 tails, (b) 7 heads at least ? [Ans. : 12, 17]

3. Assuming that half the population is vegetarian so that the chance of an individual being vegetarian is 1 / 2 and assuming that 100 investigators can take a sample of 10 individuals to see whether they are vegetarians, how many investigators would you expect to report that three people or less were vegetarians ? (M.U. 1998) [Ans. : 17]

4. An irregular six faced die is thrown. The probability that in 10 throws it will give five even numbers is twice as likely that it will give four even numbers. How many times in 10,000 sets of 10 throws, would you expect to give no even number ? (M.U. 2002)

[Ans. : $p = 5/8$; 1]

5. The probability of failure in Physics practical examination is 20%. If 25 batches of 6 students each take the examination, in how many batches 4 or more students would pass ?

(M.U. 2001) [Ans. : 20]

6. A lot contains 1% defective items. What should be the number of items in a lot so that the probability of finding at least one defective item in it, is at least 0·95. (M.U. 1999)

[Ans. : 299]

7. The probability that a bomb will hit the target is 0·2. Two bombs are required to destroy the target. If six bombs are used, find the probability that the target will be destroyed.

(M.U. 1998) [Ans. : 0·35]

8. The probability of a man hitting the target is 1/4. (i) If he fires 7 times what is the probability of his hitting the target at least twice ? (ii) How many times must be fire so that the probability of his hitting the targets at least once is greater than 2 / 3 ? (M.U. 2004)

[Ans. : (i) 0·555, (ii) 4]

9. Out of 1000 families with 4 children each, how many would you expect to have (i) 2 boys and 2 girls, (ii) at least one boy, (iii) no girl, (iv) at most 2 girls. (M.U. 1998)

[Ans. : (i) 375, (ii) 937·5, (iii) 62·5, (iv) 687·5]

10. If we take 1280 sets each of 10 tosses of a fair coin, in how many sets should we expect to get 7 heads and 3 tails ? [Ans. : 150]

11. The mean of defective blades supplied in packets of 10 is 1. In how many packets of this make out of 1000 packets would you expect to find at least 4 non-defective blades.

[Ans. : 13]

12. In a sampling of a large number of parts produced by a machine the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples how many samples would you expect to contain at least 3 defectives. [Ans. : 323]

13. Out of 800 families with 5 children each how many would you expect to have (i) 3 Boys and 2 Girls, (ii) 5 girls, (iii) 5 Boys ? (M.U. 2006) [Ans. : (i) 250, (ii) 25, (iii) 25]

14. Assuming boys and girls are equally likely find the expected number of 0 boy, 1 boy, 2 boys, ...5 boys out of 320 families with 5 children each. [Ans. : 10, 50, 100, 100, 50, 10]

15. On an average a student is present on 5 days a week. Find on how many days in a course of 100 days out of 5 students 0, 1, 2, ..., 4, 5 students will be present.

[Ans. : 0, 2, 12, 30, 37, 19]

(D) 1. Let X, Y be two independent binomial variates with parameters ($n_1 = 8$, $p = 0{\cdot}3$) and ($n_2 = 6$, $p = 0{\cdot}3$) respectively. Find $P(X+Y=2)$. [Ans. : 0·1134]

2. Let X, Y be two independent binomial variates with parameters ($n_1 = 5$, $p = 0{\cdot}4$) and ($n_2 = 7$, $p = 0{\cdot}4$). Find $P(X+Y \geq 2)$. [Ans. : 0·9804]

3. Let X, Y be two independent binomial variates with parameters ($n_1 = 9$, $p = 0{\cdot}2$) and ($n_2 = 7$, $p = 0{\cdot}2$) respectively. Find $P(X+Y \geq 2)$. [Ans. : 0·8593]

(E) 1. Five fair coins are tossed 3200 times, find the frequency distribution of number of heads obtained. Also find mean and standard deviation. (M.U. 2003)

[Ans. : 100, 500, 1000, 1000, 500, 100 ; 1600, 28·28]

2. Four fair coins are tossed 160 times and the following results were obtained.

No. of heads :	0,	1,	2,	3,	4.
Frequency :	17,	52,	54,	31,	6.

Fit a Binomial distribution.

[Ans. : $\bar{x} = \dfrac{\Sigma f_i x_i}{N} = 1{\cdot}73$, $p = \dfrac{\bar{x}}{n} = \dfrac{1{\cdot}73}{4} = 0{\cdot}43$, $q = 0$, $N = 160$

No. of heads :	0,	1,	2,	3,	4.
Frequency :	17,	51,	58,	29,	5.]

3. Fit a Binomial distribution to the following data,

X :	0,	1,	2,	3,	4,	5,	6.
f :	5,	18,	28,	12,	7,	6,	4.

(M.U. 2004, 05, 16)

[**Ans. :** $n = 6$, $N = 80$, $\bar{x} = \frac{\Sigma f_i x_i}{N} = 2{\cdot}4$ $\therefore$ $p = \frac{2{\cdot}4}{6} = 0{\cdot}4$, $q = 0{\cdot}6$

X :	0,	1,	2,	3,	4,	5,	6.
f :	4,	15,	25,	22,	11,	3,	0.]

4. In an experiment with 500 seeds in groups of 5 the following results were obtained,

X :	0,	1,	2,	3,	4,	5.	Total
f :	10,	70,	150,	160,	80,	30.	500

[**Ans. :** $\bar{x} = \frac{\Sigma f_i x_i}{N} = 2{\cdot}64$, $n = 5$, $p = \frac{\bar{x}}{n} = 0{\cdot}528$, $q = 0{\cdot}472$

x :	0,	1,	2,	3,	4,	5.
f :	12,	65,	147,	164,	92,	20.]

5. Five dice are thrown together 96 times. The number of times 4, 5 or 6 was obtained is given below.

No. of times 4, 5, or 6 was obtained :	0,	1,	2,	3,	4,	5.
Frequency :	1,	10,	24,	35,	18,	8.

Fit a Binomial distribution if (i) dice are unbiased (ii) the nature of the dice is not known.

[**Ans. :** (i) $p = \frac{3}{6} = \frac{1}{2}$, $q = \frac{1}{2}$, $n = 5$, $N = 96$.

No. of successes :	0,	1,	2,	3,	4,	5.
Frequencies :	3,	15,	30,	30,	15,	3.

(ii) $\bar{x} = \frac{\Sigma f_i x_i}{N} = 2{\cdot}86$, $p = \frac{\bar{x}}{n} = \frac{2{\cdot}86}{5} = 0{\cdot}572$, $q = 0{\cdot}428$, $n = 5$, $N = 96$.

No. of successes :	0,	1,	2,	3,	4,	5.
Frequencies :	1,	9,	25,	33,	22,	6.]

6. Eight unbiased coins are tossed 256 times. Number of heads observed in each throws are shown below.

No. of heads :	0,	1,	2,	3,	4,	5,	6,	7,	8.	Total
Observed frequency :	2,	6,	24,	63,	64,	50,	36,	10,	1.	256

Fit a Binomial distribution and find the mean and the variance of the fitted distribution. (M.U. 1996)

[**Ans. :**

No. of heads :	0,	1,	2,	3,	4,	5,	6,	7,	8.
Observed frequency :	1,	8,	28,	56,	70,	56,	28,	8,	1.

Mean $= np = 4$, S.D. $= \sqrt{npq} = \sqrt{2}$.]

7. A biased coin was tossed 6 times and the experiment was repeated 150 times. The following table gives the frequencies of 0, 1, 2, 3, ..., 6 heads.

X (No. of heads) :	0,	1,	2,	3,	4,	5,	6.	Total
Frequency :	2,	7,	20,	35,	48,	32,	6.	150

Evaluate the mean. Estimate p stating the assumptions. Fit a Binomial distribution. (M.U. 1997)

[**Ans. :** $\bar{X} = 3{\cdot}6$, $p = 0{\cdot}6$

X (No. of heads) :	0,	1,	2,	3,	4,	5,	6.	Total
Frequency :	1,	5,	21,	41,	47,	28,	7.	150]

8. Fit a Binomial distribution to the following data and calculate theoretical frequencies. (M.U. 1998)

x :	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10.
Frequency :	6,	20,	28,	12,	8,	6,	0,	0,	0,	0,	0.

[**Ans. :** $\bar{X} = 2{\cdot}175$, $n = 10$, $p = 0{\cdot}2175$, $q = 0{\cdot}7825$

X :	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10.
Frequency :	7,	19,	24,	18,	9,	3,	0,	0,	0,	0,	0.]

9. Fit a Binomial distribution to the following data. (M.U. 2004)

X :	0,	1,	2,	3,	4.
Frequency :	12,	66,	109,	59,	10.

[**Ans. :** $\bar{X} = 1{\cdot}96$, $p = 0{\cdot}49$

X :	0,	1,	2,	3,	4.
Frequency :	17,	67,	96,	61,	15.]

3. Poisson Distribution

Poisson distribution was discovered by the French Mathematician Poisson in 1837. Poisson distribution is **the limiting case of the binomial distribution** under the following conditions :

(i) n, the number of trials is infinitely large *i.e.* $n \to \infty$.

(ii) p, the probability of success in each trial is constant and infinitely small *i.e.* $p \to 0$.

(iii) np, the average success is finite say m, *i.e.* $np = m$.

Siméon Denis Poisson (1781 - 1840)

A French mathematician geometer and physicist was born in Pithviers, Loiret in France. He was a favourite student of Lagrange and was treated like a son by Laplace. Lagrange and Laplace were his doctoral advisors. In 1806 he became full professor at École Polytechnique, in Paris succeeding Fourier. His notable students were Dirichlet and Liouville. As a teacher of mathematics Poisson is said to have been extremely successful. As a scientific worker, his productivity has rarely, if ever, been equalled. Inspite of his many official duties, he published more than three hundred works, several of them extensive treatises and many of them memoires dealing with the most abtruse branches of pure mathematics, applied mathematics, mathematical physics and rational mechanics. His memoires on the theory of electricity and magnetism created a new branch of mathematical physics. He made important advances in planetary theory. Poisson is known for Poisson distribution, Poisson process, Poisson equation, Poisson Kernel, Poisson regression, Poisson summation formula, Poisson ratio, Euler-Poisson-Darboux equation, Conway-Maxwell-Poisson distribution.

(a) To Derive Poisson Distribution

Consider $p(x) = {}^nC_x p^x q^{n-x}$

$$= {}^nC_x \left(\frac{p}{q}\right)^x q^n = {}^nC_x \left(\frac{p}{1-p}\right)^x (1-p)^n$$

Putting $p = \frac{m}{n}$,

$$p(x) = \frac{n(n-1)(n-2).....(n-x+1)}{x!} \frac{(m/n)^x}{[1-(m/n)]^x}\left[1-\frac{m}{n}\right]^n$$

$$p(x) = \frac{\left[1-\frac{1}{n}\right]\left[1-\frac{2}{n}\right]....\left[1-\frac{x-1}{n}\right]m^x\left[1-\frac{m}{n}\right]^n}{x!\quad\left[1-\frac{m}{n}\right]^x}$$

Since $\lim_{n\to\infty}\left[1-\frac{m}{n}\right]^n = e^{-m}$ and $\lim_{n\to\infty}\left[1-\frac{m}{n}\right]^x = 1$

Taking the limits of both sides as $n \to \infty$

$$p(x) = \frac{m^x \cdot e^{-m}}{x!}$$

Thus, the limit of the Binomial random variable is the Poisson random variable.

(b) Definition

A random variable X is said to follow **Poisson distribution** if the probability of x is given by

$$\boxed{P(X=x) = \frac{e^{-m}m^x}{x!},\quad x = 0, 1, 2, ...}$$

and m (> 0) is called the parameter of the distribution.

Remarks ....

1. The sum of the probabilities is 1.

$$\sum_{x=0} P(X=x) = \sum_{x=0}\frac{e^{-m}m^x}{x!} = e^{-m}\sum\frac{m^x}{x!}$$

$$= e^{-m}\left[1+m+\frac{m^2}{2!}+.....\right] = e^{-m}\cdot e^m = 1.$$

2. Poisson distribution occurs where the probability of occurrence p is very small and the number of trials n is very large and where the probability of occurrence only can be known e.g. the number of accidents, the number of deaths by a disease, the number of

printing mistakes on a page etc. In these cases we can only observe the number of successes but the number of failures cannot be observed. We can observe how many accidents occur; we cannot observe how many times accidents do not occur.

(c) When do we get Poisson distribution ?

As seen before we get a Poisson distribution if the following conditions are satisfied.

1. The number of trials n is infinitely large *i.e.* $n \to \infty$.
2. A trial results in only two ways-success or failure.
3. If p and q are probabilities of success and failure, then $p + q = 1$.
4. These probabilities are mutually exclusive, exhaustive but not necessarily equally likely. (*i.e.*, p is not necessarily equal to 1 / 2.)
5. The probability p of success is very small *i.e.* $p \to 0$.
6. $n \to \infty$ and $p \to 0$ such that $np = m$, (> 0) a constant.

(d) Uses

Poisson distribution is used in problems involving :

(i) the number of deaths due to a disease such as heart attack, cancer etc.

(ii) the number of accidents during a week or a month etc.

(iii) the number of phone-calls received at a particular telephone exchange during a period of time.

(iv) the number of cars passing a particular point on a road during a period of time.

(v) the number of printing mistakes on a page of a book etc.

Note ....

Since the Poisson distribution is the limiting case of Binomial distribution, we can calculate binomial probabilities approximately by using Possion distribution whenever n is large and p is small. (See Ex. 21, page 8-32 and Ex. 22, page 8-33)

(e) Moments of the Poisson's Distribution **(M.U. 2010)**

We obtain below the first two moments about the origin.

$$\mu_1' = E(x) = \Sigma\, p_i x_i$$

$$= \sum_{x=0}\frac{e^{-m}m^x}{x!}x = \sum_{x=1}\frac{e^{-m}m^x}{(x-1)!} = m\,e^{-m}\sum\frac{m^{x-1}}{(x-1)!}$$

$$= m\,e^{-m}\left[1+m+\frac{m^2}{2!}+\frac{m^3}{3!}+.....\right]$$

$$= m\,e^{-m}\cdot e^m = m$$

Hence, $\boxed{\text{mean} = m}$

$$\mu_2' = E(x^2) = \Sigma\, p_i x_i^2 = \sum_{x=0} e^{-m}\cdot\frac{m^x}{x!}\cdot x^2$$

We can write $x^2 = x + x(x-1)$

$$\mu_2' = \sum_{x=0} e^{-m} \cdot \frac{m^x}{x!}[x + x(x-1)]$$

$$= \sum_{x=0} e^{-m} \cdot \frac{m^x \cdot x}{x!} + \sum_{x=0} e^{-m} \cdot \frac{m^x}{x!} x \cdot (x-1)$$

$$= m e^{-m} \sum_{x=1} \frac{m^{x-1}}{(x-1)!} + m^2 e^{-m} \sum_{x=2} \frac{m^{x-2}}{(x-2)!}$$

$$= m e^{-m} \cdot e^m + m^2 e^{-m}\left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \ldots\right]$$

$$= m e^{-m} \cdot e^m + m^2 e^{-m} \cdot e^m = m + m^2$$

$$\therefore \quad \mu_2 = \mu_2' - \mu_1'^2 = m + m^2 - m^2 = m$$

$$\therefore \quad \boxed{\text{Variance} = m}$$

Thus, the mean and variance of the Poisson's distribution are both equal to m.

(f) Moment Generating Function (M.U. 2014)

The m.g.f. about the origin is,

$$M_0(t) = E(e^{tx}) = \Sigma p(x) e^{tx} = \Sigma \frac{e^{-m} \cdot m^x}{x!} \cdot e^{tx}$$

$$= e^{-m} \Sigma \frac{(m e^t)^x}{x!} = e^{-m} \cdot e^{m e^t}$$

$$\boxed{M_0(t) = e^{m(e^t - 1)}} \quad \ldots\ldots\ldots (A)$$

(Note that $\Sigma \frac{k^x}{x!} = 1 + k + \frac{k^2}{2!} + \frac{k^3}{3!} + \ldots = e^k$.

We shall often meet expressions of this type in the discussion of Poisson's distribution.)

Now $\frac{d}{dt}\{M_0(t)\} = e^{m(e^t-1)} \cdot m e^t$

$$\therefore \quad \left[\frac{d}{dt} M_0(t)\right]_{t=0} = m \quad \therefore \mu_1' = m$$

$$\frac{d^2}{dt^2}\{M_0(t)\} = m\left[e^t \cdot e^{m(e^t-1)} \cdot m e^t + e^{m(e^t-1)} \cdot e^t\right]$$

$$\therefore \quad \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = m[m+1] = m^2 + m \quad \therefore \mu_2' = m^2 + m$$

$$\because \quad \mu_2 = \mu_2' - \mu^2 = m^2 + m - m^2 \quad \therefore \mu_2 = m$$

We may denote the logarithm of the m.g.f. (A) by $L(t)$. Then

$$L(t) = \log M_0(t) = \log[e^{m(e^t-1)}] \quad \therefore \quad L(t) = m(e^t - 1)$$

Differentiating both sides w.r.t. t

$$L'(t) = m e^t \quad \therefore \quad L'(0) = m$$

$$L''(t) = m e^t \quad \therefore \quad L''(0) = m$$

But as proved in (B) on page 7-31, $L'(0) = \mu$ and $L''(0) = \mu_2$.

Hence, mean $\mu = L'(0) = m$ and variance $\mu_2 = L''(0) = m$.

Example : If the moment generating function about the origin of a discrete random variable is X is $e^{4(e^t-1)}$, find $P(X = \mu + \sigma)$ where μ and σ are mean and standard deviation of X. **(M.U. 2007, 09)**

Sol. : We know that m.g.f. of a Poisson distribution with mean m is

$$M_0(t) = e^{m(e^t-1)}$$

Comparing this with the given m.g.f. we see that X is a Poisson variate with mean $m = 4$ and variance $\sigma^2 = m = 4$.

Hence, the Poisson distribution is

$$P(X = x) = \frac{e^{-4}(4)^x}{x!}$$

But $m + \sigma = 4 + 2 = 6$.

$$\therefore \quad P(X = 6) = \frac{e^{-4} \cdot 4^6}{6!} = 0.1.$$

(g) Additive property of Independent Poisson distributions

If two independent variates have Poisson distribution with means m_1 and m_2 then their sum also is a Poisson distribution with mean $m_1 + m_2$.

Proof : Let $M_1(t)$ and $M_2(t)$ be the m.g.f. s of the two Poisson variates X_1 and X_2 and let $M(t)$ be the m.g.f. of their sum.

Now $M_1(t) = E(e^{tX_1})$ and $M_2(t) = E(e^{tX_2})$

$$M_1(t) = e^{m_1(e^t-1)} \quad \text{and} \quad M_2(t) = e^{m_2(e^t-1)}$$

Since, X_1 and X_2 are independent, m.g.f. of $X_1 + X_2$ is

$$M(t) = E[e^{t(X_1+X_2)}] = E(e^{tX_1}) \cdot E(e^{tX_2})$$

$$= e^{m_1(e^t-1)} \cdot e^{m_2(e^t-1)} = e^{(m_1+m_2)(e^t-1)}$$

But this is the m.g.f. of the Poisson distribution with mean $m_1 + m_2$. Hence, the result.

Notes ....

1. Although the sum of two Poisson variates is a Poisson variate, the difference between two Poisson variates is not a Poisson variate. If X, Y are two Poisson variates then.

$$M_{(X-Y)}(t) = M_{X+(-Y)}(t) = M_X(t) \cdot M_{(-Y)}(t)$$

$$= M_X(t) \cdot M_Y(-t) \qquad [\because M_{cX}(t) = M_X(ct)]$$

$$= e^{m_1(e^t-1)} \cdot e^{m_2(e^{-t}-1)} = e^{m_1(e^t-1) + m_2(e^{-t}-1)}$$

But this cannot be put in the form $e^{m(e^t-1)}$ and hence $(X-Y)$ is not a Poisson variate.

2. If $X_1, X_2, \ldots, X_n$ are n independent Poisson variates with parameters $m_1, m_2, \ldots, m_n$ then $Y = X_1 + X_2 + \ldots + X_n$ is also a Poisson variate with parameter $m_1 + m_2 + \ldots + m_n$.

3. If X_1 and X_2 are independent Poisson variates with parameter m_1, m_2 then $Y = a_1X_1 + a_2X_2$ is **not** a Poisson variate.

(You can very easily prove these two results.)

(h) Recurrence Relation For Probabilities

We have for Poisson distribution

$$p(x) = \frac{e^{-m} \cdot m^x}{x!} \qquad \therefore \; p(x+1) = \frac{e^{-m} \cdot m^{x+1}}{(x+1)!}$$

$$\therefore \; \frac{p(x+1)}{p(x)} = \frac{m^{x+1}}{(x+1)!} \cdot \frac{x!}{m^x} = \frac{m}{x+1} \qquad \boxed{\therefore \; p(x+1) = \frac{m}{x+1} \cdot p(x)}$$

If we know $p(0) = e^{-m}$, we can find the probabilities of $x = 1, 2, 3 \ldots$

Thus, $p(1) = m \cdot p(0)$, $p(2) = \frac{m}{2} p(1)$, $p(3) = \frac{m}{3} p(2)$ and so on.

Since, expected frequency of x *i.e.* $f(x)$ is $Np(x)$, we have from the above relation.

$$Np(x+1) = \frac{m}{x+1} \cdot Np(x) \qquad \boxed{\therefore \; f(x+1) = \frac{m}{x+1} f(x)}$$

This relation can be used to find expected frequencies. This is called **fitting a Poisson distribution.**

Example 1 : Find out the fallacy if any in the following statement "The mean of a Poisson distribution is 2 and the variance is 3".

Sol. : In a Poisson distribution the mean and variance are same. Hence, the above statement is false.

Example 2 : If the mean of the Poisson distribution is 4; find $P(m - 2\sigma < X < m + 2\sigma)$. **(M.U. 2005)**

Sol. : For Poisson distribution mean = variance = m.

Hence, $m = 4$ and $\sigma = 2$

$$\therefore \; P(m - 2\sigma < X < m + 2\sigma) = P(0 < X < 8)$$
$$= P[X = 1, 2, 3, \ldots, 7]$$

But $P(X) = e^{-m} \frac{m^x}{x!} = e^{-4} \frac{4^x}{x!}$

$$\therefore \text{ Required probability} = e^{-4}\left[\frac{4}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!} + \frac{4^6}{6!} + \frac{4^7}{7!}\right] = 0.93$$

Example 3 : If the variance of a Poisson distribution is 2, find the probabilities of $r = 1, 2, 3, 4$ from the recurrence relation of Poisson distribution. **(M.U. 2002)**

Sol. : We have $P(x) = e^{-m} \frac{m^x}{x!}$

Since variance = $m = 2$ by data.

$$P(x) = e^{-2} \frac{2^x}{x!} \text{ when } x = 0, \; P(0) = e^{-2}$$

Now, the recurrence relation is

$$P(x+1) = \frac{m}{x+1} P(x)$$

Putting $x = 0$, $P(1) = \frac{2}{1} P(0) = 2e^{-2}$

Putting $x = 1$, $P(2) = \frac{2}{2} P(1) = \frac{2}{2} \cdot 2e^{-2} = e^{-2}$

Putting $x = 2$, $P(3) = \frac{2}{3} P(3) = \frac{2}{3} P(2) = \frac{2}{3} e^{-2}$

Putting $x = 3$, $P(4) = \frac{2}{4} P(3) = \frac{1}{3} e^{-2}$

Example 4 : Find out the fallacy if any in the following statement.

"If X is a Poisson variate such that $P(X = 2) = 9P(X = 4) + 90P(X = 6)$ then mean of $X = 1$." **(M.U. 1997)**

Sol. : Let m be the mean of X. $\therefore \; P(X = x) = e^{-m} \frac{m^x}{x!}$

By data $e^{-m} \cdot \frac{m^2}{2!} = 9 \cdot e^{-m} \cdot \frac{m^4}{4!} + 90 \cdot e^{-m} \cdot \frac{m^6}{6!}$

$$\therefore \; \frac{1}{2} = \frac{3m^2}{8} + \frac{m^4}{8} \qquad \therefore \; m^4 + 3m^2 - 4 = 0$$

$$\therefore \; (m^2 + 4)(m^2 - 1) = 0 \qquad \therefore \; m^2 = -4 \text{ or } m^2 = 1$$

$\therefore$ The mean is 1 since $m > 0$.

$\therefore$ The statement is correct.

Example 5 : A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as Poisson variate with mean 1·5. Calculate the proportion of days on which (i) neither car is used, (ii) some demand is refused. **(M.U. 1996, 98)**

Sol. : We have $P(x) = e^{-m} \frac{m^x}{x!} = \frac{e^{-1.5} \cdot (1.5)^x}{x!}$, $x = 0, 1, 2, \ldots$

(i) Probability that there is no demand is

$$P(X = 0) = e^{-1.5} \frac{(1.5)^0}{0!} = 0.2231$$

(ii) Probability that some demand is refused means there was demand for more than two cars.

$$\therefore \; P(X > 2) = P(X = 3) + P(X = 4) + \ldots$$
$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - \left[e^{-1\cdot5}\frac{(1\cdot5)^0}{0!} + e^{-1\cdot5}\frac{(1\cdot5)^1}{1!} + e^{-1\cdot5}\frac{(1\cdot5)^2}{2!}\right]$$

$$= 1 - [0\cdot2231 + 0\cdot3347 + 0\cdot2510]$$

$$= 0\cdot1912.$$

$\therefore$ Proportion of days on which (i) neither car is used is 0·2231.

(ii) some demand is refused is 0·1912.

Example 6 : If a random variable X follows Poisson distribution such that $P(X = 1) = 2P(X = 2)$, find the mean and the variance of the distribution. Also find $P(X = 3)$. (M.U. 2002, 05, 16)

Sol. : Let the parameter of the Poisson distribution be m.

$$\therefore \quad P(X = x) = \frac{e^{-m} \times m^x}{x!}$$

We are given that $P(X = 1) = 2P(X = 2)$

$$\therefore \quad \frac{e^{-m} \times m^1}{1!} = \frac{2e^{-m} \times m^2}{2!} \quad \therefore \; m = 1$$

$\therefore$ The mean and the variance = 1.

Now, $P(X = 3) = \dfrac{e^{-m} \cdot m^x}{x!} = \dfrac{e^{-1} \cdot 1^3}{3!} = 0\cdot0613.$

Example 7 : A hospital switch board receives an average of 4 emergency calls in a 10 minutes interval. What is the probability that (i) there are atleast 2 emergency calls, (ii) there are exactly 3 emergency call in an interval of 10 minutes ?

Sol. : We have $P(x) = \dfrac{e^{-m} \cdot m^x}{x!}$. Here, $m = 4$.

(i) $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= \frac{e^{-4} \cdot 4^0}{0!} + \frac{e^{-4} \cdot 4^1}{1!} + \frac{e^{-4} \cdot 4^2}{2!}$$

$$= e^{-4}(1 + 4 + 8) = 0\cdot238$$

(ii) $P(X = 3) = \dfrac{e^{-m} \cdot m^x}{x!} = \dfrac{e^{-4} \cdot 4^3}{3!} = 0\cdot195$

Example 8 : A variable X follows a Poisson distribution with variance 3. Calculate (i) $P(X = 2)$, (ii) $P(X \ge 4)$. (M.U. 1996)

Sol. : We have $P(X = x) = \dfrac{e^{-m} \times m^x}{x!}$, $x = 0, 1, 2, \ldots..$

$$\therefore \quad P(X = 2) = \frac{e^{-3} \times 3^2}{2!} = 0\cdot224$$

$$\therefore \quad P(X \ge 4) = 1 - [\,P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)]$$

$$= 1 - 0\cdot647 = 0\cdot353.$$

Example 9 : If X and Y are independent Poisson variates with mean m_1 and m_2, find the probability that $X + Y = k$.

Sol. : Since X, Y are independent Poisson variates with parameters m_1 and m_2, $Z = X + Y$ is also a Poisson variate with parameter $m_1 + m_2$.

$$\therefore \quad P(Z = k) = \frac{e^{-(m_1 + m_2)}(m_1 + m_2)^k}{k!}, \quad k = 0, 1, 2$$

Example 10 : If X, Y are independent Poisson variates with mean 2 and 3, find the variance of $3X - 2Y$.

Sol. : For Poisson variate mean and variance are equal. Hence, Var. $X = 2$ and Var. $Y = 3$.

Since, X, Y are independent

$$\text{Var.}\,(3X - 2Y) = 9\,\text{Var.}\,(X) + 4\,\text{Var.}\,(Y)$$

$$= 9\,(2) + 4\,(3) = 30.$$

Example 11 : If X, Y are independent Poisson variates such that $P(X = 1) = P(X = 2)$ and $P(Y = 2) = P(Y = 3)$, find the variance of $2X - 3Y$.

Sol. : Let the parameter of X and Y be m_1 and m_2

$$\therefore \quad P(X = 1) = P(X = 2) \text{ gives } \frac{e^{-m_1}\, m_1}{1} = \frac{e^{-m_1}\, m_1^2}{2!}$$

$$\therefore \quad 2e^{-m_1}\, m_1 - e^{-m_1}\, m_1^2 = 0 \quad \therefore \; e^{-m_1}\, m_1(2 - m_1) = 0 \quad \therefore \; m_1 = 2$$

$$P(Y = 2) = P(Y = 3) \text{ gives } \frac{e^{-m_2}\, m_2^2}{2!} = \frac{e^{-m_2}\, m_2^3}{3!}$$

$$\therefore \quad 3e^{-m_2} m_2^2 - e^{-m_2} m_2^3 = 0 \quad \therefore \; e^{-m_2} m_2^2(3 - m_2) = 0 \quad \therefore \; m_2 = 3$$

$$\therefore \quad \text{Var.}(X) = m_1 = 2\,;\ \text{Var.}(Y) = m_2 = 3$$

Since, X and Y are independent

$$V(2X - 3Y) = 4V(X) + 9V(Y)$$

$$= 4\,(2) + 9\,(3) = 35.$$

Example 12 : If X_1, X_2, X_3 are three independent Poisson variates with parameters $m_1 = 1$, $m_2 = 2$, $m_3 = 3$ respectively, find :

(i) $P[\,(X_1 + X_2 + X_3) \ge 3\,]$ and (ii) $P[\,X_1 = 1 / (X_1 + X_2 + X_3) = 3\,]$. (M.U. 2005)

Sol. : By additive property of Poisson distribution $Z = X_1 + X_2 + X_3$ is also a Poisson distribution with parameter $m = m_1 + m_2 + m_3 = 6$.

$$\therefore \quad P(Z \ge 3) = 1 - P(Z = \le 2)$$

$$= 1 - \sum_{Z=0}^{2} \frac{e^{-6}6^z}{z!} = 1 - \left(e^{-6} + 6e^{-6} + \frac{6^2 e^{-6}}{z!}\right)$$

$$= 1 - 25e^{-6} = 1 - 25\,(0\cdot002478) = 0\cdot938$$

By definition of conditional probability,

$$P[X_1 = 1/(X_1 + X_2 + X_3) = 3] = \frac{P(X_1 = 1 \text{ and } X_2 + X_3 = 2)}{P[(X_1 + X_2 + X_3) = 3]}$$

Now, X_1 is a Poisson variate with parameter $m_1 = 1$, $X_2 + X_3$ is a Poisson variate with parameter $m_2 + m_3 = 2 + 3 = 5$. $X_1 + X_2 + X_3$ is a Poisson variate with parameter $m_1 + m_2 + m_3 = 1 + 2 + 3 = 6$.

$$\therefore \quad P[X_1 = 1/(X_1 + X_2 + X_3) = 3] = \frac{\left(e^{-1}\cdot\frac{1}{1}\right)\left(e^{-5}\cdot\frac{5^2}{2!}\right)}{e^{-6}\cdot\frac{6^3}{3!}} = \frac{25}{72}$$

Example 13 : An insurance company found that only 0·01% of the population is involved in a certain type of accident each year. If its 1000 policy holders were randomly selected from the population, what is the probability that no more than two of its clients are involved in such accident next year ? **(M.U. 2002)**

Sol. : We have $p = \frac{0\cdot01}{100} = 0\cdot0001$, $n = 1000$.

$$\therefore \quad m = np = 1000 \times 0\cdot0001 = 0\cdot1$$

$$\therefore \quad P(X = x) = e^{-0\cdot1}\frac{(0\cdot1)^x}{x!}$$

$$\therefore \quad P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$\therefore \quad P(X \le 2) = e^{-0\cdot1}\left[\frac{(0\cdot1)^0}{0!} + \frac{(0\cdot1)^1}{1!} + \frac{(0\cdot1)^2}{2!}\right] = 0\cdot9998$$

Example 14 : Find the probability that atmost 4 defective bulbs will be found in a box of 200 bulbs if it is known that 2 percent of the bulbs are defective. (Given $e^{-4} = 0\cdot0183$).

Sol. : Since probability of a defective bulb is small we can use Poisson distribution.

We have, $\therefore \; m = np = 200 \times 0\cdot02 = 4$

$$\therefore \quad P(X = x) = \frac{e^{-4} \times 4^x}{x!}$$

$$\therefore \quad P(X \le 4) = p(0) + p(1) + p(2) + p(3) + p(4)$$

$$= \frac{e^{-4}\times 4^0}{0!} + \frac{e^{-4}\times 4^1}{1!} + \frac{e^{-4}\times 4^2}{2!} + \frac{e^{-4}\times 4^3}{3!} + \frac{e^{-4}\times 4^4}{4!}$$

$$= e^{-4}\left[1 + \frac{4}{1!} + \frac{16}{2!} + \frac{64}{3!} + \frac{256}{4!}\right]$$

$$= e^{-4} \times \frac{103}{3} = 0\cdot0183 \times \frac{103}{3} = 0\cdot6283.$$

Example 15 : Using Poisson distribution find the approximate value of

$${}^{300}C_2\,(0.02)^2\,(0\cdot98)^{298} + {}^{300}C_3\,(0\cdot02)^3\,(0\cdot98)^{297}$$

(M.U. 2004)

Sol. : Clearly the above probabilities are the probabilities of Binomial distribution. Comparing them with

$$P(X = x) = {}^nC_x\,p^x\,q^{n-x}$$

We see that $n = 300$, $p = 0\cdot02$, $q = 0\cdot98$, $x = 2$ and 3.

Now, Binomial distribution is related to Poisson distribution where $m = np$.

Hence, $m = 300 \times 0\cdot02 = 0\cdot6$.

$\therefore$ Corresponding Poisson distribution is given by

$$P(X = x) = \frac{e^{-m}\cdot m^x}{x!} = \frac{e^{-6}6^x}{x!}$$

$$\therefore \quad P(X = 2) + P(X = 3) = \frac{e^{-6}\cdot 6^2}{2} + \frac{e^{-6}\cdot 6^3}{3}$$

$$= 0\cdot04462 + 0\cdot08923 = 0\cdot1338$$

Example 16 : Find the probability that at most 4 defective bulbs will be found in a box of 200 bulbs if it is known that 2 percent of the bulbs are defective. (Given $e^{-4} = 0\cdot0183$). **(M.U. 1997)**

Sol. : Since probability of a defective bulbs is small we can use Poisson distribution.

We have, $\therefore \; m = np = 200 \times 0\cdot02 = 4$

$$\therefore \quad P(X = x) = \frac{e^{-4} \times 4^x}{x!}$$

$$\therefore \quad P(X \le 4) = p(0) + p(1) + p(2) + p(3) + p(4)$$

$$= \frac{e^{-4}\times 4^0}{0!} + \frac{e^{-4}\times 4^1}{1!} + \frac{e^{-4}\times 4^2}{2!} + \frac{e^{-4}\times 4^3}{3!} + \frac{e^{-4}\times 4^4}{4!}$$

$$\therefore \quad P(X \le 4) = e^{-4}\left[1 + \frac{4}{1!} + \frac{16}{2!} + \frac{64}{3!} + \frac{256}{4!}\right]$$

$$= e^{-4} \times \frac{103}{3} = 0\cdot0183 \times \frac{103}{3} = 0\cdot6283.$$

Example 17 : The number of accidents in a year attributed to taxi drivers in a city follows Poisson distribution with mean 3. Out of 1,000 taxi drivers, find approximately the number of drivers with (i) no accident in a year, (ii) more than 3 accidents in a year. (Given : $e^{-1} = 0\cdot3679$, $e^{-2} = 0\cdot1353$, $e^{-3} = 0\cdot0498$) **(M.U. 1998)**

Sol. : For a Poisson variate

$$P(X = x) = \frac{e^{-m} \times m^x}{x!}, \quad x = 0, 1, 2, \ldots..$$

We are given $m = 3$

$$\therefore \quad P(X = x) = \frac{e^{-3} \times (3)^x}{x!}, \quad x = 0, 1, 2, \ldots.$$

$$\therefore \quad P(X = 0) = \frac{e^{-3} \times (3)^0}{0!} = e^{-3} = 0\cdot0498$$

$$P(X = 1) = \frac{e^{-3} \times (3)^1}{1!} = 0\cdot0498 \times 3 = 0\cdot1494$$

$$P(X = 2) = \frac{e^{-3} \times (3)^2}{2!} = 0\cdot0498 \times \frac{9}{2} = 0\cdot2241$$

$\therefore$ Expected number of drivers with no accidents

$$= N \times p(0) = 1{,}000 \times 0{\cdot}0498 = 49{\cdot}8 = 50 \text{ nearly.}$$

$$\therefore\ p(0, 1, 2, 3 \text{ accidents}) = p(0) + p(1) + p(2) + p(3)$$

$$= 0{\cdot}0498 + 0{\cdot}1494 + 0{\cdot}2241 = 0{\cdot}4233$$

$$\therefore\ p(\text{more than 3 accidents}) = 1 - 0{\cdot}4233 = 0{\cdot}5767.$$

Expected number of drivers with more than 3 accidents

$$= Np = 1{,}000 \times 0{\cdot}5767$$

$$= 576{\cdot}7 = 577 \text{ nearly.}$$

Example 18 : In a certain factory turning out blades, there is a small chance 1 / 500 for any blade to be defective. The blades are supplied in packets of 10. Use the Poisson distribution to calculate the approximate number of packets containing no defective, one defective, two defective blades in a consignment of 10,000 packets. (Given $e^{-0{\cdot}02} = 0{\cdot}9802$)

Sol. : We have, $n = 10$, $p = \dfrac{1}{500}$. $\quad \therefore\ m = np = 10 \times \dfrac{1}{500} = 0{\cdot}02$

$$\therefore\ P(X = x) = \frac{e^{-0{\cdot}02} \times (0{\cdot}02)^x}{x!}$$

$$\therefore\ P(X = 0) = \frac{e^{-0{\cdot}02} \times (0{\cdot}02)^0}{0!} = e^{-0{\cdot}02} = 0{\cdot}9802$$

$$P(X = 1) = \frac{e^{-0{\cdot}02} \times (0{\cdot}02)^1}{1!} = e^{-0{\cdot}02} \times 0{\cdot}02 = 0{\cdot}0196$$

$$P(X = 2) = \frac{e^{-0{\cdot}02} \times (0{\cdot}02)^2}{2!} = e^{-0{\cdot}02} \times 0{\cdot}0002 = 0{\cdot}0002$$

$\therefore$ Expected freq. of no defective $= 10000 \times 0{\cdot}9802 = 9802$

Expected freq. of one defective $= 10000 \times 0{\cdot}0196 = 196$

Expected freq. of two defective $= 10000 \times 0{\cdot}0002 = 2.$

Example 19 : Fit a Poisson distribution to the following data

No. of deaths :	0,	1,	2,	3,	4.
Frequencies :	123,	59,	14,	3,	1.

(M.U. 2000, 01, 04)

Sol. : Fitting Poisson distribution means finding expected frequencies of $X = 0, 1, 2, 3, 4$.

Now, mean $= \dfrac{\Sigma f_i x_i}{\Sigma f_i} = m$

$$\therefore\ \text{Mean } = 123(0) + 59(1) + 14(2) + 3(3) + 1(4)$$

$$= \frac{100}{200} = 0{\cdot}5$$

$\therefore$ Poisson distribution of X is

$$P(X = x) = \frac{e^{-m} \times m^x}{x!} = \frac{e^{-0{\cdot}5} \times (0{\cdot}5)^x}{x!} \qquad \text{.....................(1)}$$

Expected frequency $= N \times p(x) = 200 \times \dfrac{e^{-0{\cdot}5} \times (0{\cdot}5)^x}{x!}$

Putting $x = 0, 1, 2, 3, 4$ we got the expected frequencies as 121, 61, 15, 2, 1.

Or putting $x = 0$ in (1).

$$P(X = 0) = e^{-0{\cdot}5}\,\frac{(0{\cdot}5)^0}{0!} = 0{\cdot}6065$$

$\therefore$ Expected frequency $f(0) = Np = 200 \times 0{\cdot}6065 = 121.$

But $\quad f(x+1) = \dfrac{m}{x+1} \cdot f(x) = \dfrac{0{\cdot}5}{x+1} \cdot f(x)$

Putting $x = 0$, $\quad f(1) = \dfrac{0{\cdot}5}{1} \cdot 121 = 61.$ $\qquad$ Putting $x = 1$, $\; f(2) = \dfrac{0{\cdot}5}{2} \cdot 60 = 15.$

Putting $x = 2$, $\quad f(3) = \dfrac{0{\cdot}5}{3} \cdot 15 = 3.$ $\qquad$ Putting $x = 3$, $\; f(4) = \dfrac{0{\cdot}5}{4} \cdot 3 = 1.$

Example 20 : Letters were received in an office on each of 100 days. Fit a Poisson distribution and find the expected frequencies for $x = 0$ and 1. (Given : $e^{-4} = 0{\cdot}0183$).

Number of letters :	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10.
Frequency :	1,	4,	15,	22,	21,	20,	8,	6,	2,	0,	1.

Sol. : $\Sigma f_i = 1 + 4 + 15 + \ldots\ldots + 1 = 100$

$$m = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{1 \times 0 + 4 \times 1 + 15 \times 2 + \ldots\ldots + 1 \times 10}{100} = \frac{400}{100} = 4$$

$$\therefore\ P(X = x) = \frac{e^{-m} \cdot m^x}{x!} = \frac{e^{-4} 4^x}{x!} \qquad \therefore\ p(0) = e^{-4} = 0{\cdot}0183$$

Now, expected frequency $f(0) = Np = 100 \times 0{\cdot}0183 = 1{\cdot}83 = 2$

But $\quad f(x+1) = \dfrac{m}{x+1} \cdot f(x) \quad \therefore\ f(1) = \dfrac{4}{1}(1{\cdot}83) = 7{\cdot}32 = 7$

Example 21 : A transmission channel has a per-digit error probability $p = 0{\cdot}01$. Calculate the probability of more than 1 error in 10 received digits using (i) Binomial Distribution, (ii) Poisson Distribution.

Also find the m.g.f. in each case. **(M.U. 2004)**

Sol. : (i) Binomial Distribution : We have $p = 0{\cdot}01$, $q = 1 - p = 0{\cdot}99$, $n = 10$.

$$\therefore\ P(X = x) = {}^nC_x\, p^x\, q^{n-x} = {}^{10}C_x\,(0.01)^x\,(0.99)^{10-x}$$

$$P(X > 1) = 1 - P(X \le 1) = 1 - P(X = 0) - P(X = 1)$$

$$= 1 - {}^{10}C_0\,(0{\cdot}01)^0\,(0{\cdot}99)^{10} - {}^{10}C_1\,(0{\cdot}01)^1\,(0{\cdot}99)^9$$

$$= 1 - 0{\cdot}9044 - 0{\cdot}09135 = 0{\cdot}00425$$

(ii) **Poisson Distribution :** We have $m = np = 10\,(0{\cdot}01) = 0{\cdot}1$.

$$\therefore\ P(X = x) = e^{-m} \cdot \frac{m^x}{x!} = e^{-0{\cdot}1}\,\frac{(0{\cdot}1)^x}{x!}$$

$$\therefore\ P(X > 1) = 1 - P(X \le 1) = 1 - P(X = 0) - P(X = 1)$$

$$= 1 - e^{-0{\cdot}1}\,\frac{(0{\cdot}1)^0}{0!} - e^{-0{\cdot}1}\,\frac{(0{\cdot}1)^1}{1!}$$

$$= 1 - 0{\cdot}9048 - 0{\cdot}0905 = 0{\cdot}0047$$

(iii) M.G.F. of Binomial Distribution is given by .

$$M_0(t) = (q + pe^t)$$ [See (A), page 8-3]

Here, $q = 0{\cdot}9,\ p = 0{\cdot}01,\ n = 10$

$$\therefore\ M_0(t) = (0{\cdot}99 + 0{\cdot}01 \cdot e^t)^{10}$$

(iv) M.G.F. of Poisson Distribution is given by

$$M_0(t) = e^{m(e^t - 1)}$$ [See (A), page 8-23]

Here, $m = 0{\cdot}1$. $\quad \therefore\ M_0(t) = e^{0.1(e^t - 1)}$

Example 22 : It is known that the probability of an item produced by a certain machine will be defective is 0·05. If the produced items are sent to the market in packets of 20, find the number of packets containing (i) at least, (ii) exactly and (iii) at most 2 defective items in a consignment of 1000 packets using (a) Binomial distribution, (b) Poisson approximation to the Binomial distribution. **(M.U. 2004, 05, 06)**

Sol. : We have P(defective) = 0·05, P(non-defective) = 0·95, $n = 20$ and $N = 1000$.

(i) ***By Binomial Distribution***

$$P(X = x) = {}^{20}C_x\,(0{\cdot}05)^x\,(0{\cdot}95)^{20-x}$$

$$P(X = 0) = {}^{20}C_0\,(0{\cdot}05)^0\,(0{\cdot}95)^{20} = 0{\cdot}36$$

$$P(X = 1) = {}^{20}C_1\,(0{\cdot}05)^1\,(0{\cdot}95)^{19} = 0{\cdot}38$$

$$P(X = 2) = {}^{20}C_2\,(0{\cdot}05)^2\,(0{\cdot}95)^{18} = 0{\cdot}19$$

No. of packets containing at least 2 defective

$$= N[\,1 - P(X = 0) - P(X = 1)\,]$$

$$= 1000\,[\,1 - 0{\cdot}36 - 0{\cdot}38\,] = 260$$

No. of packets containing exactly 2 defective

$$= N\,P(X = 2) = 1000 \times 0{\cdot}19 = 190$$

No. of packets containing at most 2 defective

$$= N[\,P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1000\,[\,0{\cdot}36 + 0{\cdot}38 + 0{\cdot}19\,] = 930$$

(ii) ***By Poisson Distribution***

Since, $m = np = 20 \times 0{\cdot}05 = 1$

$$P(X = x) = e^{-1}\frac{(1)^x}{x!},\quad x = 0, 1, 2, \ldots..$$

$$\therefore\ P(x = 0) = e^{-1}\frac{(1)^0}{0!} = 0{\cdot}37;\qquad P(x = 1) = e^{-1}\frac{(1)^1}{1!} = 0{\cdot}37;$$

$$P(x = 2) = e^{-1}\frac{(1)^2}{2!} = 0{\cdot}1839$$

$\therefore$ No. of packets containing at least 2 defective

$$= N[1 - P(X = 0) - P(X = 1)]$$

$$= 1000\,[\,1 - 0{\cdot}37 - 0{\cdot}37\,] = 260$$

No. of packets containing exactly 2 defective

$$= N \cdot P(X = 2) = 100 \times 0.1839 = 180$$

No. of packets containing atmost 2 defective

$$= N[\,P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1000\,[\,0{\cdot}37 + 0{\cdot}37 + 0{\cdot}1839\,] = 924.$$

Remark ....

As seen on page 8-21 when n is large Binomial Distribution tends to Poisson Distribution. In this example, $n = 1000$ and both distributions give nearly same frequencies.

EXERCISE - II

(A) 1. "Can we have a Poisson distribution with mean 4 and variance 5 ? Give reasoning for your answer. **[Ans. :** No **]**

2. Find the mean and the variance of the following distribution

$$P(X = x) = \frac{e^{-3} \times (3)^x}{x!},\quad x = 0, 1, 2, \ldots.$$ **[Ans. :** Mean = variance = 3 **]**

3. The mean and the variance of a probability distribution is 2. Write down the distribution. **(M.U. 2002, 05) [Ans. :** $P(X = x) = \dfrac{e^{-2} \times (2)^x}{x!},\quad x = 0, 1, 2, \ldots.$ **]**

4. In a Poisson distribution $P(x = 3)$ is $2/3$ of $P(X = 4)$. Find the mean and the standard deviation. **(M.U. 2007) [Ans. :** $m = 6, \sqrt{6}$ **]**

5. In a Poisson distribution the probability $p(x)$ for $x = 0$ is 20 percent. Find the mean of the distribution. **[Ans. :** $m = 2{\cdot}9957$ **]**

6. If X is a Poisson variate and $P(X = 0) = 6P(X = 3)$, find $P(X = 2)$. **(M.U. 1998)** **[Ans. :** $m = 1$; $e^{-1}/2$ **]**

7. If a random variable X follows Poisson distribution such that $P(X = 2) = 9\,P(X = 4) + 90\,P(X = 6)$, find the mean and the variance of X. **[Ans. :** mean = variance = $m = 1$ **]**

8. If X is a Poisson variate such that $P(X = 1) = P(X = 2)$, find $E(X^2)$. **(M.U. 2004, 17) [Ans. :** 6 **]**

9. The probability that a Poisson variable X takes a positive value is $1 - e^{-1.5}$. Find the variance and the probability that X lies between $-1{\cdot}5$ and $1{\cdot}5$.

(Hint : $P(X > 0) = 1 - P(X = 0) = 1 - e^{-1{\cdot}5}$

$$\therefore\ 1 - e^{-m} = 1 - e^{-1{\cdot}5} \qquad \therefore\ m = 1{\cdot}5$$

$$P(-1{\cdot}5 < x < 1{\cdot}5) = P(X = 0) + P(X = 1) = 2{\cdot}5\,(e^{-1{\cdot}5})\,)$$

10. If X is a Poisson variate with mean 4 and Y is a Poisson variate with mean 5, what is the mean of the variate $X + Y$? **[Ans. :** 9 **]**

11. If X and Y are Poisson variates with mean 2 and 4 respectively, find $P[(X + Y) \geq 4]$. **[Ans. :** 0·45 **]**

12. If X and Y are Poisson variates with parameters 3 and 4, find the variance of $4X - 3Y$. [Ans. : 84]

13. If X and Y are independent Poisson variates such that
$P(X = 1) = P(X = 2)$ and $P(Y = 2) = P(Y = 3)$, find the variance of $3X - 4Y$.
[Ans. : 66]

14. If the mean of the Poisson distribution is 2. Find the probabilities of $x = 1, 2, 3, 4$, from the recurrence relation of probability. (M.U. 2004)

[Ans. : $m = 2$. $p(x+1) = \dfrac{m}{x+1}\, p(x)$

$$P(0) = e^{-2},\ p(1) = 2e^{-2},\ p(2) = e^{-2},\ p(3) = \frac{2}{3}e^{-2},\ p(4) = \frac{1}{3}e^{-2}.$$]

15. If the variance of the Poisson distribution is 1·2. Find the probabilities for $r = 1, 2, 3, 4$ from the recurrence relation. (M.U. 2004) [Ans. : $m = 1{\cdot}2$, $P(0) = e^{-1\cdot 2}$ etc.]

16. In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 100 such samples, how many would you expect to contain 3 defectives (i) using the Binomial distribution, (ii) Poisson distribution. (M.U. 2004, 15)

[Ans. : (i) $p = 0{\cdot}1$, $q = 0{\cdot}9$, $p(x = 3) = {}^{20}C_3\,(0{\cdot}1)^3\,(0{\cdot}9)^{17} = 0{\cdot}1901$. No. $= Np = 190$.

(ii) $m = np = 20 \times \dfrac{1}{10} = 2$, $P(X = 3) = e^{-2} \cdot \dfrac{2^3}{3!} = 0{\cdot}18$ No. $= Np = 180$]

17. If X_1, X_2, X_3 are three independent Poisson variates with parameters 1, 2, 3 respectively, find $P[(X_1 + X_2 + X_3) \le 3]$ and $P[X_1 = 1 / (X_1 + X_2 + X_3) = 3]$
[Ans. : (i) 0·15, (ii) 25 / 72]

18. Using Poisson distribution, find the approximate value of
${}^{300}C_2\,(0{\cdot}03)^2\,(0{\cdot}97)^{298} + {}^{300}C_3\,(0{\cdot}03)^3\,(0{\cdot}97)^{297}$. [Ans. : 0·1338]

19. If 2 percent bulbs are known to be defective bulbs, find the probability that in a lot of 300 bulbs there will be 2 or 3 defective bulbs, using (i) Binomial distribution, (ii) Poisson distribution. [Ans. : (i) 0·1319, (ii) 0·1338]

(B) 1. In a certain manufacturing process 5% of the tools produced turn out to be defective. Find the probability that in a sample of 40 tools at most 2 will be defective. (Given : $e^{-2} = 0{\cdot}135$)

[Ans. : $m = np = 40 \times \dfrac{5}{100} = 2$, prob. $= 0{\cdot}675$]

2. It is 1 in 1000 that an article is defective. There are in a box 100 articles of this type. Assuming Poisson distribution, find the probability that the box contains one or more defective articles. (Given : $e^{-0\cdot 1} = 0{\cdot}9048$). [Ans. : 0·0952]

3. If the probability that an individual suffers a bad reaction from a particular injection is 0·001, determine the probability that out of 2,000 individuals (i) exactly three, (ii) more than two individuals will suffer a bad reaction.
(Given : $e^{-2} = 0{\cdot}1353$). (M.U. 2001) [Ans. : (i) 0·1804, (ii) 0·3233]

4. In a city there are a large number of street-lamps of which on an average 3 are non-working. Find the probability that on a particular night exactly two lamps are not working.
(Given : $e^{-3} = 0{\cdot}0498$). [Ans. : 0·2240]

5. The number of accidents on a particular highway in a month is a Poisson variate with parameter 5. Find the probability that more than 2 accidents have occurred on the road in a given month. [Ans. : 0·8754]

6. It is known from the past experience that in a certain plant there are on the average 4 industrial accidents per year. Find the probability that in a given year there will be less than 4 accidents. Assume Poisson distribution. (M.U. 1998) [Ans. : 0·43]

7. Find the probability that at most 5 defective fuses will be found in a box of 200 fuses, if experience shows that 2% of such fuses are defective. (M.U. 2002)
[Ans. : $m = np = 200 \times 0{\cdot}02 = 4$; 0·7851]

8. Assume that the probability of an individual coal miner being killed in a mine accident during a year is 1/ 2400. Use appropriate statistical distribution to calculate the probability that in a mine employing 200 miners there will be at least one fatal accident every year.
(M.U. 2001) [Ans. : $m = 200 / 2400$; 0·08]

9. If the variance of a Poisson distribution is 1·2. Find the probabilities of $X = 1, 2, 3, 4$ from recurrence relation. (M.U. 2004) [Ans. : 0·3614, 0·2169, 0·0867, 0·026]

10. The probability that a man aged 40 years will die within next year is 0·001. What is the probability that out of 100 such persons at least 99 will survive till next year ?
(Given : $e^{-0\cdot 1} = 0{\cdot}9048$). [Ans. : 0·9996]

11. Between the hours of 2 and 4 p.m. the average number of phone calls per minute coming into the switch board of a company is 2·5. Find the probability that during a particular minute there will be (i) no phone call at all, (ii) 4 or less calls. (M.U. 2003)
[Ans. : (i) 0·0821, (ii) 0·8909]

12. Which probability distribution is appropriate to describe the situation where 100 misprints are randomly distributed over 100 pages of a book. For this distribution find the probability that a page selected at random will contain atleast 3 misprints ?
(Given : $e^{-1} = 0{\cdot}3679$) [Ans. : Poisson $m = 1$, 0·0803]

13. A manufacture of pins knows that on an average 5% of his product is defective. He sells pins in boxes of 100 and guarantees that no more than 4 pins will be defective. What is the probability that a box will meet the guaranteed quality ? [Ans. : 0·44]

14. Suppose that a local appliances shop has found from experience that the demand for tube lights is roughly distributed as Poisson with a mean of 4 tube lights per week. If the shop keeps 6 tube lights during a particular week what is the probability that the demand will exceed the supply during that week?
(Given : $e^{-4} = 0{\cdot}0183$). (M.U. 1998) [Ans. : 0·3066]

15. It is known that in a certain plant, there are on an average 4 industrial accidents per month. Find the probability that in a given month there will be less than 4 accidents.
(Given : $e^{-4} = 0{\cdot}0183$). [Ans. : 0·4335]

16. Accidents occur on a particular stretch of highway at an average rate 3 per week. What is the probability that there will be exactly two accidents in a given week ?
(Given : $e^{-3} = 0{\cdot}0498$). [Ans. : 0·224]

17. 1% articles produced by a machine are defective. What is the probability that (i) none, (ii) two or more articles are defective in a sample of 100 ?
[Ans. : (i) 0·3679, (ii) 0·2642]

18. If 3% bulbs manufactured by a company are defective, assuming Poisson distribution find the probability that in a pack of 100 bulbs (i) none, (ii) two bulbs are defective.
[Ans. : (i) 0·0498, (ii) 0·2240]

19. In a town 10 accidents occur in a period of 50 days. Assuming Poisson distribution find the probability that there will be three or more accidents per day. [Ans. : 0·0012]

20. The average number of customers who appear at a counter of a certain bank per minute is two. Find the probability that during a given minute (i) no customer appears, (ii) three or more customers appear. [Ans. : (i) 0·1353, (ii) 0·3233]

(C) 1. A manufacturer of electric bulbs sends out 500 lots each consisting of 100 bulbs. If 5% bulbs are defective in how many lots can we expect (i) 97 or more good bulbs, (ii) less than 97 good bulbs ? (M.U. 1997) [Ans. : (i) 62, (ii) 368]

2. In sampling a large number of parts manufactured by a machine, the number of defectives in a sample of 40 is 2. Out of 1000 such samples, how many would be expected to contain 3 defective parts, by using (i) Binomial distribution, (ii) Poisson distribution ?

(M.U. 2003) [Ans. : (i) $1000 \cdot {}^{20}C_3 (0{\cdot}1)^3 (0{\cdot}9)^{17} = 190$, (ii) $1000 \cdot e^{-2} \dfrac{(2)^3}{3!} = 180$]

3. A firm produces articles, 0·1 percent of which one defective. It packs them in cases containing 500 articles. If a wholesaler purchases 100 such cases, how many cases can be expected (i) to be free from defective, (ii) to have one defective ? (M.U. 1999, 2001)
[Ans. : (i) 61, (ii) 30]

4. A manufacturer finds that the average demand per day for the mechanic to repair his new production is 1·5. Over a period of one year the demand per day is distributed as Poisson distribution. He employs two mechanics. On how many days in one year (a) both mechanics would be free, (b) some demand is refused ? (Given : $e^{-1 \cdot 5} = 0{\cdot}2231$).
[Ans. : (a) 81·4, (b) 69·8]

5. In a certain factory producing certain articles the probability that an article is defective is 1 / 500. The articles are supplied in packets of 20. Find approximately the number of packets containing no defective, one defective, two defectives in a consignment of 20,000 packets. (M.U. 1999) [Ans. : 19200, 768, 15]

6. A manufacturer of certain articles knows that on an average 5% of the articles are defective. He sells them in boxes of 100 and guarantees that no more than 4 articles will be defective. In how many boxes out of 1000 he will meet the guaranteed quality ?
[Ans. : 440]

7. In a certain factory turning out blades there is a small chance 1 / 250 for a blade to be defective. The blades are supplied in packets of 10. Calculate the approximate number of packets containing (i) no defective, (ii) one defective, (iii) two defective blades in a consignment of 10,000 packets using (a) Binomial Distribution, (b) Poisson approximation to Binomial distribution. [Ans. : (a) (i) 9607, (ii) 386, (iii) 7; (b) (i) 9608, (ii) 384, (iii) 8]

8. In a certain factory producing certain articles the probability that an article is defective is 1 / 400. The articles are supplied in packets of 10. Find approximately the number of packets in a consignment of 20,000 packets containing (i) no defective, (ii) one defective and (ii) two defective blades using (a) Binomial Distribution, (b) Poisson approximation to Binomial distribution. [Ans. : (a) (i) 19506, (ii) 489, (iii) 6; (b) (i) 19506, (ii) 488, (iii) 6]

(D) 1. Fit a Poisson distribution to the following data

X :	0,	1,	2,	3,	4.	Total
f :	192,	100,	24,	3,	1.	320

[Ans. : $m = 0{\cdot}5$, Frequencies : 194, 97, 24, 4, 1]

2. Fit a Poisson distribution to the following data.

X :	0,	1,	2,	3,	4.	Total
f :	122,	60,	15,	2,	1.	200

(M.U. 2001)
[Ans. : $m = 0{\cdot}595$, Frequencies : 121, 61, 15, 3, 0]

3. The following mistakes per page were observed in a book.

No. of mistakes :	0,	1,	2,	3,	4.	Total
No. of pages :	211,	90,	19,	5,	0.	325

Fit a Poisson distribution. (M.U. 1997, 2001)
[Ans. : $m = 0{\cdot}44$, Frequencies : 209, 92, 20, 3, 1]

4. Fit a Poisson distribution to the following data.

No. of defects per piece :	0,	1,	2,	3,	4.	Total
No. of pieces :	43,	40,	25,	10,	2.	120

(M.U. 1997)

[Ans. :

No. of defects per piece :	0,	1,	2,	3,	4.	Total
No. of pieces :	42,	44,	24,	8,	2.	120]

5. Fit a Poisson distribution to the following data

X :	0,	1,	2,	3,	4,	5,	6,	7.
f :	314,	335,	204,	86,	29,	9,	3,	0.

(M.U. 2004)

[Ans. :

X :	0,	1,	2,	3,	4,	5,	6,	7
f :	295,	354,	212,	85,	26,	6,	1,	1]

6. Fit a Poisson distribution to the following data.

X :	0,	1,	2,	3,	4,	5,	6,	7,	8
f :	56,	156,	132,	92,	37,	22,	4,	0,	1

(M.U. 2004, 07, 09, 15)

[Ans. :

X :	0,	1,	2,	3,	4,	5,	6,	7,	8
f :	70,	137,	135,	89,	44,	17,	6,	2,	0]

7. Fit a Poisson distribution to the following data.

X :	0,	1,	2,	3,	4,	5	Total
f :	142,	156,	69,	27,	5,	1	400

(M.U. 2002, 03, 04, 11)

[Ans. :

X :	0,	1,	2,	3,	4,	5	Total
f :	147,	147,	74,	24,	6,	2	400]

4. Normal Distribution

Normal distribution is one of the most important and commonly used continuous distribution. It was first developed by DeMoivre but it is credited to Gauss who referred to it first in 1809. A large number of continuous variates follow this distribution, hence, the name 'normal'.

A renowned mathematician Poincare' has remarked that "there must be something mysterious about the normal distribution because mathematicians think it is a law of nature whereas physicists are convinced that it is a mathematical theorem."

Abraham De Moivre (1667 - 1754)

A French mathematician who made important contributions to statistics, theory of probability and trigonometry The concept of statistically independent events was first developed by De Moivre. Through the use of complex number he transformed trigonometry from a branch of geometry to a branch of analysis. His treatise on probability has influenced the development of probability theory.

Karl Friedrich Gauss (1777 - 1855)

Karl Friedrich Gauss was a great German mathematician and scientist. He is called the **"prince of mathematicians"**. He is ranked with Isaac Newton and Archimedes. It is said that Gauss at the age of three had pointed out an error in calculation made by his father while preparing a payroll.

In his doctoral thesis he gave the first complete proof of the fundamental theorem of algebra that a polynomial of nth degree has n roots. He is considered to have laid the founation of number theory. We are all familiar with Gaussian probability distribution (Normal Distribution). He gave first the geometric interpretation of complex numbers, developed the theory of conformal mapping. He did fundamental work in electro-magnetism.

He knew many languages and read extensively. It is said that if Gauss had published all of his discoveries the state of mathematics would have advanced by 50 years.

1. Definition : A continuous random variable X is said to follow **normal distribution** with parameter m (called mean) and σ^2 (called variance), if its probability density function is given by

$$\boxed{f(x)=\frac{1}{\sqrt{2\pi}\cdot\sigma}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}} \qquad \begin{matrix}-\infty<x<\infty\\ -\infty<m<\infty,\ \sigma^2>0\end{matrix}$$

Remarks ....

(i) A continuous random variate X following normal distribution with mean m and standard deviation σ is referred to as $X \sim N(m, \sigma)$.

(ii) If X is a normal variate with parameter m, σ, then $\boxed{Z=\frac{X-m}{\sigma}}$ is also a normal variate with mean = 0 and standard deviation = 1.

It is called **Standard Normal Variate.**

2. Importance of Normal Distribution

(i) The variables such as height, weight, intelligence etc. follow normal distribution.

(ii) Many other distributions occurring in practice such as Binomial, Poisson etc. can be approximated by normal distribution.

(iii) Many of the distributions of sample statistic *e.g.* Sample Mean, Sample Variance tend to normal distribution for large samples.

(iv) Normal distribution has wide applications in Statistical Quality Control.

(v) Errors in measurements of physical quantities follow normal distribution.

(vi) It is also useful in psychological and educational research.

(a) Mean and Variance of the Normal Distribution

(i) By definition mean is given by

$$\text{Mean } = E(X)=\int_{-\infty}^{\infty}x\cdot f(x)\,dx$$

$$=\int_{-\infty}^{\infty}[(x-m)+m]\,f(x)\,dx$$

$$=\int_{-\infty}^{\infty}(x-m)\,f(x)\,dx+m\int_{-\infty}^{\infty}f(x)\,dx$$

But the first integral is zero, because the first moment about the mean is zero [See (2), page 7-28] and the second integral is unity.

$$\therefore \quad \boxed{\text{Mean} = m}$$

(ii) Now, $\quad \text{Var}(X)=E(X-m)^2=\int_{-\infty}^{\infty}(x-m)^2 f(x)\,dx$

$$=\int_{-\infty}^{\infty}(x-m)^2\cdot\frac{1}{\sqrt{2\pi}\cdot\sigma}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx$$

Now, put $\frac{x-m}{\sigma}=t \quad \therefore\ dx=\sigma\,dt$

$$\therefore\ \text{Var}(X)=\int_{-\infty}^{\infty}\sigma^2\cdot t^2\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{1}{2}t^2}dt=\frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t\,d\left(-e^{-\frac{1}{2}t^2}\right)$$

Now, integrating by parts

$$\text{Var}(X)=\frac{\sigma^2}{\sqrt{2\pi}}\left[\left[t\left(-e^{-\frac{1}{2}t^2}\right)\right]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}e^{-\frac{1}{2}t^2}dt\right]$$

[By Gamma Functions, $\int_{-\infty}^{\infty}e^{-\alpha x^2}=\sqrt{\frac{\pi}{\alpha}}$. Refer to Applied Mathematics - II]

$$=\frac{\sigma^2}{\sqrt{2\pi}}\left[0+\sqrt{2\pi}\right]=\sigma^2$$

$$\therefore \quad \boxed{\text{Var.}(X)=\sigma^2}$$

Note ....

A normal variate with mean m and standard deviation σ is shortly denoted as $N(m, \sigma)$.

Remark ....

Here we simply note that mean, median and mode of the normal distribution are equal to m

$$\therefore \quad \boxed{\text{Mean} = \text{Median} = \text{Mode} = m} \qquad \text{(M.U. 2009)}$$

(b) Moment Generating Function of Normal Distribution

By definition,

$$M_0(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

Putting $\frac{x-m}{\sigma} = z, \quad \frac{dx}{\sigma} = dz$

$$M_0(t) = \int_{-\infty}^{\infty} e^{t(m+\sigma z)} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}z^2} dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{mt} \cdot e^{-\frac{1}{2}(z^2 - 2t\sigma z)} dz$$

Now $z^2 - 2t\sigma z = (z - t\sigma)^2 - t^2\sigma^2$

$$\therefore \quad M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{mt} \cdot e^{-\frac{1}{2}(z-t\sigma)^2} \cdot e^{\frac{1}{2}t^2\sigma^2} dz$$

Putting $u = z - t\sigma$

$$M_0(t) = \frac{1}{\sqrt{2\pi}} \cdot e^{mt + \frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du = \frac{1}{\sqrt{2\pi}} \cdot e^{mt + \frac{t^2\sigma^2}{2}} \cdot 2\int_0^{\infty} e^{-\frac{1}{2}u^2} du$$

$$= e^{mt + \frac{t^2\sigma^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} \qquad \left[\because \int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\right]$$

$$\therefore \quad \boxed{M_0(t) = e^{mt + \frac{t^2\sigma^2}{2}}}$$

Now, $M_0(t) = e^{mt} \cdot e^{t^2\sigma^2/2} = \left[1 + mt + \frac{m^2t^2}{2!} + \ldots\right]\left[1 + \frac{t^2\sigma^2}{2!} + \frac{t^4\sigma^4}{4!} + \ldots\right]$

$\therefore \quad \mu_1' = \text{Coefficient of } t = m, \qquad \mu_2' = \text{Coefficient of } \frac{t^2}{2!} = m^2 + \sigma^2$

$\therefore \quad \text{Mean} = \mu_1' = m. \qquad \text{Variance} = \mu_2' - \mu_1'^2 = m^2 + \sigma^2 - m^2 = \sigma^2.$

Corollary : Since mean of the standard normal variate is zero and the standard deviation is unity, putting $m = 0$ and $\sigma = 1$ in the above m.g.f., we get the moment generating function of the standard normal variate as

$$\boxed{M_0(t) = e^{t^2/2}}$$

Moments of Normal Distribution : The m.g.f. about mean is given by

$$M_{\bar{x}}(t) = E[e^{t(x-m)}] = E(e^{tx} \cdot e^{-mt})$$

$$= e^{-mt} E(e^{tx}) = e^{-mt} M_0(t)$$

$$= e^{-mt} \cdot e^{mt + (t^2\sigma^2/2)} = e^{t^2\sigma^2/2}$$

$$= 1 + (t^2\sigma^2/2) + \frac{(t^2\sigma^2/2)^2}{2!} + \frac{(t^2\sigma^2/2)^3}{3!} + \ldots$$

Now, $\mu_r = \text{coefficient of } \frac{t^r}{r!}$.

Since, no odd power of t appears in the above expansion, **moments of odd powers about mean of a normal distribution are zero.**

$$\therefore \quad \boxed{\mu_{2n+1} = 0, \quad n = 0, 1, 2, \ldots} \qquad \ldots\ldots (1)$$

Moments of even power

$$\mu_{2n} = \text{coefficient of } \frac{t^{2n}}{(2n)!} = \frac{(\sigma^2/2)^n}{n!} \cdot (2n)!$$

$$= \frac{\sigma^{2n}}{2^n n!}[2n(2n-1)(2n-2)\ldots\ldots 4\cdot 3\cdot 2\cdot 1]$$

$$= \frac{\sigma^{2n}}{2^n \cdot n!}[2\cdot n(2n-1)\cdot 2(n-1)(2n-3)\ldots\ldots 4\cdot 3\cdot 2\cdot 1]$$

$$= \frac{\sigma^{2n}}{2^n \cdot n!} 2^n n!(2n-1)(2n-3)\ldots\ldots 5\cdot 3\cdot 1$$

$$\boxed{\mu_{2n} = 1\cdot 3\cdot 5\ldots.(2n-1)\cdot\sigma^{2n}} \qquad \ldots\ldots (2)$$

(c) Moment Generating Function of Standard Normal Variate **(M.U. 2006)**

We shall here find the moment generating function of Standard Normal Variate and from the m.g.f., we shall obtain the mean and variance of S.N.V. $Z = \frac{X-m}{\sigma}$.

Let $f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \qquad -\infty < x < \infty, \quad -\infty < m < \infty, \quad \sigma^2 > 0$

Now putting, $\frac{x-m}{\sigma} = z, \; dx = \sigma\,dz$, we get $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$.

$$\therefore \quad M_0(t) = E(e^{tz}) = \int_{-\infty}^{\infty} e^{tz} f(z)\,dz = \int_{-\infty}^{\infty} e^{tz} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2tz)} dz.$$

Now, $z^2 - 2tz = (z-t)^2 - t^2$

$$\therefore \quad M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2} \cdot e^{t^2/2}\, dz$$

Now, put $z - t = u$, $dz = du$

$$\therefore \quad M_0(t) = \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} \cdot du = \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_0^{\infty} 2 \cdot e^{-u^2/2} \cdot du$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \cdot 2 \cdot \sqrt{\frac{\pi}{2}} = e^{t^2/2} = 1 + \frac{t^2}{2} + \frac{(t^2/2)^2}{2!} + \ldots$$

$\therefore$ Mean = coefficient of $t = 0$. Variance = coefficient of $\frac{t^2}{2!} = 1$.

$\therefore$ Mean and standard deviation of Standard Normal Variate are zero and one respectively.

Aliter : We shall obtain mean and variance of Standard Normal Variate from the definitions.

(i) Mean $\bar{Z} = E(Z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z \cdot e^{-z^2/2}\, dz$

But the function on the r.h.s. is an odd function, hence, the definite integral is zero.

$\therefore$ Mean = 0.

(ii)

$$\mu_2' = E(z^2) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z^2 \cdot e^{-z^2/2}\, dz$$

Since the function on the r.h.s. is an even function,

$$\mu_2' = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} z^2 \cdot e^{-z^2/2}\, dz$$

Now, put $\frac{z^2}{2} = t \quad \therefore z = \sqrt{2t} \quad \therefore dz = \frac{\sqrt{2}}{2\sqrt{t}}\, dt$

$$\therefore \quad \mu_2' = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} 2t \cdot e^{-t} \frac{\sqrt{2}}{2} \cdot \frac{dt}{\sqrt{t}} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{1/2}\, dt = \frac{2}{\sqrt{\pi}} \left\lceil \frac{3}{2} \right.$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \left\lceil \frac{1}{2} \right. = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \sqrt{\pi} = 1$$

$\therefore$ Variance $= \mu_2' - \mu_1'^2 = 1 - 0 = 1$.

(d) Linear Combination (Additive Property)

Theorem : Let X_i, $i = 1, 2, 3, \ldots, n$ be n independent normal variates with mean m_i and variance σ_i^2. Let their linear combination be denoted by Y i.e.

$$Y = a_1X_1 + a_2X_2 + \ldots + a_nX_n$$

Then Y is also a normal variate with mean m and variance σ^2 where

$$m = a_1m_1 + a_2m_2 + \ldots + a_nm_n \quad \text{and} \quad \sigma^2 = a_1\sigma_1^2 + a_2\sigma_2^2 + \ldots + a_n\sigma_n^2.$$

Proof : We know that m.g.f. of a normal variate with mean m and variance σ^2 is given by

$$M_0(t) = e^{mt + t^2\sigma^2/2}$$

Further $M_Y(t) = M_{a_1X_1 + a_2X_2 + \ldots + a_nX_n}(t)$

$$= M_{a_1X_1}(t) \cdot M_{a_2X_2}(t) \ldots M_{a_nX_n}(t)$$

(since $X_1, X_2, \ldots, X_n$ are independent variates.)

$$= M_{X_1}(a_1t) \cdot M_{X_2}(a_2t) \ldots M_{X_n}(a_nt) \quad [\because M_{CX}(t) = M_X(ct)]$$

$$= e^{a_1m_1t + a_1^2\sigma_1^2t^2/2} \cdot e^{a_2m_2t + a_2^2\sigma_2^2t^2/2} \cdot \ldots$$

$$= e^{(a_1m_1 + a_2m_2 + \ldots)t + (a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \ldots)t^2/2}$$

But from its form this is m.g.f. of a normal variate whose mean is $(a_1m_1 + a_2m_2 + \ldots)$ and whose variance is $(a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \ldots)$

Remarks ....

1. This property is known as **additive property** of normal distribution.
2. If $a_3 = a_4 = \ldots = a_n = 0$ then $Y = a_1X_1 + a_2X_2$ is a normal variate with mean $a_1m_1 + a_2m_2$ and variance $= a_1^2\sigma_1^2 + a_2^2\sigma_2^2$.
3. If $a_1 = a_2 = 1$, $a_3 = a_4 = \ldots = a_n = 0$, then $Y = X_1 + X_2$ is a normal variate with mean $m_1 + m_2$ and variance $\sigma_1^2 + \sigma_2^2$.
4. If $a_1 = 1$ $a_2 = -1$, $a_3 = a_4 = \ldots = a_n = 0$ then $Y = X_1 - X_2$ is also a normal variate with mean $m_1 - m_2$ and variance $\sigma_1^2 + \sigma_2^2$.
5. Comparing Normal Distribution with Poisson Distribution we find that **sum** of two Normal or Poisson Variates is a Normal or Poisson variate. But although difference of two normal variates is a normal variate, the difference of two Poisson variates is **not** a Poisson variate.
6. If X_i; $i = 1, 2, 3, \ldots, n$ are n independent, identical normal varieties all with the same mean m and same standard deviation σ and if we put $a_1 = a_2 = a_3 = \ldots = a_n = 1/n$ then $Y = X_1 + X_2 + \ldots + X_n$ is a normal variate with mean

$$m = \frac{m + m + \ldots + m}{n} = m$$

and $$\sigma^2 = \frac{1}{n^2}\sigma^2 + \frac{1}{n^2}\sigma^2 + \ldots + \frac{1}{n^2}\sigma^2 = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

In other words, **Y is a normal variate with mean m and standard deviation $\sigma/\sqrt{n}$.**

(e) Area Property

"If X is a normal variate with mean m and variance σ^2 and Z is standard normal variate (with mean zero and variance one) then the area under the normal curve of X between $X = m$ and $X = x_1$ is equal to the area under the S.N. Curve of Z between $Z = 0$ to $Z = z_1$ (say, corresponding to x_1)."

Proof : Consider a normal variate X with mean m and variance σ^2.

Then $P(m \le X \le x_1) = \dfrac{1}{\sqrt{2\pi}\cdot\sigma}\displaystyle\int_m^{x_1} e^{-\frac{1}{2}\left(\frac{X-m}{\sigma}\right)^2} dx$

Now, put $\dfrac{X-m}{\sigma} = Z$

When $X = m$, $Z = 0$; when $X = x_1$, $Z = \dfrac{x_1 - m}{\sigma} = z_1$ say

$\therefore\ P(m \le X \le x_1) = P(0 \le Z \le z_1) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^{z_1} e^{-\frac{1}{2}z^2} dz$

Thus, the area under the normal curve from m to x_1 is equal to the area under the standard normal curve from 0 to z_1.

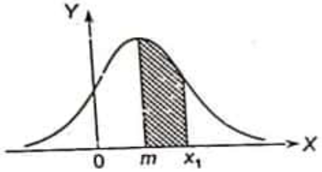

Fig. 8.1

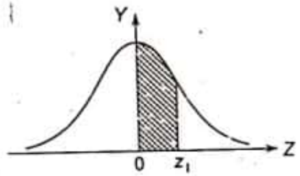

Fig. 8.2

The integral $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^{z_1} e^{-\frac{1}{2}z^2} dz$ is denoted by $\displaystyle\int_0^{z_1} \phi(z)\,dz$ and is known as **normal probability integral**. The areas under standard normal curve from $z = 0$ to various values of z_1 have been calculated and are given in the table at the end of the book.

Reverse Problem : In some problems we know that X is a normal variate with mean m and standard deviation σ. We are required to find the value of $X = x_1$ corresponding to a given probability. Suppose, we want to find of $X = x_1$ such that $P(X > x_1) = \alpha$. In this case also we consider

S.N.V. $Z = \dfrac{X-m}{\sigma}$

We now consult the area table and find the value of $Z = z_1$ for which area to the right is α *i.e.*, area between $z = 0$ to $Z = z_1$ is $(0{\cdot}5 - \alpha)$. From this value of z_1 we get X using

$$z_1 = \frac{x_1 - m}{\sigma}$$

i.e. $x_1 = m + z_1\,\sigma$.

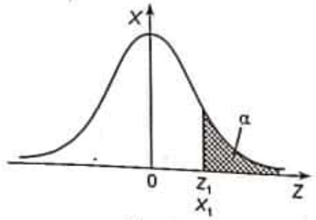

Fig. 8.3

Remarks ....

1. Since standard normal curve is symmetrical about the y-axis it is enough to find the areas to the right. The areas to the left of y-axis at equal distances will be equal.

2. The total area under the curve is unity. Hence, because of symmetry the area under S.N.V. to the right of the y-axis is 0·5.

3. To find the probability that X will lie between x_1 and x_2 $(x_1 < x_2)$, we find the corresponding values of S.N.V. Z (from $Z = \dfrac{X-m}{\sigma}$) say z_1 and z_2 and find the area from z_1 to z_2 and S.N. Curve. The required probability is this area.

$$P(x_1 \le X \le x_2) = P(z_1 \le Z \le z_2)$$
$$= \text{area between } Z = z_1 \text{ and } Z = z_2 \text{ under the S.N. curve.}$$

(f) The Quartile Deviation of a Normal Distribution is (2 / 3) S.D.

Let X be a normal variate with mean m and variance σ^2. Consider the standard normal variate

$$Z = \frac{X-m}{\sigma}$$

When $X = m$, $Z = 0$ and when $X = Q_3$ the third quartile let $Z = \dfrac{Q_3 - m}{\sigma} = z_1$.

We know that for Normal variate mean = median = mode = m. *i.e.* the mean m divides the area into two equal parts. Hence, the area from Q_1 to m is 0·25. For S.N.V. area from 0 to z_1 is 0·25. For area 0·25 we find from the table that,

$$z_1 = 0{\cdot}6745 = \frac{2}{3}.$$

$\therefore\ Q_3 = m + \sigma z_1 = m + \dfrac{2}{3}\sigma.$ Similarly, $Q_1 = m - \sigma z_1 = m - \dfrac{2}{3}\sigma.$

$$\therefore\ \boxed{\text{Quartile Deviation} = \frac{Q_3 - Q_1}{2} = \frac{2}{3}\sigma.}$$

(g) Properties of the Normal Distribution

We summaries below important properties of normal distribution which we have proved above. They are given in terms of the properties of the normal curve.

(i) The normal curve is bell-shaped and symmetrical about the maximum ordinate at $x = m$, the *mean*. In other words, the curve is divided into two equal parts by this ordinate. The curve on one side of this ordinate is the mirror image of the curve on the other side.

(ii) The curve has maximum height at $x = m$. Hence the *mode* of the distribution is also m. The ordinate $x = m$ divides the area under the curve into two equal parts. Hence the *median* of the distribution is also m. Thus, for the normal distribution,

$$\boxed{\text{mean} = \text{median} = \text{mode} = m}$$

(iii) The height of the curve goes on decreasing on either side of the ordinate at $x = m$ but never becomes zero. In other words, the curve never intersects the x-axis at any finite point. The x-axis touches it at infinity.

(iv) Since the curve is symmetrical about mean, the first quartile Q_1 and the third quartile Q_3 are at the same distance on the two sides of the mean. The distance of any quartile from the mean is 0·6745 σ units. Hence,

$Q_1 = m - 0{\cdot}6745\,\sigma$ or $\boxed{Q_1 = m - \dfrac{2}{3}\sigma}$ and

$Q_3 = m + 0{\cdot}6745$ or $\boxed{Q_3 = m + \dfrac{2}{3}\sigma}$

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Hence, middle 50 % items lie between $m - \frac{2}{3}\sigma$ and $m + \frac{2}{3}\sigma$.

Further, the quartile deviation of a normal distribution is given by

$$\text{Q.D.} = \frac{Q_3 - Q_1}{2} = \frac{\left(m + \frac{2}{3}\sigma\right) - \left(m - \frac{2}{3}\sigma\right)}{2} = \frac{2}{3}\sigma.$$

$$\boxed{\therefore\ \text{Q.D.} = \frac{2}{3}\sigma}$$

(v) The mean of the distribution is m and the standard deviation is σ.

(vi) The mean deviation is given by $\boxed{\text{M.D.} = \frac{4}{5}\sigma}$

(vii) Now, $\frac{\text{Q.D.}}{\text{M.D.}} = \frac{(2/3)\sigma}{(4/5)\sigma} = \frac{2}{3} \times \frac{5}{4} = \frac{10}{12}$

Also $\text{M.D.} = \frac{4}{5}\sigma \quad \therefore \frac{\text{M.D.}}{\sigma} = \frac{4}{5} = \frac{12}{15}$

$$\therefore \boxed{\text{Q.D. : M.D. : S.D.} = 10 : 12 : 15}$$

(viii) Odd central moments are zero i.e.,

$$\mu_{2r+1} = 0 \quad \text{for } (r = 0, 1, 2, \ldots)$$

and $\mu_{2r} = 1 \cdot 3 \cdot 5 \ldots (2r - 1)\sigma^{2r} (r = 0, 1, 2, \ldots)$

i.e., $\mu_2 = \sigma^2,\ \mu_4 = 3\sigma^4,\ \mu_6 = 15\sigma^6$

(ix) The area under the normal curve is distributed as follows.

(a) The area between $x = m - \sigma$ and $x = m + \sigma$ is 68·27 %.

(b) The area between $x = m - 2\sigma$ and $x = m + 2\sigma$ is 95·45 %.

(c) The area between $x = m - 3\sigma$ and $x = m + 3\sigma$ is 99·73 %.

These areas under the normal curve and standard normal curve are shown below.

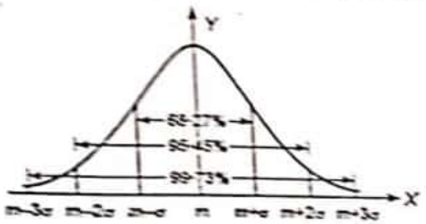


Fig. 8.4 (a)

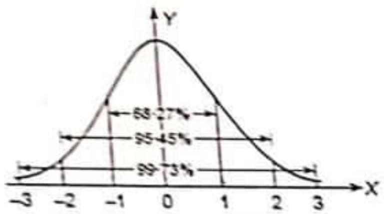


Fig. 8.4 (b)

(h) **Normal Approximation To The Binomial Distribution**

It can be proved, although we do not, that if X is a Binomial variate with parameter n and p (i.e. mean $= np$ and S.D. $= \sqrt{npq}$ where $q = 1 - p$) then

$$Z = \frac{X - np}{\sqrt{npq}}$$

is a Standard Normal Variate if $n \to \infty$ (i.e. is large) and neither p nor q is small.

Remark ....

1. Normal distribution can be used in place of Binomial distribution when np and nq are both greater than 15.

2. Normal distribution can also be obtained from Poisson distribution when the parameter $m \to \infty$.

Type I

Example 1 : For a normal distribution the mean is 50 and the standard deviation is 15. Find (i) Q_1 and Q_3, (ii) mean deviation (Also the interquartile range).

Sol. : (i) For a normal distribution

$$Q_1 = m - \frac{2}{3}\sigma = 50 - \frac{2}{3}15 = 40$$

Again $$Q_3 = m + \frac{2}{3}\sigma = 50 + \frac{2}{3}15 = 60$$

(ii) The mean deviation of the normal distribution is,

$$\text{M.D.} = \frac{4}{5}\sigma = \frac{4}{5} \times 15 = 12$$

$$\therefore \text{Interquartile range} = Q_3 - Q_1 = 60 - 40 = 20.$$

Example 2 : The first and the third quartiles of a normal distribution are 36 and 44. Fi[nd] mean, standard deviation and the mean deviation.

Sol. : We have $Q_1 = m - \frac{2}{3}\sigma$ and $Q_3 = m + \frac{2}{3}\sigma$

$$\therefore\ 36 = m - \frac{2}{3}\sigma \quad \text{and} \quad 44 = m + \frac{2}{3}\sigma$$

Adding $80 = 2m \quad \therefore m = 40$

Then, $36 = 40 - \frac{2}{3}\sigma \quad \therefore \frac{2}{3}\sigma = 4 \quad \therefore \sigma = 6$

And mean deviation $= \frac{4}{5}\sigma = \frac{4}{5} \times 6 = \frac{24}{5}$.

Example 3 : Find the probability that a random variable having the standard normal distribution will take on a value between 0·87 and 1·28.

Sol. : $P(0{\cdot}87 < Z < 1{\cdot}28)$

= area between $Z = 0{\cdot}87$ and $Z = 1{\cdot}28$.

= (area from $Z = 0$ to $Z = 1{\cdot}28$)

− (area from $Z = 0$ to $Z = 0{\cdot}87$)

$= 0{\cdot}3997 - 0{\cdot}3078 = 0{\cdot}0919$.

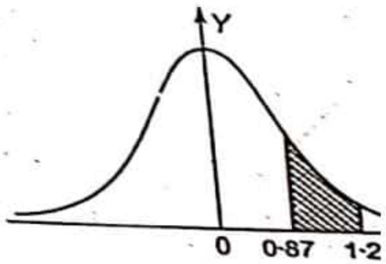


Fig. 8.5

Example 4 : Find the probability that a random variable having standard normal distribution will take a value between (i) 0·87 and 1·28, (ii) – 0·34 and 0·62.

Sol. : (i) As in the above example.

(ii) Area from $Z = -0·34$ to 0 is the same as $Z = 0$ to $Z = 0·34$ and is 0·1331

Area from $Z = 0$ to $Z = 0·62$ is 0·2324.

Required area is the sum of the two

$$\therefore \quad P(-0·34 < Z < 0·62) = 0·1331 + 0·2324 = 0·3655.$$

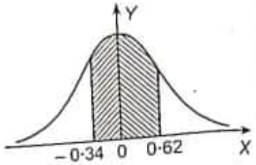


Fig. 8.6

Example 5 : For a normal variate with mean 2·5 and standard deviation 3·5, find the probability that (i) $2 \le X \le 4·5$, (ii) $-1·5 \le X \le 5·5$.

Sol. : We have S.N.V. $Z = \dfrac{X-m}{\sigma} = \dfrac{X-2·5}{3·5}$

(i) When $X = 2$, $\quad Z = \dfrac{2-2·5}{3·5} = -0·14$

When $X = 4·5$, $\quad X = \dfrac{4·5-2·5}{3·5} = 0·57$

$$P(2 \le X \le 4·5) = P(-0.14 \le Z \le 0·57)$$
$$= \text{Area between } (Z = -0·14 \text{ and } Z = 0·57)$$
$$= \text{Area between } (Z = 0 \text{ and } Z = 0·14)$$
$$+ \text{ Area between } (Z = 0 \text{ and } Z = 0·57)$$
$$= 0·0557 + 0·2157 = 0·2714.$$

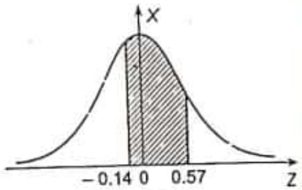


Fig. 8.7 (a)

(ii) When $X = -1·5$, $\quad Z = \dfrac{-1·5-2·5}{3·5} = -1·14$

When $X = 5·3$, $\quad Z = \dfrac{5·3-2·5}{3·5} = 0·8$

$$P(-1·5 \le X \le 5·3) = P(-1·14 \le Z \le 0·8)$$
$$= \text{Area between } (Z = -1·14 \text{ and } Z = 0·8)$$
$$= \text{Area between } (Z = 0 \text{ and } Z = 1·14)$$
$$+ \text{ Area between } (Z = 0 \text{ and } Z = 0·8)$$
$$= 0·3729 + 0·2881 = 0.6610.$$

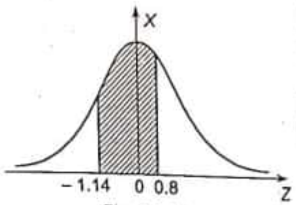


Fig. 8.7 (b)

Example 6 : If Z is a standard normal variate, find c such that

(i) $P(-c < Z < c) = 0·95$, (ii) $P(|Z| > c) = 0·01$.

If X is a normal variate with mean 120 and standard deviation 10, find c such that

(i) $P(X > c) = 0·02$, (ii) $P(X < c) = 0·05$.

Sol. : Consulting the table of S.N.V. we have to find the entry 0·475 and the corresponding value of Z. We find from the table that corresponding to the entry 0·4750, $Z = 1·96$

Again consulting the table of S.N.V. we find the entry 0·495. Corresponding to the entry 0·495, $Z = 2·58$. $\quad \therefore \; c = 2·58$.

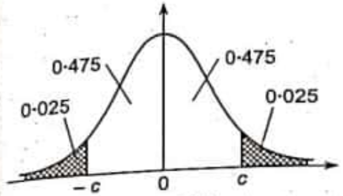


Fig. 8.8 (a)

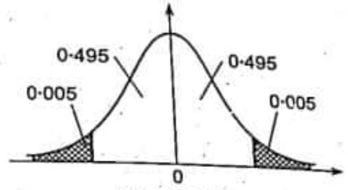


Fig. 8.8 (b)

If Z is a S.N.V. then $Z = \dfrac{X-m}{\sigma}$ $\quad \therefore \; Z = \dfrac{X-120}{10}$

$\therefore \; P(X > c) = P(Z > c) = 0·02.$

Corresponding to entry $0·5 - 0·02 = 0·48$, $Z = 2·05$.

$\therefore \; 2·05 = \dfrac{X-120}{10}$ $\quad \therefore \; X = 120 + 2·05 \times 10 = 140·5.$

Again $P(X < c) = P(Z < c) = 0·05.$

Corresponding to entry $0·5 - 0·05 = 0·45$, $Z = 1·64$.

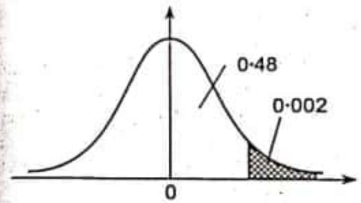


Fig. 8.8 (c)

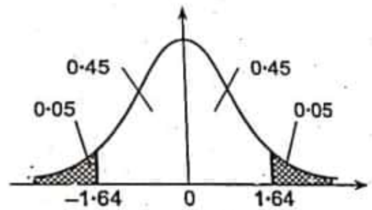


Fig. 8.8 (d)

$\therefore$ Since Z is less than c, c must be negative $\quad \therefore \; c = -1·64$.

$\therefore \; -1·64 = \dfrac{X-120}{10}$ $\quad \therefore \; X = 120 - 10 \times 1·64 = 103·6$

Example 7 : If X is a normal variate with mean 10 and standard deviation 4, find (i) $P(|X-14| < 1)$, (ii) $P(5 \le X \le 18)$, (iii) $P(X \le 12)$. **(M.U. 2002, 09, 16, 17)**

Sol. : We have $Z = \dfrac{X-m}{\sigma} = \dfrac{X-10}{4}$

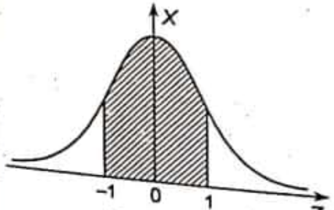


Fig. 8.9 (a)

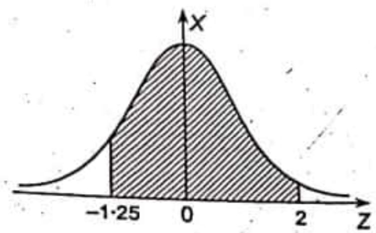


Fig. 8.9 (b)

(i) When $X = 14$, $Z = \dfrac{14-10}{4} = 1$

$\therefore$ $P(|X-14| \le 1) = P(|Z| \le 1)$

= area between ($Z = -1$ and $Z = 1$)

= 2 (area between $Z = 0$ and $Z = 1$)

= 2 (0·3413) = 0·6826

(ii) When $X = 5$, $Z = \dfrac{5-10}{4} = -1{\cdot}25$. When $X = 18$, $Z = \dfrac{18-10}{4} = 2$

$\therefore$ $P(5 \le X \le 18) = P(-1{\cdot}25 \le Z < 2)$

= area between $Z = -1{\cdot}25$ and $Z = 2$

= (area between $Z = 0$ and $Z = 1{\cdot}25$) + (are between $Z = 0$ and $Z = 2$)

= 0·3944 + 0·4772 = 0·8716

(iii) When $Z = 12$, $Z = \dfrac{12-10}{4} = 0{\cdot}5$

$\therefore$ $P(X \le 12) = P(Z \le 0{\cdot}5)$ = area upto $Z = 0 \le 0{\cdot}5$

= (area from $-\infty$ to $Z = 0$) + (area from $Z = 0$ to $Z = 0{\cdot}5$)

= 0·5 + 0·1915 = 0·6915.

Type II

Example 1 : In a factory turning out blades in mass production, it was found that in a packet of 100 blades on an average 16 blades are defective. Find the standard deviation of the defective blades. Can the distribution of defective blades be approximated to a normal distribution ? If so write its equation. **(M.U. 1999)**

Sol. : The distribution of defective blades is a Binomial distribution.

We have $n = 100$, $p = \dfrac{16}{100} = 0{\cdot}16$

$\therefore$ Mean $\overline{X} = np = 100 \times 0{\cdot}16 = 16$

$\therefore$ $q = 1 - p = 0{\cdot}84$ $\quad\therefore$ $nq = 100 \times 0{\cdot}84 = 84$.

Since both np and nq are greater than 15 as stated in the remark 1 (page 8-48), the Binomial distribution can be approximated to Normal distribution.

Now, as seen above, $\overline{X} = 16$ and $\sigma = \sqrt{npq} = \sqrt{100 \times 0{\cdot}16 \times 0{\cdot}84} = 3{\cdot}67$.

$\therefore$ The equation of the normal distribution is

$$y = \frac{1}{\sqrt{2\pi}\cdot\sigma}\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}\cdot(3{\cdot}67)}\cdot e^{-\frac{1}{2}\left(\frac{x-16}{3{\cdot}67}\right)^2}.$$

Example 2 : The marks obtained by students in a college are normally distributed with mean 65 and variance 25. If 3 students are selected at random from this college what is the probability that at least one of them would have scored more than 75 marks ? **(M.U. 2005)**

Sol. : We have S.N.V. $Z = \dfrac{X-m}{\sigma} = \dfrac{X-65}{5}$. When $X = 75$, $Z = \dfrac{75-65}{5} = 2$

$\therefore$ $P(X > 75) = P(Z > 2) = 0{\cdot}5 -$ (area from $z = 0$ to $z = 2$)

$= 0{\cdot}5 - 0{\cdot}4772 = 0{\cdot}0228$

This is the probability that a student chosen at random has scored more than 75 marks.

$\therefore$ P(a student has not scored more than 75) = 1 − 0·0228 = 0·9772

P(all three students have not scored more than 75 marks)

= 0·9772 × 0·9772 × 0·9772

= 0·93

$\therefore$ P(at least one of 3 has scored more than 75 marks) = 1 − 0·93 = 0·07.

Example 3 : For a normal variate X with mean 25 and standard deviation 10, find the area between (i) $X = 25$, $X = 35$, (ii) $X = 15$, $X = 35$ and also the area such that, (iii) $X \ge 15$, (iv) $X \ge 35$.

Sol. : S.N.V. $Z = \dfrac{X-m}{\sigma} = \dfrac{X-25}{10}$

(i) When $X = 25$, $Z = 0$, and when $X = 35$, $Z = 1$.

$\therefore$ Area (between $X = 25$ and $X = 35$) = area (between $Z = 0$ and $Z = 1$)

= 0·3413.

(ii) When $X = 15$, $Z = -1$ and when $X = 35$, $Z = 1$.

$\therefore$ Area between ($X = 15$ and $X = 35$) = area between ($Z = -1$ and $Z = 1$)

= 2 (area between $Z = 0$ and $Z = 1$)

= 2 (0·3413) = 0·6826.

(iii) When $X \ge 15$, $Z \ge -1$

$\therefore$ Area to the right of ($X = 15$)

= area to the right of ($Z = -1$)

= (area between $Z = -1$ to $Z = 0$) + (area to the right of $Z = 0$)

= 0·3413 + 0·5 = 0·8413.

(iv) When $X \ge 35$, $z \ge 1$.

$\therefore$ Area to the right of ($X = 35$)

= area to the right of ($Z = 1$)

= (area to the right of $Z = 0$) − (area between $Z = 0$ and $Z = 1$)

= 0·5 − 0·3413 = 0·1587.

Example 4 : A normal population has mean 0.1 and standard deviation 2·1. Find the probability that the value of the mean of the sample of size 900 drawn from this population will be negative. **(M.U. 2004)**

Sol. : The mean $\overline{X}$ of the sample is a S.N.V. We have $Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$.

$\because$ $\mu = 0.1$, $\sigma = 2{\cdot}1$, $n = 900$ $\quad\therefore$ $Z < -\dfrac{0{\cdot}1}{2{\cdot}1/\sqrt{900}} = -1{\cdot}43$

$\therefore$ $P(Z < -1{\cdot}43) = P(Z > 1{\cdot}43) = 0{\cdot}5 -$ (area from $Z = 0$ to $Z = 1{\cdot}43$)

$= 0{\cdot}5 - 0{\cdot}4236 = 0{\cdot}0764$.

Example 5 : A manufacturer knows from his experience that the resistance of resistors he produces is normal with $\mu = 100$ ohms and standard deviation $\sigma = 2$ ohms. What percentage of resistors will have resistance between 98 ohms and 102 ohms ? **(M.U. 1996, 10)**

Sol. : We have S.N.V. $Z = \dfrac{X-m}{\sigma} = \dfrac{X-100}{2}$

When $X = 98$, $Z = \dfrac{98-100}{2} = -1$. When $X = 102$, $Z = \dfrac{102-100}{2} = 1$.

$\therefore$ $P(98 \le X \le 102) = P(-1 \le Z \le 1)$

$=$ Area between $(Z = -1$ and $Z = 1)$

$\therefore$ $P(98 \le X \le 102) =$ Area from $(Z = -1$ to $Z = 0) +$ Area from $(Z = 0$ to $Z = 1)$

$= 2$ Area from $(Z = 0$ to $Z = 1)$

$= 2 \times 0{\cdot}3413 = 0{\cdot}6826$.

$\therefore$ % of resistors having resistance between 98 and 102 = 68·26 %.

Type III

Example 1 : Monthly salary X in a big organisation is normally distributed with mean ₹ 3000 and standard deviation of ₹ 250. What should be the minimum salary of a worker in this organisation, so that the probability that he belongs to top 5% workers ? **(M.U. 2017)**

Sol. : We have $Z = \dfrac{X-m}{\sigma} = \dfrac{X-3000}{250}$.

We want to find Z_1 such that

$$P(Z > z_1) = \frac{5}{100} = 0{\cdot}05$$

Since $0{\cdot}5 - 0{\cdot}05 = 0{\cdot}45$, and corresponding to 0·45 the entry in the area table is 1·64.

$\therefore$ $z_1 = 1{\cdot}64$.

$\therefore$ $1{\cdot}64 = \dfrac{X-3000}{250}$

$\therefore$ $X = 3000 + 250 \times 1{\cdot}64 =$ ₹ 3410.

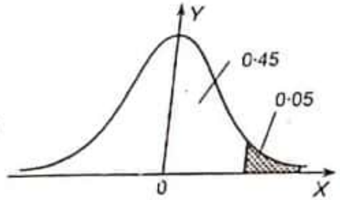


Fig. 8.10 (a)

Example 2 : The diameters of can tops produced by a machine are normally distributed with standard deviation of 0·05 cms. At what mean diameter the machine be set so that not more than 5% of the can tops produced by the machine have diameters exceeding 3 cms. ?

Sol. : Let X denote the diameter of the can tops. X is normally distributed with mean μ (unknown) and standard deviation $\sigma = 0{\cdot}05$. We are given that

$$P(Z > z_1) = 0{\cdot}05$$

Now, $0{\cdot}5 - 0{\cdot}05 = 0{\cdot}45$ and corresponding to 0·45 the entry in the area table is 1·64

$\therefore$ $z_1 = 1{\cdot}64$

$\therefore$ $Z = \dfrac{X-m}{\sigma}$ gives $1{\cdot}64 = \dfrac{3-m}{0{\cdot}01}$

$\therefore$ $m = 3 - 1{\cdot}64 \times 0{\cdot}01 = 2{\cdot}984$

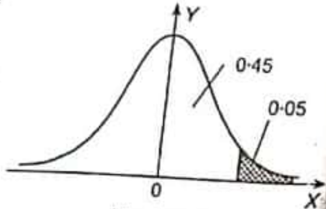


Fig. 8.10 (b)

Example 3 : If X is a normal variate with mean 25 and standard deviation 5, find the value (i) of $X = x_1$, such that $P(X \ge x_1) = 0{\cdot}32$, (ii) of $X = x_2$, such that $P(X \le x_2) = 0{\cdot}73$, (iii) of $X = x_3$ such that $P(X \le x_3) = 0{\cdot}24$.

Sol. : (i) Since 0·32 is less than 0·5, we have to find

$Z = z_1 =$ corresponding to area $= 0{\cdot}5 - 0{\cdot}32 = 0{\cdot}18$.

Now, from the table we find that corresponding to $Z = 0{\cdot}47$ the are under S.N.V. is 0·18.

$\therefore$ $Z = \dfrac{X-m}{\sigma}$ gives $0{\cdot}47 = \dfrac{X-25}{5}$

$\therefore$ $X = 25 + 5 \times 0{\cdot}47 = 25 + 2{\cdot}35 = 27{\cdot}35$

$\therefore$ $P(X \ge 27{\cdot}35) = 0{\cdot}32$ [See Fig. 8.11 (a)]

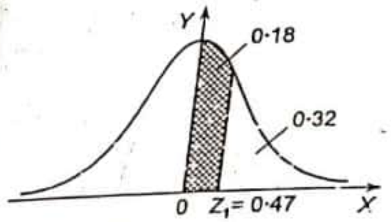


Fig. 8.11 (a)

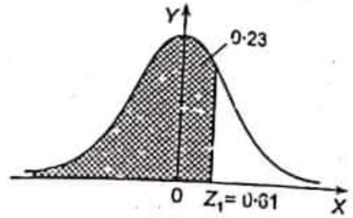


Fig. 8.11 (b)

(ii) Since 0·73 is greater than 0·5, we have to find

$Z = z_1$ corresponding to area $0{\cdot}73 - 0{\cdot}5 = 0{\cdot}23$.

Now, from the table we find that corresponding to $Z = 0{\cdot}61$ the area under S.N.V. is 0·23.

$\therefore$ $Z = \dfrac{X-m}{\sigma}$ gives $0{\cdot}61 = \dfrac{X-25}{5}$

$\therefore$ $X = 25 + 5 \times 0{\cdot}61 = 28{\cdot}05$

$\therefore$ $P(X \le 28{\cdot}05) = 0{\cdot}73$ [See Fig. 8.11 (b)]

(iii) Since 0·24 is less than 0·5, we have to find

$Z = z_1$ corresponding to are $0{\cdot}5 - 0{\cdot}24 = 0{\cdot}26$.

Now, from the table, we find that corresponding to $Z = 0{\cdot}71$, the are under the S.N.V is 0·26.

Since, we want X less than the desired value, we must take Z_1 on the left hand area *i.e.*,

$Z_1 = -0{\cdot}71$.

$\therefore$ $Z = \dfrac{X-m}{\sigma}$ gives $-0{\cdot}71 = \dfrac{X-25}{5}$

$\therefore$ $X = 25 - 5 \times 0{\cdot}71 = 25 - 3{\cdot}55 = 21{\cdot}45$

$\therefore$ $P(X \le 21{\cdot}45) = 0{\cdot}24$.

[See Fig. 10.11 (c)]

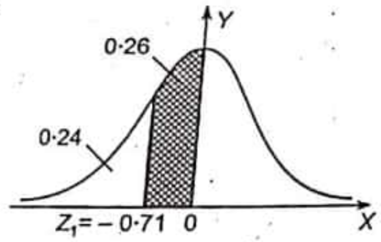


Fig. 8.11 (c)

Example 4 : The marks obtained by 1000 students in an examination are found to be normally distributed with mean 70 and standard deviation 5. Estimate the number of students whose marks will be (i) between 60 and 75, (ii) more than 75. **(M.U. 2003, 16)**

Sol. : We have S.N.V. $Z = \dfrac{X - m}{\sigma} = \dfrac{X - 70}{5}$

(i) When $X = 60$, $Z = \dfrac{60 - 70}{5} = -2$;

When $X = 75$, $Z = \dfrac{75 - 70}{5} = 1$.

$P(60 \le X \le 75) = P(-2 < Z < 1)$

= Area between ($Z = -2$ and $Z = 1$)

= Area from ($Z = 0$ to $Z = 2$) + area from ($Z = 0$ to $Z = 1$)

= 0·4772 + 0·3413 = 0·8185

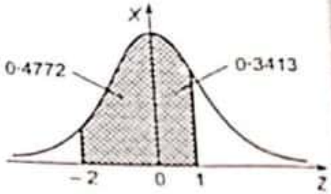


Fig. 8.12

∴ Number of students getting marks between 60 and 75

= Np = 1000 × 0·8185 = 818

(ii) $P(X \ge 75) = P(Z \ge 1)$

= Area to the right of $Z = 1$

= 0·5 – (area between $Z = 0$ and $Z = 1$)

= 0·5 – 0·3413 = 0·1587

∴ Number of students getting more than 75 marks

= Np = 1000 × 0·1587 = 159

Example 5 : In an intelligence test administered to 1000 students, the average was 42 and standard deviation was 24. Find the number of students (i) exceeding the score 50 and (ii) between 30 and 54. (M.U. 2016)

Sol. : We have S.N.V. $Z = \dfrac{X - m}{\sigma}$

By data, $m = 42$ and $\sigma = 24$.

∴ $Z = \dfrac{X - 42}{24}$.

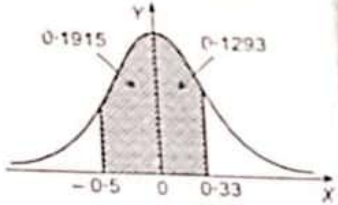


Fig. 8.13

(i) When $X = 50$, $Z = \dfrac{50 - 42}{24} = \dfrac{1}{3} = 0·33$

$P(Z \ge 50)$ = area to the right of 0·33

= 0·5 – (area between $Z = 0$ and $Z = 0·33$)

= 0·5 – 0·1293 = 0·3707

(ii) When $X = 30$ and $X = 54$, we get

$$Z = \frac{30 - 42}{24} = -0·5 \quad \text{and} \quad Z = \frac{54 - 42}{24} = 0·5$$

$P(30 \le Z \le 54)$ = area between $Z = -0·5$ to $Z = 0·5$

= 2 (area between $Z = 0$ and $Z = 0·5$)

= 2 (0·1915) = 0·3830

Number of students getting more than 50 marks

= Np = 1000 × 0·3707 = 371

Number of students getting marks between 30 and 54

= Np = 1000 × 0·383 = 383

Type IV

Example 1 : If X_1 and X_2 are two independent random variates with means 30 and 25 and variances 16 and 12 and if $Y = 3X_1 - 2X_2$, find $P(60 \le Y \le 80)$. **(M.U. 2005)**

Sol. : Since X_1, X_2 are independent normal variates with means 30 and 25 and variances 16 and 12, $Y = 3X_1 - 2X_2$ is also a normal variate with mean

$$m = a_1 \bar{X}_1 + a_2 \bar{X}_2 = 3(30) + (-2)(25) = 90 - 50 = 40.$$

and variance $\sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 = 9(16) + 4(12) = 192.$

S.N.V. $Z = \dfrac{Y - m}{\sigma} = \dfrac{Y - 40}{\sqrt{192}}$

When $Y = 60$, $Z = \dfrac{20}{\sqrt{192}} = 1·44$

When $Y = 80$, $Z = \dfrac{40}{\sqrt{192}} = 2·89$

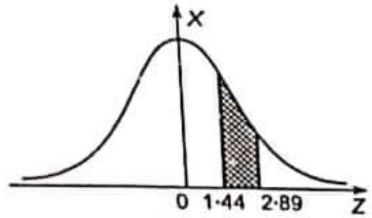


Fig. 8.14

∴ $P(60 \le Y \le 80) = P(1·44 \le Z \le 2·89)$

= area between $Z = 1·44$ and $Z = 2·89$

= (area from $Z = 0$ to $Z = 2·89$) – (are from $Z = 0$ to $Z = 1·44$)

= 0·4981 – 0.4251 = 0·0730.

Example 2 : Two independent random variates X and Y are distributed normally with mean and standard deviations (52, 3) and (50, 2) respectively. Find the probability that a randomly chosen pair of values of X and Y will differ by 1·7 or more.

Sol. : If $U = X - Y$ then by the additive property of normal variates, U is a normal variate with mean = 52 – 50 = 2 and standard deviation $\sqrt{9+4} = \sqrt{13}$ *i.e.* $N(2, \sqrt{13})$.

[See (4), page 8-44]

∴ $Z = \dfrac{U - m}{\sigma} = \dfrac{U - 2}{\sqrt{13}}$ is a S.N.V.

Now, P(X and Y will differ by 1.7 or more)

$= P(|X - Y| \ge 1·7)$

$= P(|U| \ge 1·7) = 1 - P(|U| \le 1·7)$

$= 1 - P(-1·7 \le U \le 1·7)$

Now, when $U = -1·7$, $Z = \dfrac{-1·7 - 2}{\sqrt{13}} = -1·03$

and when $U = 1·7$, $Z = \dfrac{1·7 - 2}{\sqrt{13}} = -0·08$

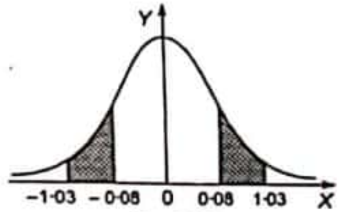


Fig. 8.15

$\therefore$ P (X and Y will differ by 1·7 or more)

$= 1 - P(-1{\cdot}03 \le Z \le -0{\cdot}08)$

$= 1 -$ (area from $Z = 0{\cdot}08$ to $Z = 1{\cdot}03$)

$= 1 -$ [(area from $Z = 0$ to $Z = 1{\cdot}03$) $-$ (area from $Z = 0$ to $Z = 0{\cdot}08$)]

$= 1 - (0{\cdot}3485 - 0{\cdot}0319) = 1 - 0{\cdot}3766$

$= 0{\cdot}6234$

Example 3 : If X and Y are two independent random variates $N(3, 4)$ and $N(8, 5)$, find the probability that a point (X, Y) will lie between the lines $5X + 3Y = 8$ and $5X + 3Y = 15$.

Sol. : Since X is N (3, 4) and Y is N (8, 5) by additive property of normal distribution $U = 5X + 3Y$ follows a normal distribution with mean

$$m = 5 \times 3 + 3 \times 8 = 39 \quad \text{and} \quad \sigma = \sqrt{25 \times 16 + 9 \times 25} = 25.$$

P (the point (X, Y) lies between the lines $5X + 3Y = 8$ and $5X + 3Y = 15$)

$= P(8 \le U \le 15))$

Now, $Z = \dfrac{U - 39}{25}$ is a S.N.V.

When $U = 8$, $\quad Z = \dfrac{8 - 39}{25} = -1{\cdot}24$

and when $U = 15$, $\quad Z = \dfrac{15 - 39}{25} = -0{\cdot}96$

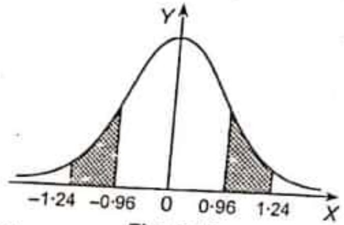


Fig. 8.16

$\therefore$ P (the point lies between the two lines)

$=$ area between $Z = -0{\cdot}96$ and $Z = -1{\cdot}24$

$=$ area between $Z = 0{\cdot}96$ and $Z = 1{\cdot}24$

$=$ (area from $z = 0$ to $z = 1{\cdot}24$) $-$ (area from $z = 0$ to $z = 0{\cdot}96$)

$= 0{\cdot}3925 - 0{\cdot}3315$

$= 0{\cdot}061$

Example 4 : If X and Y are two independent normal random variates such that their means are 8, 12 and standard deviations are 2 and $4\sqrt{3}$ respectively, find the value α such that $P[(2X - Y) \le 2\alpha] = P[(X + 2Y) \ge 3\alpha]$.

Sol. : By additive property of normal distribution $U = 2X - Y$ is a normal variate with

mean, $m = 2 \times 8 - 1 \times 12 = 4$

and standard deviation, $\sigma = \sqrt{2^2 \times 2^2 + 1^2 \times (4\sqrt{3})^2} = \sqrt{16 + 48} = 8$

$\therefore$ U is a S.N.V. with mean 4 and S.D. = 8.

$V = X + 2Y$ is a normal variate with mean $m = 1 \times 8 + 2 \times 12 = 32$ and standard deviation

$$\sigma = \sqrt{1^2 \times 2^2 + 2^2 \times (4\sqrt{3})^2} = \sqrt{4 + 192} = 14$$

$\therefore$ V is a S.N.V. with mean 32 and S.D. = 14.

Now, $P[(2X - Y) \le 2\alpha] = P[(X + 2Y) \ge 3\alpha]$

$\therefore$ $P(U \le 2\alpha) = P(V \ge 3\alpha)$

$$\therefore \quad P\left(\frac{U - 4}{8} \le \frac{2\alpha - 4}{8}\right) = P\left(\frac{V - 32}{14} \ge \frac{3\alpha - 32}{14}\right)$$

This means if Z is a S.N.V.

$$P\left(Z \le \frac{2\alpha - 4}{8}\right) = P\left(Z \ge \frac{3\alpha - 32}{14}\right)$$

By symmetry of normal distribution if $P(Z \ge z_1) = \alpha$ then $P(Z \le -z_1) = \alpha$.

$$\therefore \quad P\left(Z \le \frac{2\alpha - 4}{8}\right) = P\left(Z \le -\left(\frac{3\alpha - 32}{14}\right)\right)$$

$$\therefore \quad \frac{2\alpha - 4}{8} = -\frac{(3\alpha - 32)}{14}$$

$$\therefore \quad \frac{\alpha - 2}{4} = -\frac{3\alpha - 32}{14}$$

$\therefore$ $14a - 28 = -12a + 128$

$\therefore$ $26a = 156$

$\therefore$ $\alpha = 6$.

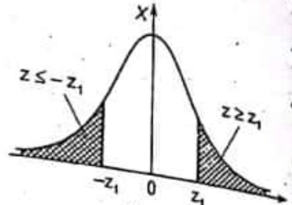


Fig. 8.17

Example 5 : In an examination marks obtained by students in Mathematics, Physics and Chemistry are normally distributed with means 51, 53 and 46 with standard deviation 15, 12, 16 respectively. Find the probability of securing total marks (i) 180 or above, (ii) 80 or below. **(M.U. 2005, 10)**

Sol. : Let X_1, X_2, X_3 denote the marks obtained in the three subjects. Then X_1, X_2, X_3 are normal variates with mean 51, 53, 46 and variance 15^2, 12^2, 16^2.

Assuming the variates to be independent, $Y = X_1 + X_2 + X_3$ is distributed normally with mean $m = 51 + 53 + 46 = 150$ and $\sigma^2 = 15^2 + 12^2 + 16^2 = 625 = 25^2$.

$$\therefore \quad \text{S.N.V. } Z = \frac{Y - m}{\sigma} = \frac{Y - 150}{25}$$

When $Y = 180$, $\quad Z = \dfrac{180 - 150}{25} = \dfrac{30}{25} = 1{\cdot}2$.

$\therefore$ $P(Y \ge 150) = P(Z \ge 1{\cdot}2)$

$=$ Area to the right of $Z = 1{\cdot}2$

$= 0{\cdot}5 -$ (area between $Z = 0$ and $Z = 1{\cdot}2$)

$= 0{\cdot}5 - 0{\cdot}3849 = 0{\cdot}1151$.

When $Y = 80$, $\quad Z = \dfrac{80 - 150}{25} = -\dfrac{70}{25} = -2{\cdot}4$

$\therefore$ $P(Y \le 80) = P(Z \le -2{\cdot}4)$

$=$ Area to the left of $Z = -2{\cdot}4$

$= 0{\cdot}5 -$ area from $Z = 0$ to $Z = 2{\cdot}4$

$= 0{\cdot}5 - 0{\cdot}4918 = 0{\cdot}0082$.

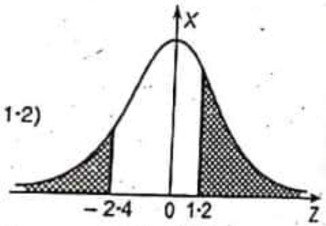


Fig. 8.18

Type V

Example 1 : The incomes of a group of 10,000 persons were found to be normally distributed with mean ₹ 520 and standard deviation ₹ 60. Find (i) the number of persons having incomes between ₹ 400 and 550, (ii) the lowest income of the richest 500.

Sol. : S.N.V. $Z = \frac{X-m}{\sigma} = \frac{X-520}{60}$

(i) When $X = 400$, $Z = \frac{400-520}{60} = -2$. When $X = 550$, $Z = \frac{550-520}{60} = 0.5$

$\therefore$ $P(400 \le X \le 550)$ = Area (from $z = -2$ to $z = 0.5$)

But, area (from $z = -2$ to $z = 0$)

= Area (from $z = 0$ to $z = 2$)

= 0·4772.

And area (from $z = 0$ to $z = 0.5$) = 0·1915.

$\therefore$ $P(400 \le x \le 550)$

= Area (from $z = -2$ to $z = 0.5$)

= 0·4772 + 0·1915 = 0·6687.

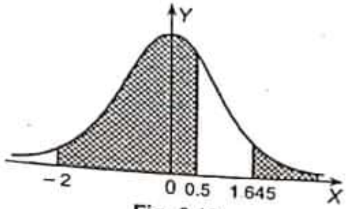


Fig. 8.19

$\therefore$ The number of persons whose incomes are between ₹ 400 and ₹ 550 = Np = 10,000 × 0·6687 = 6687.

(ii) If we have to consider the richest 500 persons then the probability that a person selected at random will be one of them

$= \frac{500}{10,000} = 0.05$

This is a reverse problem. So far we have found the probability for a given value of Z. Here, we have to find the value of Z for a given probability. We have to find the value of Z to the right of which the area is 0·05. But area to the right of $Z = 0$ is 0·5.

$\therefore$ Area from ($Z = 0$ to Z = this value) = 0·5 – 0·05 = 0·45

From the table we find that the area from $Z = 0$ to $Z = 1.645$ is 0·45

$\therefore$ The required value of $Z = 1.645$

But $Z = \frac{X-520}{60}$ $\quad\therefore\quad 1.645 = \frac{X-520}{60}$

$\therefore$ $X - 520 = 60 \times 1.645$ $\quad\therefore\quad X = 520 + 98.7 =$ ₹ 618·7.

$\therefore$ The lowest income of the richest 500 is ₹ 618·7.

Example 2 : The income of a group of 10,000 persons was found to be normally distributed with mean of ₹ 750 and standard deviation of ₹ 50. What is the lowest income of richest 250 ?

Sol. : S.N.V. $Z = \frac{X-m}{\sigma} = \frac{X-750}{50}$.

If we have to consider the richest 250 persons then the probability that a person selected at random will be one of them is $= \frac{250}{10,000} = 0.025$.

This again is a reverse problem. So far we have found the probability for a given value of Z. Here, we have to find the value of Z for a given probability. We have to find the value of Z to the right of which the area is 0·025. But the area to the right of $Z = 0$ is 0·5.

$\therefore$ Area from ($Z = 0$ to Z = this value) = 0·5 – 0·025 = 0·475.

From the table we find that the area from $Z = 0$ to $Z = 1.96$ is 0·475.

$\therefore$ The required value of $Z = 1.96$

When $Z = 1.96$, $\quad 1.96 = \frac{X-750}{50}$

$\therefore$ $X - 750 = 1.96 \times 50$ $\quad\therefore\quad X = 848$

$\therefore$ The lowest income of richest 250 persons = ₹ 848.

Example 3 : In a competitive examination the top 15% of the students appeared will get grade A, while the bottom 20% will be declared fail. If the grades are normally distributed with mean % of marks 75 and S.D. 10, determine the lowest % of marks to receive grade A and the lowest % of marks that passes. **(M.U. 2014)**

Sol. : This is a reverse problem as above.

We have $Z = \frac{X-m}{\sigma} = \frac{X-75}{10}$.

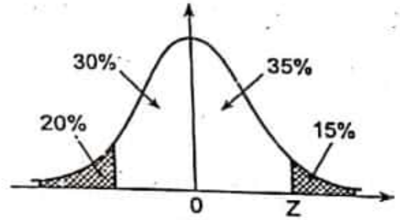


Fig. 8.20

(i) Grade A is given for 15%. We have to find the value of Z to the right of which the area is 0·15.

But the area to the right of $Z = 0$ is 0·5.

$\therefore$ Area from ($Z = 0$ to Z = this value)

= 0·5 – 0·15 = 0·35.

From the table, we find that the area between $Z = 0$ to $Z = 1.04$ is 0·35.

$\therefore$ The required value of $Z = 1.04$.

But $Z = \frac{X-75}{10}$ $\quad\therefore\quad 1.04 = \frac{X-75}{10}$ $\quad\therefore\quad X = 75 + 10.4 = 85.4$

(ii) Lowest 20% students are declared fail. We have to find the value of Z to the left of which the area is 0·20. But the area to the left of $Z = 0$ is 0·5.

$\therefore$ Area from ($Z = 0$ to Z = this value) = 0·5 – 0·2 = 0·3.

From the table we find that the area between $Z = 0$ and $Z = 0.84$ is 0·3.

But this ordinate is on the left and hence negative.

$\therefore$ The required value of $Z = -0.84$.

But $Z = \frac{X-75}{10}$ $\quad\therefore\quad -0.84 = \frac{X-75}{10}$ $\quad\therefore\quad X = 75 - 8.4 = 66.6$.

Example 4 : If the actual amount of coffee which a filling machine puts into 6 ounce jars is a random variable having normal distribution with standard deviation 0·05 ounce and if only 3% of the jars are to contain less than 6 ounce of coffee what must be the mean fill of these jars ? **(M.U. 2004, 07)**

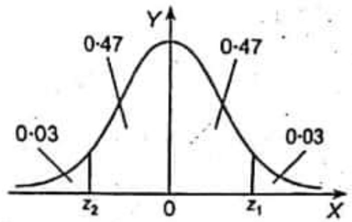


Fig. 8.21

Sol. : Let $Z = \frac{X-m}{\sigma}$

We have $\sigma = 0.05$, $X = 6$ $\quad\therefore\quad Z = \frac{6-m}{0.05}$.

We want Z such that $P(Z \leq 3) = P\left(\frac{6-m}{0.05}\right) = 0.03$.

From the table for area to be 0·47

$z_1 = 1.808 \quad \therefore z_2 = -1.808$

$\therefore \frac{6-m}{0.05} = -1.808$

$\therefore m = 6 + 0.05 \times 1.808 = 6.09$ ounce.

Type VI

Example 1 : Find the mean and the standard deviation of a normal distribution of marks in an examination where 58 % of the candidates obtained marks below 75, 4 % got above 80 and the rest between 75 and 80 (For a S.N.V. the area under the curve between $z = \pm 0.2$ is 0·16 and between $z = \pm 1.8$ is 92). (M.U. 1999)

Sol. : Let m and σ be the mean and the standard deviation of the variate.

Since 58 % students are below 75, 58 − 50 = 8 % students are between 75 and m.

Since 4 % students are above 80, 50 − 4 = 46 % students are between m and 80.

We are given that area between $Z = \pm 0.2$ is 0·16 and that between $Z = \pm 1.8$ is 9·92.

Hence, the area between $Z = 0$ and $Z = 0.2$ is $\frac{0.16}{2} = 0.8$ and that between $Z = 0$ and $Z = +1.8$ is $\frac{0.92}{2} = 0.46$.

In other words for area 0·08 (8 %), $Z = 0.2$ and for area 0·46 (46 %), $Z = 1.8$.

$\therefore \frac{75-m}{\sigma} = 0.2$ and $\frac{80-m}{\sigma} = 1.8$

$\therefore 75 - m = 0.2\,\sigma$ and $80 - m = 1.8\,\sigma$

Subtracting $-5 = -1.6\,\sigma \;\therefore\; \sigma = \frac{5}{1.6} = 3.125$

$\therefore m = 75 - 0.2\,\sigma = 75 - 3.125 \times 0.2 = 74.4$ mark.

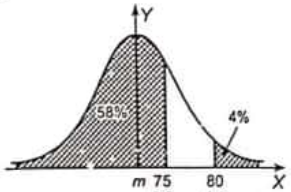


Fig. 8.22

Example 2 : Marks obtained by students in an examination follow normal distribution. If 30% of students got below 35 marks and 10% got above 60 marks, find the mean and standard deviation. (M.U. 2016)

Sol. : Let m and σ be the mean and the standard deviation of the distribution.

Since 30% students are below 35, 20% students are between 35 and m.

Since 10% students are above 60, 40% students are between m and 60.

From the table we find that,

0·2 area corresponds to $Z = 0.525$

and 0·4 area corresponds to $Z = 1.283$

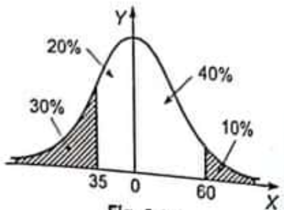


Fig. 8.23

But 0·2 area is to the left of m hence $Z = -0.525$.

$\therefore \frac{35-m}{\sigma} = -0.525; \quad \frac{60-m}{\sigma} = 1.283$

$\therefore 35 - m = -0.525\,\sigma$

$\therefore 60 - m = 1.283\,\sigma$ .... (1)

By subtraction, we get $25 = 1.808\,\sigma \;\therefore\; \sigma = \frac{25}{1.808} = 13.83$.

Putting this value of σ in (1), we get

$35 - m = -0.525\,(13.83)$

$\therefore m = 35 + 0.525\,(13.83) = 42.26$

Hence, mean = 13·83 and standard deviation, $\sigma = 42.26$.

Example 3 : In a distribution exactly normal 7 % of items are under 35 and 89 % are under 63. What are the mean and standard deviation ? (M.U. 2004)

Sol. : Since 7 % items are below 35, 50 − 7 = 43 % items are between 35 and m, and since 89 % items are below 63, 89 − 50 = 39 % items are between m, and 63.

For area 0·43, $Z = 1.48$.

Since $35 < m$, $Z = -1.48$ and for area 0·39, $Z = 1.23$.

$\therefore \frac{35-m}{\sigma} = -1.48$ and $\frac{63-m}{\sigma} = 1.23$

$\therefore 35 - m = -1.48\,\sigma$

and $63 - m = 1.23\,\sigma$.

Subtracting $28 = 2.71\sigma$

$\therefore \sigma = \frac{28}{2.71} = 10.33$

$\therefore m = 35 + 1.48\sigma = 35 + 1.48 \times 10.33$

$= 35 + 15.3 = 50.3$.

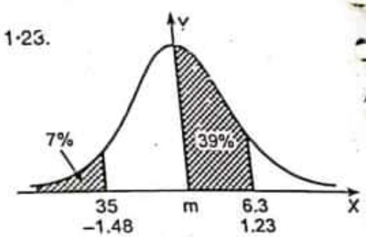


Fig. 8.24

Example 4 : In a distribution exactly normal 7% of items are under 35 and 89 % of the items are under 63. Find the probability that an item selected at random lies between 45 and 56. (M.U. 2011, 15)

Sol. : As in the above example $m = 50.3$ and $\sigma = 10.33$.

Now, $Z = \frac{X-m}{\sigma} = \frac{X-50.3}{10.33}$

When $X = 45$, $Z = \frac{45-50.3}{10.33} = -0.51$

When $X = 56$, $Z = \frac{56-50.3}{10.33} = 0.55$

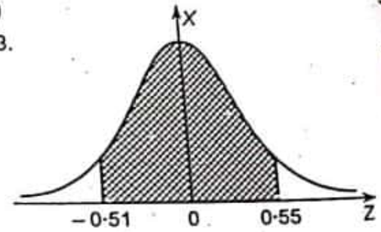


Fig. 8.25

$\therefore P(45 \leq X \leq 56) = P(-0.51 \leq Z \leq 0.55)$

= are between ($Z = -0.51$ to $Z = 0.55$)

= area from 0 to 0·51 + area from 0 to 0·55

$= 0.1950 + 0.2088 = 0.4038$

Exampl
34 percent of
months how
normal curv
and 1·1 tim
Sol. : Boca
22 and 24
correspon

Example 5 : A large number of automobile batteries have average life of 24 months. If 34 percent of them average between 22 and 26 months and 272 of them last longer than 29 months how many were in the group tested ? Assume the distribution to be normal. (For a normal curve 17 % and 36·4 % values lie between the mean and respective distance of 0·44 and 1·1 times standard deviation from the mean).

Sol. : Because of symmetry of the normal distribution and because $m = 24$, the area between 22 and 24 is equal to that between 24 and 26. This area is equal to $34/2 = 17\%$ which corresponds to S.N.V. $Z = 0{\cdot}44$ by data and $X = 26$.

$$\therefore \quad \frac{26-24}{\sigma} = 0{\cdot}44 \qquad \therefore \ \sigma = \frac{2}{0{\cdot}44} = \frac{50}{11}$$

Now when $X = 29$, $Z = \dfrac{X-m}{\sigma} = \dfrac{29-24}{50/11} = \dfrac{11}{10} = 1{\cdot}1$.

By data area between $Z = 0$ and $Z = 1{\cdot}1$ is $0{\cdot}364$.

$\therefore$ The area to the right of $Z = 1{\cdot}1$ *i.e.* $X = 29$ is $= 0{\cdot}5 - 0{\cdot}364 = 0{\cdot}136$ which is the probability that a battery will last longer than 29 months.

But $p = \dfrac{f}{N}$ and $f = 272$, $p = 0{\cdot}136$. $\quad \therefore \ N = \dfrac{f}{p} = \dfrac{272}{0{\cdot}136} = 2000$

Hence, 2000 batteries were tested.

Type VII

Example 1 : The probability that an electronic component will fail in less 1200 hours of continuous use is 0·25. Using normal approximation to Binomial distribution, find the probability that among 200 such components fewer than 45 will fail in less than 1200 hours of continuous use. **(M.U. 2005)**

Sol. : While using continuous variate in place of discrete variate we must "spread" its values over a continuous scale. This we do by taking each integer k to represent the interval $k - (1/2)$ to $k + (1/2)$.

Now, we have $n = 200$, $p = 0{\cdot}25$, $q = 0{\cdot}75$.

$$\therefore \quad m = np = 200 \times 0{\cdot}25 = 50$$

$$\sigma = \sqrt{npq} = \sqrt{200 \times 0{\cdot}25 \times 0{\cdot}75} = 6{\cdot}12$$

$$Z = \frac{X - np}{\sqrt{npq}} = \frac{X-50}{6{\cdot}12}$$

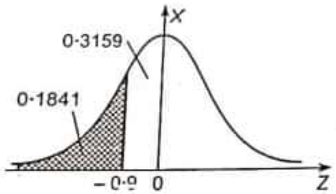


Fig. 8.26

When $X = 44{\cdot}5$ (*i.e.* $k - (1/2)$)

$$Z = \frac{44{\cdot}5 - 50}{6{\cdot}12} = -0{\cdot}8986 = -0{\cdot}9$$

For $Z = 0{\cdot}9$, $p = 0{\cdot}3159$.

$$\therefore \quad P(X \le 44{\cdot}5) = P(Z \le -0{\cdot}9)$$
$$= 0{\cdot}5 - 0{\cdot}3159 = 0{\cdot}1841.$$

Example 2 : Determine in two different ways, the probability that by guess-work a student can correctly answer 25 to 30 questions in a multiple choice quiz consisting of 80 questions. Assume that in each question with four choices only one is correct and the student has no knowledge. **(M.U. 2004)**

Sol. : (a) By using Normal approximation to Binomial Distribution

$$\text{Mean} = m = np = 80 \times (1/4) = 20$$

$$\text{S.D., } \sigma = \sqrt{npq} = \sqrt{80 \times \frac{1}{4} \times \frac{3}{4}} = 3{\cdot}873$$

$$\therefore \quad Z = \frac{X-m}{\sigma} = \frac{X-20}{3{\cdot}873}$$

Since from discrete (Binomial Distribution) we are approximating to continuous (Normal distribution) we extend the range 25 to 30 by half units on either side *i.e.*, we take the range as 24·5 to 30·5.

When $X = 24{\cdot}5$, $Z = \dfrac{24{\cdot}5 - 20}{3{\cdot}873} = 1{\cdot}16$. $\quad$ When $X = 30{\cdot}5$, $Z = \dfrac{30{\cdot}5 - 20}{3{\cdot}873} = 2{\cdot}71$

$$\therefore \quad P(24.5 \le X \le 30{\cdot}5) = P(1{\cdot}16 \le Z \le 2{\cdot}71)$$
$$= (\text{area from } Z = 0 \text{ to } Z = 2{\cdot}71) - (\text{area from } Z = 0 \text{ to } Z = 1{\cdot}16)$$
$$= 0{\cdot}4966 - 0{\cdot}6770$$
$$= 0{\cdot}1196$$

(b) By using Binomial Distribution

We have $p = \dfrac{1}{4}$, $q = \dfrac{1}{4}$, $n = 80$

$$\therefore \quad P(X = x) = {}^nC_x\, p^x\, q^{n-x} = {}^{80}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{80-x}$$

$$\therefore \text{ The required probability } = \sum_{x=25}^{30} {}^{80}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{80-x}$$
$$= 0{\cdot}0434 + 0{\cdot}0306 + 0{\cdot}0204 + 0{\cdot}0129 + 0{\cdot}0077 + 0{\cdot}0043$$
$$= 0{\cdot}1193$$

(Note that the two values differ only by 0.0003.)

Example 3 : Using normal distribution, find the probability of getting 55 heads in the toss of 100 fair coins. **(M.U. 2004)**

(Compare the result with that obtained from Binomial distribution).

Sol. : Since the coins are fair, we have $p = 1/2$, $q = 1/2$. By data $n = 100$.

$$\therefore \quad m = np = 100 \times \frac{1}{2} = 50, \qquad \sigma = \sqrt{npq} = \sqrt{100 \times \frac{1}{2} \times \frac{1}{2}} = 5$$

Hence, we have S.N.V. $Z = \dfrac{X-m}{\sigma} = \dfrac{X-50}{5}$

When $X = 54{\cdot}5$, $\quad Z = \dfrac{54{\cdot}5 - 50}{5} = 0{\cdot}9$.

When $X = 55{\cdot}5$, $\quad Z = \dfrac{55{\cdot}5 - 50}{5} = 1{\cdot}1$

$$\therefore \quad P(0{\cdot}9 < Z < 1{\cdot}1) = \text{area from } Z = 0{\cdot}9 \text{ to } Z = 1{\cdot}1$$
$$= (\text{area from } Z = 0 \text{ to } Z = 1{\cdot}1) - (\text{area from } Z = 0 \text{ to } 0{\cdot}9)$$
$$= 0{\cdot}3643 - 0{\cdot}3159 = 0{\cdot}0484$$

Now, for the second part, we have

$$P(X = x) = {}^nC_x\, p^x\, q^{n-x} = {}^{100}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{100-x}$$

$$\therefore \quad P(X = 55) = {}^{100}C_{55} \left(\frac{1}{2}\right)^{55} \left(\frac{1}{2}\right)^{45} = 0{\cdot}04847$$

EXERCISE - III

Type I

1. Find k and the mean and standard deviation of the normal distribution given by

(i) $y = k\,e^{-\left(\frac{x^2}{18} - x + \frac{9}{2}\right)}$ [Ans. : $k = \frac{1}{3\sqrt{2\pi}}$, $m = 9$, $\sigma = 3$]

(ii) $y = k\,e^{-\left(\frac{x^2}{6} - x + \frac{3}{2}\right)}$ [Ans. : $k = \frac{1}{\sqrt{6 \cdot \pi}}$, $m = 3$, $\sigma = \sqrt{3}$]

2. Write down the equation of the curve of the normal distribution with mean 50 and standard deviation 9. What is the first quartile of the distribution? [Ans. : 44]

3. Write down the equation of the normal curve with mean 10 and variance 36. What is the quartile deviation of the distribution? [Ans. : 4]

4. What is the Q.D. of a normal distribution with S.D. 9 ? [Ans. : 6]

Type II

1. If X is normally distributed with mean 10 and standard deviation 2, find $P(-3 \le X \le 12)$. [Ans. : 0·8413]

2. If X is normally distributed with mean 15 and standard deviation 6, find $P(-3 \le X \le 18)$ and $P(|X| \ge 16{\cdot}02)$. [Ans. : 0·6902; 0·5675]

3. If X is normally distributed with mean and standard deviation 4, find (i) $P(5 \le X \le 10)$, (ii) $P(X \ge 15)$, (iii) $P(10 \le X \le 15)$, (iv) $P(X \le 5)$. (M.U. 2003)

[Ans. : (i) 0·3326, (ii) 0·003, (iii) 0·1557, (iv) 0·05987]

4. A normal distribution has mean 5 and standard deviation 3. What is the probability that the deviation from the mean of an item taken at random will be negative ? (M.U. 2004)

[Ans. : 0·0575]

Type III

1. If Z is a S.N.V., find c such that (i) $P(-c < Z < c) = 0{\cdot}98$, (ii) $P(|Z| > c) = 0{\cdot}04$.

[Ans. : (i) $c = 2{\cdot}33$, (ii) $c = 2{\cdot}05$]

2. If X is a normal variate with mean 30 and standard deviation 6, find the value of $X = x_1$ such that $P(X \ge x_1) = 0{\cdot}05$. [Ans. : $z_1 = 1.64$, $x_1 = 39{\cdot}84$]

3. If X is a normal variate with mean 25 and standard deviation 5, find the value of $X = x_1$ such that $P(X \le x_1) = 0{\cdot}01$. [Ans. : $z_1 = -2{\cdot}33$, $x_1 = 13{\cdot}35$]

Type IV

1. The first and third quartiles of a normal distribution are respectively 92 and 128. Find the mean and the standard deviation. [Ans. : 110, 27]

2. For a normal distribution the first quartile is 46 and the variance is 144. Find the (i) the mode, (ii) limits of central 50% items, (iii) mean deviation.

[Ans. : (i) 54, (ii) 46; 62, (iii) 9·6]

3. The mean and the standard deviation of a normal distribution are 70 and 15. Find the quartile deviation and mean deviation. [Ans. : (i) 10, (ii) 12]

Type V

1. The weights of 4000 students are found to be normally distributed with mean 50 kgs. and standard deviation 5 kgs. Find the probability that a student selected at random will have weight (i) less than 45 kgs., (ii) between 45 and 60 kgs.

[Ans. : (i) 0·1587, (ii) 0·8185]

2. The sizes of 10,000 items are normally distributed with mean 20 cms and standard deviation 4 cm. Find the probability that an item selected at random will have size between (i) 18 cms and 23 cms, (ii) above 26 cms. [Ans. : (i) 0·4649, (ii) 0·0668]

3. The daily sales of a firm are normally distributed with mean ₹ 8000 and variance of ₹ 10,000. (i) What is the probability that on a certain day the sales will be less than ₹ 8210? (ii) What is % of days on which the sales will be between ₹ 8100 and ₹ 8200 ?

(M.U. 1999, 2001) [Ans. : (i) 0·5832, (ii) 14%]

Type VI

1. Mean and standard deviation of chest measurements of 1200 soldiers are 85 cms and 5 cms respectively. How many of them are expected to have their chest measurements exceeding 95 cms. assuming the measurements to follow the normal distribution ? (Area for S.N.V. z from $z = 0$ to $z = 2$ is 0·4772) [Ans. : 27]

2. The height of 22 year old boys is distributed normally with mean 63" and standard deviation 2·5". A boy is eligible if his height is between 62" and 56". Find the expected number of boys out of 180 who will be ineligible because of excess height. (Area for S.N.V. z from $z = 0$ to $z = 0{\cdot}4$ is 0·1554 and that from $z = 0$ to $z = -0{\cdot}8$ is 0·2881). [Ans. : 36]

3. The mean height of soldiers is 68·22" with variance 10·8". Find the expected number of soldiers in a regiment of 1000 whose height will be more than 6 feet. (Area from $z = 0$ to $z = 1{\cdot}15$ is 0·3749). (M.U. 1997, 2009) [Ans. : 125]

4. The heights of 1000 soldiers in a regiment are distributed normally with mean of 172 cms. and a standard deviation of 5 cms. How many soldiers have height greater than 180 cms ?

(Area from $z = 0$ to $z = 1{\cdot}6$ is 0·4452). [Ans. : 55]

5. The weights of 1000 students were found to be normally distributed with mean 40 kgs. and standard deviation 4 kgs. Find the expected number of students with weights (i) less than 36 kgs., (ii) more than 45 kgs. [Ans. : (i) 159, (ii) 106]

6. If the heights of 500 students is normally distributed with mean 68 inches and standard deviation 4 inches, estimate the number of students having heights (i) greater than 72 inches, (ii) less than 62 inches, (iii) between 65 and 71 inches.

[Ans. : (i) 79, (ii) 33, (iii) 273]

Type VII

1. The mean I.Q. of a large number of children of age 14 is 100 with S.D. 16. Assuming the distribution of I.Q. to be normal, find the percentage of the children having I.Q. between 70 and 120. (Area for S.N.V. z from $z = 0$ to $z = 1 \cdot 875$ is 0·4696 and that from $z = 0$ to $z = 1 \cdot 25$ is 0·3944). [Ans. : 86·4 %]

2. A sample of 100 dry battery cells is tested to find the length of life, produced the following results.

$$\bar{x} = 12 \text{ hours}, \quad \sigma = 3 \text{ hrs.}$$

Assuming normal distribution what percentage of cells is expected to have life (i) more than 15 hours, (ii) between 10 and 14 hours. (M.U. 2007) [Ans. : (i) 15·87%, (ii) 49·72 %]

3. The daily sales of a certain item are normally distributed with mean ₹ 8000 and variance ₹ 10,000. (i) What is the probability that on a certain day the sales will be less than ₹ 8210 ? (ii) What percentage of the days will the sales be between ₹ 8100 and ₹ 8210 ?

(Given : Area for S.N.V. z from $z = 0$ to $z = 2 \cdot 1$ is 0·4821; that between $z = 0$ and $z = 1$ is 0·3413). [Ans. : (i) 0·9821, (ii) 14·08 %]

4. The average selling price of houses in a city is ₹ 50,000 with standard deviation of ₹ 10,000. Assuming the distribution of selling price to be normal find (i) the percentage of houses that sell for more than ₹ 55,000, (ii) the percentage of houses selling between ₹ 45,000 and ₹ 60,000. (Area between $t = 0$ and $t = 1$ is 0·3413 and between $t = 0$ and $t = 0 \cdot 5$ is 0·1915). [Ans. : (i) 30·85 %, (ii) 53·28 %]

Type VIII

1. The income distribution of workers in a certain factory was found to be normal with mean of ₹ 500 and standard deviation equal to ₹ 50. There were 228 persons above ₹ 600. How many persons were there in all ? (Area under the S.N. curve between height at 0 and 2 is 0·4772). (M.U. 2000) [Ans. : 10,000]

2. In a factory a large number of workers have average daily income of ₹ 120. If 38·3% of them have income between ₹ 100-140 and 528 of them get more than ₹ 170, how many workers were interrogated? (Area for S.N.V. between $z = 0$ and $z = 19 \cdot 15$ is 0·5 and that between $z = 0$ and $z = 1 \cdot 25$ is 0·3944). (Hint : Find σ) [Ans. : 5,000]

3. The arithmetic mean of purchases per day by a customer in a large store is ₹ 25 with a standard deviation of ₹ 10. If on a particular day, 100 customers purchased for ₹ 37·80 or more estimate the total number of customers who purchased from the store that day. (Given that the normal area between $t = 0$ and $t = 1 \cdot 28$ is 0·4000 where t is the S.N.V.)

[Ans. : 1,000]

4. The arithmetic mean of the weights of a group of boys is 105 lbs with standard deviation of 5 lbs. If there were 456 boys having weights more than 115 lbs, how many students were there in the group?

(Given : For S.N.V. z area from $z = 0$ to $z = 2$ is 0·4772). [Ans. : 20,000]

Type IX

1. The heights of 1000 cakes baked with certain mix have a normal distribution with a mean of 5·75 cms. and a standard deviation of 0·75 cms. Find the number of cakes having heights between 5 cms. and 6·25 cms. Also find the maximum height of the flattest 200 cakes.

(For a standard variate t, the area between $t = -1$ and $t = 1$ is 0·6286, that between $t = 0$ and $t = 0 \cdot 67$ is 0·2486 and that between $t = 0$ and $t = 0 \cdot 84$ is 0·3). [Ans. : 562·9, 5·12]

2. The life of army shoes is normally distributed with mean 8 months and standard deviation 2 months. If 5000 pairs are issued, how many pairs would be expected to need replacement after 12 months ? (M.U. 2001) [Ans. : 2386]

3. In an intelligence test administered to 1000 students the average was 42 and standard deviation was 24. Find the number of students (i) exceeding 50, (ii) between 30 and 54, (iii) the least score of top 100 students. (M.U. 2003) [Ans. : (i) 371, (ii) 383, (iii) 72·72]

4. 1000 light bulbs with an average life of 120 days are installed in streets of Mumbai. Their length of life is normally distributed with variance 400 days. (i) How many will expire in less than 90 days? (ii) If it is decided to replace all the bulbs together what interval should be allowed between replacements if not more than 10 percent should expire before replacement ?

(Area between $z = 0$ and $z = 1 \cdot 5$ is 0·4332 and 80% of the area lies between $z = \pm 1 \cdot 28$). [Ans. : (i) 67, (ii) 94·4]

5. Monthly salaries of 1000 workers have a normal distribution with mean of ₹ 575 and a standard deviation of ₹ 75. Find the number of workers having salaries between ₹ 500 and ₹ 625 p.m. Also find the minimum salary of the highest paid 200 workers.

(Given : For a standard normal variate t (i) area between $t = 0$, $t = 1$ is 0·3413, (ii) area between $t = 0$ and $t = 0 \cdot 67$ is 0·2486, (iii) area between $t = 0$ and $t = 0 \cdot 84$ is 0·3). [Ans. : ₹ 590, ₹ 638]

6. The marks obtained by students in a class are normally distributed with mean 75 and standard deviations. If top 5% got grade A and bottom 25% got grade B, what are the marks of the lowest of A and what are the marks of the highest of B ? Also find the percentage of students who got marks between 60 and 70. (M.U. 2004) [Ans. : (i) 83, (ii) 72, (iii) 15·74%]

Type X

1. The local authorities in a certain city installed 10,000 electric lamps in the streets of the city. If these lamps have average life of 1000 burning hours with a standard deviation of 200 hours, what number of lamps might be excepted to fail (i) in the first 800 hours, (ii) between 800 and 1200 hours? After what period of burning hours would you expect that (i) 10% of the lamps would fail, (ii) 10% of the lamps would be still burning ? (The area between the ordinates corresponding to S.N.V. $z = 0$ and $z = 1$ is 0·34134 and 80% of the area lies between the ordinate corresponding to $z = \pm 1 \cdot 25$). (M.U. 2004)

[Ans. : (i) 1587, (ii) 6827, (i) 750 hrs., (ii) 1250 hrs.]

2. The distribution of monthly income of 3000 primary teachers confirms to a normal curve with mean equal to ₹ 600 and standard deviation equal to ₹ 100. Find (i) the percentage of teachers having monthly income of more than ₹ 800, (ii) the number of teachers having monthly income of less than ₹ 400, (iii) the highest monthly income among the lowest paid 100 teachers, and (iv) the lowest monthly income of the highest paid 100 teachers. (For a S.N.V. t = the area under the curve between $t = 0$ and $t = \pm 2$ is 0·4772 and that between $t = \pm 1 \cdot 83$ is 0·4667). [Ans. : (i) 2·28%, (ii) 68·4, (iii) ₹ 417, (iv) ₹ 783]

3. In an examination the arithmetic mean of marks scored by 10,000 students is 50 and the standard deviation is 15. Assuming the distribution to be normal find (i) the number of students who scored more than 65 marks, (ii) the number of students who scored marks between 35 and 65, (iii) the limits between which the marks of the middle 50% students lie.

(For a standard normal variate the area under the curve between $t = 0$ and $t = 1$ is 0·3413).
[**Ans. :** (i) 1587, (ii) 6826, (iii) 40-60]

4. In a test of 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average of life of 2040 hours and standard deviation of 60 hours. Estimate the number of bulbs likely to burn for (i) more than 2150 hours. (ii) less than 1950 hours.
(M.U. 2004) [**Ans. :** (i) 67, (ii) 184]

5. Assuming that the diameters of 1000 brass plugs taken consecutively from a Normal distribution with mean 0·7515 cm. and standard deviation 0·0020 cm. how many plugs are likely to be rejected if the approved diameter is $0{\cdot}752 \pm 0.004$ cms ? **(M.U. 2003, 05)**
[**Ans. :** 53]

Type XI

1. If X is normally distributed with unknown mean and standard deviation $\sigma = 4$. Find the mean if (i) not more that 6% values of X are to exceed 30. (ii) not more than 5% values of X are to be less 20.
[**Ans. :** (i) 23·8, (ii) 26·56]

2. The quantity filled in small medicine bottles is normally distributed with standard deviation of 0·04 c.c. If less than 2% of the bottles are to contain less than 4 c.c. medicine, find the mean quantity to which the machine be set up. [**Ans. :** $z_1 = -2{\cdot}05$, $m = 4{\cdot}082$]

3. The qualifying marks for a certain examination are 35 and to secure distinction one has to score more than 74. If 25 % of the students fail, whereas 6·681 % obtained distinction, determine the mean and the standard deviation assuming that the distribution of marks is normal.

(Semi-interquartile range in a normal distribution is two-third of standard deviation and in a normal distribution 43·319 % items lie between the mean and 1·5 times the standard deviation from the mean).
[**Ans. :** $m = 47$, $\sigma = 18$]

4. In a large institution 2·28 % employees receive income below ₹ 4500 and 15·87 % employees receive income above ₹ 7500.

Assuming the income to be normally distributed, find the mean and the S.D.
(M.U. 2006) [**Ans. :** $m = 6500$, $\sigma = 1000$]

5. In a normal distribution 31% items are under 45 and 8% are over 64. Find the mean and standard deviation. Find also the percentage of items lying between 30 and 75.

(Given : For S.N.V. Z area from $Z = 0$ to $Z = 0{\cdot}5$ is 0·19 and that from $Z = 0$ to $Z = 1{\cdot}4$ is 0·42). **(M.U. 1996, 98, 2003, 04)** [**Ans. :** $m = 50$, $\sigma = 10$; 0·957]

6. Of a large group of men 5% are under 60 inches in height and 40% are between 60 and 65 inches. Assuming a normal distribution, find the mean and standard deviation of the distribution. **(M.U. 2004)** [**Ans. :** $\mu = 65{\cdot}43$, $\sigma = 3{\cdot}29$]

7. The distribution of marks in a certain examination was found to be normal with 23% of the candidates scoring above 60 marks and 21% candidates scoring below 40. Find the mean and standard deviation of the distribution.

(Given : For S.N.V. Z area from $Z = 0$ to $Z = 0{\cdot}74$ is 0·27 and area from $Z = 0$ to $Z = 0{\cdot}81$ is 0·29). **(M.U. 1999)** [**Ans. :** $m = 50$ nearly and $\sigma = 13$ nearly]

8. For a normal distribution 30% items are below 45 and 8% items are above 64. Find the mean and variance of the normal distribution.
(M.U. 1998, 2001, 05)
[**Ans. :** $m = 50$, $\sigma = 10$]

9. Of a large group of men 5% are under 60 inches in height and 40% are between 60 and 65 inches in height. Assuming the distribution to be normal, find the mean and variance.
(M.U. 1998) [**Ans. :** $m = 65{\cdot}42$, $\sigma = 3{\cdot}29$]

10. Marks obtained by students in an examination follow a normal distribution. If 30% of students got below 35 marks and 10% got above 60 marks find the mean and the % of students who got marks between 40 and 50. **(M.U. 2003)** [**Ans. :** $m = 42{\cdot}25$, $\sigma = 13{\cdot}81$, 28%]

Type XII

1. The mean and standard deviation (s.d.) of marks obtained by students in Mathematics and Physics are given below.

	Mean	S.D.
Maths	50	10
Physics	55	12

Assuming the marks in the two subjects to be independent normal variates obtain the probability that a students scores marks between 100 and 130 marks in the two subjects taken together.
[**Ans. :** 0·5707]

2. In an examination marks obtained by students in Mathematics, Physics and Chemistry are distributed normally about means 40, 46, 44 with standard deviations 13, 11, 10 respectively. Find the probability of a student securing total marks (i) 180 or above, (ii) 90 or below.
[**Ans. :** (i) 0·0057, (ii) 0·0217]

Type XIII

1. A random variable has a Binomial distribution with $n = 30$ and $p = 0{\cdot}60$. Using normal approximation to Binomial distribution, find the probabilities that it will take (i) the value 14, (ii) a value less than 12.

[**Ans. :** (i) 0·0486. Find $P(13{\cdot}5 \le X \le 14{\cdot}5)$, (ii) 0·0078. Find $P(X \le 11{\cdot}5)$]

2. A random variable has a Binomial distribution with $n = 100$, $p = 0{\cdot}2$. Using normal approximation to Binomial distribution, find the probabilities that it will, take (i) a value less than 15·5, (ii) the value 15.

(**Hint :** For (i) Find $P(X \le 15{\cdot}5)$ take (i) a value less than 16.)

[**Ans. : (i)** 0·1292, **(ii)** 0·0454. For (ii) find $P(14{\cdot}5 \le X \le 15{\cdot}5)$]

3. A sample of 100 items is known to contain 40 defective items. Find in two different ways the probability that the sample will contain exactly 44 defective. **(M.U. 2007)**

[**Ans. :** (i) Binomial Distribution $p = 0{\cdot}40$, $q = 0{\cdot}6$.

$$P(x = 44) = {}^{100}C_{44}\,(0{\cdot}40)^{44}\,(0{\cdot}60)^{56} = 0{\cdot}0576.$$

(ii) Normal Distribution, $m = np = 100\,(0{\cdot}4) = 40$, $\sigma = \sqrt{npq} = 4{\cdot}9$

$$Z = \frac{x - m}{\sigma} \cdot P(43.5 < x < 44.5) = 0.060.]$$

Type XIV (Miscellaneous)

1. Find the probability of getting 30 to 35 diamond cards when cards are drawn with replacement from 100 pack of cards which is well shuffled every time before a card is drawn, using (i) Normal distribution, (ii) Binomial distribution.
[**Ans. :** (i) By Normal Approximation : 0·141, **(ii)** By Binomial : 0·140]

2. Using normal distribution find the probability that in a group of 100 persons there will be 55 males, assuming that the probability of a person being male is 1/2. [Ans. : [illegible]]

3. Using normal distribution find the probability that 65 students will pass in a group of 100 students if the probability of student's passing is 0·6. [Ans. : [illegible]]

4. Two independent random variates X and Y have normal distributions with means 45 and 44 and standard deviations 2 and 2·4 respectively. Find the probability that a randomly chosen pair of values of X and Y will (i) differ by 1·5 or more, (ii) add up to 100 or more. [Ans. : (i) 0·6541, (ii) 0·0025]

5. If X and Y are two independent normal variates both with mean 0 and standard deviations 4, find the probability that the point (X, Y) lies between the lines 4X + 3Y = 6 and 4X + 3Y = 14. [Ans. : 0·1401]

6. The time when a city-bus arrives at a certain bus-stop is distributed normally with a mean of 8·25 a.m. and standard deviation of 4 minutes. What is he least time one should arrive at this bus-stop and still have a probability of 0·99 of catching the bus ?

(**Hint** : If T is the time in minutes past 8 a.m. then T follows N(25, 4). $z = \dfrac{T-25}{4}$ is S.N.V. For probability 0·99/2 = 0·495 table value of z = 2·575. For T least, z = − 2·575. Therefore, T = 14·3 i.e. 15. He should arrive at 8·15.)

7. If X and Y are independent normal variates with the same mean μ but with variances 4 and 48 such that $P(X + 2Y \geq 4) = P(2X - Y \leq 3)$, find μ.

[**Ans.** : $U = X + 2Y$ is $N(3\mu, 10)$

and $V = 2X - Y$ is $N(\mu, 6)$. $P(U \geq 4) = P(V \leq 3)$

$$P\left(\frac{U-3\mu}{10} \geq \frac{4-10\mu}{10}\right) = P\left(\frac{V-\mu}{\sigma} \leq \frac{3-\mu}{6}\right)$$

$$\therefore -\frac{4-10\mu}{10} = \frac{3-\mu}{6} \quad \therefore \mu = \frac{27}{25} \text{]}$$

8. The number of words in a book are normally distributed with mean 800 and standard deviation 50. If 3 pages are selected at random what is the probability that none of them has between 830 and 845 words ? [Ans. : 0·75]

9. A factory turns out tubes by mass production methods. It was found that 20 tubes in a batch of 100 are defective. Find the variance of the defective tube in a batch. Also find the probability that the number of defective tubes in a batch is greater than 30.

[Ans. : var. = 16, $P(X > 30) = 0{\cdot}0062$]

10. Suppose that the length in hours, say X of light bulbs manufactured by a company A are normally distributed with mean 800 hours and standard deviation of 120 hours and those of B with mean 850 hours and standard deviation of 50 hours. One bulb is selected from the production of each company and is burned till "death". Find the probability that the length of life of the bulb from company A exceeds the length of the bulb from the company B at least by 15 hours. (M.U. 2005) [Ans. : 0·2979]

EXERCISE - IV

Theory

1. Define Binomial distribution and state its uses.

2. Explain Binomial distribution.
3. Find the first two moments about origin of Binomial distribution and hence, find mean and variance. (M.U. 2003)
4. State whether the following statements are true or false. Give justification.
   (i) If for a Binomial distribution np is an integer then the mean and mode both are equal to np.
   (ii) For a Binomial distribution mean is always greater than the variance.
   (iii) For a Poisson variate mean and variance are equal.
   (iv) If X is a Poisson variate with $P(X=2) = 3P(X=4) + 45P(X=6)$ then the variance of X is 2.
   [Ans. : All are correct statements.]
5. Show that for a Poisson distribution the mean and variance are same. (M.U. 1997)
6. Prove for the Binomial distribution that $\Sigma p(x) = 1$. (M.U. 1998)
7. For a Binomial distribution, prove that the mean and variance are np and npq. (M.U. 1997, 99)
8. Find the mean and variance of Binomial distribution. (M.U. 1994, 98, 2007)
9. Obtain the recurrence relation for Poisson distribution. (M.U. 2001)
10. State the conditions under which Binomial distribution approximates to Poisson distribution. (M.U. 1999)
11. Find moment generating function of Binomial distribution and hence, find its mean and variance. (M.U. 2002, 04, 15)
12. Define Poisson distribution and state its uses.
13. Find the mean and variance of the following distribution

$$f(x) = \frac{1}{n}, \quad x = 1, 2, 3, \ldots, n \qquad \left[\text{Ans. : (i) } \frac{n+1}{2} \text{ (ii) } \frac{n^2-1}{12}\right]$$

14. Define Poisson distribution. What are its uses ?
15. Define Poisson distribution and obtain it as the limiting case of Binomial distribution. (M.U. 2004)
16. Derive Poisson distribution. (M.U. 2009)
17. Find the mean and variance of Poisson distribution. (M.U. 2001, 03, 04, 06, 07, 09, 10)
18. State and prove additive property of Binomial distribution.
19. State and prove additive property of Poisson distribution.
20. Find the m.g.f. of Binomial distribution.
21. Derive Poisson distribution as a limiting case of binomial distribution. (M.U. 1996)
22. Find the moment generating function of a Poisson distribution. Hence, find its mean and variance. (M.U. 1998, 2004)
23. Define Normal distribution and state its important properties.
24. Define Normal distribution and find its mean and variance. (M.U. 2001, 02)
25. Find m.g.f. of Normal distribution about origin.
26. Define normal distribution and mention some of its important characteristics.

27. Explain importance of normal distribution.
28. Define Normal distribution. Define standard normal variate. Show that for standard normal variate mean is zero and standard deviation is 1. (M.U. 1998)
29. If X and Y are independent random variates having $N(2, 1)$ and $N(3, 2)$ respectively, find the distribution of (i) $2X + 3Y$ and (ii) $2X - 3Y$.
[**Ans. :** (i) $N(13, \sqrt{40})$, (ii) $N(-5, \sqrt{40})$.]
(M.U. 2004)
30. Obtain the m.g.f. of Normal Variate.
31. Obtain the m.g.f. of Standard Normal variate and hence, find its mean and variance. (M.U. 2004)
32. Find the mean and variance of standard normal variate. (M.U. 1998, 2001, 03)
33. Define moment generating function. Obtain the same for Normal distribution. (M.U. 2000, 04)
(M.U. 2003, 07)
34. Obtain moments of Normal distribution. (M.U. 2002)
35. Define Standard Normal Variate. State its properties and uses.
36. Prove that for a normal variate moments of odd power are zero and (M.U. 2003, 04)
$\mu_{2n} = 1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)\, \sigma^{2n}$.
37. What is the moment generating function of a Normal variate ? (M.U. 2004)
Also find the moment generating function of standard normal variates. (M.U. 2004, 05)
38. State the purpose for which Normal distribution is widely used.

☆☆☆

CHAPTER 9

Large Sample Tests

1. Introduction

One of the aims of statistical study of a problem is to be able to predict some characteristics concerning the problem. For example, we may be interested to know the average life or average income of an Indian. To obtain such values, information is collected from the group of the objects of study. In statistical language collecting information for statistical analysis is called **collection of data** and the aggregate of the objects of study is called the **population** or **universe**. There are two methods of collecting data **(1) Census Method** and **(2) Sampling Method**. In census method information is collected from every member of the population and in sampling method information is collected from members of a group collected from the universe by some technique. This group is called a **sample.**

A Parameter and A Statistic

A statistical measure such as mean, standard deviation calculated from the whole universe is called a **parameter**. On the other hand a statistical measure obtained from the values of a sample is called a **statistic**. Since in general, an universe is large, we use sampling method, obtain a sample, calculate from it a statistic and from the statistic we estimate the parameter.

2. Methods of Sampling

There are various methods of selecting a sample from the population. Choice of the method depends upon the information available about the population, nature of data and the object of inquiry. The methods are grouped into two classes. (1) Non-random Sampling or Non-probability Sampling, (2) Random Sampling or Probability Sampling.

(a) Non-probability Sampling

In non-probability sampling an item is included in the sample on the basis of personal judgement of the investigator. We shall study only the following method.

Deliberate Sampling also called Judgement Sampling or Purposive Sampling : The deliberate sampling is highly subjective in nature because the investigator chooses those members of the population in the sample that he thinks are the best representatives of the population. For example, if he wants to investigate expenditure pattern of students of a college having 800 students on roll he will select, say 100 students for the study at his will. There is no other rule for selection except his own will.

As is evident, although the method is very simple and easy to apply, it suffers from the drawback that it is highly subjective. Let us suppose that an investigator is studying wages of workers of a firm. If he is biased towards the workers, he will choose low paid workers and if he is baised towards the owner, he will choose highly paid workers to find out the average salary.

However, this method is useful if immediate results are required. If this is applied with proper care and skill by an experienced statistician, reliable results can be obtained.

(b) Probability Sampling

In probability sampling selection of an item in the sample is done with a certain rule. We shall study only one method.

Simple Random Sampling : Random sampling is a scientific and objective method of selection in which every item has an equal opportunity of being selected in the sample. If sample is random and sufficiently large in size then it is likely to represent population more accurately. The following methods are commonly used for random selection.

(i) Lottery Method : The method is simple and therefore popular for taking a random sample. The numbers or the names of all the members of the population are written on separate pieces of paper of the same size, shape and colour. The pieces are folded in the same manner, mixed up thoroughly in a drum and the required number of pieces are drawn blindly. All this ensures that each member of the population has equal opportunity of being selected in the sample.

Suppose we want to select a sample of 100 boys from a class of 800 boys. If lottery method is to be used the names of all the 800 boys are written on separate pieces of paper, they are folded, mixed and 100 of them are drawn from the drum blindly.

The method is used for drawing the prizes of a lottery and hence the name.

(ii) Table of random numbers : The lottery method is tedious to follow if the population is large. An alternative method is the method of random numbers. In this method all the items are given numbers. Then a book of random numbers is consulted. The book is opened at random and the numbers appearing on the page are read. The items bearing these numbers are included in the sample.

A number of statisticians have constructed tables of random numbers but Tippet's random numbers are commonly used. The table contains 10,400 numbers all in four digits arranged in a random manner. One question can be legitimately asked. Are these numbers really random ? No proof can be given, but the experience has shown that we can rely upon the tables for all practical purposes.

3. Central Limit Theorem

Central limit theorem is a very important theorem in Statistical analysis. We give below the central limit theorem in two forms, one known as Liapounoff's Form and other known as Lindberg-Levy form.

Central Limit Theorem (Liapounoff's Form)

"If $X_1, X_2, \ldots\ldots, X_n$ are independent random variates with $E(X_i) = \mu_i$ and $\mathrm{Var}(X_i) = \sigma_i^2$, $i = 1, 2, \ldots\ldots n$ then under certain general conditions $S_n = X_1 + X_2 + \ldots\ldots + X_n$ is a normal variate with $\mu = \Sigma \mu_i$ and variance $\sigma^2 = \Sigma \sigma_i^2$ an n tends to infinity. (meaning n is large.)

A particular form of the above theorem is of interest to us. The following form of the central limit theorem is known as Lindeberg-Levy theorem.

Central Limit Theorem (Lindberg-Levy Theorem)

"If $X_1, X_2, \ldots\ldots, X_n$ are **independently and identically** distributed random variates such that $E(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$, $i = 1, 2, \ldots\ldots n$ then $S_n = X_1 + X_2 + \ldots\ldots + X_n$ is a normal variate with mean μ and variance $n\sigma^2$ as n tends to infinity.

Corollary : From the above theorem, we get a very important corollary as follows.

"If $\overline{X}$ the mean of the sample of size n, taken from a population having the mean μ and variance σ^2 i.e. of

$$\overline{X} = \frac{X_1 + X_2 + \ldots\ldots + X_n}{n}$$

then $$E(\overline{X}) = \frac{n\mu}{n} = \mu \text{ and } \mathrm{Var}(\overline{X}) = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}.$$

In other words, we get the following important result as a corollary of Central Limit Theorem.

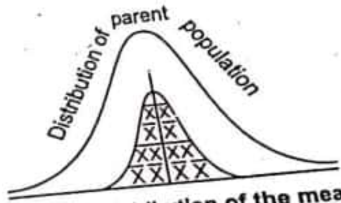
If $\overline{X}$ is the mean of the sample of size n drawn from the population with mean μ and standard deviation σ then $\overline{X}$ is normally distributed with mean μ and standard deviation $\sigma/\sqrt{n}$ i.e

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \text{ is a S.N.V. as } n \to \infty.$$

4. Sampling Distribution of Means (σ known)

If we are required to estimate a population mean, we can take a sample and use the sample mean as the estimate. Although we know that the sample mean will be close to the population mean we do not know what our error will be. We need therefore some way to know how much sample means will deviate from the population mean.

Suppose we take 1000 samples each of size 100 from the population. The difference between (i) the number of samples (1000) and (ii) the number of items in a sample *i.e.* sample size (100) should be carefully noted.



Sampling Distribution of the means

Fig. 9.1

Now for each of these 1000 samples we can calculate a separate mean $\overline{X}$, thus getting 1000 values of $\overline{X}$, the sample mean. Most of these sample means will be close to the population mean although occasionally by chance, we may get a value considerably above or below the population mean.

By constructing a histogram from the sample means, we can obtain a frequency curve. The curve will look very much like a normal curve, a 'thin' one but all the same a normal curve. Moreover, it can be proved that the mean of such a distribution *i.e.* **the mean of the sample means is equal to the true population mean.** This distribution is *called the sampling distribution of the sample means.*

The standard deviation of the mean : We saw above that the sample mean $\overline{X}$ is distributed normally with mean μ where μ is the mean of the population. But what is the standard deviation of the distribution of the sample mean? It can be shown that the standard deviation of the sample means, called **standard error** denoted by $\sigma_{\overline{x}}$ is equal to $\sigma/\sqrt{n}$ where σ is standard deviation of the population and n is the size of the sample.

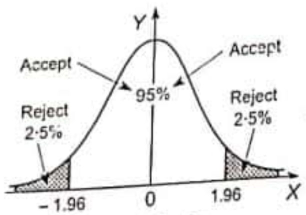
Thus, $\boxed{\sigma_{\overline{x}} = \sigma/\sqrt{n}}$ i.e., $\boxed{V(\overline{x}) = \sigma^2/n}$

Now, look at the whole thing again. **If $\overline{X}$ is the mean of the sample of size n drawn from the population with mean μ and standard deviation σ then $\overline{X}$ is normally distributed with mean μ and standard deviation $\sigma/\sqrt{n}$.**

If we put, $Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$, then Z is a standard normal variate with the mean zero and standard deviation one.

5. Critical Region

The fact that $Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$



Critical Region

Fig. 9.2

is a S.N.V. is highly useful to us in drawing statistical inference. We know that for a S.N.V. 95% area under the curve lies between $-1{\cdot}96$ and $+1{\cdot}96$, 99% between $-2{\cdot}58$ and $+2{\cdot}58$ and 99·73% between $+3$ and -3. In other words, only 5% area under the curve lies beyond $\pm 1{\cdot}96$, 1% beyond $\pm 2{\cdot}58$ and 0·27% beyond ± 3. But the areas are also the probabilities that the S.N.V. Z will exceed these values *i.e.* the probability that Z will exceed 1·96 numerically is 0·05.

$\therefore \quad P(Z < -1{\cdot}06 \text{ or } Z > 1{\cdot}96) = 0{\cdot}05$

$\therefore \quad P(|Z| > 1{\cdot}96) = 0{\cdot}05$

This means the probability that Z will lie in the shaded area is 0·05 - is very small.

Similarly, $P(|Z| > 2{\cdot}58) = 0{\cdot}01$ and $P(|Z| > 3) = 0{\cdot}0027$.

Suppose, we know the population mean is μ, the population standard deviation is σ and we take a sample of size n from this population. Let the mean of this sample be $\overline{X}_1$ and

let $\dfrac{\overline{X}_1 - \mu}{\sigma/\sqrt{n}} = z_1$.

Now, 95 out of 100 values of z_1 will be between $-1{\cdot}96$ and $+1{\cdot}96$. If $z_1 > 1{\cdot}96$ or $z_1 < -1{\cdot}96$, a rare event has taken place because the probability of such an event is very small (0·05). The relative deviation of $\overline{X}_1$ from μ is so significant that it cannot be due to sampling fluctuations alone. Similarly, if $z_1 > 2{\cdot}58$ or $-2{\cdot}58 < z_1$. A very unusual event has taken place because the probability of such an event is very remote (0·01). We again say that the deviation of $\overline{X}_1$, from μ is significant that it cannot be due to sampling fluctuations

alone. The levels marked by probabilities 0·05 or 0·01 which decide the significance of an event are called **levels of significance** and are expressed in percentages as 5% level of significance or 1% level of significance. The corresponding regions are called **critical regions**.

The limits within which we expect z to lie with specified probabilities are called **confidence limits**. Thus, $P(|z| > 1{\cdot}96) = 0{\cdot}05$ the bounding values $\pm 1{\cdot}96$ are the confidence limits, also called **fiducial limits**. This means we are confident that in 95 cases out of 100, the sample mean $\overline{X}$ will be such that z lies between $-1{\cdot}96$ and $1{\cdot}96$.

6. Procedure of Testing A Hypothesis

(1) To set up a hypothesis

A hypothesis is a statement supposed to be true till it is proved false. The hypothesis may be based on previous experience or may be derived theoretically. But a statistical hypothesis in the problems referred to above is a statement about parameters. A hypothesis can be stated in various ways *e.g.* the parameter is equal to a given value or the parameter is greater than the given value or the parameter is not equal to the given value etc.

A statistician generally sets up two hypothesis instead of one. They are called (i) Null Hypothesis and (ii) Alternative Hypothesis. They are set up in such a way that if one is true the other is false.

(i) The Null Hypothesis : The approach here is to set up the hypothesis, or assumption, that there is **no contradiction** between the believed result and the sample result and that the difference therefore can be ascribed solely to chance. Such a hypothesis is called a **null hypothesis**. The object of the test is to see whether the null hypothesis should be rejected or accepted.

(ii) Alternative Hypothesis : For example, if it is assumed that the mean of the weights of a population of boys in a college is 110 lbs., then the null hypothesis will be that the mean of the population is 110 lbs. The null hypothesis is denoted by H_o. In addition to this, one more hypothesis is stated. It is called an **alternative hypothesis** and is denoted by H_a. The alternative hypothesis generally specifies a range of values rather than one value. In the present example we may make an alternative hypothesis that the population mean is not equal to 110 lbs. These are denoted as :

$H_0 : \mu = 110$ lbs. (Null Hypothesis)

$H_a : \mu \neq 110$ lbs. (Alternative Hypothesis)

(2) To set up levels of significance

After setting up the null hypothesis we set up the limits within which we expect the null hypothesis to lie. The idea in setting up the hypothesis is to ensure that the difference between the sample value and the hypothesis should arise due to sampling fluctuations alone. If the difference does not exceed the limits the sample supports the hypothesis and it is accepted. If it exceeds the limits the sample does not support the hypothesis and it is rejected.

The limits are fixed depending upon the accuracy desired. Generally, the limits are fixed such that the probability that the difference will exceed the limits is 0·05 or 0·01. The probability that a random value of a statistic will lie in the critical region is called the **level of significance** (and is expressed in percentage as $\alpha = 5\%$ or 1% level of significance) $\alpha = 5\%$ level of significance means the probability of rejecting a true hypothesis is 0·05 and 1% level of significance means the probability of rejecting a true hypothesis is 0·01.

It should be noted that when a hypothesis is rejected it does mean that the hypothesis is disproved. It only means that the sample value does not support the hypothesis. Similarly, when a hypothesis is accepted it does not mean that the hypothesis is proved. It means that the sample value supports the hypothesis.

(3) Confidence Limits

The limits within which an hypothesis should lie with specified probability are called **confidence limits or fiducial limits.** Generally, the confidence limits are set up with 5% or 1% level of significance. If the sample value lies between the confidence limits the hypothesis is accepted, if it does not, the hypothesis is rejected at the specified level of significance.

(4) Test Statistic

After setting up the hypothesis and after fixing the level of significance we need to calculate a statistic from sample values to test the hypothesis. Depending upon the nature of the data and the nature of the problem we use normal distribution, t-distribution, χ^2-distribution, etc. From the sample we calculate sample mean or sample proportion etc. and from these we calculate the test statistic. **Test statistic can be defined as the statistic calculated on the basis of appropriate probability distribution for testing a hypothesis.** Different probability distributions are used to calculate the test statistic depending upon the size and the nature of data.

The most commonly used test statistics are :-

Z- distribution : If the parent population can be considered at least approximately as normal distribution and if the sample is large ($n \geq 30$), then we calculate the test statistic as

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$

where, $\overline{X}$ is the sample mean, μ is the population mean to be tested, σ is the standard deviation of the population, n is the size of the sample.

t - distribution : If the sample size is small ($n < 30$), then we calculate the test statistic as

$$t = \frac{\overline{X} - \mu}{s / \sqrt{n-1}}$$

where, $\overline{X}$ is the sample mean, μ is the population mean to be tested, s is the standard deviation of the sample, n is the size of the sample (< 30).

χ^2-distribution : To test the goodness of fit, to test independence of attributes etc. we calculate χ^2-statistic as

$$\chi^2 = \Sigma\left(\frac{(O-E)^2}{E}\right)$$

where, O is the observed frequency and E is the expected frequency of an event or a cell.

(5) Selection of test-statistic and its distribution

After setting up the null hypothesis and the alternative hypothesis and after deciding significance level, we construct a test criterion. Depending upon the nature of the population and size of the sample we decide the nature of the statistic and its probability distribution.

The following table shows the conditions under which z-test and t-tests are used. (t-distribution is discussed in the next chapter).

Sample size	Population s.d. σ known	Population s.d. σ unknown
$n \geq 30$ (Any population)	z-test	z-test
$n < 30$ (Population Normal or Approximately Normal)	z-test	t-test

(6) Making Decision

The next step is to compute the value of the statistic from given information and compare it with the table value for the chosen level of significance. The value of the sample statistic which separates the regions of acceptance and rejection, is called the **critical value or significant value** and denoted by z_α (or t_α or χ_α^2) where α denotes the levle of significane. The region of rejection is called the critical region. The critical region may lie on one side or both sides of the sampling distribution of the test statistic. The area of a critical region (which actually gives the probability) in the tails is equal to the level of significance.

(7) Two Tailed and One Tailed Tests

The probability distribution of a sample statistic is a normal distribution. The z-curve is symmetrical as we know and the parts of the curve at the two ends are called the **two tails of the curve.** If the rejection area lies on the two sides *i.e.* on the two tailes the test is called **two tailed tests.** If on the other hand the rejection area lies on one side only the test is called **one tailed test.**

(i) Two Tailed Test : In a two tailed hypothesis the rejection area lies on the two tails of the distribution curve. For example, while testing,

Null Hypothesis $H_0 : \mu = \mu_0$

Alternative Hypothesis : $\mu \neq \mu_0$

(*i.e.* $\mu < \mu_0$ or $\mu > \mu_0$).

We use areas on both sides and hence, it is a two tailed test.

(ii) One Tailed Test : In one tailed test we use area lying on one side of the normal curve. When an alternative hypothesis is one sided for example,

$\mu > \mu_0$ or $\mu < \mu_0$, the test is one sided.

(a) Right Tailed Test : If the region of rejection lies on the right side of the normal curve, the test is called the **right tailed test.** For example, the test for $\mu > \mu_0$ is a right tailed test.

(b) Left Tailed Test : If the region of rejection lies on the left side of the normal curve, the test is called the **left tailed test.** For example, the test $\mu < \mu_0$ is a left tailed test.

(c) *Relation between the critical values for one-tailed test and two tailed test :* Let z_α be the critical value of Z corresponding to the level of significance α in the right-tailed test then

$$P(Z \geq z_\alpha) = \alpha.$$

By symmetry of the standard normal distribution $P(Z < -z_\alpha) = \alpha$

$$\therefore \quad P(|Z| > z_\alpha) = P[(Z < z_\alpha) + (Z > z_\alpha)]$$
$$= P(Z < z_\alpha) + P(Z > z_\alpha)$$
$$= 2\alpha.$$

Thus, the critical value of Z for a one-tailed test (right or left) at level of significance (LOS) α is the same as that for a two-tailed test at the level of significance 2α.

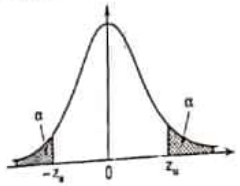


Fig. 9.3

In other words, this means for the same critical value z_α the **critical region** *i.e.* **the level of significance** *i.e.* **percentage of area** for two tailed test is **double** that of one tailed test. This means for the same critical value, say, 1·64 or 2·33 for one tailed test, the critical regions (*i.e.* the probabilities) are 5% and 1% while the critical regions for two-tailed test are 10% and 2% respectively.

Name of the test	Level of significance α			
	1 %	2 %	5 %	10 %
Two tailed test	$\lvert z_\alpha \rvert = \pm 2{\cdot}576$	$\lvert z_\alpha \rvert = \pm 2{\cdot}326$	$\lvert z_\alpha \rvert = \pm 1{\cdot}960$	$\lvert z_\alpha \rvert = \pm 1{\cdot}645$
Right tailed test	$z_\alpha = +2{\cdot}326$	$z_\alpha = +2{\cdot}054$	$z_\alpha = +1{\cdot}645$	$z_\alpha = +1{\cdot}282$
Left tailed test	$z_\alpha = -2{\cdot}326$	$z_\alpha = -2{\cdot}054$	$z_\alpha = -1{\cdot}645$	$z_\alpha = -1{\cdot}282$

The same table is recast from different point of view below.

Table of Critical Values (Normal Distribution)

Name of the test	Critical Value			
	$z_\alpha = 1{\cdot}64$	$z_\alpha = 1{\cdot}96$	$z_\alpha = 2{\cdot}33$	$z_\alpha = 2{\cdot}58$
Two tailed test (LOS)	$\alpha = 10\ \%$	$\alpha = 5\ \%$	$\alpha = 2\ \%$	$\alpha = 1\ \%$
Right tailed test (LOS)	$\alpha = 5\ \%$	$\alpha = 2{\cdot}5\ \%$	$\alpha = 1\ \%$	$\alpha = 0{\cdot}5\ \%$
Left tailed test (LOS)	$\alpha = 5\ \%$	$\alpha = 2{\cdot}5\ \%$	$\alpha = 1\ \%$	$\alpha = 0{\cdot}5\ \%$

It may be noted again that the critical value of z_α **for single-tailed test (left or right) at a level of significance α is the same as critical value of z_α for a two-tailed test at a level of significance 2α as discussed earlier.**

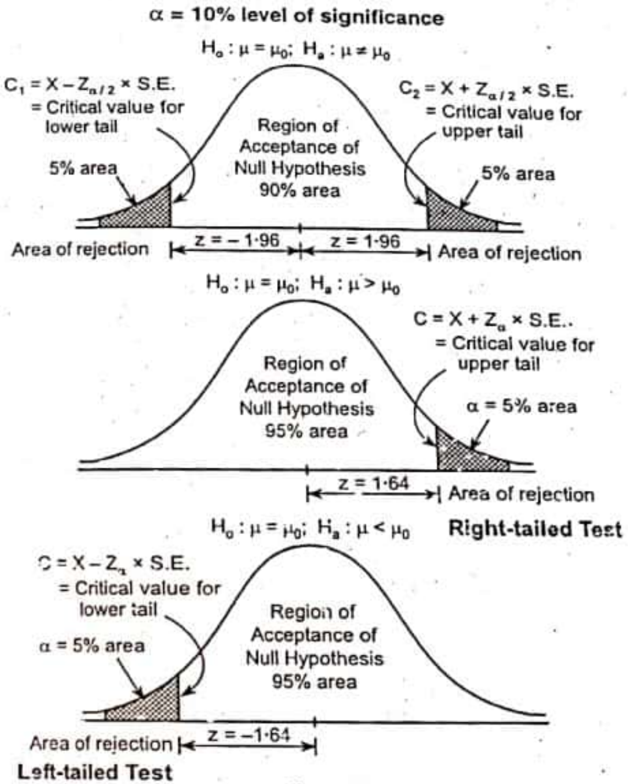


Fig. 9.4

7. Errors in Testing of Hypothesis

When a statistical hypothesis is tested there are only two results either we accept it or we reject it. **We never know** whether the hypothesis is true or false. Hence there arise four possibilities.

(i) A true hypothesis is rejected, or (ii) A true hypothesis is accepted, (iii) A false hypothesis is rejected or (iv) A false hypothesis is accepted.

Obviously, if the outcome of the test leads to the possibilities (i) and (iv) then we are committing an error of (a) rejecting a true hypothesis or (b) accepting a false one. The possibility (a) is called type I error and the possibility (b) is called type II error.

Type I Error

Type I error arises when a true hypothesis is rejected *i.e.* when the difference between the sample value and hypothetical value exceeds the confidence limits. The error can be minimised by increasing the confidence limits. But then because of this the error of type II *i.e.* of accepting a false hypothesis is increased, because we do not know whether the hypothesis is true or false in reality.

Type II Error

Type II error arises when a false hypothesis is accepted *i.e.* when the difference between the sample value and the hypothetical value lies within the limits. The error, can be minimised by decreasing the confidence limits. But then the error of type I *i.e.* of rejecting a true hypothesis is increased, because we again do not know whether the hypothesis is true or false in reality.

Thus, it seems that when error of one type is decreased that of the other is increased. The statistician therefore has to decide, depending upon the nature of the problem, as to which type of error he wishes to avoid. In certain problems type I error may prove to be serious and in certain other problems type II error may prove to be serious. Hence, the levels of significance will have to be decided on considering the practical consequences of the errors of both the types.

The four situations arising in the process of decision making can be described in the form of a table as

	H_0 is accepted	H_0 is rejected
H_0 is true	Correct decision	Type I error
H_0 is false	Type II error	Correct decision

Fig. 9.5

We shall now see how to test a hypothesis. We shall consider only two types of samples (i) Sampling of Variables, (ii) Sampling of Attributes.

8. Sampling of Variables

We have already seen that the sample mean $\overline{X}$ is normally distributed with mean μ and standard deviation $\sigma/\sqrt{n}$ where μ is the mean of the population and σ is the standard deviation of the population. This result is going to help us in testing a hypothesis about the population mean.

9. Testing the Hypothesis that the Population Mean $= \mu$

Suppose that the standard deviation σ of the population is known and let the null hypothesis to be tested be

$$H_0 = \text{mean} = \mu$$

If $\overline{X}$ is the sample mean, then $\overline{X}$ is distributed normally with mean μ and standard deviation $\sigma\sqrt{n}$. Hence, $Z = \dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is a standard normal variate with mean zero and standard deviation one.

Now we take a sample and find its mean $\overline{X}$. If this observed mean $\overline{X}$ is such that $|Z| = \left|\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right|$ (called relative deviation of sample mean $\overline{X}$ from the population mean μ) exceeds 1·96 numerically then a rare event has occurred because the probability of $|Z| = \left(\left|\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right|\right) > 1{\cdot}96$ is very small. As can be seen from the table $P(|Z| > 1{\cdot}66) = 0{\cdot}05$. If from the sample chosen we get the value of $\overline{X}$ such that

$$|Z| = \left|\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right| > 1{\cdot}96$$

we say that relative deviation of $\overline{X}$ from μ is significant and not due to sampling fluctuations alone. We, therefore, reject the hypothesis at 5% level of significance if $|Z| > 1{\cdot}96$. Similarly, we define other levels.

Note ....

If the standard deviation σ of the population is not known and if the sample size n is large (≥ 30) then we use the standard deviation s of the sample in place of σ.

Example 1 : A random sample of 50 items gives the mean 6·2 and variance 10·24. Can it be regarded as drawn from a normal population with mean 5·4 at 5% level of significance? **(M.U. 1996, 2015)**

Sol. : (i) Null Hypothesis H_o : $\mu = 5{\cdot}4$

Alternative Hypothesis H_a : $\mu \neq 5{\cdot}4$

(ii) **Test Statistic :** Since the population S.D. is unknown but sample S.D. s is known and since sample is large

$$Z = \left|\frac{\overline{X}-\mu}{s/\sqrt{n}}\right| = \left|\frac{6{\cdot}2-5{\cdot}4}{\sqrt{10{\cdot}24}/\sqrt{50}}\right| = \left|\frac{0{\cdot}8}{3{\cdot}2/7{\cdot}07}\right| = 1{\cdot}77$$

$\therefore\ |Z| = 1{\cdot}77$

(iii) **Level of significance :** $\alpha = 0{\cdot}05$

(iv) **Critical value :** The value of z_α at 5% level of significance from the table = 1·96.

(v) **Decision :** Since the computed value of $|Z| = 1{\cdot}77$ is less than the critical value $z_\alpha = 1{\cdot}96$, the null hypothesis is accepted.

$\therefore$ The sample is drawn from the population with mean 5·4.

Example 2 : A random sample of 400 members is found to have a mean of 4·45 cms. Can it be reasonably regarded as a sample from a large population whose mean is 5 cms and variance is 4 cms. **(M.U. 2016)**

Sol. : (i) Null Hypothesis H_o : $\mu = 5$

Alternative Hypothesis H_a : $\mu \neq 5$

(ii) **Test Statistic :** $Z = \left|\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right|$

Since we are given standard deviation of the population, we put

$\bar{X} = 4{\cdot}45,\ \mu = 5,\ \sigma = 2,\ n = 400.$

$$\therefore\ Z = \left|\frac{4{\cdot}45 - 5}{2/\sqrt{400}}\right| = \left|\frac{0{\cdot}55}{2/20}\right| = 5{\cdot}5$$

(iii) **Level of Significance :** $\alpha = 0{\cdot}05$.

(iv) **Critical value :** The value of z_α at 5% level of significance from the table = 1·96.

(v) **Decision :** Since the computed value of $Z = 5{\cdot}5$ is greater than the critical value $z_\alpha = 1{\cdot}96$, the null hypothesis is rejected and the alternative hypothesis is accepted.

$\therefore$ The sample is not drawn from the above population.

Example 3 : Can it be concluded that the average life-span of an Indian is more than 70 years, if a random sample of 100 Indians has an average life span of 71·8 years with standard deviation of 8·9 years ? (M.U. 2004)

Sol. : (i) **Null Hypothesis** $H_0 : \mu = 70$ years.

Alternative Hypothesis $H_a : \mu \neq 70$ years

(ii) **Test Statistic :** $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$

Since we are given standard deviation of the sample, we put

$\bar{X} = 71{\cdot}8,\ \mu = 70,\ \sigma = 8{\cdot}9,\ n = 100.$

$$\therefore\ Z = \frac{71{\cdot}8 - 70}{8{\cdot}9/\sqrt{100}} = 2{\cdot}02$$

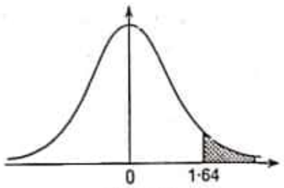


Fig. 9.6

(iii) **Level of Significance :** $\alpha = 0{\cdot}05$. 1.96

(iv) **Critical value :** The value of z_α at 5% level of significance is 1·96.

(v) **Decision :** Since the computed value of $|Z| = 2{\cdot}02$ is greater than the critical value $z_\alpha = 1{\cdot}96$, the null hypothesis is rejected.

$\therefore$ The hypothesis is rejected.

Example 4 : A tyre company claims that the lives of tyres have mean 42,000 kms with S.D. of 4000 kms. A change in the production process is believed to result inbetter product. A test sample of 81 new tyres has a mean life of 42,500 kms. Test at 5% level of significance that the new product is significantly better than the old one. (M.U. 2006, 09)

Sol. : (i) **Null Hypothesis** $H_0 : \mu = 42000$

Alternative Hypothesis $H_a : \mu \neq 42000$

(ii) **Test Statistic :** Since the population S.D. is unknown but sample S.D. s is known and since the sample is large

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{42500 - 42000}{4000/\sqrt{81}} = 1{\cdot}125$$

$\therefore\ |Z| = 1{\cdot}125$

(iii) **Level of Significance :** $\alpha = 0{\cdot}05$

(iv) **Critical value :** The value of z_α at 5% level of significance is 1·96.

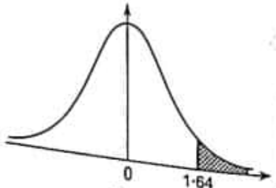


Fig. 9.7

(v) **Decision :** Since the computed value of $|Z| = 1{\cdot}125$ is less than the critical value $z_\alpha = 1{\cdot}96$, the null hypothesis is accepted.

$\therefore$ There is no improvement.

EXERCISE - I

1. A machine is claimed to produce nails of mean length 5 cm. and standard deviation of 0·45 cm. A random sample of 100 nails gave 5·1 cm. as their average length. Does the performance of the machine justify the claim? Mention the level of significance you apply. (M.U. 2014) [**Ans. :** $|Z| = 2{\cdot}2$, No. at 5%]

2. The mean height of random sample of 100 individuals from a population is 160. The S.D. of the sample is 10. Would it be reasonable to suppose that the mean height of the population is 165 ? (M.U. 2005) [**Ans. :** $|Z| = 5$, No]

3. A sample of 50 pieces of certain type of string was tested. The mean breaking strength turned out to be 14·5 pounds. Test whether the sample is from a batch of a string having a mean breaking strength of 15·6 pounds and standard deviation of 2·2 pounds. [**Ans. :** $|Z| = 3{\cdot}53$, No]

4. The mean breaking strength of cables suplied by a manufacturer is 1800 with standard deviation 100. By a new technique in the manufacturing process it is claimed that the breaking strength of the cable has increased. In order to test the claim a sample of 50 cables is tested. It is found that the mean breaking strength is 1850. Can we support the claim at 1 % level of significance. (M.U. 2007, 09) [**Ans. :** $|Z| = 3.54$. Right lait test. Yes]

5. A random sample of size 36 has 53 as mean and sum of squares of deviations from mean is 150. Can this sample be regarded as drawn from the population having 54 as mean? [**Ans. :** $s = 2{\cdot}04$, $|Z| = 2{\cdot}94$, Yes at 0·27%]

6. A distribution with unknown μ has variance 1·5. Use central limit theorem to find how large a sample should be taken from the distribution in order that the probability will be at least 0·95 that the sample will be within 0·5 of the population mean. (M.U. 2004)

(**Hint :** $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} = 1{\cdot}96$ *i.e.*, $\dfrac{0{\cdot}5}{\sqrt{1{\cdot}5}/\sqrt{n}} = 1{\cdot}96$) [**Ans. :** $n = 23$]

7. A random sample of 900 items is found to have a mean of 65·3 cms. Can it be regarded as a sample from a large population whose mean is 66·2 cms. and standard deviation is 5 cms. at 5% level of significance ? (M.U. 2014) [**Ans. :** $|z| = 5{\cdot}4$, No]

8. A machine is set to produce metal plates of thickness 1·5 cms with standard deviation of 0·2 cms. A sample of 100 plates produced by the machine gave an average thickness of 1·52 cms. Is the machine fulfilling the purpose ? [**Ans. :** $|Z| = 1$, Yes]

10. Testing the Difference Between Means

Sometimes we may have two distinct populations and we may want to test whether they have equal means. For example, we may want to know whether the average I.Q. of students of Mumbai University is equal to average I.Q. of students of Pune University. If we actually take samples from two populations, it is unlikely that the two sample means would be identical. Even if they are equal, how are we to know whether the two samples came from populations having equal means or that the two samples came from the same population ? In other words, how can we test the hypothesis that the two populations have equal means? The procedure to test this hypothesis is discussed below.

(1) Distribution of the difference between means

Suppose two populations have equal means. Suppose further we draw a large number of pairs of samples from the two populations. Let us take difference between the pairs of sample means for all these pairs, always subtracting the sample mean of the second population from the sample mean of the first population. If these differences are graphed we would find that the distribution would follow a normal curve. If $\overline{X}_1$ and $\overline{X}_2$ denote the means of the samples drawn from the first and the second population respectively having means μ_1, μ_2 and standard deviations σ_1 and σ_2 and if the sizes of the samples are n_1 and n_2. Then we can prove that **the distribution of the difference between the means $\overline{X}_1 - \overline{X}_2$ is normally distributed with mean $\mu_1 - \mu_2$ and standard deviation given by**

$$s = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

i.e. $$Z = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \text{ is a S.N.V.}$$

Further, under the hypothesis $\mu_1 = \mu_2$ we see that

$$Z = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \text{ is a S.N.V.} \qquad \text{.................... (i)}$$

If the samples are drawn from the same population, so that $\sigma_1 = \sigma_2 = \sigma$, we see from the above expression, that

$$Z = \frac{\overline{X}_1 - \overline{X}_2}{\sigma\sqrt{(1/n_1) + (1/n_2)}} \text{ is a S.N.V.} \qquad \text{.................... (ii)}$$

(i) If samples are large then $\sigma_1^2 = s_1^2$, $\sigma_2^2 = s_2^2$ asymptotically and

$$\text{S.E.} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

(ii) If samles are large and $\sigma_1 = \sigma_2 = \sigma$, then the combined standard deviation of the two samples is given by

$$s^2 = \frac{\Sigma(x_{i1} - \bar{x}_1)^2 + \Sigma(x_{2i} - \bar{x}_2)^2}{n_1 + n_2}$$

But $$\frac{\Sigma(x_{i1} - \bar{x}_1)^2}{n_1} = s_1^2 \text{ and } \frac{\Sigma(x_{2i} - \bar{x}_2)^2}{n_2} = s_2^2$$

$$\therefore \quad s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}$$

s^2 is asymptotially unbiased estimator of σ^2.

$$\therefore \quad \text{S.E.} = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}} \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$= \sqrt{\frac{n_1 s_1^2 + n_2^2 s_2^2}{n_1 + n_2} \cdot \frac{n_1 + n_2}{n_1 n_2}} = \sqrt{\frac{s_1^2}{n_2} + \frac{s_2^2}{n_1}}.$$

(2) Procedure to test the hypothesis : $\mu_1 = \mu_2$:

Suppose we take two samples of size n_1 and n_2 with means $\overline{X}_1$ and $\overline{X}_2$ from the populations with means μ_1, μ_2 and standard deviation σ_1, σ_2. To test the hypothesis that $\mu_1 = \mu_2$.

(i) we calculate $\overline{X}_1 - \overline{X}_2$

(ii) then we calculate the standard error,

$$\text{S.E., } s = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \qquad \text{.................... (1)}$$

(iii) Then we calculate, $|Z| = \left|\frac{\overline{X}_1 - \overline{X}_2}{s}\right|$ and take the decision as explained in 2 (i) above.

Remarks ....

1. If the samples are drawn from the populations with common S.D. σ *i.e.* if $\sigma_1 = \sigma_2 = \sigma$ (known) then standard error

$$\text{S.E., } s = \sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \qquad \text{.................... (2)}$$

In this case we calculate the value of

$$|Z| = \left|\frac{\overline{X}_1 - \overline{X}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}\right|$$

2. If population standard deviations σ_1 and σ_2 are not known ($\sigma_1 \neq \sigma_2$) and if the sample sizes n_1 and n_2 are sufficiently large, (since sample standard deviation s is asymptotically unbiased estimator of standard deviation of the population) we replace σ_1 by s_1 and σ_2 by s_2 and find the S.E. s from,

$$\text{S.E., } s = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \qquad \text{.................... (3)}$$

In this case $|Z| = \left|\frac{\overline{X}_1 - \overline{X}_2}{s}\right|$

3. If the standard deviations of the two populations σ_1 and σ_2 are equal to σ, say, and are unknown and if s_1 and s_2 are the standard deviations of the samples of sizes n_1 and n_2, then

$$\text{S.E., } s = \sqrt{\frac{s_1^2}{n_2} + \frac{s_2^2}{n_1}} \qquad \text{.................... (4)}$$

Carefully note the difference between the denominators of the formulae (3) and (4).

Note ....

Formulae for s given in (1), (2), (3) and (4) are to be taken into consideration for testing the hypothesis $\mu_1 = \mu_2$ as well as for estimating the interval of $\mu_1 - \mu_2$.

Testing of Hypothesis

Example 1 : The means of two samples of sizes 1000 and 2000 respectively are 67·50 and 68·0 inches. Can the samples be regarded as drawn from the same population of standard deviation 2·5 inches ? **(M.U. 2015)**

σ p

Sol. : (i) Null Hypothesis $H_o : \mu_1 = \mu_2$

Alternative Hypothesis $H_a : \mu_1 \neq \mu_2$

(ii) **Calculation of Statistic :** $\bar{X}_1 - \bar{X}_2 = 67{\cdot}5 - 68{\cdot}0 = -0{\cdot}5$

Since S.D. of the population is known,

$$\text{S.E. } s = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$= (2{\cdot}5)\sqrt{\frac{1}{1000} + \frac{1}{2000}} = 0{\cdot}097$$

$$\therefore \quad Z = \frac{\bar{X}_1 - \bar{X}_2}{s \cdot E} = -\frac{0{\cdot}5}{0{\cdot}097} = -5{\cdot}15 \qquad \therefore \quad |Z| = 5{\cdot}15.$$

(iii) **Level of significance :** $\alpha = 0{\cdot}27\%$

(iv) **Critical Value :** The value of z_α at 0·27% level of significance from the table is 3.

(v) **Decision :** Since the computed value of $|Z| = 5{\cdot}15$ is greater than the critical value $z_\alpha = 3$, the hypothesis is rejected.

$\therefore$ The samples cannot be regarded as drawn from the same population. σ_1

Example 2 : The average of marks scored by 32 boys is 72 with standard deviation 8 while that of 36 girls is 70 with standard deviation 6. Test at 1% level of significance whether the boys perform better than the girls. σ_2 **(M.U. 2004, 09, 10, 15, 16)**

Sol. : (i) Null Hypothesis $H_o : \mu_1 = \mu_2$

Alternative Hypothesis $H_a : \mu_1 \neq \mu_2$

(ii) **Calculation of Statistic :** $\bar{X}_1 - \bar{X}_2 = 72 - 70 = 2$

$$\text{S.E.} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{64}{32} + \frac{36}{36}} = \sqrt{3}$$

$$\therefore \quad Z = \frac{\bar{X}_1 - \bar{X}_2}{\text{S.E.}} = \frac{2}{\sqrt{3}} = 1{\cdot}15 \qquad \therefore \quad |Z| = 1{\cdot}15$$

(We assume that the standard deviations σ_1 and σ_2 of the two populations are not equal.)

(iii) **Level of significance :** $\alpha = 1\%$

(iv) **Critical Value :** The value of z_α at 1% level of significance from the table is 2·58.

(v) **Decision :** Since the computed value of $|Z| = 1{\cdot}15$ is less than the critical value $z_\alpha = 2{\cdot}56$, the hypothesis is accepted.

$\therefore$ Boys do not perform better than the girls.

Example 3 : Test the significance of the difference between the means of two normal population with the same standard deviation from the following data.

	Size	Mean	S.D.
Sample I	100	64	6
Sample II	200	67	8

Sol. : (i) Null Hypothesis $H_o : \mu_1 = \mu_2$

Alternative Hypothesis $H_a : \mu_1 \neq \mu_2$

(ii) **Calculation of Statistic :** $\bar{X}_1 - \bar{X}_2 = 67 - 64 = 3$

Since the standard deviations of the two populations are equal but unknown

$$\text{S.E } s = \sqrt{\frac{s_1^2}{n_2} + \frac{s_2^2}{n_1}} = \sqrt{\frac{36}{200} + \frac{64}{100}}$$

$$= \sqrt{0{\cdot}18 + 0{\cdot}64} = \sqrt{0{\cdot}82} = 0{\cdot}91$$

$$\therefore \quad Z = \frac{\bar{X}_1 - \bar{X}_2}{s} = \frac{3}{0{\cdot}8} = 3{\cdot}3 \qquad \therefore \quad |Z| = 3{\cdot}3$$

(iii) **Level of Significance :** $\alpha = 5\ \%$

(iv) **Critical value :** z_α at 5% LOS is 1·96.

(v) **Decision :** Since the computed value of $|Z| = 3{\cdot}3$ is greater than the critical value $z_\alpha = 1{\cdot}96$, the null hypothesis is rejected.

$\therefore$ The samples do not support the hypothesis that the two populations have the same mean although they may have the same standard deviation.

Example 4 : Two samples drawn from two different populations gave the following results.

	Size	Mean	S.D.
Sample I	125	340	25
Sample II	150	380	30

Test the hypothesis at 5% LOS that the difference of the means of the two populations is 45.

Sol. : (i) Null Hypothesis $H_o : \mu_1 - \mu_2 = 45$

Alternative Hypothesis $H_a : \mu_1 \neq \mu_2$

(ii) **Calculation of Statistic :** $\bar{X}_1 - \bar{X}_2 = 40, \quad \mu_1 - \mu_2 = 35$

$$S.E\ s = \sqrt{\frac{s_1^2}{n_2} + \frac{s_2^2}{n_1}} = \sqrt{\frac{25^2}{125} + \frac{30^2}{150}} = 3.32$$

$$\therefore\ Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s} = \frac{40 - 35}{3.32} = 1.5 \qquad \therefore\ |Z| = 1.5$$

(iii) **Level of Significance :** $\alpha = 5\ \%$

(iv) **Critical value :** z_α at 5% LOS is 1·96.

(v) **Decision :** Since the computed value of $|Z| = 1.5$ is less than the critical value $z_\alpha = 1.96$.

$\therefore$ The data supports the hypothesis that the difference between the means of the two populations may be 45.

Example 5 : Two populations have the same mean but the standard deviation of one is twice that of the other. Show that in samples, each of size 500, drawn under simple random conditions the difference of the means, in all probability, will not exceed 0.3σ, where σ is the smaller standard deviation. **(M.U. 2007)**

Sol. : We have $\mu_1 = \mu, \mu_2 = \mu$; $\sigma_1 = \sigma$, $\sigma_2 = 2\sigma$; $n_1 = 500$, $n_2 = 500$.

$$\text{S.E. } s = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{\sigma^2}{500} + \frac{4\sigma^2}{500}}$$

$$= \sqrt{\frac{5\sigma^2}{500}} = \sqrt{\frac{\sigma^2}{100}} = \frac{\sigma}{10}$$

We want $|\bar{X}_1 - \bar{X}_2| < 0.3\sigma$

Dividing both sides by S.E.,

$$\left|\frac{\bar{X}_1 - \bar{X}_2}{\sigma/10}\right| < \frac{0.3\sigma}{\sigma/10} \qquad [\because\ \sigma \text{ is positive}]$$

$$\therefore\ \left|\frac{\bar{X}_1 - \bar{X}_2}{\text{S.E.}}\right| < 3$$

For S.N.V. $Z = 3$, 99·73 % area lies under the curve. The event is almost impossible. Hence, for $\mu_1 = \mu_2$; $|\bar{X}_1 - \bar{X}_2|$ in all probability will be $< 0.3\sigma$.

EXERCISE - II

1. A sample of 200 fish of a particular kind taken at random from one end of a lake had mean weight of 20 lbs. and standard deviation of 2 lbs. At the other end of the lake, a sample of 80 fish of the same kind had mean weight of 20·5 lbs. and standard deviation of 2 lbs. also. Is the difference between the mean weights significant ? **[Ans. : $|Z| = 1.89$, No]**

2. A man buys 100 electric bulbs of each of two well-known makes taken at random from stock for testing purpose. He finds that 'make A' has a mean life of 1300 hours with a S.D. of 82 hours, and 'make B' has a mean life of 1248 hours with S.D. of 93 hours. Discuss the significance of these results. **[Ans. : $|Z| = 4.19$, Significant]**

3. The mean consumption of food grains among 400 sampled middle class consumers is 380 grams per day per person with a standard deviation of 120 grams. A similar sample survey of 600 working class consumers gave a mean of 410 grams with standard deviation of 80 grams. Are we justified in saying that the the difference between the averages orf the two classcs is 40 ? Use 5% level of significance. **[Ans. : $|z| = 5.86$, No]**

4. Two groups consisting of 400 and 500 persons have mean heights 68·5 inches and 66·1 inches and variance 6·4 and 6·0 respectively. Examine whether the difference is significant. **[Ans. : $|z| = 14.54$, Yes]**

5. A potential buyer of light bulbs bought 50 bulbs each of 2 brands. Upon testing the bulbs, he found that brand A had a mean life of 1,282 hours with a S.D. of 80 hours; brand B had a mean life of 1,208 hours with S.D. of 94 hours. Can the buyer be quite certain that the means of the two brands do differ in quality ? **[Ans. : $|z| = 4.24$, Yes]**

6. Average height of a sample of 6400 persons from one population was found to be 67·85 inches with a S.D. of 2·56 inches. Average height of a sample of 1600 persons from another population was found to be 68 inches with a S.D. of 2·52 inches. Is the difference between the mean heights of the two samples significant ? **[Ans. : $|z| = 2.12$, No at 1%]**

7. Intelligence tests of two groups of boys and girls obtained from two normal populations having the same standard deviations gave the following results.

	Mean	S.D.	No.
Girls	84	10	121
Boys	81	12	81

Is the difference between the means significant ? **[Ans. : $|Z| = 1.93$, No]**

8. The mean life of a sample of 100 electric light bulbs was found to be 1456 hours with S.D. 400. A second sample of 225 bulbs chosen from a different batch showed a mean life of 1400 hours with standard deviation of 144 hours. Assuming that the two populations have same standard deviation find, if there any significant difference between the mean of two batches ? **[Ans. : $|Z| = 1.84$, No]**

EXERCISE - III

Theory

1. Distinguish between :-
   (i) Census Survey and Sample Survey.
   (ii) Sample and population
   (iii) Standard deviation and standard error.
   (iv) Null Hypothesis and Alternative Hypothesis. *(M.U. 2007)*
   (v) Type I error and type II error.
   (vi) Point estimation and interval estimation. *(M.U. 2007)*
   (vii) Statistic and parameter
   (viii) One tailed test and two tailed test.
2. Explain the following terms *(M.U. 1998)*
   (i) Statistic and Parameter. (ii) Test Statistic.
   (iii) Level of Significance. (iv) Null of Hypothesis.

3. What are the principles of sampling ?
4. What are the principles of sampling ?
5. Explain simple random sampling method. State its merits and demerits. (M.U. 2005)
6. Explain the use of random numbers in selection of a sample. Indicate the method used and the principles adopted. (M.U. 1997)
7. Define the following terms
   (i) Sampling, (ii) Standard Error, (iii) Level of Significance. (M.U. 2002)
8. Define finite population correction factor.
9. If $\overline{X}$ is the mean of a random sample of size n taken from the population of size n having mean μ and variance σ^2 then, prove that the mean of $\overline{X}$ is m and the variance of $\overline{X}$ is $\sigma / \sqrt{n}$.
10. Prove that the sample mean $\bar{x}$ is an unbiased estimator of the population mean μ. (M.U. 2005)
11. Prove that sample variance $s^2 = \dfrac{\Sigma(x_i - \bar{x})^2}{n}$ is not an unbiased estimator of population variance σ^2.
12. If $S^2 = \dfrac{1}{n-1}\Sigma(x_i - \bar{x})^2$ then prove that S^2 is an unbiased estimator of σ^2.
13. Explain the following terms : (i) Critical Region, (ii) Fiducial Limits. (M.U. 2006)
14. Write short note on
    (i) Null Hypothesis and alternative hypothesis.
    (ii) Type I error and type II error. (M.U. 2006)
    (iii) Level of significance and Confidence interval. (M.U. 2006)
    (iv) One tailed and two tailed tests. (M.U. 2004, 07)
15. Sumarise various steps in testing a statistical hypothesis in a systematic manner. (M.U. 2004)
16. Describe briefly the steps used in testing of a statistical hypothesis. (M.U. 2003)
    (See § 6, page 9-5)
17. State Central Limit Theorem.
18. Derive the formulae for sample size for testing (i) mean and (ii) proportion. (M.U. 2001)
19. Describe the test of significance of difference between sample mean and population mean. (M.U. 2004)

☆☆☆

CHAPTER 10

Small Sample Tests

1. Introduction

In the previous chapter we have seen that if the samples are large (≥ 30) then the sampling distribution of a statistic is normal. But if the samples are small (< 30) then the above result does not hold good and for estimation of the parameter as well as for testing a hypothesis we cannot use the above methods.

If we take a large number of samples of small (< 30) size, calculate the mean of each sample, plot the frequencies and obtain the frequency curve we will find that the resulting sampling distribution of the mean is not normal but is the student's t-distribution.

2. Student's t - distribution

Theoretical work on t - distribution was done by Irish Statistician W. S. Gosset. W. S. Gosset was working with Guinness Brewery in Dublin which did not allow its employees to publish their reserach work under their own names. So he adopted the pen-name "Student" and published his research work under that name in early period of 20th century. Hence, this distribution is known as **Student's t - distribution** or simply **t - distribution.**

The t - distribution is used when **(i) the sample size is 30 or less and (ii) population standard deviation is not known.**

The "t - statistic" is defined as

$$t = \frac{\overline{X} - \mu}{S / \sqrt{n}} \quad \text{where,} \quad S = \sqrt{\frac{\Sigma(X - \overline{X})^2}{n-1}}$$

The curve is given by

$$y = C\left(1 + \frac{t^2}{\nu}\right)^{-\frac{(\nu+1)}{2}} \qquad \text{where,} \quad t = \frac{\overline{X} - \mu}{S / \sqrt{n}}$$

C = constant required to make the area under the curve unity,

$\nu = n - 1$, the number of degrees of freedom.

The t - distribution has been derived mathematically **under the hypothesis that parent population is distributed normally.**

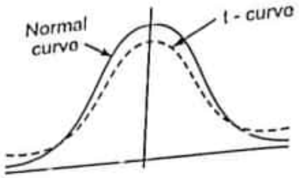
3. Properties of t-distribution

(i) As the normal curve, this curve also extends from $-\infty$ to $+\infty$.

(ii) The constant C depends upon the ν (pronounced as 'nu'), the degrees of freedom.

(iii) Like the normal distribution, the t-distribution also is symmetrical and has a mean zero.

(iv) The variance of t-distribution is greater than unity and approaches unity as the number of degrees of freedom and therefore the size of the sample becomes large.



1. Assumptions for t - test : (i) Samples are drawn from **normal population** and they are random.

(ii) For testing the equality of means of two populations their **variances are assumed to be equal.**

2. t - table : An interval estimate of the population mean is given by $\overline{X} \pm t\,\hat{\sigma}_{\overline{X}}$. It should be noted that t table gives the probability that population parameter will **not** lie within the desired confidence interval *i.e.* the parameter will lie **outside** the confidence interval. For making an estimate, say, at 95% confidence level we consult the column under the head 0·05 (100% – 95% = 5% = 0·05) of the t-table. Similarly for 98%, 99% confidence level we consult the column under the head 0·02 and 0·01.

3. Degree of freedom : Degree of freedom means the number of values we are free to choose. Suppose the sum of *three* numbers is 15. How many numbers we are free to choose such that the sum is 15? Certainly not all the three. We can choose two numbers at our will but the third will be given by 15 – (sum of the two chosen numbers). Thus, we are free to choose only two numbers. Hence, the degrees of freedom here are two.

4. Uses of t - distribution : The t - distribution has a wide number of applications. Some important of them are :

(i) To estimate the population mean μ from the sample mean $\overline{X}$.

(ii) To test the hypothesis that the population mean is μ with the help of the sample mean $\overline{X}$.

(iii) To test the hypothesis that two populations have the same mean with the help of the sample means.

4. Distribution of Sample Mean

If $\overline{X}$ is the sample mean and μ is the population mean then

$$t = \frac{\overline{X} - \mu}{S/\sqrt{n}} \quad \text{where, } S^2 = \frac{\Sigma(X_i - \overline{X})^2}{n-1}$$

follows Student's t - distribution with $n - 1$ degrees of freedom.

Remark ....

We know that the sample variance is given by

$$s^2 = \frac{\Sigma(X_i - \overline{X})^2}{n} \qquad \therefore\ ns^2 = \Sigma(X_i - \overline{X})^2 = (n-1)S^2$$

$$\therefore\ \frac{S^2}{n} = \frac{s^2}{(n-1)} \qquad \therefore\ \frac{S}{\sqrt{n}} = \frac{s}{\sqrt{n-1}}$$

Hence, $t = \dfrac{\overline{X} - \mu_o}{S/\sqrt{n}}$ .... (1) or $t = \dfrac{\overline{X} - \mu_o}{s/\sqrt{n-1}}$ .... (2)

We use (2) when sample standard deviation s is given and use (1) when sample values $x_1, x_2, \ldots, x_n$ are given or when $S^2 = \Sigma(X_i - \overline{X})^2/(n-1)$ is obtained or an unbiased estimates of standard deviation σ is given.

Thus, we have the following results when sample is small.

For Small Samples

1. If the standard deviation σ of the parent population is known, then $Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$ is a S.N.V.

2. If the standard deviation of the parent population is not known and the parent population is normal, then $t = \dfrac{\overline{X} - \mu}{S/\sqrt{n}}$ is a t - distribution where $S^2 = \dfrac{\Sigma(X_i - \overline{X})^2}{n-1}$.

3. If the standard deviation of the parent population is not known and if the parent population is normal, then $t = \dfrac{\overline{X} - \mu}{s/\sqrt{n-1}}$ is a t - distribution where $s^2 = \dfrac{\Sigma(X_i - \overline{X})^2}{n}$.

[Note the cases (2) and (3) carefully.]

5. Testing the Hypothesis that the Population Mean is μ

To test the hypothesis that the population mean is μ when the sample is small we follow the steps as for large sample but use the t-distribution instead of normal distribution and use the unbiased estimator $s/\sqrt{n-1}$ of the standard deviation of the population and not $s/\sqrt{n}$.

Example 1 : A soap manufacturing company was distributing a particular brand of soap through a large number of retail soaps. Before a heavy advertisement campaign, the mean sales per week per shop was 140 dozens. After the campaign a sample of 26 shops was taken and the mean sale was found to be 147 dozens with standard deviation of 16. Can you consider the advertise-ment effective ?

Sol. : (i) The null Hypothesis H_o : $\mu = 140$

Alternative Hypothesis H_a : $\mu \neq 140$

(ii) Calculation of test statistic : Since the sample is small and S. D. of the population is not known, we use t - distributions.

$$t = \frac{\overline{x} - \mu}{s/\sqrt{n-1}} = \frac{147 - 140}{16/\sqrt{26-1}} = \frac{7}{3 \cdot 2} = 2 \cdot 19 \qquad \therefore\ |t| = 2 \cdot 19.$$

(iii) Level of significance : $\alpha = 0 \cdot 05$.

(iv) Critical value : The value of t_α for 5% level of significance and degrees of freedom $\nu = 26 - 1 = 25$ from the table is $t = 2 \cdot 06$.

(v) Decision : Since the computed value of $|t| = 2 \cdot 19$ is greater than the critical value $t_\alpha = 2 \cdot 06$ the null hypothesis is rejected.

$\therefore$ The advertisement may have changed the average sales.

Example 2 : A random sample of size 16 from a normal population showed a mean of 103·75 cm. and sum of squares of deviations from the mean 843·75 cm^2. Can we say that the population has a mean of 108·75 cm.? **(M.U. 2004)**

Sol. : We first calculate sample standard deviation

$$s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n} = \frac{843{\cdot}75}{16} = 52{\cdot}73$$

(i) **The null hypothesis** $H_o : \mu = 108{\cdot}75$

Alternative hypothesis $H_a : \mu \neq 108{\cdot}75$

(ii) **Calculation of test statistic :**

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{103{\cdot}75 - 108{\cdot}75}{\sqrt{52{\cdot}74}/\sqrt{15}} = -\frac{5}{1{\cdot}875} = -2{\cdot}67 \qquad \therefore \ |t| = 2{\cdot}67$$

(iii) **Level of significance :** $\alpha = 0{\cdot}05$.

(iv) **Critical value :** The value of t_α for 5% level of significance and degrees of freedom $\nu = 16 - 1 = 15$ from the table is 2·131.

(v) **Decision :** Since the computed value of $|t| = 2{\cdot}67$ is greater than the table value $t_\alpha = 2{\cdot}131$, the null hypothesis is rejected.

$\therefore$ We cannot say that the population mean is 108·75.

Example 3 : Nine items of a sample had the following values

45, 47, 50, 52, 48, 47, 49, 53, 51

Does the mean of 9 items differ significantly from the assumed population mean 47·5? **(M.U. 2002, 10)**

Sol. : We first calculate sample mean $\bar{X}$ and sample standard deviation s^2 (by assumed mean method).

Calculation of $\bar{X}$ and s^2

X	45	47	50	52	48	47	49	53	51	Sum
$d_i = x_i - 48$	−3	−1	2	4	0	−1	1	5	3	10
$d_i^2 = (x_i - 48)^2$	9	1	4	16	0	1	1	25	9	66

$$\bar{X} = a + \frac{\Sigma d_i}{n} = 48 + \frac{10}{9} = 49{\cdot}11$$

$$\Sigma(X_i - \bar{X})^2 = \Sigma d_i^2 - \frac{(\Sigma d_i)^2}{n} = 66 - \frac{100}{9} = 54{\cdot}89$$

$$\therefore \ s^2 = \frac{\Sigma(X_i - \bar{X})^2}{n} = \frac{54{\cdot}89}{9} = 6{\cdot}099$$

(i) **The null hypothesis** $H_o : \mu = 47{\cdot}5$

Alternative hypothesis $H_a : \mu \neq 47{\cdot}5$

(ii) **Calculation of test statistic :** Since the sample size is small, we use t-distribution.

$$t = \frac{\bar{X} - \mu}{s/\sqrt{n-1}} = \frac{49{\cdot}11 - 47{\cdot}5}{\sqrt{6{\cdot}099}/\sqrt{8}} = 1{\cdot}84 \qquad \therefore \ |t| = 1{\cdot}84.$$

(iii) **Level of significance :** $\alpha = 0{\cdot}05$.

(iv) **Critical value :** The value of t_α at 5% level of significance for $\nu = 9 - 1 = 8$ degrees of freedom is 2·306.

(v) **Decision :** Since the calculated value of $|t| = 1{\cdot}84$ is less than the table value $t_\alpha = 2{\cdot}306$, the null hypothesis is accepted.

$\therefore$ The mean of nine items does not differ significantly from assumed population mean 47·5.

Example 4 : Ten individuals are chosen at random from a population and their heights are found to be 63, 63, 64, 65, 66, 69, 69, 70, 70, 71 inches. Discuss the suggestion that the mean height of the universe is 65 inches. **(M.U. 2003, 15, 16)**

Sol. : We first calculate sample mean $\bar{X}$ and sample standard deviation s^2.

Calculation of $\bar{X}$ and s^2

X	63	63	64	65	66	69	69	70	70	71	Sum
$d_i = x_i - 66$	−3	−3	−2	−1	0	3	3	4	4	5	10
$d_i^2 = (x_i - 66)^2$	9	9	4	1	0	9	9	16	16	25	98

$$\bar{X} = a + \frac{\Sigma d_i}{n} = 66 + \frac{10}{10} = 67$$

$$\Sigma(X_i - \bar{X})^2 = \Sigma d_i^2 - \frac{(\Sigma d_i)^2}{n} = 98 - \frac{100}{10} = 88$$

$$\therefore \ s^2 = \frac{\Sigma(X_i - \bar{X})^2}{n} = \frac{88}{10} = 8{\cdot}8$$

(i) **The null hypothesis** $H_o : \mu = 65$

Alternative hypothesis $H_a : \mu \neq 65$

(ii) **Calculation of test statistic :**

$$t = \frac{\bar{X} - \mu}{s/\sqrt{n-1}} = \frac{67 - 65}{\sqrt{8{\cdot}8}/\sqrt{9}} = \frac{6}{2.97} = 2{\cdot}02 \qquad \therefore \ |t| = 2{\cdot}02.$$

(iii) **Level of significance :** $\alpha = 0{\cdot}05$.

(iv) **Critical value :** The value of t_α at 5% level of significance for $\nu = 10 - 1 = 9$ degrees of freedom is 2·6.

(v) **Decision :** Since the calculated value of $t = 2{\cdot}02$ is less than the table value $t_\alpha = 2{\cdot}6$, the null hypothesis is accepted.

$\therefore$ The mean height of the universe may be 65 inches.

Example 5 : Tests made on breaking strength of 10 pieces of a metal wire gave the following results.

578, 572, 570, 568, 572, 570, 570, 572, 596 and 584 in kgs.

Test if the breaking strength of the metal wire can be assumed to be 577 kg. ? **(M.U. 2002, 04, 05, 06, 15)**

Sol. : First we calculate the mean and standard deviation s^2 of the sample.

Calculation of $\overline{X}$ and s^2

X	578	572	570	568	572	570	570	572	596	584	Sum
$d_i = x_i - 580$	−2	−8	−10	−12	−8	−10	−10	−8	6	4	−48
$d_i^2 = (x_i - 580)^2$	4	64	100	144	64	100	100	64	256	16	864

$$\overline{X} = a + \frac{\Sigma d_i}{n} = 580 + \frac{48}{10} = 575.2$$

$$\Sigma(X_i - \overline{X})^2 = \Sigma d_i^2 - \frac{(\Sigma d_i)^2}{n} = 912 - \frac{(-48)^2}{10} = 633.6$$

$$\therefore \quad s^2 = \frac{\Sigma(X_i - \overline{X})^2}{n} = \frac{681.6}{10} = 68.16$$

(i) **The null hypothesis** $H_o : \mu = 577$

Alternative hypothesis $H_a : \mu \neq 577$

(ii) **Calculation of test statistic**

$$t = \frac{\overline{X} - \mu}{s/\sqrt{n-1}} = \frac{575.2 - 577}{\sqrt{68.16}/\sqrt{10-1}} = -0.65 \qquad \therefore \ |t| = 0.65.$$

(iii) **Level of significance :** $\alpha = 0.05$.

(iv) **Critical value :** The value of t_α at 5% level of significance for $v = 10 - 1 = 9$ degrees of freedom is 2·25.

(v) **Decision :** Since the calculated value of $|t| = 0.65$ is less than the table value $t_\alpha = 2.25$, the null hypothesis is accepted.

$\therefore$ The mean is 577.

EXERCISE - I

1. Vanaspati oil is marketed in tins of 10 kgs. A sample of 20 tins showed the mean weight as 9·5 kg. with standard deviation of 3 kgs. Does the sample justify the claim that the mean weight is 10 kg. Mention the level of significance, you use.

[**Ans. :** $t = 0.726$, Yes at 5%]

2. A random sample of 16 observations has mean 103.75 cm. The sum of the squares of the deviations from the mean is 843.75 cm. Can this sample be regarded as coming from the population having 108.75 cm as the mean? ($t_{15} = 2.131$ and $t_{16} = 2.120$ at 5% level.)

(M.U. 2004) [**Ans. :** $t = 2.67$, No]

3. A machine is designed to pack edible oil in tins of 5 kgs. A random sample of 10 tins gave the average weight of a tin as 4·8 kg. and standard deviation of 2 kgs. Is the machine working properly ? Value of t for 9 degrees of freedom at 5% level of significance is 2·262.

[**Ans. :** $t = 3$, No]

4. A company supplies tooth-paste in a packing of 100 gm. A sample of 10 packings gave the following weights in gms.

100·5, 100·3, 100·1, 99·8, 99·7, 99·7, 100·3, 100·4, 99·2, 99·3.

Does the sample support the claim of the company that the packing weighs 100 gms ?

[**Ans. :** $\overline{X} = 99.93$, $s^2 = 0.2112$, $t = 0.48$, Yes]

5. A machine is designed to produce insulating washers for electrical devices of average thickness of 0·025 cms. A random sample of 10 washers was found to have average thickness of 0·024 cms., with standard deviation of 0·002 cms. Test the significance of the deviation.

(M.U. 2004) [**Ans. :** $|t| = 1.5$; Accept H_o]

6. A certain drug administered to 12 patients resulted in the following change in their Blood Pressure.

5, 2, 8, −1, 3, 0, 6, −2, 1, 5, 0, 4

Can we conclude that the drug increases the blood pressure ? **(M.U. 2005, 09, 10, 14)**

[**Ans. :** $t = 2.89$, One tailed test $\overline{X} > \mu$ is to be accepted. There is increase in B.P.]

6. Testing the Difference Between Means

We have seen how to test the difference between the means of two samples when they are large. We shall now see how to test the difference between the means of two samples when the samples are small.

(a) Case I : Independent Samples

If the sample size $(n_1 + n_2 - 2)$ is small, an unbiased estimate of the common population standard deviation σ is obtained by pooling the data with the help of the following formula.

$$\boxed{s_p = \sqrt{\frac{\Sigma(x_{i1} - \overline{x}_1)^2 + \Sigma(x_{i2} - \overline{x}_2)^2}{n_1 + n_2 - 2}}} \qquad \text{.......... (1)}$$

If we are given **unbiased standard deviations** of the two samples,

$$S_1 = \sqrt{\frac{\Sigma(x_{i1} - \overline{x}_1)^2}{n_1 - 1}} \quad \text{and} \quad S_2 = \sqrt{\frac{\Sigma(x_{i2} - \overline{x}_2)^2}{n_2 - 1}}$$

then, we get, from (1)

$$\boxed{s_p = \sqrt{\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}}} \qquad \text{.......... (2)}$$

On the other hand, if we are given **standard deviations of the two samples**

$$s_1 = \sqrt{\frac{\Sigma(x_{i1} - \overline{x}_1)^2}{n_1}} \quad \text{and} \quad s_2 = \sqrt{\frac{\Sigma(x_{i2} - \overline{x}_2)^2}{n_2}}$$

then, we get, from (1)

$$\boxed{s_p = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}}} \qquad \text{.......... (3)}$$

The standard error of the difference between the two means is then given by

$$\boxed{\text{S.E.} = s_p \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

The test statistic is then computed as

$$\boxed{t = \frac{\overline{X}_1 - \overline{X}_2}{\text{S.E.}}}$$

The statistic t so computed follows Student's t - distribution.

Note ...

It is assumed that the two populations have the same standard deviation σ. If we cannot assume that $\sigma_1 = \sigma_2$, then the problem is beyond the scope of this book.

Example 1 : If two independent random samples of sizes 15 and 8 have respectively the following means and population standard deviations,

$\overline{X}_1 = 980$ $\overline{X}_2 = 1012$

$\sigma_1 = 75$ $\sigma_2 = 80$

Test the hypothesis that $\mu_1 = \mu_2$ at 5% level of significance.

(Assume the population to be normal.) **(M.U. 2015)**

Sol. : When population standard deviations σ_1 and σ_2 are known, we can assume $\overline{X}_1 - \overline{X}_2$ to be normal with mean zero and S.E. $= \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$ and hence, use Z-distribution. (See page 9-14).

(i) **Null Hypothesis $H_o : \mu_1 = \mu_2$**

Alternative Hypothesis $\mu_1 \neq \mu_2$

(ii) **Calculation of test statistic :**

$$\text{S.E} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{75^2}{15} + \frac{80^2}{8}} = \sqrt{375 + 800} = \sqrt{1175} = 34 \cdot 28$$

$$\therefore \quad z = \frac{\overline{X}_1 - \overline{X}_2}{\text{S.E.}} = \frac{980 - 1012}{34 \cdot 28} = -0 \cdot 93 \qquad \therefore \quad |Z| = 0 \cdot 93$$

(iii) **Level of significance :** $\alpha = 0 \cdot 05$

(iv) **Critical value :** The table value of z at $\alpha = 0 \cdot 05$ is $z_\alpha = 1 \cdot 96$.

(We use Z-test because we have assumed the population to be normal and population S.D. is known. See table on page 9-7.)

(v) **Decision :** Since the computed value $|Z| = 0 \cdot 93$ is less than the table value 1·96, the hypothesis is accepted.

$\therefore$ The population means are equal $\mu_1 = \mu_2$.

Example 2 : A sample of 8 students of 16 years each shown up a mean systolic blood pressure of 118·4 mm of Hg with S.D. of 12·17 mm. While a sample of 10 students of 17 years each showed the mean systolic B.P. of 121·0 mm with S.D. of 12·88 during an investigation. The investigator feels that the systolic B.P. is related to age.

Do you think that the data provides enough reasons to support investigators feeing at 5% LOS? (Assume the distribution of systolic B.P. to be normal.) **(M.U. 2014)**

Sol. : We are given $n_1 = 8$, $n_2 = 10$; $\overline{X}_1 = 118.4$, $\overline{X}_2 = 121.0$; $s_1 = 12 \cdot 17$, $s_2 = 12 \cdot 88$.

(i) **Null Hypothesis $H_o : \mu_1 = \mu_2$**

Alternative Hypothesis $\mu_1 \neq \mu_2$

(ii) **Calculation of test statistic :**

$$s_p = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{8(12 \cdot 17)^2 + 10(12 \cdot 88)^2}{8 + 10 - 2}}$$

$$\therefore \quad s_p = \sqrt{\frac{1184 \cdot 87 + 1658 \cdot 94}{16}} = 13 \cdot 33$$

$$\text{S.E.} = s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 13 \cdot 33 \sqrt{\frac{1}{8} + \frac{1}{10}} = 6 \cdot 32$$

$$t = \frac{\overline{X}_1 - \overline{X}_2}{\text{S.E.}} = \frac{118 \cdot 4 - 121 \cdot 0}{6 \cdot 32} = -0 \cdot 41 \qquad \therefore \quad |t| = 0 \cdot 41$$

(iii) **Level of significance :** $\alpha = 0 \cdot 05$

(iv) **Critical value :** The table value of t at $\alpha = 2 \cdot 12$ for $8 + 10 - 2 = 16$ d.f.

(v) **Decision :** Since the computed value $|t| = 0 \cdot 41$ is less than the table value 2·12, the hypothesis is accepted. $\mu_1 = \mu_2$

(Although the population is normal, since the population S.D. is not known, we use t-test. See table on page 9-7.)

Example 3 : The means of two random samples of size 9 and 7 are 196·42 and 198·82 respectively. The sum of the squares of the deviations from the means are 26·94 and 18·73 respectively. Can the samples be considered to have been drawn from the same population? **(M.U. 2004, 15)**

Sol. : (i) **Null Hypothesis $H_o : \mu_1 = \mu_2$**

Alternative Hypothesis $H_a : \mu_1 \neq \mu_2$

(ii) **Calculations of test statistic :** Unbiased estimate of common population standard deviation is

$$s_p = \sqrt{\frac{\Sigma(X_i - \overline{X})^2 + \Sigma(Y_i - \overline{Y})^2}{n_1 + n_2 - 2}} = \sqrt{\frac{26 \cdot 94 + 18 \cdot 73}{9 + 7 - 2}} = \sqrt{\frac{14 \cdot 67}{14}} = 1 \cdot 81$$

Standard error of the difference between the means

$$\text{S.E.} = s_p \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 1 \cdot 81 \sqrt{\frac{1}{9} + \frac{1}{7}} = 0 \cdot 91$$

$$\therefore \quad t = \frac{\overline{X}_1 - \overline{X}_2}{\text{S.E.}} = \frac{196 \cdot 42 - 198 \cdot 82}{0 \cdot 91} = -2 \cdot 64 \qquad \therefore \quad |t| = 2 \cdot 64$$

(iii) **Level of significance :** $\alpha = 0 \cdot 05$.

(iv) **Critical value :** The table value of t at $\alpha = 0 \cdot 05$ for $\nu = 9 + 7 - 2 = 14$ degrees of freedom is 2·145.

(v) **Decision :** Since the computed value of $|t| = 2 \cdot 64$ is greater than the table value $t_\alpha = 2 \cdot 145$, the null hypothesis is rejected.

$\therefore$ The samples cannot be considered to have been drawn from the same population.

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Example 4 : Six guinea pigs injected with 0·5 mg. of a medication took on an average 15·4 secs. to fall asleep with an unbiased standard deviation 2·2 secs., while six other guinea pigs injected with 1·5 mg. of the medication took on an average 11·2 secs. to fall asleep with an unbiased standard deviation 2·6 cms. Use 5% level of significance to test the null hypothesis that the difference in dosage has no effect.

Sol. : We have $\bar{X}_1 = 15\cdot4$, $\bar{X}_2 = 11\cdot2$, $S_1 = 2\cdot2$, $S_2 = 2\cdot6$, $n_1 = 6$, $n_2 = 6$.

(i) **Null Hypothesis** $H_0 : \mu_1 = \mu_2$

Alternative Hypothesis $H_a : \mu_1 \neq \mu_2$

(ii) **Calculation of test statistic :** We are given **unbiased** standard deviations.

$\therefore$ The unbiased estimate of the common population is given by

$$s_p = \sqrt{\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{5 \times (2\cdot2^2) + 5 \times (2\cdot6)^2}{6 + 6 - 2}}$$

$$= \sqrt{\frac{58}{10}} = \sqrt{5\cdot8} = 2\cdot408$$

The standard error of the difference between the two means is given by

$$S.E. = s_p \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 2\cdot408 \times \sqrt{\frac{1}{6} + \frac{1}{6}} = 1\cdot39$$

$$\therefore \quad t = \frac{\bar{X}_1 - \bar{X}_2}{S.E.} = \frac{15\cdot4 - 11\cdot2}{1\cdot39} = 3\cdot02 \qquad \therefore \ |t| = 3\cdot02$$

(iii) **Level of significance :** $\alpha = 0\cdot05$.

(iv) **Critical values :** The table value of t at $\alpha = 0\cdot05$ for $v = 6 + 6 - 2 = 10$ degrees of freedom is $t_\alpha = 2\cdot228$.

(v) **Decision :** Since the computed value of $|t| = 3\cdot02$ is greater than the table value $t_\alpha = 2\cdot28$ the null hypothesis is rejected.

$\therefore$ The difference is significant.

Example 5 : Samples of two types of electric bulbs were tested for length of life and the following data were obtained.

	Type I	Type II
No. of samples	8	7
Mean of the samples (in hours)	1134	1024
Standard deviation (in hours)	35	40

Test at 5% level of significance whether the difference in the sample means is significant. (Table value of t for 13 d.f. is 2·16, for 14 d.f. is 2·15 and for 15 d.f. is 2·13) **(M.U. 2004, 06)**

Sol. : We have $\bar{X}_1 = 1134$, $\bar{X}_2 = 1024$, $s_1 = 35$, $s_2 = 40$, $n_1 = 8$, $n_2 = 7$.

(i) **Null Hypothesis** $H_0 : \mu_1 = \mu_2$

Alternative Hypothesis $H_a : \mu_1 \neq \mu_2$

(ii) **Calculation of test statistic :** Since the sizes of the samples are small we use t-distribution.

The unbiased estimate of the common population is given by

$$s_p = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{8 \times 35^2 + 7 \times 40^2}{8 + 7 - 2}} = \sqrt{\frac{21000}{13}}$$

$$= \sqrt{1615\cdot38} = 40\cdot19$$

The standard error of the difference between the two means is given by

$$S.E. = s_p \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 40\cdot19\sqrt{\frac{1}{8} + \frac{1}{7}} = 20\cdot8$$

$$\therefore \quad t = \frac{\bar{X}_1 - \bar{X}_2}{S.E.} = \frac{1134 - 1024}{20\cdot8} = \frac{110}{20\cdot8} = 5\cdot288 \qquad \therefore \ |t| = 5\cdot288.$$

(iii) **Level of significance :** $\alpha = 0\cdot05$.

(iv) **Critical value :** The table value of t at $\alpha = 0\cdot05$ for $v = 8 + 7 - 2 = 13$ degrees of freedom is $t_\alpha = 2\cdot16$.

(v) **Decision :** Since the computed value $|t| = 5\cdot68$ is greater than the table value $t_\alpha = 2\cdot16$, the hypothesis is rejected.

$\therefore$ The difference is significant.

Example 6 : The heights of six randomly chosen sailors are in inches : 63, 65, 68, 69, 71 and 72. The heights of ten randomly chosen soldiers are : 61, 62, 65, 66, 69, 69, 70, 71, 72 and 73.

Discuss in the light that these data throw on the suggestion that the soldiers on an average are taller than sailors. **(M.U. 1997, 2002)**

Sol. : We first calculate the mean and standard deviation of the heights of both sailors and soldiers.

Sailors			Soldiers		
Height X_1	d_i $(x_1 - \bar{x}_1)$	d_i^2 $(x_1 - \bar{x}_1)^2$	Height X_2	d_i $(x_2 - \bar{x}_2)$	d_i^2 $(x_2 - \bar{x}_2)^2$
63	−5	25	61	−6·8	46·24
65	−3	9	62	−5·8	33·64
68	0	0	65	−2·8	7·84
69	1	1	66	−1·8	3·24
71	3	9	69	1·2	1·44
72	4	16	69	1·2	1·44
			70	2·2	4·84
			71	3·2	10·24
			72	4·2	17·64
			73	5·2	27·04
$\Sigma x_1 = 408$	0	$\Sigma (x_1 - \bar{x}_1)^2 = 60$	$\Sigma x_2 = 678$	0	$\Sigma (x_2 - \bar{x}_2)^2 = 163{\cdot}60$

$$\text{Now,} \quad \bar{X}_1 = \frac{\Sigma \bar{X}_1}{N} = \frac{408}{6} = 68, \quad \bar{X}_2 = \frac{\Sigma X_2}{N} = \frac{678}{10} = 67\cdot8$$

The unbiased estimate of the common population

$$s_p = \sqrt{\frac{\Sigma(X_1-\bar{X}_1)^2+\Sigma(X_2-\bar{X}_2)^2}{n_1+n_2-2}} = \sqrt{\frac{60+153.60}{6+10-2}} = \sqrt{\frac{213.6}{14}} = \sqrt{15.26} = 3.9$$

(i) **Null Hypothesis** $H_0 : \mu_1 = \mu_2$

Alternative Hypothesis $H_a : \mu_1 \neq \mu_2$

(ii) **Calculation of test statistic**

$$t = \frac{\bar{X}_1 - \bar{X}_2}{\text{S.E.}}. \quad \text{Now, } \bar{X}_1 = 68,\ \bar{X}_2 = 67.8$$

$$\therefore \text{S.E.} = s_p \times \sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = 3.9 \times \sqrt{\frac{1}{6}+\frac{1}{10}} = 2.014$$

$$\therefore t = \frac{\bar{X}_1-\bar{X}_2}{\text{S.E.}} = \frac{68-67.8}{2.014} = 0.099$$

(iii) **Level of significance :** $\alpha = 0.05$.

(iv) **Critical value :** The table value of t at $\alpha = 0.05$ for $\nu = 6 + 10 - 2 = 14$ degrees of freedom is $t_\alpha = 2.145$.

(v) **Decision :** Since the computed value $|t| = 0.099$ is smaller than the table value $t_\alpha = 2.145$, the hypothesis is accepted

$\therefore$ The means are equal *i.e.* the suggestion that the soldiers on the average are taller than sailors cannot be accepted.

EXERCISE - II

1. Two independent samples of sizes 8 and 7 gave the following results.

Sample 1 : 19 17 15 21 16 18 16 14

Sample 2 : 15 14 15 19 15 18 16

Is the difference between sample means significant ? **(M.U. 2003, 04, 14)**

[**Ans. :** $\bar{X}_1 = 17$, $\bar{X}_2 = 16$, $s_1 = 2.12$, $s_2 = 1.69$,

$$s_p = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}} \ ; \ \text{S.E.} = s_p \times \sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = 1.073$$

$\therefore$ $t = 0.93$. Accept H_0]

2. The mean and standard deviation of heights of 8 randomly chosen soldiers are 166·9 cms. and 8·29 cms. respectively. The corresponding values for 6 randomly chosen sailors are 170·3 cms. and 8·5 cms. respectively. Based on this data can we conclude that the soldiers, in general, are soldiers shorter than the sailors ? Find 95% confidence limits for the statistic used. **(M.U. 2005)** [**Ans. :** $|t| = 0.6967$. No]

3. Two independent random samples of sizes 8 and 10 have the means 950 and 1000. The standard deviations of the two populations are 80 and 100. Test the hypothesis that the populations have the same mean. [**Ans. :** $|z| = 1.178$, Accepted]

4. Two independent random sample have the following data

$\bar{x}_1 = 110$, $\sigma_1 = 25$, $n_1 = 16$

$\bar{x}_2 = 120$, $\sigma_2 = 30$, $n_2 = 9$

If σ_1 and σ_2 are the standard deviations of the populations. Test the hypothesis that $\mu_1 = \mu_2$ at 5% level of significance. (State the assumption you make if any.)

[**Ans. :** $|z| = 0.848$, Accepted]

5. Samples of electric tubes of two companies were tested for lengths of their life and the following information was obtained,

	Company A	**Company B**
No. of sample	8	7
Mean life (in hrs.)	1210	1314
Standard deviation (in hrs)	36	42

Test at 5% level of significance whether the difference in the sample means is significant. (Table value of t for $\nu = 13$ is 2·16, for $\nu = 14$ is 2·15 and for $\nu = 15$ is 2·13)

[**Ans. :** $|t| = 4.81$, Reject]

6. A medicine was found to be effective for 9 patients in 8 days on an average with standard deviation of 2·2 days. Another medicine administered to another group of 8 patients was found to be effective in 6 days on an average with standard deviation of 2·6 days. Use 5% level of significance to test the null hypothesis that the two medicines are equally affective.

[**Ans. :** $|t| = 1.614$, Accept]

7. Two types of anti-biotics were tested on two groups of patients for curing a particular disease and the following data were obtained.

	Type A	**Type B**
No. of patients	6	6
Mean period (in days)	13·55	10·10
Unbiased standard deviation (in days)	3·2	2·8

Use 5% level of significance to test the null hypothesis that the difference in the mean period of the two drugs is significant. [**Ans. :** $|t| = 1.40$, Reject]

8. Two kinds of manures were used in seventeen plots of the same size other conditions being the same. The yields in quintals are given below.

Manure I : 35, 42, 40, 42, 34, 24, 42.

Manure II : 34, 44, 32, 40, 52, 41, 50, 40, 42, 45.

Test at 5% level of significance whether the two manures differ as regards their mean yields. (Table value of t at 5% level of significance for 15 degrees of freedom is 2·131).

[**Ans. :** $|t| = 1.68$. Difference is not significant]

9. The following are the gain in weights of cows fed on two types of diets X and Y.

Diet X : 30, 37, 35, 37, 29, 19, 37.

Diet Y : 29, 39, 27, 35, 47, 37, 45, 35, 37, 40.

Test at 5% level of significance whether the two diets differ as regards their effect on mean increase in weight. (Table value of t for 15 degrees of freedom at 5% level of significance is 2·131). [**Ans. :** $|t| = 1.62$. Do not differ]

10. The means of two random samples of size 9 and 7 are 196 and 199 respectively. The sum of the squares of the deviations from the mean are 27 and 19 respectively. Can the samples be regarded to have been drawn from the same normal population ?

[**Ans. :** $|t| = 3.30$, No]

(b) Case II : Samples not independent

In the previous test it was assumed that the two samples were independent. For example, guinea pigs to which sleeping drugs were administered in the two group were different. The cows fed on two diets were different. The plots on which two different manures were used were different. But in some cases the samples may not be different. We may test the effectiveness of a drug on the same group of persons. We may test the effectiveness of coaching on the same batch of students. In such cases the sample is the same for two tests. The samples are not independent and the above formula for testing of hypothesis cannot be used. In such cases we calculate the t - statistic as explained below.

We first find the differences of the corresponding values of the two sets of data then find the mean difference $\overline{X}$ and standard deviation of the differences s. We then define

$$t = \frac{\overline{X} - \mu}{s / \sqrt{n-1}} = \frac{\overline{X} - 0}{s / \sqrt{n-1}} \quad \text{or} \quad t = \frac{\overline{X} - \mu}{S / \sqrt{n}} = \frac{\overline{X}}{S / \sqrt{n}}$$

where, $\mu = 0$ is the null hypothesis, s = S.D. of the sample,

S = unbiased estimator of σ.

Note ....

Taking the null hypothesis $\mu = 0$ for differences amounts to the null hypothesis of equality of means $\mu_1 = \mu_2$ of the two populations.

Example 1 : A certain injection administered to 12 patients resulted in the following changes of blood pressure :

5, 2, 8, −1, 3, 0, 6, −2, 1, 5, 0, 4

Can it be concluded that the injection will be in general accompanied by an increase in blood pressure ? **(M.U. 1998, 2004, 11, 16)**

Sol. : We first calculate $\overline{X}$ and s^2.

Calculation of $\overline{X}$ and s^2

x	5	2	8	−1	3	0	6	−2	1	5	0	4
$d_i = x_i - 2$	3	0	6	−3	1	−2	4	−4	−1	3	−2	2
$d_i^2 = (x_i - 2)^2$	9	0	36	9	1	4	16	16	1	9	4	4

$$\overline{X} = a + \frac{\Sigma d_i}{n} = 2 + \frac{7}{12} = 2{\cdot}58$$

$$\Sigma (X_i - \overline{X})^2 = \Sigma d_i^2 - \frac{(\Sigma d_i)^2}{n} = 109 - \frac{49}{12} = 104{\cdot}92$$

$$\therefore \quad s^2 = \frac{\Sigma (X_i - \overline{X})^2}{n} = \frac{104{\cdot}92}{12} = 8{\cdot}74$$

(i) **The null hypothesis $H_o : \mu = 0$**

Alternative hypothesis $H_a : \mu \neq 0$

(ii) **Calculation of test statistic :** Since the sample size is small, we use students t-distribution.

$$t = \frac{\overline{X} - \mu}{s / \sqrt{n-1}} = \frac{2{\cdot}58 - 0}{\sqrt{8{\cdot}74} / \sqrt{11}} = 2{\cdot}89 \qquad \therefore \ |t| = 2{\cdot}89$$

(iii) **Level of significance :** $\alpha = 0{\cdot}05$.

(iv) **Critical value :** The value of t_α at 5% level of significance for $v = 12 - 1 = 11$ degrees of freedom = 2·201.

(v) **Decision :** Since the calculated value of $t = 2{\cdot}89$ is greater than the critical value $t_\alpha = 2{\cdot}201$, the hypothesis is rejected.

$\therefore$ There is rise in B.P.

Example 2 : Ten school boys were given a test in Statistics and their scores were recorded. They were given a months special coaching and a second test was given to them in the same subject at the end of the coaching period. Test if the marks given below give evidence to the fact that the students are benefitted by coaching.

Marks in Test I : 70, 68, 56, 75, 80, 90, 68, 75, 56, 58.

Marks in Test II : 68, 70, 52, 73, 75, 78, 80, 92, 54, 55. **(M.U. 2004)**

Sol. : We first calculate the differences between marks in test II and marks in test I = X and from these we calculate $\overline{X}$ and s^2.

Calculation of $\overline{X}$ and s^2

X	−2	2	−4	−2	−5	−12	12	17	−2	−3
$d_i = x_i - 2$	−4	0	−6	−4	−7	−14	10	15	−4	−5
$d_i^2 = (x_i - 2)^2$	16	0	36	16	49	196	100	225	16	25

$$\overline{X} = a + \frac{\Sigma d_i}{n} = 2 + \left(-\frac{19}{10}\right) = 0{\cdot}1$$

$$\Sigma (X_i - \overline{X})^2 = \Sigma d_i^2 - \frac{(\Sigma d_i)^2}{n} = 679 - \frac{0{\cdot}01}{10} = 678{\cdot}999$$

$$\therefore \quad s^2 = \frac{\Sigma (X_i - \overline{X})^2}{n} = 67{\cdot}90$$

(i) **The null hypothesis $H_o : \mu = 0$**

Alternative hypothesis $H_a : \mu \neq 0$

(ii) **Calculation of test statistic :** Since the sample size is small, we use students t-distribution.

$$t = \frac{\overline{X} - \mu}{s / \sqrt{n-1}} = \frac{0{\cdot}1 - 0}{\sqrt{67{\cdot}90} / \sqrt{9}} = 0{\cdot}036 \qquad \therefore \ |t| = 0{\cdot}036.$$

(iii) **Level of significance :** $\alpha = 0{\cdot}05$.

(iv) **Critical value :** The value of t_α at 5% level of significance for $v = 10 - 1 = 9$ degrees of freedom = 2·262.

(v) **Decision :** Since the calculated value of $|t| = 0{\cdot}036$ is less than the critical value $t_\alpha = 2{\cdot}262$, the hypothesis is accepted.

$\therefore$ The students are not benefitted by coaching.

Example 3 : In a certain experiment to compare two types of pig-foods A and B, the following results of increasing weights were obtained.

Pig Number	:	1	2	3	4	5	6	7	8
Increase in weight X kg by A	:	49	53	51	52	47	50	52	53
Increase in weight Y kg by B	:	52	55	52	53	50	54	54	53

(i) Assuming that the two sample of pigs are independent, can we conclude that food B is better than food A.

(ii) Examine the case if the same set of pigs were used in both the cases.

(M.U. 2004, 06)

Sol. : (a) We first calculate $\overline{X}_1$ and $\overline{X}_2$.

Calculation of $\overline{X}_1$ and $\overline{X}_2$ etc.

Food A			Food B		
X_1	$d_1 = x_1 - 51$	$d_1^2 = (x_1 - 51)^2$	X_2	$d_2 = x_2 - 53$	$d_2^2 = (x_2 - 53)^2$
49	−2	4	52	−1	1
53	2	4	55	2	4
51	0	0	52	−1	1
52	1	1	43	0	0
47	−4	16	50	−3	9
50	−1	1	54	1	1
52	1	1	54	1	1
53	2	4	53	0	0
	−1	31		−1	17

$$\overline{X}_1 = a + \frac{\Sigma d_i}{n} = 51 + \frac{-1}{8} = 50.875$$

$$\Sigma(X_1 - \overline{X})^2 = \Sigma d_1^2 - \frac{(\Sigma d_1)^2}{n} = 31 - \frac{(-1)^2}{8} = 30.875$$

and $$\overline{X}_2 = a + \frac{\Sigma d_i}{n} = 53 - \frac{1}{8} = 52.875$$

$$\Sigma(X_2 - \overline{X})^2 = \Sigma d_2^2 - \frac{(\Sigma d_2)^2}{n} = 17 - \frac{(-1)^2}{8} = 16.875$$

(i) **The null hypothesis H_o :** $\mu_1 = \mu_2$

Alternative hypothesis H_a : $\mu_1 \neq \mu_2$

(ii) **Calculation of test statistic**

$$s_p = \sqrt{\frac{\Sigma(X_1 - \overline{X}_1)^2 + \Sigma(X_2 - \overline{X}_2)^2}{n_1 + n_2 - 2}} = \sqrt{\frac{30.875 + 16.875}{8 + 8 - 2}} = \sqrt{3.41}$$

$$\text{S.E.} = s_p \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = \sqrt{3.41} \times \sqrt{\frac{1}{8} + \frac{1}{8}} = 0.92$$

$$\therefore \quad t = \frac{\overline{X}_1 - \overline{X}_2}{\text{S.E.}} = \frac{50.875 - 82.875}{0.92} = -2.17 \qquad \therefore \ |t| = 2.17$$

(iii) **Level of significance :** $\alpha = 0.05$.

(iv) **Critical value :** The value of t at $\alpha = 0.05$ for $v = 8 + 8 - 2 = 14$ degrees of freedom $= 2.145$.

(v) **Decision :** Since computed value $t = 2.17$ is greater than the table value $t_\alpha = 2.145$, the hypothesis is rejected at 5% level of significance.

$\therefore$ Food B is superior to food A.

(b) If the same set of pigs were used in the two tests : We first calculate the differences between the weights in the two tests and from these we calculate $\overline{X}$ and s^2.

Calculation of $\overline{X}$ and s^2

X_1	49	53	51	52	47	50	52	53	Total
X_2	52	55	52	53	50	54	54	53	
X	−3	−2	−1	−1	−3	−4	−2	0	
$d_i = x_i - 2$	−5	−4	−3	−3	−5	−6	−3	−2	−32
$d_i^2 = (x_i - 2)^2$	25	16	9	9	25	36	16	4	140

$$\overline{X} = a + \frac{\Sigma d_i}{n} = 2 - \frac{32}{8} = -2$$

$$\Sigma(X_i - \overline{X})^2 = \Sigma d_i^2 - \frac{(\Sigma d_i)^2}{n} = 140 - \frac{(-32)^2}{8} = 12$$

$$\therefore \quad s^2 = \frac{\Sigma(X_i - \overline{X})^2}{n} = \frac{12}{8} = 1.5$$

(i) **The null hypothesis H_o :** $\mu = 0$

Alternative hypothesis H_a : $\mu \neq 0$

(ii) **Calculation of test statistic**

$$t = \frac{\overline{X} - \mu}{s / \sqrt{n-1}} = \frac{-2 - 0}{\sqrt{1.5/7}} = -4.32 \qquad \therefore \ |t| = 4.32$$

(iii) **Level of significance :** $\alpha = 0.05$.

(iv) **Critical value :** The value of t_α at 5% level of significance for $v = 8 - 1 = 7$ degrees of freedom $= 2.365$.

(v) **Decision :** Since the calculated value of $t = 4.32$ is greater than the critical value $t_\alpha = 2.365$, the hypothesis is rejected.

$\therefore$ Food B is superior to food A.

EXERCISE - III

1. A drug was administered to 5 persons and the systolic blood pressure before and after was measured. The results are given below

Candidates	:	I,	II,	III,	IV,	V.
B.P. before	:	140,	130,	132,	150,	140
B.P. after	:	132,	126,	133,	144,	133.

Test whether the drug is effective in lowering the systolic blood pressure.

(M.U. 2007, 09) [**Ans.** : $|t| = 3$. Accept $\mu_1 = \mu_2$. The drug is not effective]

2. A drug was administered to 10 patients and the changes in the sugar content in the blood was recorded as under 10, 8, – 6, – 4, 2, – 8, 6, – 5, – 3, – 6. Is it reasonable to believe that the drug has no effect on change of sugar? (Use 5% level of t. For 9 d.f. $t = 2{\cdot}262$).

[**Ans.** : $|t| = 0{\cdot}289$. Accept $\mu_1 = \mu_2$. Drug has no effect.]

3. Ten accountants were given intensive coaching and tests were conducted before and after coaching. The scores of tests are given below.

Sr. No.	: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Marks in test before coaching	: 50, 42, 51, 42, 60, 41, 70, 55, 62, 38.
Marks in test after coaching.	: 62, 40, 61, 52, 68, 51, 64, 63, 72, 50.

Does the score show an improvement? Test at 5% level of significance. (The value of t for $\nu = 9$ at 5% level for one tail test is 1·8333 and for two tail test is 2·262).

[**Ans.** : $|t| = 3{\cdot}72$. Reject $\mu_1 = \mu_2$. There is improvement.]

4. Ten students were given intensive coaching for a month in Statistics. The scores obtained in tests are given below.

Sr. No.	: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Marks in 1st test	: 50, 52, 53, 60, 65, 67, 48, 69, 72, 80.
Marks in 2nd test	: 65, 55, 65, 65, 60, 67, 49, 82, 74, 86.

Does the score from test 1 to test 2 shows an improvement? Test at 5% level of significance. (The value of t for 9 d.f. at 5% level of significance is 1·833 for one tailed test and 2·262 for two tailed test.)

[**Ans.** : $t = -2{\cdot}57 < -1{\cdot}833$ for one tailed test. t falls in rejection area. Hypothesis $\mu_1 = \mu_2$ is rejected for one tailed test. Coaching is effective.]

5. The sales-data of an item in six shops before and after a special promotional compaign are as under.

Shops	:	A	B	C	D	E	F
Before compaign	:	53	28	31	48	50	42
After compaign	:	58	29	30	55	56	45

Can the campaign be judged to be a success at 5% level of significance ?
(Use one tailed test.)

[**Ans.** : $|t| = 3{\cdot}14$, Yes]

6. The following data relates to the marks obtained by 11 students in two tests, one held at the beginning of the year and the other at the end of the year after giving intensive coaching.

Test I	:	19	23	16	24	17	18	20	18	21	19	20
Test II	:	17	24	20	24	20	22	20	20	18	22	18

Do the data indicate that the students are benifited by coaching ? **(M.U. 2004)**

[**Ans.** : $|t| = 1{\cdot}20$, No]

7. The following data represent the marks obtained by 12 students in 2 tests, one held before coaching and the other after coaching.

Test I	:	55	60	65	75	49	25	18	30	35	51	61	72
Test II	:	63	70	70	81	54	29	21	38	32	50	70	80

Do the data indicate that the coaching was effective in improving the performance of the students ? **(M.U. 2004)** [**Ans.** : $t = 4$, Yes]

8. An I.Q. test was administered to 5 persons before and after training. The results are given below.

		1	2	3	4	5
I.Q. Before Training	:	110	120	123	132	125
I.Q. After Training	:	120	118	125	136	121

Test whether there is any change in I.Q. after training programme. Use 1% level of significance. **(M.U. 2006)**

[**Ans.** : $|t| = 0{\cdot}82$.

The value of t for $\nu = 4$ at 1% level of significance = 4·6. H_0 accepted.]

7. Non-parametric Tests

So far we have dealt with the problems of testing an hypothesis about a parameter. Such tests which deal with the parameter of the population are called **parametric tests.**

On the other hand tests which do not deal with the parameter are called **non-parametric tests.** One such test, which we are going to study is χ^2-test (pronounced as 'ki' square test - 'ki' as in kite).

8. Definition of χ^2

Suppose we are given a dic and we want to know whether it is biased or unbiased. Or suppose in a cholera epidemic we inoculate a group and we want to know whether inoculation is effective in preventing the attack of cholera. In such situations Chi-square test is used to test the hypothesis *e.g.* the die is not biased or the inoculation is not effective. χ^2 is calculated on the assumption of **status-quo** *i.e.* there is no change.

To test such a hypothesis we toss the die for say, 138 times and observe how many times we get 1, 2, 3, 4, 5, 6. These are observed frequencies 0. We can calculate expected frequencies of 1, 2, 3, 4, 5, 6 in 138 tosses. These are expected frequencies E. Then we find the value of Chi-square from the following.

The statistic χ^2-pronounced as ki-square (ki as in 'kite') and first used by Karl Pearson is defined by

$$\boxed{\chi^2 = \Sigma\left(\frac{(O-E)^2}{E}\right)}$$

where, O = observed frequency, E = expected frequency.

We calculate expected frequencies on certain assumptions such as (i) the coin or a die is unbiased, (ii) there is not association between the attributes, (iii) the accident occur evenly on all days, (iv) errors occur evenly on all pages, (v) the events occur in the given ratio, (vi) the events occur according to the given distribution (Binomial, Poisson, Normal). This is called testing goodness of fit.

We now compare the calculated value of χ^2 with the table value for the given degrees of freedom and at a specified level of significance.

If the **calculated value of χ^2 is greater than the table value** we conclude that the **difference** between the observed values and expected frequencies **is significant** and the **hypothesis is rejected.** If on the other hand the calculated value of χ^2 is less than the table value, we conclude that the difference between the observed values and the expected frequencies is not significant and the hypothesis is accepted.

Note ....

The value of χ^2 will be zero if the observed and expected frequencies coincide. The value of χ^2 is always positive. As observed frequencies depart from expected frequencies χ^2 goes on increasing.

(a) Analysis of $r \times c$ table (Contengency Table)

Chi-square criterion is based on observed frequencies O and expected frequencies E. Assuming that there is no association between the given attributes, we calculate frequencies in each cell. This frequency is called expected frequency of the cell. We denote the given frequency called observed frequency of the (i, j)th cell by O_{ij} and the expected frequency of (i, j)th cell by E_{ij}. If the table giving the observed frequencies of two attributes has r-rows and c-columns, there will be $r \times c$ cells in the table. Such a table is called **contingency table.** If $A_1, A_2, \ldots, A_r$ are totals of r-rows and $B_1, B_2, \ldots, B_c$ are the totals of c-columns obtained from given frequencies, then expected frequency $(i, j)$the cell is given by

$$E_{(i,j)} = \frac{(A_i \times B_j)}{N}$$

i.e. $$E_{(1,1)} = \frac{(A_1 B_1)}{N}, \quad E_{(1,2)} = \frac{(A_1 B_2)}{N}, \quad E_{(1,3)} = \frac{(A_1 B_3)}{N} \ldots\ldots \text{ etc.}$$

The statistic χ^2 is now defined by

$$\chi^2 = \Sigma\Sigma\left(\frac{O_{ij} - E_{ij}}{E_{ij}}\right)^2$$

with $(r-1) \times (c-1)$ degrees of freedom.

								Total
	$\frac{A_1 \times B_1}{N}$	$\frac{A_1 \times B_2}{N}$		$\frac{A_i \times B_j}{N}$				A_1 A_2 $\vdots$ A_i $\vdots$ A_r
Total	B_1	B_2		B_j			B_c	N

If calculated value of χ^2 is less than the table value of χ^2 the hypothesis that there is **no** association between the attributes is accepted.

9. Degrees of Freedom

While comparing the calculated value of χ^2 with the table value we must know the degrees of freedom. The term degrees of freedom means the number of values which can be chosen arbitrarily under the given restrictions. For example, if we have to choose 5 numbers whose sum is 50, we **cannot** choose all the five numbers arbitrarily because of the restriction. We can choose four numbers arbitrarily and the fifth number will have to be 50 – (sum of four). Thus, the degrees of freedom are four and not five. If more restrictions are placed on the choice of number, the degrees of freedom will be less. In general if there are n numbers **to be chosen and k independent constraints then the degrees of freedom denoted by** d.f. are given by **d.f. = $n - k$.**

If the table has r rows and c columns then in each row we can select $(c-1)$ elements and in each column we can select $(r-1)$ elements freely. This means for each row we have $(c-1)$ degrees of freedom and for each column we have $(r-1)$ degrees of freedom. Hence, for the whole table we have $(r-1) \times (c-1)$ degrees of freedom.

Thus, if there are two rows and two columns then d.f. = $(2-1)(2-1) = 1$. There is only one degree of freedom. If there are three rows and three columns then d.f. = $(3-1)(3-1) = 4$. There are four degrees of freedom and so on.

10. Conditions for χ^2 Test

The following conditions should be satisfied while applying χ^2 test.

1. N, the total number of observations must be sufficiently large. Preferably N should be greater than 50.

2. Frequency of every cell must be **greater than 10.** If any frequency is less than 10 it is combined with neighbouring frequencies so that the **combined frequency is greater than 10** and the degrees of freedom are reduced accordingly. (See Ex. 7 and 9, page 10-35 and 10-37)

3. The number of classes n must not be too small nor too large. Preferably we should have $4 \leq n \leq 16$.

Note ....

It may be noted that the χ^2-test depends upon (i) the observed frequencies, (ii) the expected frequencies, (iii) the number of observations only. It does not make any assumption regarding the nature of parent population.

11. Yate's Correction

In a 2×2 table the degrees of freedom is $(2-1)(2-1) = 1$. If any of the cell frequency is less than 5, we have to use pooling method. But this will result in χ^2 with zero degrees of freedom. This is meaningless. In this case Yates in 1934 suggested to use

$$\chi^2 = \Sigma\left[\frac{\{|O - E| - 0{\cdot}5\}^2}{E}\right]$$

He showed that by taking χ^2 as defined above, χ^2 approximation is improved. (See Ex. 5, page 10-27).

12. Uses of χ^2 Test

We shall consider only two uses of χ^2 distribution.

1. To test independence of attributes : χ^2 - test is widely used to test whether there is association between two or more attributes. For example, χ^2 - test can be used to determine whether there is association between the colour of mother's eye and daughter's eye, between inoculation and prevention of a disease. In such cases we proceed on the **null hypothesis that there is no association between the attributes.** If the calculated value of χ^2 at a certain level of significance is less than the table value, the hypothesis is accepted otherwise rejected.

In the same way χ^2 - test is also used to test if a characteristic is dependent upon another characteristic. For example, χ^2 - test can be used to test whether the performance of students workers in a factory is dependent on the traning or to test whether performance of students in a particular subject is dependent on the performance in another subject. Using χ^2 - distribution in this way to test the independence of one attribute on another is called **test of independence.**

2. To test the goodness of fit : χ^2 - test is very commonly known as χ^2 - **test of goodness of fit** because it enables us to ascertain how well the theoretical distributions such as Binomial, Poisson or Normal fit the observed frequencies. In such cases we proceed on the **null hypothesis that the theory supports the observations *i.e.* the fit is good.** For example, suppose we toss 3 fair coins 200 times and observe the frequencies of 0, 1, 2, 3 heads. We can also calculate the expected frequencies by using Binomial Distribution. χ^2 - test can be used to ascertain whether Binomial Distribution fits well. If the calculated value of χ^2 at a certain level of significance is less than the table value, the fit is supposed to be good otherwise the fit is supposed to be poor.

3. To test the discrepancies between observed frequencies and expected frequencies : χ^2 - test can also be used to ascertain whether the difference between observed frequencies and the expected frequencies is purely due to chance or whether due to inadequacy in the theory applied.

4. To test equality of several proportions : χ^2 - test can also be used to test whether the proportions p_1, p_2, p_3, p_4 in different populations are equal *i.e.* χ^2 - test can also be used to test the **null hypothesis that** $p_1 = p_2 = p_3 = p_4$.

5. To test the hypothesis about σ^2 : χ^2 is also used to test the population variance. (See Ex. 10, page 10-38)

Type I : Independence of Attributes

Example 1 : Investigate the association between the darkness of eye colour in father and son from the following data.

Colour of father's eyes

Colour of son's eyes	Dark	Not dark	Total
Dark	48	90	138
Not dark	80	782	862
Total	128	872	1000

(M.U. 2010)

Sol. : (i) **Null Hypothesis H_o :** There is no association between the darkness of eye colour in father and son.

Alternative Hypothesis H_a : There is an association.

(ii) **Calculation of test statistic :** On the basis of this hypothesis the expected frequency of dark eyed sons with dark eyed fathers

$$= \frac{A \times B}{N}$$

where, A = number of dark eyed fathers (total of first column)

B = number of dark eyed sons (total of first row)

N = total number of observations

$$\therefore \quad \text{Expected frequency} = \frac{128 \times 138}{1000} = 18$$

(This is because if there is no association, since the ratio of dark eyed fathers to the total is 128 / 1000 out of 138 dark eyed sons there will be $\frac{138 \times 128}{100}$ dark eyed sons.)

Having obtained the expected frequency in the first cell, since the totals remain the same, the figures in other cells can be easily obtained as $138 - 18 = 120$, $128 - 18 = 110$, $872 - 120 = 752$.

We thus get the following table.

Colour of father's eyes

Colour of son's eyes	Dark	Not dark	Total
Dark	18	120	138
Not dark	110	752	862
Total	128	872	1000

Calculation of $(O - E)^2 / E$

O	E	$(O-E)^2$	$(O-E)^2/E$
48	18	900	50·00
80	110	900	8·18
90	120	900	7·50
782	752	900	1·20
		Total	$\chi^2 = 66{\cdot}88$

(iii) **Level of significance :** $\alpha = 0{\cdot}05$.

Degrees of freedom $= (r-1)(c-1) = (2-1)(2-1) = 1$.

(r = number of rows, c = number of columns.)

(iv) **Critical value :** For 1 d.f. at 5% level of significate the table value of χ^2 is 3·84.

(v) **Decision :** Since the calculated value of $\chi^2 = 66{\cdot}68$ is much greater than the table value of $\chi^2 = 3{\cdot}84$, the null hypothesis is rejected.

$\therefore$ There is an association between darkness of colour of fathers and sons.

Example 2 : The following table gives the number of accounting clerks not committing errors among trained and untrained clerks working in an organisation.

	No. of clerks committing errors	No. of clerks not committing errors	Total
Trained	70	530	600
Untrained	155	745	900
Total	225	1275	1500

Test the effectiveness of training in preventing errors. (Table value of χ^2 for 1 d.f., 2 d.f., 3 d.f., 4 d.f. are 3·84, 5·99, 7·81, 4·99 respectively.)

Sol. : (i) **Null Hypothesis** H_o : There is no association between training and errors.

Alternative Hypothesis H_a : There is an association between training and errors.

(ii) **Calculation of test statistic :** On the basis of this hypothesis, the number in the first cell

$$= \frac{A \times B}{N}$$

where A = number of clerks committing error *i.e.* the total in the first column,

B = number of trained clerks *i.e.* the total in the first row,

N = Total number of observations.

(This is so because ratio of clerks committing errors to the total is $\frac{225}{1500}$. If there is no association out of total of 600 trained clerks $\frac{225 \times 600}{1500}$ will commit errors.)

$\therefore$ The number in the first cell $= \frac{225 \times 600}{1500} = 90$.

Having obtained the expected frequency in the first cell, since the totals remain the same, the figures in the other cells are $600 - 90 = 510$, $225 - 90 = 135$, $1275 - 510 = 765$.

We, thus, get the following table.

Table of calculated frequencies

	No. of clerks committing errors	No. of clerks not committing errors	Total
Trained	90	510	600
Untrained	135	765	900
Total	225	1275	

Calculation of $(O - E)^2 / E$

O	E	$(O-E)^2$	$(O-E)^2/E$
70	90	400	4·44
530	510	400	0·78
155	135	400	2·96
745	765	400	0·52
		Total	$\chi^2 = 8·7$

(iii) **Level of significance :** $\alpha = 0·05$.

Degrees of freedom $= (r - 1)(c - 1) = (2 - 1)(2 - 1) = 1$.

(iv) **Critical value :** For 1 d.f. and 5% level of significance, the table value of $\chi^2 = 3·81$.

(v) **Decision :** Since the calculated value of $\chi^2 = 8·7$ is greater than the table value of $\chi^2 = 3·81$, the null hypothesis is rejected.

$\therefore$ The training is effective in preventing errors.

Example 3 : A sample of 400 students of under-graduate and 400 students of post-graduate classes was taken to know their opinion about autonomous colleges. 290 of the under-graduate and 310 of the post-graduate students favoured the autonomous status. Present these facts in the form of a table and test at 5% level, that the opinion regarding autonomous status of colleges is independent of the level of classes of students.

(M.U. 2016)

Sol. :

Opinion about autonomous colleges

	Favoured	Not-favoured	Total
Under-graduate	290	110	400
Post-graduate	310	90	400
Total	600	200	800

(i) **Null Hypothesis** H_o **:** Opinion is independent of the level of classes (There is no association between the classes and the opinion).

Alternative Hypothesis H_a **:** There is association.

(ii) **Calculation of test statistic :** On the basis of this hypothesis the expected frequency of the undergraduate students who favoured autonomy

$$= \frac{A \times B}{N}$$

where, A = Number of under-graduate students.

B = Number who favoured,

N = Total number of students.

$\therefore$ Expected frequency $= \frac{400 \times 600}{800} = 300$.

This is the frequency in the first cell.

The frequencies in the remaining cells are

$400 - 300 = 100$, $600 - 300 = 300$, $400 - 300 = 100$.

Thus, we get the following table.

Opinion about the autonomous colleges

	Favoured	Not-favoured	Total
Under-graduate	300	100	400
Post-graduate	300	100	400
Total	600	200	800

Calculation of $(O-E)^2/E$

O	E	$(O-E)^2$	$(O-E)^2/E$
290	300	100	0·33
310	300	100	0·33
110	100	100	1·00
90	100	100	1·00
		Total	2·66

(iii) **Level of significance :** $\alpha = 0.05$

Degrees of freedom $= (r-1)(c-1) = (2-1)(2-1) = 1$.

(iv) **Critical value :** For 1 d.f. at 5% level of significance, the table value of χ^2 is 3·84.

(v) **Decision :** Since the calculated value of $\chi^2 = 2{\cdot}66$ is less than the table value of $\chi^2 = 3{\cdot}84$, the null hypothesis is accepted.

$\therefore$ There is no association between the opinion and the level of classes.

Example 4 : In a survey of 200 boys of which 75 were intelligent, 40 had educated fathers, while 85 of the unintelligent boys had uneducated fathers. Do these figures support the hypothesis that educated fathers have intelligent boys.

Sol. : (i) Null Hypothesis H_o : These is no association between education of fathers and intelligence of sons.

Alternative Hypothesis H_a : There is association.

(ii) **Calculation of test statistic :** We first tabulate the given information as follows.

	Intelligent sons	Unintelligent sons	Total
Educated fathers	40	35	75
Uneducated fathers	35	85	125
Total	75	125	200

On the basis of this hypothesis, the number in the first cell $= \dfrac{A \times B}{N}$

where, A = number of boys who are intelligent *i.e.* the total in the first column

B = number of boys having educated fathers *i.e.* the total in the first row

N = Total number of observations.

$\therefore$ The number in the first cell $= \dfrac{75 \times 75}{200} = 28$.

Having obtained the expected frequency in the first cell, since the totals remain the same the figures in the remaining cells are 75 − 28 = 47, again 75 − 28 = 47 and 125 − 47 = 78.

We thus, get the following table.

Table of Calculated Frequencies

	Intelligent sons	Unintelligent sons	Total
Educated fathers	28	47	75
Uneducated fathers	47	78	125
Total	75	125	200

Calculation of $(O-E)^2/E$

O	E	$(O-E)^2$	$(O-E)^2/E$
40	28	144	5·14
35	47	144	3·06
35	47	144	3·06
85	78	49	0·63
		Total	$\chi^2 = 11{\cdot}89$

(iii) **Level of significance :** $\alpha = 0{\cdot}05$.

The number of degrees of freedom $= (r-1)(c-1) = (2-1)(2-1) = 1$.

(iv) **Critical value :** For 1 d.f. at 5% level of significance the table value of $\chi^2 = 3{\cdot}84$.

(v) **Decision :** Since the calculated value of $\chi^2 = 11{\cdot}89$ is much greater than the table value of $\chi^2 = 3{\cdot}84$, the null hypothesis is rejected.

$\therefore$ There is association between education of fathers and intelligence of sons.

Example 5 : Two batches of 12 animals each are given test of inoculation. One batch was inoculated and the other was not. The number of dead and surviving animals are given in the following table for both cases. Can the inoculation be regarded as effective against the disease at 5% level of significance. (Make Yates correction)

	Dead	Surviving	Total
Inoculated	2	10	12
Not-inoculated	8	4	12
Total	10	14	24

(M.U. 2004, 11)

Sol. : (i) Null Hypothesis H_o : There is **no association** between inoculation and death.

Alternative Hypothesis H_a : There is association between inoculation and death.

(ii) **Calculation of test statistic :** On the basis of this hypothesis the number in the first cell

$$= \frac{A \times B}{N}$$

where, A = total in the first column,

B = total in the first row,

N = Total number of observations.

$\therefore$ The number in the first cell $= \dfrac{10 \times 12}{24} = 5$.

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The remaining cell frequencies may be calculated in the same manner or may be obtained by subtracting this frequency from the row total and column total. We then apply Yates correction and prepare the table.

Calculation of χ^2

O	E	$\lvert O-E\rvert-0.5$	$\frac{\{\lvert O-E\rvert-0.5\}^2}{E}$
2	$\frac{10\times 12}{24}=5$	2·5	$\frac{6.25}{5}=1.25$
10	$12-5=7$	2·5	$\frac{6.25}{7}=0.89$
8	$10-5=5$	2·5	$\frac{6.25}{5}=1.25$
4	$12-5=7$	2·5	$\frac{6.25}{7}=0.89$
		Total	$\chi^2=4.29$

(iii) **Level of significance :** $\alpha = 0.05$.

Degree of freedom $= (r-1)(c-1) = (2-1)(2-1) = 1$.

(iv) **Critical value :** For 1 degree of freedom at 5% level of significance the table value of $\chi^2 = 3.81$.

(v) **Decision :** Since the calculated value of $\chi^2 = 4.29$ is greater than the table value $\chi^2 = 3.81$, the hypothesis is rejected.

$\therefore$ There as association between inoculation and death *i.e.* inoculation is effective against the disease.

Remark ....

We would arrive at the same conclusion even if Yate's correction is not made.

Example 6 : To test the effect of a new drug, a controlled experiment was conducted. 300 patients were given the new drug while 200 patients were given no drug. On the basis of examination of these persons, the following results were obtained.

	Cured	Condition worsoned	No effect	Total
Given the new drug	200	40	60	300
Not given the drug	120	30	50	200
Total	320	70	110	500

Use χ^2 test to find the effect of the new drug.

Sol. : (i) **Null Hypothesis H_o :** The drug is not effective.

Alternative Hypothesis H_a : The drug is effective.

(ii) On the basis on this hypothesis the number in the first cell

$$= \frac{A \times B}{N}$$

where, A = total in the first column,

B = total in the first row,

N = Total number of observations.

$\therefore$ The number in the first cell $= \frac{320 \times 300}{500} = 192$.

Similarly, the number in the second cell $= \frac{70 \times 300}{500} = 42$.

Since the totals remain the same, the numbers in the remaining cells are

$320 - 192 = 128$, $\quad 70 - 42 = 28$,

$300 - (192 + 42) = 66$, $\quad 110 - 66 = 44$.

We, thus, get the following table.

Table of Calculated Frequencies

	Cured	Condition worsoned	No effect	Total
Given the new drug	192	42	66	300
Not given the drug	128	28	44	200
Total	320	70	110	500

Calculation of $(O-E)^2/E$

O	E	O − E	$(O-E)^2$	$(O-E)^2/E$
200	192	8	64	0·333
40	42	−2	4	0·095
60	66	−6	36	0·545
120	128	−8	64	0·500
30	28	2	4	0·143
50	44	6	36	0·818
			Total	$\chi^2 = 2.434$

(iii) **Level of significance :** $\alpha = 0.05$.

Degrees of freedom $= (r-1)(c-1) = (2-1)(3-1) = 2$.

(iv) **Critical value :** For 2 degrees of freedom at 5% level of significance, the table value of $\chi^2 = 5.991$.

(v) **Decision :** Since the calculated value of $\chi^2 = 2.434$ is less than the table value $\chi^2 = 5.991$, the hypothesis is accepted.

$\therefore$ The new drug is **not** effective.

Example 7 : The following table gives the result of opinion pole for three parties A, B and C. Test whether the age and the choice of the party are independent at 5% level of significance using χ^2 test.

Age	Party			Total
	A	B	C	
20 - 35	25	20	25	70
35 - 50	20	25	35	80
Above 50	25	25	30	80
Total	70	70	90	230

Sol. : (i) **Null Hypothesis H_0 :** There is no relation between the age and the choice of the party.

Alternative Hypothesis H_a : There is a relation between the two.

(ii) On the basis on this hypothesis the number in the first cell

$$= \frac{A \times B}{N}$$

where, A = total in the first column,

B = total in the first row,

N = Total number of observations.

$\therefore$ The number in the first cell of the first row $= \frac{70 \times 70}{230} = 21 \cdot 3$.

Similarly, the number in the second cell of the first row $= \frac{70 \times 70}{230} = 21 \cdot 3$.

The number in the first cell of the second row $= \frac{70 \times 80}{230} = 24 \cdot 3$.

The number in the second cell of the second row $= \frac{70 \times 80}{230} = 24 \cdot 3$.

Since the totals remain the same, the numbers in the remaining cells are

$70 - (21 \cdot 3 + 21 \cdot 3) = 27 \cdot 4$,

$80 - (24 \cdot 3 + 24 \cdot 3) = 31 \cdot 4$.

We, thus, get the following table.

Table of Calculated Frequencies

Age	Party			Total
	A	B	C	
20 - 35	21·3	21·3	27·4	70·0
35 - 50	24·3	24·3	31·4	80·0
Above 50	24·4	24·4	31·2	80·0
Total	70·0	70·0	90·0	230·0

Calculation of $(O - E)^2 / E$

O	E	O – E	$(O - E)^2$	$(O - E)^2 / E$
25	21·3	3·7	13·69	0·643
20	21·3	– 1·3	1·69	0·079
25	27·4	– 2·4	5·76	0·210
20	24·3	– 4·3	18·49	0·685
25	24·3	0·7	0·49	0·020
35	31·4	– 3·6	12·96	0·413
25	24·4	0·6	0·36	0·015
25	24·4	0·6	0·36	0·015
30	31·2	– 1·2	1·44	0·046
			Total	$\chi^2 = 2 \cdot 126$

(iii) **Level of significance :** $\alpha = 0 \cdot 05$.

Degrees of freedom $= (r - 1)(c - 1) = (3 - 1)(3 - 1) = 4$.

(iv) **Critical value :** For 4 degrees of freedom at 5% level of significance, the table value of $\chi^2 = 9 \cdot 488$.

(v) **Decision :** Since the calculated value of $\chi^2 = 2 \cdot 126$ is less than the table value $\chi^2 = 9 \cdot 488$, the hypothesis is accepted.

$\therefore$ There is no relation between the age and the choice of the party.

Type II : Goodness of Fit

Example 1 : The following table gives the number of accidents in a city during a week. Find whether the accidents are uniformly distributed over a week.

Day :	Sun.,	Mon.,	Tue.,	Wed.,	Thu.,	Fri.,	Sat.	Total
No. of accidents :	13,	15,	9,	11,	12,	10,	14.	84

(M.U. 2017)

Sol. : (i) **Null Hypothesis H_0 :** Accidents are equally distributed over all the days of a week.

Alternative Hypothesis H_a : Accidents do not occur equally.

(ii) **Calculation of test statistic :** If the accidents occur equally on all days of a week, there will be (84) / 7 = 12 accidents per day.

$$\therefore \chi^2 = \Sigma \frac{(O - E)^2}{E} = \frac{(13 - 12)^2}{12} + \frac{(15 - 12)^2}{12} + \frac{(9 - 12)^2}{12} + \frac{(11 - 12)^2}{12} + \frac{(12 - 12)^2}{12} + \frac{(10 - 12)^2}{12} + \frac{(14 - 12)^2}{12}$$

$$= \frac{1}{12}[1 + 9 + 9 + 1 + 0 + 4 + 4] = \frac{28}{12} = 2 \cdot 33$$

(iii) **Level of significance :** $\alpha = 0 \cdot 05$.

Degrees of freedom $= n - 1 = 7 - 1 = 6$.

(iv) **Critical value :** For 6 degrees of freedom at 5% level of significance table value of χ^2 is 12·59.

(v) **Decision :** Since the calculated value of χ^2 is less than the table value. The hypothesis is accepted.

$\therefore$ The accidents occur equally on all working days.

Example 2 : A die was thrown 132 times and the following frequencies were observed.

No. obtained :	1,	2,	3,	4,	5,	6.	Total
Frequency :	15,	20,	25,	15,	29,	28.	132

Test the hypothesis that the die is unbiased. **(M.U. 2010, 15, 17)**

Sol. : (i) **Null Hypothesis H_o :** The die is unbiased.

Alternative Hypothesis H_a : The die is not unbiased.

(ii) **Calculation of test statistic :** On the hypothesis that the die is unbiased we should expect the frequency of each number to be 132 / 6 = 22.

Calculation of $(O-E)^2/E$

No.	O	E	$(O-E)^2$
1	15	22	49
2	20	22	4
3	25	22	9
4	15	22	49
5	29	22	49
6	28	22	36
		Total	196

$$\therefore \chi^2 = \Sigma\frac{(O-E)^2}{E} = \frac{196}{22} = 8{\cdot}91$$

(iii) **Level of significance :** $\alpha = 0{\cdot}05$.

Number of degrees of freedom $= n-1 = 6-1 = 5$.

(iv) **Critical value :** For 5 d.f. at 5% level of significance the table value of χ^2 is 11·07.

(v) **Decision :** Since the calculated value of $\chi^2 = 8{\cdot}91$ is less than the table value of $\chi^2 = 11{\cdot}07$, the null hypothesis is accepted.

$\therefore$ The die is unbiased.

Example 3 : The number of car accidents in a metropolitan city was found to be 20, 17, 12, 6, 7, 15, 8, 5, 16 and 14 per month respectively. Use χ^2 - test to check whether these frequencies are in agreement with the belief that occurrence of accidents was the same during 10 months period. Test at 5% level of significance. (Table value of χ^2 at 9 d.f. is 16·9) **(M.U. 2001, 14)**

Sol. : (i) **Null Hypothesis H_o :** Accidents occur equally on all months.

Alternative Hypothesis H_a : Accidents do not occur equally on all months.

(ii) **Calculation of test statistic :** On the basis of this hypothesis, the number of accidents per month = (total) / 10 = (20 + 17 + 12 + 6 + 7 + 15 + 8 + 5 + 16 + 14) / 10 = 120 / 10 = 12.

$$\therefore \quad \chi^2 = \Sigma\frac{(O-E)^2}{E}$$

$$= \frac{(20-12)^2}{12} + \frac{(17-12)^2}{12} + \frac{(12-12)^2}{12} + \frac{(6-12)^2}{12} + \frac{(7-12)^2}{12} + \frac{(15-12)^2}{12} + \frac{(8-12)^2}{12} + \frac{(5-12)^2}{12} + \frac{(16-12)^2}{12} + \frac{(14-12)^2}{12}$$

$$= \frac{1}{12}[64 + 25 + 0 + 36 + 25 + 9 + 16 + 49 + 16 + 4]$$

$$= \frac{244}{12} = 20{\cdot}33.$$

(iii) **Level of significance :** $\alpha = 0{\cdot}05$.

Number of degrees of freedom $= n-1 = 10-1 = 9$.

(iv) **Critical value :** For 9 d.f. at 5% level of significance, the table value of χ^2 is 16·92.

(v) **Decision :** Since the calculated value of $\chi^2 = 20{\cdot}33$ is greater than the table value of $\chi^2 = 16{\cdot}92$, the null hypothesis is rejected.

$\therefore$ Accidents do not occur equally on all months.

Example 4 : 300 digits were chosen at random from a table of random numbers. The frequency of digits was as follows.

Digit :	0	1	2	3	4	5	6	7	8	9	**Total**
Frequency :	28	29	33	31	26	35	32	30	31	25	300

Using χ^2 - test examine the hypothesis that the digits were distributed in equal numbers in the table. **(M.U. 1996)**

Sol. : (i) **Null Hypothesis H_o :** The digits are distributed **equally.**

Alternative Hypothesis H_a : The digits are not distributed equally.

(ii) **Calculation of test statistic :** On the basis of the hypothesis the frequency of each digit.

$$= \frac{\text{Total}}{10} = \frac{28+29+33+31+26+35+32+30+31+25}{10}$$

$$\therefore \quad E = \frac{300}{10} = 30.$$

$$\therefore \quad \chi^2 = \Sigma\frac{(O-E)^2}{E}$$

$$= \frac{(28-30)^2}{30} + \frac{(29-30)^2}{30} + \frac{(33-30)^2}{30} + \frac{(31-30)^2}{30} + \frac{(26-30)^2}{30} + \frac{(35-30)^2}{30} + \frac{(32-30)^2}{30} + \frac{(30-30)^2}{30} + \frac{(31-30)^2}{30} + \frac{(25-30)^2}{30}$$

$$\therefore \quad \chi^2 = \frac{1}{30}[4+1+9+1+16+25+4+0+1+25]$$

$$= \frac{86}{30} = 2.87$$

(iii) Level of significance : $\alpha = 0.05$.

Number of degrees of freedom $= n - 1 = 10 - 1 = 9$.

(iv) Critical value : For 9 d.f. at 5% level of significance the table value of χ^2 is 16·92.

(v) Decision : Since the calculated value of $\chi^2 = 2.87$ is less than the table value of χ^2, the null hypothesis is accepted.

$\therefore$ Digits are equally distributed in the table.

Example 5 : Theory predicts that the proportion of beans in the four groups A, B, C, D should be 9 : 3 : 3 : 1. In an experiment among 1600 beans the numbers in the four groups were 882, 313, 287 and 118. Does the experimental results support the theory ? **(M.U. 2001, 06)**

Sol. : (i) Null Hypothesis H_o : The proportion of the beans in the four groups A, B, C, D is the given proportion 9 : 3 : 3 : 1.

Alternative Hypothesis H_a : The proportion is not as given above.

(ii) Calculation of test statistic : On the basis of the above hypothesis, since the sum is 9 + 3 + 3 + 1 = 16, the number of beans in the four groups will be

$$A = \frac{9}{16} \times 1600 = 900, \quad B = \frac{3}{16} \times 1600 = 300$$

$$C = \frac{3}{16} \times 1600 = 300, \quad D = \frac{1}{16} \times 1600 = 100$$

$$\therefore \quad \chi^2 = \Sigma \frac{(O-E)^2}{E} = \frac{(882-900)^2}{900} + \frac{(313-300)^2}{300} + \frac{(287-300)^2}{300} + \frac{(118-100)^2}{100}$$

$$= 0.36 + 0.56 + 0.56 + 3.24$$

$$= 4.72$$

(iii) Level of significance : $\alpha = 0.05$.

Degrees of freedom $= n - 1 = 4 - 1 = 3$.

(iv) Critical value : For 3 degrees of freedom at 5% level of significance, the table value of χ^2 is 7·81.

(v) Decision : Since the calculated value of $\chi^2 = 4.72$ is less than the table value of $\chi^2 = 7.81$, the null hypothesis is accepted.

$\therefore$ The proportion 9 : 3 : 3 : 1 is correct.

Example 6 : In an experiment on pea breeding the following frequencies were obtained.

Round and yellow	Wrinkled and yellow	Round and green	Wrinkled and green	Total
315	101	108	32	556

Theory predicts that the frequencies should be in proportion of 9 : 3 : 3 : 1.

Examine the correspondence between theory and experiment using Chi-square Test. **(M.U. 2016)**

Sol. : (i) Null Hypothesis H_o : The proportion of the peas in the four groups say A, B, C, D is in the given proportion 9 : 3 : 3 : 1.

Alternative Hypothesis H_a : The proportion is not as given above.

(ii) Calculation of test statistic : On the basis of the above hypothesis since the sum is 9 + 3 + 3 + 1 = 16, the number of peas in the four groups will be

$$A = \frac{9}{16} \times 556 = 312.75 = 313, \quad B = \frac{3}{16} \times 556 = 104.25 = 104,$$

$$C = \frac{3}{16} \times 556 = 104.25 = 104, \quad D = \frac{1}{16} \times 556 = 34.75 = 35.$$

$$\therefore \quad \chi^2 = \Sigma \frac{(O-E)^2}{E} = \frac{(315-313)^2}{313} + \frac{(101-104)^2}{104} + \frac{(108-104)^2}{104} + \frac{(32-35)^2}{35}$$

$$= 0.013 + 0.086 + 0.154 + 0.257 = 0.51$$

(iii) Level of significance : $\alpha = 0.05$

Degree of freedom $= n - 1 = 4 - 1 = 3$.

(iv) Critical value : For 3 degrees of freedom at 5% level of significance table value of χ^2 is 7·81.

(v) Decision : Since the calculated value of $\chi^2 = 0.51$ is less than the table value of $\chi^2 = 7.81$, the null hypothesis is accepted.

$\therefore$ The proportion 9 : 3 : 3 : 1 is correct.

Example 7 : The figures given below are (a) the observed frequencies of a distribution, (b) the frequencies of the normal distribution, having the same mean, standard deviation and the total frequency as in (a).

(a) 1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1.

(b) 2, 15, 66, 210, 484, 799, 943, 799, 484, 210, 66, 15, 2.

Apply χ^2 test of goodness of fit. **(M.U. 2004)**

Sol. : Since the frequencies at the beginning and end are less than 10, we group them and then apply the χ^2 test.

Calculation of χ^2

O	: 1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1
E	: $\underbrace{2,\ 15}$, 66, 210, 484, 799, 943, 799, 484, 210, 66, $\underbrace{15,\ 2}$
$(O-E)^2/E$	: 0·94, 0·0, 0·48, 0·25, 0·06, 0·38, 0·06, 0·25, 0·48, 0, 0·94

$$\therefore \quad \chi^2 = \Sigma \frac{(O-E)^2}{E} = 3.84.$$

(i) Null Hypothesis H_o : The fit is good.

Alternative Hypothesis H_a : The fit is not good.

(ii) Calculation of test statistic : $\chi^2 = 3.84$

(iii) **Level of significance :** $\alpha = 0{\cdot}05$.

Number of degrees of freedom : There are originally 13 classes. Since they are reduced to 11 by grouping twice, the degrees of freedom is reduced by 2. Further, since the mean, the standard deviation and the total frequency of original data are used, three constraints are introduced, reducing the degree of freedom by 3. (For calculating the mean and the standard deviation three sums $\sum f_i$, $\sum f_i x_i$ and $\sum f_i x_i^2$ are required. Hence, the degree of freedom is reduced by 3.) Thus, the d.f. $= 13 - (2) - (3) = 8$.

(iv) **Critical value :** For 8 d.f. at 5% level of significance, the table value of $\chi^2 = 15{\cdot}51$.

(v) **Decision :** Since the calculated value of $\chi^2 = 3{\cdot}84$ is less than the table value $\chi^2 = 15{\cdot}51$, the null hypothesis is accepted.

$\therefore$ The fit is good.

Example 8 : The number of defects in printed circuit board is hypothesised to follow Poisson distribution. A random sample of 60 printed boards showed the following data.

Number of defects	0	1	2	3
Observed frequency	32	15	9	4

Does the hypothesis of Poisson distribution seem appropriate. **(M.U. 2004)**

Sol. : (i) **Null Hypothesis H_o :** The defects follow Poisson distribution.

Alternative Hypothesis H_a : The defects do not follow Poisson distribution.

(ii) **Calculation of test statistic :** The expected frequencies of Poisson's distribution are given by

$$\text{Expected frequency} = Np = N \times \frac{e^{-m} m^x}{x!}$$

where, m = mean of the distribution, x = random variable, N = number of observations.

$$\text{Here, } m = \frac{\Sigma f x}{\Sigma f} = \frac{32 \times 0 + 15 \times 1 + 9 \times 2 + 4 \times 3}{32 + 15 + 9 + 4} = \frac{45}{60} = 0.75$$

$$\text{Exp. Freq.} = 60 \times \frac{e^{-0.75}(0.75)^x}{x!}, \quad x = 0, 1, 2, 3$$

Of zero defects = 28.32, Of one defect = 21.25, Of two defects = 7.97,

Of three defects = 60 − (Sum of the above frequencies)
= 60 − 57·54 = 2·46.

Calculation of $(O - E)^2 / E$

No. of defects	O	E	$(O - E)^2 / E$
0	32	28·32	0·4782
1	15	21·25	1·8382
2	9	7·97	0·6332
3	4	2·46	
		Total	2·9494

$$\therefore \quad \chi^2 = \Sigma \frac{(O - E)^2}{E} = 2{\cdot}95$$

(iii) **Level of significance :** $\alpha = 0{\cdot}05$.

Degrees of freedom $= 4 - (1 + 2) = 1$

(The number of degrees of freedom for each class is one. There are originally 4 classes. Hence, the degrees of freedom originally is 4. But we reduced the classes by one, thus, reducing the degree by one. Further, while calculating the parameter m, we used two sums Σf_i and $\Sigma f_i x_i$, thus, reducing the degree of freedom by two.)

(iv) **Critical value :** For 1 degree of freedom at 5% level of significance the table value of χ^2 is 3·84.

(v) **Decision :** Since the calculated value of $\chi^2 = 2{\cdot}95$ is less than the table value of $\chi^2 = 3{\cdot}84$, the null hypothesis is accept.

$\therefore$ The defects follow Poisson's distribution.

Example 9 : The following mistakes per page were observed in a book.

No. of mistakes per page	0	1	2	3	4
No. of pages	211	90	19	5	0

Fit a Poisson distribution and test the goodness of fit.

Sol. : (i) **Null Hypothesis H_o :** The mistakes **follow Poisson's distribution.** The fit is good.

Alternative Hypothesis H_a : The mistakes do not follow Poisson's distribution.

(ii) **Calculation of test statistic :** The expected frequencies by Poisson's distribution are given by

$$\text{Expected frequency} = Np = N \times \frac{e^{-m} m^x}{x!}$$

where, m = mean of the distribution, x = random variable,
N = number of observations.

$$\text{Here, } m = \frac{\Sigma f x}{\Sigma f} = \frac{211(0) + 90(1) + 19(2) + 5(3) + 0(4)}{211 + 90 + 19 + 5 + 0} = \frac{143}{325} = 0{\cdot}44$$

$x = 0, 1, 2, 3, 4$; $N = 325$.

$$\text{Exp. Freq.} = \frac{325 \times e^{-0{\cdot}44}(0{\cdot}44)^x}{x!}$$

Since $N \times e^{-m}$ appears in every calculation we calculate it first.

$$N \times e^{-m} = 325 \times e^{-0{\cdot}44} = 209{\cdot}31$$

Now, the expected frequencies are

$$\text{Of zero mistakes} = 325 \times \frac{e^{-0{\cdot}44}(0{\cdot}44)^0}{0!} = 209{\cdot}31(1) = 209{\cdot}31$$

$$\text{Of one mistakes} = 325 \times \frac{e^{-0{\cdot}44}(0{\cdot}44)^1}{1!} = 209{\cdot}31(0{\cdot}44) = 92{\cdot}1$$

Of two mistakes $= 325 \times \frac{e^{-0.44}(0.44)^2}{2!} = 209.31(0.0968) = 20.26$

Of three mistakes $= 325 \times \frac{e^{-0.44}(0.44)^3}{3!} = 209.31(0.0142) = 2.97$

Of four mistakes $= 325 -$ (sum of the above frequencies)

$= 325 - 324.64 = 0.36.$

Calculation of $(O - E)^2 / E$

No. of mistakes	O	E	$(O-E)^2$	$(O-E)^2/E$
0	211	209.31	2.87	0.014
1	90	92.10	4.41	0.048
2	19	20.26		
3	5	2.97	0.17	0.007
4	0	0.36		
			Total	0.069

$\therefore \chi^2 = \Sigma \frac{(O-E)^2}{E} = 0.069$

(iii) Level of significance : $\alpha = 0.05$.

Degrees of freedom $= 5 - 4 = 1$.

(The number of degrees of freedom is 1 for each class. There are 5 classes originally. Hence, the degrees of freedom originally is 5. But we reduced the classes by two, thus reduced the degrees of freedom by 2. Further, while calculating the parameter m, we used two sums Σf_i and $\Sigma f_i x_i^2$, thus, reducing the degree of freedom again by 2.)

Hence, the number of degrees of freedom $= 5 - (2 + 2) = 1$.

(iv) Critical value : For 1 degree of freedom at 5% level of significance the table value of χ^2 is 3.84.

(v) Decision : Since the calculated value of $\chi^2 = 0.069$ is less than the table value of $\chi^2 = 3.84$ we accept the hypothesis.

$\therefore$ The mistakes follow Poisson's distribution.

Example 10 : Weights in kgs. of 10 students are given below.

38, 40, 45, 53, 47, 43, 55, 48, 52, 49.

Can we say that the variance of the normal distribution from which the above sample is drawn is 20 kg. ? (M.U. 2004)

Sol. : **Computation of $(x_i - \bar{x})^2$**

x : 38, 40, 45, 53, 47, 43, 55, 48, 52, 49.

$(x_i - 47)^2$: 81, 49, 4, 36, 0, 16, 64, 1, 25, 04.

$\bar{x} = \frac{\Sigma x_i}{n} = \frac{470}{10} = 47$; $\quad \Sigma(x_i - \bar{x})^2 = 280.$

(i) **Null Hypothesis H_0 :** $\sigma = \sqrt{20}$

Alternative Hypothesis H_a : $\sigma \neq \sqrt{20}$

(ii) **Calculation of test statistic :** $\chi^2 = \frac{\Sigma(x_i - \bar{x})^2}{\sigma^2} = \frac{280}{20} = 14.$

(iii) **Level of significance :** $\alpha = 0.05$.

Degree of freedom : $10 - 1 = 9$.

(iv) **Critical value :** For 9 degrees of freedom at 5% level of significance, the table value of χ^2 is 16.99.

(v) **Decision :** Since the calculated value of $\chi^2 = 14$ is less than the critical value $\chi^2 = 16.99$, the hypothesis is accepted.

$\therefore$ The sample was drawn from the normal population with variance 20.

Example 11 : Five dice were thrown 192 times and the number of times 4, 5 or 6 were obtained are as follows.

No. of dice showing 4, 5, 6	: 5, 4, 3, 2, 1, 0.
Frequency	: 6, 46, 70, 48, 20, 2.

Calculate χ^2. (M.U. 2002)

Sol. : Assuming that all 5 dice are fair, probability of getting 4, 5 or 6 in throw of single die is 1/2.

$\therefore$ By Binomial Theorem, probability distribution is given by

$$^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}, \; x = 0, 1, 2, 3, 4, 5.$$

The number of times getting $x = 0, 1, 2, 3, 4, 5$ is given by

$$192 \cdot {}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}$$

Thus, the expected frequencies are given as

No. of successes	: 5,	4,	3,	2,	1,	0.
Exp. frequency	: 6,	30,	60,	60,	30,	6.
Obse. frequency	: 6,	46,	70,	48,	20,	2.
$(O-E)^2$	: 0,	256,	100,	144,	100,	16.
$\frac{(O-E)^2}{E}$	: 0,	8.53,	1.67,	2.40,	1.67,	2.67.

$\therefore \chi^2 = \Sigma \frac{(O-E)^2}{E} = 16.94.$

13. Hypothesis Concerning Several Proportions

If we want to test the equality of parameter p of several Binomial distributions we have to use χ^2 - test as explained below.

For example, if we want to test the consumer responses to several products of the same kind, to test the proportion of defective parts produced on different machines. We assume that the proportion of items having a particular characteristic is same in all samples. On the basis of this assumption, we calculate the expected E frequency of each cell. Then as usual we calculate

$$\chi^2 = \Sigma \frac{(O-E)^2}{E}.$$

We compare this calculated value with the table value. If the calculated value is less than the table value, we accept the hypothesis that the proportion is same otherwise we reject it.

Example 1 : Samples of three shipments A, B, C of defective items gave the following results.

	Shipment A	Shipment B	Shipment C	Total
Defective :	5	8	9	21
Non-defective :	35	42	51	129
Total :	40	50	60	150

Test whether the proportion of defective items is same in the three shipments at 0·05 level of significance.

Sol. : (i) Null Hypothesis $H_o : p_1 = p_2 = p_3$.

Alternative Hypothesis $H_a : p_1 \neq p_2 \neq p_3$.

(ii) Calculation of test statistic : If the proportion of defective item is same in the three shipment then there will be $\frac{40 \times 21}{150} = 5{\cdot}6$ defective items in shipment A and the remaining $40 - 5{\cdot}6 = 34{\cdot}4$ non-defective. In the same way defectives in B $= \frac{50 \times 21}{150} = 7$ and remaining $50 - 7 = 43$ non-defective. Defective in C $= \frac{60 \times 21}{150} = 8{\cdot}4$ and remaining $60 - 8{\cdot}4 = 51{\cdot}6$ non-defective.

Calculation of χ^2

O	E	$(O-E)^2$	$(O-E)^2/E$
5	5·6	0·36	0·064
35	34·4	0·36	0·029
8	7·0	1·00	0·143
42	43·0	1·00	0·023
9	8·4	0·36	0·043
51	51·6	0·36	0·007
			$\chi^2 = 0{\cdot}309$

(iii) Level of significance : $\alpha = 0{\cdot}05$.

Degrees of freedom $= (r-1)(c-1) = (2-1)(3-1) = 2$.

(iv) Critical value : For 2 degrees of freedom and 5% level of significance, the table value of $\chi^2 = 5{\cdot}991$.

(v) Decision : Since the calculated value of $\chi^2 = 0{\cdot}0309$ is less than the table value of $\chi^2 = 5{\cdot}991$, the hypothesis is accepted.

∴ Proportion of defective is same in all shipments.

Example 2 : An item is produced on four machines and inspection of samples of these items show the following results.

	M - 1	M - 2	M - 3	M - 4	Total
Grade I :	30	42	32	45	149
Grade II :	20	18	18	15	71
Total :	50	60	50	60	220

Test at 0·05 level of significance whether the proportion of Grade I items is same in the production of all machines.

Sol. : (i) Null Hypothesis $H_o : p_1 = p_2 = p_3 = p_4$.

Alternative Hypothesis $H_a : p_1 \neq p_2 \neq p_3 \neq p_4$.

(ii) Calculation of test statistic : If the proportion of Grade I item is same in the production of all machines then there will be $\frac{50 \times 149}{220} = 33{\cdot}86$ items of grate I in the production of machine M - 1 and the remaining $50 - 33{\cdot}86 = 16{\cdot}14$. In the same way,

Grade I items in B $= \frac{60 \times 149}{220} = 40{\cdot}64$ and remaining $60 - 40{\cdot}64 = 19{\cdot}36$.

Grade I items in C $= \frac{50 \times 149}{220} = 33{\cdot}86$, remaining $50 - 33.86 = 16{\cdot}14$.

Grade I items in D $= \frac{60 \times 149}{220} = 40{\cdot}64$ and remaining $60 - 40{\cdot}64 = 19{\cdot}36$.

Calculation of χ^2

O	E	$(O-E)^2$	$(O-E)^2/E$
30	33·86	14·90	0·30
20	16·14	14·90	0·92
42	40·64	1·85	0·05
18	19·36	1·85	0·09
32	33·86	3·46	0·10
18	16·14	3·46	0·21
45	40·64	19·01	0·47
15	19·36	19·01	0·98
			$\chi^2 = 3{\cdot}12$

(iii) Level of significance : $\alpha = 0{\cdot}05$.

Degrees of freedom $= (r-1)(c-1) = (2-1)(4-1) = 3$.

(iv) Critical value : For 3 degrees of freedom and 5% level of significance, the table value of $\chi^2 = 7{\cdot}815$.

(v) **Decision :** Since the calculated value of $\chi^2 = 3{\cdot}12$ is less than the table value of $\chi^2 = 7{\cdot}815$, the hypothesis is accepted.

$\therefore$ Proportion of Grade I items is same for all machines.

EXERCISE - IV

Type I : Independence of Attributes

1. In an experiment on immunisation of cattle from Tuberculosis the following results were obtained.

	Affected	Not affected	Total
Inoculated	267	27	294
Not inoculated	757	155	912
Total	1024	182	1206

Use χ^2 - test to determine the efficacy of vaccine in preventing tuber-culosis. **(M.U. 2015)**

[**Ans. :** $\chi^2 = 10{\cdot}19$, d.f. = (2 − 1) (2 − 1) = 1, Table value of $\chi^2 = 3{\cdot}84$, Effective]

2. Based on the following data determine if there is a relation between literacy and smoking.

	Smokers	Non-smokers
Literates	83	57
Illiterates	45	68

(M.U. 2006) [**Ans. :** $\chi^2 = 9{\cdot}19$. Yes]

3. A total of 3759 individuals were interviewed in a public opinion survey on a political proposal. Of them 1872 were men and the rest well women. A total of 2257 individuals were in favour of the proposal and 917 were opposed to it. A total of 243 men were undecided and 442 women were opposed to it. Do you justify or contradict the hypothesis that there is no association between sex and attitude at 5% level of significance. **(M.U. 2007, 09, 14)**

[**Ans. :** $\chi^2 = 18{\cdot}76$, Sex and attitude are associated.]

4. A sample of 300 students of under-graduate and 300 students of post-graduate classes of a university were asked to give their opinion on autonomy of colleges. 190 of the under graduate and 210 of the post-graduate students favoured autonomous states.

Present the above facts in the form of a frequency table and test at 5% level the opinions of under-graduate and post-graduate students on autonomous status of colleges are independent. (Table value of χ^2 at 5% level for 1 d.f. is 3·84)

[**Ans. :** $\chi^2 = 3$, Independent]

5. Out of 800 persons 25% were literate and 300 had travelled beyond the limits of the district 40% of the literates were among those who had not travelled. Prepare a 2 × 2 table and test at 5% level of significance whether there is any relation between travelling and literacy.

[**Ans. :** $\chi^2 = 57{\cdot}6$, There is a relation]

6. In the contingency table given in adjoining table, use χ^2 - test for independence of hair-colour and eye-colour of persons.

Eye colour	Hair colour		
	Light	Dark	Total
Blue	26	9	35
Brown	7	18	25
Total	33	27	60

[**Ans. :** $\chi^2 = 13{\cdot}59$, d.f. = 1, Table value of $\chi^2 = 3{\cdot}84$, Attributes are associated.]

7. Calculate the expected frequencies for the following data presuming the two attributes viz. condition of home and condition of child independent.

Condition of child	Condition of Home	
	Clean	Dirty
Clean	70	50
Fairly clean	80	20
Dirty	35	45

Use χ^2 - test at 5% level to find whether the two attributes are independent. (Table value of χ^2 at 5% for 2, 3, 4 d.f.s. are 5·991, 7·815 and 9·488 respectively).

[**Ans. :** $\chi^2 = 26{\cdot}25$, d.f. = 2, Not independent]

8. A certain drug is claimed to be effective in curing fever in an experiment on 164 persons with fever. Half of them were given the drug and half were given sugar pills. The results obtained are shown in the following table. Test the hypothesis that the drug is effective in curing fever.

	Helped	Harmed	No effect	Total
Drug	52	10	20	82
Sugar pills	44	12	26	82
Total	96	22	46	164

[**Ans. :** $\chi^2 = 1{\cdot}86$, d.f. = $(r-1)(c-1) = (3-1)(2-1) = 2$. Table value of $\chi^2 = 5{\cdot}99$. Not effective.]

9. The following table gives the information regarding the colour of hair and the colour of eye.

Eye colour	Hair colour			
	Black	Fair	Brown	Total
Brown	10	22	32	64
Blue	15	28	29	72
Grey	25	20	19	64
Total	50	70	80	200

Use χ^2 - test to check whether there is any association between the hair colour and eye colour.

[**Ans. :** $\chi^2 = 10{\cdot}64$, d.f. = 4, Table value of $\chi^2 = 9{\cdot}48$, No]

10. In an industry 200 workers employed for a specific job were classified according to their performance and training received to test independence of training received and performance. The data are summarised as follows.

Performance	Trained	Untrained	Total
Good	100	20	150
Not good	50	30	50
Total	150	50	200

Use χ^2 - test for independence at 5% level of significance and write your conclusion.

(Given $\chi^2 = 3{\cdot}84$ for 1 d.f.)

[**Ans. :** $\chi^2 = 11{\cdot}11$, Training and performance are not independent]

11. Table below shows the performances of students in Mathematics and Physics. Test the hypothesis that the performance in Mathematics is independent of performance is Physics.

Grades in Physics	Grades in Maths		
	High	Medium	Low
High	56	71	12
Medium	47	163	38
Low	14	42	81

(M.U. 2014)

[**Ans. :** $\chi^2 = 132{\cdot}31$, Table value of χ^2 at 5% level of significance for $v = 4$ is 9·49. Reject the hypothesis.]

12. The result of a certain survey shows that out of 50 ordinary shops of small size 35 are managed by men of which 17 are in cities. 12 shops in villages are run by women. Can it be inferred that shops run by women are relatively more in villages than in cities ?

Use χ^2 - test. (Table value of χ^2 for 1 d.f. is 3·84.) [**Ans. :** $\chi^2 = 3{\cdot}57$, No]

Type II : Goodness of Fit

1. The following figures show the distribution of digits in numbers chosen at random from a telephone directory.

Digit :	0,	1,	2,	3,	4,	5,	6,	7,	8,	9.
Freq. :	1026,	1107,	997,	966,	1075,	933,	1107,	972,	964,	853.

Test at 5% level whether digits may be taken to occur equally frequently in the directory. (Table value of χ^2 at 9 d.f. is 16·92.) [**Ans. :** $\chi^2 = 59{\cdot}36$, No]

2. The following table gives the number of accidents in a district during a week. Apply χ^2 - test to find whether the accidents are uniformly distributed over the week.

Day :	Sun.,	Mon.,	Tues.,	Wed.,	Thur.,	Fri.,	Sat.
No. of accidents :	13,	12,	11,	9,	15,	10,	14.

[**Ans. :** $\chi^2 = 2{\cdot}33$, d.f. $= 7 - 1 = 6$. Table value of $\chi^2 = 12{\cdot}59$, Yes]

3. The following data represent the monthly sales in ₹ of a certain retail store in a leap year. Examine if daily sales are uniform throughout the year.

6100, 6600, 6350, 6050, 6250, 6200,

6300, 6250, 5800, 6000, 6150, 6150.

[**Ans. :** $\overline{X} = 73200 / 366 =$ ₹ 200 per day.

$E_1 = 200 \times 31 = 6200$ for Jan., $E_2 = 200 \times 29 = 5800$ for Feb. etc.

$\chi^2 = 40{\cdot}6$. Reject the hypothesis.]

4. According to a theory the proportion of a commodity in the four classes A, B, C, D should be 9 : 4 : 2 : 1. In a survey of 1600 items of this commodity the numbers in the four classes were 882, 432, 168 and 118. Does the survey support the theory ?

(**Hint :** Expected frequencies in the four classes are

$$A = \frac{9}{16} \times 1600 = 900, \quad B = 400, \quad C = 200, \quad D = 100.)$$

[**Ans. :** $\chi^2 = 16{\cdot}06$, d.f. $= 4 - 1 = 3$, Table value of $\chi^2 = 7{\cdot}81$, No]

5. Of the 64 offsprings of a certain cross between guinea pigs, 34 were red, 10 were black and 20 were white. According to generic model these numbers should be in the ratio 9 : 3 : 4. Use χ^2 - test to check whether the data are consistent with the model.

[**Ans. :** $\chi^2 = 1{\cdot}31$, d.f. = 2, Table value of $\chi^2 = 5{\cdot}99$, Consistent]

6. The following table shows the number of heads obtained when 5 coins were tossed 3200 times.

Heads :	0	1	2	3	4	5
Frequency :	80	570	1100	900	500	50

Fit a Binomial distribution and test whether the coins are unbiased.

(**Hint :** Expected Frequencies are

$$= 3200 \cdot {}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}, \quad x = 0, 1, 2, 3, 4, 5.$$

$= 100, 500, 1000, 1000, 500, 100.$)

[**Ans. :** $\chi^2 = 58{\cdot}8$, d.f. $= 5 - 1 = 4$, Table value of $\chi^2 = 11{\cdot}07$, No]

7. Twelve dice were thrown 4096 times and the number of appearance of 6 each time was noted. The results were recorded as follows :

No. of successes :	0,	1,	2,	3,	4,	5,	6 and above.
Frequency :	447,	1145,	1181,	796,	380,	115,	32.

Fit a Binomial distribution and test whether the dice are unbiased.

(**Hint :** Expected frequency $= 4096 \times {}^{12}C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{12-x}$

$= 459, 1103, 1209, 803, 363, 117, 42.$

where, $x = 0, 1, 2, 3, 4, 5$. The last frequency is obtained by subtracting the sum from 4096.)

[**Ans. :** $\chi^2 = 6{\cdot}18$, d.f. $= 7 - 1 = 6$, Table value of $\chi^2 = 12{\cdot}59$, Yes]

8. A survey of 320 families with 5 children revealed the distribution of boys as given below. Is the result consistent with the hypothesis that male and female births are equally probable ?

No. of boys	:	0	1	2	3	4	5
No. of families	:	8	40	88	110	56	18

(**Hint :** Expected frequencies $= 320 \cdot {}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x} = 10, 50, 100, 100, 50, 10.$)

[**Ans. :** $\chi^2 = 11{\cdot}93$, d.f. $= 6 - 1 = 5$, Table value of $\chi^2 = 11{\cdot}07$, No]

9. The following mistakes per page were observed in a book.

No. of mistakes per page	:	0	1	2	3	4	Total
No. of pages	:	17167	1861	124	2	1	19155

Fit a Poisson distribution and test the goodness of fit. **(M.U. 2005)**

(**Hint :** $\overline{X} = 0{\cdot}11$,

Expected frequencies $= 19155 \cdot e^{-0{\cdot}11} \frac{(0{\cdot}11)^x}{x!} = 17150, 1887, 104, 4, 0.$)

[**Ans. :** $\chi^2 = 3{\cdot}7$, d.f. $= 5 - 2 - 2 = 1$, Table value of $\chi^2 = 3{\cdot}84$. Fit is not good]

10. A list of wars of modern civilisation provided the following data for the period 1550-1981. Fit a Poisson distribution to the data and test the goodness of fit.

No. of outbreaks in year	:	0,	1,	2,	3,	4,	5	Total
No. of years	:	223,	142,	48,	15,	4,	0	432

(**Hint :** $\overline{X} = \frac{\Sigma f x}{\Sigma f} = 0{\cdot}69$,

Expected frequencies $= 320 \cdot e^{-0{\cdot}69} \frac{(0{\cdot}69)^x}{x!} = 217, 149, 52, 12, 2, 0.$)

[**Ans. :** $\chi^2 = 2{\cdot}6$, d.f. $= 6 - 4 = 2$, Table value of $\chi^2 = 5{\cdot}99$, Fit is good. Combine the last three groups]

EXERCISE - VI

Theory

1. Write a short note on degrees of freedom in connection with the applications of χ^2 - test.
2. Write two uses of χ^2 - test.
3. What are the purposes for which χ^2 - tests of significance are used ? Explain and illustrate.
4. What is meant by parametric and non parametric tests ?
5. Explain $r \times c$ table.
6. Discuss the importance of χ^2 - test. How is it used to test independence of attributes ? **(M.U. 2003)**
7. Define student's t-distribution and state its properties. **(M.U. 2001)**
8. Define χ^2 distribution and state its uses.

9. State the uses of χ^2 test and the conditions for these uses. **(M.U. 2005)**
10. What is test of "independence" ? **(M.U. 1998)**
11. Why the test-statistics are different for large samples and small samples ? **(M.U. 2005)**
12. What is the necessity of level of significance ? **(M.U. 2005)**
13. What are type I and type II errors and how are they minimised ? **(M.U. 2005)**

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CHAPTER 11

Simplex Method

1. Introduction

It is obvious that the graphical method cannot be used for solving an L.P.P. if there are more than two variables in the objective function. In such cases a simple and widely used method is the **simplex method**. The method was developed by G. Dantzing in 1947.

In simplex method we move step by step from one vertex of the region of feasible solutions to another in such a way that the value of the objective function is more (or less) at the next vertex than that at the preceeding vertex. The procedure of jumping from one vertex to another is repeated. Since there are finite number of vertices, by this procedure we arrive at an optimal solution or at an unbounded solution.

Georege Dantzig (1914 - 2005)

Georege Dantzig was a great American mathematician known for his pioneering work in Operation Research, Computer Science, Economics and Statistics. His father was a mathematician and linguist and his mother was a linguist. His interest in mathematics was nurtured by his father, challenging him with complicated problems particularly in projective geometry. He earned his doctorate in 1946 of University of California, Berkaley. He was the recipient of many honours, including the first Joh von Neumann Theory Prize in 1974, the National Medal Of Science in 1975, an honorary doctorate from the university of Mary land in 1976. His work in operations research is used in airline industry, in shipping industry, in oil industry, in revenue management, in telecommunications, in architecture, in circuit design and in many other areas.

2. General Linear Programming Problem

Any L.P. problem has three aspects (i) objective function, (ii) constraints, (iii) non-negative restrictions. At this stage we need to learn a few new terms. Any L.P.P. with more than two variables can be expressed as given below.

Find the values of $x_1, x_2, \ldots, x_n$ which maximise (or minimise) the objective function

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \qquad \ldots\ldots (1)$$

subject to the restrictions

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\le b_1 \ (\text{or} \ge b_1) \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\le b_2 \ (\text{or} \ge b_2) \\ a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n &\le b_3 \ (\text{or} \ge b_3) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\le b_m \ (\text{or} \ge b_m) \end{aligned} \qquad \ldots\ldots (2)$$

and which satisfy the non-negativity restrictions.

$$x_1, x_2, x_3, \ldots x_n \ge 0 \qquad \ldots\ldots (3)$$

The constraints in (2) may be of greater than or equal to type (≥), or less than or equal to type (≤) or simply equal to type (=).

Definition 1 : The coefficients $c_1, c_2, \ldots, c_n$ of $x_1, x_2, \ldots, x_n$ in the object functions are called **cost coefficients.**

Definition 2 : The constants on the right hand side of the constraints *i.e.* $b_1, b_2, \ldots, b_m$ are called the **stipulations.**

Definition 3 : The constants a_{ij} $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$ which are the coefficients of x's in the constraints are called **structural coefficients.**

Definition 4 : Any set of values $x_1, x_2, \ldots, x_n$ which satisfies the constraints given in (2) is called a **solution** of the L.P.P.

Note that here we have made no reference to non-negativity restrictions (3) or objective function (1).

Definition 5 : Any solution which satisfies the non-negativity restrictions is called a **feasible solution.**

Note that we have not yet made any reference to objective function.

Definition 6 : Any feasible solution which maximises (or minimises) the objective function is called the **optimal solution** of the given L.P.P.

Definition 7 : If the constraints of an L.P.P. are of less than or equal to type (≤) then we can add new non-negative variables so that the constraints can be expressed as equalities.

If we have $\qquad \sum_{j=1}^{n} a_{ij}x_j \le b_i \quad (i = 1, 2, \ldots, k)$

then we can add non-negative variables s_i, so that

$$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i \quad (i = 1, 2, \ldots, k)$$

These new variables s_i are called **slack variables.** It is clear that the **number of slack variables will be equal to the number of constraints.**

Definition 8 : If the constraints of an L.P.P. are of greater than or equal to type (≥) then we can **subtract** new non-negative variables so that the constraints can be expressed as equalities.

If we have $\qquad \sum_{j=1}^{n} a_{ij}x_j \ge b_i \quad (i = 1, 2, \ldots, k)$

then we can subtract non-negative variables s_i, so that

$$\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i \quad (i = 1, 2, \ldots, k)$$

These new variables s_i are called surplus variables.

Note ....

In definition 8 we have defined the variable to be subtracted in "greater than" type constraints as surplus. But we may consider it as *"negative slack variable"* and call it a slack variable. Hence, here after we shall call all variables to be added as slack variables treating surplus variables to be subtracted as negative slack variables.

We can express the above problem in matrix form as follows. Let

$$A_{(m\times n)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$x_{(n\times 1)} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b_{(m\times 1)} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$c_{(1\times n)} = [c_1 \;\; c_2 \;\; \cdots \;\; c_n]$$

i.e., A is an $(m \times n)$ matrix, x is an $(n \times 1)$ column matrix, b is an $(m \times 1)$ column matrix and c is a $(1 \times n)$ row matrix, then we can write the L.P.P. as

Maximise $\quad Z = cx$

subject to $\quad Ax\ (\le, =, \ge)\ b.$

A is called the **coefficient matrix**, x is the **decision vector**, b is the **requirement vector** and c is the **cost or price vector** of the L.P.P.

Thus, we have in matrix form

$$Z = [c_1 \;\; c_2 \;\; \cdots \;\; c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

subject to

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \le, =, \ge \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

3. Canonical and Standard Forms of L.P.P.

After formulating the L.P.P. but before proceeding to find its solution, the problem is required to be put in a certain form. We shall here consider two forms. (i) The **canonical** *form*, (ii) The **standard form**.

(a) The Canonical Form

A general linear programming problem can be put in the following form.

Maximise $\quad z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the constraints

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &\le b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &\le b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &\le b_m \end{aligned}$$

Or in short $z = \sum_{i=1}^{n} c_i x_i$

subject to $\sum_{i=1}^{n} a_{ij} x_i \le b_j, \quad j = 1, 2, \ldots, m$

$x_i \ge 0, \quad i = 1, 2, \ldots, n.$

This form of L.P.P. is called the **Canonical Form.**

The characteristics of this form are :

(i) The objective function is of the maximisation type : If the objective function is of the minimisation type, we can, by multiplying throughout by (-1), convert it into maximisation type.

For example, the function minimise $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ can be written as maximise $z' = -z = -c_1 x_1 - c_2 x_2 \ldots - c_n x_n$

(ii) The right-hand side of each constraint should be non-negative : If the right-hand side of a constraint is negative, multiply throughout by (-1) and make the right-hand side constraint non-negative.

For example, if we have $\quad a_1 x_1 + a_2 x_2 \ge -b_1$

we can write it as $\quad -a_1 x_1 - a_2 x_2 \le b_1$

(iii) An equality is changed to an inequality : If a constraint is in the form of an equation then it needs to be expressed as an inequality.

For example, we can write $a_1 x_1 + a_2 x_2 = b$ as $a_1 x_1 + a_2 x_2 \le b$ and *i.e.* $a_1 x_1 + a_2 x_2 \ge b$ *i.e.*, $-a_1 x_1 - a_2 x_2 \le -b$.

(Or we can also express the above equality $a_1 x_1 + a_2 x_2 = b$ as $a_1 x_1 + a_2 x_2 \le b$ and $a_1 x_1 + a_2 x_2 \ge b$.)

(iv) All the variables are non-negative : We should have $x_i \ge 0$. If any variable is unrestricted *i.e.* if it can be positive or negative or zero we can write it as the difference between two non-negative variables. For example, if x_j is unrestricted we can write $x_j = x_j' - x_j''$ where both x_j' and x_j'' are both non-negative *i.e.* $x_j' \ge 0$ and $x_j'' \ge 0$. (For instance we can write $5 = 7 - 2$ say, we can write $0 = x_j' - x_j''$ etc.)

(b) Standard Form

In the standard form we introduce slack variables and express the objective function as well as all constraints in the form of equalities. The L.P.P. given in § 2, page 11-1, 11-2 can be expressed in the standard form as follows.

Maximise $z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n + 0s_1 + 0s_2 + \ldots\ldots + 0s_m$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n + s_1 + 0s_2 + \ldots\ldots + 0s_m = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n + 0s_1 + s_2 + \ldots\ldots + 0s_m = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n + 0s_1 + 0s_2 + \ldots\ldots + s_m = b_m$$
$$x_1, x_2, \ldots\ldots, x_n, s_1, s_2, \ldots\ldots, s_m \geq 0$$

This form is called the **Standard Form.**

The characteristics of the standard form are as follows :

(i) All the constraints are expressed in the form of equations : (except the non-negativity restrictions which remain as inequalities ≥ 0)

This is done by introducing slack variables. For example, we write

$a_1x_1 + a_2x_2 \leq b_1$

as $a_1x_1 + a_2x_2 + s_1 = b_1$

and $a_3x_1 + a_4x_2 \geq b_2$

as $a_3x_1 + a_4x_2 - s_2 = b_2$.

(ii) The right hand sides of all constraints are non-negative : If any constraints has right hand side negative we can multiply throughout by (–1) and change the right hand side from negative to positive.

For example, the constraint $2x_1 + 3x_2 - s_3 = -4$

can be changed to $-2x_1 - 3x_2 + s_3 = 4$

(iii) The objective function should be of maximisation type : If the objective function is of minimisation type, it should be converted to maximisation type by multiplying throughout by (–1).

(iv) All decision variables and also slack variables are non-negative : This means if $x_1, x_2 \ldots\ldots, x_n$ are decision variables and $s_1, s_2, \ldots\ldots, s_n$ are slack variables, then $x_1, x_2, \ldots\ldots s_1, s_2, \ldots\ldots \geq 0$.

4. Types of Solutions

1. Basic Solution : A solution obtained by setting any n variables out of $m + n$ variables equal to zero and solving for remaining m variables, provided the determinant of the coefficients of these m variables is non-zero is called a **basic solution.** Such m variables (some of which may turn out to be zero) are called **basic variables** and the remaining n zero-valued variables are called **non-basic variables**.

Since there are $m + n$ variables and since we solve for m variables the number of basic solutions will be $^{m+n}C_m = \dfrac{(m+n)!}{m!\,n!}$ which is the number of combinations of $m + n$ things taken m at a time.

2. Basic Feasible Solution (BFS) : A basic solution which also satisfies non-negativity restrictions (*i.e.* feasibility restrictions) is called **basic feasible solution.**

The basic feasible solutions are classified into two classes.

(a) Non-degenerate BFS : If in the basic feasible solution obtained all m values of the basic variables x_i $(i = 1, 2, \ldots\ldots, m)$ are positive, the basic feasible solution is called **non-degenerate basic feasible solution**. In other words in a non-degenerate BFS, all m basic variables are positive and the remaining n variables are zero.

(b) Degenerate BFS : If in the basic feasible solution obtained one or more values of the m basic variable are zero, then the basic feasible solution is called **degenerate basic feasible solution.**

Example 1 : Convert the following L.P.P. in the standard form.

Maximise $z = 3x_1 + 5x_2$

subject to $3x_1 + 2x_2 \leq 15$

$2x_1 + 5x_2 \geq 12$

$x_1, x_2 \geq 0.$

Sol. : Introducing the slack variables the problem can be converted to standard form as :

Maximise $z = 3x_1 + 5x_2 + 0s_1 + 0s_2$

subject to $3x_1 + 2x_2 + s_1 + 0s_2 = 15$

$2x_1 + 5x_2 + 0s_1 - s_2 = 12$

$x_1, x_2, s_1, s_2 \geq 0.$

Example 2 : Convert the following L.P.P. in the standard form.

Minimise $z = -3x_1 + 2x_2 - x_3$

subject to $x_1 - 3x_2 + 2x_3 \geq -6$

$3x_1 + 4x_3 \leq 3$

$-3x_1 + 5x_2 \leq 4$

$x_1, x_2 \geq 0$, x_3 is unrestricted.

Sol. : Since the problem is of minimisation type we write $z' = -z$, so that the objective function is of maximisation type. Since in the first constraint the right hand side is negative, we multiply it by (–1), so that it becomes positive and of less than type. Hence, we add slack variable s_1 (≥ 0). Since the second and the third constraints are of less than type we add slack variables s_2 and s_3 both (≥ 0). Since x_3 is unrestricted we write $x_3 = x_3' - x_3''$ where $x_3' \geq 0$, $x_3'' \geq 0$.

Now, the problem becomes :

Maximise $z' = -z = 3x_1 - 2x_2 + x_3' - x_3'' + 0s_1 + 0s_2 + 0s_3$

subject to $-x_1 + 3x_2 - 2x_3' + 2x_3'' + s_1 + 0s_2 + 0s_3 = 6$

$3x_1 + 0x_2 + 4x_3' - 4x_3'' + 0s_1 + s_2 + 0s_3 = 3$

$-3x_1 + 5x_2 + 0x_3' - 0x_3'' + 0s_1 + 0s_2 + s_3 = 4$

$x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \geq 0.$

Example 3 : Convert the following L.P.P. to the standard form.

Maximise $z = 3x_1 + 4x_2 - 2x_3$

subject to

$$6x_1 - 4x_2 \le 5$$
$$3x_1 + x_2 + 4x_3 \ge 11$$
$$4x_1 + 3x_2 \le 2$$
$$x_1 \ge 0,\ x_2 \ge 0.$$

(M.U. 2003)

Sol. : Since the third decision variable is unrestricted we write $x_3 = x_3' - x_3''$ where $x_3' \ge 0, x_3'' \ge 0$. Introducing slack variables $s_1, s_2, s_3\ (\ge 0)$, we write the problem as

Maximise $z = 3x_1 + 4x_2 - 2x_3' + 2x_3'' + 0s_1 + 0s_2 + 0s_3$

subject to

$$6x_1 - 4x_2 + 0x_3' - 0x_3'' + s_1 + 0s_2 + 0s_3 = 5$$
$$3x_1 + x_2 + 4x_3' - 4x_3'' + 0s_1 - s_2 + 0s_3 = 11$$
$$4x_1 + 3x_2 + 0x_3' - 0x_3'' + 0s_1 + 0s_2 + s_3 = 2.$$
$$x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \ge 0.$$

Example 4 : Convert the following L.P.P. to the standard form.

Maximise $z = 3x_1 + 2x_2 + 5x_3$

subject to

$$2x_1 - 3x_2 \le 3$$
$$x_1 + 2x_2 + 3x_3 \ge 5$$
$$3x_1 + 2x_3 \le 2$$
$$x_1, x_2 \ge 0.$$

(M.U. 2004)

Sol. : Since x_3 is unrestricted we write x_3 as $x_3 = x_3' - x_3''$ where $x_3' \ge 0, x_3'' \ge 0$. Introducing slack variables s_1, s_2, s_3, we write the problem as-

Maximise $z = 3x_1 + 2x_2 + 5x_3' - 5x_3'' + 0s_1 + 0s_2 + 0s_3$

subject to

$$2x_1 - 3x_2 + 0x_3' - 0x_3'' + s_1 + 0s_2 + 0s_3 = 3$$
$$x_1 + 2x_2 + 3x_3' - 3x_3'' + 0s_1 - s_2 + 0s_3 = 5$$
$$3x_1 + 0x_2 + 2x_3' - 0x_3'' + 0s_1 + 0s_2 + s_3 = 2$$
$$x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \ge 0.$$

Example 5 : Convert the following L.P.P. in the standard form.

Minimise $z = 2x_1 + 3x_2$

subject to

$$2x_1 - 3x_2 - x_3 = -4$$
$$3x_1 + 4x_2 - x_4 = -6$$
$$2x_1 + 5x_2 + x_5 = 10$$
$$4x_1 - 3x_2 + x_6 = 18$$
$$x_3, x_4, x_5, x_6 \ge 0.$$

(M.U. 2001)

Sol. : Carefully observing the given problem we first note that the decision variables x_1, x_2 are unrestricted. The variables x_3, x_4, x_5, x_6 are slack variables.

Hence, we put $x_1 = y_1 - y_2,\ x_2 = y_3 - y_4,\ x_3 = y_5,\ x_4 = y_6,\ x_5 = y_7$ and $x_6 = y_8$.

The problem then becomes :

Minimise $z = 2y_1 - 2y_2 + 3y_3 - 3y_4$

subject to

$$2y_1 - 2y_2 - 3y_3 + 3y_4 - y_5 = -4$$
$$3y_1 - 3y_2 + 4y_3 - 4y_4 - y_6 = -6$$
$$2y_1 - 2y_2 + 5y_3 - 5y_4 + y_7 = 10$$
$$4y_1 - 4y_2 - 3y_3 + 3y_4 + y_8 = 18$$

Multiply the object function, the first and the second constraints by (-1), then the given problem in the standard form becomes,

Maximise $z' = -z = -2y_1 + 2y_2 - 3y_3 + 3y_4 + 0y_5 + 0y_6 + 0y_7 + 0y_8$

subject to

$$-2y_1 + 2y_2 + 3y_3 - 3y_4 + y_5 + 0y_6 + 0y_7 + 0y_8 = 4$$
$$-3y_1 + 3y_2 - 4y_3 + 4y_4 + 0y_5 + y_6 + 0y_7 + 0y_8 = 6$$
$$2y_1 - 2y_2 + 5y_3 - 5y_4 + 0y_5 + 0y_6 + y_7 + 0y_8 = 10$$
$$4y_1 - 4y_2 - 3y_3 + 3y_4 + 0y_5 + 0y_6 + 0y_7 + y_8 = 18$$
$$y_i \ge 0 \text{ for all } i = 1, 2, \ldots, 8.$$

Example 6 : Express the following L.P.P. in the standard matrix form.

Maximise $z = 2x_1 + 3x_2 + 6x_3$

subject to

$$3x_1 - 2x_2 + 4x_3 \le 5$$
$$2x_1 + 5x_2 = 10$$
$$x_1 + 2x_2 + x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0.$$

Sol. : Introducing slack variables s_1 and s_3 both (≥ 0), we have

Maximise $z = 2x_1 + 3x_2 + 6x_3 + 0s_1 + 0s_2$

subject to

$$3x_1 - 2x_2 + 4x_3 + s_1 + 0s_2 = 5$$
$$2x_1 + 5x_2 + 0x_3 + 0s_1 + 0s_2 = 10$$
$$x_1 + 2x_2 + x_3 + 0s_1 + 0s_2 = 2.$$

The constraints can be written in matrix form as

$$\begin{bmatrix} 3 & -2 & 4 & 1 & 0 \\ 2 & 5 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 2 \end{bmatrix} \text{ i.e. } \mathbf{A\,x = b}$$

Thus, the problem in the matrix form becomes :

Maximise $z = \mathbf{c\,x}$

subject to $\mathbf{A\,x = b,\ x \ge 0}$.

where, $\mathbf{A} = \begin{bmatrix} 3 & -2 & 4 & 1 & 0 \\ 2 & 5 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & -1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 10 \\ 2 \end{bmatrix}$, $\mathbf{c} = [2\ \ 3\ \ 6\ \ 0\ \ 0]$

EXERCISE - I

Convert the following problems to standard form.

1. Maximise $z = 2x_1 + 3x_2 + 5x_3$
   subject to $3x_1 - 2x_2 \le 6$
   $2x_1 + 3x_2 + 5x_3 \ge 12$
   $3x_1 + 4x_3 \le 5$
   $x_1, x_2 \ge 0$. x_3 unrestricted.

   [**Ans. :** Maximise $z = 2x_1 + 3x_2 + 5x_3' - 5x_3'' + 0s_1 + 0s_2 + 0s_3$
   subject to $3x_1 - 2x_2 + 0x_3' - 0x_3'' + s_1 + 0s_2 + 0s_3 = 6$
   $2x_1 + 3x_2 + 5(x_3' - x_3'') + 0s_1 - s_2 + 0s_3 = 12$
   $3x_1 + 0x_2 + 4(x_3' - x_3'') + 0s_1 + 0s_2 + s_3 = 5$
   $x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \ge 0.$]

2. Minimise $z = 3x_1 + 2x_2 - x_3$
   subject to $2x_1 + 3x_2 \le 10$
   $3x_1 + 4x_2 \ge 7$
   $4x_1 + 6x_2 + 3x_3 \le 12$
   $x_1, x_2 \ge 0$. x_2 unrestricted.

   [**Ans. :** Maximise $z' = -3x_1 - 2(x_2' - x_2'') + x_3 + 0s_1 + 0s_2 + 0s_3$
   subject to $2x_1 + 3(x_2' - x_2'') + 0x_3 + s_1 + 0s_2 + 0s_3 = 10$
   $3x_1 + 4(x_2' - x_2'') + 0x_3 + 0s_1 - s_2 + 0s_3 = 7$
   $4x_1 + 6(x_2' - x_2'') + 3x_3 + 0s_1 + 0s_2 + s_3 = 12$
   $x_1, x_2', x_2'', x_3, s_1, s_2, s_3 \ge 0.$]

3. Maximise $z = 2x_1 + 3x_2 + 4x_3$
   subject to $-2x_1 + 3x_2 \le -5$
   $x_1 + 2x_2 + x_3 \ge 6$
   $3x_1 + 2x_3 \le 7,$
   $x_1, x_2, x_3 \ge 0.$

   [**Ans. :** Maximise $z = 2x_1 + 3x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$
   subject to $2x_1 - 3x_2 + 0x_3 - s_1 + 0s_2 + 0s_3 = 5$
   $x_1 + 2x_2 + x_3 + 0s_1 - s_2 + 0s_3 = 6$
   $3x_1 + 0x_2 + 2x_3 + 0s_1 + 0s_2 + s_3 = 7$
   $x_1, x_2, x_3, s_1, s_2, s_3 \ge 0.$]

4. Minimise $z = -3x_1 + 2x_2 + x_3$
   subject to $x_1 - 3x_2 + 2x_3 \le 13$
   $-4x_1 + 2x_2 + x_3 \ge 5$
   $2x_1 - x_3 = -1,$
   $x_1, x_2 \ge 0,$ $x_3 \ge$ or $\le 0.$

   [**Ans. :** Maximise $z' = 3x_1 - 2x_2 - x_3' + x_3'' + 0s_1 + 0s_2$
   subject to $x_1 - 3x_2 + 2(x_3' - x_3'') + s_1 + 0s_2 = 13$
   $-4x_1 + 2x_2 + (x_3' - x_3'') + 0s_1 - s_2 = 5$
   $-2x_1 + 0x_2 + (x_3' - x_3'') + 0s_1 + 0s_2 = 1$
   $x_1, x_2, x_3', x_3'', s_1, s_2 \ge 0.$]

5. Maximise $z = 2x_1 - x_2 + x_3$
   subject to $x_1 + x_2 \le 10$
   $2x_1 - x_2 - x_3 \ge -4$
   $x_2 - x_1 \le 7$
   $x_1, x_2 \ge 0.$ (M.U. 1999)

   [**Ans. :** Maximise $z = 2x_1 - 2x_2 + x_3 + 0s_1 + 0s_2 + 0s_3$
   subject to $x_1 + x_2 + 0x_3 + s_1 + 0s_2 + 0s_3 = 10$
   $-2x_1 + x_2 + x_3 + 0s_1 + s_2 + 0s_3 = 4$
   $-x_1 + x_2 + 0x_3 + 0s_1 + 0s_2 + s_3 = 7$
   $x_1, x_2, x_3, s_1, s_2, s_3 \ge 0.$]

6. Minimise $z = x_1 + 2x_2 + 3x_3$
   subject to $3x_1 + 4x_2 \le 5$
   $5x_1 + x_2 + 6x_3 = 7$
   $8x_1 + 9x_2 \le 9;$ $x_1, x_2, x_3 \ge 0.$ (M.U. 2001)

   Also put the problem in matrix form.

   [**Ans. :** Maximise $z' = -x_1 - 2x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_3$
   subject to $3x_1 + 0x_2 + 4x_3 + s_1 + 0s_2 + 0s_3 = 5$
   $5x_1 + x_2 + 6x_3 + 0s_1 + s_2 + 0s_3 = 7$
   $8x_1 + 9x_2 + 0x_3 + 0s_1 + 0s_2 - s_3 = 9$
   $x_1, x_2, x_3, s_1, s_2, s_3 \ge 0.$

   In matrix form, the problem will be -
   Maximise $z' = \boldsymbol{cx}$
   subject to $\boldsymbol{Ax} = \boldsymbol{b}$
   where, $\boldsymbol{x} = [x_1, x_2, x_3, s_1, s_2, s_3]$

$$\boldsymbol{A} = \begin{bmatrix} 3 & 0 & 4 & 1 & 0 & 0 \\ 5 & 1 & 6 & 0 & 0 & 0 \\ 8 & 9 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{b} = [5,\ 7,\ 9]', \quad \boldsymbol{c} = [-1,\ -2,\ -3]$$

5. Simplex Method

Let us consider again the general L.P.P. in the standard form.

Maximise $z = c_1x_1 + c_2x_2 + \dots + c_nx_n + 0s_1 + 0s_2 + \dots + 0s_m$ .... (1)

subject to $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 + 0s_2 + \dots + 0s_m = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + 0s_1 + s_2 + \dots + 0s_m = b_2$ .... (2)

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + 0s_1 + 0s_2 + \dots + s_m = b_m$$

and $\quad x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_m \geq 0. \quad \dots\dots\dots (3)$

For pedagogical reasons we shall repeat the definitions given earlier and then define some more terms.

(i) Solution : Any set of values $x_1, x_2, \dots, x_n$ which satisfies the constraints given in (2) is called a **solution** of the L.P.P.

Note that we have not yet made any reference to non-negativity restrictions (3) or objective function (1).

(ii) Feasible Solution : Any solution which satisfies non-negativity restrictions is called **feasible** (possible, likely) **solution** of the L.P.P.

Note that we have not yet made any reference to the objective function (3).

(iii) Optimal Solution : Any feasible solution which maximises (minimises) objective function is called **optimal** (most favourable) **solution**.

(iv) Basic Variables : When there are m constraints and $m + n$ (decision and slack) variables, we start with setting any n variables equal to zero and solve the remaining m equations. The n variables which are equated to zero are called **non-basic variables**. The remaining m variables are called **basic variables**. The solution thus obtained by putting any n variables equal to zero and solving the m equations, is called a **basic solution**. Since there are in all $m + n$ variables and m equations, there will be ${}^{m+n}C_m$ basic solutions.

(v) Basic Feasible Solution : If a basic solution satisfies the non-negativity restriction it is called a **basic feasible solution**. If a basic solution contains negative values it is called **a basic infeasible solution.**

(vi) Degenerate and Non-degenerate solutions : If all the m values obtained in a basic feasible solution are non-zero, the solution is called the **non-degenerate basic feasible solution**. On the other hand if some of the values obtained in a basic feasible solution are zero, the solution is called **degenerate basic feasible solution.**

(vii) Outgoing variable and incoming variable : From one basic feasible solution we may obtain another new basic feasible solution by equating one of the basic variables to zero and replacing it by another non-basic variable. The eliminated variable is called the **outgoing variable** and the new replacing variable is called the **incoming variable.**

(viii) Optimal Solution : That basic feasible solution which optimises the objective function (1) [also satisfying conditions (2) and (3)] is called the **optimal solution**.

Example 1 : Determine all basic solutions to the following problem.

Maximise $\quad z = x_1 - 2x_2 + 4x_3$

subject to $\quad x_1 + 2x_2 + 3x_3 = 7$

$\quad 3x_1 + 4x_2 + 6x_3 = 15$ **(M.U. 1999, 2009)**

Sol. : Since there $(m + n =)$ three variables and $(m =)$ two constraints, a basic solution can be obtained by putting $3 - 2 = 1$ variable equal to zero. And there will be ${}^{m+n}C_m = {}^3C_2 = 3$ basic solutions.

Table 11.5.1

No. of basic solutions	Non-basic variables = 0	Basic Variables	Equations And the values of the basic variables	Is the solution feasible $x_i \geq 0$?	Is the solution degenerate ?
1.	$x_3 = 0$	x_1, x_2	$x_1 + 2x_2 = 7$ $3x_1 + 4x_2 = 15$ $x_1 = 1, x_2 = 3$	Yes	No
2.	$x_2 = 0$	x_1, x_3	$x_1 + 3x_3 = 7$ $3x_1 + 6x_3 = 15$ $x_1 = 1, x_3 = 2$	Yes	No
3.	$x_1 = 0$	x_2, x_3	$2x_2 + 3x_3 = 7$ $4x_2 + 6x_3 = 15$ unbounded solution	—	—

Note ....

In the second solution x_2 is the outgoing variable and x_3 is the incoming variable. In the third solution x_1 is the outgoing variable and x_2 is the incoming variable.

Example 2 : Find all the basic solutions of the following system of equations. Identify in each case basic and non-basic variables.

$$2x_1 + x_2 + 4x_3 = 11, \quad 3x_1 + x_2 + 5x_3 = 14.$$

Investigate whether the basic solutions are degenerate or not. Find also the basic feasible solutions. **(M.U. 2004)**

Sol. :

Table 11.5.2

No. of basic solutions	Non-basic variables = 0	Basic Variables	Equations And the values of the basic variables	Is the solution feasible ?	Is the solution degenerate ?
1.	$x_3 = 0$	x_1, x_2	$2x_1 + x_2 = 11$ $3x_1 + x_2 = 14$ $x_1 = 3, x_2 = 5$	Yes	No
2.	$x_2 = 0$	x_1, x_3 outgoing x_2 incoming x_3	$2x_1 + 4x_3 = 11$ $3x_1 + 5x_3 = 14$ $x_1 = 1/2, x_3 = 5/2$	Yes	No
3.	$x_1 = 0$	x_2, x_3 outgoing x_1 incoming x_2	$x_2 + 4x_3 = 11$ $x_2 + 5x_3 = 14$ $x_2 = -1, x_3 = 3$	No	No

Example 3 : Maximise $z = x_1 + 3x_2 + 3x_3$

subject to $x_1 + 2x_2 + 3x_3 = 4$

$2x_1 + 3x_2 + 5x_3 = 7$

Find all basic solutions to the above problem. Which of them are basic feasible, non-degenerate, infeasible basic and optimal basic feasible solutions ? **(M.U. 2003, 07, 15)**

Sol. : *For solution see Table 11.5.3 on page 11-14.*

Example 4 : Consider the following problem.

Maximise $z = 2x_1 - 2x_2 + 4x_3 - 5x_4$

subject to $x_1 + 4x_2 - 2x_3 + 8x_4 \le 2$

$-x_1 + 2x_2 + 3x_3 + 4x_4 \le 1$

$x_1, x_2, x_3, x_4 \ge 0.$

Determine (i) all basic solutions,

(ii) all feasible basic solutions,

(iii) optimal feasible basic solution. **(M.U. 1997)**

Sol. : *For solution see Table 11.5.4 on page 11-15.*

Example 5 : Use matrix vector notation and show that the following system of linear equations has degenerate solutions.

$2x_1 + x_2 - x_3 = 2;\quad 3x_1 + 2x_2 + x_3 = 3.$ **(M.U. 2010)**

Sol. : The system of equations can be written as

$$\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{.................... (1)}$$

Since A is of order 2×3, there can be ${}^3C_2 = \frac{3!}{2!\,1!} = 3$ sub-matrices of order 2×2. They are

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

The variables associated with there sub-matrices are respectively x_1, x_2; x_1; x_3 and x_2, x_3. These are basic variables for these sub-matrices. The variables not associated with these sub-matrices are x_3, x_2 and x_1. These are non-basic variables for these sub-matrices.

Putting $x_3 = 0$ in (1), we get

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \therefore\ 2x_1 + x_2 = 2,\ 3x_1 + 2x_2 = 3.$$

Solving these equations, we get,

$x_1 = 1,\ x_2 = 0$ (basic) and $x_3 = 0$ (non-basic)

Since, one of the basic variables x_2 is zero, this is a degenerate basic solution.

Similarly, putting $x_2 = 0$ in (1), we get,

$$\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \therefore\ 2x_1 - x_3 = 2,\ 3x_1 + x_3 = 3$$

Table 13.5.3 (Sol. of Ex. 3)

No. of basic solutions	Non-basic variables = 0	Basic Variables	Equations And the values of the basic variables	Is the solution feasible ?	Is the solution degenerate ?	Value of z	Is the solution optimal ?
1.	$x_3 = 0$	x_1, x_2	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $x_1 = 2, x_2 = 1$	Yes	No	5	Yes
2.	$x_2 = 0$	x_1, x_3	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $x_1 = 1, x_3 = 1$	Yes	No	4	No
3.	$x_1 = 0$	x_2, x_3	$2x_2 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$ $x_2 = -1, x_3 = 2$	No	No	—	—

Remark : In the second solution, x_2 is the outgoing variable and x_3 is the incoming variable. In the third solution, x_1 is the outgoing variable and x_2 is the incoming variable.

Table 13.5.4 (Sol. of Ex. 4)

No. of basic solutions	Non-basic variables = 0	Basic Variables	Equations And the values of the basic variables	Is the solution feasible ?	Is the solution degenerate ?	Value of z	Is the solution optimal ?
1.	$x_3 = 0$ $x_4 = 0$	x_1, x_2	$x_1 + 4x_2 = 2$ $-x_1 + 2x_2 = 1$ $x_1 = 0, x_2 = 1/2$	Yes	Yes	-1.5	No
2.	$x_2 = 0$ $x_4 = 0$	x_1, x_3 outgoing x_2 incoming x_1	$x_1 - 2x_3 = 2$ $-x_1 + 3x_3 = 1$ $x_1 = 8, x_3 = 3$	Yes	No	28	Yes
3.	$x_1 = 0$ $x_4 = 0$	x_2, x_3 outgoing x_1 incoming x_2	$4x_2 - 2x_3 = 2$ $2x_2 + 3x_3 = 1$ $x_3 = 0, x_2 = 1/2$	Yes	Yes	-1	No
4.	$x_2 = 0$ $x_3 = 0$	x_1, x_4 outgoing x_2 incoming x_1	$x_1 + 8x_4 = 2$ $-x_1 + 4x_4 = 1$ $x_1 = 0, x_4 = 1/4$	Yes	Yes	-1.25	No
5.	$x_1 = 0$ $x_3 = 0$	x_2, x_4 outgoing x_1 incoming x_2	$4x_2 + 8x_4 = 2$ $2x_2 + 4x_4 = 1$ unbounded	—	—	—	—
6.	$x_1 = 0$ $x_2 = 0$	x_3, x_4 outgoing x_2 incoming x_3	$-2x_3 + 8x_4 = 2$ $3x_3 + x_4 = 12$ $x_3 = 0, x_4 = 1/4$	Yes	Yes	-12.5	No

Sol. of Ex. 5 contd....

Solving these equations, we get,

$x_1 = 1, x_3 = 0$ (basic) and $x_2 = 0$ (non-basic)

Since one of the basic variables x_3 is zero, this is a degenerate solution.

Similarly, putting $x_1 = 0$ in (1), we get $\quad \therefore x_2 - x_3 = 2, \quad 2x_2 + x_3 = 3$

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\therefore x_2 = 5/3, x_3 = -1/3$ (basic), $x_2 = 0$ (non-basic)

Since, one of the basic variables x_3 is negative, this is not a feasible solution.

The other two solutions viz. $x_1 = 1, x_2 = 0, x_3 = 0$ and $x_1 = 1, x_2 = 0, x_3 = 0$ are degenerate basic feasible solutions.

Example 6 : Determine all basic feasible solutions of the equations **(M.U. 2010)**

$2x_1 + 6x_2 + 2x_3 + x_4 = 3; \quad 6x_1 + 4x_2 + 4x_3 + 6x_4 = 2.$

Sol. : Since there are $(m + n =)$ four variables and $(m =)$ two constraints, basic solution can be obtained by putting $4 - 2 = 2$ variables equal to zero. And there will be

$${}^{m+n}C_m = {}^4C_2 = \frac{4 \cdot 3}{2} = 6 \text{ basic solutions.}$$

Table 11.5.5

No. of basic solutions	Non-basic variables	Basic Variables	Equations And the values of the basic variables	Is the solution feasible ?
1.	$x_3 = 0$ $x_4 = 0$	x_1, x_2	$2x_1 + 6x_2 = 3$ $6x_1 + 4x_2 = 2$ $x_1 = 0, x_2 = 1/2$	Yes
2.	$x_2 = 0$ $x_4 = 0$	x_1, x_3	$2x_1 + 2x_3 = 3$ $6x_1 + 4x_3 = 2$ $x_1 = -2, x_3 = 7/2$	No
3.	$x_1 = 0$ $x_4 = 0$	x_2, x_3	$6x_2 + 2x_3 = 3$ $4x_2 + 4x_3 = 2$ $x_2 = 1/2, x_3 = 0$	Yes
4.	$x_1 = 0$ $x_3 = 0$	x_2, x_4	$6x_2 + x_4 = 3$ $4x_2 + 6x_4 = 2$ $x_2 = 1/2, x_4 = 0$	Yes
5.	$x_2 = 0$ $x_3 = 0$	x_1, x_4	$2x_1 + x_4 = 3$ $6x_1 + 6x_4 = 2$ $x_1 = 8/3, x_4 = -7/3$	No
6.	$x_1 = 0$ $x_2 = 0$	x_3, x_4	$2x_3 + x_4 = 3$ $4x_3 + 6x_4 = 2$ $x_3 = 2, x_4 = -1$	No

Example 7 : Consider the following problem.

Maximise $z = 2x_1 - 3x_2 + 4x_3 - 5x_4$

subject to $x_1 + 4x_2 - 2x_3 + 8x_4 = 2$

$-x_1 + 2x_2 + 3x_3 + 4x_4 = 1$

$x_1, x_2, x_3, x_4 \geq 0.$

Determine using matrix method (i) all the basic solution, (ii) feasible basic solutions, (iii) optimal feasible basic solutions.

Sol. : The given system of equations can be written in the matrix form $A\,x = b$ where

$$A = \begin{bmatrix} 1 & 4 & -2 & 8 \\ -1 & 2 & 3 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Since, the rank of A is 2, the maximum number of linearly independent solutions is 2. Hence, $4 - 2 = 2$ variables are to be taken zero at a time.

(i) Putting $x_3 = 0, x_4 = 0$, we have

$$\begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{By} \quad R_2 + R_1 \begin{bmatrix} 1 & 4 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\therefore\ x_1 + 4x_2 = 2,\ 6x_2 = 3 \quad \therefore\ x_2 = 1/2 \quad \therefore\ x_1 = 0.$

(ii) Putting $x_2 = 0, x_4 = 0$, we have,

$$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{By} \quad R_2 + R_1 \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\therefore\ x_1 - 2x_3 = 2,\ x_3 = 3 \quad \therefore\ x_1 = 8.$

(iii) Putting $x_1 = 0, x_4 = 0$, we have,

$$\begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{By} \quad R_2 - (1/2)\,R_1 \begin{bmatrix} 4 & -2 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$\therefore\ 4x_2 - 2x_3 = 2,\ x_3 = 0 \quad \therefore\ x_2 = 1/2$

(iv) Putting $x_2 = 0, x_3 = 0$, we have,

$$\begin{bmatrix} 1 & 8 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{By} \quad R_2 + R_1 \begin{bmatrix} 1 & 8 \\ 0 & 12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\therefore\ x_1 + 8x_4 = 2,\ 12x_4 = 3 \therefore\ x_4 = 1/4,\ x_1 = 0.$

(v) Putting $x_1 = 0, x_3 = 0$, we have

$$\begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{By} \quad R_2 - (1/2)\,R_1 \begin{bmatrix} 4 & 8 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\therefore\ 4x_2 + 8x_4 = 2,\ 0x_2 + 0x_4 = 1$

$\therefore$ This leads to infinitely many solutions and hence, its solution is unbounded.

(vi) Putting $x_1 = 0, x_2 = 0$, we have

$-2x_3 + 8x_4 = 2$

$3x_3 + 4x_4 = 1$

$$\therefore\ \begin{bmatrix} -2 & 8 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{By} \quad R_2 - (1/2)\,R_1 \begin{bmatrix} -2 & 8 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$\therefore\ -2x_3 + 8x_4 = 2,\quad 4x_3 = 0,\quad x_4 = 1/4.$

Thus, we have

(i) $x_3 = 0, x_4 = 0, x_1 = 0, x_2 = 1/2, z = -3/2$

(ii) $x_2 = 0, x_4 = 0, x_1 = 8, x_3 = 3, z = 28$

(iii) $x_1 = 0, x_4 = 0, x_2 = 1/2, x_3 = 0, z = -1$

(iv) $x_2 = 0, x_3 = 0, x_1 = 0, x_4 = 1/4, z = -5/4$

(v) $x_1 = 0, x_3 = 0$, unbounded solution.

(vi) $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 1/4, z = -5/4.$

Hence, there are six basic solutions.

All the solutions except (v) are feasible basic solutions.

The solution (ii) is the optimal basic solution.

Example 8 : Is the solution $x_1 = 1, x_2 = 1/2, x_3 = 0, x_4 = 0, x_5 = 0$ a basic solution of the following problem ?

$x_1 + 2x_2 + x_3 + x_4 = 2$

$x_1 + 2x_2 + (1/2)\,x_3 + x_5 = 2.$

Sol. : We can write the given system in matrix form as

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & 2 & 1/2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Putting $x_3 = 0, x_4 = 0, x_5 = 0$, we get

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \therefore\ x_1 + 2x_2 = 2,\ x_1 + 2x_2 = 2$$

The system gives only one equation $\quad x_1 + 2x_2 = 2.$

Geometrically this is a single line and there are infinite pairs (x_1, x_2) which satisfy this equation. The given pair $x_1 = 1, x_2 = 1/2$ is only one of them. All the points lying on the segment of the line $x_1 + 2x_2 = 2$ lying in the first quadrant *i.e.* (x_1, x_2) where $(0 \leq x_1 \leq 2)$ and $(0 \leq x_2 \leq 1)$ give the solution. This is called unbounded solution.

EXERCISE - II

1. Find all the basic feasible solutions of the following system of equations,

$2x_1 + x_2 + x_3 = 3$

$x_1 + x_2 - x_3 = 2$

[**Ans. :** $x_1 = 1, x_2 = 1$ (Basic), $\quad x_3 = 0$ (Non-basic)

$x_2 = 5/2, x_3 = 1/2$ (Basic) $\quad x_1 = 0$ (Non-basic)]

2. Find all the basic feasible solutions of the following system of equations. Also indicate those which are degenerate solutions.

$$2x_1 + x_2 + x_3 = 2$$
$$3x_1 + x_2 - x_3 = 3$$

[**Ans.** : Only basic feasible solutions are degenerate solutions.

$x_1 = 1, x_3 = 0$ (Basic), $x_2 = 0$ (Non-basic)

$x_1 = 1, x_0 = 0$ (Basic) $x_3 = 0$ (Non-basic)]

3. Find all the basic solutions to the following system of equations.

$$2x_1 + 3x_2 + x_3 = 8$$
$$x_1 + 2x_2 + 3x_3 = 5$$

[**Ans.** : (i) $x_1 = 1, x_2 = 2$ (Basic), $x_3 = 0$ (Non-basic)

(ii) $x_1 = 1{\cdot}2, x_3 = 0{\cdot}4$ (Basic), $x_2 = 0$ (Non-basic)

(iii) $x_2 = 2{\cdot}375, x_3 = 0{\cdot}875$ (Basic), $x_1 = 0$ (Non-basic)]

4. Find all the basic solutions to the following problem. State which are not feasible.

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5$$

[**Ans.** : (i) $x_1 = 2, x_2 = 1$ (Basic), $x_3 = 0$ (Non-basic)

(ii) $x_1 = 5, x_3 = -1$ (Basic), $x_2 = 0$ (Non-basic) This solution is not feasible.

(iii) $x_2 = 5/3, x_3 = 2/3$ (Basic), $x_1 = 0$ (Non-basic)]

5. Find all the basic solutions to the following problem.

Maximize $z = 2x_1 + 3x_2 + 4x_3 + 7x_4$

subject to $2x_1 + 3x_2 + x_3 + 4x_4 = 8$

$$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$$
$$x_i \geq 0, \ i = 1, 2, 3, 4.$$

[**Ans.** : (i) $x_1 = 1, x_2 = 2$ (Basic), $x_3 = 0, x_4 = 0$ (Non-basic)

(ii) $x_1 = 22/9, x_4 = 7/9$ (Basic), $x_2 = 0, x_3 = 0$ (Non-basic)

(iii) $x_2 = 51/20, x_3 = 7/20$ (Basic), $x_1 = 0, x_4 = 0$ (Non-basic)

(iv) $x_3 = 44/31, x_4 = 51/31$ (Basic), $x_1 = 0, x_2 = 0$ (Non-basic)

(v) $x_2 = 44/13, x_4 = -7/13$ (Basic), $x_1 = 0, x_3 = 0$ (Non-basic)

Not feasible. $x_2 = 0, x_4 = 0$ (Non-basic)

(vi) $x_1 = 51/11, x_3 = -14/11$ (Basic), $x_2 = 0, x_4 = 0$ (Non-basic)

Not feasible.]

6. Solve the following L.P.P. by simplex method. **(M.U. 2001)**

Maximise $z = x_1 + x_2 + 3x_3$

subject to $x_1 + 2x_2 + 3x_3 = 9$

$$3x_1 + 2x_2 + 2x_3 = 15$$

[**Ans.** : (i) $x_1 = 3, x_2 = 3, x_3 = 0, z_{Max} = 6$

(ii) $x_1 = 27/7, x_2 = 0, x_3 = 12/7, z_{Max} = 9$

(iii) $x_1 = 0, x_2 = 27/2, x_3 = -6$. Not feasible]

6. Procedure of Simplex Method

Step I : Express The Problem In Standard Form

Write the object function as

$$z - c_1x_1 - c_2x_2 - c_3x_3 + 0s_1 + 0s_2 + 0s_3 = 0 \text{ (objective equation)} \quad \text{.............. (1)}$$

(Since the sign of zero can be taken as positive or negative, we prefer to write the equation (1) with positive signs for zeros.)

Express all the constraints in the form of equalities by adding slack variables. Thus, we have

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + s_1 + 0s_2 + 0s_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + 0s_1 + s_2 + 0s_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + 0s_1 + 0s_2 + s_3 &= b_3 \end{aligned}\right\} \text{(constraint equations)} \quad \text{.............. (2)}$$

This form is called the **Standard Form.**

We shall also refer to equation (1) as the **objective equation** and the equations (2) as **constraint equations.**

Step II : Find the Initial Basic Solution

The slack variables s_1, s_2, s_3 which form the **basis** are called **basic variables.** The variables x_1, x_2, x_3 are called **non-basic variables.** We start with the initial basic solution $x_1 = 0, x_2 = 0, x_3 = 0$ and $s_1 = b_1, s_2 = b_2, s_3 = b_3$. We put the equations (1) and (2) in the form of a table called Simplex Table as shown below. The matrix formed by the coefficients of x_1, x_2, x_3 is called the **body matrix** and the matrix formed by the coefficients of s_1, s_2, s_3 is obviously a **unit matrix.**

Simplex Table

Iteration Number	Basic Variables	Coefficients of x_1	x_2	x_3	s_1	s_2	s_3	R.H.S. (b) Solution	Ratio
1	z	$-c_1$	$-c_2$	$-c_3$	0	0	0	0	
	s_1	a_{11}	a_{12}	a_{13}	1	0	0	b_1	b_1 / a_{12}
s_2 leaves	s_2	a_{21}	a_{22}^{*}	a_{23}	0	1	0	b_2	b_2 / a_{22} ←
x_2 enters	s_3	a_{31}	a_{32}	a_{33}	0	0	1	b_3	b_3 / a_{32}

Body matrix ↑ Unit matrix

In the first row *i.e.* the row of z we find the most minimum value and mark this column by an arrow-head at the bottom of that column. We also put this column in a rectangular box. This column is called the **key column** or **pivot column.** *Suppose* $-c_2$ is the most negative value then the column under $-c_2$ is the key column. We now obtain the ratios of b_1, b_2, b_3 by dividing them by the corresponding elements in the key column and put them under a new column of ratios. In this column **neglecting the negative (and infinite) ratios** if there are any, we find the minimum finite positive ratio and mark the row containing this (minimum positive) ratio by an arrow head on the right. We also put this row in a rectangular box. Suppose b_2 / a_{22} is the smallest positive number in the column of ratios. This row is called the **key row** or **pivot row.** The element in the square box *i.e.*, the element common to both the key row and the key column is marked with an asterisk. It is called the **key element** or **pivot element.** In the present case a_{22} is the key element.

The element on the left of the **key row** (in the present case s_2) is the **outgoing element.** The element at the top of the **key column** (in the present case x_2) is the **incoming element.** We record this result in the first column as "s_2 leaves and x_2 enters".

We now prepare the second part of the simplex table. First we replace s_2 by x_2 and divide all the element of the second row by the key element a_{22}. Now the key element is replaced by 1. We now perform elementary row operations. We, now, subtract the proper multiples of the elements of the (new) second row from the elements of the other rows, so that the other elements in the **key column** are zero. This operation is performed on the elements of the column of b's (*i.e.* solution column) also but not on the elements of the last column of ratios.

Step III : Iteration

The above procedure is repeated till all the elements in the row above the dotted line *i.e.* all the coefficients in the objective equation are **non-negative.** When this situation is obtained, the optimum solution is arrived at.

The column of the basic variables states the variables in the optimal solution and the column of solution gives their values. The value in this column in the row of z is the optimum value of z.

Note ....

If in any iteration all the values in the ratio column are negative, the problem has **unbounded solution.** See Ex. 14, page 11-38, Ex. 16, page 11-40.

Example 1 : Solve the following L.P.P. by simplex method.

Maximise $z = x_1 + 4x_2$

subject to $2x_1 + x_2 \le 3$

$3x_1 + 5x_2 \le 9$

$x_1 + 3x_2 \le 5,$

$x_1, x_2 \ge 0.$ **(M.U. 2014)**

Sol. : We first express the problem in the standard form. Since there are three constraints, there will be three slack variables.

Maximise $z = x_1 + 4x_2 - 0s_1 - 0s_2 - 0s_3$

i.e. $z - x_1 - 4x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

subject to $2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 3$

$3x_1 + 5x_2 + 0s_1 + s_2 + 0s_3 = 9$

$x_1 + 3x_2 + 0s_1 + 0s_2 + s_3 = 5$

We now put this in the table given on the next page.

The table is prepared as follows.

First column is the column of iteration number, second is of Basic Variables s_1, s_2, s_3. Then we have the columns of $x_1, x_2, x_3, s_1, s_2, s_3$. Then we have the columns of R.H.S. (b's) (solution) and that of Ratio. Below the first row, we have the row of z. In this row we enter the values of the coefficients of $x_1, x_2, x_3, \ldots, s_1, s_2, s_3 \ldots$ appearing in the objective equation. In the column of Basic Variables we enter the variables s_1, s_2, s_3 and in their rows we enter

Simplex Table

Iteration Number	Basic Variables	x_1	x_2	s_1	s_2	s_3	R.H.S. Solution	Ratio
	z	-1	-4	0	0	0	0	
0	s_1	2	1	1	0	0	3	$3/1 = 3$
s_3 leaves	s_2	3	5	0	1	0	9	$9/5 = 1{\cdot}8$
x_2 enters	s_3	1	3^*	0	0	1	5	$5/3 = 1{\cdot}67$ ←

the coefficients of $x_1, x_2, \ldots, s_1, s_2, \ldots$ appearing in these constraint equations *i.e.* the values of the coefficients in the constraints.

Since in the first row -4 is the most **minimum** value, the column headed by this element is the **key-column** or **pivot-column.** We put an arrow head at the bottom of this column. Also we put this column in a rectangular box. We now divide each element in the column of solution (or R.H.S.) by the corresponding element in the key-column and put the results in a new column of ratios. Without taking into consideration, negative (and infinite) ratios appearing in this column, we mark the **minimum positive** ratio by an arrow-head on the right. In the present case the minimum positive ratio is 1·67. This is the **key-row** or **pivot-row.** We put this row in a rectangular box. The element common to the key-column and key-row is the **key-element** or **pivot-element.** The key element in this case is 3. We mark it by an asterisk. The variable on the left of the key-row is the **outgoing element** (s_3) and the element at the top of the key-column is the **incoming element** (x_2). We record this fact in the first column.

Using the **key element** and by using **elementary row operations** we bring zeros at **all other places in the key-column.** For this we first divide the elements in the **key row** by 3 and write the resulting row **in another table.** Thus, we get the new row (See the fourth row in the at the bottom).

Divide by 3*

x_2	1 / 3	1	0	0	1 / 3	5 / 3

Using **this row,** we bring zeros at all other places in the key - column. So multiply the elements of **this row by 4** and add to the corresponding elements of the row above the dotted line. Thus, we get the new row above the dotted line (See the first row of the table at the bottom).

Multiply by 4 and add to Z

1	z	1 / 3	0	0	0	4 / 3	20 / 3

Now, multiply the elements of **the same** row by 1 and subtract them from the corresponding elements of the row of s_1. Thus, we get the new row of s_1 (See the second row of the table at the bottom).

s_1	5 / 3	0	1	0	$-1/3$	4 / 3

Similarly, multiply the elements of the **same row** by 5 and subtract them from the corresponding elements of the row of s_2. Thus, we get the new row s_2 (See the third row of the table at the bottom).

s_2	4 / 3	0	0	1	− 5 / 3	2 / 3

The result is shown in the following table.

Table (For Example 1)

Iteration Number	Basic Variables	x_1	x_2	s_1	s_2	s_3	R.H.S. Solution
1	z	1 / 3	0	0	0	4 / 3	20 / 3
	s_1	5 / 3	0	1	0	− 1 / 3	4 / 3
	s_2	4 / 3	0	0	1	− 5 / 3	2 / 3
	x_2	1 / 3	1	0	0	1 / 3	5 / 3

Since, all the coefficients in the objective equation (above the dotted line) are positive, the optimum solution is obtained. The required results are given by the last column. Thus, since x_1 does not appear in the second column, $x_1 = 0$, $x_2 = 5/3$, $z_{Max} = 20/3$.

Example 2 : Using Simplex Method solve the following L.P.P.

Maximise $z = 10x_1 + x_2 + x_3$

subject to
$$x_1 + x_2 - 3x_3 \leq 10$$
$$4x_1 + x_2 + x_3 \leq 20$$
$$x_1, x_2, x_3 \geq 0.$$

Sol. : We first express the given problem in standard form
$$z - 10x_1 - x_2 - x_3 + 0s_1 + 0s_2 = 0$$
$$x_1 + x_2 - 3x_3 + s_1 + 0s_2 = 10$$
$$4x_1 + x_2 + x_3 + 0s_1 + s_2 = 20$$

We now put this information in tabular form. Since there are two constraints, there will be two stack variables.

Simplex Table

Iteration Number	Basic Variables	x_1	x_2	x_3	s_1	s_2	R.H.S. Solution	Ratio
0	z	− 10	−1	−1	0	0	0	
s_2 leaves	s_1	1	1	−3	1	0	10	10/1 = 10
x_1 enters	s_2	4*	1	3	0	1	20	20/4 = 5 ←

The **most minimum value** in the row of z is − 10. The column headed by − 10 is the **key-column** or **pivot column**. We put an arrow-head below this column and put this column in a rectangular box. We now divide the elements in the column of R.H.S. by the corresponding

elements in the key column and obtain the column of ratios. Neglecting the negative values, we find the **minimum positive ratio**. It is 5. The row having the element 5 is the **key row** or **pivot row**. We put an arrow head on the right and also put this row in a rectangular box. The element common to the key-column and the key-row is the **key-element** or **pivot element**. It is 4. The element on the left of key row is the **outgoing element**. Thus, s_2 leaves. The element, at the top of the key column is the **incoming element**. Thus, x_1 enters.

Using the **key element** and by using **elementary row operations**, we bring **zeros at all other places in the key column**. For this we divide the elements of the key row by 4 and write the resulting row in another table. Thus, we get the following row. (See the third row of the next table)

x_1	1	1 / 4	3 / 4	0	1 / 4	5

Using this row, we bring zeros at all other places in the key-column. So multiply these elements of this row by 10 and add them to the corresponding elements of the first row. Thus, we get new row above the dotted line. (See the first row of the next table)

z	0	3 / 2	13 / 2	0	5 / 2	50

Now multiply the elements of the **same** row by 1 and subtract them from the corresponding elements of the second row. Thus, we get the new row. (See the second row of the next table)

s_1	0	3 / 4	− 15 / 4	1	− 1 / 4	5

Table (For Example 2)

Iteration Number	Basic Variables	x_1	x_2	x_3	s_1	s_2	R.H.S. Solution
1	z	0	3 / 2	13 / 2	0	5 / 2	50
	s_1	0	3 / 4	−15 / 4	1	− 1 / 4	5
	x_1	1	1 / 4	3 / 4	0	1 / 4	5

Since all the coefficients in the objective equation in the row of z are positive this is a optimal solution. The values of the variables and of z are given by the last column. Since x_2, x_3 do not appear in the second column, they are zero.

$\therefore$ $x_1 = 5$, $x_2 = 0$, $x_3 = 0$, $z_{Max} = 50$.

Example 3 : Solve the following L.P.P. by Simplex Method.

Maximise $z = 5x_1 + 4x_2$

subject to
$$6x_1 + 4x_2 \leq 24$$
$$x_1 + 2x_2 \leq 6$$
$$-x_1 + x_2 \leq 1$$
$$x_2 \leq 2; \quad x_1, x_2 \geq 0.$$

Sol. : We first express the given problem in standard form

$$z - 5x_1 - 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 = 0$$
$$6x_1 + 4x_2 + s_1 + 0s_2 + 0s_3 + 0s_4 = 24$$
$$x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 + 0s_4 = 6$$
$$-x_1 + x_2 + 0s_1 + 0s_2 + s_3 + 0s_4 = 1$$
$$0x_1 + x_2 + 0s_1 + 0s_2 + 0s_3 + s_4 = 2$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

We now put this information in tabular form.

Simplex Table

Iteration Number		Basic Var.	Coefficients of x_1	x_2	s_1	s_2	s_3	s_4	R.H.S. Solution	Ratio
0		z	−5	−4	0	0	0	0	0	
	s_1 leaves	s_1	6*	4	1	0	0	0	24	24 / 6 = 4 ←
	x_1 enters	s_2	1	2	0	1	0	0	6	6 / 1 = 6
		s_3	−1	1	0	0	1	0	1	1 / −1 = −1
		s_4	0	1	0	0	0	1	2	2 / 0 = ∞
1		z	0	−2/3	5/6	0	0	0	20	
	s_2 leaves	x_1	1	2/3	1/6	0	0	0	4	4 ÷ 2/3 = 6
	x_2 enters	s_2	0	4/3	−1/6	1	0	0	2	2 ÷ 4/3 = 1·5 ←
		s_3	0	5/3	1/6	0	1	0	5	5 ÷ 5/3 = 3
		s_4	0	1	0	0	0	1	2	2 ÷ 1 = 2
2		z	0	0	3/4	1/2	0	0	21	
		x_1	1	0	1/4	−1/2	0	0	3	
		x_2	0	1	−1/8	3/4	0	0	3/2	
		s_3	0	0	3/8	−5/4	1	0	5/2	
		s_4	0	0	1/8	−3/4	0	1	1/2	

$\therefore \; x_1 = 3, \quad x_2 = \frac{3}{2}, \quad z_{max} = 21.$

Example 4 : Solve the following L.P.P. by Simplex Method

Maximise $z = 4x_1 + 10x_2$

subject to $2x_1 + x_2 \leq 50$

$2x_1 + 5x_2 \leq 100$

$2x_1 + 3x_2 \leq 90$

$x_1, x_2 \leq 0.$

Sol. : We first express the given problem in standard form

$$z - 4x_1 - 10x_2 + 0s_1 + 0s_2 + 0s_3 = 0$$
$$2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 50$$
$$2x_1 + 5x_2 + 0s_1 + s_2 + 0s_3 = 100$$
$$2x_1 + 3x_2 + 0s_1 + 0s_2 + s_3 = 90$$

Simplex Table

Iteration Number		Basic Variables	Coefficients of x_1	x_2	s_1	s_2	s_3	R.H.S. Solution	Ratio
0		z	−4	−10	0	0	0	0	
		s_1	2	1	1	0	0	50	50/1 = 50
	s_2 leaves	s_2	2	5*	0	1	0	100	100/5 = 20 ←
	x_2 enters	s_3	2	3	0	0	1	90	90/3 = 30
1		z	0	0	0	2	0	200	
		s_1	8 / 5	0	1	−1 / 5	0	30	
		x_2	2 / 5	1	0	1 / 5	0	20	
		s_3	4 / 5	0	0	−3 / 5	1	30	

$\therefore \; x_1 = 0, \; x_2 = 20, \; z_{Max} = 200.$

Example 5 : Using simplex method solve the following L.P.P.

Maximise $z = x_1 + 3x_2$

subject to $x_1 + 2x_2 \leq 10$

$0 \leq x_1 \leq 5$

$0 \leq x_2 \leq 4$

Sol. : We first express the problem in standard form

$$z - x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3 = 0 \qquad \text{.....................} (1)$$
$$x_1 + 2x_2 + s_1 + 0s_2 + 0s_3 = 10 \qquad \text{.....................} (2)$$
$$x_1 + 0x_2 + 0s_1 + s_2 + 0s_3 = 5 \qquad \text{.....................} (3)$$
$$0x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 4 \qquad \text{.....................} (4)$$

As discussed earlier in graphical method we start at the origin and move from one corner to another corner successively, improving upon the solution. At the origin $x_1 = 0$, $x_2 = 0$. Putting these values in equations (1), (2), (3), (4) we get immediately the following solution.

$$z = 0, \quad s_1 = 10, \quad s_2 = 5, \quad s_3 = 4.$$

This information is supplied by the second column of our table (under basic variables) and the last but one column (under R.H.S. solution).

Because we have expressed the objective function with right hand side zero as $z - x_1 - 3x_2 = 0$, the non-basic variable with the ***most negative*** coefficient is the entering variable. Since the coefficients of s_1, s_2, s_3 are zero obviously the entering variable will be one of x_1, x_2, x_3, x_4 ..... In this case x_2. To determine the leaving variable, we calculate the intercepts of all constraints on the ***non-negative*** direction of ***entering*** variable x_2. These intercepts are obviously the **ratios** of the equations 2, 3, 4 (R.H.S. solution column) to the corresponding constraint coefficients under the entering variable x_2. The ratios *i.e.* the intercepts made by the constraints on the x_2 axis are shown below.

Basic Variables	Entering x_2	R.H.S. Solution	Ratio (intercept)
s_1	2	10	10 / 2 = 5
s_2	0	5	5 / 0 = ∞ (Ignore)
s_3	1	4	4 / 1 = 4 ← Minimum

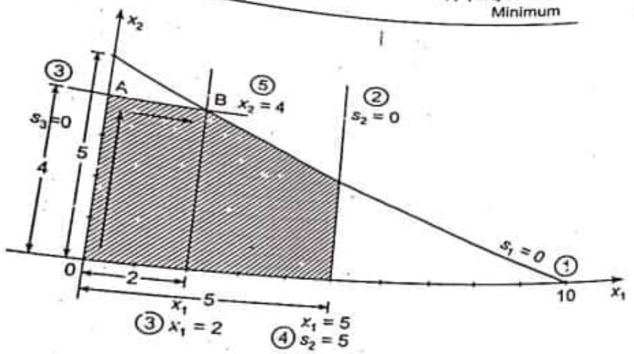

Fig. 11.1

Referring to the figure, we see that intercept made on the x_2-axis by the line $x_1 + 2x_2 = 10$ corresponding to $s_1 = 0$ is 5, that by $x_1 = 5$ corresponding to $s_2 = 0$ is infinity (the line $s_2 = 0$ is parallel to the x_2-axis), that by $x_2 = 4$ corresponding to $s_3 = 0$ is 4. The variable with minimum ratio *i.e.* minimum intercept leaves. Hence, s_3 leaves.

We now find the solution at the point *A*. The method used for finding the solution is **Gauss-Jorden** interation method which we have studied already.

1. New pivot row

New pivot row = Current pivot row ÷ Pivot element

2. New Rows

New row = Current row – (its pivot column coefficient) × (New pivot row)

In the above problem the second part of the table is obtained as follows.

New pivot row (s_3 which leaves and where x_2 enters)

= (Current pivot row ÷ Pivot element 1)

Thus, in this case we have the following.

1. New pivot row x_2 row = Current pivot s_3 row ÷ 1
2. New z row = Current z row – (– 3) × New pivot row
3. New s_1 row = Current s_1 row – (2) New pivot row
4. New s_2 row = Current s_2 row – (0) New pivot row

Thus, at *A* we have $s_1 = 2$ .... (5) and $s_2 = 5$ .... (6)

$x_2 = 4$ .... (7) $z = 12$

See the simplex table given on the next page.

This is not an optimum solution because the coefficient of x_1 in the z-row is negative. So we repeat the whole procedure at the point *A*.

The most negative coefficient is –1. Hence, the entering variable is x_1. To determine the leaving variable we calculate the intercepts of all constraints on the non-negative direction of entering variable x_1. These intercepts are obviously the ratios of the equations 5, 6, 7 (R.H.S. solution column) to the corresponding constraint coefficients (first column) under the entering variable x_1.

The ratios *i.e.* the intercepts made by the constraints on the x_1 axis are

Basic Variables	Entering x_1	R.H.S. Solution	Ratio (intercept)
s_1	1	2	2 / 1 = 2
s_2	1	5	5 / 1 = 5
x_2	0	4	4 / 0 = —

1. New pivot row x_1 row = Current pivot row s_1 row ÷ 1
2. New z row = Current z row – (–1) New pivot row
3. New s_2 row = Current s_2 row – (1) New pivot row
4. New x_2 row = Current x_2 row – (0) New pivot row.

Thus, at *B*, we have $x_1 = 2$, $s_2 = 3$, $x = 4$ and $z = 12$.

This gives the solution.

Gauss-Jordan Method

For clearer understanding of the process of finding the optimal solution, we shall solve the above problem by Gauss-Jordan Method.

The problem is :-

$$\begin{aligned} z - x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3 &= 0 \\ x_1 + 2x_2 + s_1 + 0s_2 + 0s_3 &= 10 \\ x_1 + 0x_2 + 0s_1 + s_2 + 0s_3 &= 5 \\ 0x_1 + x_2 + 0s_1 + 0s_2 + s_3 &= 4 \end{aligned}$$

We write these equations in matrix from, starting with z, s_1, s_2, s_3 and using the technique of leaving variable and entering variable. We shall use Gauss-Jordan method to find the solution.

Simplex Table (For Example 5)

Iteration Number	Basic Variables	x_1	x_2	s_1	s_2	s_3	R.H.S. Solution	Ratio
0	z	-1	-3	0	0	0	0	
s_3 leaves	s_1	1	2	1	0	0	10	$10/2=5$
x_2 enters	s_2	1	0	0	1	0	5	$5/0=-$
	s_3	0	1^*	0	0	1	4	$4/1=4 \leftarrow$
			$\uparrow$					
1	z	-1	0	0	0	3	12	
s_1 leaves	s_1	1^*	0	1	0	-2	2	$2/1=2 \leftarrow$
x_1 enters	s_2	1	0	0	1	0	5	$5/1=5$
	x_2	0	1	0	0	1	4	$4/0=-$
		$\uparrow$						
2	z	0	0	1	0	1	14	
	x_1	1	0	1	0	-2	2	
	s_2	0	0	-1	1	2	3	
	x_2	0	1	0	0	1	4	

$\therefore \ x_1 = 2, \ x_2 = 4, \ z_{Max} = 14.$

$$\begin{array}{l} s_3 \text{ leaves} \\ x_2 \text{ enters} \end{array} \begin{bmatrix} -1 & -3 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 5 \\ 4 \end{bmatrix} \quad \text{(First Part of The Table)}$$

$$\begin{array}{l} \text{By } R_1 - (-3)R_4 \\ R_2 - 2R_4 \\ R_3 - 0R_4 \\ s_1 \text{ leaves} \\ x_1 \text{ enters} \end{array} \begin{bmatrix} -1 & 0 & 0 & 0 & 3 \\ 1 & 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ s_1 \\ s_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \\ 5 \\ 4 \end{bmatrix} \quad \text{(Second Part of The Table)}$$

$$\begin{array}{l} \text{By } R_1 - (-1)R_2 \\ R_3 - R_2 \end{array} \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ x_1 \\ s_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{(Third Part of The Table)}$$

Since the first row contains all non-negative values, this is the optimum solution.

$x_1 = 2, \ x_2 = 4, \ z = 14.$

Thus, the simplex method is an ingenious combination of graphical method and Gauss-Jordan method.

The simplex table gives some additional information of the problem which is not available in the graphical method. A resource is designated as **scares** if it is used completely. Otherwise it is supposed to be **abundant**.

In the present problem the last table gives,

Resource	Slack value	Status
R_1	$s_1 = 0$	Scare
R_2	$s_2 = 3$	Abundand
R_3	$s_3 = 0$	Scare

Example 6 : Solve the following L.P.P. by simplex method.

Maximise $\quad z = 3x_1 + 2x_2$

subject to $\quad 3x_1 + 2x_2 \le 18 ; \quad 0 \le x_1 \le 4 ;$

$\quad 0 \le x_2 \le 6 ; \quad x_1, x_2 \ge 0$ **(M.U. 1998, 2010, 16)**

Sol. : We first express the problem in standard form.

$$z - 3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 = 0$$
$$3x_1 + 2x_2 + s_1 + 0s_2 + 0s_3 = 18$$
$$x_1 + 0x_2 + 0s_1 + s_2 + 0s_3 = 4$$
$$0x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 6$$

We now express the above informations in tabular form.

Simplex Table

Iteration Number	Basic Variables	x_1	x_2	s_1	s_2	s_3	R.H.S. Solution	Ratio
0	z	-3	-2	0	0	0	0	
s_2 leaves	s_1	3	2	1	0	0	18	6
x_1 enters	s_2	1^*	0	0	1	0	4	4 $\leftarrow$
	s_3	0	1	0	0	1	6	—
		$\uparrow$						
1	z	0	-2	0	3	0	12	
s_1 leaves	s_1	0	2^*	1	-3	0	6	3 $\leftarrow$
x_2 enters	x_1	1	0	0	1	0	4	–
	s_3	0	1	0	0	1	6	8
			$\uparrow$					
2	z	0	0	1	0	0	18	
	x_2	0	1	1/2	$-3/2$	0	3	
	x_1	1	0	0	1	0	4	
	s_3	0	0	$-1/2$	3/2	1	3	

$\therefore \ x_1 = 4, \ x_2 = 3, \ z_{Max} = 18.$

Example 7 : Solve the following L.P.P. by simplex method.

Maximise $z = 6x_1 - 2x_2 + 3x_3$

subject to $2x_1 - x_2 + 2x_3 \le 2$

$x_1 + 4x_3 \le 4$

$x_1, x_2, x_3 \ge 0$ (M.U. 1997, 2001, 15)

Sol. : We first express the problem in standard form.

$$z - 6x_1 + 2x_2 - 3x_3 + 0s_1 + 0s_2 = 0$$

$$2x_1 - x_2 + 2x_3 + s_1 + 0s_2 = 2$$

$$x_1 + 0x_2 + 4x_3 + 0s_1 + s_2 = 4$$

We now express the above informations in tabular form.

Simplex Table

Iteration Number	Basic Variables	Coefficients of x_1	x_2	x_3	s_1	s_2	R.H.S. Solution	Ratio
0	z	−6	2	−3	0	0	0	
s_1 leaves	s_1	2*	−1	2	1	0	2	1 ←
x_1 enters	s_2	1	0	4	0	1	4	4
		↑						
1	z	0	−1	3	3	0	6	
s_2 leaves	x_1	1	−1/2	1	1/2	0	1	−2
x_2 enters	s_2	0	1/2*	3	−1/2	1	3	6 ←
			↑					
2	z	0	0	9	2	2	12	
	x_1	1	0	4	0	1	4	
	x_2	0	1	6	−1	2	6	

$\therefore \ x_1 = 4, \ x_2 = 6, \ x_3 = 0, \ z_{Max} = 12.$

Example 8 : Solve the following L.P.P. by simplex method.

Maximise $z = 3x_1 + 2x_2 + 5x_3$

subject to $x_1 + 2x_2 + x_3 \le 430$

$3x_1 \quad + 2x_3 \le 460$

$x_1 + 4x_2 \quad \le 420$

$x_1, x_2, x_3 \ge 0$ (M.U. 1996, 2000, 04, 06, 10)

Sol. : We first express the problem in standard form.

$$z - 3x_1 - 2x_2 - 5x_3 + 0s_1 + 0s_2 + 0s_3 = 0$$

$$x_1 + 2x_2 + x_3 + s_1 + 0s_2 + 0s_3 = 430$$

$$3x_1 + 0x_2 + 2x_3 + 0s_1 + s_2 + 0s_3 = 460$$

$$x_1 + 4x_2 + 0x_3 + 0s_1 + 0s_2 + s_3 = 420$$

We put this information in tabular form as follows.

Simplex Table

Iteration Number	Basic Var.	Coefficients of x_1	x_2	x_3	s_1	s_2	s_3	R.H.S. Solution	Ratio
0	z	−3	−2	−5	0	0	0	0	
s_2 leaves	s_1	1	2	1	1	0	0	430	430 / 1 = 430
x_3 enters	s_2	3	0	2*	0	1	0	460	460 / 2 = 230 ←
	s_3	1	4	0	0	0	1	420	420 / 0 = –
				↑					
1	z	9 / 2	−2	0	0	5/2	0	1150	
s_1 leaves	s_1	− 1/2	2	0	1	− 1/2	0	200	200 / 2 = 100 ←
x_2 enters	x_3	3/2	0	1	0	1/2	0	230	230 / 0 = –
	s_3	1	4	0	0	0	0	420	420 / 4 = 105
			↑						
2	z	4	0	0	1	0	0	1350	
	x_2	−1/4	1	0	1/2	− 1/4	0	100	
	x_3	3/2	0	1	0	1/2	0	230	
	s_3	2	0	0	− 2	1	0	20	

$\therefore \ x_1 = 0, \ x_2 = 100, \ x_3 = 230, \ z_{Max} = 1350.$

Example 9 : Solve the following L.P.P by simplex method.

Maximise $z = 3x_1 + 2x_2$

subject to $x_1 + x_2 \le 4$

$x_1 - x_2 \le 2$

$x_1, x_2 \ge 0$ (M.U. 2011)

Sol. : We first express the problem in standard form.

$$z - 3x_1 - 2x_2 + 0s_1 + 0s_2 = 0$$
$$x_1 + x_2 + s_1 + 0s_2 = 4$$
$$x_1 - x_2 + 0s_1 + s_2 = 2$$

We now express the above information in tabular form.

Simplex Table

Iteration Number	Basic Variables	Coefficients of x_1	x_2	s_1	s_2	R.H.S. Solution	Ratio
0	z	−3	−2	0	0	0	
s_2 leaves	s_1	1	1	1	0	4	4 / 1 = 4
x_1 enters	s_2	1*	−1	0	1	2	2 / 1 = 2 ←
1	z	0	−5	0	3	6	
s_1 leaves	s_1	0	2*	1	−1	2	2 / 2 = 1 ←
x_2 enters	x_1	1	−1	0	1	2	2 / −1 = −
2	z	0	0	5/2	1/2	11	
	x_2	0	1	1/2	−1/2	1	
	x_1	1	0	1/2	1/2	3	

$\therefore$ $x_1 = 3,\ x_2 = 1,\ z_{Max} = 11.$

Example 10 : Solve the following L.P.P. by Simplex Method.

Maximise $z = 4x_1 + 2x_2 + 5x_3$

subject to $12x_1 + 7x_2 + 9x_3 \le 1260$

$22x_1 + 18x_2 + 16x_3 \le 19008$

$2x_1 + 4x_2 + 3x_3 \le 396.$

Sol. : We first express the problem in standard form.

$$z - 4x_1 - 2x_2 - 5x_3 + 0s_1 + 0s_2 + 0s_3 = 0$$
$$12x_1 + 7x_2 + 9x_3 + s_1 + 0s_2 + 0s_3 = 1260$$
$$22x_1 + 18x_2 + 16x_3 + 0s_1 + s_2 + 0s_3 = 19008$$
$$2x_1 + 4x_2 + 3x_3 + 0s_1 + 0s_2 + s_3 = 393$$

We put this information in tabular form as follows.

Simplex Table

Iteration Number	Basic Variables	Coefficients of x_1	x_2	x_3	s_1	s_2	s_3	R.H.S. Solution	Ratio
0	z	−4	−2	−5	0	0	0	0	
s_3 leaves	s_1	12	7	9	1	0	0	1260	1260 / 9 = 140
x_3 enters	s_2	22	18	16	0	1	0	19008	19008 / 16 = 1188
	s_3	2	4	3*	0	0	1	396	396 / 3 = 132 ←
1	z	−2/3	14/3	0	0	0	5/3	660	
s_1 leaves	s_1	6*	−5	0	1	0	−3	72	72 / 6 = 12 ←
x_1 enters	s_2	34/3	−10/3	0	0	1	−16/3	16896	$16896 \times \frac{3}{34} = 1491$
	x_3	2/3	4/3	1	0	0	1/3	132	$132 \times \frac{3}{2} = 198$
2	z	0	37/9	0	1/9	0	4/3	668	
	x_1	1	−5/6	0	1/6	0	−1/2	12	
	s_2	0	55/9	0	−17/9	1	1/3	16760	
	x_3	0	17/9	1	−1/9	0	2/3	124	

$\therefore$ $x_1 = 12,\ x_2 = 0,\ x_3 = 124$ and $z_{Max} = 668.$

Example 11 : Solve the following L.P.P. by simplex method.

Minimise $z = x_1 - 3x_2 + x_3$

subject to
$$3x_1 - x_2 + 2x_3 \le 7$$
$$2x_1 + 4x_2 \ge -12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10.$$
(M.U. 2004, 07)

Sol. : We first express the problem in standard form.

Maximise $z' = -x_1 + 3x_2 - x_3 + 0s_1 + 0s_2 + 0s_3$

i.e. $z' + x_1 - 3x_2 + x_3 + 0s_1 + 0s_2 + 0s_3 = 0$

subject to
$$3x_1 - x_2 + 3x_3 + s_1 + 0s_2 + 0s_3 = 7$$
$$-2x_1 - 4x_2 + 0x_3 + 0s_1 + s_2 + 0s_3 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + 0s_1 + 0s_2 + s_3 = 10$$

We put this information in tabular form as follows.

Simplex Table

Iteration Number	Basic Variables	x_1	x_2	x_3	s_1	s_2	s_3	R.H.S. Solution	Ratio
0	z'	1	−3	1	0	0	0	0	
s_2 leaves	s_1	3	−1	3	1	0	0	7	7 / −1 = - -
x_2 enters	s_2	−2	4*	0	0	1	0	12	12 / 4 = 3 ←
	s_3	−4	3	8	0	0	1	10	10 / 3 = 3.33
			↑						
1	z'	−1/2	0	1	0	3/4	0	9	
s_1 leaves	s_1	5/2*	0	3	1	1/4	0	10	$10 \times \frac{2}{5} = 4$ ←
x_1 enters	x_2	−1/2	1	0	0	1/4	0	3	$3 \times -2 = -6$
	s_3	−5/2	0	8	0	−3/4	1	1	$1 \times -\frac{2}{5} = -\frac{2}{5}$
		↑							
2	z'	0	0	8/5	1/5	4/5	0	11	
	x_1	1	0	6/5	2/5	1/10	0	4	
	x_2	0	1	3/5	1/5	3/10	0	5	
	s_3	0	0	11	1	−1/2	1	11	

$\therefore \; x_1 = 4, \; x_2 = 5, \; x_3 = 0, z_{Min} = -z'_{Max} = -11.$

Example 12 : Solve the following L.P.P. by simplex method.

Maximise $z = 100x_1 + 50x_2 + 50x_3$

subject to
$$4x_1 + 3x_2 + 2x_3 \le 10$$
$$3x_1 + 8x_2 + x_3 \le 8$$
$$4x_1 + 2x_2 + x_3 \le 6$$
$$x_1, x_2, x_3 \le 0.$$
(M.U. 2006)

Sol. : We first express the problem in standard form.
$$z - 100x_1 - 50x_2 - 50x_3 + 0s_1 + 0s_2 + 0s_3 = 0$$
$$4x_1 + 3x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 10$$
$$3x_1 + 8x_2 + x_3 + 0s_1 + s_2 + 0s_3 = 8$$
$$4x_1 + 2x_2 + x_3 + 0s_1 + 0s_2 + s_3 = 6$$

We put this information in tabular form as follows.

Simplex Table

Iteration Number	Basic Variables	x_1	x_2	x_3	s_1	s_2	s_3	R.H.S. Solution	Ratio
0	z	−100	−50	−50	0	0	0	0	
s_3 leaves	s_1	4	3	2	1	0	0	10	10 / 4 = 2·5
x_1 enters	s_2	3	8	1	0	1	0	8	8 / 3 = 2·67
	s_3	4*	2	1	0	0	1	6	6 / 4 = 1·5 ←
		↑							
1	z	0	0	−25	0	0	25	150	
s_1 leaves	s_1	0	1	1*	1	0	−1	4	4 / 1 = 4 ←
x_3 enters	s_2	0	13/2	1/4	0	1	−3/4	7/2	$\frac{7}{2} \times \frac{4}{1} = 14$
	x_1	1	1/2	1/4	0	0	1/4	3/2	$\frac{3}{2} \times \frac{4}{1} = 6$
				↑					
2	z	0	25	0	25	0	0	250	
	x_3	0	1	1	1	0	−1	4	
	s_2	0	25/4	0	−1/4	1	−1/2	5/2	
	x_1	1	1/4	0	−1/4	0	1/2	1/2	

$\therefore \; x_1 = 1/2, \; x_2 = 0, \; x_3 = 4, \; z_{Max} = 250.$

Example 13 : Solve the following L.P.P. by simplex method.

Maximise $z = 3x_1 + 5x_2 + 4x_3$

subject to $2x_1 + 3x_2 \le 8$

$2x_2 + 5x_3 \le 10$

$3x_1 + 2x_2 + 4x_3 \le 15$

$x_1, x_2, x_3 \le 0.$ (M.U. 2004)

Sol. : We first express the given problem in standard form.

$$z - 3x_1 - 5x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3 = 0$$
$$2x_1 + 3x_2 + 0x_3 + s_1 + 0s_2 + 0s_3 = 8$$
$$0x_1 + 2x_2 + 5x_3 + 0s_1 + s_2 + 0s_3 = 10$$
$$3x_1 + 2x_2 + 4x_3 + 0s_1 + 0s_2 + s_3 = 15$$

We put this information in tabular form as follows.

Simplex Table

Iteration Number	Basic Variables	x_1	x_2	x_3	s_1	s_2	s_3	R.H.S. Solution	Ratio
0	z	−3	−5	−4	0	0	0	0	
s_1 leaves	s_1	2	3*	0	1	0	0	8	8 / 3 = 2·67 ←
x_2 enters	s_2	0	2	5	0	1	0	10	10 / 2 = 5·00
	s_3	3	2	4	0	0	1	15	15 / 2 = 7·5
			↑						
1	z	1/3	0	−4	5/3	0	0	40/3	
s_2 leaves	x_2	2/3	1	0	1/3	0	0	8/3	—
x_3 enters	s_2	−4/3	0	5*	−2/3	1	0	14/3	14/15 ←
	s_3	5/3	0	4	−2/3	0	1	29/3	29/12
				↑					
2	z	−11/15	0	0	17/15	4/5	0	256/15	
s_3 leaves	x_2	2/3	1	0	1/3	0	0	8/3	4
x_1 enters	x_3	−4/15	0	1	−2/15	1/5	0	14/15	—
	s_3	41/15*	0	0	−2/15	−4/5	1	89/15	89/41 ←
		↑							
3	z	0	0	0	45/41	24/41	11/41	765/41	
	x_2	0	1	0	15/41	8/41	−10/41	50/41	
	x_3	0	0	1	−6/41	5/41	4/41	62/41	
	x_1	1	0	0	−2/41	−12/41	15/41	89/41	

$$\therefore\ x_1 = \frac{89}{41},\ x_2 = \frac{50}{41},\ x_3 = \frac{62}{41},\ z_{Max} = \frac{765}{41}.$$

Example 14 : Solve the following L.P.P. by simplex method.

Maximise $z = 4x_1 + x_2 + 3x_3 + 5x_4$

subject to $-4x_1 + 6x_2 + 5x_3 + 4x_4 \le 20$

$-3x_1 - 2x_2 + 4x_3 + x_4 \le 10$

$-8x_1 - 3x_2 + 3x_3 + 2x_4 \le 20$

$x_1, x_2, x_3, x_4 \le 0.$ (M.U. 1997, 2016)

Sol. : We first express the given problem in standard form.

$$z - 4x_1 - x_2 - 3x_3 - 5x_4 + 0s_1 + 0s_2 + 0s_3 = 0$$
$$-4x_1 + 6x_2 + 5x_3 + 4x_4 + s_1 + 0s_2 + 0s_3 = 20$$
$$-3x_1 - 2x_2 + 4x_3 + x_4 + 0s_1 + s_2 + 0s_3 = 10$$
$$-8x_1 - 3x_2 + 3x_3 + 2x_4 + 0s_1 + 0s_2 + s_3 = 20$$

We put this information in tabular form as follows.

Simplex Table

Iteration Number	Basic Variables	x_1	x_2	x_3	x_4	s_1	s_2	s_3	R.H.S. Solution	Ratio
0	z	−4	−1	−3	−5	0	0	0	0	
s_1 leaves	s_1	−4	6	5	4*	1	0	0	20	5 ←
x_4 enters	s_2	−3	−2	4	1	0	1	0	10	10
	s_3	−8	−3	3	2	0	0	1	20	10
					↑					
1	z	−9	13/2	13/4	0	5/4	0	0	25	
	x_1	−1	3/2	5/4	1	1/4	0	0	5	−5
	s_2	−2	−7/2	−11/4	0	−1/4	1	0	5	−5/2
	s_3	−6	−6	1/2	0	−1/2	0	1	10	−5/3
		↑								

Since all entries in the ratio column are negative the problem has unbounded solution.

(See note on the page 11-21)

Applied Mathemat

Example 15

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Example 15 : Solve the following L.P.P. by Simplex Method. (M.U. 2002, 15)

Maximise $z = 4x_1 + 3x_2 + 6x_3$

subject to
$$2x_1 + 3x_2 + 2x_3 \le 440$$
$$4x_1 + 3x_3 \le 470$$
$$2x_1 + 5x_2 \le 430$$
$$x_1, x_2, x_3 \le 0.$$

Sol. : We first express the given problem in standard form.

Maximise $z = 4x_1 + 3x_2 + 6x_3 + 0s_1 + 0s_2 + 0s_3$

i.e. $z - 4x_1 - 3x_2 - 6x_3 + 0s_1 + 0s_2 + 0s_3 = 0$

subject to
$$2x_1 + 3x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 440$$
$$4x_1 + 0x_2 + 3x_3 + 0s_1 + s_2 + 0s_3 = 470$$
$$2x_1 + 5x_2 + 0x_3 + 0s_1 + 0s_2 + s_3 = 430$$

We put this information in tabular form as follows.

Simplex Table

Iteration Number	Basic Variables	x_1	x_2	x_3	s_1	s_2	s_3	R.H.S. Solution	Ratio
0	z	−4	−3	−6	0	0	0	0	
s_2 leaves	s_1	2	3	2	1	0	0	440	220
x_3 enters	s_2	4	0	3*	0	1	0	470	156·67 ←
	s_3	2	5	0	0	0	1	430	—
1	z	4	−3	0	0	2	0	940	
s_1 leaves	s_1	−2/3	3*	0	1	−2/3	0	380/3	380/9 ←
x_2 enters	x_3	4/3	0	1	0	1/3	0	470/3	—
	s_3	2	5	0	0	0	1	430	86
2	z	10/3	0	0	1	4/3	0	3200/3	
	x_2	−2/9	1	0	1/3	−2/9	0	380/9	
	x_3	4/3	0	1	0	1/3	0	470/3	
	s_3	28/9	0	0	−5/3	10/9	1	1970/9	

$$\therefore \quad x_1 = 0, \quad x_2 = \frac{380}{9}, \quad x_3 = \frac{470}{3}, \quad z_{max} = \frac{3200}{3}.$$

Example 16 : Solve the following L.P.P. by simplex method.

Maximise $z = 107x_1 + x_2 + 2x_3$

subject to
$$14x_1 + x_2 - 6x_3 + 3x_4 = 7$$
$$16x_1 + (1/2)x_2 - 6x_3 \le 5$$
$$3x_1 - x_2 - x_3 \le 0$$
$$x_1, x_2, x_3, x_4 \le 0.$$

Sol. : We first express the given problem in standard form, noticing that x_4 is a slack variable and its coefficient must be unity.

Maximise $z - 107x_1 - x_2 - 2x_3 + 0x_4 + 0s_1 + 0s_2 = 0$
$$\frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 + 0s_1 + 0s_2 = \frac{7}{3}$$
$$16x_1 + \frac{1}{2}x_2 - 6x_3 + 0x_4 + s_1 + 0s_2 = 5$$
$$3x_1 - x_2 - x_3 + 0x_4 + 0s_1 + s_2 = 0$$

We put this information in tabular form as follows.

Simplex Table

Iteration Number	Basic Variables	x_1	x_2	x_3	x_4	s_1	s_2	R.H.S. Solution	Ratio
0	z	−107	−1	−2	0	0	0	0	
s_2 leaves	x_4	14/3	1/3	−2	1	0	0	7/3	1/2
x_1 enters	s_1	16	1/2	−6	0	1	0	5	5/16
	s_2	3*	−1	−1	0	0	1	0	0 ←
1	z	0	−110/3	−113/3	0	0	107/3	0	
	x_4	0	17/9	−4/9	1	0	−14/9	7/3	−ve
	s_1	0	35/6	−2/3	0	1	−16/3	5	−ve
	x_1	1	−1/3	−1/3	0	0	1/3	0	∞

Here −113 / 3 is the least number and as such x_3 will be incoming variable. But all ratios in the last ratio column are negative or infinite and hence, no variable can leave and x_3 cannot enter.

$\therefore$ The solution to the problem is unbounded.

(See the note on the page 11-21)

Example 17 : Solve the following L.P.P. by simplex method.

(M.U. 2004)

Maximise $z = 4x_1 + 10x_2$

subject to $2x_1 + x_2 \le 10$

$2x_1 + 5x_2 \le 20$

$2x_1 + 3x_2 \le 18$

Sol. : We first express the given problem in standard form.

Maximise $z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$

i.e. $z - 4x_1 - 10x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

subject to $2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 10$

$2x_1 + 5x_2 + 0s_1 + s_2 + 0s_3 = 20$

$2x_1 + 3x_2 + 0s_1 + 0s_2 + s_3 = 18.$

We put this information in tabular form as follows.

Simplex Table

Iteration Number	Basic Variables	Coefficients of x_1	x_2	s_1	s_2	s_3	R.H.S. Solution	Ratio
0	z	−4	−10	0	0	0	0	
s_2 leaves	s_1	2	1	1	0	0	10	10
x_2 enters	s_2	2	5*	0	1	0	20	4 ←
	s_3	2	3	0	0	1	18	6
			↑					
1	z	0	0	0	2	0	40	
s_1 leaves	s_1	8/5*	0	1	−1/5	0	6	15/4 ←
x_1 enters	x_2	2/5	1	0	1/5	0	4	10
	s_3	4/5	0	0	−3/5	1	6	15/2
		↑						

$\therefore \quad x_1 = 0, \; x_2 = 4, \; z_{Max} = 40$

But further considerations show that s_1 may leave and x_1 may enter.

2	z	0	0	0	−1/5	0	40
	x_1	1	0	5/8	−1/8	0	15/4
	x_2	0	1	−1/4	1/5	0	5/2
	s_3	0	0	−1/2	−1/2	1	3

$\therefore \quad x_1 = 15/4, \; x_2 = 5/2, \; z_{Max} = 40$

This is an alternative solution. But this does not improve the above optimal solution.

Thus, we have two solutions $x_1 = 0, x_2 = 4, z_{Max} = 40$

and $x_1 = 15/4, x_2 = 5/2, z_{Max} = 40$

If there are two solutions to a problem then there are infinite number solutions.

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $X_1 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $X_2 = \begin{bmatrix} 15/4 \\ 5/2 \end{bmatrix}$, then $X = \lambda X_1 + (1-\lambda) X_2$ for $0 \le \lambda \le 1$

i.e. $$X = \begin{bmatrix} \frac{15}{4}(1-\lambda) \\ 4 + \frac{5}{2}(1-\lambda) \end{bmatrix}$$

gives infinite number of feasible solutions, all giving $z_{Max} = 40$.

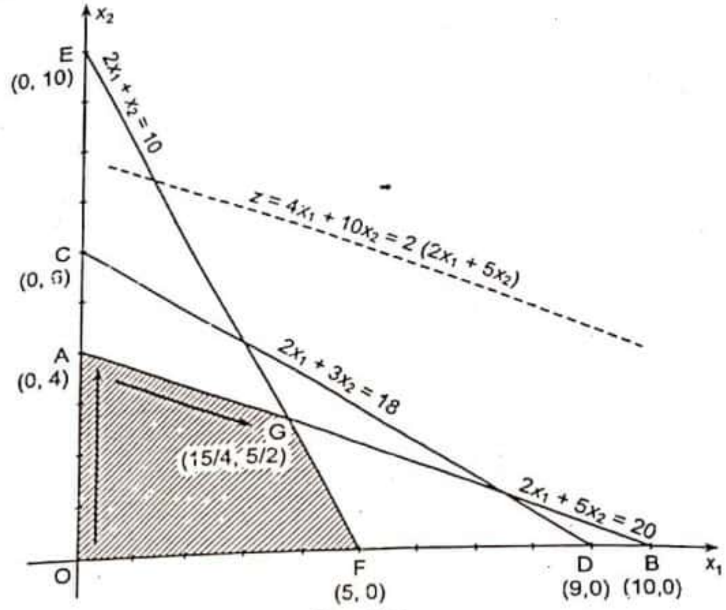

Fig. 11.2.

If we solve the problem graphically, we shall get better understanding of the nature and solution of such problems having **alternate solutions.**

By plotting the points $A(0, 4)$ and $B(10, 0)$ and joining them, we get the line $2x_1 + 5x_2 = 20$. By plotting the points $C(0, 6)$ and $D(9, 0)$ and joining them, we get the line $2x_1 + 3x_2 = 18$. By plotting the points $E(0, 10)$ and $F(5, 0)$ and joining them we get the line $2x_1 + x_2 = 10$.

Thus, we get the feasible region $OFGA$ where G (15/4, 5/2) is the point of intersection of the lines $2x_1 + 5x_2 = 20$ and $2x_1 + x_2 = 10$.

The values of $z = 4x_1 + 10x_2$ at the vertices of the region are :-

At O (0, 0), $\quad z = 0 + 0 = 0$

At A (0, 4), $\quad z = 0 + 40 = 40$

...mber solutions.

$) x_2$ for $0 \le \lambda \le 1$

At G $\left(\frac{15}{4}, \frac{5}{2}\right)$, $z = 4 \times \frac{15}{4} + 10 \times \frac{5}{2} = 40$

At F (5, 0), $z = 20 + 0 = 20$

Thus, we get two points A (0, 4) and G (15/4, 5/2) giving the same maximum value of z (= 40).

Observe that the two points lie on the same line AB. This is not coincidence. Note carefully that the slope of the objective function $4x_1 + 10x_2 = 2(2x_1 + 5x_2) = k$ is equal to the slope of the line of second constraint *i.e.* to the slope of the line AB $(2x_1 + 5x_2 = 20)$. When this happens *i.e.* **when the objective function is parallel to one of the constraints**, any point on the constraint (line AB) gives the optimum solution.

EXERCISE - III

Solve the following linear programming problems by simplex method.

1. Maximise $z = 7x_1 + 5x_2$
subject to $-x_1 - 2x_2 \ge -6$
$4x_1 + 3x_2 \le 12$
$x_1, x_2 \ge 0.$
[**Ans. :** $x_1 = 3,\ x_2 = 0,\ z = 21$]

2. Maximise $z = 5x_1 + 7x_2$
subject to $x_1 + x_2 \le 4$
$3x_1 - 8x_2 \le 24$
$10x_1 + 7x_2 \le 35$
$x_1, x_2 \ge 0.$
[**Ans. :** $x_1 = 0,\ x_2 = 4,\ z = 28$]

3. Maximise $z = 5x_1 + 3x_2$
subject to $x_1 + x_2 \le 2$
$5x_1 + 2x_2 \le 10$
$3x_1 + 8x_2 \le 12$
$x_1, x_2 \ge 0.$
[**Ans. :** $x_1 = 2,\ x_2 = 0,\ z = 10$]

4. Maximise $z = 3x_1 + 5x_2$
subject to $3x_1 + 2x_2 \le 18$
$x_1 \le 4,\ x_2 \le 6$
$x_1, x_2 \ge 0.$
[**Ans. :** $x_1 = 2,\ x_2 = 6,\ z = 36$]

5. Maximise $z = 3x_1 + 5x_2$
subject to $x_1 + x_2 \le 450$
$2x_1 + x_2 \le 600$
$x_1, x_2 \ge 0.$
[**Ans. :** $x_1 = 0,\ x_2 = 450,\ z = 2250$]

6. Maximise $z = 3x_1 + 4x_2$
subject to $2x_1 + x_2 \le 40$
$2x_1 + 5x_2 \le 180$
$x_1, x_2 \ge 0.$
[**Ans. :** $x_1 = 2{\cdot}5,\ x_2 = 35,\ z = 147{\cdot}5$]

7. Maximise $z = 2x_1 + x_2$
subject to $x_1 - x_2 \le 10$
$2x_1 - x_2 \le 40$
$x_1, x_2 \ge 0.$
[**Ans. :** Unbounded solution.]

8. Maximise $z = x_1 + x_2 + 3x_3$
subject to $3x_1 + 2x_2 + x_3 \le 3$
$2x_1 + x_2 + 2x_3 \le 2$
$x_1, x_2, x_3 \ge 0.$
[**Ans. :** $x_1 = 0, x_2 = 0, x_3 = 1, z = 3$]

9. Maximise $z = x_1 + x_2 + x_3$
subject to $4x_1 + 5x_2 + 3x_3 \le 15$
$10x_1 + 7x_2 + x_3 \le 12$
$x_1, x_2, x_3 \ge 0.$
[**Ans. :** $x_1 = 0, x_2 = 0, x_3 = 5, z = 5$]

10. Maximise $z = 10x_1 + 40x_2 + 50x_3$
subject to $x_1 + 2x_2 + 2x_3 \le 14$
$3x_1 + 2x_2 \le 14$
$x_1, x_2, x_3 \ge 0.$
[**Ans. :** $x_1 = 0, x_2 = 7, x_3 = 0, z = 280$]

11. Maximise $z = 10x_1 + x_2 + 2x_3$
subject to $x_1 + x_2 - 3x_3 \le 10$
$4x_1 + x_2 + x_3 \le 20$
$x_1, x_2, x_3 \ge 0.$
[**Ans. :** $x_1 = 5, x_2 = 0, x_3 = 0, z = 50$]

12. Maximise $z = 100x_1 + 40x_2 + 50x_3$
subject to $4x_1 + 3x_2 + 2x_3 \le 10$
$3x_1 + 8x_2 + x_3 \le 8$
$4x_1 + 2x_2 + x_3 \le 6$
$x_1, x_2, x_3 \ge 0.$
[**Ans. :** $x_1 = 1/2, x_2 = 0, x_3 = 4, z = 250$]

13. Maximise $z = x_1 + x_2 + 2x_3$
subject to $3x_1 + 2x_2 + x_3 \le 3$
$2x_1 + x_2 + 2x_3 \le 2$
$x_1, x_2, x_3 \ge 0.$
[**Ans. :** $x_1 = 0, x_2 = 0, x_3 = 1, z = 2$]

14. Maximise $z = 10x_1 + 6x_2 + 5x_3$
subject to $x_1 + x_2 + x_3 \le 100$
$10x_1 + 4x_2 + 5x_3 \le 600$
$2x_1 + 2x_2 + 6x_3 \le 300$
$x_1, x_2, x_3 \ge 0.$
[**Ans. :** $x_1 = 100/3, x_2 = 200/3, x_3 = 0,$
$z = 2200/3$]

15. Maximise $z = 3x_1 + 2x_2 + 5x_3$
subject to $x_1 + x_2 + x_3 \le 9$
$2x_1 + 3x_2 + 5x_3 \le 30$
$2x_1 - x_2 - x_3 \le 8$
$x_1, x_2, x_3 \ge 0.$
[**Ans. :** $x_1 = 5, x_2 = 0, x_3 = 4, z_{Max} = 35$]

16. Minimise $z = 4x_1 + x_2 + 3x_3 + 5x_4$
subject to $4x_1 - 6x_2 - 5x_3 - x_4 \ge -2$
$-3x_1 - 2x_2 + 4x_3 + x_4 \le 10$
$-8x_1 - 3x_2 + 3x_3 + 2x_4 \le 20$
$x_1, x_2, x_3, x_4 \ge 0.$ **(M.U. 2004)**
[**Ans. :** Unbounded solution]

17. Maximise $z = 2x_1 + 3x_2 + 4x_3 + x_4$
subject to $x_1 + 5x_2 + 9x_3 - 6x_4 \ge -2$
$3x_1 - x_2 + x_3 + 3x_4 \le 10$
$-2x_1 - 3x_2 + 7x_3 - 8x_4 \ge 0$
$x_1, x_2, x_3, x_4 \ge 0.$
[**Ans. :** Unbounded solution]

18. Maximise $z = 5x_1 + 3x_2 + 7x_3$
subject to $x_1 + x_2 + 2x_3 \le 26$
$3x_1 + 2x_2 + x_3 \le 26$
$x_1 + x_2 + x_3 \le 18$
$x_1, x_2, x_3 \ge 0$ **(M.U. 2007)**
[**Ans. :** $z = \frac{494}{5},\ x_1 = \frac{26}{5},\ x_2 = 0,\ x_3 = \frac{52}{5}$]

19. Maximise $z = 4x_1 + 10x_2$
subject to $2x_1 + x_2 \le 20$
$2x_1 + 5x_2 \le 40$
$2x_1 + 3x_2 \le 6$
$x_1, x_2 \ge 0.$
[**Ans. :** (i) $x_1 = 0, x_2 = 4,$
(ii) $x_1 = 15/4, x_2 = 5/2,$
$z_{Max} = 80$ in both cases.
Infinite number of solutions giving $z_{Max} = 80$
The line $4x_1 + 10x_2 = 2(2x_1 + 5x_2) = k$
is parallel to the line of second constraint
$2x_1 + 5x_2 = 40.$]

20. Maximise $z = 100x_1 + 40x_2$
subject to $10x_1 + 4x_2 \le 2000$
$3x_1 + 2x_2 \le 900$
$6x_1 + 12x_2 \le 3000$
Has it an alternative optima ? **(M.U. 2005)**
[**Ans. :** $x_1 = 200, x_2 = 0$;
Alternative optima $x_1 = 125, x_2 = 187.5,$
$z_{max} = 20000.$
The line
$100x_1 + 40x_2 = 10(10x_1 + 4x_2) = k$
is parallel to the line of first constraint
$10x_1 + 4x_2 = 2000$]

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EXERCISE - IV

Theory

1. Explain the following terms. (M.U. 1999, 2000)
   (i) Slack and surplus variables.
   (ii) Standard form and canonical form.
2. Explain the following terms. (M.U. 2003)
   (i) Canonical form of an L.P.P.
   (ii) Degenerate solution of L.P.P.
3. Define the following terms with reference to the Simplex method of an L.P.P. (M.U. 2003, 07)
   (i) Basic solution
   (ii) Degenerate solution.
4. Define the following terms
   (i) Feasible Solution
   (ii) Optimal Solution
   (iii) Basic Variables
   (iv) Non-basic Variables
   (v) Basic Feasible Solution
   (vi) Degenerate Basic Feasible Solution. (M.U. 2000)
5. Explain the terms - Optimality and feasibility conditions.
6. In solving an L.P.P. how do you decide the number of non-basic variables ?
7. In solving an L.P.P. how do you know that the problem has unbounded solution. (M.U. 2000)
8. For the simplex method in L.P.P. define :
   (i) basic solution,
   (ii) degenerate solution,
   (iii) optimal solution.
9. Define the following terms.
   (i) Unbounded solution.
   (ii) Degenerate solution
   (iii) Slack and surplus variables. (M.U. 2004)
10. Define the following terms.
    (i) Slack variables
    (ii) Feasible solution
    (iii) Standard form of L.P.P. (M.U. 2001)

☆☆☆

CHAPTER 12

Penalty And Duality

1. Introduction

In this chapter we are going to study some other techniques of simplex method viz. the Big-M method or penalty method, and dual simplex method. Later on we shall study post optimal analysis *i.e.* analysis by which we can estimate the changes in the constants viz. b's in the constraints and c's in the object function without changing the optimality of the solution.

2. Artificial Variable Technique

So far we have seen the problems in which all constraints were of the less than or equal to (≤) type. The coefficients of slack variables in the constraints (in the standard form) were either 1 or 0. But if the constraints are of "greater than or equal to" (≥) or "equal to" (=) type the above method cannot be used. The coefficients of some slack variables in such cases will be −1 or will be 0. We, thus, will not get the unit matrix to start with. So, we have to introduce a new type of variables called **"artificial variable"**. As the name suggests these variables are fictitious and do not have any physical meaning. These variables are used only to get the starting basic feasible solution in the required form.

The Big M-method

The method is due to Charnes and is based on the following considerations.

If any one or some constraints are of greater than type then we have to subtract surplus variables *i.e.* we have to add $-s_1, -s_2, \ldots\ldots$ etc. to convert "the greater than or equal to" type inequality to equality. But then we would not get a unit matrix. To overcome this difficulty, we introduce in addition to surplus variables $-s_1, -s_2, \ldots\ldots$ (and slack variables) artificial variables $A_1, A_2, \ldots\ldots$ with positive sign in these constraints. An artificial variable is introduced even when the constraint is of equality type. In the objective function we assign big penalty by subtracting $MA_1, MA_2, \ldots\ldots$ if the objective function is of maximisation type.

Consider the problem.

$$\text{Maximise} \quad z = c_1x_1 + c_2x_2 + c_3x_3$$
$$\text{subject to} \quad a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \geq b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \leq b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \leq b_3$$
$$x_1, x_2, x_3 \geq 0.$$

Since the first constraint is of "greater than or equal to type" (≥), we subtract s_1 and add an artificial variable A_1. In the object function, we assign big penalty for this artifical variable A_1 *i.e.* we subtract MA_1 from the objective function. Thus, we have

Maximise $z = c_1 x_1 + c_2 x_2 + c_3 x_3 - 0s_1 - 0s_2 - 0s_3 - MA_1$

subject to $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 - s_1 + 0s_2 + 0s_3 + A_1 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + 0s_1 + s_2 + 0s_3 + 0A_1 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + 0s_1 + 0s_2 + s_3 + 0A_1 = b_3$

$x_1, x_2, x_3, s_1, s_2, s_3, A_1 \geq 0.$

We now write the object function free from artifical variable *i.e.*, to eliminate the term MA by adding M times, the first constraint to the object function. Now, the object function becomes

$$z = (c_1 + Ma_{11})\, x_1 + (c_2 + Ma_{12})\, x_2 + (c_3 + Ma_3)\, x_3 - Ms_1 - 0s_2 - 0s_3 - Mb_1$$

$$\therefore \quad z - (c_{11} + Ma_{11})\, x_1 - (c_2 + Ma_{12})\, x_2 - (c_3 + Ma_3)\, x_3 + Ms_1 + 0s_2 + 0s_3 = -Mb_1$$

Now, we follow all the usual steps of simplex method as before. After the required number of iterations, we will find one of the following situations.

1. The artificial variables leave the process and the optimality condition is satisfied by the basic variables. This is, then, the optimal basic feasible solution. (See Ex. 1 below.)
2. Atleast one of the artificial variables remains in the basis with zero value and the optimality condition is satisfied. This is the optimal basic feasible solution (though degenerate) to the given problem.
3. Atleast one of the artificial variables remains in the basis with non-zero value and the optimality condition is satisfied. This solution though satisfies optimality conditions is not an optimal solution since it contains large penalty M. This is not a solution but a pseudo-solution. (See Ex. 5 below.)

Example 1 : Using Penalty (Big-M or Charne's) method solve the following L.P.P.

Maximise $z = 3x_1 - x_2$

subject to $2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 3$

$x_2 \leq 4$

$x_1, x_2 \geq 0.$ **(M.U. 1999)**

Sol. : As explained above we introduce the artificial variable A_1 in the first constraint and big penalty in the object function.

Maximise $z = 3x_1 - x_2 - 0s_1 - 0s_2 - 0s_3 - MA_1$ .... (1)

subject to $2x_1 + x_2 - s_1 + 0s_2 + 0s_3 + A_1 = 2$ .... (2)

$x_1 + 3x_2 + 0s_1 + s_2 + 0s_3 + 0A_1 = 3$ .... (3)

$0x_1 + x_2 + 0s_1 + 0s_2 + s_3 + 0A_1 = 4$ .... (4)

We now eliminate the term $-MA_1$ from (1) by adding M times the first constraint to it.

$$\therefore \quad z = (3 + 2M)\, x_1 + (-1 + M)\, x_2 - Ms_1 - 0s_2 - 0s_3 - 0A_1 - 2M$$

$$\therefore \quad z - (3 + 2M)\, x_1 - (-1 + M)\, x_2 + Ms_1 + 0s_2 + 0s_3 + 0A_1 = -2M$$

Setting decision variables $x_1 = 0$, $x_2 = 0$ and $s_1 = 0$ the basic feasible solution is $A_1 = 2$, $s_2 = 3$, $s_3 = 4$. A_1 is greater than zero, in this case 2. But it must not appear in the final solution. To achieve this we assign a large penalty $(-M)$ to A_1 in the object function (1).

Simplex Table

Iteration Number	Basic Var.	x_1	x_2	s_1	s_2	s_3	A_1	R.H.S. Sol.	Ratio
0	z	$-3-2M$	$1-M$	M	0	0	0	$-2M$	
A_1 leaves	A_1	2*	1	−1	0	0	1	2	$2/2 = 1$ ←
x_1 enters	s_2	1	3	0	1	0	0	3	$3/1 = 3$
	s_3	0 ↑	1	0	0	1	0	4	$1/0 = \ldots$
1	z	0	5/2	−3/2	0	0		3	
s_2 leaves	x_1	1	1/2	−1/2	0	0		1	−2
s_1 enters	s_2	0	5/2	1/2*	1	0		2	4 ←
	s_3	0	1	0 ↑	0	1		4	—
2	z	0	10	0	3	0		9	
	x_1	1	3	0	1	0		3	
	s_1	0	5	1	2	0		4	
	s_3	0	1	0	0	1		4	

$\therefore \quad x_1 = 3, \; x_2 = 0, \; z_{Max} = 9.$

Example 2 : Using Penalty (Big-M or Charne's) method solve the following L.P.P.

Maximise $z = 3x_1 - x_2$

subject to $2x_1 + x_2 \leq 2$

$x_1 + 3x_2 \geq 3$

$x_2 \leq 4$

$x_1, x_2 \geq 0.$ **(M.U. 2001, 09)**

Sol. : As explained above we introduce the artificial variable A_2 in the second constraints and big penalty in the object function.

Maximise $z = 3x_1 - x_2 - 0s_1 - 0s_2 - 0s_3 - MA_2$ .... (1)

subject to $2x_1 + x_2 + s_1 + 0s_2 + 0s_3 + 0A_2 = 2$ .... (2)

$x_1 + 3x_2 + 0s_1 - s_2 + 0s_3 + A_2 = 3$ .... (3)

$0x_1 + x_2 + 0s_1 + 0s_2 + s_3 + 0A_2 = 4$ .... (4)

$x_1, x_2, s_1, s_2, A_2 \geq 0$

We now eliminate the term $-MA_2$ from the object function by adding M times the second constraint to it.

$$\therefore \quad z = (3 + M)\, x_1 - (1 - 3M)\, x_2 - 0s_1 - Ms_2 - 0s_3 - 0A_2 - 3M$$

$$\therefore \quad z - (3 + M)\, x_1 + (1 - 3M)\, x_2 + 0s_1 + Ms_2 + 0s_3 + 0A_2 = -3M$$

with constraints as above.

Setting decision variables $x_1 = 0$, $x_2 = 0$ and $s_2 = 0$, the basic feasible solution is $s_1 = 2$, $A_2 = 3$, $s_3 = 4$. A_2 is greater than zero, in this case 3. But it must not appear in the final solution. To achieve this, we assign a large penalty $(-M)$ to A_2 in the object function (1).

Simplex Table

Iteration Number	Basic Var.	x_1	x_2	s_1	s_2	s_3	A_2	R.H.S. Sol.	Ratio
0	z	$-3-M$	$1-3M$	0	M	0	0	$-3M$	
	s_1	2	1	1	0	0	0	2	$2/1=2$
A_2 leaves	A_2	1	3^*	0	-1	0	1	3	$3/3=1$ ←
x_2 enters	s_3	0	1	0	0	1	0	4	$4/1=4$
1	z	$-10/3$	0	0	$1/3$	0		-1	
s_1 leaves	s_1	$5/3^*$	0	1	$1/3$	0		1	$3/5$ ←
x_1 enters	x_2	$1/3$	1	0	$-1/3$	0		1	3
	s_3	$-1/3$	0	0	$1/3$	1		3	-9
2	z	0	0	2	1	0		1	
	x_1	1	0	$3/5$	$1/5$	0		$3/5$	
	x_2	0	1	$-1/5$	$-2/5$	0		$4/5$	
	s_3	0	0	$1/5$	$2/5$	1		$16/5$	

$\therefore\ x_1 = \frac{3}{5},\ x_2 = \frac{4}{5},\ z_{\text{Max}} = 1.$

Example 3 : Use the Big M-method to solve the following L.P.P.

Maximise $z = 5x_1 - 2x_2 + 3x_3$

subject to $2x_1 + 2x_2 - x_3 \geq 2$

$3x_1 - 4x_2 \leq 3$

$x_2 + 3x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$ **(M.U. 2004)**

Sol. : As explained above we introduce the artificial variable A_1 in the first constraint and big penalty in the object function.

Maximise $z = 5x_1 - 2x_2 + 3x_3 - 0s_1 - 0s_2 - 0s_3 - MA_1$

subject to $2x_1 + 2x_2 - x_3 - s_1 + 0s_2 + 0s_3 + A_1 = 2$

$3x_1 - 4x_2 - 0x_3 + 0s_1 + s_2 + 0s_3 + 0A_1 = 3$

$0x_1 + x_2 + 3x_3 + 0s_1 + 0s_2 + s_3 + 0A_1 = 5$

We now eliminate the term $-MA_1$ from the object function by adding M times the first constraint to it.

$$\therefore\ z = 5x_1 - 2x_2 + 3x_3 + 2Mx_1 + 2Mx_2 - Mx_3 - Ms_1 - 0s_2 - 0s_3 + 0A_1 - 2M$$

$$\therefore\ z - 5x_1 - 2Mx_1 + 2x_2 - 2Mx_2 - 3x_3 + Mx_3 + Ms_1 + 0s_2 + 0s_3 + 0A_1 = -2M$$

with constraints as above.

Simplex Table

Iteration Number	Basic Var.	x_1	x_2	x_3	s_1	s_2	s_3	A_1	R.H.S. Sol.	Ratio
0	z	$-5-2M$	$2-2M$	$-3+M$	M	0	0	0	$-2M$	
A_1 leaves	A_1	2^*	2	-1	-1	0	0	1	2	$\frac{2}{2}=1$ ←
x_1 enters	s_2	3	-4	0	0	1	0	0	3	$\frac{3}{3}=1$
	s_3	0	1	3	0	0	1	0	5	$\frac{5}{0}=\ldots$
1	z	0	7	$-\frac{11}{2}$	$-\frac{5}{2}$	0	0		5	
s_2 leaves	x_1	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0		1	-2
x_3 enters	s_2	0	-7	$\frac{3}{2}^*$	$\frac{3}{2}$	1	0		0	0 ←
	s_3	0	1	3	0	0	1		5	$5/3$
2	z	0	$-\frac{56}{3}$	0	3	$\frac{11}{3}$	0		5	
s_3 leaves	x_1	1	$-\frac{4}{3}$	0	0	$\frac{1}{3}$	0		1	$-\frac{3}{4}$
x_2 enters	x_3	0	$-\frac{14}{3}$	1	1	$\frac{2}{3}$	0		0	0
	s_3	0	15^*	0	-3	-2	1		5	$\frac{5}{15}=\frac{1}{3}$ ←
3	z	0	0	0	$-\frac{11}{15}$	$\frac{53}{45}$	$\frac{56}{45}$		$\frac{101}{9}$	
x_3 leaves	x_1	1	0	1	$-\frac{4}{15}$	$\frac{7}{45}$	$\frac{4}{45}$		$\frac{13}{9}$	—
s_1 enters	x_3	0	0	1	$\frac{1}{15}^*$	$\frac{2}{45}$	$\frac{14}{45}$		$\frac{14}{9}$	$\frac{70}{3}$ ←
	x_2	0	1	0	$-\frac{1}{5}$	$-\frac{2}{15}$	$\frac{1}{15}$		$\frac{1}{3}$	—
4	z	0	0	11	0	$\frac{5}{3}$	$\frac{42}{9}$		$\frac{85}{3}$	
	x_1	1	0	4	0	$\frac{1}{3}$	$\frac{4}{3}$		$\frac{23}{3}$	
	s_1	0	0	15	1	$\frac{2}{3}$	$\frac{14}{3}$		$\frac{70}{3}$	
	x_2	0	1	3	0	0	1		5	

$\therefore\ x_1 = \frac{23}{3},\ x_2 = 5,\ x_3 = 0,\ z_{\text{Max}} = \frac{85}{3}.$

Example 4 : Use penalty (Big M) method to solve the following L.P.P.

Maximise $z = 6x_1 + 4x_2$

subject to $2x_1 + 3x_2 \leq 30$

$3x_1 + 2x_2 \leq 24$

$x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$

Is the solution unique ? If not, find another solution. If the requirement vector $\begin{bmatrix} 30 \\ 24 \\ 3 \end{bmatrix}$ is changed to $\begin{bmatrix} 24 \\ 30 \\ 3 \end{bmatrix}$, is the solution still optimal ? **(M.U. 2004, 05)**

Sol. : As explained earlier, we introduce slack variables s_1, s_2 in the first two constraints, surplus variable s_3 and the artificial variable A_3 in the third constraint, and big penalty in the object function.

Maximise $z = 6x_1 + 4x_2 - 0s_1 - 0s_2 - 0s_3 - MA_3$

subject to $2x_1 + 3x_2 + s_1 + 0s_2 + 0s_3 + 0A_3 = 30$

$3x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 + 0A_3 = 24$

$x_1 + x_2 + 0s_1 + 0s_2 - s_3 + A_3 = 3$

We now eliminate the term $-MA_3$ from the object function by adding M times the third constraint to the object function.

$\therefore \quad z = 6x_1 + Mx_1 + 4x_2 + Mx_2 - 0s_1 - 0s_2 - Ms_3 + 0A_3 - 3M$

$\therefore \quad z - (6 + M)\,x_1 - (4 + M)\,x_2 + 0s_1 + 0s_2 + Ms_3 + 0A_3 = -3M$

Simplex Table (For Example 4)

Iteration Number		Basic Var.	x_1	x_2	s_1	s_2	s_3	A_3	R.H.S. Sol.	Ratio
0		z	$-6-M$	$-4-M$	0	0	M	0	$-3M$	
	A_3 leaves	s_1	2	3	1	0	0	0	30	$\frac{30}{2} = 15$
	x_1 enters	s_2	3	2	0	1	0	0	24	$\frac{24}{3} = 8$
		A_3	1*	1	0	0	-1	1	3	$\frac{3}{1} = 3 \leftarrow$

Simplex Table contd. on the next page ....

..... Simplex Table contd. from previous page

Iteration Number		Basic Var.	x_1	x_2	s_1	s_2	s_3	A_3	R.H.S. Sol.	Ratio
1		z	0	2	0	0	-6		18	
	s_2 leaves	s_1	0	1	1	0	2		24	$\frac{24}{2} = 12$
	s_3 enters	s_2	0	-1	0	1	3*		15	$\frac{15}{3} = 5 \leftarrow$
		x_1	1	1	0	0	-1		3	$\frac{3}{-1} = -3$
2		z	0	0	0	2	0		48	
	s_1 leaves	s_1	0	$\frac{5}{3}$*	1	$-\frac{2}{3}$	0		14	$14 \times \frac{3}{5} = \frac{42}{5}$
	x_2 enters	s_3	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	1		5	$1 \times (-3) = ...$
		x_1	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0		8	$8 \times \frac{3}{2} = 12$

$\therefore \quad x_1 = 8, \; x_2 = 0, \; z_{Max} = 48.$

In the first table the value of x_2 comes out to be zero, which indicates the existence of an alternative solution. To find the alternative solution we select the column of x_2 as 'key column'. Then calculating the ratios as usual we find that the row of s_1 comes out to be 'key row'. The element 5/3 is the key element. Thus, we get the following third iteration.

Iteration Number	Basic Var.	x_1	x_2	s_1	s_2	s_3	A_3	R.H.S. Sol.	Ratio
3	z	0	0	0	2	0		48	
	x_2	0	1	3/5	$-2/5$	0		42/5	
	s_3	0	0	1/5	1/5	1		39/5	
	x_1	1	0	$-2/5$	3/5	0		12/5	

$\therefore$ The alternate optimal basic feasible solution is

$$x_1 = \frac{12}{5}, \; x_2 = \frac{42}{5}, \; z_{Max} = 48.$$

Further, if the requirement vector [30, 24, 3]' is changed to [24, 30, 3]', we see in the first table that the entries in the ratio-column will be 12, 15 and 3. Since the smallest number remains 3 in the third row, the row of A_3 will be the key row as before. The key element will remain the same. Hence, as a result of this the solution will not change.

Example 5 : Using Penalty (Big M) method solve the following L.P.P.

Minimise $z = x_1 + 2x_2 + x_3$

subject to $x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \le 1$

$\frac{3}{2}x_1 + 2x_2 + x_3 \ge 8$

$x_1, x_2, x_3 \ge 0.$ (M.U. 2001)

Sol. : Since the given problem is of minimisation type we convert it into maximisation type.

Maximize $z' = -z = -x_1 - 2x_2 - x_3$. .... (1)

We have

Maximise $z' = -x - 2x_2 - x_3 - 0s_1 - 0s_2 - MA_2$ .... (2)

subject to $x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 + s_1 + 0s_2 + 0A_2 = 1$ .... (3)

$\frac{3}{2}x_1 + 2x_2 + x_3 + 0s_1 - s_2 + A_2 = 8$

Multiply (3) by M and add to (1).

$(-1 + \frac{3}{2}M)x_1 + (-2+2M)x_2 + (-1+M)x_3 + 0s_1 - Ms_2 + OA_2 - 8M$

$\therefore$ Maximise $z' = \left(-1+\frac{3}{2}M\right)x_1 + (-2+2M)x_2 + (-1+M)x_3 + 0s_1 - Ms_2 + OA_2 - 8M$

$\therefore z' + \left(1 - \frac{3}{2}M\right)x_1 + (2-2M)x_2 + (1-M)x_2 + 0s_1 + Ms_2 + 0s_3 + 0A_2 = -8M$

Simplex Table

Iteration Number	Basic Var.	x_1	x_2	x_3	s_1	s_2	A_2	R.H.S. Sol.	Ratio
0	z'	$1-\frac{3}{2}M$	2-2M	$1-M$	0	M	0	$-8M$	
s_1 leaves	s_1	1	1/2*	1/2	1	0	0	1	2 ←
x_2 enters	A_2	3/2	2	1	0	−1	1	4	4
1	z'	$-3+\frac{5}{2}M$	0	$-1+M$	$-4+4M$	M	0	$-4-4M$	
	x_2	2	1	1	2	0	0	2	
	A_2	−5/2	0	−1	−4	−1	1	0	

Since all entries in the row of z are positive (M is a big positive number) A_2 appears with a positive values. The given problem has no feasible solution.

Example 6 : Use Penalty method to solve the following L.P.P.

Minimise $z = 2x_1 + 3x_2$

subject to $x_1 + x_2 \ge 5$

$x_1 + 2x_2 \ge 6$

$x_1, x_2 \ge 0.$ (M.U. 2000, 04, 06, 11, 16)

Sol. : Since the given problem is of minimisation type we convert it into maximisation type.

Maximise $z' = (-z) = -2x_1 - 3x_2$

subject to $x_1 + x_2 \ge 5$

$x_1 + 2x_2 \ge 6$

We now introduce slack and artificial variables and the penalties in the object function.

Maximise $z' = -2x_1 - 3x_2 - 0s_1 - 0s_2 - MA_1 - MA_2$

subject to $x_1 + x_2 - s_1 + 0s_2 + A_1 + 0A_2 = 5$

$x_1 + 2x_2 + 0s_1 - s_2 + 0A_1 + A_2 = 6$

We now eliminate the terms $-MA_1$ and $-MA_2$ from the object function by adding M times the first and second constraints to the object function.

$\therefore z' = -2x_1 - 3x_2 - 0s_1 - 0s_2 - MA_1 - MA_2$

$+ Mx_1 + Mx_2 - Ms_1 + MA_1 - 5M$

$+ Mx_1 + 2Mx_2 - Ms_2 + MA_2 - 6M$

$\therefore z' = (-2+2M)x_1 + (-3+3M)x_2 - Ms_1 - Ms_2 - 0A_1 - 0A_2 - 11M$

$\therefore z' + (2-2M)x_1 + (3-3M)x_2 + Ms_1 + Ms_2 + 0A_1 + 0A_2 = -11M$

Simplex Table

Iteration Number	Basic Var.	x_1	x_2	s_1	s_2	A_1	A_2	R.H.S. Sol.	Ratio
0	z'	$2-2M$	$3-3M$	M	M	0	0	$-11M$	
A_2 leaves	A_1	1	1	−1	0	1	0	5	5 / 1 = 5
x_2 enters	A_2	1	2*	0	−1	0	1	6	6 / 2 = 3 ←
1	z'	$\frac{1-M}{2}$	0	M	$\frac{3-M}{2}$	0		$-9-2M$	
A_1 leaves	A_1	1/2*	0	−1	1/2	1		2	4 ←
x_1 enters	x_2	1/2	1	0	−1/2	0		3	6
2	z'	0	0	1	1			−11	
	x_1	1	0	−2	1			4	
	x_2	0	1	1	−1			1	

$\therefore x_1 = 4, \; x_2 = 1$ and $z' = -11$.

$\therefore z_{Min} = 11.$

Example 7 : Using the Penalty (Big M) method solve the following L.P.P.

Maximise $z = x_1 + 2x_2 + 3x_3 - x_4$

subject to $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 10$

Sol. : Introducing three artificial variables A_1, A_2, A_3 in the three equalities and assigning big penalty $-M$ in the object function for A_1, A_2, A_3. We have,

Maximise $z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2 - MA_3$

subject to $x_1 + 2x_2 + 3x_3 + 0x_4 + A_1 + 0A_2 + 0A_3 = 15$

$2x_1 + x_2 + 5x_3 + 0x_4 + 0A_1 + A_2 + 0A_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 + 0A_1 + 0A_2 + A_3 = 10$

Adding M times the first, second and the third constraints to the object function, we get

$$z = (1 + 4M)\,x_1 + (2 + 5M)\,x_2 + (3 + 9M)\,x_3 + (-1 + M)\,x_4 + 0A_1 + 0A_2 + 0A_3 - 45M$$

$$\therefore\ z - (1 + 4M)\,x_1 - (2 + 5M)\,x_2 - (3 + 9M)\,x_3 + (1 - M)\,x_4 + 0A_1 + 0A_2 + 0A_3 = -45M$$

Simplex Table

Iteration Number	Basic Var.	x_1	x_2	x_3	x_4	A_1	A_2	A_3	R.H.S. Sol.	Ratio
0	z	$-(1+4M)$	$-(2+5M)$	$-(3+9M)$	$1-M$	0	0	0	$-45M$	
A_2 leaves	A_1	1	2	3	0	1	0	0	15	5
x_3 enters	A_2	2	1	5*	0	0	1	0	20	4 ←
	A_3	1	2	1	1	0	0	1	10	10
				↑						
1	z	$\frac{1-2M}{5}$	$\frac{-7-16M}{5}$	0	$1-M$	0		0	$12-9M$	
A_1 leaves	A_1	$-1/5$	$7/5$*	0	0	1		0	3	15/7 ←
x_2 enters	x_3	2/5	1/5	1	0	0		0	4	20
	A_3	3/5	9/5	0	1	0		1	6	10/3
			↑							
2	z	$-\frac{6M}{7}$	0	0	$1-M$			0	$\frac{105-15M}{7}$	
A_3 leaves	x_2	$-1/7$	1	0	0			0	15/7	-15
x_1 enters	x_3	3/7	0	1	0			0	25/7	25/3
	A_3	6/7*	0	0	1			1	15/7	5/2 ←
		↑								
3	z	0	0	0	1				15	
	x_2	0	1	0	1/6				5/2	
	x_3	0	0	1	$-1/2$				5/2	
	x_1	1	0	0	7/6				5/2	

$$\therefore\ x_1 = \frac{5}{2},\quad x_2 = \frac{5}{2},\quad x_3 = \frac{5}{2},\quad z_{Max} = 15.$$

Example 8 : Using Penalty (Big M) method solve the following L.P.P.

Minimise $z = 2x_1 + x_2$

subject to $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0.$ **(M.U. 2000, 17)**

Sol. : We have

Maximise $z' = -z = -2x_1 - x_2 - 0s_2 - 0s_3 - MA_1 - MA_2$ .... (1)

subject to $3x_1 + x_2 + 0s_2 + 0s_3 + A_1 + 0A_2 = 3$ .... (2)

$4x_1 + 3x_2 - s_2 + 0s_3 + 0A_1 + A_2 = 6$ .... (3)

$x_1 + 2x_2 + 0s_2 + s_3 + 0A_1 + 0A_2 = 3$ .... (4)

Multiply (2) and (3) by M and to (1).

$$\therefore\ \text{Maximise}\quad z' = (-2 + 7M)\,x_1 + (-1 + 4M)\,x_2 - Ms_2 + 0s_3 - A_1 - 0A_2 - 9M$$

$$\therefore\ z' + (2 - 7M)\,x_1 + (1 - 4M)\,x_2 + Ms_2 + 0s_3 + 0A_1 + 0A_2 = -9M$$

Simplex Table

Iteration Number	Basic Var.	x_1	x_2	s_2	s_3	A_1	A_2	R.H.S. Sol.	Ratio
0	z'	$2-7M$	$1-4M$	M	0	0	0	$-9M$	
A_1 leaves	A_1	3*	1	0	0	1	0	3	1 ←
x_1 enters	A_2	4	3	-1	0	0	1	6	1·5
	s_3	1	2	0	1	0	0	3	3
		↑							
1	z'	0	$\frac{1}{3} - \frac{5M}{3}$	M	0		0	$-2-2M$	
A_2 leaves	x_1	1	1/3	0	0		0	1	3
x_2 enters	A_2	0	5/3*	-1	0		1	2	6/5 ←
	s_3	0	5/3	0	1		0	2	6/5
			↑						
2	z'	0	0	1/5	0			$-12/5$	
	x_1	1	0	1/5	0			3/5	
	x_2	0	1	$-3/5$	0			6/5	
	s_3	0	0	1	1			0	

$$\therefore\ x_1 = \frac{3}{5},\quad x_2 = \frac{6}{5},\quad z'_{Max} = -\frac{12}{5}\qquad \therefore\ z_{Min} = \frac{12}{5}.$$

Example 9 : Using the Penalty (Big M) method solve the following L.P.P. (M.U. 1997)

Maximise $z = 3x_1 + 2x_2$

subject to $2x_1 + x_2 \le 2$ .... (1)

$3x_1 + 4x_2 \ge 12$ .... (2)

$x_1, x_2 \ge 0.$ .... (3)

Sol. : We have

Maximise $z = 3x_1 + 2x_2 - 0s_1 - 0s_2 - MA_2$

subject to $2x_1 + x_2 + s_1 + 0s_2 + 0A_2 = 2$

$3x_1 + 4x_2 + 0s_1 - s_2 + A_2 = 12$

Multiply (3) by M and add to 1.

$\therefore\ z = (3 + 3M)\, x_1 + (2 + 4M)\, x_2 - 0s_1 - Ms_2 - 0A_2 - 12M$

$\therefore\ z - (3 + 3M)\, x_1 - (2 + 4M)\, x_2 + 0s_1 + Ms_2 + 0A_2 = -12M$

Simplex Table

Iteration Number	Basic Var.	x_1	x_2	s_1	s_2	A_2	R.H.S. Sol.	Ratio
0	z	$-3-3M$	$-2-4M$	0	M	0	$-12M$	
s_1 leaves	s_1	2	1^*	1	0	0	2	2 ←
x_2 enters	A_2	3	4	0	-1	1	12	3
			↑					
1	z	$1+5M$	0	$2+4M$	M	0	$4-4M$	
	x_2	2	1	1	0	0	2	
	A_2	-5	0	-4	-1	1	4	

Since the artificial variable A_2 appears not at zero level and all entries in the row of z have M with positive coefficient, feasible solution does not exit. The solution is called pseudo-optimum basic feasible solution.

Example 10 : Using the Penalty (Big M) method solve the following L.P.P. (M.U. 2009)

Minimise $z = 4x_1 + x_2$

subject to $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \ge 6$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0.$

Sol. : We first write the problem in the standard form

Maximise $z' = -z = -4x_1 - x_2$

subject to $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \ge 6$

$x_1 + 2x_2 \le 4$

$\therefore\ z' = -4x_1 - x_2 - 0s_1 - 0s_2 - MA_1 - MA_2$ .... (1)

$3x_1 + x_2 + 0s_1 + 0s_2 + A_1 + 0A_2 = 3$ .... (2)

$4x_1 + 3x_2 - s_1 + 0s_2 + 0A_1 + A_2 = 6$ .... (3)

$x_1 + 2x_2 + 0s_1 + s_2 + 0A_1 + 0A_2 = 4$ .... (4)

Multiply (2) and (3) by M and add to (1)

$\therefore\ z' = (-4 + 7M)\, x_1 + (-1 + 4M)\, x_2 - Ms_1 - 0s_2 - 0A_1 - 0A_2 - 9M$

$\therefore\ z' + (4 - 7M)\, x_1 + (1 - 4M)\, x_2 + Ms_1 + 0s_2 + 0A_1 + 0A_2 = -9M$

Simplex Table

Iteration Number	Basic Var.	x_1	x_2	s_1	s_2	A_1	A_2	R.H.S. Sol.	Ratio
0	z'	$4-7M$	$1-4M$	M	0	0	0	$-9M$	
A_1 leaves	A_1	3^*	1	0	0	1	0	3	1 ←
x_1 enters	A_2	4	3	-1	0	0	1	6	1·5
	s_2	1	2	0	1	0	0	4	4
		↑							
1	z'	0	$-\dfrac{1+5M}{3}$	M	0		0	$-(4+2M)$	
A_2 leaves	x_1	1	$1/3$	0	0		0	1	3
x_2 enters	A_2	0	$5/3^*$	-1	0		1	2	$6/5$ ←
	s_2	0	$5/3$	0	1		0	3	$9/5$
			↑						
2	z'	0	0	$-1/5$	0			$-18/5$	
s_2 leaves	x_1	1	0	$1/5$	0			$3/5$	3
s_1 enters	x_2	0	1	$-3/5$	0			$6/5$	—
	s_2	0	0	1^*	1			1	1 ←
				↑					
3	z'	0	0	0	$1/5$			$-17/5$	
	x_1	1	0	0	$-1/5$			$2/5$	
	x_2	0	1	0	$3/5$			$9/5$	
	s_1	0	0	1	1			1	

$\therefore\ x_1 = \dfrac{2}{5},\quad x_2 = \dfrac{9}{5},\quad z'_{Max} = -\dfrac{17}{5} \quad \therefore\ z_{Min} = \dfrac{17}{5}$

Example 11 : Using the Big-M method solve the following L.P.P. (M.U. 2006)

Minimise $z = 10x_1 + 3x_2$

subject to $x_1 + 2x_2 \ge 3$

$x_1 + 4x_2 \ge 4\ ;\qquad x_1, x_2 \ge 0.$

Sol. : We first write the given problem in standard form.

Maximise $z' = -z = -10x_1 - 3x_2$

subject to $x_1 + 2x_2 \ge 3$

$x_1 + 4x_2 \ge 4$

$\therefore\ z' = -10x_1 - 3x_2 - 0s_1 - 0s_2 - MA_1 - MA_2$ .... (1)

$x_1 + 2x_2 - s_1 + 0s_2 + A_1 + 0A_2 = 3$ .... (2)

$x_1 + 4x_2 + 0s_1 - s_2 + 0A_1 + A_2 = 4$ .... (3)

We now eliminate $-MA_1$ and $-MA_2$ from the object function by adding M times the first and the second constraints to the object function.

$$\therefore\ z' = -10x_1 - 3x_2 - 0s_1 - 0s_2 - MA_1 - MA_2 + Mx_1 + 2Mx_2 - Ms_1 + MA_1 - 3M + Mx_1 + 4Mx_2 - Ms_2 + MA_2 - 4M$$

$$= (-10 + 2M)x_1 + (-3 + 6M)x_2 - Ms_1 - Ms_2 + 0A_1 + 0A_2 - 7M$$

$$\therefore\ z' + (10 - 2M)x_1 + (3 - 6M)x_2 + Ms_1 + Ms_2 + 0A_1 + 0A_2 = -7M$$

with constraints as above.

Setting $x_1 = 0, x_2 = 0, s_1 = 0, s_2 = 0$, we have $A_1 = 3, A_2 = 4$.

Simplex Table

Iteration Number	Basic Var.	x_1	x_2	s_1	s_2	A_1	A_2	R.H.S. Sol.	Ratio
0	z'	$10 - 2M$	$3 - 6M$	M	M	0	0	$-7M$	
A_2 leaves	A_1	1	2	-1	0	1	0	3	$3/2 = 1.5$
x_2 enters	A_2	1	4^*	0	-1	0	1	4	$4/4 = 1$ ←
1	z'	$\frac{37}{4} - \frac{M}{2}$	0	M	$\frac{3}{4} - \frac{M}{2}$	0		$-M-3$	
A_1 leaves	A_1	$1/2^*$	0	-1	$1/2$	0		1	2 ←
s_2 enters	x_2	$1/4$	1	0	$-1/4$	0		1	4
2	z'	$17/2$	0	$3/2$	0			$-9/2$	
	s_2	1	0	-2	1			2	
	x_2	$1/2$	0	$-1/2$	0			$3/2$	

$$\therefore\ x_1 = 0,\ x_2 = \frac{3}{2},\ z' = -\frac{9}{2}\quad \therefore\ z = \frac{9}{2}.$$

EXERCISE - I

Using the method of penalties (Big M) method solve the following L.P.P.

1. Maximise $z = 5x_1 - 2x_2 + 7x_3$
subject to $2x_1 + 2x_2 - x_3 \geq 2$
$3x_1 - 4x_2 \leq 3$
$x_2 + 3x_3 \leq 5$
$x_1, x_2, x_3 \geq 0$ **(M.U. 1996)**
[**Ans. :** $x_1 = 1, x_2 = 0, x_3 = 5/3, z_{Max} = 40/3$]

2. Maximise $z = 6x_1 + 4x_2$
subject to $2x_1 + 3x_2 \leq 30$
$3x_2 + 2x_2 \leq 24$
$x_1 + x_2 \geq 3$
$x_1, x_2 \geq 0$ **(M.U. 1997)**
[**Ans. :** $x_1 = 3, x_2 = 0, z_{Max} = 18$]

3. Maximise $z = 4x_1 + 5x_2 + 2x_3$
subject to $2x_1 + x_2 + x_3 \leq 10$
$x_1 + 3x_2 + x_3 \leq 12$
$x_1 + x_2 + x_3 = 6$
$x_1, x_2, x_3 \geq 0$ **(M.U. 1998, 2000)**
[**Ans. :** $x_1 = 3, x_2 = 3, x_3 = 0, z_{Max} = 27$]

4. Minimise $z = x_1 + x_2$
subject to $2x_1 + x_2 \geq 4$
$x_1 + 7x_2 \geq 7$
$x_1, x_2 \geq 0$ **(M.U. 1999)**
[**Ans. :** $x_1 = 21/13, x_2 = 10/13, z_{Max} = 31/13$]

5. Minimise $z = 4x_1 + x_2$
subject to $3x_1 + x_2 = 3$
$4x_1 + 3x_2 \geq 6$
$x_1 + 2x_2 \leq 4$
$x_1, x_2 \geq 0$ **(M.U. 2002)**
[**Ans. :** $x_1 = 3/5, x_2 = 6/5, z_{Min} = 18/5$]

6. Minimise $z = x_1 - 3x_2 - 2x_3$
subject to $3x_1 - x_2 + 2x_3 \geq 7$
$-2x_1 + 4x_2 \leq 12$
$-4x_1 + 3x_2 + 8x_3 \leq 10$
$x_1, x_2, x_3 \geq 0$
[**Ans. :** $x_1 = 78/25, x_2 = 114/25, x_3 = 11/10, z_{Min} = -319/25$]

7. Maximise $z = 2x_1 + 3x_2$
subject to $4x_1 + 3x_2 \geq 4$
$x_1 + 2x_2 \leq 1$
$x_1, x_2 \geq 0$
[**Ans. :** $x_1 = 1, x_2 = 0, z_{Max} = 2$]

8. Minimise $z = x_1 + 2x_2 + x_3$
subject to $2x_1 + x_2 + x_3 \leq 2$
$3x_1 + 4x_2 + 2x_3 \geq 10$
$x_1, x_2, x_3 \geq 0$
[**Ans. :** No feasible solution.]

9. Minimise $z = 2x_1 + 5x_2$
subject to $x_1 - 2x_2 \leq 4$
$2x_1 + 3x_2 \geq 6$
x_1, x_2 are both unrestricted. **(M.U. 2005)**

(Hint : Minimise $z = 2x_1' - 2x_1'' + 5x_2' - 5x_2''$
subject to $x_1' - x_1'' - 2x_2' + 2x_2' \leq 4$
$2x_1' - 2x_1' + 3x_2' - 3x_2'' \geq 4$

[**Ans. :** $x_1' = 24/7, x_1'' = 0, x_2' = 0, x_2'' = 2/7, x_1 = 24/7, x_2 = -2/7, z_{Min} = 38/7$]

10. Maximise $z = 3x_1 + 2x_2 + 3x_3$
subject to $3x_1 + 4x_2 + 2x_3 \geq 8$
$2x_1 + x_2 + x_3 \leq 2$
$x_1, x_2, x_3 \geq 0$
[**Ans. :** $x_1 = 0, x_2 = 2, x_3 = 0, z_{Max} = 4$]

11. Minimise $z = 3x_1 + 2x_2$
subject to $5x_1 + x_2 \geq 10$
$2x_1 + x_2 \geq 12$
$x_1 + 4x_2 \geq 12$; $x_1, x_2 \geq 0$
[**Ans. :** $x_1 = 1, x_2 = 5, z_{Min} = 13$]

12. Minimise $z = 2x_1 + 9x_2 + x_3$
subject to $x_1 + 4x_2 + 2x_3 \geq 5$
$3x_1 + x_2 + 2x_3 \geq 4$
$x_1, x_2, x_3 \geq 0$
[**Ans. :** $x_1 = 0, x_2 = 0, x_3 = 5/2, z_{Max} = 5/2$]

13. Minimise $z = 3x_1 + 8x_2$
subject to $x_1 + x_2 = 200$
$x_1 \leq 80, x_2 \geq 60$
$x_1, x_2 \geq 0$ **(M.U. 2007)**
[**Ans. :** $x_1 = 80, x_2 = 120, z_{Min} = 1200$]

14\. Maximise $z = 2x_1 + x_2 + 3x_3$ (M.U. 2007)
subject to $x_1 + x_2 + 2x_3 \le 5$
$2x_1 + 3x_2 + 4x_3 = 12$; $x_1 > 0$.

[Ans. : $x_1 = 3, x_2 = 2, x_3 = 0, z_{max} = 8$]

3. Duality

You are already familiar with the following results in set theory.

1. $A \cap A = A$	1. $A \cup A = A$
2. $A \cap B = B \cap A$	2. $A \cup B = B \cup A$
3. $A \cap (B \cap C) = (A \cap B) \cap C$	3. $A \cup (B \cup C) = (A \cup B) \cup C$
4. $(A \cap B)' = A' \cup B'$	4. $(A \cup B)' = A' \cap B'$
5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

The results on the right can be simply obtained by interchanging $\cap$ and $\cup$ in the results on the left and vice versa. Such results are called **duals** of each other. If one result is dual of another result, then the first is called the **primal**. Of course, primal and dual are relative terms. If duality exists then any one of the two results can be called primal and the other can be called as dual.

It is interesting to see that duality is found in linear programming problems also. Given a problem in linear programming we can obtain another, closely related linear programming problem from the same set of data and with the same solution. One is called the primal and the other is called the dual. The concept of duality is important because if we get a solution of one, the solution of the other can be immediately written down. For further analysis also duality is highly useful.

Consider the following problems.

Problem A

Maximise $z = 6x_1 + 10x_2$
subject to $3x_1 + 4x_2 \le 18$
$2x_1 + x_2 \le 8$
$x_1 + 3x_2 \le 20$
$x_1, x_2 \ge 0$.

Problem B

Minimize $w = 18y_1 + 8y_2 + 20y_3$
subject to $3y_1 + 2y_2 + y_3 \ge 6$
$4y_1 + y_2 + 3y_3 \ge 10$
$y_1, y_2, y_3 \ge 0$

The following way of writing the problems will enable you to understand the relation between the two problems.

Problem A

6	10	
x_1	x_2	
3	4	18
2	1	8
1	3	20

Problem B

18	8	20	
y_1	y_2	y_3	
3	2	1	6
4	1	3	10

It is now easy to see the relationship between the two problems. (i) In the two problems rows and column are interchanged *i.e.* the matrix of coefficients in one is the transpose of the matrix in the other. (ii) The column of the r.h.s. in one problem is the row of the coefficients in the objective function of the other. (iii) If the constraints in one problem are of "less than or equal to" type then the constraints in the other are "greater than or equal to" type. (iv) Further, the objective in one is of "maximisation", while the objective in the other is of "minimisation".

Thus, we can obtain the dual of a given problem by

(i) taking the transpose of the coefficient matrix,
(ii) interchanging the constraints on r.h.s. and the coefficients of the objective function,
(iii) reversing the inequalities,
(iv) interchanging the objective function from "minimisation" to "maximisation" and vice versa.

Definition : The phenomenon occurring in linear programming that given a problem there exists another closely related problem with the same set of data and with the same solution is called the **principle of duality**.

Formulation of Dual Problem

Consider a problem.

Maximise $z = c_1 x_1 + c_2 x_2 + \ldots\ldots c_n x_n$
subject to $a_{11} x_1 + a_{12} x_2 + \ldots\ldots + a_{1n} x_n \le b_1$
$a_{21} x_1 + a_{22} x_2 + \ldots\ldots + a_{2n} x_n \le b_2$
$\ldots\ldots\ldots\ldots\ldots\ldots$
$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots + a_{mn} x_n \le b_m$
$x_1, x_2, \ldots\ldots x_n \ge 0$.

To obtain dual of this problem, we proceed as follows.

(i) The maximisation problem in the primal becomes minimisation problem in the dual and *vice versa*.
(ii) The "less than or equal to" type of constraints in the primal become" greater than or equal to" constraints in the dual and *vice versa*.
(iii) The coefficients $c_1, c_2, \ldots\ldots, c_n$ in the objective function in the primal become the right hand side constants of the constraints of the dual and the right hand side constants $b_1, b_2, \ldots\ldots, b_m$ of the constraints in the primal become the coefficients in the objective function in the dual and *vice versa*.
(iv) If the primal has n decision variables and m constraints then the dual has m decision variables and n constraints and *vice versa*.
(v) The transpose of the body matrix (coefficient matrix) in the primal is the body matrix (coefficient matrix) in the dual and *vice versa*.
(vi) The variables in both the primal and dual are non-negative.

Following the above rules we can obtain the dual of the above problem as :-

Minimise $w = b_1 y_1 + b_2 y_2 + \ldots\ldots + b_m y_m$
subject to $a_{11} y_1 + a_{21} y_2 + \ldots\ldots + a_{m1} y_m \ge c_1$
$a_{12} y_1 + a_{22} y_2 + \ldots\ldots + a_{m2} y_m \ge c_2$
$\ldots\ldots\ldots\ldots\ldots\ldots$
$a_{1n} y_1 + a_{2n} y_2 + \ldots\ldots + a_{mn} y_m \ge c_n$
$y_1, y_2, \ldots\ldots y_m \ge 0$.

Procedure of writing dual from its primal

1. First write the objective function in 'maximisation type', if not.
2. Write all the constraints in less than or equal to type ($\leq$). If any constraint is in 'greater than or equal to' type ($\geq$), multiply the inequality throughout by (-1) and change the inequality sign to $\leq$.
3. If a constraint is given in the equality type, it must be expressed in the form of two inequalities as follows.

   First express the given equality in the form of two inequalities $\geq$ and $\leq$ going in opposite directions. Then, change $\geq$ to $\leq$ by multiplying through by (-1).

   For example, $3x_1 + 2x_2 = 5$ is equivalent to $3x_1 + 2x_2 \leq 5$ and $3x_1 + 2x_2 \geq 5$. Now, change the second inequality by multiplying throughout by (-1) as $-(3x_1 + 2x_2) \leq -5$.

   Thus, the given equality $3x_1 + 2x_2 = 5$ is now equivalent to $3x_1 + 2x_2 \leq 5$ and $-3x_1 - 2x_2 \leq -5$.

   If we want to express the given equality in the form of $\geq$, we multiply $3x_1 + 2x_2 \leq 5$ throughout by (-1).

   $\therefore \quad -(3x_1 + 2x_2) \geq -5$

   Thus, the given equality $3x_1 + 2x_2 = 5$ is now equivalent to $-3x_1 - 2x_2 \geq -5$ and $3x_1 + 2x_2 \geq 5$.
4. All decision variables must be greater than or equal to type. If any variable say x_2 is unrestricted express it as $x_2 = x_2' - x_2''$ where $x_2' \geq 0$, $x_2'' \geq 0$.

   For example, a negative number, say, -5 can be expressed as say $-5 = 8 - 13$; zero can be expressed as, say $0 = 7 - 7$.
5. Thus, in the primal in the standard form, (i) the objective function is of maximisation type with the constraints in "less than or equal to" type or (ii) the objective function is of minimisation type with the constraints in "greater than or equal to" type.
6. Now, write the dual by -

   (i) Changing the form of objective function from maximisation to minimisation.

   (ii) transposing the coefficient matrix.

   (iii) interchanging the coefficients $c_1, c_2, \ldots, c_n$ in the objective function with the right hand side $b_1, b_2, \ldots, b_m$ of the constraints.

   (iv) Changing the signs of inequalities of the constraints from $\geq$ to $\leq$ and vice versa.

Note ....

If the given linear programming problem is in **canonical form**, *then the primal-dual pair is said to be symmetric.*

Example 1 : Write the dual of the following L.P.P.

$$\text{Maximise} \quad z = 2x_1 - x_2 + 4x_3$$

$$\text{subject to} \quad \begin{aligned} x_1 + 2x_2 - x_3 &\leq 5 \\ 2x_1 - x_2 + x_3 &\leq 6 \\ x_1 + x_2 + 3x_3 &\leq 10 \\ 4x_1 + x_3 &\leq 12 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Sol. : Since in the given L.P.P. there are $m = 3$ variables and $n = 4$ constraints the dual will have $m = 4$ variables and $n = 3$ constraints.

Further the coefficients of x_1, x_2, x_3 in the objective function $c_1 = 2$, $c_2 = -1$, $c_3 = 4$ become the constraints $b_1 = 2$, $b_2 = -1$, $b_3 = 4$ in the dual. The right hand side constants $b_1 = 5$, $b_2 = 6$, $b_3 = 10$, $b_4 = 12$ become the coefficients c_1, c_2, c_3, c_4 of y_1, y_2, y_3, y_4 in the objective function.

Further, the objective function of the dual must be of minimisation type and the constraints must be of greater than or equal to type.

We have seen on page 11-3 that an L.P. problem can be written in matrix form as

$$\text{Maximise} \quad Z = cx$$

$$\text{subject to} \quad A\,x = b$$

where c is the matrix of the coefficients of the decision variables, x is the matrix of the decision variable, A is the matrix of the coefficients of $x_1, x_2, \ldots, x_n$ and b is the matrix of the r.h.s. of the constraints.

Let $c_{(1,n)} = [\, c_1, c_2, \ldots, c_n \,]$

$$x_{(n \times 1)} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A_{(m,n)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad b_{(m,1)} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The problem then can be expressed as

$$\text{Maximise} \quad z = c\,x$$

$$\text{subject to} \quad A\,x \leq b$$

We can write the given problem as

$$\text{Maximise} \quad z = c\,x \quad \text{where } c = [\, 2, -1, 4 \,]$$

$$\text{subject to} \quad A\,x = b$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 3 \\ 4 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 6 \\ 10 \\ 12 \end{bmatrix}$$

As discussed above, in the dual the following changes take place. **b** become new **c'**, **c** becomes new b', the transpose of A *i.e.* A^T becomes new A. Thus, the dual becomes

$$\text{Minimise} \quad W = c'\,Y$$

$$\text{subject to} \quad A^T\,Y = b'$$

In terms of the matrices A, b, c of the primal the dual becomes

$$\text{Minimise} \quad W = bY$$

$$\text{subject to} \quad A^T\,Y = c$$

$$\text{Minimise} \quad W = [\, 5, 6, 10, 12 \,] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\text{subject to} \quad \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -1 & 1 & 0 \\ -1 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Thus, the dual of the given problem becomes

$$\begin{aligned} &\text{Minimise} && w = 5y_1 + 6y_2 + 10y_3 + 12y_4 \\ &\text{subject to} && y_1 + 2y_2 + y_3 + 4y_4 \ge 2 \\ &&& 2y_1 - y_2 + y_3 \ge -1 \\ &&& -y_1 + y_2 + 3y_3 + y_4 \ge 4 \\ &&& y_1, y_2, y_3, y_4 \ge 0. \end{aligned}$$

Example 2 : Construct the dual of the following problem.

$$\begin{aligned} &\text{Minimise} && z = x_2 + 3x_3 \\ &\text{subject to} && 2x_1 + x_2 \le 3 \\ &&& x_1 + 2x_2 + 6x_3 \ge 5 \\ &&& -x_1 + x_2 + 2x_3 = 2; \quad x_1, x_2, x_3 \ge 0. \end{aligned}$$

(M.U. 2002)

Sol. : Since the objective function is of "minimisation" type, the constraints must be of "greater than or equal to" type. Hence, we multiply the first constraint by (–1) and get $-2x_1 - x_2 \ge -3$.

Since, the third constraints is of "equal to" type we write it as

$$-x_1 + x_2 + 2x_3 \ge 2 \quad \text{and} \quad -x_1 + x_2 + 2x_3 \le 2$$

$$\text{i.e.} \quad -(-x_1 + x_2 + 2x_3) \ge -2 \quad \text{i.e.,} \quad x_1 - x_2 - 2x_3 \ge -2$$

Thus, the given problem becomes

$$\begin{aligned} &\text{Minimise} && z = 0x_1 + x_2 + 3x_3 \\ &\text{subject to} && -2x_1 - x_2 + 0x_3 \ge -3 \\ &&& x_1 + 2x_2 + 6x_3 \ge 5 \\ &&& -x_1 + x_2 + 2x_3 \ge 2 \\ &&& x_1 - x_2 - 2x_3 \ge -2; \quad x_1, x_2, x_3 \ge 0. \end{aligned}$$

Since the last given equality type constraint is now expressed in the form of two constraints, we write y_3 as $y_3' - y_3''$.

$\therefore$ The dual of the problem is

$$\begin{aligned} &\text{Maximise} && w = -3y_1 + 5y_2 + 2y_3' - 2y_3'' \\ &\text{subject to} && -2y_1 + y_2 - y_3' + y_3'' \le 0 \\ &&& -y_1 + 2y_2 + y_3' - y_3'' \le 1 \\ &&& 0y_1 + 6y_2 + 2y_3' - 2y_3'' \le 3. \end{aligned}$$

Replacing $y_3' - y_3''$ by y_3 which now is unrestricted, the dual becomes,

$$\begin{aligned} &\text{Maximise} && w = -3y_1 + 5y_2 + 2y_3 \\ &\text{subject to} && -2y_1 + y_2 - y_3 \le 0 \\ &&& -y_1 + 2y_2 + y_3 \le 1 \\ &&& 6y_2 + 2y_3 \le 3. \quad y_1, y_2 \ge 0, \ y_3 \text{ is unrestricted.} \end{aligned}$$

Example 3 : Obtain the dual of the following L.P.P.

$$\begin{aligned} &\text{Minimise} && z = 3x_1 - 2x_2 + x_3 \\ &\text{subject to} && 2x_1 - 3x_2 + x_3 \le 5 \\ &&& 4x_1 - 2x_2 \ge 9 \\ &&& -8x_1 + 4x_2 + 3x_3 = 8 \\ &&& x_1, x_2 \ge 0, \ x_3 \text{ unrestricted.} \end{aligned}$$

(M.U. 2004)

Sol. : Since x_3 is unrestricted we put $x_3 = x_3' - x_3''$ where $x_3' \ge 0$, $x_3'' \ge 0$ and since the problem is of minimisation type we rewrite the given problem with all constraints in "greater than or equal to form".

Now, the first constraint is of less than or equal to type. We multiply it by (–1).

$$\therefore \quad -2x_1 + 3x_2 - (x_3' - x_3'') \ge -5$$

The last constraint is in the form of equality we write it as

$$-8x_1 + 4x_2 + 3x_3 \ge 8 \quad \text{and} \quad -8x_1 + 4x_2 + 3x_3 \le 8.$$

$$\text{i.e.} \quad -8x_1 + 4x_2 + 3x_3 \ge 8 \quad \text{and} \quad 8x_1 - 4x_2 - 3x_3 \ge -8$$

$$\therefore \quad -8x_1 + 4x_2 + 3(x_3' - x_3'') \ge 8$$

$$\text{and} \quad 8x_1 - 4x_2 - 3(x_3' - x_3'') \ge -8$$

Thus, the given problem becomes

$$\begin{aligned} &\text{Minimise} && z = 3x_1 - 2x_2 + x_3' - x_3'' \\ &\text{subject to} && -2x_1 + 3x_2 - x_3' + x_3'' \ge -5 \\ &&& 4x_1 - 2x_2 + 0x_3' - 0x_3'' \ge 9 \\ &&& -8x_1 + 4x_2 + 3x_3' - 3x_3'' \ge 8 \\ &&& 8x_1 - 4x_2 - 3x_3' + 3x_3'' \ge -8 \\ &&& x_1', x_2', x_3', x_3'' \ge 0. \end{aligned}$$

If we use y_1, y_2, y_3', y_3'' as the associated non-negative variables, then the dual of the problem is

$$\begin{aligned} &\text{Maximise} && w = -5y_1 + 9y_2 + 8y_3' - 8y_3'' \\ &\text{subject to} && -2y_1 + 4y_2 - 8y_3' + 8y_3'' \le 3 \\ &&& 3y_1 - 2y_2 + 4y_3' - 4y_3'' \le -2 \\ &&& -y_1 + 0y_2 + 3y_3' - 3y_3'' \le 1 \\ &&& y_1 + 0y_2 - 3y_3' + 3y_3'' \le -1 \end{aligned}$$

Replacing $y_3' - y_3''$ by y_3 where y_3 is the unrestricted and expressing the last two constraints as an equality, the dual of the given problem is

$$\begin{aligned} &\text{Maximise} && w = -5y_1 + 9y_2 + 8y_3 \\ &\text{subject to} && -2y_1 + 4y_2 - 8y_3 \le 3 \\ &&& 3y_1 - 2y_2 + 4y_3 \le -2 \\ &&& -y_1 + 3y_3 = 1 \\ &&& y_1, y_2 \ge 0, \ y_3 \text{ unrestricted.} \end{aligned}$$

Example 4 : Construct the dual of the following L.P.P.

Maximise $z = 3x_1 + 17x_2 + 9x_3$

subject to $x_1 - x_2 + x_3 \ge 3$

$-3x_1 + 2x_3 \le 1$

$2x_1 + x_2 - 5x_3 = 1$

$x_1, x_2, x_3 \ge 0$ (M.U. 2004)

Sol. : Since the problem is of maximisation type, all the constraints must be expressed in less than or equal to form (≤). Hence, we multiply the first constraint by −1 and get

$-x_1 + x_2 - x_3 \le 3.$

Since, the third constraint is in the form of an equality, we write it as

$2x_1 + x_2 - 5x_3 \ge 1$ and $2x_1 + x_2 - 5x_3 \le 1$

$\therefore -2x_1 - x_2 + 5x_3 \le -1$ and $2x_1 + x_2 - 5x_3 \le 1.$

Thus, the given problem becomes,

Maximise $z = 3x_1 + 17x_2 + 9x_3$

subject to $-x_1 + x_2 - x_3 \le -3$

$-3x_1 + 0x_2 + 2x_3 \le 1$

$2x_1 + x_2 - 5x_3 \le 1$

$-2x_1 - x_2 + 5x_3 \le -1$

$x_1, x_2, x_3 \ge 0.$

Since the last constraint in the primal is an equality, y_3 must be unrestricted. Let y_1, y_2, y_3', y_3'' be the associated non-negative variables of the dual. Then the dual is

Minimise $w = -3y_1 + y_2 + y_3' - y_3''$

subject to $-y_1 - 3y_2 + 2y_3' - 2y_3'' \ge 3$

$y_1 + 0y_2 + y_3' - y_3'' \ge 17$

$-y_1 + 2y_2 - 5(y_3' - y_3'') \ge 19.$

Putting $y_3' - y_3'' = y_3$ where y_3 is unrestricted, the required dual is

Minimise $w = -3y_1 + y_2 + y_3$

subject to $-y_1 - 3y_2 + 2y_3 \ge 3$

$y_1 \qquad + y_3 \ge 17$

$-y_1 + 2y_2 - 5y_3 \ge 19$

$y_1, y_2 \ge 0$, y_3 is unrestricted.

Example 5 : Find the dual of the following L.P.P.

Maximise $z = 2x_1 - x_2 + 3x_3$

subject to $x_1 - 2x_2 + x_3 \ge 4$

$2x_1 + x_3 \le 10$

$x_1 + x_2 + 3x_3 = 20$

$x_1, x_3 \le 0$, x_2 unrestricted. (M.U. 2004, 06, 11, 15)

Sol. : We first express the given problem in the standard form. Since x_2 is unrestricted we put $x_2 = x_2' - x_2''$.

Maximise $z = 2x_1 - (x_2' - x_2'') + 3x_3$

subject to $-[x_1 - 2(x_2' - x_2'') + x_3] \le -4$

$2x_1 + 0x_2 + x_3 \le 10$

$x_1 + (x_2' - x_2'') + 3x_3 \le 20$

and $-[x_1 + (x_2' - x_2'') + 3x_3] \le -20.$

Hence, we have

Maximise $z = 2x_1 - x_2' + x_2'' + 3x_3$

subject to $-x_1 + 2x_2' - 2x_2'' - x_3 \le -4$

$2x_1 + 0x_2' - 0x_2'' + x_3 \le 10$

$x_1 + x_2' - x_2'' + 3x_3 \le 20$

$-x_1 - x_2' + x_2'' - 3x_3 \le -20$

$x_1, x_2', x_2'', x_3 \ge 0.$

If we use y_1, y_2, y_3', y_3'' as the associated variables then the dual of the problem is

Minimise $w = -4y_1 + 10y_2 + 20y_3' - 20y_3''$

subject to $-y_1 + 2y_2 + y_3' - y_3'' \ge 2$

$2y_1 + 0y_2 + y_3' - y_3'' \ge -1$

$-2y_1 + 0y_2 - y_3' + y_3'' \ge 1$

$-y_1 + y_2 + 3y_3' - 3y_3'' \ge 3.$

i.e. Minimise $w = -4y_1 + 10y_2 + 20y_3$

subject to $-y_1 + 2y_2 + y_3 \ge 2$

$2y_1 \qquad + y_3 = -1$

$-y_1 + y_2 + 3y_3 \ge 3$

$y_1, y_2 \ge 0$, y_3 unrestricted.

Example 6 : Obtain the dual from the following primal

Minimise $z = x_1 - 3x_2 - 2x_3$

subject to $3x_1 - x_2 + 2x_3 \le 7$

$2x_1 - 4x_2 \qquad \ge 12$

$-4x_1 + 3x_2 + 8x_3 = 10$

$x_1, x_2 \ge 0$, x_3 is unrestricted. (M.U. 1996)

Sol. : First we express the given problem in standard form.

Minimise $z = x_1 - 3x_2 - 2x_3$

subject to $-3x_1 + x_2 - 2x_3 \ge -7$

$2x_1 - 4x_2 + 0x_3 \ge 12$

$-4x_1 + 3x_2 + 8x_3 \ge 10$

and $-4x_1 + 3x_2 + 8x_3 \le 10$

i.e. $4x_1 - 3x_2 - 8x_3 \ge -10.$

Since, x_3 is unrestricted, we put $x_3 = x_3' - x_3''$

$\therefore$ Minimise $z = x_1 - 3x_2 - 2x_3' + 2x_3''$

subject to $-3x_1 + x_2 - 2x_3' + 2x_3'' \geq -7$

$2x_1 - 4x_2 + 0x_3' - 0x_3'' \geq 12$

$-4x_1 + 3x_2 + 8x_3' - 8x_3'' \geq 10$

$4x_1 - 3x_2 - 8x_3' + 8x_3'' \geq -10$

$x_1, x_2, x_3', x_3'' \geq 0.$

If y_1, y_2, y_3', y_3'' are the dual variables and w is the function of the dual then dual of the given problem will be

Maximise $w = -7y_1 + 12y_2 + 10y_3' - 10y_3''$

subject to $-3y_1 + 2y_2 - 4y_3' + 4y_3'' \leq 1$

$y_1 - 4y_2 + 3y_3' - 3y_3'' \leq -3$

$-2y_1 \quad + 8y_3' - 8y_3'' \leq -2$

$2y_1 \quad - 8y_3' + 8y_3'' \leq 2$

Putting $y_3' - y_3'' = y_3$, we get

Maximise $w = -7y_1 + 12y_2 + 10y_3$

subject to $-3y_1 + 2y_2 - 4y_3 \leq 1$

$y_1 - 4y_2 + 3y_3 \leq -3$

$-2y_1 + 8y_3 = -2$

$y_1, y_2 \geq 0$, y_3 unrestricted.

Example 7 : Find the dual fo the following L.P.P.

Maximise $z = x_1 - 2x_2 + 3x_3$

subject to $-2x_1 + x_2 + 3x_2 = 2$

$2x_1 + 3x_2 + 4x_3 = 1$

$x_1, x_2, x_3 \geq 0.$

Sol. : Since the problem is of maximisation type, the constraints must be expressed in less than or equal to form.

$\therefore \quad -2x_1 + x_2 + 3x_3 \geq 2$ and $(-2x_1 + x_2 + 3x_3) \leq 2$

i.e. $-(-2x_1 + x_2 + 3x_3) \leq -2$ i.e. $2x_1 - x_2 - 3x_3 \leq -2$

and $\quad -2x_1 + x_2 + 3x_3 \leq 2$

Also $\quad 2x_1 + 3x_2 + 4x_3 \geq 1$ and $2x_1 + 3x_2 + 4x_3 \leq 1$

i.e. $-(2x_1 + 3x_2 + 4x_3) \leq -1$ i.e. $-2x_1 - 3x_2 - 4x_3 \leq -1$

and $\quad 2x_1 + 3x_2 + 4x_3 \leq 1$

Hence, the given problem becomes,

Maximise $z = x_1 - 2x_2 + 3x_3$

subject to $-2x_1 + x_3 + 3x_3 \leq 2$

$2x_1 - x_2 - 3x_3 \leq -2$

$2x_1 + 3x_2 + 4x_3 \leq 1$

$-2x_1 - 3x_2 - 4x_3 \leq -1$

$x_1, x_2, x_3 \geq 0.$

Now, if y_1', y_1'', y_2', y_2'' are the associated variables then the dual of the given problem is

Minimise $w = 2y_1' - 2y_1'' + y_2' - y_2''$

subject to $-2y_1' + 2y_1'' + 2y_2' - 2y_2'' \geq 1$

$y_1' - y_1'' + 3y_2' - 3y_2'' \geq -2$

$3y_1' - 3y_1'' + 4y_2' - 4y_2'' \geq 3$

Putting $y_1' - y_1'' = y_1$ and $y_2' - y_2'' = y_2$ the problem becomes.

Minimise $w = 2y_1 + y_2$

subject to $-2y_1 + 2y_2 \geq 1$

$y_1 + 3y_2 \geq -2$

$3y_1 + 4y_2 \geq 3.$

EXERCISE - II

Write the duals of the following L.P.P.'s.

1. Maximise $z = 2x_1 - x_2 + 4x_3$

subject to $x_1 + x_2 + 2x_3 \leq 12$

$2x_1 \quad - x_3 \leq 4$

$2x_1 - x_2 - 3x_3 \leq 5$

$x_1, x_2, x_3 \geq 0.$

[Ans. :

Minimise $w = 12y_1 + 4y_2 + 5y_3$

subject to $y_1 + 2y_2 + 2y_3 \geq 2$

$y_1 \quad - y_3 \geq -1$

$2y_1 - y_2 - 3y_3 \geq 4$

$y_1, y_2, y_3 \geq 0$]

2. Minimise $z = 2x_1 - 3x_2 + 4x_3$

subject to $3x_1 + 4x_2 + 5x_3 \geq 10$

$5x_1 + x_2 - 2x_3 \geq 6$

$6x_1 - 2x_2 - x_3 \leq 12$

$x_1 + 2x_2 + 6x_3 \geq 3$

$3x_1 + 7x_2 - 2x_3 \geq 2$

$x_1, x_2, x_3 \geq 0.$

[Ans. :

Maximise $w = 10y_1 + 6y_2 - 12y_3 + 3y_4 + 2y_5$

subject to $3y_1 + 5y_2 - 6y_3 + y_4 + 3y_5 \leq 2$

$4y_1 + y_2 + 2y_3 + 2y_4 + 7y_5 \leq -3$

$5y_1 + 2y_2 + y_3 + 6y_4 - 2y_5 \leq 4$

$y_1, y_2, y_3 \geq 0$]

3. Maximise $z = 3x_1 + 5x_2$

subject to $2x_1 + 6x_2 \leq 50$

$3x_1 + 2x_2 \leq 35$

$5x_1 - 3x_2 \leq 10$

$x_2 \leq 20,$

$x_1 \geq 0.$ **(M.U. 2007)**

[Ans. :

Minimise $w = 50y_1 + 35y_2 + 10y_3 + 20y_4$

subject to $2y_1 + 3y_2 + 5y_3 \geq 3$

$6y_1 + 2y_2 - 3y_3 + y_4 \geq 5$

$y_1, y_2 \geq 0$]

4. Maximise $z = 2x_1 + x_2 + x_3$

subject to $x_1 + x_2 + x_3 \geq 6$

$3x_1 - 2x_2 + 3x_3 = 3$

$-4x_1 \quad + x_3 \leq 10$

$x_1, x_3 \geq 0$, x_2 unrestricted.

(M.U. 2000)

[Ans. :

Minimise $w = -6y_1 + 3y_2 + 10y_3$

subject to $-y_1 + 3y_2 - 4y_3 \geq 2$

$-y_1 - 2y_2 = 1$

$-y_1 + 3y_2 + y_3 \geq 1$

$y_1, y_3 \geq 0$, y_2 unrestricted.]

5. Maximise $z = 2x_1 + 9x_2 + 11x_3$

subject to
$$x_1 - x_2 + x_3 \geq 3$$
$$-3x_1 + 2x_3 \leq 1$$
$$2x_1 + x_2 - 5x_3 = 1$$
$$x_1, x_2, x_3 \geq 0.$$

[Ans. : Minimise $w = -3y_1 + y_2 + y_3$

subject to
$$-y_1 - 3y_2 + 2y_3 \geq 2$$
$$y_1 \qquad + y_3 \geq 9$$
$$-y_1 + 2y_2 - 5y_3 \geq 11$$
$$y_1, y_2 \geq 0,\ y_3 \text{ unrestricted.}]$$

6. Maximise $z = 3x_1 + 10x_2 + 2x_3$

subject to
$$2x_1 + 3x_2 + 2x_3 \leq 8$$
$$3x_1 - 2x_2 + 4x_3 = 4$$
$$x_1, x_2, x_3 \geq 0.$$

[Ans. : Minimise $w = 8y_1 + 4y_2$

subject to
$$2y_1 + 3y_2 \geq 3$$
$$3y_1 - 2y_2 \geq 10$$
$$2y_1 + 4y_2 \geq 2$$
$$y_1 \geq 0,\ y_2 \text{ unrestricted.}]$$

7. Maximise $z = 3x_1 + 2x_2 + x_3$

subject to
$$x_1 + x_2 + x_3 \geq 6$$
$$3x_1 - 2x_2 + 3x_3 = 3$$
$$-4x_1 + 3x_2 - 6x_3 = 1$$
$$x_1, x_2, x_3 \geq 0.$$

[Ans. : Minimise $w = -6y_1 + 3y_2 + y_3$

subject to
$$-y_1 + 3y_2 - 4y_3 \geq 3$$
$$-y_1 - 2y_2 + 3y_3 \geq 2$$
$$-y_1 + 3y_2 - 6y_3 \geq 1$$
$$y_1 \geq 0,\ y_2, y_3 \text{ unrestricted.}]$$

8. Maximise $z = x_1 + 3x_2 - 2x_3 + 5x_4$

subject to
$$3x_1 - x_2 + x_3 - 4x_4 = 6$$
$$5x_1 + 3x_2 - x_3 - 2x_4 = 4$$
$$x_1,\ x_2 \geq 0,\ x_3,\ x_4$$

unrestricted.

[Ans. : Minimise $w = 6y_1 + 4y_2$

subject to
$$3y_1 + 5y_2 \geq 1$$
$$-y_1 + 3y_2 \geq 3$$
$$y_1 - y_2 = -2$$
$$-4y_1 - 2y_2 = 5\ ;\ y_1, y_2 \geq 0]$$

9. Minimise $z = 7x_1 + 5x_2 + 8x_3$

subject to
$$8x_1 + 2x_2 + x_3 \geq 3$$
$$3x_1 + 6x_2 + 4x_3 \geq 4$$
$$4x_1 + x_2 + 5x_3 \geq 1$$
$$x_1 + 5x_2 + 2x_3 \geq 7$$
$$x_1, x_2, x_3 \geq 0.$$

[Ans. : Maximise $w = 3y_1 + 4y_2 + y_3 + 7y_4$

subject to
$$8y_1 + 3y_2 + 4y_3 + y_4 \leq 7$$
$$2y_1 + 6y_2 + y_3 + 5y_4 \leq 5$$
$$y_1 + 4y_2 + 5y_3 + 2y_4 \leq 8$$
$$y_1, y_2, y_3, y_4 \geq 0]$$

4. Relationship Between Primal And Dual Optimal Solutions

1. If the primal possesses a finite optimal solution then the dual also possesses a finite optimal solution and $\boxed{z_{Max} = w_{Min}.}$

2. If the primal variable corresponds to *slack starting* variable in the dual then the *optimal value* of the primal variable is given by the coefficient of the slack variable in the dual. The value of z is given by the value of w and *vice versa*. (See Ex. 1 below).

3. If the primal variable corresponds to an *artificial starting variable* in the dual then the value of the primal variable is given by the coefficient of the artificial variable in the dual after deleting the constant M and changing its sign and vice versa. (See Ex. 2 below)

4. If the primal has an unbounded solution then the dual will not have a feasible solution and vice versa. (See Ex. 3 below)

Example 1 : Solve the L.P.P. from its primal as well as from its dual.

Minimise $z = 0{\cdot}7x_1 + 0{\cdot}5x_2$

subject to
$$x_1 \geq 4,\ x_2 \geq 6$$
$$x_1 + 2x_2 \geq 20$$
$$2x_1 + x_2 \geq 18$$
$$x_1,\ x_2 \geq 0$$

(M.U. 2010)

Sol. : We first write the given L.P.P. in standard form. Since the given problem is of minimisation type, we convert it into maximisation type.

Maximise
$$z' = -z = -0{\cdot}7x_1 - 0{\cdot}5x_2 - 0s_1 - 0s_2 - 0s_3 - 0s_4 - MA_1 - MA_2 - MA_3 - MA_4 \qquad \text{.......... (1)}$$

subject to
$$x_1 + 0x_2 - s_1 + 0s_2 + 0s_3 + 0s_4 + A_1 + 0A_2 + 0A_3 + 0A_4 = 4 \qquad \text{.......... (2)}$$
$$0x_1 + x_2 + 0s_1 - s_2 + 0s_3 + 0s_4 + 0A_1 + A_2 + 0A_3 + 0A_4 = 6 \qquad \text{.......... (3)}$$
$$x_1 + 2x_2 + 0s_1 + 0s_2 - s_3 + 0s_4 + 0A_1 + 0A_2 + A_3 + 0A_4 = 20 \qquad \text{.......... (4)}$$
$$2x_1 + x_2 + 0s_1 + 0s_2 + 0s_3 - s_4 + 0A_1 + 0A_2 + 0A_3 + A_4 = 18 \qquad \text{.......... (5)}$$

Multiply (2), (3), (4) and (5) by M and add to (1)

$$\therefore\quad z' = (-0{\cdot}7 + 4M)\,x_1 + (-0{\cdot}5 + 4M)\,x_2 - Ms_1 - Ms_2 - Ms_3 - Ms_4 + 0A_1 + 0A_2 + 0A_3 + 0A_4 - 48M$$

$$\therefore\quad z' + (0{\cdot}7 - 4M)\,x_1 + (0{\cdot}5 - 4M)\,x_2 + Ms_1 + Ms_2 + Ms_3 + Ms_4 + 0A_1 + 0A_2 + 0A_3 + 0A_4 = -48M$$

Simplex Table (Primal)

Iteration Number	Basic Var.	x_1	x_2	s_1	s_2	s_3	s_4	A_1	A_2	A_3	A_4	R.H.S. Sol.	Ratio
		Coefficients of											
0	z	$0{\cdot}7-4M$	$0{\cdot}5-4M$	M	M	M	M	0	0	0	0	$-18M$	
A_2 leaves	A_1	1	0	−1	0	0	0	1	0	0	0	4	—
x_2 enters	A_2	0	1*	0	−1	0	0	0	1	0	0	6	6 ←
	A_3	1	2	0	0	−1	0	0	0	1	0	20	10
	A_4	2	1	0	0	0	−1	0	0	0	1	18	18
		↑											
	z	$0{\cdot}7-4M$	0	M	$0{\cdot}5-3M$	M	M	0		0	0	$-3-24M$	
A_1 leaves	A_1	1*	0	−1	0	0	0	1		0	0	4	4 ←
x_1 enters	x_2	0	1	0	−1	0	0	0		0	0	6	—
	A_3	1	0	0	2	−1	0	0		1	0	8	8
	A_4	2	0	0	1	0	−1	0		0	1	12	6
		↑											
	z	0	0	$0{\cdot}7-3M$	$0{\cdot}5-3M$	M	M			0	0	$-5{\cdot}8-8M$	
A_3 leaves	x_1	1	0	−1	0	0	0			0	0	4	—
s_2 enters	x_2	0	1	0	−1	0	0			0	0	6	—
	A_3	0	0	1	2*	−1	0			1	0	4	2 ←
	A_4	0	0	2	1	0	−1			0	1	4	4
					↑								

Simplex Table continued on the next page

Simplex Table (For Ex. 1) continued from the previous page.

Iteration Number	Basic Var.	x_1	x_2	s_1	s_2	s_3	s_4	A_1	A_2	A_3	A_4	R.H.S. Sol.	Ratio
	z	0	0	$0{\cdot}45-\frac{3}{2}M$	0	$\frac{0{\cdot}5}{2}-\frac{M}{2}$	M				0	$-6{\cdot}8-2M$	
A_4 leaves	x_1	1	0	−1	0	0	0				0	4	—
s_1 enters	x_2	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0				0	8	16
	s_2	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0				0	2	4
	A_4	0	0	$\frac{3}{2}$*	0	$\frac{1}{2}$	−1				0	2	$\frac{4}{3}$ ←
				↑									
	z	0	0	0	0	0·1	0·3					−7·4	
	x_1	1	0	0	0	$\frac{1}{3}$	$-\frac{2}{3}$					16/3	
	x_2	0	1	0	0	$-\frac{2}{3}$	$\frac{1}{3}$					22/3	
	s_2	0	0	0	1	$-\frac{2}{3}$	$\frac{1}{3}$					4/3	
	s_1	0	0	1	0	$\frac{1}{3}$	$-\frac{2}{3}$					4/3	

(See page 14-30)

$\therefore \quad x_1 = \frac{16}{3}, \quad x_2 = \frac{22}{3}, \quad z'_{Max} = -7.4 \quad \therefore \quad z_{Min} = 7.4$

The dual of the given problem is

Maximise $\quad w = 4y_1 + 6y_2 + 20y_3 + 18y_4$

subject to $\quad y_1 + 0y_2 + y_3 + 2y_4 \le 0.7$

$\quad 0y_1 + y_2 + 2y_3 + y_4 \le 0.5$

Introducing slack variables the problem becomes :

Maximise $\quad w = 4y_1 + 6y_2 + 20y_3 + 18y_4 + 0s_1 + 0s_2$

$\therefore \quad w - 4y - 6y_2 - 20y_3 - 18y - 0s_1 - 0s_2 = 0$

subject to $\quad y_1 + 0y_2 + y_3 + 2y_4 + s_1 + 0s_2 = 0.7$

$\quad 0y_1 + y_2 + 2y_3 + y_4 + 0s_1 + s_2 = 0.5$

$\quad y_1, y_2, y_3, y_4 \ge 0.$

Simplex Table (Dual)

Iteration Number		Basic Var.	Coefficients of y_1	y_2	y_3	y_4	s_1	s_2	R.H.S. Sol.	Ratio
0		w	-4	-6	-20	-18	0	0	0	
	s_2 leaves	s_1	1	0	1	2	1	0	0.7	0.7
	y_3 enters	s_2	0	1	2*	1	0	1	0.5	0.25 ←
1		w	-4	4	0	-8	0	10	5	
	s_1 leaves	s_1	1	$-\frac{1}{2}$	0	$\frac{3}{2}$*	1	$-\frac{1}{2}$	0.45	0.3 ←
	y_4 enters	y_3	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0.25	0.5
2		w	$\frac{4}{3}$	$\frac{4}{3}$	0	0	$\frac{16}{3}$	$\frac{22}{3}$	7.4	
		y_4	$\frac{2}{3}$	$-\frac{1}{3}$	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0.3	
		y_3	$-\frac{1}{3}$	$\frac{2}{3}$	1	0	$-\frac{1}{2}$	$\frac{2}{3}$	0.1	

In the second iteration of the simplex table we see that in the row of w, the coefficient of s_1 is 16/3 and that of s_2 is 22/3. These are the values of x_1 and x_2 in the primal.

Hence, $\quad x_1 = \frac{16}{3}, \quad x_2 = \frac{22}{3}$

and $\quad w_{Max} = 7.4$ *i.e.* $z_{Min} = 7.4$.

Now, once again read (2) of § 4. (page 12-27)

Example 2 : Solve the following L.P.P. from its primal as well as from its dual.

Maximise $\quad z = 2x_1 + x_2$

subject to $\quad -x_1 + 2x_2 \le 2$ **(M.U. 2014)**

$\quad x_1 + x_2 \le 4$

$\quad x_1 \le 3$ and $x_1, x_2 \ge 0.$

Sol. : To solve the problem from its primal we write it in the standard form.

Maximise $\quad z = 2x_1 + x_2 - 0s_1 - 0s_2 - 0s_3$

i.e. $\quad z - 2x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

subject to $\quad -x_1 + 2x_2 + s_1 + 0s_2 + 0s_3 = 2$

$\quad x_1 + x_2 + 0s_1 + s_2 + 0s_3 = 4$

$\quad x_1 + 0x_2 + 0s_1 + 0s_2 + s_3 = 3$

Simplex Table (Primal)

Iteration Number		Basic Var.	Coefficients of x_1	x_2	s_1	s_2	s_3	R.H.S. Sol.	Ratio
0		z	-2	-1	0	0	0	0	
	s_3 leaves	s_1	-1	2	1	0	0	2	—
	x_1 enters	s_2	1	1	0	1	0	4	4
		s_3	1*	0	0	0	1	3	3 ←
1		z	0	-1	0	0	2	6	
	s_2 leaves	s_1	0	2	1	0	1	5	2.5
	x_2 enters	s_2	0	1*	0	1	-1	1	2 ←
		x_1	1	0	0	0	1	3	—
2		z	0	0	0	1	3	7	
		s_1	0	0	1	-2	3	7	
		x_2	0	1	0	1	-1	1	
		x_1	1	0	0	0	1	3	

$\therefore \quad x_1 = 3, \quad x_2 = 1, \quad z_{max} = 7.$

The dual of the above problem clearly is

Minimise $\quad w = 2y_1 + 4y_2 + 3y_3$

subject to $\quad -y_1 + y_2 + y_3 \ge 2$

$\quad 2y_1 + y_2 + 0y_3 \ge 1$

$\quad y_1, y_2, y_3 \ge 0.$

The dual in the standard form will be

Maximise $w' = -w = -2y_1 - 4y_2 - 3y_3 - 0s_1 - 0s_2 - MA_1 - MA_2$ .... (1)

subject to $-y_1 + y_2 + y_3 - s_1 + 0s_2 + A_1 + 0A_2 = 2$ .... (2)

$2y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1$ .... (3)

Multiply (2) and (3) M and add to (1).

Maximise $w' = -2y_1 - 4y_2 - 3y_3 + My_1 + 2My_2 + My_3 - Ms_1 - Ms_2 - 0A_1 - 0A_2 - 3M$

i.e. $w' + (2 - M)y_1 + (4 - 2M)y_2 + (3 - M)y_3 + Ms_1 + Ms_2 + 0A_1 + 0A_2 = -3M$

subject to $-y_1 + y_2 + y_3 - s_1 + 0s_2 + A_1 + 0A_2 = 2$

$2y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1$

Simplex Table (Dual)

Iteration Number	Basic Var.	y_1	y_2	y_3	s_1	s_2	A_1	A_2	R.H.S. Sol.	Ratio
0	w'	$2-M$	$4-2M$	$3-M$	M	M	0	0	$-3M$	
	A_1	-1	1	1	-1	0	1	0	2	2
A_2 leaves y_2 enters	A_2	2	1^*	0	0	-1	0	1	1	1 ←
1	w'	$-6+3M$	0	$3-M$	M	$4-M$	0	$-4+2M$	$-4-M$	
A_1 leaves y_3 enters	A_1	-3	0	1^*	-1	1	1	-1	1	1 ←
	y_2	2	1	0	0	-1	0	1	1	—
2	w'	3	0	0	3	1	$-3+M$	$-1+M$	-7	
	y_3	-3	0	1	-1	1	1	-1	1	
	y_2	2	1	0	0	-1	0	1	1	

Deleting M from the coefficients of A_1 and A_2 in the final table and then changing the signs of these coefficients we see that A_1 and A_2 correspond to 3 and 1.

These are the values of x_1 and x_2 in the primal.

Hence, $x_1 = 3$, $x_2 = 1$ and $w'_{Max} = -7$ $\therefore$ $w_{Min} = 7$

Now, once again read (3) of § 4. (page 12-27)

***Example 3 :** When the primal L.P.P is infeasible, its dual is unbounded. Verify if this is true for the following problem using the simplex method.*

Maximise $z = 8x_1 + 6x_2$

subject to $x_1 - x_2 \le 0{\cdot}6$

$x_1 - x_2 \ge 2$

$x_1, x_2 \ge 0.$

(M.U. 2004)

Sol. : We first write the primal in the standard form.

Maximise $z = 8x_1 + 6x_2 + 0s_1 + 0s_2 - MA_2$ .... (1)

subject to $x_1 - x_2 + s_1 + 0s_2 = 0{\cdot}6$ .... (2)

$x_1 - x_2 + 0s_1 - s_2 + A_2 = 2$ .... (3)

Multiply (3) by M and add it to (1).

Maximise $z = 8x_1 + 6x_2 + Mx_1 - Mx_2 + 0s_1 - Ms_2 - 0A_2 - 2M$

i.e. $z - (8 + M)x_1 - (6 - M)x_2 + 0s_1 + Ms_2 + 0A_2 = -2M$

subject to $x_1 - x_2 + s_1 + 0s_2 + 0A_2 = 0{\cdot}6$

$x_1 - x_2 + 0s_1 - s_2 + A_2 = 2$

Simplex Table (Primal)

Iteration Number	Basic Var.	x_1	x_2	s_1	s_2	A_2	R.H.S. Sol.	Ratio
0	z	$-8-M$	$-6+M$	0	0	0	$-2M$	
s_1 leaves	s_1	1^*	-1	1	0	0	3/5	3/5 ←
x_1 enters	A_2	1	-1	0	-1	1	2	2
1	z	0	-14	$8+M$	0	0	$\frac{24-7M}{5}$	
	x_1	1	-1	1	0	0	3/5	$-5/3$
	A_2	0	0	-1	-1	1	7/5	∞

Since the ratios in the last column are either negative or infinite i.e. no finite positive ratio, the solution to the primal is unbounded.

Now, to find the dual of the above problem we write it as

Maximise $z = 8x_1 + 6x_2$

subject to $x_1 - x_2 \le 0{\cdot}6$

$-x_1 + x_2 \le -2$

$x_1, x_2 \ge 0.$

Hence, the dual of the above problem is

Minimise $w = 0{\cdot}6y_1 - 2y_2$

subject to $y_1 - y_2 \ge 8$

$-y_1 + y_2 \ge 6$

Introducing the slack and artificial variables, we have

Maximise $w' = -w = -0{\cdot}6y_1 + 2y_2 - MA_1 - MA_2$ .... (1)

subject to $y_1 - y_2 - s_1 + 0s_2 + A_1 + 0A_2 = 8$ .... (2)

$-y_1 + y_2 + 0s_1 - s_2 + 0A_1 + A_2 = 6$ .... (3)

$A_2 - 2M$ .... (1)

$A_2 = -2M$ .... (2)

.... (3)

Ratio

3/5 ←

2

ositive

Multiply (2) and (3) by M and add to (1).

Maximise $w' = -0{\cdot}6y_1 + 2y_2 - Ms_1 - Ms_2 - 0A_1 - 0A_2 - 14M$

i.e. $w' + 0{\cdot}6y_1 - 2y_2 + Ms_1 + Ms_2 + 0A_1 + 0A_2 = -14M$

subject to $y_1 - y_2 - s_1 + 0s_2 + A_1 + 0A_2 = 8$ .... (2)

$-y_1 + y_2 + 0s_1 - s_2 + 0A_1 + A_2 = 6$ .... (3)

Simplex Table (Dual)

Iteration Number	Basic Var.	Coefficients of y_1	y_2	s_1	s_2	A_1	A_2	R.H.S. Sol.	Ratio
0	w'	3/5	-2	M	M	0	0	$-14M$	
	A_1	1	-1	-1	0	0	0	8	—
A_2 leaves y_2 enters	A_2	-1	1^*	0	-1	1	1	6	6 ←
1	w'	$-7/5$	0	M	$M-2$	2		$-14M + 12$	
	A_1	0	0	-1	-1	1		14	∞
	y_2	-1	1	0	-1	1		6	-6

Since in the column of ratios the ratios are either negative or infinite, there is no leaving variable. Since the artificial variable appears as a basic variable, the solution is infeasible.

Example 4 : Using Duality solve the following L.P.P.

Minimise $z = 4x_1 + 3x_2 + 6x_3$

subject to $x_1 + x_3 \geq 2$

$x_2 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0.$ **(M.U. 1998)**

Sol. : *We first write the given problem as :*

Minimise $z = 4x_1 + 3x_2 + 6x_3$

subject to $x_1 + 0x_2 + x_3 \geq 2$

$0x_1 + x_2 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0.$

The dual of this problem is :

Maximise $w = 2y_1 + 5y_2$

subject to $y_1 + 0y_2 \leq 4$

$0y_1 + y_2 \leq 3$

$y_1 + y_2 \leq 6$

$y_1, y_2 \geq 0$

Maximise $w = 2y_1 + 5y_2 - 0s_1 - 0s_2 - 0s_3$

$\therefore$ i.e. $w - 2y_1 - 5y_2 + 0s_1 + 0s_2 + 0s_3 = 0$

subject to $y_1 + 0y_2 + s_1 + 0s_2 + 0s_3 = 4$

$0y_1 + y_2 + 0s_1 + s_2 + 0s_3 = 3$

$y_1 + y_2 + 0s_1 + 0s_2 + s_3 = 6$

Simplex Table

Iteration Number	Basic Var.	Coefficients of y_1	y_2	s_1	s_2	s_3	R.H.S. Sol.	Ratio
0	w	-2	-5	0	0	0	0	
	s_1	1	0	1	0	0	4	—
s_2 leaves y_2 enters	s_2	0	1^*	0	1	0	3	3 ←
	s_3	1	1	0	0	1	6	6
1	w	-2	0	0	5	0	15	
	s_1	1	0	1	0	0	4	4
s_3 leaves y_1 enters	y_2	0	1	0	1	0	3	—
	s_3	1^*	0	0	-1	1	3	3 ←
2	w	0	0	0	3	2	21	
	s_1	0	0	1	1	-1	1	
	y_2	0	1	0	1	0	3	
	y_1	1	0	0	-1	1	3	

$s_1 = 0,\ s_2 = 3,\ s_3 = 2,\ w_{Max} = 21$

$\therefore\ x_1 = 0,\ x_2 = 3,\ x_3 = 2,\ z_{Min} = 21.$

Example 5 : Using Duality solve the following L.P.P.

Minimise $z = 5x_1 + 8x_2$

subject to $x_1 + x_2 \leq 2$

$x_1 + 2x_2 \geq 0$

$-x_1 + 4x_2 \leq 1$ **(M.U. 1999)**

$x_1, x_2 \geq 0.$

Sol. : We first write the given problem with all constraints in greater than or equal to form.

Minimise $z = 5x_1 + 8x_2$

subject to $-x_1 - x_2 \geq -2$

$x_1 + 2x_2 \geq 0$

$x_1 - 4x_2 \geq -1$

$x_1, x_2 \geq 0$

$\therefore$ The Dual of this problem is

Maximise $w = -2y_1 + 0y_2 - y_3$

subject to $-y_1 + y_2 + y_3 \le 5$

$-y_1 + 2y_2 - 4y_3 \le 8, \quad y_1, y_2, y_3 \ge 0.$

Introducing slack variables the problem becomes

Maximise $w = -2y_1 + 0y_2 - y_3 - 0s_1 - 0s_2$

i.e. $w + 2y_1 + 0y_2 + y_3 + 0s_1 + 0s_2 = 0$

subject to $-y_1 + y_2 + y_3 + s_1 + 0s_2 = 5$

$-y_1 + 2y_2 - 4y_3 + 0s_1 + s_2 = 8$

Simplex Table

Iteration Number	Basic Variable	Coefficients of y_1	y_2	y_3	s_1	s_2	R.H.S. Sol.	Ratio
0	w	2	0	1	0	0	0	
	s_1	–1	1	1	1	0	5	—
	s_2	–1	2	–4	0	1	8	—

Since there is no negative entry in the row of w this is the optimal solution.

$\therefore s_1 = 0, s_2 = 0.$

$\therefore$ The solution of the primal is $x_1 = 0,\ x_2 = 0,\ z_{Min} = 0.$

Example 6 : Using Duality solve the following L.P.P.

Minimise $z = 430x_1 + 460x_2 + 420x_3$

subject to $x_1 + 3x_2 + x_3 \ge 3$

$2x_1 + 4x_3 \ge 2$

$x_1 + 2x_2 \ge 5$

$x_1, x_2, x_3 \ge 0.$

(M.U. 2000)

Sol. : The Dual of the above problem is :

Maximise $w = 3y_1 + 2y_2 + 5y_3$

subject to $y_1 + 2y_2 + y_3 \le 430$

$3y_1 + 0y_2 + 2y_3 \le 460$

$y_1 + 4y_2 + 0y_3 \le 420$

$y_1, y_2, y_3 \ge 0$

Introducing the slack variables, the problem becomes

Maximise $w = 3y_1 + 2y_2 + 5y_3 - 0s_1 - 0s_2 - 0s_3$

i.e. $w - 3y_1 - 2y_2 - 5y_3 + 0s_1 + 0s_2 + 0s_3 = 0$

subject to $y_1 + 2y_2 + y_3 + s_1 + 0s_2 + 0s_3 = 430$

$3y_1 + 0y_2 + 2y_3 + 0s_1 + s_2 + 0s_3 = 460$

$y_1 + 4y_2 + 0y_3 + 0s_1 + 0s_2 + s_3 = 420$

Simplex Table

Iteration Number	Basic Var.	Coefficients of y_1	y_2	y_3	s_1	s_2	s_3	R.H.S. Sol.	Ratio
0	w	–3	–2	–5	0	0	0	0	
s_2 leaves	s_1	1	2	1	1	0	0	430	430
y_3 enters	s_2	3	0	2*	0	1	0	460	230 ←
	s_3	1	4	0	0	0	1	420	—
1	w	9/2	–2	0	0	5/2	0	1150	
s_1 leaves	s_1	–1/2	2*	0	1	–1/2	0	200	100 ←
y_2 enters	y_3	3/2	0	1	0	1/2	0	230	—
	s_3	1	4	0	0	0	1	430	105
2	w	4	0	0	1	2	0	1350	
	y_2	–1/4	1	0	1/2	–1/4	0	100	
	y_3	3/2	0	1	0	1/2	0	230	
	s_3	2	0	0	–2	1	1	30	

$\therefore\ s_1 = 1,\ s_2 = 2,\ s_3 = 0,\ w_{Max} = 1350$

$\therefore\ x_1 = 1,\ x_2 = 2,\ x_3 = 0,\ z_{Min} = 1350$

Example 7 : Using Duality solve the following problem.

Maximise $z = 2x_1 + x_2$

subject to $2x_1 - x_2 \le 2$

$x_1 + x_2 \le 4$

$x_1 \le 3$

$x_1, x_2 \ge 0.$

(M.U. 1998, 2011)

Sol. : The dual of the above problem is

Minimise $w = 2y_1 + 4y_2 + 3y_3$

subject to $2y_1 + y_2 + y_3 \ge 2$

$-y_1 + y_2 + 0y_3 \ge 1$

$y_1, y_2, y_3 \ge 0.$

Introducing the slack and artificial variables

Minimise $w = 2y_1 + 4y_2 + 3y_3$

$\therefore$ Maximise $w' = -w = -2y_1 - 4y_2 - 3y_3 - 0s_1 - 0s_2 - MA_1 - MA_2$ .... (1)

subject to $2y_1 + y_2 + y_3 - s_1 - 0s_2 + A_1 + 0A_2 = 2$ .... (2)

$-y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1$ .... (3)

Multiply (2) and (3) by M and add to (1).

$\therefore \quad w' = -2y_1 - 4y_2 - 3y_3 + My_1 + 2My_2 + My_3 - Ms_1 - Ms_2 + 0A_1 + 0A_2 - 3M$

$\therefore \quad w' + (2-M)\,y_1 + (4-2M)\,y_2 + (3-M)\,y_3 + Ms_1 + Ms_2 + 0A_1 + 0A_2 = -3M$

Simplex Table

Iteration Number	Basic Var.	y_1	y_2	y_3	s_1	s_2	A_1	A_2	R.H.S. Sol.	Ratio
0	w'	$2-M$	$4-2M$	$3-M$	M	M	0	0	$-3M$	
A_2 leaves	A_1	2	1	1	-1	0	1	0	2	2
y_2 enters	A_2	-1	1*	0	0	-1	0	1	1	1 ←
1	w'	$6-3M$	0	$3-M$	M	$4-M$	0	$-4+2M$	$-4-M$	
A_1 leaves	A_1	3*	0	1	-1	1	1	-1	1	1/3 ←
y_1 enters	y_2	-1	1	0	0	-1	0	1	1	—
2	w'	0	0	1	2	2	$M-2$	$M-2$	-6	
	y_1	1	0	1/3	$-1/3$	1/3	1/3	$-1/3$	1/3	
	y_2	0	1	1/3	$-1/3$	$-2/3$	1/3	2/3	4/3	

$\therefore \quad s_1 = 2,\ s_2 = 2,\ w'_{Max} = -6,\ w_{Min} = 6.$

$\therefore \quad x_1 = 2,\ x_2 = 2,\ z_{Max} = 6.$

Example 8 : Using duality solve the following L.P.P.

Maximise $\quad z = 3x_1 + 2x_2$

subject to $\quad 2x_1 + x_2 \le 5$

$x_1 + x_2 \le 3$

$x_1, x_2 \ge 0.$

(M.U. 1997, 99)

Sol. : The dual of the given problem is

Minimise $\quad w = 5y_1 + 3y_2$

subject to $\quad 2y_1 + y_2 \ge 3$

$y_1 + y_2 \ge 2$

$y_1, y_2 \ge 0$

Introducing the slack and artificial variables, the problem becomes ;

Maximise $\quad w' = -w = -5y_1 - 3y_2$

i.e. $\quad w' = -5y_1 - 3y_2 - 0s_1 - 0s_2 - MA_1 - MA_2$ .... (1)

subject to $\quad 2y_1 + y_2 - s_1 - 0s_2 + A_1 + 0A_2 = 3$ .... (2)

$y_1 + y_2 + 0s_1 - s_2 + 0A_1 + A_2 = 2$ .... (3)

Muliply (2) and (3) by M and add to (1)

$\therefore \quad w' = -5y_1 - 3y_2 + 3My_1 + 2My_2 - Ms_1 - Ms_2 - 0A_1 - 0A_2 - 5M$

$\therefore \quad w' + (5-3M)\,y_1 + (3-2M)\,y_2 + Ms_1 + Ms_2 + 0A_1 + 0A_2 = -5M$

Simplex Table

Iteration Number	Basic Var.	y_1	y_2	s_1	s_2	A_1	A_2	R.H.S. Sol.	Ratio
0	w'	$5-3M$	$3-2M$	M	M	0	0	$-5M$	
A_1 leaves	A_1	2*	1	-1	0	1	0	3	3/2 ←
y_1 enters	A_2	1	1	0	-1	0	1	2	2
1	w'	0	$\frac{1-M}{2}$	$\frac{5-M}{2}$	M	$\frac{3M-5}{2}$	0	$-\frac{15+M}{2}$	
A_2 leaves	y_1	1	1/2	$-1/2$	0	1/2	0	3/2	3
y_2 enters	A_2	0	1/2*	1/2	-1	$-1/2$	1	1/2	1 ←
2	w'	0	0	2	1	$M-2$	$M-1$	-8	
	y_1	1	0	-1	1	1	-1	1	
	y_2	0	1	1	-2	-1	2	1	

Since, $s_1 = 2,\ s_2 = 1,\ w'_{Max} = -8. \quad \therefore\ w_{Min} = 8$

$\therefore \quad x_1 = 2,\ x_2 = 1,\ z_{Max} = 8.$

Example 9 : Using the principle of duality solve the following L.P.P.

Minimise $\quad z = 4x_1 + 14x_2 + 3x_3$

subject to $\quad -x_1 + 3x_2 + x_3 \ge 3$

$2x_1 + 2x_2 - x_3 \ge 2$

$x_1, x_2, x_3 \ge 0.$

(M.U. 2006, 09)

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Sol. : The dual of this problem is

Maximise $w = 3y_1 + 2y_2$

subject to $-y_1 + 2y_2 \le 4$

$3y_1 + 2y_2 \le 14$

$y_1 - y_2 \le 3$

$y_1, y_2 \ge 0.$

We write this problem in standard form.

Maximise $w = 3y_1 + 2y_2 - 0s_1 - 0s_2 - 0s_3$

i.e. $w - 3y_1 - 2y_2 + 0s_1 + 0s_2 + 0s_3 = 0$

subject to $-y_1 + 2y_2 + s_1 + 0s_2 + 0s_3 = 4$

$3y_1 + 2y_2 + 0s_1 + s_2 + 0s_3 = 14$

$y_1 - y_2 + 0s_1 + 0s_2 + s_3 = 3$

Simplex Table

Iteration Number	Basic Var.	Coefficients of y_1	y_2	s_1	s_2	s_3	R.H.S. Solution	Ratio
0	w	-3	-2	0	0	0	0	
s_3 leaves	s_1	-1	2	1	0	0	4	$3 \div (-1) = —$
y_1 enters	s_2	3	2	0	1	0	14	$14 \div 3 = 4 \cdot \bar{6}6$
	s_3	1^*	-1	0	0	1	3	$3 \div 1 = 3$ ←
		↑						
1	w	0	-5	0	0	3	9	
s_2 leaves	s_1	0	1	1	0	1	7	$7 \div 1 = 7$
y_2 enters	s_2	0	5	0	1	-3	5	$5 \div 5 = 1$ ←
	y_1	1	-1	0	0	1	3	$3 \div (-1) = —$
			↑					
2	w	0	0	0	1	0	14	
	s_1	0	0	1	$-1/5$	$8/5$	6	
	y_2	0	1	0	$1/5$	$-3/5$	1	
	y_1	1	0	0	$1/5$	$2/5$	4	

$\therefore$ The optimal solution of the dual is $s_1 = 0, s_2 = 1, s_3 = 0, w = 14.$

$\therefore$ The optimal solution of the primal is $x_1 = 0,\ x_2 = 1,\ x_3 = 0,\ z = 14.$

Example 10 : Using duality solve the following L.P.P.

Maximise $z = 5x_1 - 2x_2 + 3x_3$

subject to $2x_1 + 2x_2 - x_3 \ge 2$

$3x_1 - 4x_2 \le 3$

$x_2 + 3x_3 \le 5$

$x_1, x_2, x_3 \ge 0$

(M.U. 2001, 06, 15)

Sol. : Since the problem is of maximisation type. We first write the constraints in less than or equal to form.

Maximise $z = 5x_1 - 2x_2 + 3x_3$

subject to $-2x_1 - 2x_2 + x_3 \le -2$

$3x_1 - 4x_2 + 0x_3 \le 3$

$0x_1 + x_2 + 3x_3 \le 5$

$x_1, x_2, x_3 \ge 0$

Its dual will be

Minimise $w = -2y_1 + 3y_2 + 5y_3$

i.e. Maximise $w' = -w = 2y_1 - 3y_2 - 5y_3$

subject to $-2y_1 + 3y_2 + 0y_3 \ge 5$

$-2y_1 - 4y_2 + y_3 \ge -2$

i.e. $2y_1 + 4y_2 - y_3 \le 2$

$y_1 + 0y_2 + 3y_3 \ge 3$

$y_1, y_2, y_3 \ge 0$

$\therefore$ Introducing slack and artificial variables, the problem becomes

Maximise $w' = -w = 2y_1 - 3y_2 - 5y_3 - 0s_1 - 0s_2 - 0s_3 - MA_1 - MA_3$ .... (1)

subject to $-2y_1 + 3y_2 + 0y_3 - s_1 + 0s_2 + 0s_3 + A_1 + 0A_3 = 5$ .... (2)

$2y_1 + 4y_2 - y_3 + 0s_1 + s_2 + 0s_3 + 0A_1 + 0A_3 = 2$ .... (3)

$y_1 + 0y_2 + 3y_3 + 0s_1 + 0s_2 - s_3 + 0A_1 + A_3 = 3$ .... (4)

Multiply (2) and (4) by M and add to (1).

$$\therefore \quad w' = 2y_1 - 3y_2 - 5y_3 - 0s_1 - 0s_2 - 0s_3 - Ms_1 - Ms_3 - My_1 + 3My_2 + 3My_3 - 0A_1 - 0A_3 - 8M$$

$$\therefore \quad w' = (2 - M)\,y_1 + (-3 + 3M)\,y_2 + (-5 + 3M)\,y_3 - Ms_1 - 0s_2 - Ms_3 - 0s_3 - 0A_1 - 0A_3 - 8M$$

$$\therefore \quad w' + (-2 + M)\,y_1 + (3 - 3M)\,y_2 + (5 - 3M)\,y_3 + Ms_1 + 0s_2 + Ms_3 + 0A_1 + 0A_3 = -8M$$

Simplex Table

Iteration Number		Basic Variables	y_1	y_2	y_3	s_1	s_2	s_3	A_1	A_3	R.H.S. Sol.	Ratio
0		w'	$-2+M$	$3-3M$	$5-3M$	M	0	M	0	0	$-8M$	
	s_2 leaves	A_1	-2	3	0	-1	0	0	1	0	5	$\frac{5}{3}$
	y_2 enters	s_2	2	4^*	-1	0	1	0	0	0	2	$\frac{1}{2} \leftarrow$
		A_3	1	0	3	0	0	-1	0	1	3	—
				$\uparrow$								
1		w'	$\frac{-7+5M}{2}$	0	$\frac{23-15M}{4}$	M	$\frac{-3+3M}{4}$	M	0	0	$\frac{-3-13M}{2}$	
	A_3 leaves	A_1	$-\frac{7}{2}$	0	$\frac{3}{4}$	-1	$-\frac{3}{4}$	0	1	0	$\frac{7}{2}$	$\frac{14}{3}$
	y_3 enters	y_2	$\frac{1}{2}$	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	$\frac{1}{2}$	-2
		A_3	1	0	3^*	0	0	-1	0	1	3	$1 \leftarrow$
					$\uparrow$							

Simplex Table continued on the next page.

Simplex Table continued from the previous page.

Iteration Number		Basic Variables	y_1	y_2	y_3	s_1	s_2	s_3	A_1	A_3	R.H.S. Sol.	Ratio
2		w'	$\frac{-65+45M}{12}$	0	0	M	$\frac{-3+3M}{4}$	$\frac{23-3M}{12}$	0		$\frac{-29-11M}{4}$	
	A_1 leaves	A_1	$-\frac{15}{4}$	0	0	-1	$-\frac{3}{4}$	$\frac{1}{4}^*$	1		$\frac{11}{4}$	$11 \leftarrow$
	s_3 enters	y_2	$\frac{7}{12}$	1	0	0	$\frac{1}{4}$	$-\frac{1}{12}$	0		$\frac{3}{4}$	—
		y_3	$\frac{1}{3}$	0	1	0	0	$-\frac{1}{3}$	0		1	—
								$\uparrow$				
3		w'	$\frac{70}{3}$	0	0	$\frac{23}{3}$	5	0	$\frac{-23+3M}{3}$		$-\frac{85}{3}$	
		s_3	-15	0	0	-4	-3	1	4		11	
		y_2	$-\frac{2}{3}$	1	0	$-\frac{1}{3}$	0	0	$\frac{1}{3}$		$\frac{5}{3}$	
		y_3	$-\frac{14}{3}$	0	1	$-\frac{4}{3}$	-1	0	$\frac{4}{3}$		$\frac{14}{3}$	

$\therefore \quad s_1 = \frac{23}{3},\ s_2 = 5,\ s_3 = 0,\ W'_{Max} = -\frac{85}{3} \quad \therefore W_{Min} = \frac{85}{3}$

$\therefore \quad x_1 = \frac{23}{3},\ x_2 = 5,\ x_3 = 0,\ z_{Max} = \frac{85}{3}$

EXERCISE - III

(A) Using duality solve the following L.P.P.

1. Maximise $z = 3x_1 - 2x_2$
   subject to $x_1 + x_2 \le 5$
   $x_2 \ge 1$
   $x_1 \le 4$
   $x_2 \le 6$
   $x_1, x_2 \ge 0$
   [**Ans. :** $x_1 = 4,\ x_2 = 1,\ z_{Max} = 10$]

2. Maximise $z = 3x_1 + 4x_2$
   subject to $x_1 - x_2 \le 1$
   $x_1 + x_2 \ge 4$
   $x_1 - 3x_2 \le 3$
   $x_1, x_2 \ge 0$
   [**Ans. :** No feasible solution for the dual]

3. Maximise $z = 5x_1 + 8x_2$
   subject to $x_1 + x_2 \le 2$
   $x_1 - 2x_2 \ge 0$
   $-x_1 + 4x_2 \le 1$
   $x_1, x_2 \ge 0$ **(M.U. 1999)**
   [**Ans. :** $x_1 = 7/5,\ x_2 = 3/5,\ z_{Max} = 59/5$]

4. Minimise $z = x_1 + x_2$
   subject to $2x_1 + x_2 \ge 4$
   $x_1 + 7x_2 \ge 7$
   $x_1, x_2 \ge 0$
   [**Ans. :** $x_1 = 21/13,\ x_2 = 10/13,\ z_{Min} = 31/13$]

5. Minimise $z = 10x_1 + 15x_2 + 30x_3$
   subject to $x_1 + 3x_2 + x_3 \ge 90$
   $2x_1 + 5x_2 + 3x_3 \ge 120$
   $x_1 + x_2 + x_2 \ge 60$
   $x_1, x_2, x_3 \ge 0$
   (M.U. 2007)
   [**Ans. :** $x_1 = 45,\ x_2 = 15,\ x_3 = 0,\ z_{Min} = 625$]

6. Minimise $z = 0.007x_1 + 0{\cdot}05x_2$
   subject to $0{\cdot}1x_1 \ge 0{\cdot}4$
   $0{\cdot}1x_2 \ge 0{\cdot}6$
   $0{\cdot}1x_1 + 0{\cdot}2x_2 \ge 2{\cdot}0$
   $0{\cdot}2x_1 + 0{\cdot}1x_2 \ge 1{\cdot}8$
   $x_1, x_2 \ge 0$
   [**Ans. :** $x_1 = 1{\cdot}6/3,\ x_2 = 2{\cdot}2/3,\ z_{Min} = 0{\cdot}074$]

7. Minimise $z = 2x_1 + 2x_2$
   subject to $2x_1 + 4x_2 \ge 1$
   $x_1 + 2x_2 \ge 1$
   $x_1, x_2 \ge 0$ **(M.U. 2005)**
   [**Ans. :** $x_1 = 1/3,\ x_2 = 1/3,\ z_{Min} = 4/3$]

(B) Solve the following L.P.P. by Simplex method. Also read the solution to the dual from the final table.

1. Maximise $z = 6x_1 - 2x_2 + 3x_3$
   subject to $2x_1 - x_2 + 2x_3 \le 2$
   $x_1 + 0x_2 + 4x_3 \le 4$
   $x_1, x_2, x_3 \ge 0$
   [**Ans. :** $x_1 = 4,\ x_2 = 6,\ x_3 = 0,\ z_{Max} = 12$
   $y_1 = 2,\ y_2 = 2,\ w_{Min} = 12$]

2. Maximise $z = 30x_1 + 23x_2 + 29x_3$
   subject to $6x_1 + 5x_2 + 3x_3 \le 26$
   $4x_1 + 2x_2 + 5x_3 \le 7$
   $x_1, x_2, x_3 \ge 0$
   [**Ans. :** $x_1 = 0,\ x_2 = 7/2,\ x_3 = 0,\ z_{Max} = 161/2$
   $y_1 = 0,\ y_2 = 23/2,\ w_{Min} = 161/2$]

5. The Dual Simplex Method

We have seen chapter IX how to solve an L.P.P. by simplex method. In this method we start with a basic feasible but not optimal solution, starting with $x_1 = 0,\ x_2 = 0$, ... etc. and work towards the optimal solution. In the dual simplex method we start with the optimal but infeasible solution and work downwards the feasible solution.

The constraints are converted into less than or equal to type ($\le$) and the objective function is converted into minimisation if not so already.

Working Procedure for Dual Simplex Method

1. Convert the problem into minimisation type.
2. Convert all constraints in less than or equal to type. If any constraint is of greater than or equal to type multiply throughout by (–1) and change the inequality sign.
3. Convert the inequality constraints into equalities by adding slack variables.
4. Put this information in a table.
5. If all the coefficients in the row of z are negative and all the right-hand side constants *i.e.*, b's are positive then basic feasible solution is obtained.

   If all the coefficients in the row of z are negative and atleast one 'b' is negative then go to step 6.

   (If all the coefficient in the row of z are positive the method fails.)
6. Look at the last right hand side column. Select the row which contains the smallest (negative) number in the column of b's. Denote it by an arrow-head like ←. This is the key row and the corresponding variable on the left is the outgoing variable.
7. Find the ratios of the elements **in the row of z to the corresponding elements in the key row.** Write these **ratios in another row below the tables.**
8. **Now select the incoming variable :** Select the column which contains the smallest ratio and denote it by an arrow-head like ↑. This is the key column. The element where the key row and the key column intersect is the key element.
9. Construct another table as in the usual simplex method. Divide each element in the key row by key element, so that key element will be converted into unity. Now, make all other elements in the key column zero by subtracting proper multiples of the elements in the key row from the corresponding elements in the other rows as in the simplex method.
10. Continue the procedure till all the b's are positive. These are the required values of the decision variables.

Comparison between the Regular Simplex Method and the Dual Simplex Method

1. In regular simplex method the objective function is of maximisation type while in the dual simplex method the object function is of minimisation type.
2. In the regular simplex method we start with basic feasible but not optimal solution. In the dual simplex method we start with basic infeasible but optimal solution.
3. In the regular simplex method we arrive at the optimal solution in the end if it exists. In the dual simplex method we arrive at the feasible solution in the end.
4. In the regular simplex method we first decide incoming variable and from the ratio column we then decide the outgoing variable. In the dual simplex method, we first decide the outgoing variable and then from ratio row, we decide the incoming variable.

5. In the regular simplex method artificial variables are required when the constraints are of greater than type. In the dual simplex method artificial variables are not required.
6. The dual simplex method is more convenient than the regular simplex method.

Example 1 : Use the dual simplex method to solve the following L.P.P.

Minimise $z = 2x_1 + 2x_2 + 4x_3$

subject to $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_2, x_3 \geq 0.$

(M.U. 2001, 03, 06, 09, 15)

Sol. : We first express the given problem using $\leq$ in the first constraint.

Minimise $z = 2x_1 + 2x_2 + 4x_3$

subject to $-2x_1 - 3x_2 - 5x_3 \leq -2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

Introducing the slack variables s_1, s_2, s_3, we have

Minimise $z = 2x_1 + 2x_2 + 4x_3 - 0s_1 - 0s_2 - 0s_3$

i.e. $z - 2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3 = 0$

subject to $-2x_1 - 3x_2 - 5x_3 + s_1 + 0s_2 + 0s_3 = -2$

$3x_1 + x_2 + 7x_3 + 0s_1 + s_2 + 0s_3 = 3$

$x_1 + 4x_2 + 6x_3 + 0s_1 + 0s_2 + s_3 = 5$

Simplex Table

Iteration Number	Basic Variables	Coefficients of x_1	x_2	x_3	s_1	s_2	s_3	R.H.S. Solution
0	z	−2	−2	−4	0	0	0	0
s_1 leaves	s_1	−2	−3*	−5	1	0	0	−2 ←
x_2 enters	s_2	3	1	7	0	1	0	3
	s_3	1	4	6	0	0	1	5
Ratio		1	2/3 ↑	4/5				
1	z	−2/3	0	−2/3	−2/3	0	0	4/3
	x_2	2/3	1	5/3	−1/3	0	0	2/3
	s_2	7/3	0	16/3	1/3	1	0	7/3
	s_3	−5/3	0	−2/3	4/3	0	1	7/3

$\therefore\ x_1 = 0,\ x_2 = \frac{2}{3},\ x_3 = 0,\ z_{Min} = \frac{4}{3}.$

Example 2 : Use the dual simplex method to solve the following L.P.P.

Maximise $z = -3x_1 - 2x_2$

subject to $x_1 + x_2 \geq 1$

$x_1 + x_2 \leq 7$

$x_1 + 2x_2 \leq 10$

$x_2 \leq 3$

$x_1, x_2 \geq 0.$

(M.U. 1998, 2004, 09, 16)

Sol. : We first express the given problem in minimisation type and use $\leq$ in the first and third constraints.

$\therefore$ Minimise $z' = -z = 3x_1 + 2x_2$

subject to $-x_1 - x_2 \leq -1$

$x_1 + x_2 \leq 7$

$-x_1 - 2x_2 \leq -10$

$x_2 \leq 3$

Introducing the slack variables s_1, s_2, s_3 and s_4, we have

Minimise $z' = 3x_1 + 2x_2 - 0s_1 - 0s_2 - 0s_3 - 0s_4$

i.e. $z' - 3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 = 0$

subject to $-x_1 - x_2 + s_1 + 0s_2 + 0s_3 + 0s_4 = -1$

$x_1 + x_2 + 0s_1 + s_2 + 0s_3 + 0s_4 = 7$

$-x_1 - 2x_2 + 0s_1 + 0s_2 + s_3 + 0s_4 = -10$

$0x_1 + x_2 + 0s_1 + 0s_2 + 0s_3 + s_4 = 3$

See Simplex Table on the next page.

Simplex Table

Iteration Number	Basic Variables	x_1	x_2	s_1	s_2	s_3	s_4	R.H.S. Solution
0	z'	-3	-2	0	0	0	0	0
s_3 leaves	s_1	-1	-1	1	0	0	0	-1
x_2 enters	s_2	1	1	0	1	0	0	7
	s_3	-1	-2^*	0	0	1	0	-10 ←
	s_4	0	1	0	0	0	1	3
Ratio		3	1 ↑					
1	z'	-2	0	0	0	-1	0	10
s_4 leaves	s_1	$-1/2$	0	1	0	$-1/2$	0	4
x_1 enters	s_2	$1/2$	0	0	1	$1/2$	0	2
	x_2	$1/2$	1	0	0	$-1/2$	0	5 ←
	s_4	$-1/2^*$	0	0	0	$1/2$	1	-2
Ratio		4 ↑						
2	z'	0	0	0	0	-3	-4	18
	s_1	0	0	1	0	-1	-1	6
	s_2	0	0	0	1	1	1	0
	x_2	0	1	0	1	-1	1	3
	x_1	1	0	0	0	-1	-2	4

$\therefore\ x_1 = 4,\ x_2 = 3,\ z'_{Min} = 18,\ z_{Max} = -18.$

Example 3 : Use the dual simplex method to solve the following L.P.P.

Minimise $z = 6x_1 + 3x_2 + 4x_3$

subject to $x_1 + 6x_2 + x_3 = 10$

$2x_1 + 3x_2 + x_3 = 15$

$x_1, x_2, x_3 \geq 0.$ (M.U. 2004)

Sol. : We shall first express the equality constraints in less than or equal to type.

Hence, we write the two constraints as

$x_1 + 6x_2 + x_3 \leq 10$ and $x_1 + 6x_2 + x_3 \geq 10$ .... (1)

$2x_1 + 3x_2 + x_3 \leq 15$ and $2x_1 + 3x_2 + x_3 \geq 10$ .... (2)

and multiply the second constraints in (1) and (2) by (−1).

$\therefore\quad x_1 + 6x_2 + x_3 \leq 10$ and $-x_1 - 6x_2 - x_3 \leq -10$

$2x_1 + 3x_2 + x_1 \leq 15$ and $-2x_1 - 3x_2 - x_3 \leq -15$

We now introduce the slack variables s_1, s_2, s_3, s_4 and write the object function and the constraints in equality form.

Minimise $z = 6x_1 + 3x_2 + 4x_3 - 0s_1 - 0s_2 - 0s_3 - 0s_4$

i.e. $z - 6x_1 - 3x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3 + 0s_4 = 0$

subject to $x_1 + 6x_2 + x_3 + s_1 + 0s_2 + 0s_3 + 0s_4 = 10$

$-x_1 - 6x_2 - x_3 + 0s_1 + s_2 + 0s_3 + 0s_4 = -10$

$2x_1 + 3x_2 + x_3 + 0s_1 + 0s_2 + s_3 + 0s_4 = 15$

$-2x_1 - 3x_2 - x_3 + 0s_1 + 0s_2 + 0s_3 + s_4 = -15$

Simplex Table

Iteration Number	Basic Var.	x_1	x_2	x_3	s_1	s_2	s_3	s_4	R.H.S. Sol.
0	z	-6	-3	-4	0	0	0	0	0
s_4 leaves	s_1	1	6	1	1	0	0	0	10
x_2 enters	s_2	-1	-6	-1	0	1	0	0	-10
	s_3	2	3	1	0	0	1	0	15
	s_4	-2	-3^*	-1	0	0	0	1	-15 ←
Ratio		3	1 ↑	4					
1	z	-4	0	-3	0	0	0	-1	15
s_4 leaves	s_1	-3^*	0	-1	1	0	0	2	-20 ←
x_1 enters	s_2	3	0	0	0	1	0	-2	20
	s_3	0	0	0	0	0	1	1	0
	x_2	$2/3$	1	$1/3$	0	0	0	$-1/3$	5
Ratio		$4/3$ ↑		3					
2	z'	0	0	$-5/3$	$-4/3$	0	0	$-11/3$	$125/3$
	x_1	1	0	$1/3$	$-1/3$	0	0	$-2/3$	$20/3$
	s_2	0	0	0	1	1	0	0	0
	s_3	0	0	0	0	0	1	1	0
	x_2	0	0	$1/2$	$2/9$	0	0	$1/9$	$5/9$

$\therefore\ x_1 = \frac{20}{3},\ x_2 = \frac{5}{9},\ x_3 = 0,\ z = \frac{125}{3}.$

Example 4 : Use the dual simp

Minimise $z = 6x_1 + x$

subject to $2x_1 + x_2 \geq$

$x_1 - x_2 \geq$

Sol. : We first express the given

Minimise $z = 6x_1$

subject to $-2x_1$

$-x_1 +$

Introducing the slack va

Minimise $z =$

i.e. $z -$

subject to $-$

Iteration Number	
0	
s_1 leav	
x_2 ent	
Ra	

E

Example 4 : Use the dual simplex method to solve the following L.P.P. (M.U. 1997)

Minimise $z = 6x_1 + x_2$

subject to $2x_1 + x_2 \geq 3$

$x_1 - x_2 \geq 0$; $x_1, x_2 \geq 0$.

Sol. : We first express the given problem using $\leq$ in the given constraints as

Minimise $z = 6x_1 + x_2$

subject to $-2x_1 - x_2 \leq -3$

$-x_1 + x_2 \leq 0$

Introducing the slack variables s_1, s_2, we have

Minimise $z = 6x_1 + x_2 - 0s_1 - 0s_2$

i.e. $z - 6x_1 - x_2 + 0s_1 + 0s_2 = 0$

subject to $-2x_1 - x_2 + s_1 + 0s_2 = -3$

$-x_1 + x_2 + 0s_1 + s_2 = 0$

Simplex Table

Iteration Number	Basic Variables	Coefficients of x_1	x_2	s_1	s_2	R.H.S. Solution
0	z	−6	−1	0	0	0
s_1 leaves	s_1	−2	−1*	1	0	−3
x_2 enters	s_2	−1	1	0	1	0
Ratio		3	1 ↑			
1	z	−4	0	−1	0	3
s_2 leaves	x_2	2	1	−1	0	3
x_1 enters	s_2	−3*	0	1	1	−3
Ratio		4/3 ↑				
2	z	0	0	−7/3	−4/3	7
	x_2	0	1	−1/3	2/3	1
	x_1	1	0	−1/3	−1/3	1

$\therefore\ x_1 = 1,\ x_2 = 1,\ z_{Min} = 7.$

Example 5 : Use the dual simplex method to solve the following L.P.P.

Minimise $z = x_1 + x_2$

subject to $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$; $x_1, x_2 \geq 0$. (M.U. 2007, 10)

Sol. : We first express the given problem using $\leq$ in the given constraints.

Minimise $z = x_1 + x_2$

subject to $-2x_1 - x_2 \leq -2$

$x_1 + x_2 \leq -1$

Introducing the slack variables s_1, s_2, we have

Minimise $z = x_1 + x_2 - 0s_1 - 0s_2$

i.e. $z - x_1 - x_2 + 0s_1 + 0s_2 = 0$

subject to $-2x_1 - x_2 + s_1 + 0s_2 = -2$

$x_1 + x_2 + 0s_1 + s_2 = -1$

Simplex Table

Iteration Number	Basic Variables	Coefficients of x_1	x_2	s_1	s_2	R.H.S. Solution
0	z	−1	−1	0	0	0
s_1 leaves	s_1	−2*	−1	1	0	−2
x_1 enters	s_2	1	1	0	1	−1
Ratio		1/2 ↑	1			
1	z	0	−1/2	−1/2	0	1
	x_1	1	1/2	−1/2	0	1
	s_2	0	1/2	1/2	1	−2
Ratio		—	−1	−1	—	

Since s_2 row is negative, s_2 leaves. But since all ratios are negative, the L.P.P. has no feasible solution.

Example 6 : Use the dual simplex method to solve the following L.P.P.

Minimise $z = 3x_1 + 2x_2 + x_3 + 4x_4$

subject to $2x_1 + 4x_2 + 5x_3 + x_4 \geq 10$

$3x_1 - x_2 + 7x_3 - 2x_4 \geq 2$

$5x_1 + 2x_2 + x_3 + 6x_4 \geq 15$

$x_1, x_2, x_3, x_4 \geq 0$.

Sol. : We first express the given problem using $\leq$ in the given constraints.

Minimise $z = 3x_1 + 2x_2 + x_3 + 4x_4$

subject to $-2x_1 - 4x_2 - 5x_3 - x_4 \leq -10$

$-3x_1 + x_2 - 7x_3 + 2x_4 \leq -2$

$-5x_1 - 2x_2 - x_3 - 6x_4 \leq -15$

$x_1, x_2, x_3, x_4 \geq 0$.

Introducing the slack variables s_1, s_2, and s_3, we have

Minimise $\quad z = 3x_1 + 2x_2 + x_3 + 4x_4 - 0s_1 - 0s_2 - 0s_3$

i.e. $\quad z - 3x_1 - 2x_2 - x_3 - 4x_4 + 0s_1 + 0s_2 + 0s_3 = 0$

subject to
$$-2x_1 - 4x_2 - 5x_3 - x_4 + s_1 + 0s_2 + 0s_3 = -10$$
$$-3x_1 + x_2 - 7x_3 + 2x_4 + 0s_1 + s_2 + 0s_3 = -2$$
$$-5x_1 - 2x_2 - x_3 - 6x_4 + 0s_1 + 0s_2 + s_3 = -15$$

Simplex Table

Iteration Number	Basic Var.	x_1	x_2	x_3	x_4	s_1	s_2	s_3	R.H.S. Sol.
0	z	-3	-2	-1	-4	0	0	0	0
s_3 leaves	s_1	-2	-4	-5	-1	1	0	0	-10
x_1 enters	s_2	-3	1	-7	2	0	1	0	-2
	s_3	-5^*	-2	-1	-6	0	0	1	-15 ←
Ratio		$\frac{3}{5}$ ↑	1	1	$\frac{2}{3}$				
1	z	0	$-\frac{4}{5}$	$-\frac{2}{5}$	$-\frac{2}{5}$	0	0	$-\frac{3}{5}$	9
s_1 leaves	s_1	0	$-\frac{16}{5}$	$-\frac{23}{5}^*$	$\frac{7}{5}$	1	0	$-\frac{2}{5}$	-4 ←
x_3 enters	s_2	0	$\frac{11}{5}$	$-\frac{32}{5}$	$\frac{28}{5}$	0	1	$-\frac{3}{5}$	7
	x_1	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{6}{5}$	0	0	$-\frac{1}{5}$	3
Ratio			$\frac{1}{4}$	$\frac{2}{23}$ ↑	$\frac{2}{7}$				
2	z	0	$-\frac{12}{23}$	0	$-\frac{12}{23}$	$-\frac{2}{23}$	0	$-\frac{13}{23}$	$\frac{215}{23}$
	x_3	0	$\frac{16}{23}$	1	$-\frac{7}{23}$	$-\frac{5}{23}$	0	$\frac{2}{23}$	$\frac{20}{23}$
	s_2	0	$\frac{153}{23}$	0	$\frac{84}{23}$	$-\frac{32}{23}$	1	$-\frac{1}{23}$	$\frac{289}{23}$
	x_1	1	$\frac{6}{23}$	0	$\frac{29}{23}$	$\frac{1}{23}$	0	$-\frac{5}{23}$	$\frac{65}{23}$

$\therefore\ x_1 = \frac{65}{23},\ x_2 = 0,\ x_3 = \frac{20}{23},\ z_{Min} = \frac{215}{23}.$

EXERCISE - IV

Use the dual simplex method to solve the following L.P.P.

1. Minimise $z = 2x_1 + x_2$
   subject to $3x_1 + x_2 \ge 3$
   $4x_1 + 3x_2 \ge 6$
   $x_1 + 2x_2 \le 3$
   $x_1, x_2 \ge 0$
   (M.U. 2000, 04)
   [Ans. : $x_1 = 3/5,\ x_2 = 6/5,\ z_{Min} = 12/5$ **]**

2. Minimise $z = 20x_1 + 16x_2$
   subject to $x_1 + x_2 \ge 12$
   $2x_1 + x_2 \ge 17$
   $x_1 \ge 2.5$
   $x_2 \ge 6;\ x_1, x_2 \ge 0$
   (M.U. 2001, 09)
   [Ans. : $x_1 = 5,\ x_2 = 7,\ z_{Min} = 212$ **]**

3. Minimise $z = 3x_1 + x_2$
   subject to $x_1 + x_2 \ge 1$
   $2x_1 + 3x_2 \ge 2$
   $x_1, x_2 \ge 0$
   [Ans. : $x_1 = 0,\ x_2 = 1,\ z_{Min} = 1$ **]**

4. Minimise $z = 10x_1 + 6x_2 + 2x_3$
   subject to $-x_1 + x_2 + x_3 \ge 1$
   $3x_1 + x_2 - x_3 \ge 2$
   $x_1, x_2, x_3 \ge 0$
   [Ans. : $x_1 = 1/4,\ x_2 = 5/4,\ x_3 = 0,\ z_{Min} = 10$ **]**

5. Minimise $z = 6x_1 + 7x_2 + 3x_3 + 5x_4$
   subject to $2x_1 + 5x_2 + x_3 + x_4 \ge 8$
   $x_2 + 5x_3 - 6x_4 \ge 10$
   $5x_1 + 6x_2 - 3x_3 + 4x_4 \ge 12$
   $x_1, x_2, x_3, x_4 \ge 0$ **(M.U. 2007)**
   [Ans. : $x_1 = 0,\ x_2 = 30/11,\ x_3 = 16/11,\ x_4 = 0,\ z_{Min} = 258/11$ **]**

6. Maximize $z = 2x_1 - x_2$
   subject to $x_1 + x_2 - x_3 \ge 5$
   $x_1 - 2x_2 + 4x_3 \ge 8$
   $x_1, x_2, x_3 \ge 0$
   [Ans. : $x_1 = 0,\ x_2 = 14,\ x_3 = 9,\ z_{max} = -9$ **]**

EXERCISE - IV

Theory

1. Explain the following terms and their uses. **(M.U. 2007)**
   (i) Artificial variable, (ii) Big-M method, (iii) Dual Simplex Method.
2. What is the difference between "Simplex Method" and "Dual Simplex Method" ? **(M.U. 1999)**
3. Compare the regular simplex method and the dual simplex method.
4. Explain how you will obtain the dual from the primal of an L.P.P.
5. Explain how you come to know that an L.P.P. has
   (i) infinite solution, **(ii)** unbounded solution,
   (iii) infeasible solution by Simplex Method.

☆☆☆

$20x_1 + 16x_2$
$x_2 \geq 12$
$x_2 \geq 17$
5
$x_1, x_2 \geq 0$
01, 09)
Min = 212]
$x_2 + 2x_3$
≥ 1
≥ 2

Min = 10]

9]

CHAPTER 13

Non-Linear Programming

1. Introduction

So far we have considered optimisation problems in which the object function as well as the constraints were linear. However, in many practical cases we come across optimisation problems in which the object function and / or some or all constraints are non-linear.

Definition : *An optimisation problem in which either the object function and / or some or all constraints are non-linear is called a* **non-linear programming problem.**

It is abbreviated as N.L.P.P. For example, the following problems are N.L.P.P.

(i) Optimise $z = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 3x_2x_3 + 100$

subject to $x_1 + x_2 + x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

Here, the object function is non-linear

(ii) Optimise, $z = 3x_1 + 4x_2 + 7x_3$

subject to $x_1^2 + 2x_2^2 \geq 20$

$3x_1 + x_2 + 2x_3 \leq 10$

$x_1, x_2, x_3 \geq 0.$

Here, although the object function is linear, one of the constraints is non-linear.

(iii) Optimise, $z = x_1^2 + x_2^2 - 3x_1 x_2$

subject to $x_1^2 + x_1x_2 + 2x_3 = 35$

$x_1, x_2 \geq 0$

Here, both the object function as well as the constraints are non-linear.

Before going to study the methods of solving optimisation problems of quadratic function with or without restrictions, we shall recall briefly the method of solving the problems of maxima and minima in one and two independent revariables.

If $y = f(x)$ is differentiable function of real variable x upto second order then $f'(x) = 0$ gives the stationary values of x. Suppose $f'(x) = 0$, for $x = x_0$.

Now, x_0 is a stationary point, and if $f''(x_0) > 0$ then x_0 is a **minima**, if $f''(x_0) < 0$ then x_0 is a **maxima**, if $f''(x_0) = 0$, no conclusion can be drawn about the nature of $f(x)$ at x_0. It is called the point of **inflexion.**

Geometrically speaking if a curve before a point is decreasing at the point is neither decreasing nor increasing and after the point is increasing, then such a point is called a **minima.** If a curve before a point is increasing, at the point is neither increasing nor decreasing and after that point is decreasing, then such a point is called a **maxima.** Since at the point of maxima or minima the curve is neither increasing nor decreasing they are also called **stationary** points.

Further, if x_0 is the only stationary point in the interval (a, b) then it is called **local maxima (or local minima).** On the other hand if in an interval (a, b), $f(x)$ has several (local) maximas and minimas then the greatest of these maximas is called the **global maxima** and the least of these minimas is called the **global minima.** In the following figure we have shown local minimas, local maximas, global minima, global maxima, point of inflexion. It may noted that the $f(x)$ may have maxima or minima at end points also.

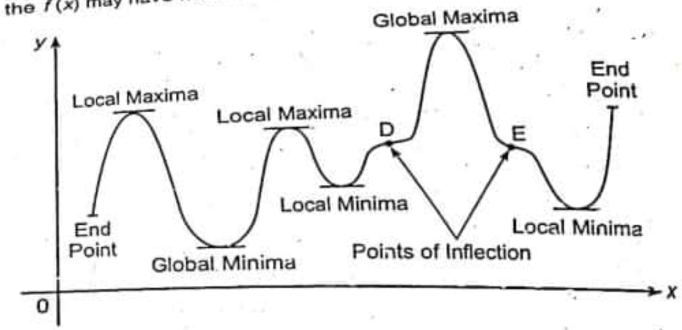

Fig. 13.1

Further, if $z = f(x, y)$ is a function of two real variables x and y such that the second order partial derivatives exist then the stationary points are given by

$$f_x(x, y) = 0 \text{ and } f_y(x, y) = 0$$

If $X_0(x_0, y_0)$ is a stationary point and $f_{xx} f_{yy} - f_{xy}^2 > 0$.

then $X_0(x_0, y_0)$ is a **minima** if $f_{xx} > 0$ and $X_0(x_0, y_0)$ is a **maxima**, if $f_{xx} < 0$ at (x_0, y_0).

If $f_{xx} f_{yy} - f_{xy}^2 < 0$, then (x_0, y_0) is called a **saddle point** and if $f_{xx} f_{yy} - f_{xy}^2 = 0$, then **no definite conclusion** can be drawn about the nature of $f(x, y)$ at (x_0, y_0).

Saddle Point : The term being new needs some explanation. If $u = f(x_1, x_2, \ldots, x_n)$ is a function of n real variables $x_1, x_2, \ldots, x_n$ then at maxima, with respect to small increase in $x_1, x_2, \ldots, x_n$, u decreases and at minima, with respect to small increase in $x_1, x_2, \ldots, x_n$, u increases in the neighbourhood of $X_0(x_{10}, x_{20}, \ldots, x_{n0})$. But the situation is different at saddle point. If X_0 is a saddle point then with respect to small changes in some variables u increases, while with respect to small changes in other variables u decreases. If $z = f(x, y)$ is a function of two variables *i.e.* in three dimensional space, the surface $z = f(x, y)$ in the neighbourhood of a saddle point takes the form of a saddle (a seat for a rider on a horse.) Hence, the name saddle point.

We shall now consider three types of Non-linear Programming Problems.

(1) Quadratic Programming Problems with no constraints,

(2) Quadratic Programming Problems with linear equality constraints,

(3) Quadratic Programming Problems with linear inequality constraints.

2. Quadratic Programming Problems

In this type of problems the object function is the sum of quadratic functions and linear functions of the decision variables $x_1, x_2, \ldots, x_n$ and there are no constraints on $x_1, x_2, \ldots, x_n$. The object function looks like

$$z = a_{11} x_1^2 + a_{22} x_2^2 + \ldots + a_{nn} x_n^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \ldots$$
$$+ a_{23}x_2x_3 + a_{24}x_2x_4 + \ldots + c_1x_1 + c_2x_2 + \ldots + c_nx_n.$$

(a) Method of Solution

We assume that all the first order and second order partial derivatives

i.e., $\dfrac{\partial f}{\partial x_i}$ and $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ exist for all i and j.

The points of maxima and minima are obtained by solving the equations .... (1)

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \quad \ldots\ldots, \quad \frac{\partial f}{\partial x_n} = 0.$$

Suppose by solving these equations we get the point $X_0\ (x_1, x_2, \ldots, x_n)$.

Now, we consider the Hessian matrix defined below.

Definition : If $f(x_1, x_2, \ldots, x_n)$ is a quadratic function of $x_1, x_2, \ldots, x_n$ and if all second order partial derivatives of $f(x_1, x_2, \ldots, x_n)$ exist then matrix H of second ordered partial derivatives defined as given below is called **Hessian Matrix** and is denoted by H.

$$H = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \ldots\ldots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \ldots\ldots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \ldots\ldots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Hessian matrix is highly useful for determining the nature of stationary points *i.e.* for determining whether the stationary point x_0 is a maxima or a minima.

We accept the following theorem without proof.

Theorem : A sufficient condition for a stationary point x_0 to be extremum is that the Hessian matrix H evaluated at x_0 is

(i) positive definite when x_0 is a minimum point.

(ii) negative definite when x_0 is a maximum point.

Note ....

Note that these conditions are only sufficient. In other words, we may get a maxima or a minima even when the conditions are not satisfied.

But a quadratic form is positive definite when all its principal minor determinants are positive and negative definite when its principle minor determinants $D_1, D_3, \ldots$ are negative and $D_2, D_4, \ldots$ are positive. **(See Appendix)**

Combining the above theorem and the above results, we get the following criteria for determining the maxima or minima at x_0 from Hessian matrix.

1. If all the principal minor determinants of Hessian matrix at x_0 are positive then X_0 is a minima.

2. If the principal minor determinants $D_1, D_3, \ldots$ are negative and $D_2, D_4, \ldots$ are positive, X_0 is a maxima.

3. In general if Hessian matrix is indefinite at X_0, X_0 is a saddle point *i.e.* neither a maxima nor a minima.

If there are two variables x_1 and x_2, then the Hessian matrix will be of the form

$$\begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \text{ say } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

(i) If the determinants $A_1 = |a_{11}|$, $A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ are both positive at (x_0, y_0) then we have a minima at (x_0, y_0).

(ii) If the determinant A_1 is negative and A_2 is positive at (x_0, y_0), then we have a maxima at (x_0, y_0).

If there are three variables x_1, x_2, x_3 the Hessian matrix will be of the form

$$\begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_3} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_3} \\ \dfrac{\partial^2 f}{\partial x_3 \partial x_1} & \dfrac{\partial^2 f}{\partial x_3 \partial x_2} & \dfrac{\partial^2 f}{\partial x_3^2} \end{bmatrix} \text{ say } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(i) If all the determinants

$$A_1 = |a_{11}|, \quad A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

are positive at (x_0, y_0), then we get a minima at (x_0, y_0).

(ii) If A_1 is negative, A_2 is positive and A_3 is negative at (x_0, y_0), then we get a maxima at (x_0, y_0).

Example 1 : Find the relative maximum or minimum of the function

$$z = x_1^2 + x_2^2 + x_3^2 - 4x_1 - 8x_2 - 12x_3 + 100 \qquad \textbf{(M.U. 2016)}$$

Sol. : We have $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 4x_1 - 8x_2 - 12x_3 + 100.$

The stationary points are given by

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \quad \frac{\partial f}{\partial x_3} = 0.$$

X_0 are positive then
and $D_2, D_4, \ldots$ are
point i.e. neither
the form

y_0) then
ave a

Now, $\frac{\partial f}{\partial x_1} = 2x_1 - 4 \quad \therefore 2x_1 - 4 = 0 \quad \therefore x_1 = 2$

$\frac{\partial f}{\partial x_2} = 2x_2 - 8 \quad \therefore 2x_2 - 8 = 0 \quad \therefore x_2 = 4$

$\frac{\partial f}{\partial x_3} = 2x_3 - 12 \quad \therefore 2x_3 - 12 = 0 \quad \therefore x_3 = 6.$

$\therefore$ Thus, X_0 (2, 4, 6) is the stationary point.

To check whether the point is a point of minima or a point of maxima, we apply the sufficiency condition.

Now, $\frac{\partial^2 f}{\partial x_1^2} = 2, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$

Similarly, we get $\frac{\partial^2 f}{\partial x_2^2} = 2, \quad \frac{\partial^2 f}{\partial x_3^2} = 2.$

Now, consider the Hessian matrix at X_0 (2, 4, 6).

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The principal minors of this matrix are

$$[2], \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The values of their determinants are 2, 4, 8.

Since all these determinants are positive, z is minimum at X_0 (2, 4, 6).

$\therefore \quad z_{Min} = 2^2 + 4^2 + 6^2 - 4 \times 2 - 8 \times 4 - 12 \times 6 + 100 = 44.$

Example 2 : Optimise $z = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 8x_2 - 10x_3$. **(M.U. 2004, 09)**

Sol. : We have $\quad f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 8x_2 - 10x_3$

The stationary points are given by

$$\frac{\partial f}{\partial x_1} = 0, \ \frac{\partial f}{\partial x_2} = 0, \ \frac{\partial f}{\partial x_3} = 0.$$

Now, $\frac{\partial f}{\partial x_1} = 2x_1 - 6 \quad \therefore 2x_1 - 6 = 0 \quad \therefore x_1 = 3$

$\frac{\partial f}{\partial x_2} = 2x_2 - 8 \quad \therefore 2x_2 - 8 = 0 \quad \therefore x_2 = 4$

$\frac{\partial f}{\partial x_3} = 2x_3 - 10 \quad \therefore 2x_3 - 10 = 0 \quad \therefore x_3 = 5.$

$\therefore$ Thus, X_0 (3, 4, 5) is the stationary point.

To check whether the point is a point of minima or a point of maxima, we apply the sufficiency condition.

As before $\frac{\partial^2 f}{\partial x_1^2} = 2, \ \frac{\partial^2 f}{\partial x_2^2} = 2, \ \frac{\partial^2 f}{\partial x_3^2} = 2$ and all other second order partial derivatives are zero.

Now, consider the Hessian matrix at X_0 (3, 4, 5).

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The principal minors of this matrix are $[2], \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$

The values of their determinants are 2, 4, 8.

Since all these determinants positive, z is minimum at X_0 (3, 4, 5).

$\therefore \quad z_{Min} = f(3, 4, 5)$

$= 3^2 + 4^2 + 5^2 - 6 \times 3 - 8 \times 4 - 10 \times 5$

$= -50.$

Example 3 : Obtain the relative maximum or minimum (if any) of the function

$$z = x_1 + 2x_3 + x_2x_3 - x_1^2 - x_2^2 - x_3^2$$ **(M.U. 1996, 98, 2010)**

Sol. : We have $\quad f(x_1, x_2, x_3) = x_1 + 2x_3 + x_2x_3 - x_1^2 - x_2^2 - x_3^2$

The stationary points are given by

$$\frac{\partial f}{\partial x_1} = 0, \ \frac{\partial f}{\partial x_2} = 0, \ \frac{\partial f}{\partial x_3} = 0.$$

Now, $\frac{\partial f}{\partial x_1} = 1 - 2x_1 \quad \therefore 1 - 2x_1 = 0 \quad \therefore x_1 = \frac{1}{2}$

$\frac{\partial f}{\partial x_2} = x_3 - 2x_2 \quad \therefore x_3 - 2x_2 = 0$

$\frac{\partial f}{\partial x_3} = 2 + x_2 - 2x_3 \quad \therefore 2 + x_2 - 2x_3 = 0$

Solving the last two simultaneous equation, we get $x_2 = 2/3$, $x_3 = 4/3$.

$\therefore$ Thus, X_0 (1/2, 2/3, 4/3) is the stationary point.

To check whether the point is a point of minima or a point of maxima, we apply the sufficiency test.

Clearly $\frac{\partial^2 f}{\partial x_1^2} = -2, \ \frac{\partial^2 f}{\partial x_2^2} = -2, \ \frac{\partial^2 f}{\partial x_3^2} = -2, \ \frac{\partial^2 f}{\partial x_3 \partial x_2} = 1$

and the remainging second order partial derivatives are zero.

Now, consider the Hessian matrix at X_0 (1/2, 2/3, 4/3).

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

The principal minors of this matrix are

$$[-2], \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

The values of their determinants are $-2, 4, -6$.

Since the values of the determinants are alternately negative, positive and negative, X_0 (1/2, 2/3, 4/3) is a maxima.

$$\therefore \quad z_{Max} = \frac{1}{2} + 2 \cdot \frac{4}{3} + \frac{2}{3} \cdot \frac{4}{3} - \left(\frac{1}{2}\right)^2 - \left(\frac{2}{3}\right)^2 - \left(\frac{4}{3}\right)^2 = \frac{19}{12}$$

Example 4 : Obtain the relative maximum or minimum (if any) of the function

$$z = x_1 x_2 + 9x_1 + 6x_3 - x_1^2 - x_2^2 - x_3^2.$$

Sol. : We have $\quad f(x_1, x_2, x_3) = x_1 x_2 + 9x_1 + 6x_3 - x_1^2 + x_2^2 - x_3^2$

The stationary points are given by

$$\frac{\partial f}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} = 0, \frac{\partial f}{\partial x_3} = 0.$$

Now, $\frac{\partial f}{\partial x_1} = x_2 + 9 - 2x_1 \quad \therefore x_2 - 2x_1 = -9$

$$\frac{\partial f}{\partial x_2} = x_1 - 2x_2 \quad \therefore x_1 - 2x_2 = 0$$

$$\frac{\partial f}{\partial x_3} = 6 - 2x_3 \quad \therefore 6 - 2x_3 = 0$$

Solving this equations, we get $\quad x_1 = 6, x_2 = 3, x_3 = 3$.

Thus, X_0 (6, 3, 3) is the stationary point.

To check whether the point is a point of minima or a point of maxima, we apply the sufficiency test.

Now, consider the Hessian matrix at X_0 (6, 3, 3).

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The principal minors of this matrix are $[-2], \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

The values of their determinants are $-2, 3, -6$.

Since the values of the determinants are alternately negative, positive and negative, X_0 (6, 3, 3) is a maxima.

$$\therefore \quad z_{Max} = 18 + 54 + 18 - 36 - 9 - 9 = 36.$$

EXERCISE - I

1. Find the relative maximum or minimum of the function
$z = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 10x_2 - 14x_3 + 103$
(M.U. 2009)
[**Ans. :** $x_1 = 3, x_2 = 5, x_3 = 7, z_{Min} = 20$]

2. Find the relative maximum or minimum of the function
$z = x_1^2 + x_2^2 + x_3^2 - 8x_1 - 10x_2 - 12x_3 + 100$ [**Ans. :** $x_1 = 4, x_2 = 5, x_3 = 6, z_{Max} = 23$]

3. Obtain the relative maximum or minimum, if any, of the function
$z = 2x_1 + 6x_3 + 9x_2x_3 - 4x_1^2 - 9x_2^2 - 9x_3^2$
[**Ans. :** $x_1 = 1/4, x_2 = 2/9, x_3 = 4/9, z_{Max} = 19/12$]

4. Obtain the relative maximum or minimum, if any, of the function
$z = 2x_1 + x_3 + 3x_2x_3 - x_1^2 - 3x_2^2 - 3x_3^2 + 17$
[**Ans. :** $x_1 = 1, x_2 = 1/9, x_3 = 2/9, z_{Max} = 18$]

3. Optimisation With Equality Constraints

A general non-linear programming problem in which the object function is non-linear but the constraints are linear and in the form of equalities, takes the following form.

Optimise $\quad z = f(x_1, x_2, \ldots\ldots, x_n)$

subject to $\quad g_1(x_1, x_2, \ldots\ldots, x_n) = b_1,$

$g_2(x_1, x_2, \ldots\ldots, x_n) = b_2,$

$\ldots\ldots\ldots\ldots\ldots\ldots$

$g_n(x_1, x_2, \ldots\ldots, x_n) = b_m.$

$x_1, x_2, \ldots\ldots, x_n \geq 0$

where, $f(x_1, x_2, \ldots\ldots, x_n)$ is a non-linear function and $g_1(x_1, x_2, \ldots\ldots, x_n)$, $g_2(x_1, x_2, \ldots\ldots, x_n)$ ..... $g_n(x_1, x_2, \ldots\ldots, x_n)$ are linear functions and $m < n$.

The problem of this type is solved by forming what is called **Lagrangian Function with Lagrange's multiplier λ.**

Two cases need to be considered separately (i) when there is only one constraint and (ii) when there are two constraints.

(a) Non-linear Programming Problem with n-variables and One Equality Constraint

Optimise $\quad z = f(x_1, x_2, \ldots\ldots, x_n)$

subject to $\quad g(x_1, x_2, \ldots\ldots, x_n) = b,$

$x_1, x_2, \ldots\ldots, x_n \geq 0.$

We first express the constraints with r.h.s. equal to zero and denote it by h.

The problem then becomes

Optimize $z = f(x_1, x_2, \ldots, x_n)$

subject to $h(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) - b = 0$

$x_1, x_2, \ldots, x_n \geq 0.$

Now, we construct a new function called **Lagrangian Function** using the multiplier called the **Lagrangian multiplier** as

$$L(x_1, x_2, \ldots, x_n, \lambda) = f(x_1, x_2, \ldots, x_n) - \lambda\, h(x_1, x_2, \ldots, x_n) \quad \ldots\ldots (1)$$

The necessary conditions for maxima or minima subject to the constraint $h(x_1, x_2, \ldots, x_n) = 0$ are

$$\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0, \ldots, \frac{\partial L}{\partial x_n} = 0,\ \frac{\partial L}{\partial \lambda} = 0 \quad \ldots\ldots (2)$$

Now, from (1), we get,

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1}\ ;\ \frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} \ldots\ldots$$

$$\frac{\partial L}{\partial x_n} = \frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n}\ ;\ \frac{\partial L}{\partial \lambda} = -\lambda h$$

Using (2), we get the following $(n+1)$ necessary conditions.

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0\ ;\ \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0 \ldots\ldots$$

$$\frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n} = 0\ ;\ h(x_1, x_2, \ldots, x_n) = 0$$

Solving these $(n+1)$ equations we can find $x_1, x_2 \ldots x_n$ and λ. Thus, the point of maxima or minima can be obtained.

To determine whether the point obtained above is a maxima or a minima, we find the value of the following determinant of order $(n+1)$ at X_0.

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \ldots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \ldots & \frac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \ldots & \frac{\partial^2 f}{\partial x_2 \partial x_n} - \lambda \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_2} & \ldots & \frac{\partial^2 f}{\partial x_n^2} - \lambda \frac{\partial^2 h}{\partial x_n^2} \end{vmatrix}$$

If the signs of the principal minors $\Delta_3, \Delta_4, \Delta_5, \ldots$ are **alternately positive and negative** *i.e.*, $\Delta_3 > 0, \Delta_4 < 0, \Delta_5 > 0 \ldots$ etc. then the points X_0 **is a maxima.**

If all the principal minors $\Delta_3, \Delta_4, \Delta_5, \ldots \Delta_{n+1}$ are **negative** *i.e.* $\Delta_3 < 0, \Delta_4 < 0, \ldots$ etc. then the point X_0 **is a minima.**

Particular Cases

1. If z is a function of two variables only then we get only one determinant of third order Δ_3. **If Δ_3 is positive X_0 is a maxima and Δ_3 is negative X_0 is a minima.** (See Ex. 1 and 2, page 13-11 and 13-12.)

2. If z is a function of three variables only then the above determinant is of fourth order Δ_4. If in this case if **Δ_3 and Δ_4 are both negative then X_0 is a minima and if Δ_3 is positive and Δ_4 is negative then X_0 is a maxima.** (See Ex. 1, 2, 3, 4 on pages 13-14 to 13-17.)

(b) Non-linear Programming Problem with Two Variables and One Equality Constraint

Optimise $z = f(x_1, x_2)$

subject to $g(x_1, x_2) = b_2,$

$x_1, x_2 \geq 0$

We first express the constraint with r.h.s. equal to zero.

The problem then becomes,

Optimize $z = f(x_1, x_2)$

subject to $h(x_1, x_2) = g(x_1, x_2) - b = 0$

$x_1, x_2 \geq 0$

We now construct a new function called **Lagrangian function**, using the multiplier λ called **Lagragian multiplier.**

$$L(x_1, x_2, \lambda) \equiv f(x_1, x_2) - \lambda \cdot h(x_1, x_2) \quad \ldots\ldots (1)$$

The necessary conditions for maxima or minima subject to the condition $h(x_1, x_2) = 0$ are

$$\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0,\ \frac{\partial L}{\partial \lambda} = 0 \quad \ldots\ldots (2)$$

Now, from (1), we get

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1}\ ;\ \frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2}\ ;\ \frac{\partial L}{\partial \lambda} = -\lambda h \quad \ldots\ldots (3)$$

Using (2), we get from (3) the following three necessary conditions.

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0\ ;\ \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0\ ;\ h(x_1, x_2) = 0$$

Solving these three equations, we can find x_1, x_2 and λ. Thus, the point of maxima or minima can be obtained.

To determine whether the point obtained above is a maxima or minima we consider the following determinant.

$$\Delta_3 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} \end{vmatrix}.$$

If Δ_3 is positive at X_0, then X_0 is a maxima and if Δ_3 is negative then X_0 is a minima.

Example 1 : Using the method of Lagrange's multipliers, solve the following N.L.P.P. (M.U. 2010)

Optimise $z = 6x_1^2 + 5x_2^2$

subject to $x_1 + 5x_2 = 7,$

$x_1, x_2 \geq 0.$

Sol. : We have the Lagrangian function,

$$L(x_1, x_2, \lambda) = (6x_1^2 + 5x_2^2) - \lambda(x_1 + 5x_2 - 7)$$

We, now, obtain the following partial derivatives

$$\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0,\ \frac{\partial L}{\partial \lambda} = 0.$$

$\therefore\ 12x_1 - \lambda = 0,\ 10x_2 - 5\lambda = 0,\ x_1 + 5x_2 - 7 = 0$

$\therefore\ \lambda = 12x_1,\ \lambda = 2x_2,\ x_1 + 5x_2 = 7$

$\therefore\ \lambda = 12x_1 = 2x_2 \qquad \therefore\ x_2 = 6x_1$

$\therefore\ x_1 + 30x_1 = 7 \qquad \therefore\ x_1 = 7/31$

and $x_2 = 42/31$ and $\lambda = 12x_1 = 84/31.$

(Or, multiply the second by 6 and add the result to the first.

$\therefore\ 12x_1 + 60x_2 - \lambda - 30\lambda = 0 \qquad \therefore\ 12(x_1 + 5x_2) = 31\lambda$

But $x_1 + 5x_2 = 7.\ \lambda \times 84 = 31 \quad \therefore\ \lambda = 84/31.$)

Hence, X_0 is $(7/31, 42/31)$.

Now, $h(x_1, x_2) = x_1 + 5x_2 - 7 = 0$

$\therefore\ \frac{\partial h}{\partial x_1} = 1,\ \frac{\partial h}{\partial x_2} = 5$ and all other partial derivatives are zero.

And $f(x_1, x_2) = 6x_1^2 + 5x_2^2$

$$\therefore\ \frac{\partial f}{\partial x_1} = 12x_1,\ \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0,\ \frac{\partial^2 f}{\partial x_1^2} = 12$$

$$\frac{\partial f}{\partial x_2} = 10x_2,\ \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0,\ \frac{\partial^2 f}{\partial x_2^2} = 10$$

$$\therefore\ \Delta = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda\frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda\frac{\partial^2 h}{\partial x_1 \partial x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda\frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda\frac{\partial^2 h}{\partial x_2^2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 5 \\ 1 & 12 & 0 \\ 5 & 0 & 10 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 \\ 5 & 10 \end{vmatrix} + 5\begin{vmatrix} 1 & 12 \\ 5 & 0 \end{vmatrix} = -10 - 300 = -310$$

Since, Δ is negative, X_0 is a minima.

Hence, $x_1 = \frac{7}{31},\ x_2 = \frac{42}{31},\ z_{Min} = \frac{294}{31}.$

Example 2 : Using the method of Lagrange's multipliers solve the following N.L.P.P.

Optimise $z = 4x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $x_1 + x_2 = 4,$

$x_1, x_2 \geq 0.$

Sol. : We have the Lagrangian function

$$L(x_1, x_2, \lambda) = (4x_1 + 8x_2 - x_1^2 - x_2^2) - \lambda(x_1 + x_2 - 4)$$

We, now, obtain the following partial derivatives

$$\frac{\partial L}{\partial x_1} = 4 - 2x_1 - \lambda,\ \frac{\partial L}{\partial x_2} = 8 - 2x_2 - \lambda,\ \frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 4)$$

Solving the equations $\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0,\ \frac{\partial L}{\partial \lambda} = 0$, we get

$\therefore\ 4 - 2x_1 - \lambda = 0,\ 8 - 2x_2 - \lambda = 0,\ x_1 + x_2 = 4$

Adding the first two, we get

$12 - 2(x_1 + x_2) - 2\lambda = 0 \qquad \therefore\ 12 - 8 = 2\lambda \qquad \therefore\ \lambda = 2.$

Hence, from the first equation, we get

$4 - 2x_1 - 2 = 0 \qquad \therefore\ 2x_1 = 2 \qquad \therefore\ x_1 = 1$

And from the second equation, we get

$8 - 2x_2 - 2 = 0 \qquad \therefore\ 2x_2 = 6 \qquad \therefore\ x_2 = 3.$

Hence, X_0 is (1, 3).

Now, $h(x_1, x_2) = x_1 + x_2 - 4 = 0$

$\therefore\ \frac{\partial h}{\partial x_1} = 1,\ \frac{\partial h}{\partial x_2} = 1$ and all other partial derivatives are zero.

And $f(x_1, x_2) = 4x_1 + 8x_2 - x_1^2 - x_2^2$

$$\therefore\ \frac{\partial f}{\partial x_1} = 4 - 2x_1,\ \frac{\partial^2 f}{\partial x_1 x_2} = 0,\ \frac{\partial^2 f}{\partial x_1^2} = -2,$$

$$\frac{\partial f}{\partial x_2} = 8 - 2x_2,\ \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0,\ \frac{\partial^2 f}{\partial x_2^2} = -2.$$

$$\therefore\ \Delta = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda\frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda\frac{\partial^2 h}{\partial x_1 \partial x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda\frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda\frac{\partial^2 h}{\partial x_2^2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} = 2 + 2 = 4$$

Since, Δ is positive, X_0 is a maxima.

Hence, $x_1 = 1,\ x_2 = 3,\ z_{Max} = 18.$

EXERCISE - II

Using Lagrange's multipliers, solve the following N.L.P.P.

1. Optimise $z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$
   subject to $x_1 + 2x_2 = 2$
   $x_1, x_2 \geq 0$. [Ans. : $x_1 = 1/3, x_2 = 5/6, z_{Max} = 77/18$]
2. Optimise $z = 5x_1 + x_2 - 2x_1x_2 - x_1^2 - x_2^2$
   subject to $x_1 + x_2 = 4$
   $x_1, x_2 \geq 0$. [Ans. : $x_1 = 5/2, x_2 = 3/2, z_{Max} = 13$]
3. Optimise $z = 10x_1 + 4x_2 + 4x_1x_2 - x_1^2 - 5x_2^2$
   subject to $x_1 + x_2 = 6$
   $x_1, x_2 \geq 0$. [Ans. : $x_1 = 9/2, x_2 = 3/2, z_{Max} = 93/2$]
4. Optimise $z = 7x_1^2 + 5x_2^2$
   subject to $2x_1 + 5x_2 = 7$
   $x_1, x_2 \geq 0$. [Ans. : $x_1 = 14/39, x_2 = 49/39, z_{Min} = 343/39$]
5. Optimise $z = 6x_1^2 + 5x_2^2$
   subject to $x_1 + 5x_2 = 3$
   $x_1, x_2 \geq 0$. [Ans. : $x_1 = 3/31, x_2 = 18/31, z_{Max} = 54/31$]
6. Optimise $z = 4x_1 + 8x_2 - x_1^2 - x_2^2$
   subject to $x_1 + x_2 = 2$
   $x_1, x_2 \geq 0$. [Ans. : $x_1 = 0, x_2 = 2, z_{Max} = 12$]
7. Optimise $z = 6x_1^2 + 5x_2^2$
   subject to $x_1 + 5x_2 = 11$
   $x_1, x_2 \geq 0$. [Ans. : $x_1 = 11/31, x_2 = 66/31, z_{Max} = 726/31$]
8. Optimise $z = 3x_1^2 + x_2^2 + 2x_1x_2 + 6x_1 + 2x_2$
   subject to $2x_1 + x_2 = 4$
   $x_1, x_2 \geq 0$. [Ans. : $x_1 = 1, x_2 = 2, z_{Min} = 28$]
9. Optimise $z = 3x_1^2 + 2x_2^2 + 4x_1 + 2x_2$
   subject to $3x_1 + 5x_2 = 11$
   $x_1, x_2 \geq 0$. [Ans. : $x_1 = 1/3, x_2 = 2, z_{Min} = 41/3$]
10. Optimise $z = 2x_1 + 6x_2 - x_1^2 - x_2^2 + 14$
    subject to $x_1 + x_2 = 4$
    $x_1, x_2 \geq 0$. [Ans. : $x_1 = 1, x_2 = 3, z_{Max} = 24$]

(c) Non-linear Programming Problems with Three Variables and One Equality Constraints

If we have three variables we get a determinant of order four.

If Δ_3 and Δ_4 are both negative then X_0 is a minima.

If Δ_3 is positive and Δ_4 is negative X_0 is a maxima. [See page 13-9]

Example 1 : Using the method of Lagrange's multipliers, solve the following L.P.P.

Optimise $z = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$ **(M.U. 1998, 2004, 09)**

subject to $x_1 + x_2 + x_3 = 7;\ x_i \geq 0.$

Sol. : We have the Lagrangian function,

$$L(x_1, x_2, x_3, \lambda) = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3 - \lambda(x_1 + x_2 + x_3 - 7)$$

We, now, obtain the following partial derivatives

$$\frac{\partial L}{\partial x_1} = 2x_1 - 10 - \lambda,\quad \frac{\partial L}{\partial x_2} = 2x_2 - 6 - \lambda,$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - 4 - \lambda,\quad \frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 7)$$

Solving the equations, $\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0,\ \frac{\partial L}{\partial x_3} = 0,\ \frac{\partial L}{\partial \lambda} = 0$, we get

$$\therefore\ 2x_1 - 10 - \lambda = 0,\ 2x_2 - 6 - \lambda = 0,\ 2x_3 - 4 - \lambda = 0,\ x_1 + x_2 + x_3 = 7$$

Adding the first three equations, we get

$$2(x_1 + x_2 + x_3) - 20 - 3\lambda = 0. \qquad \text{But } x_1 + x_2 + x_3 = 7$$

$$\therefore\ 14 - 20 - 3\lambda = 0 \qquad \therefore\ 3\lambda = -6 \qquad \therefore\ \lambda = -2$$

Hence, $x_1 = 4,\ x_2 = 2,\ x_3 = 1$ $\therefore$ X_0 is (4, 2, 1).

Now,
$$\Delta_4 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda\frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda\frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} - \lambda\frac{\partial^2 h}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda\frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda\frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} - \lambda\frac{\partial^2 h}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 f}{\partial x_3 \partial x_1} - \lambda\frac{\partial^2 h}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} - \lambda\frac{\partial^2 h}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} - \lambda\frac{\partial^2 h}{\partial x_3^2} \end{vmatrix}$$

where, $f \equiv x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

and $h \equiv x_1 + x_2 + x_3 - 7$.

Now,
$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix}$$

By $C_2 - C_4,\ C_3 - C_4$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & -2 & -2 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & -2 & -2 \end{vmatrix}$$

By $C_2 - 2C_1$, $$= (-1)\begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & -4 & -2 \end{vmatrix} = (-1)[4 + 8] = -12$$

Now, taking the first three rows and the first three columns from Δ_4

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

By $C_2 - C_3$, $\quad \Delta_3 = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -2 - 2 = -4$

Since both Δ_3 and Δ_4 are negative, X_0 is a minima.

$\therefore\ x_1 = 4,\ x_2 = 2,\ x_3 = 1. \quad \therefore\ z_{Min} = 16 + 4 + 1 - 40 - 12 - 4 = -35.$

Example 2 : Determine the optimal solution for the following problem and check whether it maximises or minimises the objective function.

Optimise $\quad z = 2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 260$

subject to $\quad x_1 + x_2 + x_3 = 11$

$\quad x_1, x_2, x_3 \geq 0.$ **(M.U. 1998)**

Sol. : We have the Lagrangian function

$$L(x_1, x_2, x_3, \lambda) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 260 - \lambda(x_1 + x_2 + x_3 - 11)$$

We, now, obtain the following partial derivatives.

$$\frac{\partial L}{\partial x_1} = 4x_1 - 24 - \lambda, \quad \frac{\partial L}{\partial x_2} = 4x_2 - 8 - \lambda,$$

$$\frac{\partial L}{\partial x_3} = 4x_3 - 12 - \lambda, \quad \frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 11$$

Solving the equations, $\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0,\ \frac{\partial L}{\partial x_3} = 0,\ \frac{\partial L}{\partial \lambda} = 0$, we get

$\therefore\ 4x_1 - 24 - \lambda = 0,\ 4x_2 - 8 - \lambda = 0,$

$4x_3 - 12 - \lambda = 0,\ x_1 + x_2 + x_3 - 11 = 0$

Adding the first three equations, we get,

$4(x_1 + x_2 + x_3) - 44 - 3\lambda = 0. \quad$ But $x_1 + x_2 + x_3 = 11$

$\therefore\ 44 - 44 - 3\lambda = 0 \quad \therefore\ \lambda = 0.$

Hence, $4x_1 = 24 \quad \therefore\ x_1 = 6;\ 4x_2 = 8 \quad \therefore\ x_2 = 2.$

$\therefore\ 4x_4 = 12 \quad \therefore\ x_3 = 3; \quad \therefore\ X_0$ is $(6, 2, 3)$.

Now, $\Delta_4 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} - \lambda \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} - \lambda \frac{\partial^2 h}{\partial x_3^2} \end{vmatrix}$

where $f = 2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 260$ and $h \equiv x_1 + x_2 + x_3 - 11.$

Now, $\quad \Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix}$

By $C_2 - C_4,\ C_3 - C_4$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 4 & -4 & -4 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 4 \\ 1 & -4 & -4 \end{vmatrix} = -[(16) - 4(-8)] = -48$$

Now, taking the first three rows and the first three columns from Δ_4,

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix} = -1(4) + 1(-4) = -8$$

Since, Δ_3, Δ_4 are both negative, $X_0 \equiv (6, 2, 3)$ is the minima.

$\therefore\ x_1 = 6,\ x_2 = 2,\ x_3 = 3$

and $\quad z_{Min} = 2(36) + 2(4) + 2(9) - 1244 - 16 - 36 + 260 = 162.$

Example 3 : Use the method of Lagranges multipliers to solve the following N.L.P.P.

Optimise $\quad z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

subject to $\quad x_1 + x_2 + x_3 = 20$

$\quad x_1, x_2, x_3 \geq 0.$ **(M.U. 1996, 99, 2004, 06)**

Sol. : We have the Lagrangian function.

$$L(x_1, x_2, x_3, \lambda) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 - \lambda(x_1 + x_2 + x_3 - 20)$$

We, now, obtain the following partial derivatives.

$$\frac{\partial L}{\partial x_1} = 4x_1 + 10 - \lambda, \quad \frac{\partial L}{\partial x_2} = 2x_2 + 8 - \lambda,$$

$$\frac{\partial L}{\partial x_3} = 6x_3 + 6 - \lambda, \quad \frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 20)$$

Solving the equations, $\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0,\ \frac{\partial L}{\partial x_3} = 0,\ \frac{\partial L}{\partial \lambda} = 0$, we get

$\therefore\ 4x_1 + 10 - \lambda = 0, \quad 2x_2 + 8 - \lambda = 0$

$6x_3 + 6 - \lambda = 0, \quad x_1 + x_2 + x_3 = 20.$

Multiply the first by 3, second by 6, third by 2 and add.

$\therefore\ 12(x_1 + x_2 + x_3) + (30 + 48 + 12) - 11\lambda = 0$

But $\quad x_1 + x_2 + x_3 = 20$

$\therefore\ 240 + 90 = 11\lambda \quad \therefore\ 11\lambda = 330 \quad \therefore\ \lambda = 30.$

Hence, $4x_1 = 20 \quad \therefore\ x_1 = 5; \quad 2x_2 = 22, \quad \therefore\ x_2 = 11;$

$6x_3 = 24 \quad \therefore\ x_3 = 4. \quad \therefore\ X_0$ is $(5, 11, 4)$.

$$\text{Now,}\quad \Delta_4 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 f}{\partial x_3 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} - \lambda \frac{\partial^2 h}{\partial x_3^2} \end{vmatrix}$$

where $f = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$ and $h = x_1 + x_2 + x_3 - 20$.

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix}$$

By $C_2 - C_4$, $C_3 - C_4$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & -6 & -6 & 6 \end{vmatrix} = -(1) \begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 2 \\ 1 & -6 & -6 \end{vmatrix}$$

By $C_2 - 4C_1$, $$= (-1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 1 & -10 & -6 \end{vmatrix} = (-1)[24 + 20] = -44$$

Now, taking the first three rows and the first columns from Δ_4, we get

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

By $C_2 - C_3$, $$\Delta_3 = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 4 & 0 \\ 1 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix} = -2 - 4 = -6$$

Since, Δ_3 and Δ_4 are both negative, $X_0\,(5, 11, 4)$ is a minima.

$\therefore\ x_1 = 5,\ x_2 = 11,\ x_3 = 4$

and $\quad z_{Min} = 50 + 121 + 48 + 50 + 88 + 24 - 100 = 281.$

Example 4 : Using the method of Lagrange's multipliers solve the following N.L.P.P.

Optimise $\quad z = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$

subject to $\quad x_1 + x_2 + x_3 = 10$

$\quad x_1, x_2, x_3 \geq 0.$ (M.U. 2007, 11)

Sol. : We have the Lagrangian function.

$$L(x_1, x_2, x_3, \lambda) = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23 - \lambda(x_1 + x_2 + x_3 - 10)$$

We, now, obtain the following partial derivatives.

$$\frac{\partial L}{\partial x_1} = 12 - 2x_1 - \lambda,\quad \frac{\partial L}{\partial x_2} = 8 - 2x_2 - \lambda,$$

$$\frac{\partial L}{\partial x_3} = 6 - 2x_3 - \lambda,\quad \frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 10$$

Solving the equations, $\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0,\ \frac{\partial L}{\partial x_3} = 0,\ \frac{\partial L}{\partial \lambda} = 0$, we get

$\therefore\ 12 - 2x_1 - \lambda = 0,\ 8 - 2x_2 - \lambda = 0$

$6 - 2x_3 - \lambda = 0,\ x_1 + x_2 + x_3 - 10 = 0$

Adding the first three equations, we get

$26 - 2(x_1 + x_2 + x_3) - 3\lambda = 0.$

But $x_1 + x_2 + x_3 = 10$

$\therefore\ 26 - 20 = 3\lambda \quad \therefore\ 6 = 3\lambda \quad \therefore\ \lambda = 2$

Hence, $x_1 = 5,\ x_2 = 3,\ x_3 = 2.$

$$\text{Now,}\quad \Delta_4 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 f}{\partial x_3 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} - \lambda \frac{\partial^2 h}{\partial x_3^2} \end{vmatrix}$$

where, $f \equiv 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$ and $h = x_1 + x_2 + x_3 - 10.$

$$\therefore\ \Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{vmatrix}$$

$\therefore$ By $C_2 - C_4$, $C_3 - C_4$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 2 & 2 & -2 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \\ 1 & 2 & 2 \end{vmatrix}$$

By $C_2 + 2C_1$ $$= (-1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -2 \\ 1 & 4 & 2 \end{vmatrix} = (-1)[4 + 8] = -12$$

Now, taking the first three rows and the first three columns, we get

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix}$$

By $C_2 - C_3$, $\Delta_3 = \begin{vmatrix} 0 & 0 & 1 \\ 1 & -2 & 0 \\ 1 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} = 2 + 4 = 4$

Since Δ_3 is positive and Δ_4 is negative X_0 is a maxima.

$\therefore\ x_1 = 5,\ x_2 = 3,\ x_3 = 2.$

and $z_{Max} = 60 + 24 + 9\ 12 - 25 - 9 - 4 - 23 = 35.$

EXERCISE - III

Using the Lagrangian multipliers solve the following N.L.P.P.

1. Optimise $z = 2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 196$
   subject to $x_1 + x_2 + x_3 = 11$
   $x_1, x_2, x_3 \geq 0.$ [Ans. : $x_1 = 6,\ x_2 = 2,\ x_3 = 3,\ z_{Min} = 98$]
2. Optimise $z = x_1^2 + x_2^2 + x_3^2$
   subject to $4x_1 + x_2^2 + 2x_3 - 14 = 0$
   $x_1, x_2, x_3 \geq 0.$ [Ans. : $x_1 = 2,\ x_2 = 2,\ x_3 = 1,\ z_{Min} = 9$]
3. Optimisa $z = 3x_1^2 + x_2^2 + x_3^2$
   subject to $x_1 + x_2 + x_3 = 2$
   $x_1, x_2, x_3 \geq 0.$ [Ans. : $x_1 = 0{\cdot}81,\ x_2 = 0{\cdot}35,\ x_3 = 0{\cdot}28,\ z_{Min} = 0{\cdot}84$]

4. Non-Linear Programming Problem With n-variables And More Than One (m) Equality Constraints ($m < n$)

Consider the following non-linear programming problem with n-variables and m-equality constraints.

Optimise $z = f(x_1, x_2, \ldots\ldots, x_n)$

subject to
$h_1(x_1, x_2, \ldots\ldots, x_n) = 0$
$h_2(x_1, x_2, \ldots\ldots, x_n) = 0$
$\cdots\cdots\cdots\cdots\cdots\cdots$
$h_m(x_1, x_2, \ldots\ldots, x_n) = 0$
$x_1, x_2, \ldots\ldots x_n \geq 0.$

We first construct a new function called Lagrangian function, using the m multipliers $\lambda_1, \lambda_2, \ldots\ldots, \lambda_m$ as

$$L(x_1, x_2, \ldots\ldots x_n, \lambda_1, \lambda_2, \ldots\ldots, \lambda_m) = f(x_1, x_2, \ldots\ldots, x_n) - \lambda_1 h_1(x_1, x_2, \ldots\ldots, x_n) - \lambda_2 h_2(x_1, x_2, \ldots\ldots, x_n) \ldots\ldots - \lambda_m h_m(x_1, x_2, \ldots\ldots, x_n) = 0.$$

The necessary conditions for maxima or minima are obtained by differentiating the Lagrangian function partially with respect to $x_1, x_2, \ldots\ldots, x_n, \lambda_1, \lambda_2, \ldots\ldots, \lambda_m$ and then equating these partial derivatives to zero.

Thus, we get, the following $m + n$ equations

$$\frac{\partial L}{\partial x_i} = 0,\ i = 1, 2, \ldots\ldots, n \quad ; \quad \frac{\partial L}{\partial \lambda_j} = 0,\ j = 1, 2, \ldots\ldots, m.$$

Solving these equations, we get the stationary points.

To obtain the sufficient conditions for maxima or minima *i.e.*, to decide whether a point obtained by solving the above equations is a maxima or a minima we consider a new matrix called **bordered Hessian matrix**, denoted by H^B. It is defined as follows.

$$H^B = \begin{bmatrix} O & \vdots & P \\ \cdots & \cdots & \cdots \\ P' & \vdots & Q \end{bmatrix}_{(m+n)\times(m+n)}$$

where, O is an $m \times m$ null matrix

i.e. $O = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times m}$; $P = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}_{m\times n}$

P' = Transpose of P and ; $Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}$

Let $X_0(x_1, x_2, \ldots\ldots, x_n)$ be a stationary point with $\lambda_{1,0}, \lambda_{2,0}, \ldots\ldots, \lambda_{m,0}$ as the corresponding values of $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$. We now obtain the bordered Hessian matrix for these values of $x_1, x_2, \ldots\ldots, x_n$ and for these values of λ's.

Conditions for Maxima and Minima

The nature of the function at X_0 whether a maxima or a minima is determined by the signs of $(n - m)$ principal minors of the matrix H^B.

Starting with the principal minor of order $(2m + 1)$ we check the signs of $(n - m)$ principal minors. If these signs alternate, starting with the sign $(-1)^{m+n}$ of $(2m+1)^{st}$ minor then X_0 is a maxima.

For example, if $n = 10$ and $m = 4$, we check the signs of $n - m = 6$ minors, starting with $2m + 1 = 9^{th}$ principal minor *i.e.* we check the signs of 9^{th}, 10^{th}, 11^{th}, 12^{th}, 13^{th} and 14^{th} principal minors.

(a) If the signs of these minors form a pattern of alternately positive and negative signs starting with $(-1)^{m+n}$ then x_0 is a maxima.

(b) If the signs of these minors are $(-1)^m$, then x_0 is a minima.

For example, if $n = 10$ and $m = 4$ then we check the signs of 6 principal minors of order 9, 10, 11, 12, 13 and 14. If the sign of 9^{th} minor is $(-1)^{m+n} = (-1)^{10+4} = +1$ and the sign of the remaining minors are alternately positive and negative then X_0 is a maxima. On the other hand if all these minors have the sign of $(-1)^m = (-1)^4 = +1$, then X_0 is a minima.

Example 1 : Using the method of Lagrangian multipliers solve the following non-linear programming problem.

Maximise $z = 6x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $4x_1 + 3x_2 = 16,$

$3x_1 + 5x_2 = 15$

$x_1, x_2 \geq 0.$ **(M.U. 1999, 2000, 03, 04, 10)**

Sol. : We have $f(x_1, x_2) = 6x_1 + 8x_2 - x_1^2 - x_2^2$

$h_1(x_1, x_2) = 4x_1 + 3x_2 - 16 \; ; \; h_2(x_1, x_2) = 3x_1 + 5x_2 - 15$

Now, the Lagrangian function is,

$$L(x_1, x_2, \lambda_1, \lambda_2) = f(x_1, x_2) - \lambda_1 h_1(x_1, x_2) - \lambda_2 h_2(x_1, x_2)$$
$$= 6x_1 + 8x_2 - x_1^2 - x_2^2 - \lambda_1(4x_1 + 3x_2 - 16) - \lambda_2(3x_1 + 5x_2 - 15)$$

We now obtain the following partial derivatives.

$$\frac{\partial L}{\partial x_1} = 6 - 2x_1 - 4\lambda_1 - 3\lambda_2, \quad \frac{\partial L}{\partial x_2} = 8 - 2x_2 - 3\lambda_1 - 5\lambda_2$$

$$\frac{\partial L}{\partial \lambda_1} = -[4x_1 + 3x_2 - 16], \quad \frac{\partial L}{\partial \lambda_2} = -[3x_1 + 5x_2 - 15]$$

We, then, solve the following equations.

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial \lambda_1} = 0, \quad \frac{\partial L}{\partial \lambda_2} = 0.$$

$\therefore \; 6 - 2x_1 - 4\lambda_1 - 3\lambda_2 = 0$ .... (1)

$8 - 2x_2 - 3\lambda_1 - 5\lambda_2 = 0$ .... (2)

$4x_1 + 3x_2 = 16$ .... (3)

$3x_1 + 5x_2 = 15$ .... (4)

Multiply (1) by (4), (2) by 3 (coefficients of λ_1) and add.

$$24 - 8x_1 - 16\lambda_1 - 12\lambda_2 + 24 - 6x_2 - 9\lambda_1 - 15\lambda_2 = 0$$

$\therefore \; 48 - 2(4x_1 + 3x_2) - 25\lambda_1 - 27\lambda_2 = 0$

By (3), $48 - 32 - 24\lambda_1 - 27\lambda_2 = 0$

$\therefore \; 25\lambda_1 + 27\lambda_2 = 16$ .... (5)

Multiply (1) by 3, (2) by 5 (coefficients of λ_2) and add.

$$18 - 6x_1 - 12\lambda_1 - 9\lambda_2 + 40 - 10x_2 - 15\lambda_1 - 25\lambda_2 = 0$$

$\therefore \; 58 - 2(3x_1 + 5x_2) - 27\lambda_1 - 34\lambda_2 = 0$

By (4), $58 - 30 - 27\lambda_1 - 34\lambda_2 = 0 \quad \therefore \; 27\lambda_1 + 34\lambda_2 = 28$ .... (6)

Solving (5) and (6), we get $\lambda_1 = -\dfrac{212}{121}, \; \lambda_2 = \dfrac{268}{121}.$

Now, from (1), we have, $2x_1 = 6 - 4\lambda_1 - 3\lambda_2$

$$\therefore \; 2x_1 = 6 + 4 \times \frac{212}{121} - \frac{3 \times 268}{121} = \frac{770}{121} = \frac{70}{11} \quad \therefore \; x_1 = \frac{35}{11}$$

From (2), we have, $2x_2 = 8 - 3\lambda_1 - 5\lambda_2$

$$\therefore \; 2x_2 = 8 + 3 \times \frac{212}{121} - 5 \times \frac{268}{121} = \frac{264}{121} = \frac{24}{11} \quad \therefore \; x_2 = \frac{12}{11}$$

Since $n = 2, m = 2, n - m = 0$, the above method fails.

The criterion of bordered Hessian matrix cannot be applied when $m = n$ or $m \geq n$ i.e., when the number of constraints is equal to or greater than the number of variables.

Note ....

If there are two unknowns and two (linear) constraints, bordered Hessian matrix H^B does not help us.

Instead we find the signs of two principal minors of

$$\begin{vmatrix} \dfrac{\partial^2 z}{\partial x_1^2} & \dfrac{\partial^2 z}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 z}{\partial x_2 \partial x_1} & \dfrac{\partial^2 z}{\partial x_2^2} \end{vmatrix}$$ and use the criterion given on page 13-4.

Now; $z = 6x_1 + 8x_2 - x_1^2 - x_2^2$

$$\therefore \; \frac{\partial z}{\partial x_1} = 6 - 2x_1, \qquad \frac{\partial^2 z}{\partial x_1 \partial x_2} = 0$$

$$\frac{\partial^2 z}{\partial x_1^2} = -2, \qquad \frac{\partial z}{\partial x_2} = 8 - 2x_2$$

$$\frac{\partial^2 z}{\partial x_2 \partial x_1} = 0, \qquad \frac{\partial^2 z}{\partial x_2^2} = -2.$$

$$\therefore \; \begin{vmatrix} \dfrac{\partial^2 z}{\partial x_1^2} & \dfrac{\partial^2 z}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 z}{\partial x_2 \partial x_1} & \dfrac{\partial^2 z}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4$$

$$\therefore \; A_1 = |a_{11}| = -2 \quad \text{and} \quad A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 4$$

Since A_1 is negative and A_2 is positive (x_0, y_0) is maxima, therefore, $X_0\left(\dfrac{35}{11}, \dfrac{12}{11}\right)$ is a maxima and $z_{Max} = 6\left(\dfrac{35}{11}\right) + 8\left(\dfrac{12}{11}\right) - \left(\dfrac{35}{11}\right)^2 - \left(\dfrac{12}{11}\right)^2 = 16 \cdot 504.$

Example 2 : Using the method of Lagrangian multipliers solve the following problem.

Optimise $z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

subject to $x_1 + x_2 + x_3 = 15$

$2x_1 - x_2 + 2x_3 = 20$

$x_1, x_2, x_3 \geq 0.$ **(M.U. 2001, 02, 04, 05, 09)**

Sol. : We have $f(x_1, x_2, x_3) \equiv 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

$h_1(x_1, x_2, x_3) \equiv x_1 + x_2 + x_3 - 15 \; ;$

$h_2(x_1, x_2, x_3) \equiv 2x_1 - x_2 + 2x_3 - 20$

Now, the Lagrangian function is,

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = f(x_1, x_2, x_3) - \lambda_1 h_1(x_1, x_2, x_3) - \lambda_2 h_2(x_1, x_2, x_3)$$
$$= 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2 - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

We now obtain the following partial derivatives.

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2, \quad \frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2$$
$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2, \quad \frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + x_3 - 15),$$
$$\frac{\partial L}{\partial \lambda_2} = -(2x_1 - x_2 + 2x_3 - 20)$$

We, then, solve the following equations.

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial x_3} = 0, \quad \frac{\partial L}{\partial \lambda_1} = 0, \quad \frac{\partial L}{\partial \lambda_2} = 0.$$

$\therefore\ 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0$ .... (1)

$4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0$ .... (2)

$2x_3 - \lambda_1 - 2\lambda_2 = 0$ .... (3)

$x_1 + x_2 + x_3 = 15$ .... (4)

$2x_1 - x_2 + 2x_3 = 20$ .... (5)

We have to solve these five equations to find the values of $x_1, x_2, x_3, \lambda_1, \lambda_2$.

Add 4 times (3) to (1).

$\therefore\ 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 + 8x_3 - 4\lambda_1 - 8\lambda_2 = 0$

$\therefore\ 4(2x_1 - x_2 + 2x_3) = 5\lambda_1 + 10\lambda_2$

By (5), we get $5\lambda_1 + 10\lambda_2 = 80$ .... (6)

Now, multiply (1) by 2, (2) by 3 and (3) by 2 and add.

$$16x_1 - 8x_2 - 2\lambda_1 - 4\lambda_2 + 12x_2 - 12x_1 - 3\lambda_1 + 3\lambda_2 + 4x_3 - 2\lambda_1 - 4\lambda_2 = 0$$

$\therefore\ 4(x_1 + x_2 + x_3) - 7\lambda_1 - 5\lambda_2 = 0$

By (4), we get $7\lambda_1 + 5\lambda_2 = 60$ .... (7)

[Alternatively you may solve (1) and (2) for x_1 and x_2, (3) for x_3 and substitute these values in (4) and (5). After that you may find λ_1 and λ_2.]

Solving (6) and (7), we get

$\lambda_1 = 40/9, \lambda_2 = 52/9.$

Adding (1) and (2), we get

$4x_1 = 2\lambda_1 + \lambda_2.$

Adding 2 times (2) to (1), we get $4x_2 = 3\lambda_1.$

And from (3), we get $2x_3 = \lambda_1 + 2\lambda_2$

$$\therefore\ x_1 = \frac{11}{3}, \quad x_2 = \frac{10}{3}, \quad x_3 = 8.$$

Now, $\dfrac{\partial h_1}{\partial x_1} = 1, \quad \dfrac{\partial h_1}{\partial x_2} = 1, \quad \dfrac{\partial h_1}{\partial x_3} = 1$

$$\frac{\partial h_2}{\partial x_1} = 2, \quad \frac{\partial h_2}{\partial x_2} = -1, \quad \frac{\partial h_2}{\partial x_3} = 2$$
$$\frac{\partial^2 L}{\partial x_1^2} = 8, \quad \frac{\partial^2 L}{\partial x_1 \partial x_2} = -4, \quad \frac{\partial^2 L}{\partial x_1 \partial x_3} = 0$$
$$\frac{\partial^2 L}{\partial x_2 \partial x_1} = -4, \quad \frac{\partial^2 L}{\partial x_2^2} = 4, \quad \frac{\partial^2 L}{\partial x_2 \partial x_3} = 0$$
$$\frac{\partial^2 L}{\partial x_3 \partial x_1} = 0, \quad \frac{\partial^2 L}{\partial x_3 \partial x_2} = 0, \quad \frac{\partial^2 L}{\partial x_3^2} = 2.$$

Hence, $P = \begin{bmatrix} \dfrac{\partial h_1}{\partial x_1} & \dfrac{\partial h_1}{\partial x_2} & \dfrac{\partial h_1}{\partial x_3} \\ \dfrac{\partial h_2}{\partial x_1} & \dfrac{\partial h_2}{\partial x_2} & \dfrac{\partial h_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}$

$$\therefore\ P' = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} \dfrac{\partial^2 L}{\partial x_1^2} & \dfrac{\partial^2 L}{\partial x_1 \partial x_2} & \dfrac{\partial^2 L}{\partial x_1 \partial x_3} \\ \dfrac{\partial^2 L}{\partial x_2 \partial x_1} & \dfrac{\partial^2 L}{\partial x_2^2} & \dfrac{\partial^2 L}{\partial x_2 \partial x_3} \\ \dfrac{\partial^2 L}{\partial x_3 \partial x_1} & \dfrac{\partial^2 L}{\partial x_3 \partial x_2} & \dfrac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence, bordered Hessian matrix at X_0 (33/9, 10/3, 8) and $\lambda_1 = 40/9, \lambda_2 = 52/9$ is

$$H_0^B = \begin{bmatrix} O & \vdots & P \\ \cdots & \cdots & \cdots \\ P' & \vdots & Q \end{bmatrix} = \begin{bmatrix} 0 & 0 & \vdots & 1 & 1 & 1 \\ 0 & 0 & \vdots & 2 & -1 & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \vdots & 8 & -4 & 0 \\ 1 & -1 & \vdots & -4 & 4 & 0 \\ 1 & 2 & \vdots & 0 & 0 & 2 \end{bmatrix}$$

Since $n = 3$, $m = 2$, $n - m = 1$ and $2m + 1 = 5$, we have to check whether the principal minor of order 5 has the sign of $(-1)^2$.

By Laplace method (see the note below) the determinant H^B is

[We multiply the determinant $\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$ by the determinant obtained by deleting the rows and the columns in which these elements lie *i.e.* by $\begin{vmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 2 \end{vmatrix}$. The sign is determined by $(-1)^{3+4}$ as the elements of $\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$ lie in the third and fourth columns. Similarly we take the products of all other determinants with proper sign. Thus, we have]

$$H^B = (-1)^{3+4+1}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}\begin{vmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 2 \end{vmatrix} + (-1)^{3+5+1}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}\begin{vmatrix} 1 & 2 & -4 \\ 1 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix}$$

$$+ (-1)^{4+5+1}\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}\begin{vmatrix} 1 & 2 & 8 \\ 1 & -1 & -4 \\ 1 & 2 & 0 \end{vmatrix}$$

Now, we evaluate each of the above six determinants.

$$\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = (-1-2) = -3$$

$$\begin{vmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 2 \end{vmatrix} = 2(2+1) = 6 \qquad \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & -4 \\ 1 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix} = \text{Not required} \qquad \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 2+1 = 3$$

$$\begin{vmatrix} 1 & 2 & 8 \\ 1 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix} = 8(3) - 4(0)$$

$\therefore \quad H^B = (-3)(6) + 0 + 3(24) = -18 + 72 = 54$

Since $n = 3$, $m = 2$, $n + m = 5$ and $2m + 1 = 5$, we solve the principle minor of order 5. For maxima the sign of this minor should be $(-1)^{m+1} = (-1)^3$ *i.e.*, negative and for minima the sign should be $(-1)^m = (-1)^2$ *i.e.*, positive.

Since, the determinant is positive X_0 (33/9, 10/9, 8) is a minima.

$$z_{Min} = 4\left(\frac{11}{3}\right)^2 + 2\left(\frac{10}{3}\right)^2 + 8^2 - 4\left(\frac{11}{3}\right)\left(\frac{10}{3}\right) = \frac{820}{9}.$$

[**Note on Laplace Method :** Consider

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix}$$

We shall denote the determinant $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ by (a_1, b_2). In the same way we denote other determinants by their principal diagonal. For example, we shall denote the determinant $\begin{vmatrix} c_3 & c_4 & c_5 \\ d_3 & d_4 & d_5 \\ e_3 & e_4 & e_5 \end{vmatrix}$ by (c_3, d_4, e_5). In the expansion of Δ the sum of the terms which contain $a_1 b_2$

is $a_1 b_2 (c_3, d_4, e_5)$ and the sum of the terms which contain $a_2 b_1$ is $-a_2 b_1 (c_3, d_4, e_5)$. The sum of all these terms is $(a_1, b_2, a_2, b_1)(c_3, d_4, e_5)$ *i.e.*, $(a_1, b_2)(c_3, d_4, e_5)$. Further, every term in the product $(a_1, b_2)(c_3, d_4, e_5)$ appears in the expansion of the given determinant Δ.

Now, Δ is the sum of all the **ten** products, each with proper sign, of the form $(a_1, b_2)(c_3, d_4, e_5)$, where the first factor is any minor formed from the elements of the first two rows (or columns) and the second is the complementary minor. The sign is determined by $(-1)^{r+s+1}$ where r and s denote the number of columns in which the two elements in the first two rows lie.]

Remark ....

Geometrically speaking the constraints are the two planes. They intersect in a line if they do.

Note ....

If there are three variables and two constraints we simplify the bordered Hessian H^B. If its sign is negative, X_0 is a maxima and if its sign is positive X_0 is a minima.

Example 3 : Using the method of Lagrangian multipliers solve the following problem

Optimise $\quad z = x_1^2 + x_2^2 + x_3^2$

subject to $\quad x_1 + x_2 + 3x_3 = 2$

$\quad 5x_1 + 2x_2 + x_3 = 5; \quad x_1, x_2, x_3 \geq 0.$ **(M.U. 1997, 2001)**

Sol. : We have $f(x_1, x_2, x_3) \equiv x_1^2 + x_2^2 + x_3^2$

$h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 2;$

$h_2(x_1, x_2, x_3) = 5x_1 + 2x_2 + x_3 - 5$

Now, the Lagrangian function is,

$$L(x_1, x_2, x_3\ \lambda_1, \lambda_2) = f(x_1, x_2, x_3) - \lambda_1 h_1(x_1, x_2, x_3) - \lambda_2 h_2(x_1, x_2, x_3)$$
$$= x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$$

We, now, obtain the following partial derivatives.

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2, \quad \frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda_2$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - \lambda_2, \quad \frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + 3x_2 - 2),$$

$$\frac{\partial L}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5).$$

We, then, solve the following equations.

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial x_3} = 0, \quad \frac{\partial L}{\partial \lambda_1} = 0, \quad \frac{\partial L}{\partial \lambda_2} = 0.$$

$\therefore \quad 2x_1 - \lambda_1 - 5\lambda_2 = 0 \qquad$ .... (1)

$2x_2 - \lambda_1 - 2\lambda_2 = 0 \qquad$ .... (2)

$2x_3 - 3\lambda_1 - \lambda_2 = 0 \qquad$ .... (3)

$x_1 + x_2 + x_3 - 2 = 0 \qquad$ .... (4)

$5x_1 + 2x_2 + x_3 - 5 = 0 \qquad$ .... (5)

We have to solve these equations to find the value of $x_1, x_2, x_3, \lambda_1, \lambda_2$.

Add 3 times (3) to the sum of (1) and (2)

$$2x_1 - \lambda_1 - 5\lambda_2 + 2x_2 - \lambda_1 - 2\lambda_2 + 6x_3 - 9\lambda_1 - 3\lambda_2 = 0$$

$$\therefore \quad 2x_1 + 2x_2 + 6x_3 - 11\lambda_1 - 10\lambda_2 = 0$$

$$\therefore \quad 2(x_1 + x_2 + 3x_3) - 11\lambda_1 - 10\lambda_2 = 0$$

By (4), we get $\quad 11\lambda_1 + 10\lambda_2 = 4 \qquad \text{.................... (6)}$

Now multiply (1) by 5, (2) by 2 and (3) by 1 and add

$$10x_1 - 5\lambda_1 - 25\lambda_2 + 4x_2 - 2\lambda_1 - 4\lambda_2 + 2x_3 - 3\lambda_1 - \lambda_2 = 0$$

$$\therefore \quad 10x_1 + 4x_2 + 2x_3 - 10\lambda_1 - 30\lambda_2 = 0$$

$$\therefore \quad 2(5x_1 + 2x_2 + x_3) - 10\lambda_1 - 30\lambda_2 = 0$$

By (5), we get $\quad 10\lambda_1 + 30\lambda_2 = 10$ i.e. $\lambda_1 + 3\lambda_2 = 1 \qquad \text{.................... (7)}$

Now solving (6) and (7), we get $\lambda_1 = 2/23, \lambda_2 = 7/23$.

From (1), (2) and (3), we get $x_1 = \dfrac{37}{46} = 0.804,\ x_2 = \dfrac{16}{46} = 0.348,\ x_3 = \dfrac{13}{46} = 0.283.$

Now
$$\frac{\partial h_1}{\partial x_1} = 1,\ \frac{\partial h_1}{\partial x_2} = 1,\ \frac{\partial h_1}{\partial x_3} = 3$$

$$\frac{\partial h_2}{\partial x_1} = 5,\ \frac{\partial h_2}{\partial x_2} = 2,\ \frac{\partial h_2}{\partial x_3} = 1$$

$$\frac{\partial^2 L}{\partial x_1^2} = 2,\ \frac{\partial^2 L}{\partial x_1 \partial x_2} = 0,\ \frac{\partial^2 L}{\partial x_1 \partial x_3} = 0$$

$$\frac{\partial^2 L}{\partial x_2 \partial x_1} = 0,\ \frac{\partial^2 L}{\partial x_2^2} = 2,\ \frac{\partial^2 L}{\partial x_2 \partial x_3} = 0$$

$$\frac{\partial^2 L}{\partial x_3 \partial x_1} = 0,\ \frac{\partial^2 L}{\partial x_3 \partial x_2} = 0,\ \frac{\partial^2 L}{\partial x_3^2} = 2$$

Hence
$$P = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 1 \end{bmatrix},\quad P' = \begin{bmatrix} 1 & 5 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence, the bordered Hessian matrix at X_0 (0·804, 0·348, 0·283) and $\lambda_1 = \dfrac{2}{23}, \lambda_2 = \dfrac{7}{23}$ is

$$H_0^B = \begin{bmatrix} O & \vdots & P \\ \cdots & \cdots & \cdots \\ P' & \vdots & Q \end{bmatrix} = \begin{bmatrix} 0 & 0 & \vdots & 1 & 1 & 3 \\ 0 & 0 & \vdots & 5 & 2 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 5 & \vdots & 2 & 0 & 0 \\ 1 & 2 & \vdots & 0 & 2 & 0 \\ 3 & 1 & \vdots & 0 & 0 & 2 \end{bmatrix}$$

Since $n = 3$, $m = 2$, $n - m = 1$ and $2m + 1 = 5$, we have to check for minima whether $(-1)^m$ = the principal minor of order 5 has the sign of $(-1)^2$ and for maxima the sign of $(-1)^{m+1} = (-1)^3$.

By Laplace method the determinant

$$H_0^B = (-1)^{3+4+1}\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix}\begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} + (-1)^{3+5+1}\begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix}\begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix}$$

$$+ (-1)^{4+5+1}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}\begin{vmatrix} 1 & 5 & 2 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{vmatrix}$$

$$= (2-5)(+2)(2-5) + (-1)(1-15)(-2)(1-15) + (1-6)(+2)(1-6)$$

$$= 460$$

Since, the determinant is positive X_0 (0·804, 0·348, 0·283) is a minima.

$$\therefore \quad z_{Min} = (0.804)^2 + (0.348)^2 + (0.283)^2 = 0.8476.$$

EXERCISE - IV

Using the Lagrangian multipliers solve the following N.L.P.P.

1. Optimise $z = 4x_1 + 9x_2 - x_1^2 - x_2^2$
   subject to $4x_1 + 3x_2 = 15$
   $3x_1 + 5x_2 = 14$
   $x_1, x_2 \geq 0.$ **(M.U. 2007)**
   [**Ans. :** $x_1 = 3, x_2 = 1, z = 11$]

2. Optimise $z = 5x_1 + 9x_2 - x_1^2 - x_2^2$
   subject to $4x_1 + 5x_2 = 13$
   $3x_1 + 4x_2 = 10$
   $x_1, x_2 \geq 0.$
   [**Ans. :** $x_1 = 2, x_2 = 1, z = 14$]

3. Optimise $z = 8x_1 + 9x_2 - x_1^2 - x_2^2$
   subject to $2x_1 + 3x_2 = 12$
   $5x_1 + 2x_2 = 19$
   $x_1, x_2 \geq 0.$
   [**Ans. :** $x_1 = 3, x_2 = 2, z = 29$]

4. Optimise $z = 8x_1 + 7x_2 - x_1^2 - x_2^2$
   subject to $4x_1 + 3x_2 = 22$
   $3x_1 + x_2 = 14$
   $x_1, x_2 \geq 0.$
   [**Ans. :** $x_1 = 4, x_2 = 2, z = 26$]

5. Optimise $z = 4x_1^2 - x_2^2 - x_3^2 - 4x_1 x_2$
   subject to $x_1 + x_2 + x_3 = 15$
   $2x_1 - x_2 + 2x_3 = 20$ **(M.U. 2000)**
   [**Ans. :** $x_1 = 5.95, x_2 = 3.33, x_3 = 5.71, H_0^B = 54, z_{Min} = 83.87$]

6. Optimise $z = x_1^2 + x_2^2 + x_3^2$
   subject to $x_1 + x_2 + x_3 = 13$
   $3x_1 + x_2 + x_3 = 27$
   $x_1, x_2, x_3 \geq 0$
   [**Ans. :** $x_1 = 7, x_2 = 3, x_3 = 3,$
   $\lambda_1 = 2, \lambda_2 = 4, z_{Min} = 67$]

7. Optimise $z = 2x_1^2 + 3x_2^2 + x_3^2$
   subject to $x_1 + x_2 + 2x_3 = 13$
   $2x_1 + x_2 + x_3 = 10$
   $x_1, x_2, x_3 \geq 0$
   [**Ans. :** $x_1 = 2, x_2 = 1, x_3 = 5,$
   $\lambda_1 = 4, \lambda_2 = 2, z_{Min} = 36$]

5. Non-Linear Programming Problem With Inequality Constraints (Kuhn-Tucker Conditions)

Consider the following non-linear programming problem with n-variables and one inequality constraint.

Maximise $z = f(x_1, x_2, \ldots\ldots, x_n)$

subject to $g(x_1, x_2, \ldots\ldots, x_n) \leq b$

$x_1, x_2, \ldots\ldots, x_n \geq 0$

We first change the inequality constraint to equality type by introducing a slack variable s in the form of s^2, so that it is non-negative.

Now, the constraint changes to

$$h(x_1, x_2, \ldots\ldots, x_n) + s^2 = 0$$

where, $h(x_1, x_2, \ldots\ldots, x_n) = g(x_1, x_2, \ldots\ldots, x_n) - b \leq 0.$

The problem now becomes

Maximise $z = f(x_1, x_2, \ldots\ldots, x_n)$

subject to $h(x_1, x_2, \ldots\ldots, x_n) + s^2 = 0$

$x_1, x_2, \ldots\ldots, x_n \geq 0$

There are now $(n + 1)$ variables and one equality constraint.

We construct the following Lagrangian function.

$$L(x_1, x_2, \ldots\ldots, x_n, S, \lambda) = f(x_1, x_2, \ldots\ldots, x_n) - \lambda\,[\,h(x_1, x_2, \ldots\ldots, x_n) + s^2\,] \quad \ldots\ldots\ldots (1)$$

The necessary conditions for a stationary point are $\quad \ldots\ldots\ldots (2)$

$$\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0, \ldots\ldots, \frac{\partial L}{\partial x_n} = 0,\ \frac{\partial L}{\partial \lambda} = 0,\ \frac{\partial L}{\partial s} = 0$$

Now, from (1), we get,

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1};\quad \frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2};\ \ldots\ldots$$

$$\frac{\partial L}{\partial x_n} = \frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n};\quad \frac{\partial L}{\partial \lambda} = -[h(x_1, x_2, \ldots, x_n) + s^2]$$

and $\dfrac{\partial L}{\partial s} = -2s\lambda$

Using (2), we get the following $(n + 2)$ necessary conditions.

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0,\quad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0,\ \ldots$$

$$\frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n} = 0,\quad h(x_1, x_2, \ldots, x_n) + s^2 = 0$$

and $-2s\lambda = 0.$

Now, from $-2s\lambda = 0$, we get either $s = 0$ or $\lambda = 0$. If $s = 0$, then from the condition $h(x_1, \ldots\ldots, x_n) + s^2 = 0$, we get $h(x_1, x_2, \ldots\ldots, x_n) = 0$. Thus, either $\lambda = 0$ or $h(x_1, x_2, \ldots\ldots, x_n) = 0$ i.e. $\lambda\, h(x_1, x_2, \ldots\ldots, x_n) = 0.$

Since s^2 is positive and $h(x_1, x_2, \ldots\ldots, x_n) + s^2 = 0$, we have

$$h(x_1, x_2, \ldots\ldots, x_n) < 0.$$

Thus, when $\lambda = 0$, $\quad h(x_1, x_2, \ldots\ldots, x_n) < 0$

and when $\lambda > 0$, $\quad h(x_1, x_2, \ldots\ldots, x_n) = 0$

Thus, the necessary conditions for maxima are :-

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0,\quad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0,\ \ldots$$

$$\frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n} = 0,\quad \lambda\, h(x_1, x_2, \ldots, x_n) = 0$$

$$h(x_1, x_2, \ldots\ldots, x_n) \leq 0,\quad \lambda \geq 0.$$

These conditions are called **Kuhn-Tucker** conditions.

If the problem is of **minimisation**, then the last condition charges to

$$\boxed{\lambda < 0}$$

Harold W. Kuhn (1925-2014)

Harold W. Kuhn is a great American mathematician who made important contributions in game theory, non-linear programming. He worked as Professor Emeritus of Mathematics at Princeton University. In 1980 he was awarded John von Neumann prize. He was a life-long friend of John Forbes Nash who got Nobel prize in Economics in 1994. One of his three sons is a professor of mathematics at University of Virginia. He is known for the Kuhn-tucker conditions in non-linear programming and the Hungarian method for the assignment problems.

Albert W. Tucker (1905-1995)

Albert William Tucker was a great Canadian-born American mathematician who made important contributions in topology, game theory and non-linear programming. He got his B.A.. from Toronto University and Ph.D. from Princeton University in 1928. For some time he was a National Research Fellow at Cambridge, Harward and the University of Chicago. From 1933 to 1970 he was on faculty of Princeton University and head of the department of mathematics for twenty years. On a number of papers and models he worked with Harold W. Kuhn. One of his Ph.D. students John Forbes Nash got Nobel prize in Economics in 1994. He is well known for Karush-Kuhn-Tucker conditions in non-linear programming (Historically W. Karush was the first to develop these KKT conditions as part of M.S. thesis at the university of Chicago in 1939. The same conditions were developed independently in 1951 by Kuhn and Tucker.)

Example 1 : Solve the following N.L.P.P.

Maximise $z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

subject to $2x_1 + x_2 \leq 5$

$x_1, x_2 \geq 0.$ **(M.U. 2010, 15, 16)**

Sol. : We rewrite the given problem as

$$f(x_1, x_2) = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$$

and $h(x_1, x_2) = 2x_1 + x_2 - 5.$

Now, Kuhn-Tucker conditions are

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0,$$

$$\lambda h(x_1, x_2) = 0, \quad h(x_1, x_2) \le 0, \quad \lambda \ge 0$$

$\therefore$ We get

$10 - 4x_1 - 2\lambda = 0$ .... (1) $\qquad 4 - 2x_2 - \lambda = 0$ .... (2)

$\lambda(2x_1 + x_2 - 5) = 0$ .... (3) $\qquad 2x_1 + x_2 - 5 \le 0$ .... (4)

$x_1, x_2, \lambda \ge 0$ .... (5)

From (3), we get either $\lambda = 0$ or $2x_1 + x_2 - 5 = 0.$

Case 1 : If $\lambda = 0$, from (1) and (2), we get

$4x_1 = 10$ *i.e.* $x_1 = 5/2$ and $2x_2 = 4$ *i.e.* $x_2 = 2.$

Putting these values of x_1 and x_2 in l.h.s. of (4), we get

l.h.s. $= 5 + 2 - 5 = 2 \le 0$

Thus, these values do not satisfy (4). Hence, $\lambda = 0$ does not yield a feasible solution.

We reject these values.

Case 2 : If $\lambda \ne 0$, $2x_1 + x_2 - 5 = 0$ .... (6)

We now solve the equations (1), (2) and (6). Subtracting twice the second equation from the first.

$10 - 4x_1 - 8 + 4x_2 = 0 \quad \therefore 2x_1 - 2x_2 = 1$

Multiply (6) by 2, $\quad \therefore 4x_1 + 2x_2 = 10$

$\therefore$ By addition, we get $6x_1 = 11 \quad \therefore x_1 = 11/6$

$\therefore x_2 = 5 - 2x_1 = 5 - (11/3) = 4/3$

$\therefore$ From (2), $\lambda = 4 - 2x_2 = 4 - (8/3) = 4/3.$

These values satisfy all the necessary conditions.

$\therefore$ The optimal solution is $x_1 = 11/6,\ x_2 = 4/3.$

$$\therefore z_{Max} = 10 \times \frac{11}{6} + 4 \times \frac{4}{3} - 2\left(\frac{11}{6}\right)^2 - \left(\frac{4}{3}\right)^2 = \frac{91}{6}.$$

Example 2 : Use the Kuhn-Tucker conditions to solve the following N.L.P.P.

Maximise $\quad z = 2x_1^2 - 7x_2^2 + 12x_1x_2$

subject to $\quad 2x_1 + 5x_2 \le 98$

$\quad x_1, x_2 \ge 0.$ **(M.U. 2004, 07, 11, 14)**

Sol. : We rewrite the given problem as

$$f(x_1, x_2) = 2x_1^2 - 7x_2^2 + 12x_1x_2$$

and $h(x_1, x_2) = 2x_1 + 5x_2 - 98.$

Now, Kuhn-Tucker conditions are

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0,$$

$$\lambda h(x_1, x_2) = 0, \quad h(x_1, x_2) \le 0, \quad \lambda \ge 0$$

$\therefore$ We get

$4x_1 + 12x_2 - 2\lambda = 0$ .... (1) $\qquad -14x_2 + 12x_1 - 5\lambda = 0$ .... (2)

$\lambda(2x_1 + 5x_2 - 98) = 0$ .... (3) $\qquad 2x_1 + 5x_2 - 98 \le 0$ .... (4)

$x_1, x_2, \lambda \ge 0$ .... (5)

From (3), we get either $\lambda = 0$ or $2x_1 + 5x_2 - 98 = 0.$

Case 1 : If $\lambda = 0$, from (1) and (2), we get

$4x_1 + 12x_2 = 0$ and $12x_1 - 14x_2 = 0$

Solving these equation we find that $x_1 = 0,\ x_2 = 0$. This solution gives $z = 0$.

Hence, for $\lambda = 0$, feasible solution is not obtained.

We reject these values.

Case 2 : If $\lambda \ne 0$, $2x_1 + 5x_2 - 98 = 0$ .... (6)

To find x_1, x_2 we obtain one more relation between x_1, x_2 by eliminating λ from (1) and (2).

Now, multiply (1) by 5, (2) by 2 and subtract.

$\therefore 20x_1 + 60x_2 + 28x_2 - 24x_1 = 0$

$\therefore -4x_1 + 88x_2 = 0 \qquad \therefore -x_1 + 22x_2 = 0$ .... (7)

Putting $x_1 = 22x_2$ in (6), we get

$44x_2 + 5x_2 = 98 \qquad \therefore x_2 = 2$ and $x_1 = 44.$

Now, from (1), $\quad 176 + 24 = 2\lambda \qquad \therefore \lambda = 100,\ \lambda > 0.$

These values satisfy all the necessary conditions.

$\therefore$ The optimal solution is $x_1 = 44,\ x_2 = 2.$

$\therefore z_{Max} = 2(1936) - 7(4) + 12(44)(2) = 4900.$

Example 3 : Use Kuhn-Tucker conditions to solve the following N.L.P.P.

Maximise $\quad z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

subject to $\quad 3x_1 + 2x_2 \le 6$

$\quad x_1, x_2 \ge 0.$ **(M.U. 2006, 15)**

Sol. : We rewrite the problem as

$$f(x_1, x_2) = 8x_1 + 10x_2 - x_1^2 - x_2^2$$

and $h(x_1, x_2) = 3x_1 + 2x_2 - 6.$

Now, Kuhn-Tucker conditions are

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0, \quad h(x_1, x_2) \le 0$$

$$\lambda h(x_1, x_2) = 0, \quad \lambda \ge 0$$

$\therefore$ We get

$8 - 2x_1 - 3\lambda = 0$ .... (1) $\qquad 10 - 2x_2 - 2\lambda = 0$ .... (2)

$$\lambda(3x_1 + 2x_2 - 6) = 0 \qquad \text{.........(3)} \qquad 3x_1 + 2x_2 - 6 \le 0 \qquad \text{.........(4)}$$

$$x_1, x_2, \lambda \ge 0 \qquad \text{.........(5)}$$

From (3), we get either $\lambda = 0$ or $3x_1 + 2x_2 - 6 = 0$.

Case 1 : If $\lambda = 0$, from (1) and (2), we get

$$8 - 2x_1 = 0 \text{ and } 10 - 2x_2 = 0 \qquad \therefore\ x_1 = 4,\ x_2 = 5.$$

But then for $x_1 = 4$, $x_2 = 5$, the condiiion (3) is not satisfied.

$\therefore$ Hence, $\lambda = 0$ does not yield a feasible solution.

We reject these values.

Case 2 : If $\lambda \ne 0$, $3x_1 + 2x_2 - 6 = 0$ ....(6)

To find x_1, x_2 we obtain one more relation between x_1, x_2 by eliminating λ from (1) and (2).

Now, multiply (1) by 2, (2) by 3 and subtract.

$$\therefore\ 16 - 4x_1 - 30 + 6x_2 = 0 \qquad \therefore\ -2x_1 + 3x_2 - 7 = 0$$

Multiply (6) by 3 and (7) by 2 and subtract.

$$\therefore\ 9x_1 - 18 + 4x_1 + 14 = 0 \qquad \therefore\ 13x_1 = 4 \qquad \therefore\ x_1 = 4/13$$

$$\therefore \text{ From (7)},\quad -\frac{8}{13} + 3x_2 - 7 = 0 \quad \therefore\ 3x_2 = \frac{99}{13} \quad \therefore\ x_2 = \frac{33}{13}$$

$$\text{Now from (1)},\quad 4 \times \frac{4}{13} + 12 \times \frac{33}{13} = 2\lambda \quad \therefore\ 2\lambda = \frac{412}{33} \quad \therefore\ \lambda > 0$$

These values satisfy all the necessary conditions.

$\therefore$ The optimal solution is $x_1 = \frac{4}{13}$, $x_2 = \frac{33}{13}$.

$$\therefore\ z_{Max} = 8\left(\frac{4}{13}\right) + 10\left(\frac{33}{13}\right) - \frac{16}{169} - \frac{1089}{169} = \frac{3601}{169} = \frac{277}{13} = 21{\cdot}3$$

Example 4 : Solve the following N.L.P.P.

Miminise $\quad z = x_1^3 - 4x_1 - 2x_2$

subject to $\quad x_1 + x_2 \le 1$

$\quad x_1, x_2 \ge 0.$

Sol. : We rewrite the problem as

$$f(x_1, x_2) = x_1^3 - 4x_1 - 2x_2$$

and $\quad h(x_1, x_2) = x_1 + x_2 - 1.$

The Kuhn-Tucker conditions for minima are

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\lambda\, h(x_1, x_2) = 0, \quad h(x_1, x_2) \le 0, \quad \lambda \le 0.$$

$\therefore$ We get

$$3x_1^2 - 4 - \lambda = 0 \qquad \text{.........(1)} \qquad -2 - \lambda = 0 \qquad \text{.........(2)}$$

$$\lambda(x_1 + x_2 - 1) = 0 \qquad \text{.........(3)} \qquad x_1 + x_2 - 1 \le 0 \qquad \text{.........(4)}$$

$$\lambda \le 0 \qquad \text{.........(5)}$$

From (2), we got $\lambda = -2$ and from (3), we got, since $\lambda \ne 0$, $x_1 + x_2 - 1 = 0$.

From (1), we got $\quad 3x_1^2 = 4 + \lambda = 4 - 2 = 2 \qquad \therefore\ x_1 = \sqrt{2/3}$.

Since $x_1 + x_2 = 1$, $\quad x_2 = 1 - x_1 = 1 - \sqrt{2/3}$.

These values satisfy all the above conditions (1) to (5).

$\therefore$ The optimal solution is $x_1 = \sqrt{2/3}$, $\quad x_2 = 1 - \sqrt{2/3}$.

$$\therefore\ z_{Min} = \frac{2}{3}\sqrt{\frac{2}{3}} - 4 \cdot \sqrt{\frac{2}{3}} - 2\left(1 - \sqrt{\frac{2}{3}}\right)$$

$$= \frac{2}{3}(0{\cdot}82) - 4(0{\cdot}81) - 2(1 - 0{\cdot}82) = -3{\cdot}093.$$

EXERCISE - V

Solve the following non-linear programming problems.

1. Maximise $\quad z = x_1 + 3x_1^2 - x_2^3 + 2x_2 + 10$

   subject to $\quad x_1 + x_2 \le 0$

   $\quad x_1, x_2 \ge 0 \qquad$ [**Ans. :** $x_1 = 1 + \sqrt{2/3}$, $x_2 = 2 - \sqrt{2/3}$, $z_{Max} = 18{\cdot}09$]

2. Maximise $\quad z = 16x_1 + 6x_2 - 2x_1^2 - x_2^2 - 17$

   subject to $\quad 2x_1 + x_2 \le 8$

   $\quad x_1, x_2 \ge 0 \qquad$ [**Ans. :** $x_1 = 3$, $x_2 = 2$, $z_{Max} = 21$]

3. Maximise $\quad z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

   subject to $\quad 3x_1 + 2x_2 \le 6$

   $\quad x_1, x_2 \ge 0 \qquad$ [**Ans. :** $x_1 = 17/13$, $x_2 = 46/13$, $z_{Max} = 411/13$]

4. Maximise $\quad z = 2x_1^2 - 7x_2^2 - 16x_1 + 2x_2 + 12x_1x_2 + 7$

   subject to $\quad 2x_1 + 5x_2 \le 105$

   $\quad x_1, x_2 \ge 0 \qquad$ [**Ans. :** $x_1 = 45$, $x_2 = 3$, $z_{Max} = 4900$]

5. Maximise $\quad z = 6x_1^2 + 5x_2^2$

   subject to $\quad x_1 + 5x_2 \le 3$

   $\quad x_1, x_2 \ge 0 \qquad$ [**Ans. :** $x_1 = 3/31$, $x_2 = 18/31$, $z_{Max} = 54/31$]

6. Kuhn-Tucker Conditions for A General Non-linear Programming Problem

Consider the problem :-

Maximise $\quad z = f(X)$

subject to $\quad g_i(X) \le b_i, \quad i = 1, 2, \ldots\ldots, n$

$\quad X \ge 0.$

where, $f(X) = f(x_1, x_2, \ldots\ldots, x_n)$.

We first write the constraints as $h_i(X) = g_i(X) - b_i \le 0$.

These constraints can be written in the form of equalities by introducing slack variables as

$$h(X) + s_i^2 = 0, \quad i = 1, 2, ..., m.$$

We then construct Lagrangian function.

$$L(X, s, \lambda) = f(X) - \sum_{i=1}^{n} \lambda_i [h_i(X) + s_i^2] \quad ....(1)$$

The necessary conditions for maxima are

$$\frac{\partial L}{\partial x_j} = \frac{\partial f(X)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h_i(X)}{\partial x_j} = 0 \quad ....(2)$$

$$\frac{\partial L}{\partial \lambda_i} = -[h_i(X) + s_i^2] = 0 \quad ....(3)$$

$$\frac{\partial L}{\partial s_i} = -2 s_i \lambda_i = 0$$

$i = 1, 2, ..., n;\ j = 1, 2, ..., m.$

The condition $\frac{\partial L}{\partial s_i} = 0$ i.e. $s_i \lambda_i = 0$ gives either $s_i = 0$ or $\lambda_i = 0$.

If $s_i = 0$, from (2), we get $h_i(X) = 0$. Thus, either $\lambda_i = 0$ or $h_i(X) = 0$, which means ....(4)

$$\lambda_i h_i(X) = 0 \quad ....(5)$$

Since s_i^2 are non-negative, from (2), we get $h_i(X) \le 0$

From (4) and (5), we can infer that

when $h_i(X) < 0$, $\lambda_i = 0$ and when $\lambda_i > 0$, $h_i(X) = 0$

Thus, we get the following conditions.

$$\left.\begin{array}{l} \frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} - ..... - \lambda_n \frac{\partial h_n}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} - ..... - \lambda_n \frac{\partial h_n}{\partial x_2} = 0 \\ .... \\ \frac{\partial f}{\partial x_n} - \lambda_1 \frac{\partial h_1}{\partial x_n} - \lambda_2 \frac{\partial h_2}{\partial x_n} - ..... - \lambda_n \frac{\partial h_n}{\partial x_n} = 0 \end{array}\right\} \quad ....(1)$$

$$\left.\begin{array}{l} \lambda_1 h_1(x_1, x_2, ..., x_n) = 0,\ \lambda_2 h_2(x_1, x_2, ..., x_n) = 0, \\ ......,\ \lambda_n h_n(x_1, x_2, ..., x_n) = 0 \end{array}\right\} \quad ....(2)$$

$$\left.\begin{array}{l} h_1(x_1, x_2, ..., x_n) \le 0,\ h_2(x_1, x_2, ..., x_n) \le 0, \\ ......,\ h_n(x_1, x_2, ..., x_n) \le 0 \end{array}\right\} \quad ....(3)$$

$$x_1, x_2, ..., x_n \ge 0 \quad ....(4)$$

$$\lambda_1, \lambda_2, ..., \lambda_n \ge 0 \quad ....(5)$$

Kuhn-Tucker Conditions For minimisation Problem

For minimisation the last condition changes to

$$\boxed{\lambda_1, \lambda_2, ..., \lambda_n \le 0}$$

Note ....

It should be carefully noted that the constraints for maxima as well as for minima should be expressed as

$$\boxed{g_i(X) \le b_i}$$

Example 1 : Find the optimum value of the object function

$$z = 10x_1 + 10x_2 - x_1^2 - x_2^2$$

subject to $x_1 + x_2 \le 14$

$-x_1 + x_2 \le 6;\ x_1, x_2 \ge 0.$

Sol. : We write the given problem as

$$f(x_1, x_2) = 10x_1 + 10x_2 - x_1^2 - x_2^2$$

$$h_1(x_1, x_2) = x_1 + x_2 - 14$$

$$h_2(x_1, x_2) = -x_1 + x_2 - 6$$

The Kuhn-Tucker conditions are

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0 \qquad \therefore\ 10 - 2x_1 - \lambda_1 + \lambda_2 = 0 \quad ....(1)$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0 \qquad \therefore\ 10 - 2x_2 - \lambda_1 - \lambda_2 = 0 \quad ....(2)$$

$\lambda_1 (x_1 + x_2 - 14) = 0$ ....(3) $\quad \lambda_2(-x_1 + x_2 - 6) = 0$ ....(4)

$x_1 + x_2 - 14 \le 0$ ....(5) $\quad -x_1 + x_2 - 6 \le 0$ ....(6)

$x_1, x_2 \ge 0$ ....(7)

For maxima $\lambda_1, \lambda_2 \ge 0$ ....(8)

For minima $\lambda_1, \lambda_2 \le 0$ ....(9)

Now, depending upon the values of λ_1 and λ_2, the following cases arise.

Case 1 : $\lambda_1 = 0, \lambda_2 = 0$

From (1) and (2), we get $10 - 2x_1 = 0 \quad \therefore\ x_1 = 5$

$10 - 2x_2 = 0 \quad \therefore\ x_2 = 5$

These values satisfy (5), (6) and (7). They also satisfy (8).

We cannot immediately conclude that $x_1 = 5, x_2 = 5$ is a maxima because $\lambda_1 = 0, \lambda_2 = 0$ can also give a minima.

(The conditions for minima are $\lambda_1 \le 0, \lambda_2 \le 0$ equality sign included.)

Hence, we test the Hessian matrix for the given objective function

$$H = \begin{bmatrix} \frac{\partial^2 z}{\partial x_1^2} & \frac{\partial^2 z}{\partial x_1 \partial x_2} \\ \frac{\partial^2 z}{\partial x_2 \partial x_1} & \frac{\partial^2 z}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

Now, the principal minors are $A_1 = |a_{11}| = -2$ and $A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4.$

Since the principal mi

$\therefore$ z is a maxima at

$\therefore$ $z_{Max} = 10$ (5)

Case 2 : $\lambda_1 = 0, \lambda_2 \ne 0$

To find x_1, x_2 in thi

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Adding the two e

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When $x_1 = 2$,

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Case 3 : $\lambda_1 \ne 0$,

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Case 4 : W

Now

Add

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Since the principal minors are alternately positive and negative (x_0, y_0) is a maxima.

$\therefore$ z is a **maxima** at $x_1 = 5,\ x_2 = 5$.

$\therefore \quad z_{Max} = 10\,(5) + 10\,(5) - 5^2 - 5^2 = 50.$

Case 2 : $\lambda_1 = 0,\ \lambda_2 \neq 0$

To find x_1, x_2 in this case we first eliminate λ_2 from

$$10 - 2x_1 + \lambda_2 = 0 \text{ and } 10 - 2x_2 - \lambda_2 = 0$$

Adding the two equalities, we get,

$$20 - 2x_1 - 2x_2 = 0 \qquad \therefore\ x_1 + x_2 = 10$$

Since $\lambda_2 \neq 0$, we get from (4), $\qquad -x_1 + x_2 = 6$

Adding the two, $\quad 2x_2 = 16 \quad \therefore\ x_2 = 8 \quad \therefore\ x_1 = x_2 - 6 = 2$

When $x_1 = 2$, we have $\lambda_2 = -6$.

Since λ_2 is negative, we reject this pair.

Case 3 : $\lambda_1 \neq 0,\ \lambda_2 = 0$

To find x_1, x_2 in this case, we first eliminate λ_1 from

$$10 - 2x_1 - \lambda_1 = 0 \text{ and } 10 - 2x_2 - \lambda_1 = 0$$

By subtraction, we get $\quad -2x_1 + 2x_2 = 0 \quad \therefore\ x_1 = x_2$

Since $\lambda_1 \neq 0$, we get from (3)

$$x_1 + x_2 = 14 \quad \therefore\ 2x_2 = 14 \quad \therefore\ x_2 = 7 \quad \therefore\ x_1 = 7$$

When $x_1 = 7,\ x_2 = 7$, we get $\lambda_1 = -4$.

Again since λ is negative, we reject this pair.

Case 4 : When $\lambda_1 \neq 0,\ \lambda_2 \neq 0$.

Now, we get from (3) and (4), $\quad x_1 + x_2 = 14$ and $-x_1 + x_2 = 6$

Adding we get $\quad 2x_2 = 20 \quad \therefore\ x_2 = 10 \quad \therefore\ x_1 = 14 - 10 = 4.$

For these values of x_1, x_2 we get from (1) and (2)

$$-\lambda_1 + \lambda_2 = -10 + 2x_1 = -2 \quad \text{and} \quad -\lambda_1 - \lambda_2 = -10 + 2x_2 = 10$$

Adding the two, we get

$$-2\lambda_1 = 8 \qquad \therefore\ \lambda_1 = -4 \quad \text{and} \quad \lambda_2 = \lambda_1 - 2 = -6$$

Since both λ_1 and λ_2 are negative we reject this pair.

$\therefore \quad z_{Max} = 50.$

Example 2 : Using the Kuhn-Tucker conditions solve the following problem.

Maximise $\quad z = 10x_1 + 10x_2 - x_1^2 - x_2^2$

subject to $\quad x_1 + x_2 \leq 8$

$\qquad\qquad -x_1 + x_2 \leq 5.$

Sol. : *We write the given problem as*

$$f(x_1, x_2) = 10x_1 + 10x_2 - x_1^2 - x_2^2$$

$$h_1(x_1, x_2) = x_1 + x_2 - 8$$

$$h_2(x_1, x_2) = -x_1 + x_2 - 5$$

The Kuhn-Tucker conditions for maxima are

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0 \qquad \therefore\ 10 - 2x_1 - \lambda_1 + \lambda_2 = 0 \quad \text{.......... (1)}$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0 \qquad \therefore\ 10 - 2x_2 - \lambda_1 - \lambda_2 = 0 \quad \text{.......... (2)}$$

$$\lambda_1 (x_1 + x_2 - 8) = 0 \quad \text{.......... (3)} \qquad \lambda_2 (-x_1 + x_2 - 5) = 0 \quad \text{.......... (4)}$$

$$x_1 + x_2 - 8 \leq 0 \quad \text{.......... (5)} \qquad -x_1 + x_2 - 5 \leq 0 \quad \text{.......... (6)}$$

$$x_1, x_2 \geq 0 \quad \text{.......... (7)} \qquad \lambda_1, \lambda_2 \geq 0 \quad \text{.......... (8)}$$

Now, depending upon the values of λ_1 and λ_2 the following cases arise.

Case 1 : $\lambda_1 = 0,\ \lambda_2 = 0$

From (1) and (2), we get $\quad 10 - 2x_1 = 0 \quad \therefore\ x_1 = 5$

$\qquad\qquad\qquad\qquad\qquad 10 - 2x_2 = 0 \quad \therefore\ x_2 = 5$

But these values do not satisfy (5).

Hence, we reject this pair.

Case 2 : $\lambda_1 = 0,\ \lambda_2 \neq 0$

To find x_1, x_2 we first eliminate λ_2 from

$$10 - 2x_1 + \lambda_2 = 0 \quad \text{and} \quad 10 - 2x_2 - \lambda_2 = 0$$

Adding the two equalities, we get

$$20 - 2x_1 - 2x_2 = 0 \qquad \therefore\ x_1 + x_2 = 10$$

Since $\lambda_2 \neq 0$, we get from (4), $\qquad -x_1 + x_2 = 5.$

Adding the two, we get $\quad 2x_2 = 15$

$\therefore\ x_2 = 7{\cdot}5 \quad \therefore\ x_1 = 10 - 7{\cdot}5 = 2{\cdot}5$

But these values do not satisfy (5). Hence, we reject this pair.

Case 3 : $\lambda_1 \neq 0,\ \lambda_2 = 0$

To find x_1, x_2 in this case, we first eliminate λ_1 from

$$10 - 2x_1 - \lambda_1 = 0 \quad \text{and} \quad 10 - 2x_2 - \lambda_1 = 0$$

By subtraction, we get $\quad -2x_1 + 2x_2 = 0 \quad \therefore\ x_1 = x_2$

Since $\lambda_1 \neq 0$, we get from (3),

$$x_1 + x_2 = 8 \quad \therefore\ 2x_1 = 8 \quad \therefore\ x_1 = 4 \quad \therefore\ x_2 = 4.$$

Now, from $10 - 2x_1 - \lambda_1 = 0$, we get $\quad \lambda_1 = 10 - 2x_1 = 2.$

Also $x_1 = 4,\ x_2 = 4$ satisfy the conditions (5) and (6).

The condition (8) also is satisfied.

Hence, z is a maxima at $x_1 = 4$ and $x_2 = 4$.

$\therefore \quad z_{Max} = 10\,(4) + 10\,(4) - 16 - 16 = 48.$

Case 4 : $\lambda_1 \neq 0,\ \lambda_2 \neq 0$

Now, we get from (3) and (4)

$$x_1 + x_2 = 8 \quad \text{and} \quad -x_1 + x_2 = 5$$

Adding the two, we get $\quad 2x_2 = 13$

$\therefore\ x_2 = 6{\cdot}5 \quad \therefore\ x_1 = 8 - 6{\cdot}5 = 1{\cdot}5.$

For these values of x_1 and x_2, we get from (1) and (2)

$$-\lambda_1 + \lambda_2 = -10 + 2x_1 = -7$$

$$-\lambda_1 - \lambda_2 = -10 + 2x_2 = 3$$

Adding the two, we get $-2\lambda_1 = -4$ $\quad \therefore \lambda_1 = 2$

and $\lambda_2 = -\lambda_1 - 3 = -5.$

We reject this pair, $\therefore z_{Max} = 48.$

Example 3 : Using the Kuhn-Tucker conditions, solve the following N.L.P.P.

Maximise $z = x_1^2 + x_2^2$

subject to $x_1 + x_2 - 4 \leq 0$

$2x_1 + x_2 - 5 \leq 0$

$x_1, x_2 \geq 0.$

Sol. : We rewrite the problem as

$$f(x_1, x_2) = x_1^2 + x_2^2$$

$$h_1(x_1, x_2) = x_1 + x_2 - 4$$

$$h_2(x_1, x_2) = 2x_1 + x_2 - 5$$

Kuhn-Tucker conditions for maxima are

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0 \quad \therefore 2x_1 - \lambda_1 - 2\lambda_2 = 0 \quad \text{.......... (1)}$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_2}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0 \quad \therefore 2x_2 - \lambda_1 - \lambda_2 = 0 \quad \text{.......... (2)}$$

$\lambda_1 (x_1 + x_2 - 4) = 0$ .... (3) $\quad \lambda_2 (2x_1 + x_2 - 5) = 0$ .... (4)

$x_1 + x_2 - 4 \leq 0$ .... (5) $\quad 2x_1 + x_2 - 5 \leq 0$ .... (6)

$x_1, x_2 \geq 0$ .... (7) $\quad \lambda_1, \lambda_2 \geq 0$ .... (8)

Now, depending upon the values of λ_1, λ_2 the following cases arise.

Case 1 : $\lambda_1 = 0$ and $\lambda_2 = 0$

In this case from (1) and (2), we get,

$2x_1 = 0 \quad \therefore x_1 = 0$ and $2x_2 = 0 \quad \therefore x_2 = 0$

This is a trivial solution.

Case 2 : $\lambda_1 = 0$ and $\lambda_2 \neq 0$

From (1) and (2), we get $2x_1 = 2\lambda_2$ and $2x_2 = \lambda_2$

From (4), we get $2x_1 + x_2 = 5$

$$\therefore 2\lambda_2 + \frac{\lambda_2}{2} = 5 \quad \therefore \frac{5\lambda_2}{2} = 5 \quad \therefore \lambda_2 = 5$$

$$\therefore x_1 = 5, \; x_2 = \frac{5}{2}.$$

But these values do not satisfy (5) and (6).

Hence, we reject this pair.

Case 3 : $\lambda_1 \neq 0, \; \lambda_2 = 0$

From (1) and (2), we get $2x_1 = \lambda_1, \quad 2x_2 = \lambda_1 \quad \therefore x_1 = x_2$

From (3), we get, $x_1 + x_2 = 4.$

Putting $x_2 = x_1, \; 2x_1 = 4 \quad \therefore x_1 = 2$

$\therefore x_2 = x_1 = 2 \quad \therefore \lambda_1 = 2x_1 = 4$

Since these values satisfy (5), (6), (7) and (8), we get,

$$z_{max} = x_1^2 + x_2^2 = 4 + 4 = 8.$$

Case 4 : $\lambda_1 \neq 0, \; \lambda_2 \neq 0$

Then we get from (3) and (4), $x_1 + x_2 = 4$ and $2x_1 + x_2 = 5$

and solving these equations, we get $x_1 = 1$ and $x_2 = 3.$

Putting these values in (1) and (2), we get

$$\lambda_1 + 2\lambda_2 = 2 \text{ and } \lambda_1 + \lambda_2 = 6$$

Solving these we get $\lambda_2 = -4$ and $\lambda_1 = 10$

$\therefore \lambda_1$ is positive and λ_2 is negative.

Hence, we reject this pair.

The required solution is $x_1 = 2, \; x_2 = 2$ and $z_{max} = 8.$

Example 4 : Using the Kuhn-Tucker conditions, solve the following N.L.P.P.

Minimise $z = x_1^2 - 2x_1 - x_2$

subject to $2x_1 + 3x_2 \leq 6$

$2x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0.$

Sol. : We write the given problem as

$$f(x_1, x_2) = x_1^2 - 2x_1 - x_2$$

$$h_1(x_1, x_2) = 2x_1 + 3x_2 - 6$$

$$h_2(x_1, x_2) = 2x_1 + x_2 - 4$$

Kuhn-Tucker conditions for minima are

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0 \quad \therefore 2x_1 - 2 - 2\lambda_1 - 2\lambda_2 = 0 \quad \text{.......... (1)}$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0 \quad \therefore -1 - 3\lambda_1 - \lambda_2 = 0 \quad \text{.......... (2)}$$

$\lambda_1 (2x_1 + 3x_2 - 6) = 0$ .... (3) $\quad \lambda_2 (2x_1 + x_2 - 4) = 0$ .... (4)

$2x_1 + 3x_2 - 6 \leq 0$ .... (5) $\quad 2x_1 + x_2 - 4 \leq 0$ .... (6)

$x_1, x_2 \geq 0$ .... (7) $\quad \lambda_1, \lambda_2 \leq 0$ .... (8)

Now, depending upon the values of λ_1, λ_2 the following cases arise.

Case 1 : $\lambda_1 = 0, \lambda_2 = 0$

In this case from (1) and (2), we get

$$2x_1 - 2 = 0 \text{ and } -1 = 0 \text{ which is absurd.}$$

Hence, we reject this pair.

Case 2 : $\lambda_1 = 0, \lambda_2 \neq 0$

From (2), we get $\lambda_2 = -1$.

Putting this value of $\lambda_2 = -1$ in (1), we get

$$2x_1 - 2 + 2 = 0 \quad \therefore x_1 = 0$$

Now, from (4), we get

$$2x_1 + x_2 - 4 = 0 \quad \therefore x_2 = 4.$$

But these values although satisfy (6), do not satisfy (5).

Hence, we reject this pair.

Case 3 : $\lambda_1 \neq 0, \ \lambda_2 = 0$

From (2), we get $-1 - 3\lambda_1 = 0 \quad \therefore \lambda_1 = -1/3$.

Putting this value in (1), we get

$$2x_1 - 2 + \frac{2}{3} = 0 \quad \therefore 2x_1 = \frac{4}{3} \quad \therefore x_1 = \frac{2}{3}$$

Then, from (3), we get,

$$2x_1 + 3x_2 = 6 \quad \therefore \frac{4}{3} + 3x_2 = 6 \quad \therefore 3x_2 = \frac{14}{3} \quad \therefore x_2 = \frac{14}{9}.$$

These values satisfy the conditions (5), (6), (7) and (8).

$$\therefore z_{Min} = \left(\frac{2}{3}\right)^2 - 2\left(\frac{2}{3}\right) - \frac{14}{9} = -\frac{22}{9}.$$

Case 4 : $\lambda_1 \neq 0, \lambda_2 \neq 0$

If $\lambda_1 \neq 0, \lambda_2 \neq 0$, we get from (3) and (4)

$$2x_1 + 3x_2 - 6 = 0 \quad \text{and} \quad 2x_1 + x_2 - 4 = 0.$$

Solving these two equations, we get $x_1 = 3/2, \ x_2 = 1$.

For those values of x_1, x_2, (5), (6) and (7) are satisfied.

Now, from (1) and (2), we get

$$2\lambda_1 + 2\lambda_2 = 2x_1 - 2 = 1 \quad \text{and} \quad 3\lambda_1 + \lambda_2 = -1$$

$$\therefore \lambda_1 = -\frac{3}{4}, \quad \lambda_2 = \frac{5}{4}$$

We reject this pair. Hence, the solution is $z_{Min} = -\dfrac{22}{9}$.

Example 5 : Use the Kuhn-Tucker conditions to solve the following N.L.P.P.

Maximise $z = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$

subject to $x_1 + 3x_2 \leq 6$

$5x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0.$ (M.U. 1997, 99, 2003, 10)

Sol. : We rewrite the given problem as,

$$f(x_1, x_2) = 2x_1 + 3x_2 - x_1^2 - 2x_2^2,$$
$$h_1(x_1, x_2) = x_1 + 3x_2 - 6,$$
$$h_2(x_1, x_2) = 5x_1 + 2x_2 - 10.$$

Kuhn-Tucker conditions for maxima are

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0 \quad \therefore 2 - 2x_1 - \lambda_1 - 5\lambda_2 = 0 \quad \text{.......... (1)}$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0 \quad \therefore 3 - 4x_2 - 3\lambda_1 - 2\lambda_2 = 0 \quad \text{.......... (2)}$$

$$\lambda_1(x_1 + 3x_2 - 6) = 0 \quad \text{.......... (3)} \qquad \lambda_2(5x_1 + 2x_2 - 10) = 0 \quad \text{.......... (4)}$$

$$x_1 + 3x_2 - 6 \leq 0 \quad \text{.......... (5)} \qquad 5x_1 + 2x_2 - 10 \leq 0 \quad \text{.......... (6)}$$

$$x_1, x_2 \geq 0 \quad \text{.......... (7)} \qquad \lambda_1, \lambda_2 \geq 0 \quad \text{.......... (8)}$$

Now, depending upon the values of λ_1 and λ_2 the following cases arise.

Case 1 : $\lambda_1 = 0$ and $\lambda_2 = 0$

In this case from (1) and (2), we get,

$$2 - 2x_1 = 0 \quad \therefore x_1 = 1 \text{ and } 3 - 4x_2 = 0 \quad \therefore x_2 = 3/4$$

These values satisfy (5), (6) and (7). They also satisfy (8). But we cannot immediately conclude that $x_1 = 1, x_2 = 3/4$ is a maxima, because $\lambda_1 = 0, \lambda_2 = 0$ can also give a minima.

(The conditions for minima are $\lambda_1 \leq 0, \lambda_2 \leq 0$, the equality sign included.)

Hence, we test the Hessian matrix for the objective function

$$H = \begin{bmatrix} \dfrac{\partial^2 z}{\partial x_1^2} & \dfrac{\partial^2 z}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 z}{\partial x_2 \partial x_1} & \dfrac{\partial^2 z}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

Now, the principal minors are

$$A_1 = |a_{11}| = -2 \quad \text{and} \quad A_2 = \begin{vmatrix} -2 & 0 \\ 0 & -4 \end{vmatrix} = 8$$

Since the principal minors are alternately negative, positive $x_1 = 1, x_2 = 3/4$ gives a maxima

$$z_{max} = 2 + 3\left(\frac{3}{4}\right) - 1 - 2\left(\frac{9}{16}\right) = \frac{17}{8}.$$

Case 2 : $\lambda_1 = 0$ and $\lambda_2 \neq 0$

To find x_1, x_2 in this case, we have first to eliminate λ_2 from

$$2 - 2x_1 - 5\lambda_2 = 0 \text{ and } 3 - 4x_2 - 2\lambda_2 = 0.$$

Multiply the first by 2, the second by 5 and subtract.

$$\therefore 4 - 4x_1 - 15 + 20x_2 = 0 \quad \therefore 4x_1 - 20x_2 = -11.$$

Since $\lambda_2 \neq 0$, from (4), we get, $5x_1 + 2x_2 = 10.$

Solving the two equations, we get

$$x_1 = \frac{89}{54} = 1.648 \text{ and } x_2 = \frac{95}{108} = 0.880$$

Now, for $x_1 = 89/54$ and $x_2 = 95/108$ and $\lambda_1 = 0$, we get from (1)

$$2 - 2 \times \frac{89}{54} - 5\lambda_2 = 0 \quad \therefore 5\lambda_2 = \frac{108 - 178}{54} = -\frac{70}{54}, -\text{ve}$$

Since λ_2 is negative, we reject this pair.

Case 3 : $\lambda_1 \neq 0$ and $\lambda_2 = 0$

To find x_1, x_2 in this case, we have, first to eliminate λ_1 from

$$2 - 2x_1 - \lambda_1 = 0 \quad \text{and} \quad 3 - 4x_2 - 3\lambda_1 = 0$$

Multiply the first by 3 and from it subtract the second.

$$\therefore \; 6 - 6x_1 - 3 + 4x_2 = 0 \qquad \therefore \; 6x_1 - 4x_2 = 3$$

Now, since $\lambda_1 \neq 0$ from (3) we have, $x_1 + 3x_2 = 6$.

Solving the two equations, we get $x_1 = 3/2$, $x_2 = 3/2$.

But these values, although satisfy (5), do not satisfy (6).

Hence, we reject these values.

Case 4 : $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$

From the equations (3) and (4), we get,

$$x_1 + 3x_2 = 6 \quad \text{and} \quad 5x_1 + 2x_2 = 10$$

Solving the two equations, we get,

$$x_1 = \frac{20}{13} = 1.538 \,;\; x_2 = \frac{18}{13} = 1.385$$

Now, with these values of x_1, x_2, we solve (1) and (2) for λ_1 and λ_2.

$$\therefore \; \lambda_1 + 5\lambda_2 = -\frac{14}{13} \quad \text{and} \quad 3\lambda_1 + 2\lambda_2 = -\frac{33}{13}$$

Solving the two equations, we get,

$$\lambda_1 = -\frac{137}{169} = -0.81 \quad \text{and} \quad \lambda_2 = -\frac{9}{169} = -0.053$$

Since λ_1, λ_2 are negative, we reject this pair.

Hence, $\quad z_{max} = \frac{17}{8}$.

Example 6 : Using Kuhn-Tucker conditions solve the following N.L.P.P.

Maximise $\quad z = -2x_1^2 - 2x_2^2 + 12x_1 + 21x_2 + 2x_1x_2$

subject to $\quad x_1 + x_2 \leq 10$

$$x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

Sol. : We rewrite the given problem as

$$f(x_1, x_2) = -2x_1^2 - 2x_2^2 + 12x_1 + 21x_2 + 2x_1x_2$$

$$h_1(x_1, x_2) = x_1 + x_2 - 10$$

$$h_2(x_1, x_2) = x_2 - 8$$

Kuhn-Tucker conditions for maxima are

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0 \qquad \therefore \; -4x_1 + 12 + 2x_2 - \lambda_1 + 0 = 0 \quad \text{.......... (1)}$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0 \qquad \therefore \; -4x_2 + 21 + 2x_1 - \lambda_1 - \lambda_2 = 0 \quad \text{.......... (2)}$$

$$\lambda_1(x_1 + x_2 - 10) = 0 \quad \text{.......... (3)} \qquad \lambda_2(x_2 - 8) = 0 \quad \text{.......... (4)}$$

$$x_1 + x_2 - 10 \leq 0 \quad \text{.......... (5)} \qquad x_2 - 8 \leq 0 \quad \text{.......... (6)}$$

$$x_1, x_2 \geq 0 \quad \text{.......... (7)} \qquad \lambda_1, \lambda_2 \geq 0 \quad \text{.......... (8)}$$

Now, depending upon the values of λ_1 and λ_2 the following cases arise.

Case 1 : $\lambda_1 = 0$ and $\lambda_2 = 0$.

In this case from (1) and (2), we get

$$-4x_1 + 12 + 2x_2 = 0$$

and $\quad -4x_2 + 2x_1 + 21 = 0$

Solving these equations, we get, $\quad x_2 = 9$ and $x_1 = 15/2$.

But $\quad x_1 + x_2 - 10 = 9 + \frac{15}{2} - 10 = 6.5 \not\leq 0$

Since the condition (5) is not satisfied, we reject this pair.

Case 2 : $\lambda_1 = 0$ and $\lambda_2 \neq 0$

$\therefore$ From (1) and (2), we get

$$-4x_1 + 12 + 2x_2 = 0$$

and $\quad -4x_2 + 21 + 2x_1 - \lambda_2 = 0$

From (4), we get, since $\lambda_2 \neq 0$,

$$x_2 - 8 = 0 \qquad \therefore \; x_2 = 8.$$

$$\therefore \; 4x_1 = 12 + 2x_2 = 28 \qquad \therefore \; x_1 = 7$$

But $\quad x_1 + x_2 - 10 = 8 + 7 - 10 = 5 \not\leq 0$

Since, the condition (5) is not satisfied, we reject this pair.

Case 3 : $\lambda_1 \neq 0$ and $\lambda_2 = 0$

From (1) and (2), we get

$$-4x_1 + 12 + 2x_2 - \lambda_1 = 0 \quad \text{.......... (8)}$$

$$-4x_2 + 21 + 2x_1 - \lambda_1 = 0 \quad \text{.......... (9)}$$

From (3), we get $\quad x_1 + x_2 = 10 \quad$ .... (10)

Also from the above two equations (8), (9) by subtraction, we get,

$$6x_2 - 6x_1 = 9 \quad \text{.......... (11)}$$

Solving (10) and (11), we get

$$x_2 = \frac{23}{4} \qquad \therefore \; x_1 = \frac{17}{4}.$$

Now, from (8) $\quad \lambda_1 = -4x_1 + 12 + 2x_2$

$$\therefore \; \lambda_1 = -17 + 12 + \frac{23}{2} = \frac{13}{2} > 0$$

Since these values satisfy (5), (6), (7) and (8), we accept the solution.

$$\therefore \; z_{max} = -2\left(\frac{17}{4}\right)^2 - 2\left(\frac{23}{4}\right)^2 + 12\left(\frac{17}{4}\right) + 21\left(\frac{23}{4}\right) + 2\left(\frac{17}{4}\right)\left(\frac{23}{4}\right)$$

$$= \frac{1734}{16} = 108.375$$

Case 4 : $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$

In this case from (3) and (4), we get

$x_1 + x_2 = 10$ and $x_2 = 8$ $\therefore x_1 = 2,\ x_2 = 8.$

Putting these values in (1) and (2), we get

$$\lambda_1 = -4x_1 + 12 + 2x_2 = -8 + 12 + 16 = 20$$

and $$\lambda_2 = 2x_1 + 21 - 4x_2 - \lambda_1 = 4 + 21 - 32 - 20 = -27.$$

Since this violates the condition (8) we reject this pair.

$\therefore\ z_{Max} = 108{\cdot}375.$

Example 7 : Using the Kuhn-Tucker conditions solve the following N.L.P.P.

Minimise $z = 7x_1^2 + 5x_2^2 - 6x_1$

subject to $x_1 + 2x_2 \le 10$ (M.U. 1999)

$x_1 + 3x_2 \le 9$

$x_1, x_2 \ge 0.$

Sol. : We rewrite the given problem as,

$$f(x_1, x_2) = 7x_1^2 + 5x_2^2 - 6x_1,$$
$$h_1(x_1, x_2) = x_1 + 2x_2 - 10,$$
$$h_2(x_1, x_2) = x_1 + 3x_2 - 9.$$

Kuhn-Tucker conditions for maxima are

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0 \quad \therefore\ 14x_1 - 6 - \lambda_1 - \lambda_2 = 0 \quad \text{.......... (1)}$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0 \quad \therefore\ 10x_2 - 2\lambda_1 - 3\lambda_2 = 0 \quad \text{.......... (2)}$$

$$\lambda_1 (x_1 + 2x_2 - 10) = 0 \quad \text{.......... (3)} \qquad \lambda_2 (x_1 + 3x_2 - 9) = 0 \quad \text{.......... (4)}$$

$$x_1 + 2x_2 - 10 \le 0 \quad \text{.......... (5)} \qquad x_1 + 3x_2 - 9 \le 0 \quad \text{.......... (6)}$$

$$x_1, x_2 \ge 0 \quad \text{.......... (7)} \qquad \lambda_1, \lambda_2 \ge 0 \quad \text{.......... (8)}$$

Now, depending upon the values of λ_1 and λ_2 the following cases arise.

Case 1 : $\lambda_1 = 0$ and $\lambda_2 = 0$.

In this case from (1) and (2), we get,

$$14x_1 - 6 = 0 \quad \therefore\ x_1 = 3/7 \text{ and } 10x_2 = 0 \quad \therefore\ x_2 = 0.$$

These values satisfy (5), (6) and (7). They also satisfy (8). But we cannot immediately conclude that $x_1 = 3/7$ and $x_2 = 0$ is a maxima, because $\lambda_1 = 0$ and $\lambda_2 = 0$ can also give a minima.

(The conditions for minima are $\lambda_1 \le 0$, $\lambda_2 \le 0$, the equality sign included.)

Hence, we test the Hessian matrix for the objective function.

$$H = \begin{bmatrix} \dfrac{\partial^2 z}{\partial x_1^2} & \dfrac{\partial^2 z}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 z}{\partial x_2 \partial x_1} & \dfrac{\partial^2 z}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 10 \end{bmatrix}$$

Now, the principal minors are

$$A_1 = |a_{11}| = 14 \quad \text{and} \quad A_2 = \begin{vmatrix} 14 & 0 \\ 0 & 10 \end{vmatrix} = 140$$

Since the principal minors are both positive $x_1 = 3/7$, $x_2 = 0$ is a minima.

$$\therefore\ z_{min} = 7\left(\frac{9}{49}\right) + 5(0) - 6\left(\frac{3}{7}\right) = -\frac{9}{7}.$$

Case 2 : $\lambda_1 = 0$ and $\lambda_2 \neq 0$

To find x_1, x_2 in this case, we have first to eliminate λ_2 from

$$14x_1 - 6 - \lambda_2 = 0 \text{ and } 10x_2 - 3\lambda_2 = 0.$$

Multiply the first by 3 and subtract the second from it.

$$42x_1 - 18 - 10x_2 = 0 \quad i.e.\ 21x_1 - 5x_2 = 9.$$

Now, from (4), we have, $x_1 + 3x_2 = 9.$

Solving the two equations, we get

$$x_2 = \frac{45}{17} \text{ and } x_1 = \frac{18}{17}$$

To solve (1) and (2) for λ_1 and λ_2, multiply (1) by 2 and subtract the result from (2).

$$\therefore\ \lambda_2 = -28x_1 + 10x_2 + 12 = \frac{150}{17} \text{ and } \lambda_1 = 0.$$

Since λ_2 is positive, we reject this pair.

Case 3 : $\lambda_1 \neq 0$ and $\lambda_2 = 0$

To find x_1, x_2 in this case, we have, first to eliminate λ_1 from

$$14x_1 - 6 - \lambda_1 = 0 \text{ and } 10x_2 - 2\lambda_1 = 0$$

Multiply the first by 2 and subtract the result from the second.

$$\therefore\ 10x_2 - 28x_1 + 12 = 0 \quad i.e.\ 5x_2 - 14x_1 = -6$$

Now, from (3) we have, $x_1 + 2x_2 = 10.$

Solving the two equations, we get

$$x_1 = \frac{62}{33} = 1{\cdot}879, \quad x_2 = \frac{134}{33} = 4{\cdot}06$$

To find λ_1 from (1) and (2), multiply (1) by 3 and subtract it from (2).

$$\therefore\ 10x_2 - 42x_1 + 18 = -\lambda_1 \quad \therefore\ \lambda_1 = \frac{670}{33}$$

Since λ_1 is positive we reject this pair.

Case 4 : $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$

In this case, from (3) and (4), we get,

$$x_1 + 2x_2 = 10 \text{ and } x_1 + 3x_2 = 9$$

Solving the two equations, we get,

$$x_1 = 12 \text{ and } x_2 = -1.$$

Since the condition 7 is not-satisfied, we reject this solution.

Hence, $z_{min} = -\dfrac{9}{7}.$

Example 8 : Using the Kuhn-Tucker conditions solve the following N.L.P.P.

Maximise $z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

subject to $x_1 + x_2 \le 2$

$2x_1 + 3x_2 \le 12$ **(M.U. 1996, 2000, 04, 14, 17)**

$x_1, x_2 \ge 0.$

Sol. : We rewrite the given problem as,

$f(x_1, x_2) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

$h_1(x_1, x_2) = x_1 + x_2 - 2$

$h_2(x_1, x_2) = 2x_1 + 3x_2 - 12.$

Kuhn-Tucker conditions for maxima are

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0 \qquad \therefore\ -2x_1 + 4 - \lambda_1 - 2\lambda_2 = 0 \qquad \text{.......... (1)}$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0 \qquad \therefore\ -2x_2 + 6 - \lambda_1 - 3\lambda_2 = 0 \qquad \text{.......... (2)}$$

$$\frac{\partial f}{\partial x_3} - \lambda_1 \frac{\partial h_1}{\partial x_3} - \lambda_2 \frac{\partial h_2}{\partial x_3} = 0 \qquad \therefore\ x_3 = 0 \qquad \text{.......... (3)}$$

$\lambda_1 (x_1 + x_2 - 2) = 0$ .... (4) $\quad \lambda_2 (2x_1 + 3x_2 - 12) = 0$ .... (5)

$x_1 + x_2 - 2 \le 0$ .... (6) $\quad 2x_1 + 3x_2 - 12 \le 0$ .... (7)

$x_1, x_2, x_3 \ge 0$ .... (8) $\quad \lambda_1, \lambda_2 \ge 0$ .... (9)

Now, depending upon the values of λ_1 and λ_2 the following cases arise.

Case 1 : $\lambda_1 = 0$ and $\lambda_2 = 0.$

In this case from (1), (2) and (3), we get,

$-2x_1 + 4 = 0$ and $-2x_2 + 6 = 0$, $x_3 = 0$

$\therefore\ x_1 = 2,\ x_2 = 3$, and $x_3 = 0.$

But these values, do not satisfy the conditions (6) and (7).

Hence, we reject these values.

Case 2 : $\lambda_1 = 0$ and $\lambda_2 \ne 0$

To find x_1, x_2 in this case, we have first to eliminate λ_2 from

$-2x_1 + 4 - 2\lambda_2 = 0$ and $-2x_2 + 6 - 3\lambda_2 = 0.$

Multiply the first by 3, second by 2 and subtract.

$-6x_1 + 12 + 4x_2 - 12 = 0 \quad \therefore\ 2x_2 = 3x_1.$

Now, from (5), we have, $2x_1 + 3x_2 - 12 = 0.$

$$\therefore\ 2x_1 + 3 \cdot \frac{3x_1}{2} = 12 \quad \therefore\ \frac{13x_1}{2} = 12 \quad \therefore\ x_1 = \frac{24}{13}$$

$$\therefore\ x_2 = \frac{3}{2},\quad x_1 = \frac{36}{13},\quad x_3 = 0$$

But these values, although satisfy (7), do not satisfy the condition (6).

Hence, we reject these values.

Case 3 : $\lambda_1 \ne 0$ and $\lambda_2 = 0$

To find x_1, x_2 in this case, we have, first to eliminate λ_1 from

$-2x_1 + 4 - \lambda_1 = 0$ and $-2x_2 + 6 - \lambda_1 = 0.$

Subtracting the second from the first, we get,

$-2x_1 + 4 + 2x_2 - 6 = 0 \qquad \therefore\ -x_1 + x_2 = 1.$

Now, from (4) we have, $x_1 + x_2 = 2.$

Adding we get $x_2 = 3/2$ and subtracting, we get $x_1 = 1/2.$

Further from (1), $\lambda_1 = 3.$

Hence, these values satisfy the conditions (6), (7), (8) and (9).

$\therefore\ x_1 = \dfrac{1}{2},\ x_2 = \dfrac{3}{2},\ x_3 = 0$ is a feasible solution

and $\quad z_{\text{Max}} = -\left(\dfrac{1}{2}\right)^2 - \left(\dfrac{3}{2}\right)^2 + 4\left(\dfrac{1}{2}\right) + 6\left(\dfrac{3}{2}\right) = \dfrac{17}{2}$

Case 4 : $\lambda_1 \ne 0$ and $\lambda_2 \ne 0$

If $\lambda_1 \ne 0$ and $\lambda_2 \ne 0$, we get from (3) and (4)

$x_1 + x_2 - 2 = 0$ and $2x_1 + 3x_2 - 12 = 0$

Solving these equations, we get,

$x_1 = -6,\ x_2 = 8$ and $x_3 = 0.$

But these values do not satisfy the non-negativity restrictions (8) on the decision variables as $x_1 \le 0.$

Hence, we reject the solution.

$\therefore$ The solution is $x_1 = \dfrac{1}{2},\ x_2 = \dfrac{3}{2},\ x_3 = 0,\ z_{\max} = \dfrac{17}{2}.$

EXERCISE - VI

Using Kuhn-Tucker conditions solve the following N.L.P.P

1. Maximise $z = 2x_1 + 3x_2 - x_1^2 - x_2^2$

   subject to $x_1 + x_2 \le 1$

   $2x_1 + 3x_2 \le 6$

   $x_1, x_2 \ge 0$ **(M.U. 2009)**

   [**Ans. :** $x_1 = 1/4,\ x_2 = 3/4,\ \lambda_1 = 3/2,\ \lambda_2 = 0,\ z_{\text{Max}} = 17/8$]

2. Maximise $z = 7x_1^2 + 5x_2^2 + 6x_1$

   subject to $x_1 + 2x_2 \le 10$

   $x_1 - 3x_2 \le 9$

   $x_1, x_2 \ge 0$

   [**Ans. :** $x_1 = 48/5,\ x_2 = 1/5,\ \lambda_1 = 2116/25,\ \lambda_2 = 1394/5,\ z_{\text{Max}} = 702{\cdot}92$]

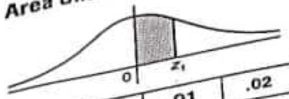
EXERCISE - VII

Theory

1. Explain what you understand by N.L.P.P. (M.U. 1999)
2. Define the following terms.
   (a) Point of Inflexion, (b) Saddle Point.
3. Define the following terms.
   (a) Hessian matrix, (b) Bordered Hessian matrix. (M.U. 1999, 2004, 07)
4. Explain the method of solving a non-linear programming problem with two variables and one equality constraints.
5. What is the bordered Hessian matrix ? Explain how it is used to determine the maxima and minima of N.L.P.P. (M.U. 2004)
6. What are Kuhn-Tucker conditions for solving an N.L.P.P. (M.U. 2005)

☆☆☆

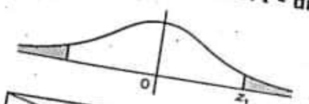
Area Under Standard Normal Curve



The table gives the area under the standard normal curve from $z = 0$ to $z = z_1$ which is the probability that z will lie between $z = 0$ and $z = z_1$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.09 10	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4[illegible]	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4415	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4841	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4560	.4961	.4962	.4963	.496
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.497
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.498
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.498
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.49

Percentage Points of t - distribution



Example

For $\Phi = 10$ d. o. f.

$P(|t| > 1.812) = 0.1$

Φ \ P	0.20	0.10	0.05	0.02	0.01
1	3.078	6.314	12.706	31.812	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.287
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
40	1.303	1.684	2.021	2.423	2.704
60	1.296	1.671	2.000	2.390	2.660
120	1.289	1.658	1.980	2.358	2.617
∞	1.282	1.645	1.960	2.325	2.576

Percentage Points of χ^2 - Distribution

Example

For $\Phi = 10$ d. o. f.

$P(\chi^2 > 15.99) = 0.10$

Φ \ P	0 = .99	0.95	0.50	0.10	0.05	0.02	0.01
1	.000157	.00393	.455	2.706	3.841	5.214	6.635
2	.0201	.103	1.386	4.605	5.991	7.824	9.210
3	.115	.352	2.366	6.251	7.815	9.837	11.341
4	.297	.711	3.357	7.779	9.488	11.668	13.277
5	.554	1.145	4.351	9.236	11.070	13.388	15.086
6	.872	1.635	5.348	10.645	12.592	15.033	16.812
7	1.339	2.167	6.346	12.017	14.067	16.622	18.475
8	1.646	2.733	7.344	13.362	15.507	18.168	20.090
9	2.088	3.325	8.343	14.684	16.919	19.679	21.666
10	2.558	3.940	9.340	15.987	18.307	21.161	23.209
11	3.053	4.575	10.341	17.275	19.675	22.618	24.725
12	3 571	5.226	11.340	18.549	21.026	24.054	26.217
13	4.107	5.892	12.340	19.812	22.362	25.472	27.688
14	4.660	6.571	13.339	21.064	23.685	26.873	29.141
15	4.229	7.261	14.339	22.307	24.996	28.259	30.578
16	5.812	7.962	15.338	23.542	26.296	29.633	32.000
17	6.408	8.672	16.338	24.769	27.587	30.995	33.409
18	7.015	9.390	17.338	25.989	28.869	32.346	34.805
19	7.633	10.117	18.338	27.204	30.144	33.687	36.191
20	8.260	10.851	19.337	28.412	31.410	35.020	37.566
21	8.897	11.591	20.337	29.615	32.671	36.349	38.932
22	9.542	12.338	21.337	30.813	33.924	37.659	40.289
23	10.196	13.091	22.337	32.007	35.172	38.968	41.638
24	10.856	13.848	23.337	32.196	36.415	40.270	42 980
25	11.524	14.611	24.337	34.382	37.652	41.566	44.314
26	12.198	15.379	25.336	35.363	38.885	41.856	45.642
27	12.879	16.151	26.336	36.741	40.113	44.140	46.963
28	13.565	16.928	27.336	37.916	41.337	45.419	48.278
29	14.256	17.708	28.336	39.087	42.557	46.693	49.588
30	14.953	18 493	29.336	40.256	43.773	47.962	50.892