Time Series Analysis

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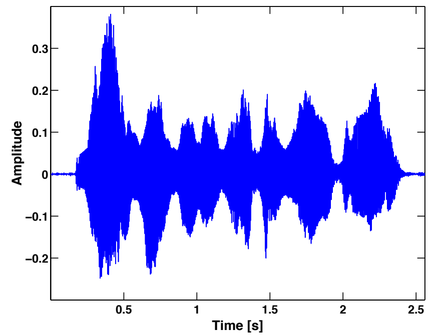
Contents:

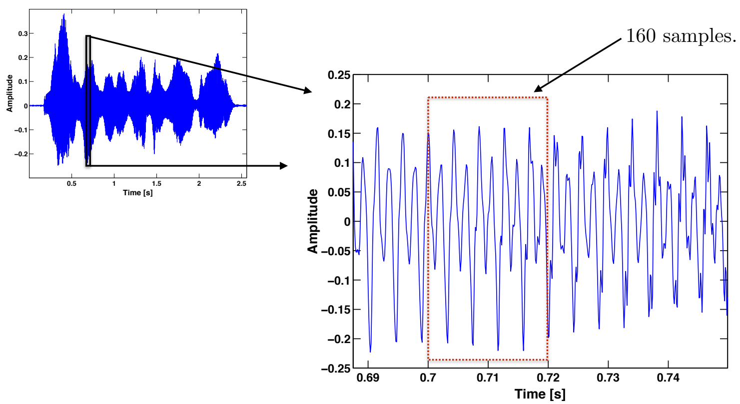
- Stochastic processes
- Estimating the mean and the ACF.

Stochastic processes



A female voice uttering: “Why were you away a year, Roy?”

Stochastic processes



The human voice may be well modelled as being stationary over about 20 ms intervals. If sampled at 8 kHz, this results in a measurement of 160 samples.

Stochastic processes

In this course, we will restrict our attention to *wide-sense stationary* (WSS) processes. For such processes, the statistical properties of the process do not change over time. Furthermore,

(i) The mean of the process is constant and finite.

(ii) The autocovariance $C\{y_s, y_t^*\}$ only depends on the difference $(s-t)$, and not on the actual values of s and t.

(iii) The variance of the process is finite, i.e., $E\{|y_t|^2\} < \infty$.

Furthermore, we will assume that all considered processes are *ergodic*. Essentially, this implies that it is possible to estimate the characteristic statistical features of the process.

Stochastic processes

To describe the dependencies of the process over time, we introduce the *auto-covariance function*, $r_y(k)$, which is defined as

$$r_y(k) \equiv C\left\{y_t, y_{t-k}^*\right\} = E\left\{[y_t - m_y]\,[y_{t-k} - m_y]^*\right\} = E\left\{y_t y_{t-k}^*\right\} - m_y m_y^*$$

where $m_y = E\{y_t\}$ and $V\left\{y_t\right\} = r_y(0)$.

The autocovariance function of a WSS process satisfies:

(i) It is conjugate symmetric, i.e., $r_y(k) = r_y^*(-k)$.

(ii) The variance is always non-negative, i.e.,

$$r_y(0) = E\left\{|y_t - m_y|^2\right\} \geq 0.$$

(iii) It takes its largest value at lag 0, i.e.,

$$r_y(0) \geq |r_y(k)|\,, \forall k.$$

Stochastic processes

We also define the *auto-correlation* function (ACF)

$$\rho_y(k) = \frac{r_y(k)}{r_y(0)}$$

which will be bounded as $|\rho_y(k)| \leq 1$.

Similarly, we define the *cross-correlation* function between the x_t and y_t as

$$\rho_{x,y}(k) = \frac{r_{x,y}(k)}{\sqrt{r_x(0) r_y(0)}}$$

where $r_{x,y}(k) = C\left\{x_t, y_{t-k}^*\right\}$.

Stochastic processes

What is the smallest possible value for $\rho_{x,y}(k)$?

- There is no lower limit.
- -1
- 0
- 1
- We need more information to answer this.

Estimating the mean

Clearly, one will typically not know the statistical properties of the process, and will need to estimate these from the observed realization. The most natural estimator of the mean is

$$\hat{m}_y = \frac{1}{N}\sum_{t=1}^{N} y_t$$

This is an *unbiased* estimate of the true mean, m_y, as

$$E\left\{\hat{m}_y\right\} = \frac{1}{N}\sum_{t=1}^{N} E\left\{\hat{y}_t\right\} = m_y$$

The estimate is also *consistent*. This is a highly desirable property of an estimator, implying that the estimate is (asymptotically) unbiased and that the variance decrease to zero as the data length grows. The variance of the mean may be expressed as

$$V\left\{\hat{m}_y\right\} = \frac{1}{N^2}\sum_{t=1}^{N}\sum_{s=1}^{N} r_y(t-s) = \frac{r_y(0)}{N}\sum_{k=-N+1}^{N-1}\frac{N-|k|}{N}\rho_y(k) \to 0, \quad N \to \infty$$

Estimating the mean

In case y_t is a Gaussian process, with mean m_y and ACF $r_y(k)$, then

$$\lim_{N\to\infty} NV\{\hat{m}_y\} = \sum_{k=-\infty}^{\infty} r_y(k)$$

Thus, $\hat{m}_y$ is a consistent estimate to m_y. For large N, an often useful approximation is

$$V\{\hat{m}_y\} \approx \frac{1}{N}\sum_{k=-\infty}^{\infty} r_y(k)$$

In the special case of a white process,

$$V\{\hat{m}_y\} \approx r_y(0)/N$$

Estimating the ACF

When computing the ACF, one needs to average the available components for each lag. This is done using the estimator

$$\hat{r}_y(k) = \frac{1}{P}\sum_{t=k+1}^{N} \left(y_t - \hat{m}_y\right)\left(y_{t-k} - \hat{m}_y\right)^*$$

where P is a scaling parameter. How many terms are there in the sum?

- N-k-1
- N-k
- N
- N+k

Estimating the ACF

The two standard ways to estimate the ACF can be expressed as

$$\hat{r}_y^u(k) = \frac{1}{N-k}\sum_{t=k+1}^{N} \left(y_t - \hat{m}_y\right)\left(y_{t-k} - \hat{m}_y\right)^*$$

$$\hat{r}_y^b(k) = \frac{1}{N}\sum_{t=k+1}^{N} \left(y_t - \hat{m}_y\right)\left(y_{t-k} - \hat{m}_y\right)^* = \frac{1}{N}\psi_k$$

for $0 \leq k \leq N-1$. Which is the better estimate?

It can be shown that $E\{\psi_k\} \approx (N-k)\left(r_y(k) - V\{\hat{m}_y\}\right)$, which implies that

$$E\{\hat{r}_y^u(k)\} = \frac{1}{N-k}E\{\psi_k\} = r_y(k) - V\{\hat{m}_y\}$$

$$E\{\hat{r}_y^b(k)\} = \frac{1}{N}E\{\psi_k\} = r_y(k) - \frac{k}{N}r_y(k) - \frac{N-k}{N}V\{\hat{m}_y\}$$

Both estimators are thus biased.

Estimating the ACF

For a *zero-mean* process

$$E\{\psi_k\} = \sum_{t=k+1}^{N} r_y(k) = (N-k)r_y(k)$$

In this case, this implies that

$$E\{\hat{r}_y^u(k)\} = \frac{1}{N-k}E\{\psi_k\} = r_y(k)$$

$$E\{\hat{r}_y^b(k)\} = \frac{1}{N}E\{\psi_k\} = \frac{N}{N-k}r_y(k)$$

In this case, $r_y^u(k)$ is unbiased, whereas $r_y^b(k)$ is only *asymptotically* unbiased.

We will always use the *biased* estimator!

Why?

Estimating the ACF

Estimating the ACF

For a zero-mean white Gaussian process with variance σ_e^2, and

$$\hat{\rho}_e(k) = \frac{\hat{r}_e(k)}{\hat{r}_e(0)}$$

where $\hat{r}_e(k)$ is the *biased* estimate of the ACF, then, for $k \neq 0$,

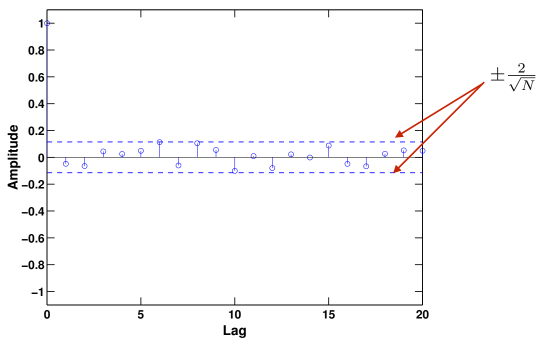
$$E\{\hat{\rho}_e(k)\} = 0$$
$$V\{\hat{\rho}_e(k)\} = \frac{1}{N}$$

Furthermore, $\hat{\rho}_e(k)$ is asymptotically Normal distributed for $k > 0$.

This implies that a 95% confidence interval can be formed as

$$\hat{\rho}_e(k) \approx 0 \pm \frac{2}{\sqrt{N}}$$

Estimating the ACF



As far as we can tell, this is a *white* noise. This assumes that $\hat{\rho}_e(k) \in \mathcal{N}$.

About 5% of the ACF to be (slightly) outside the confidence intervall. As a rule of thumb, only estimate $r_y(k)$ for lags up to (at most) $N/4$.

Testing the model

$$y_t \longrightarrow \boxed{\text{Model}^{-1}} \longrightarrow e_t$$

Is the residual white?