# Time Series Analysis

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- Determining model orders.
- Modeling example.

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# The ARMAX process

One may sometimes improve the modeling of a system by considering additional relevant time series. To do so, we introduce two ways to model single-input single-output (SISO) system. The first model is the so-called ARMAX processes, given by

$$A(z)y_t = B(z)x_{t-d} + C(z)e_t$$

where $x_{t-d}$ is the $d$-step delayed input signal influencing the process $y_t$, and

$$A(z) = 1 + a_1 z^{-1} + \ldots + a_s z^{-p}$$
$$C(z) = 1 + c_1 z^{-1} + \ldots + c_s z^{-q}$$
$$B(z) = b_0 + b_1 z^{-1} + \ldots + b_s z^{-s}$$

If $A(z) = 1$ or $C(z) = 1$, the corresponding processes are termed MAX and ARX processes, respectively.

Note that the same $A(z)$ polynomial affects both $x_t$ and $e_t$.

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# The Box-Jenkins process

A more flexible model is the so-called Box-Jenkins (BJ) model, formed as

$$y_t = \frac{B(z)}{A_2(z)}x_{t-d} + \frac{C_1(z)}{A_1(z)}e_t$$

where

$$A_1(z) = 1 + a_1^{(1)} z^{-1} + \ldots + a_p^{(1)} z^{-p}$$
$$A_2(z) = 1 + a_1^{(2)} z^{-1} + \ldots + a_r^{(2)} z^{-r}$$

with both polynomials having all roots within the unit circle to ensure stability, and

$$C_1(z) = 1 + c_1 z^{-1} + \ldots + c_q z^{-q}$$
$$B(z) = b_0 + b_1 z^{-1} + \ldots + b_s z^{-s}$$

We will in the following focus on the BJ model, striving to identify the relevant model orders $(p, q, d, r, s)$.

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# The Box-Jenkins process

To do so, we rewrite $y_t$ as

$$y_t = \frac{B(z)z^{-d}}{A_2(z)}x_t + \tilde{e}_t = H(z)x_t + \tilde{e}_t$$

where the input, $x_t$, is assumed to be uncorrelated of the noise process, $\tilde{e}_t$, and

$$H(z) = \sum_{k=-\infty}^{\infty} h_k z^{-k}$$

is the transfer function between the input and the output signals, with the co-efficients $h_k$ being the impulse response weights. We assume this to be a stable, causal, and linear filter.

If we initially ignore the noise entirely, then

$$A_2(z)H(z) = B(z)z^{-d}$$

Thus, if we knew $H(z)$, we ought to be able to determine the delay and the model orders of $A_2(z)$ and $B(z)$, i.e., $(d, r, s)$.

## Determining the BJ model orders

Writing out the resulting polynomial multiplication

$$A_2(z)H(z) = B(z)z^{-d}$$

and identifying the corresponding $z^{-k}$ terms, imply that

$$
\begin{aligned}
h_k &= 0, & k < d \\
h_k &= a_1^{(2)} h_{k-1} + \ldots + a_r^{(2)} h_{k-r} + b_0, & k = d \\
h_k &= a_1^{(2)} h_{k-1} + \ldots + a_r^{(2)} h_{k-r} - b_{k-d}, & k = d+1, d+2, \ldots, d+s \\
h_k &= a_1^{(2)} h_{k-1} + \ldots + a_r^{(2)} h_{k-r}, & k > d+s
\end{aligned}
$$

Note that, if we know $h_k$, the first equation gives us information about $d$. How can we reach $r$ and $s$?

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## Determining the BJ model orders

We assume that $r$ only takes the values 0, 1, or 2. Then, we get three cases:

1. When $r = 0$, such that $A_2(z) = 1$, the transfer function only contains a finite number of non-zero impulse response weights, starting with $h_d = b_0$ and ending with $h_{d+s} = -b_s$.

2. When $r = 1$, such that $A_2(z) = 1 + a_1^{(2)} z - 1$, the impulse response will start at $h_d$, but will exhibits an exponential decay starting from $h_{d+s}$.

3. When $r = 2$, the impulse response starting at $h_d$ will exhibits either a damped exponential or sinusoidal behavior; if the roots of $A_2(z)$ are real, it will follow an exponential decay, and if they are complex, it will follow a damped sinusoidal behavior. The decay will start from $h_{d+s}$.

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| $(d,r,s)$ | Typical impulse weights |
|---|---|

Finite impulse response $(2,0,0)$, $(2,0,1)$, $(2,0,2)$ — Delay $d$ steps

Exponential decay $(2,1,0)$, $(2,1,1)$, $(2,1,2)$ — Decay starts at $d+s$

Damped exponential or sinusoidal $(2,2,0)$, $(2,2,1)$, $(2,2,2)$ — Decay starts at $d+s$

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## Determining the BJ model orders

As $H(z)$ is assumed to be a causal filter,

$$y_t = h_0 x_t + h_1 x_{t-1} + \ldots + \tilde{e}_t$$

Assuming that $m_y = m_x = 0$, and that $x_t$ is uncorrelated with $\tilde{e}_t$,

$$r_{y,x}(k) = E\{y_t x_{t-k}^*\} = h_0 r_x(k) + h_1 r_x(k-1) + \ldots$$

given that $r_{\tilde{e},x}(k) = 0$ for all $k$, imply that

$$\rho_{y,x}(k) = \frac{r_{y,x}(k)}{\sigma_y \sigma_x} = \frac{\sigma_x}{\sigma_y}\Big(h_0 \rho_x(k) + h_1 \rho_x(k-1) + \ldots\Big)$$

with $\sigma_x^2$ and $\sigma_y^2$ denoting the variance of $x_t$ and $y_t$, respectively. If $x_t$ is a white process,

$$\rho_{y,x}(k) = \frac{\sigma_x}{\sigma_y} h_k \rho_x(0) = \frac{\sigma_x}{\sigma_y} h_k$$

implying that

$$h_k = \frac{\sigma_y}{\sigma_x} \rho_{y,x}(k)$$

For a white input, the impulse response can thus be estimated using the cross-correlation function.

## Example

*Example:* Consider the noise-free first-order (real-valued) system

$$y_t - a_1 y_{t-1} = x_t$$

where $|a_1| < 1$ and $x_t$ is a zero-mean white noise with variance $\sigma_x^2$. To form the crosscorrelation, note that

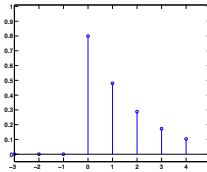$$y_t = \frac{1}{1 - a_1 z^{-1}} x_t = x_t + a_1 x_{t-1} + a_1^2 x_{t-2} + \ldots$$

and thus, using that both $x_t$ and $y_t$ are zero-mean,

$$r_{y,x}(k) = E\{y_t x_{t-k}\} = \begin{cases} a_1^k \sigma_x^2, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

yielding the crosscorrelation

$$\rho_{y,x}(k) = \frac{r_{y,x}(k)}{\sigma_y \sigma_x} = \begin{cases} a_1^k \sqrt{1 - a_1^2}, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

as $\sigma_y^2 = \sigma_x^2 / (1 - a_1^2)$. For example, $a_1 = 0.4$. Note that the result is just as expected given that $d = 0$ and $r = 1$.

## Determining the BJ model orders

In the case when the input is not white, we cannot directly estimate $h_k$ using $\rho_{y,x}(k)$, as

$$\rho_{y,x}(k) = \frac{r_{y,x}(k)}{\sigma_y \sigma_x} = \frac{\sigma_x}{\sigma_y}\left( h_0 \rho_x(k) + h_1 \rho_x(k-1) + \ldots \right)$$

To handle this, we proceed to model $x_t$ as an ARMA process, such that

$$A_3(z) x_t = C_3(z) w_t$$

where $w_t$ is a white process. Thus, we view the BJ process as

$$y_t = \frac{B(z) z^{-d}}{A_2(z)} \frac{C_3(z)}{A_3(z)} w_t + \frac{C_1(z)}{A_1(z)} e_t$$

Multiplying both sides with $A_3(z)/C_3(z)$ yields

$$\epsilon_t = \frac{A_3(z)}{C_3(z)} y_t = \frac{B(z) z^{-d}}{A_2(z)} w_t + \frac{A_3(z)}{C_3(z)} \frac{C_1(z)}{A_1(z)} e_t = H(z) w_t + v_t$$

where

$$v_t = \frac{A_3(z)}{C_3(z)} \frac{C_1(z)}{A_1(z)} e_t$$

## Determining the BJ model orders

We thus have that

$$\epsilon_t = H(z) w_t + v_t$$

with $w_t$ being uncorrelated with $v_t$. This implies that we can estimate $h_k$ as $\rho_{\epsilon,w}(k)$. As

$$V\{\rho_{\epsilon,w}(k)\} \approx \frac{1}{N - k} \approx \frac{1}{N}$$

we may use $\pm 2/\sqrt{N}$ as our 95% confidence interval of $\rho_{\epsilon,w}(k)$.

## Determining the BJ model orders

This suggests the following procedure:

1. Form an ARMA model for $x_t$, and construct

$$\epsilon_t = \frac{A_3(z)}{C_3(z)} y_t$$

$$w_t = \frac{A_3(z)}{C_3(z)} x_t$$

2. Estimate the impulse response, $h_k$, using the significant coefficients in $\rho_{\epsilon,w}(k)$. Determine suitable model orders $d$, $r$, and $s$.

3. Using these model orders, form initial estimates of $B(z)$ and $A_2(z)$ by treating $\tilde{e}_t$ as a white noise.

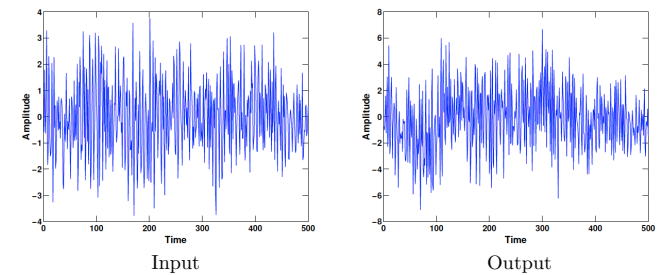## Determining the BJ model orders

Then,

4. Using this preliminary transfer function, form the residual

$$\tilde{e}_t = y_t - \frac{B(z)z^{-d}}{A_2(z)} x_t$$

which is then modeled using an ARMA model, identifying the proper model orders of $A_1(z)$ and $C_1(z)$.

5. Estimate the coefficients for all polynomials jointly.

6. Examine the resulting model residual to determine if this is white.
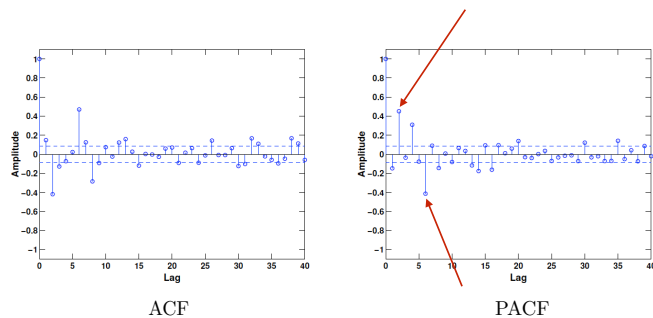
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## Example



| Input | Output |

Lets examine an example. We begin with forming a model for the input, $x_t$, such that

$$A_3(z)x_t = C_3(z)w_t$$

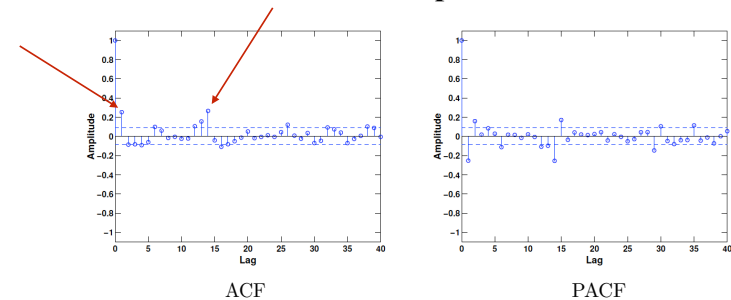To do so, lets examine the ACF and PACF of $x_t$.

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## Example



| ACF | PACF |

There seems to be strong dependencies for order 2 and 6, as well as, perhaps, at 4. In order to have a simple model, we begin with trying

$$A_3(z) = 1 + a_2 z^{-2} + a_6 z^{-6}$$
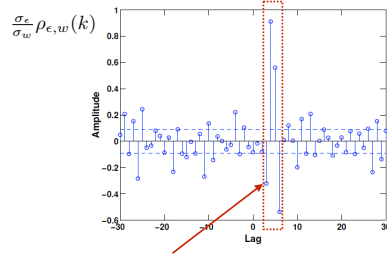
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## Example



| ACF | PACF |

We estimate the parameters of this model and examine the ACF and PACF of the residual (above). Looking at the ACF, there seems to be strong dependencies at lag 1 and 14. We thus modify our model to

$$A_3(z) = 1 + a_2 z^{-2} + a_6 z^{-6}$$
$$C_3(z) = 1 + c_1 z^{-1} + c_{14} z^{-14}$$

We re-estimating *all* parameters and examine the residuals. These are now deemed to be white.

## Example



$$\frac{\sigma_\epsilon}{\sigma_w}\rho_{\epsilon,w}(k)$$

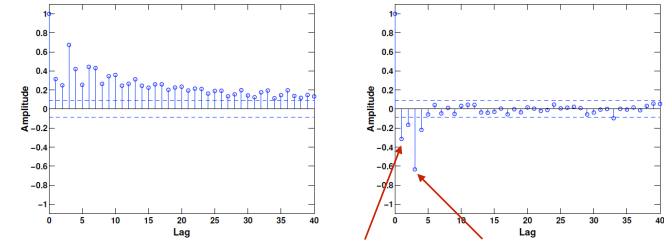We form the "white" input signal and the corresponding output

$$w_t = \frac{A_3(z)}{C_3(z)}x_t \quad \text{and} \quad \epsilon_t = \frac{A_3(z)}{C_3(z)}y_t$$

and then estimate the transfer function from $w_t$ to $\epsilon_t$ as

$$h_k = \frac{\sigma_\epsilon}{\sigma_w}\rho_{\epsilon,w}(k)$$

The delay suggests $d = 3$. The impulse response seems to "ring", so we try $r = 2$. There seems to be 4 dominant components, i.e., $s = 3$.

## Example



We pretend that the additive noise is white, and estimate the parameters detailing the model

$$y_t = \frac{B(z)z^{-d}}{A_2(z)}x_t + \tilde{e}_t$$

where $B(z)$ and $A_2(z)$ are of order $s$ and $r$, respectively. We then compute the ACF and PACF of the residual $\tilde{e}_t$ (above). We form a model of the residual, beginning with using just $a_1$ and $a_3$.

## Example

Examining the resulting residual suggest that we also needs $c_1$. This yields

$$A_1(z) = 1 - 0.31z^{-1} - 0.59z^{-3}$$
$$C_1(z) = 1 - 0.32z^{-1}$$

Examining the resulting residual, we note that it is white. Using the found model, we re-estimate all the polynomials jointly, obtaining

$$A_1(z) = 1 - 0.31z^{-1} - 0.60z^{-3}$$
$$C_1(z) = 1 - 0.31z^{-1}$$
$$A_2(z) = 1 - 0.21z^{-1} + 0.40z^{-2}$$
$$B(z) = -0.39 + 0.90z^{-1} + 0.48z^{-2}$$

with $d = 3$. This process yields a white modeling residual.