Time Series Analysis

Andreas Jakobsson

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The power spectral density

The *power spectral density* (PSD) of a process is defined as

$$\phi_y(\omega) = \sum_{k=-\infty}^{\infty} r_y(k) e^{-i\omega k}$$

over the frequencies $-\pi < \omega \leq \pi$. The inverse transform recovers $r_y(k)$

$$r_y(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_y(\omega) e^{i\omega k} \, d\omega$$

It is worth noting that

$$r_y(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_y(\omega) \, d\omega$$

which, for a zero-mean process, measures the *power* of y_t, i.e.,

$$r_y(0) = E\{|y_t|^2\}$$

The PSD is *real-valued* and *non-negative*. For a real-valued process, the PSD is *symmetric*. For a complex-valued, it is *non-symmetric*.

The power spectral density

Under the weak assumption that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=-N}^{N} |k| \, |r_y(k)| = 0$$

the PSD can be expressed equivalently as

$$\phi_y(\omega) = \lim_{N\to\infty} E\left\{ \frac{1}{N} \left| \sum_{t=1}^{N} y_t e^{-i\omega t} \right|^2 \right\}$$

This suggests two natural estimators, namely as the *periodogram*

$$\hat{\phi}_y^p(\omega) = \frac{1}{N} \left| \sum_{t=1}^{N} y_t e^{-i\omega t} \right|^2$$

and the *correlogram*

$$\hat{\phi}_y^c(\omega) = \sum_{k=-N+1}^{N-1} \hat{r}_y(k) e^{-i\omega k}$$

Note that $\hat{r}_y(k)$ should be the *biased* estimator.

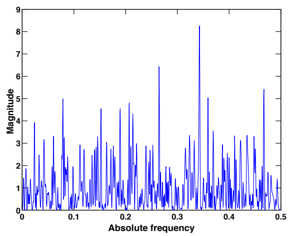
The power spectral density

Example:
As a white noise is uncorrelated,

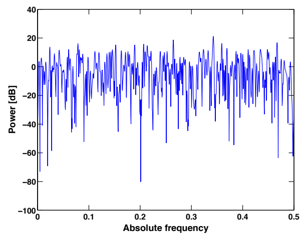
$$r_e(k) = \sigma_e^2 \delta_K(k)$$

where $\delta_K(k)$ is the Kronecker delta. Thus, the PSD of a white noise is

$$\phi_e(\omega) = \sum_{k=-\infty}^{\infty} r_e(k) e^{-i\omega k} = \sigma_e^2$$



$\hat{\phi}_e(\omega)$



$\hat{\phi}_e^{dB}(\omega) = 10 \log\left\{\hat{\phi}_e(\omega)\right\}$

The power spectral density

Example:
Consider a real-valued sinusoidal signal

$$y_t = A\cos(\omega_0 t + \phi) + e_t$$

where e_t is an additive white noise. The ACF of y_t is then

$$r_y(k) = \frac{A^2}{2}\cos(\omega_0 k) + \sigma_e^2 \delta_K(k)$$

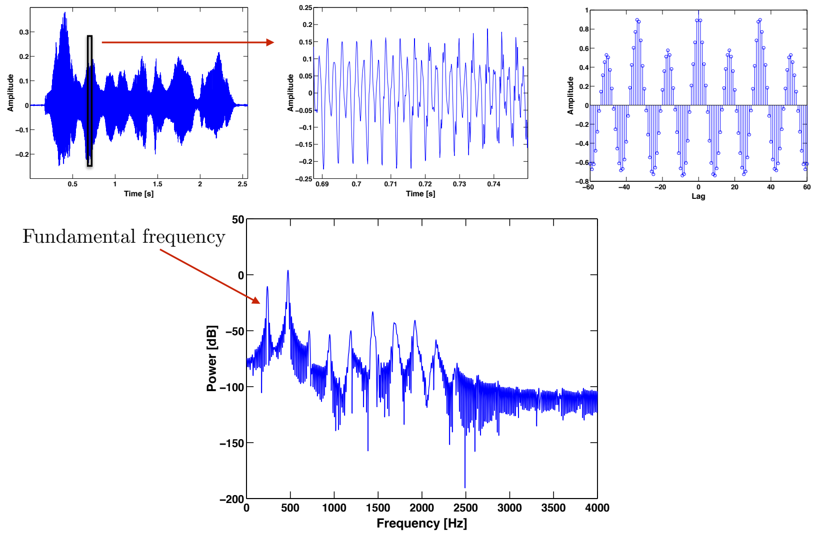
This implies that the PSD of y_t is

$$\phi_y(\omega) = \frac{A^2}{4}\delta_D(\omega - \omega_\ell) + \frac{A^2}{4}\delta_D(\omega + \omega_\ell) + \sigma_e^2$$

where $\delta_D(\omega)$ is the Dirac delta, satisfying

$$f(a) = \int f(x)\delta_D(x-a)\,dx$$

The power spectral density



Filtering a stochastic process

$$x_t \longrightarrow \boxed{\quad h_k \quad} \longrightarrow y_t$$

Consider a stochastic process x_t that is filtered though a linear (stable) filter h_k, such that

$$y_t = \sum_{k=-\infty}^{\infty} h_k x_{t-k}$$

Then,

$$m_y = E\left\{\sum_{k=-\infty}^{\infty} h_k x_{t-k}\right\} = m_x \sum_{k=-\infty}^{\infty} h_k = m_x H(0)$$

where

$$H(\omega) = \sum_{k=-\infty}^{\infty} h_k e^{-i\omega k}$$

Filtering a stochastic process

Similarly, the ACF may be formed as

$$r_y(k) = \sum_{m=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} h_m^* h_\ell r_x(m+k-\ell) = r_x(k) \star h_k \star h_{-k}^*$$

where $\star$ denotes the convolution operator. Expressed in the frequency domain,

$$\phi_y(\omega) = |H(\omega)|^2\,\phi_x(\omega)$$

Defining the cross spectral density of two stationary processes, x_t and y_t, as the DFT of the cross-covariance function, i.e.,

$$\phi_{x,y}(\omega) = \sum_{k=-\infty}^{\infty} r_{x,y}(k)e^{-i\omega k}$$

Then,

$$\phi_{x,y}(\omega) = H(\omega)\phi_x(\omega)$$

This is the so-called Wiener-Hopf equation.