Properties

If λ is an eigen value of A, then

i) $\bar{\lambda}$ is eigen value of A^{θ}.

ii) $k\lambda$ is eigen value of kA. (Here eigen vector of λ & $k\lambda$ are same).

iii) λ^{-1} is eigen value of A^{-1}. [Eigen vector of λ & λ^{-1} are same]

iv) λ^{n} is eigen value of A^{n}. (Eigen vector of λ & λ^{n} are same)

v) $\frac{|A|}{\lambda}$ is eigen value of adj A. [eigen vector of λ & $\frac{|A|}{\lambda}$ are same].

vi) $f(\lambda)$ is eigen value of $f(A)$. (Here eigen vector of λ & $f(\lambda)$ are same).

Problems :-

(1) Find eigen value & eigen vector of A^{-1}, A^{3}, adj A $\quad A = \begin{pmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{pmatrix}$

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$$\lambda^3 - 6\lambda^2 + 4\lambda - 6 = 0$$

$$\lambda = 1, 2, 3.$$

When $\lambda = 1$ $\quad x_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$

When $\lambda = 2$ $\quad x_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

When $\lambda = 3$ $\quad x_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

To find eigen values & vectors of A^{-1}

λ^{-1} is eigen value of A^{-1}

as $\lambda = 1, 2, 3,$ $\quad \lambda^{-1} = 1, 1/2, 1/3.$

Eigen values of A^{-1} are $1, 1/2, 1/3$ and corresponding eigen vectors are $x_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$

$x_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ $\quad x_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

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To find eigen values & eigen vectors of A^3

$\lambda^3 = 1^3 = 1$ $\quad X_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$

$\lambda^3 = 2^3 = 8$ $\quad X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\lambda^3 = 3^3 = 27$ $\quad X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

To find eigen values & eigen vectors of adj A.

$|A| = 6$.

Eigen values of adj A can be calculated using $\frac{|A|}{\lambda}$.

$\lambda = 1 \Rightarrow \frac{6}{1} = 6$ $\quad X_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$

$\lambda = 2 \Rightarrow \frac{6}{2} = 3$ $\quad X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\lambda = 3 \Rightarrow \frac{6}{3} = 2$. $\quad X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

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② If $A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{pmatrix}$ find eigen values

& eigen vectors of $B = I - 6A^{-1}$.

Soln:- Since the given matrix is triangular,

eigen values are $\lambda = 1, 3, -2$.

To find Eigen vectors solve,

$$(A - \lambda I)x = 0$$

$$\begin{pmatrix} 1-\lambda & 0 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 1$

$$\begin{pmatrix} 0 & 0 & -3 \\ 0 & 2 & 2 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 + 2x_2 + 2x_3 = 0$$

$$0x_1 + 0x_2 - 3x_3 = 0$$

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By cramer's rule

$$\frac{x_1}{-6+0} = \frac{-x_2}{0+0} = \frac{x_3}{0+0} = k \text{ (say)}$$

$$\frac{x_1}{-6} = \frac{x_2}{0} = \frac{x_3}{0} = k.$$

$$X_1 = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 3}$

$$\begin{pmatrix} -2 & 0 & -3 \\ 0 & 0 & 2 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + 0x_2 - 3x_3 = 0$$

$$0x_1 + 0x_2 + 2x_3 = 0$$

By cramer's rule,

$$\frac{x_1}{0+0} = \frac{-x_2}{-4+0} = \frac{x_3}{0+0} = k \text{ (say)}.$$

$$\frac{x_1}{0} = \frac{-x_2}{-4} = \frac{x_3}{0} = k \text{ (say)}$$

$$X_2 = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

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$\lambda = -2$

$$\begin{pmatrix} 3 & 0 & -3 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x_1 + 0x_2 - 3x_3 = 0$$

$$0x_1 + 5x_2 + 2x_3 = 0$$

By cramer's rule

$$\frac{x_1}{0+15} = \frac{-x_2}{6+0} = \frac{x_3}{15-0} = k \text{ (say)}$$

$$\frac{x_1}{15} = \frac{x_2}{-6} = \frac{x_3}{15} = k \text{ (say)}$$

$$X_3 = \begin{pmatrix} 15 \\ -6 \\ 15 \end{pmatrix}$$

$$B = I - \cancel{A}6A^{-1} = f(A)$$

$f(\lambda)$ is eigen value of $f(A)$

$$f(\lambda) = 1 - 6\lambda^{-1}$$

λ	$f(\lambda) = 1 - 6\lambda^{-1}$
1	$1 - 6(1)^{-1} = 1 - 6 = -5$
3	$1 - 6(3)^{-1} = 1 - 6/3 = -1$
-2	$1 - 6(-2)^{-1} = 1 + 6/2 = 4$

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& corresponding eigen vectors are

$$x_1 = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 15 \\ -6 \\ 15 \end{pmatrix}$$

Exercise

① Find eigen values & eigen vectors of $A^3 - 3A^2 + A$ & adj A where $A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix}$

② If $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ find characteristic roots (eigen values) & characteristic vectors (eigen vectors) of $A^3 + I$.

③ If $A = \begin{pmatrix} 2 & 4 \\ 0 & 3 \end{pmatrix}$ find eigen values & eigen vectors of $6A^{-1} + 8A^3 + 2I$.

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If λ is eigen value of A then prove that 1) $k\lambda$ is eigen value of kA.

2) λ^n is eigen value of A^n.

3) λ^{-1} is eigen value of A^{-1}.

Soln:-

As λ is eigen value of A then there exist $X \neq 0$ such that $(A - \lambda I)X = 0 \Rightarrow AX - \lambda IX = 0$

$\Rightarrow AX = \lambda X \rightarrow (1)$.

i) Multiply (1) by k we get

$(kA)X = (k\lambda)X$

$\Rightarrow k\lambda$ is eigen value of kA (& eigen vector of X remains same).

ii) Multiply (1) by A we get

$AAX = A\lambda X$

$\Rightarrow A^2X = \lambda AX \Rightarrow \lambda \cdot \lambda X$ — from (1) $\because AX = \lambda X$

$\Rightarrow A^2X = \lambda^2 X \longrightarrow (2)$

$\Rightarrow \lambda^2$ is eigen value of A^2 (& eigen vector X remains same).

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Multiply ~~by~~ A.

$AA^2X = A\lambda^2X$

$\Rightarrow A^3X = \lambda^2AX = \lambda^2\lambda X$

$\Rightarrow A^3X = \lambda^3X$.

$\Rightarrow \lambda^3$ is eigen value of A^3 (& eigen vector X remains same).

Similarly we get,

$A^nX = \lambda^nX$

$\Rightarrow \lambda^n$ is eigen value of A^n (& eigen vector X remains same).

iii) Multiply ① by A^{-1}

$A^{-1}AX = A^{-1}\lambda X$

$\Rightarrow IX = \lambda A^{-1}X$

$\Rightarrow \frac{1}{\lambda}X = A^{-1}X$

$\Rightarrow A^{-1}X = \lambda^{-1}X$

$\Rightarrow \lambda^{-1}$ is eigen value A^{-1} (& eigen vector X remains same).

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