Lotka-Volterra Equations

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What are the Lotka-Volterra Equations?

- These equations describe the population dynamics between predators and preys.
- They are a pair of nonlinear ordinary differential equations.

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dx}{dt} = (\alpha - \beta y)x$$

$\frac{dP}{dt} = kP$ (Simple population model)
$\frac{dP}{dt} = (B - D)P$ (Population model based on birth and death rates)

x = number of prey
y = number of predators
α = rate of prey population increase
β = predation rate coefficient
δ =reproduction rate of the predators per prey eaten
γ =predator mortality rate

$$\frac{dy}{dt} = \delta xy - \gamma y$$

$$\frac{dy}{dt} = (\delta x - \gamma)y$$

Finding the Numerical Solution (Euler's Method)

$$\frac{dx}{dt} = ax - \beta x$$

$$\frac{x_{n+1} - x_n}{\Delta t} = ax_n - \beta x_n y_n$$

$$\boxed{x_{n+1} = x_n + \Delta t(ax_n - \beta x_n y_n)}$$

$$\frac{dy}{dt} = \delta xy - \gamma y$$

$$\frac{y_{n+1} - y_n}{\Delta t} = \delta x_n y_n - \gamma y_n$$

$$\boxed{y_{n+1} = y_n + \Delta t(\delta x_n y_n - \gamma y_n)}$$

Finding the Numerical Solution (Heun's Method)

$$x_{n+1} = x_n + \Delta t(ax - \beta x_n y_n)$$

$$Area\ of\ Trapezoid = \frac{1}{2} h * (b_1 + b_2)$$

$$\boxed{x_{n+1} = x_n + \frac{\Delta t}{2}[ax_n - \beta x_n y_n + ax - \beta x_{n+1} y_{n+1}]}$$

$$y_{n+1} = y_n + \Delta t(\delta x_n y_n - \gamma y_n)$$

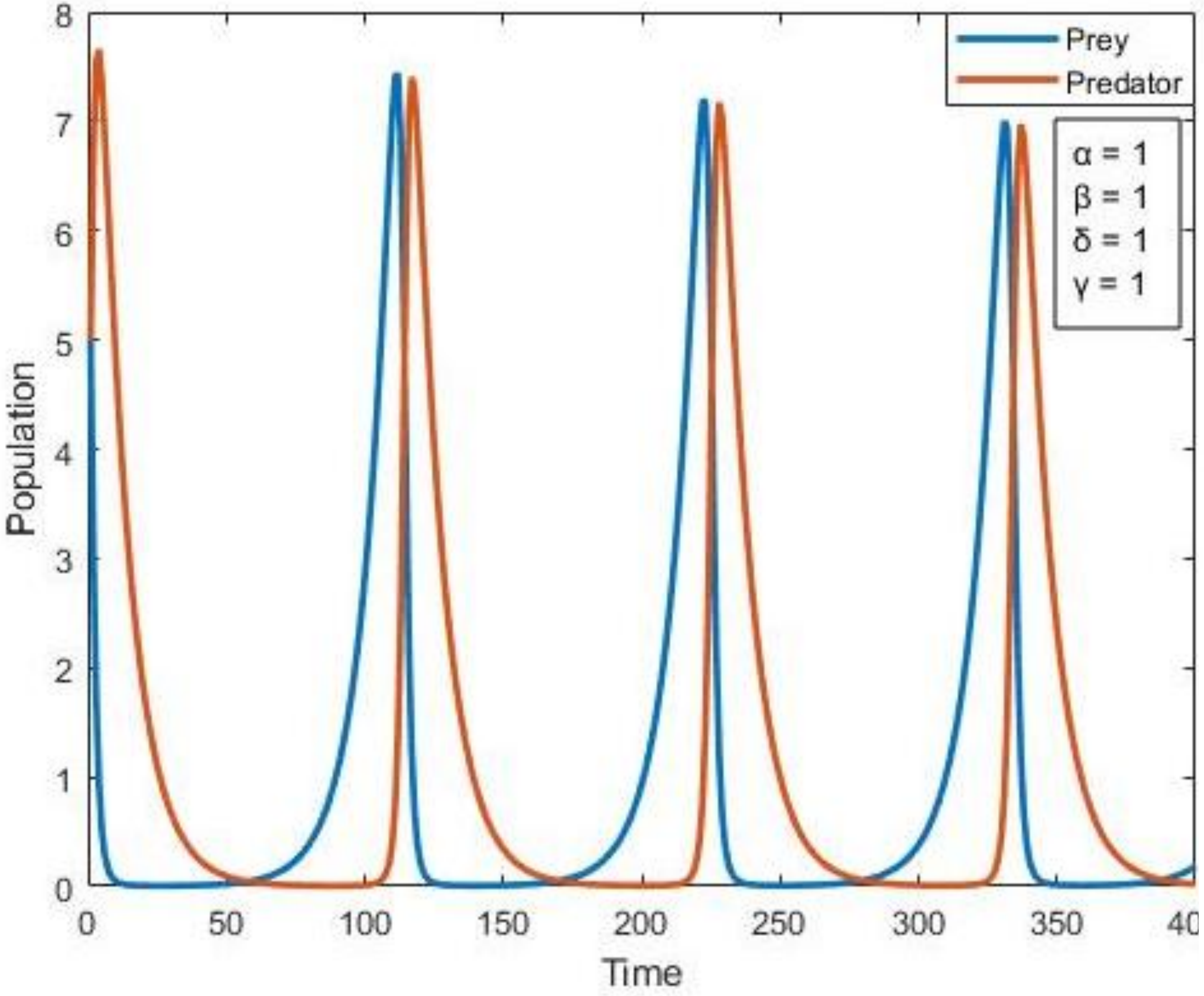
$$Area\ of\ Trapezoid = \frac{1}{2} h * (b_1 + b_2)$$

$$\boxed{y_{n+1} = y_n + \frac{\Delta t}{2}[\delta x_n y_n - \gamma y_n + \delta x_{n+1} y_{n+1} - \gamma y_{n+1}]}$$

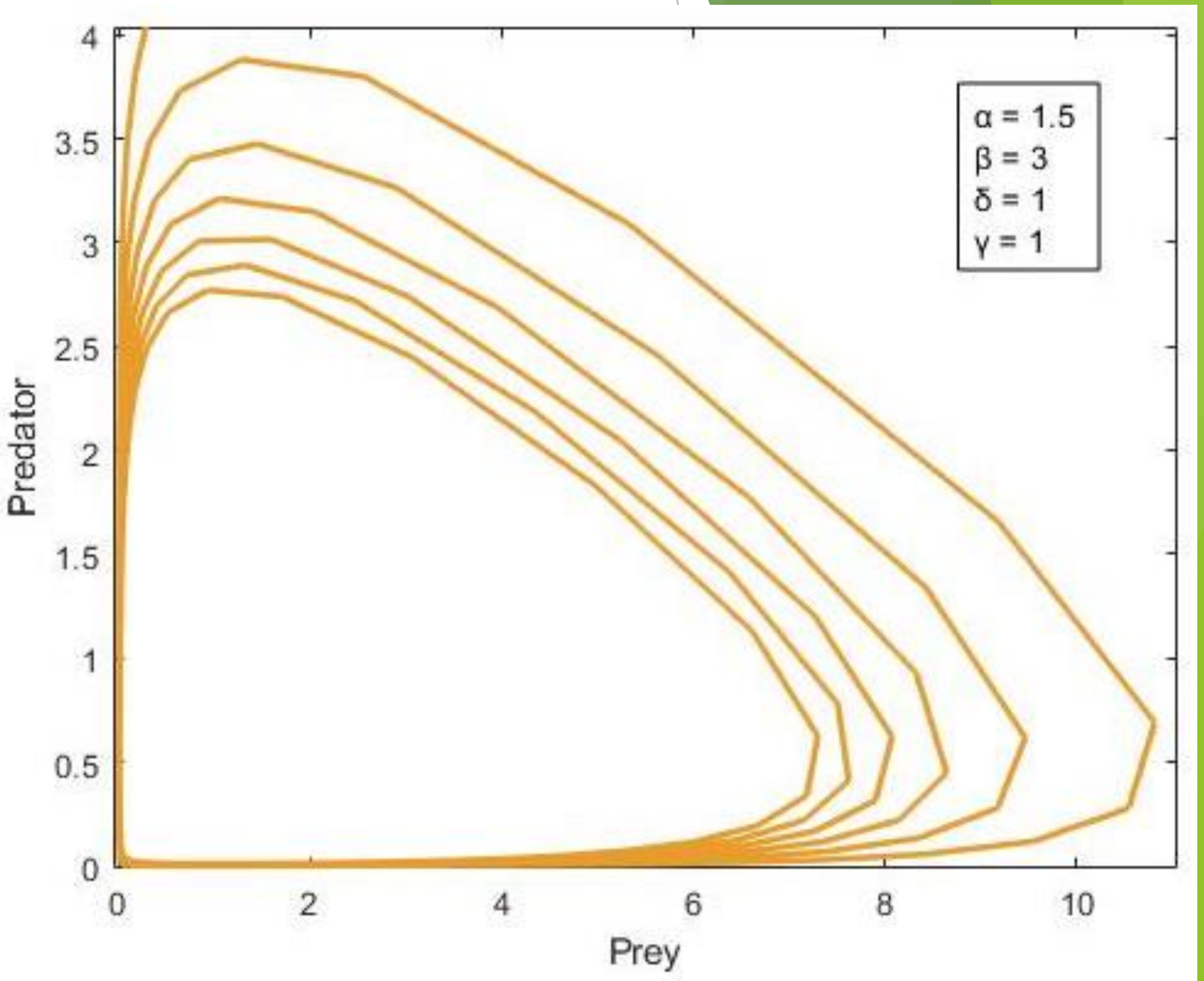
C++ Code Using Heun's Method

```
for ( int i = 0; i < n ; i++)
{
    prey << x[i] << " ";
    predator << y[i] << " ";

    x_star = x[i] + dt * fx(x[i], y[i]);
    y_star = y[i] + dt * fy(x[i], y[i]);
    x[i + 1] = x[i] + dt * 1 / 2 * (fx(x[i], y[i]) + fx(x_star, y_star));
    y[i + 1] = y[i] + dt * 1 / 2 * (fy(x[i], y[i]) + fy(x_star, y_star));

}
```

Base Graph



* We will discuss this plot later on in the presentation*



$$\frac{dP}{dt} = kP$$

$P(t) = P_o e^{kt}$

Initial values:
x(0) = 5
y(0) = 5

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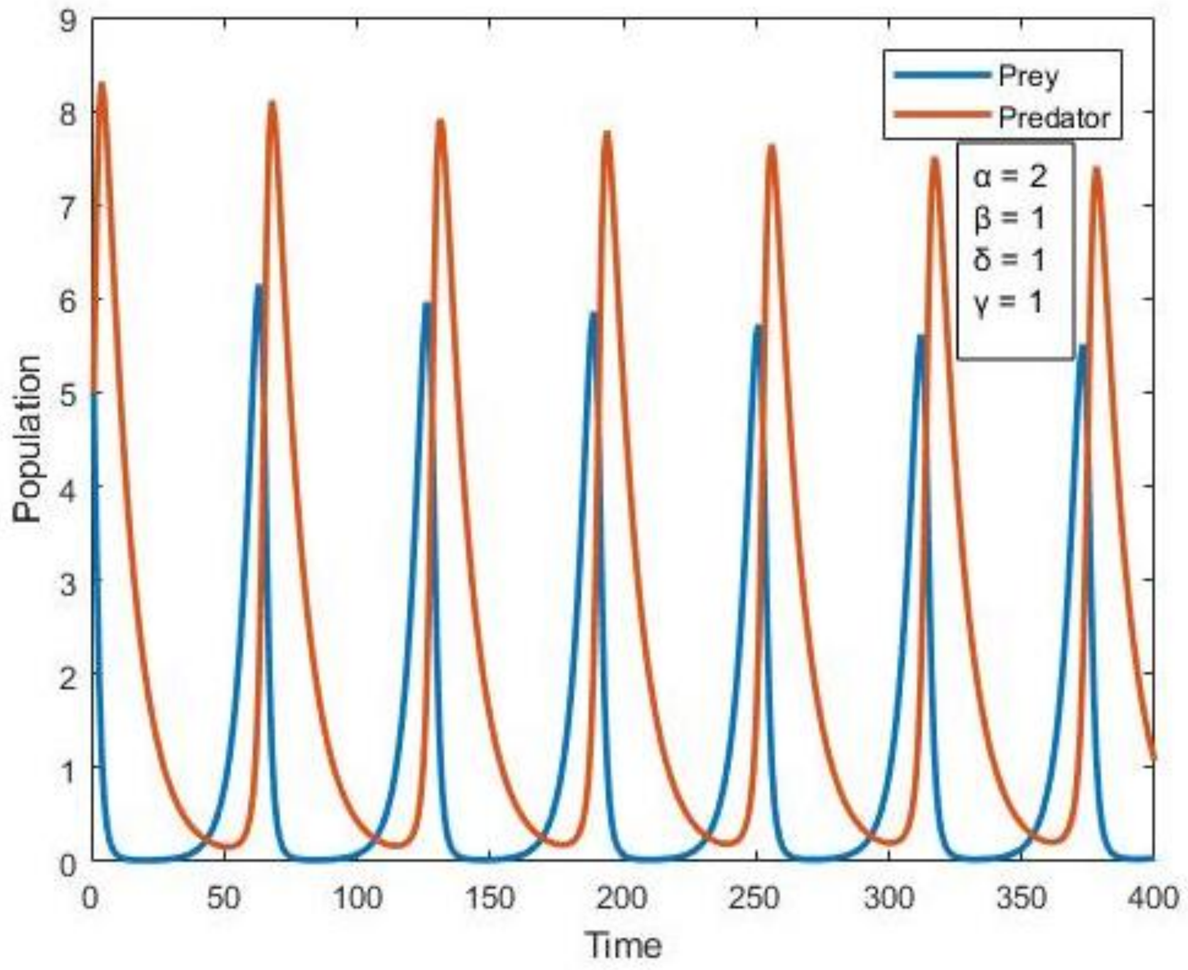
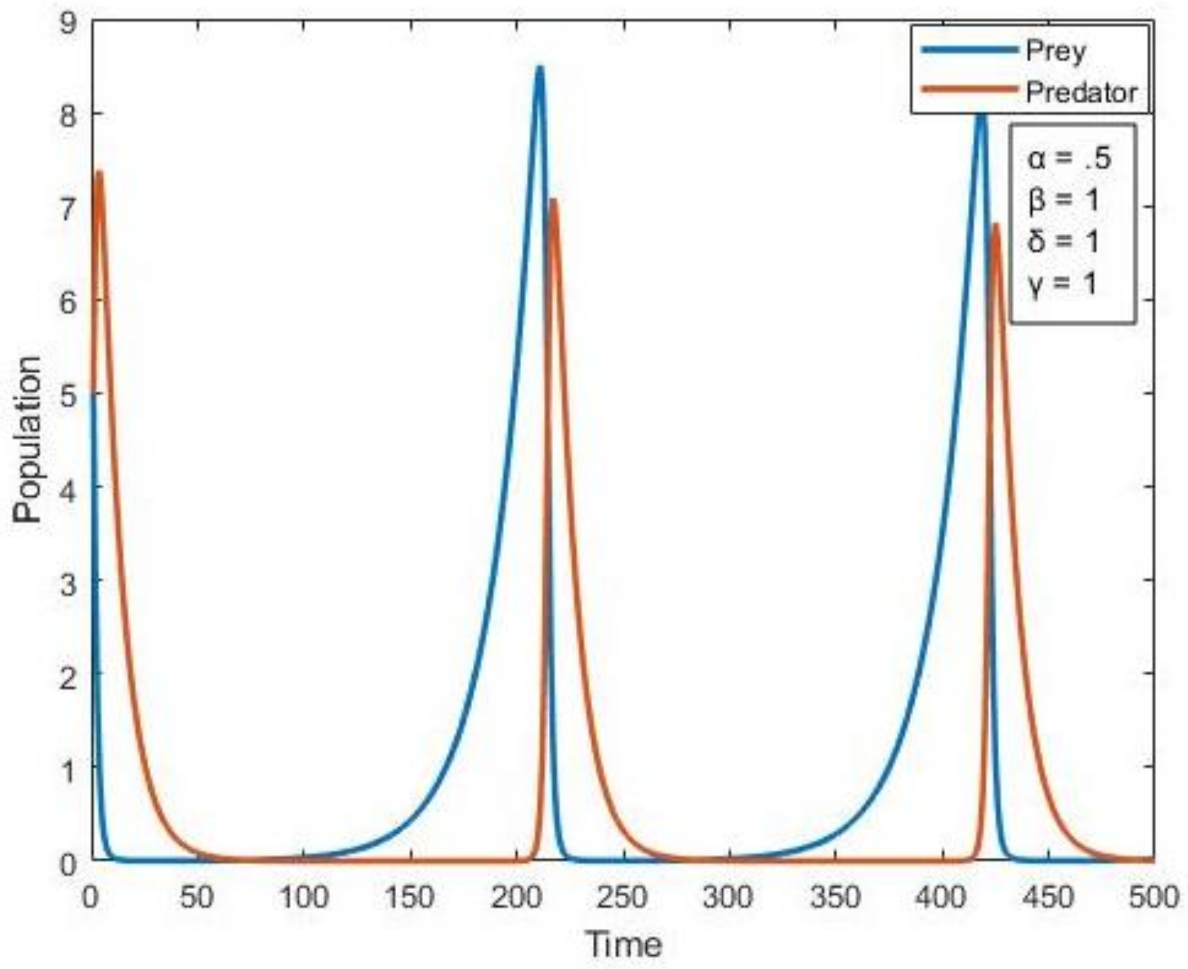
x(0) = 5
y(0) = 5

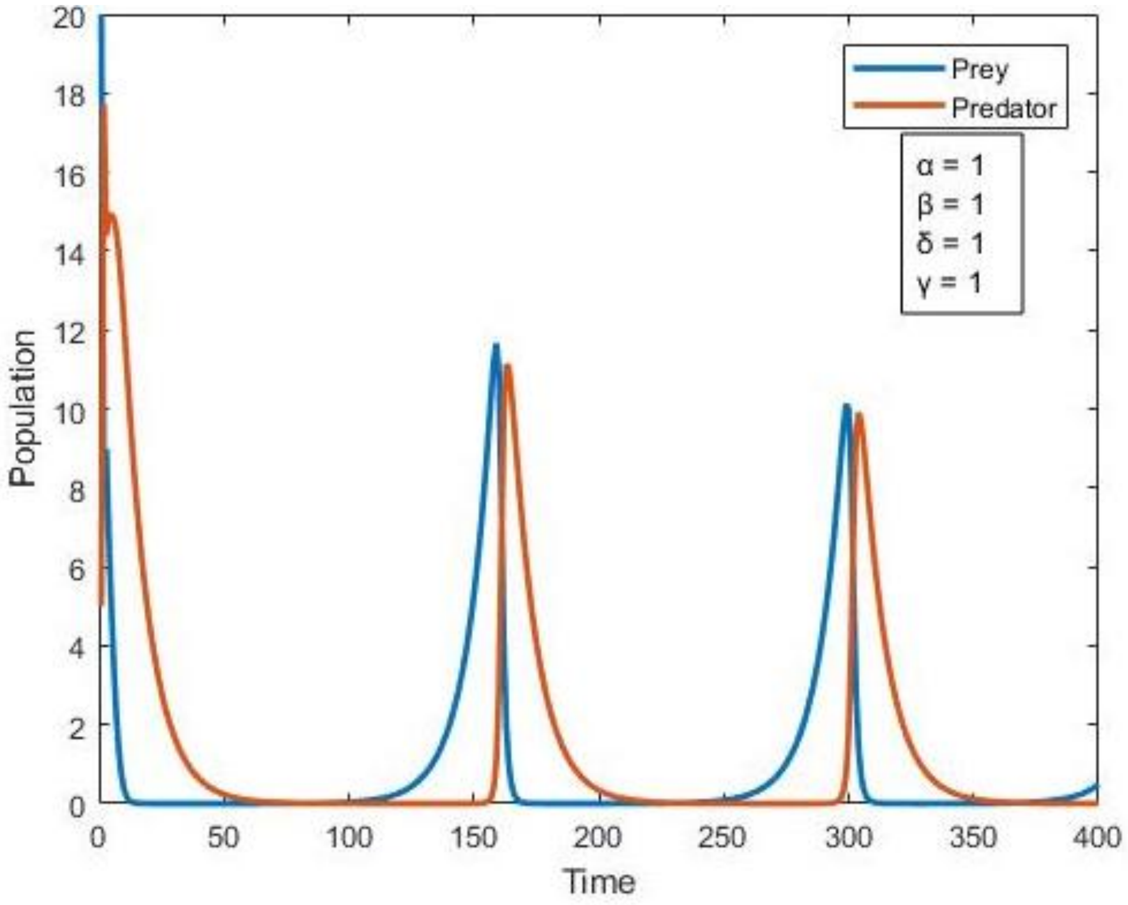
α = rate of prey population increase

$$\frac{dx}{dt} = (\alpha - \beta y)x$$
$$\frac{dy}{dt} = (\delta x - \gamma)y$$

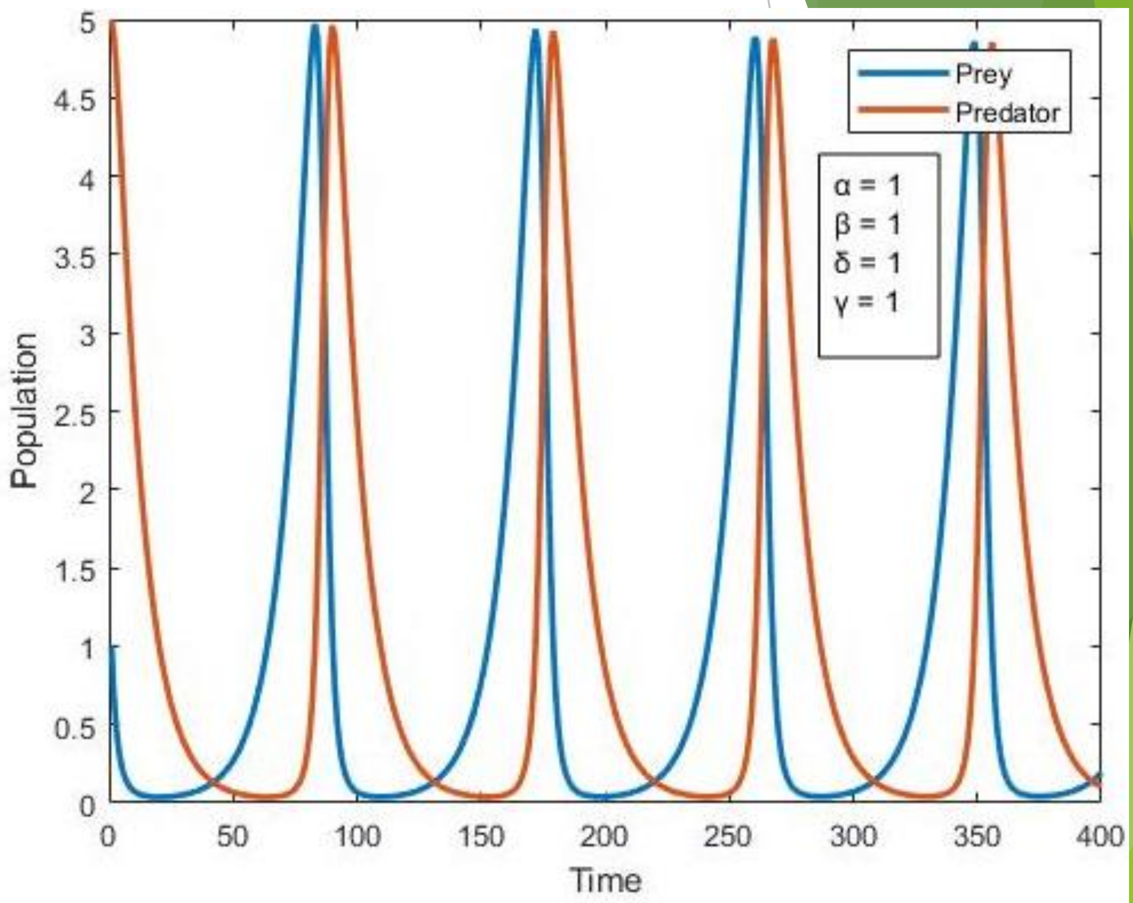




x = number of prey



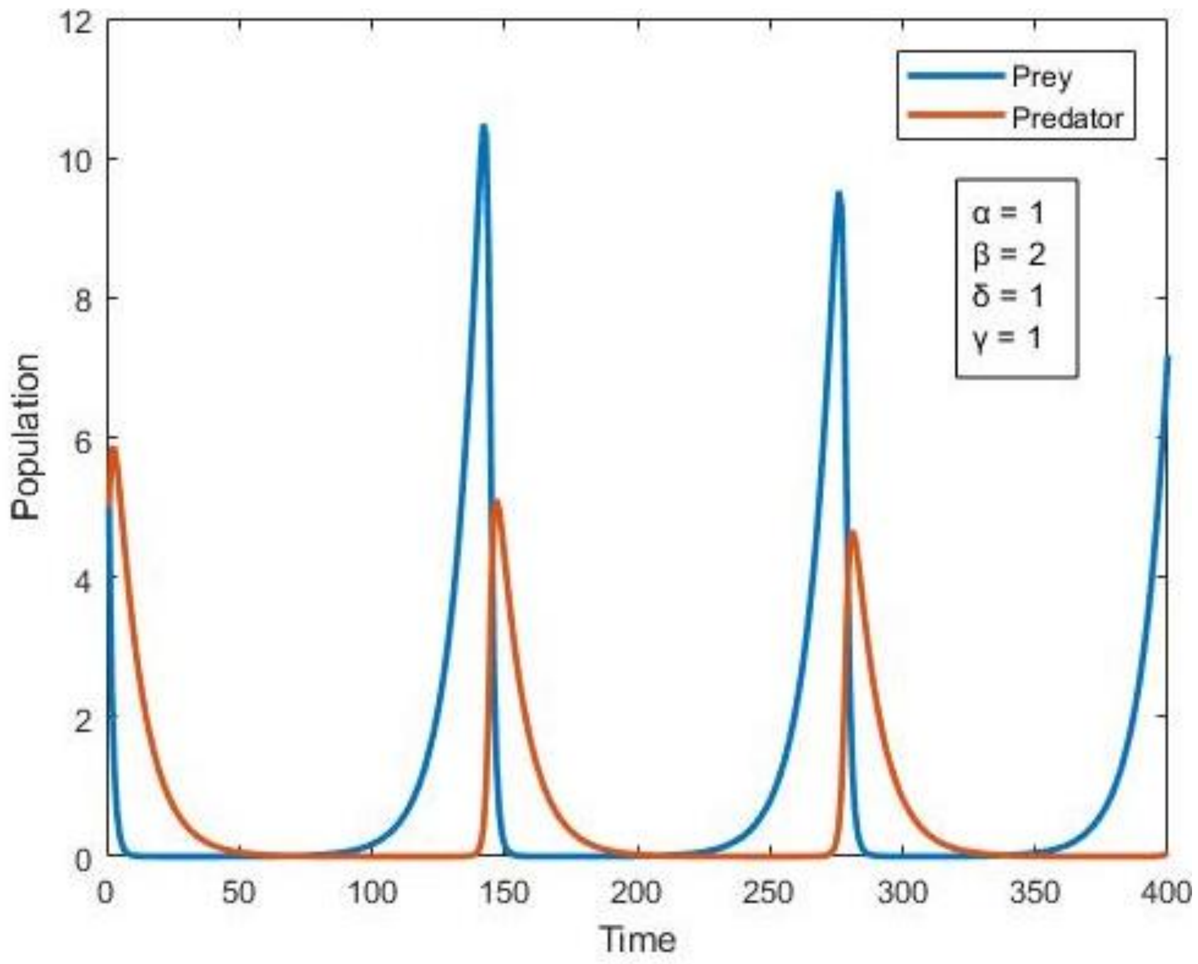
Initial values:
x(0) = 20
y(0) = 5

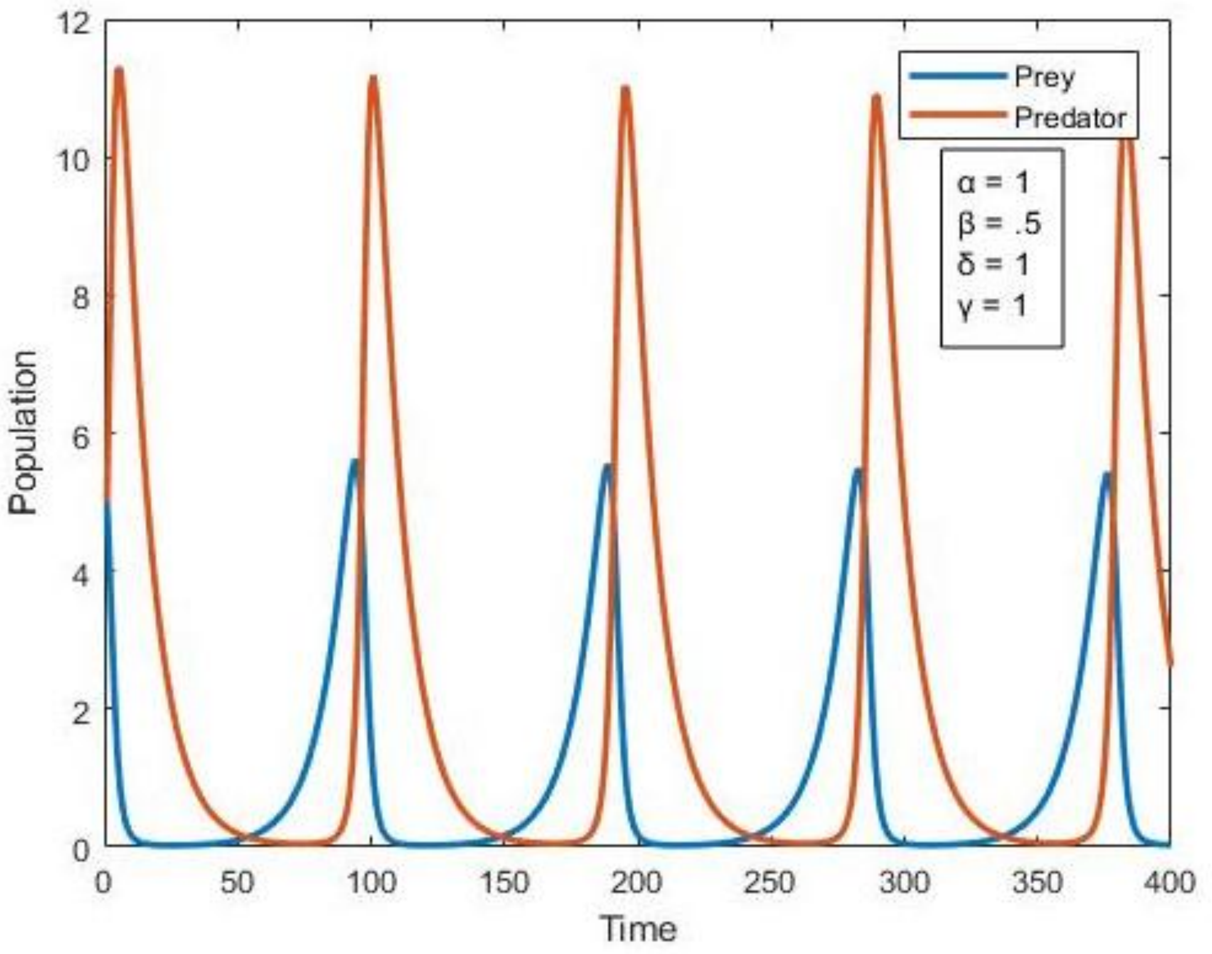


Initial values:
x(0) = 1
y(0) = 5

β = predation rate coefficient

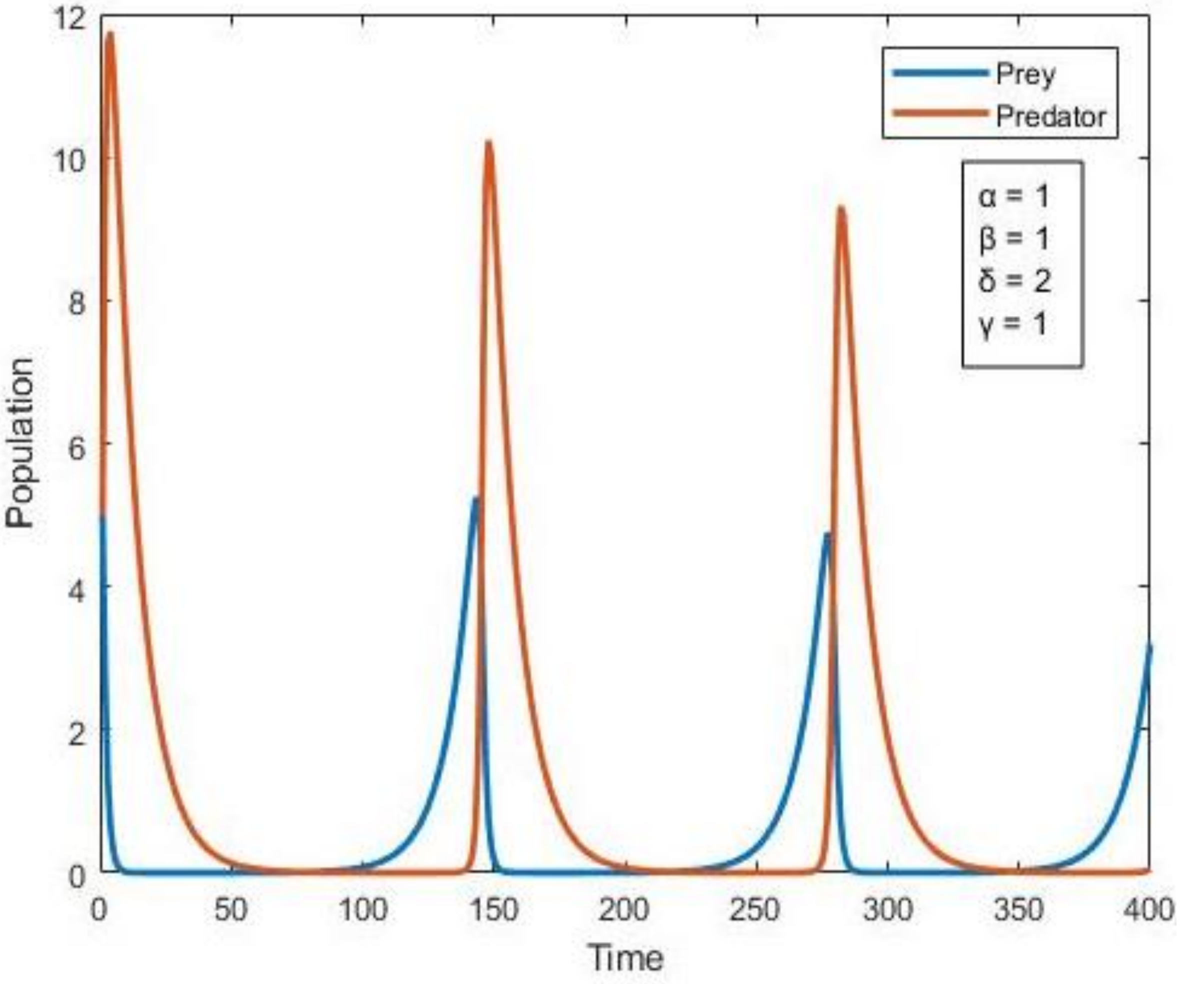
$$\frac{dx}{dt} = (\alpha - \beta y)x$$
$$\frac{dy}{dt} = (\delta x - \gamma)y$$

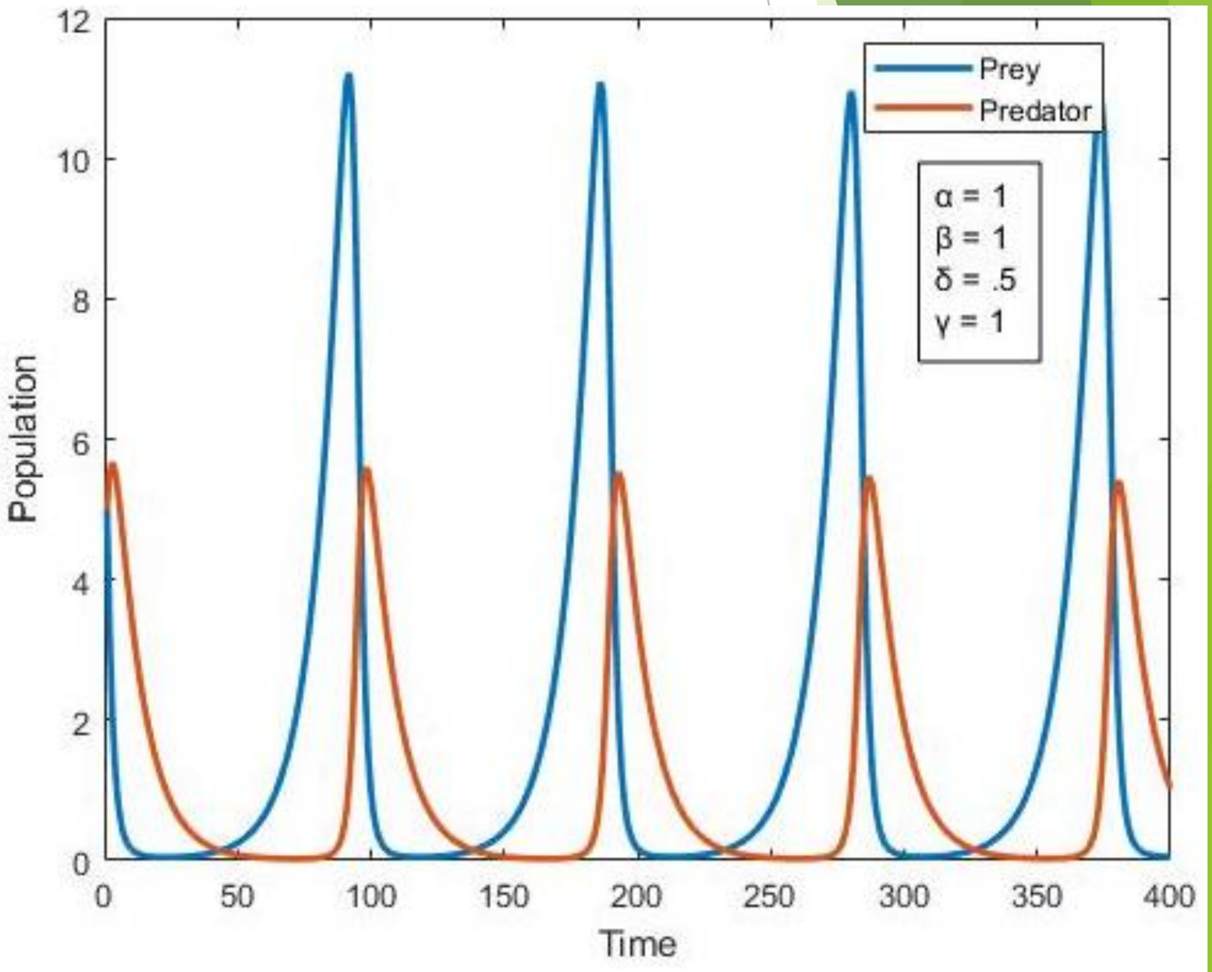




δ =reproduction rate of the predators per prey eaten

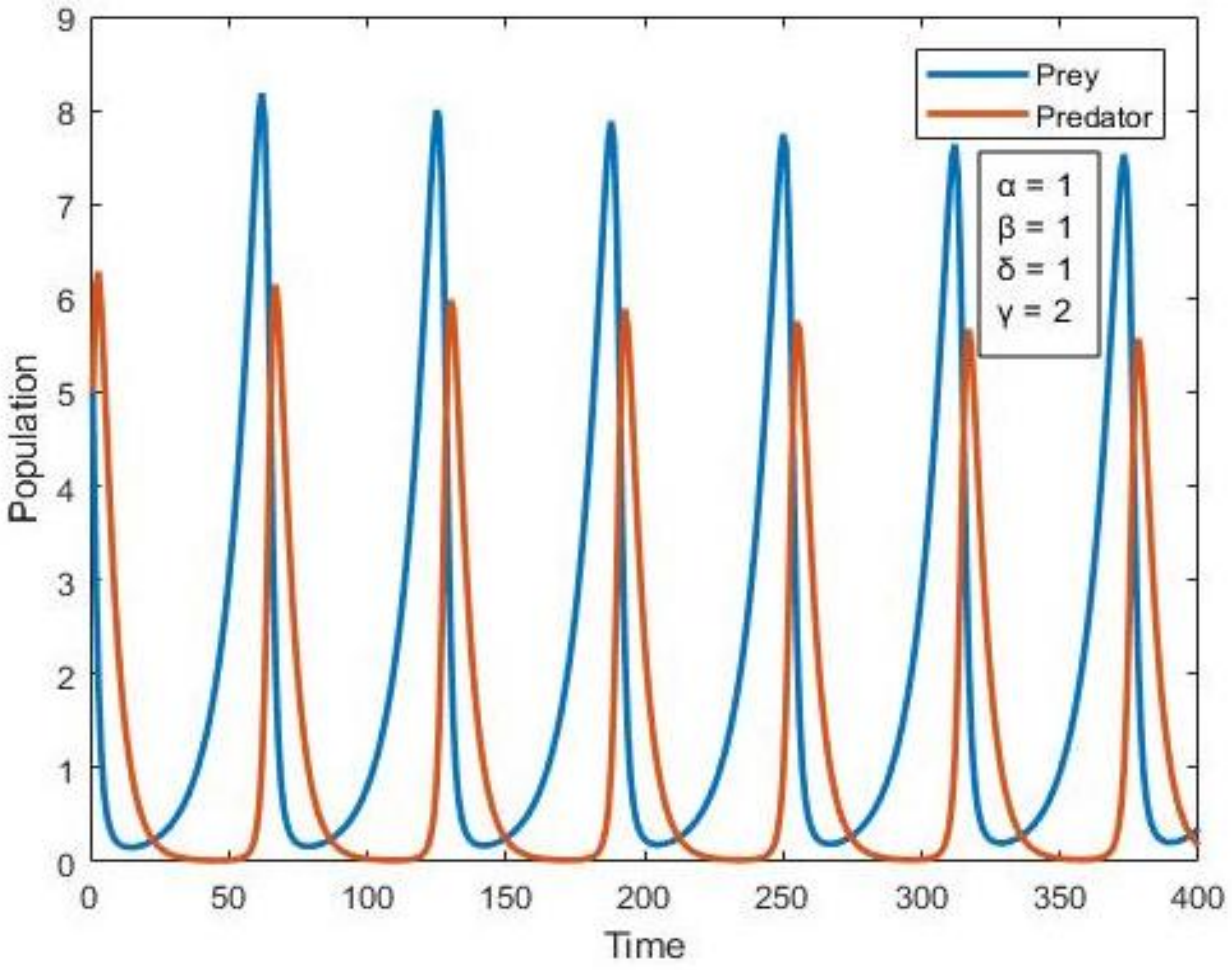
$$\frac{dx}{dt} = (\alpha - \beta y)x$$
$$\frac{dy}{dt} = (\delta x - \gamma)y$$

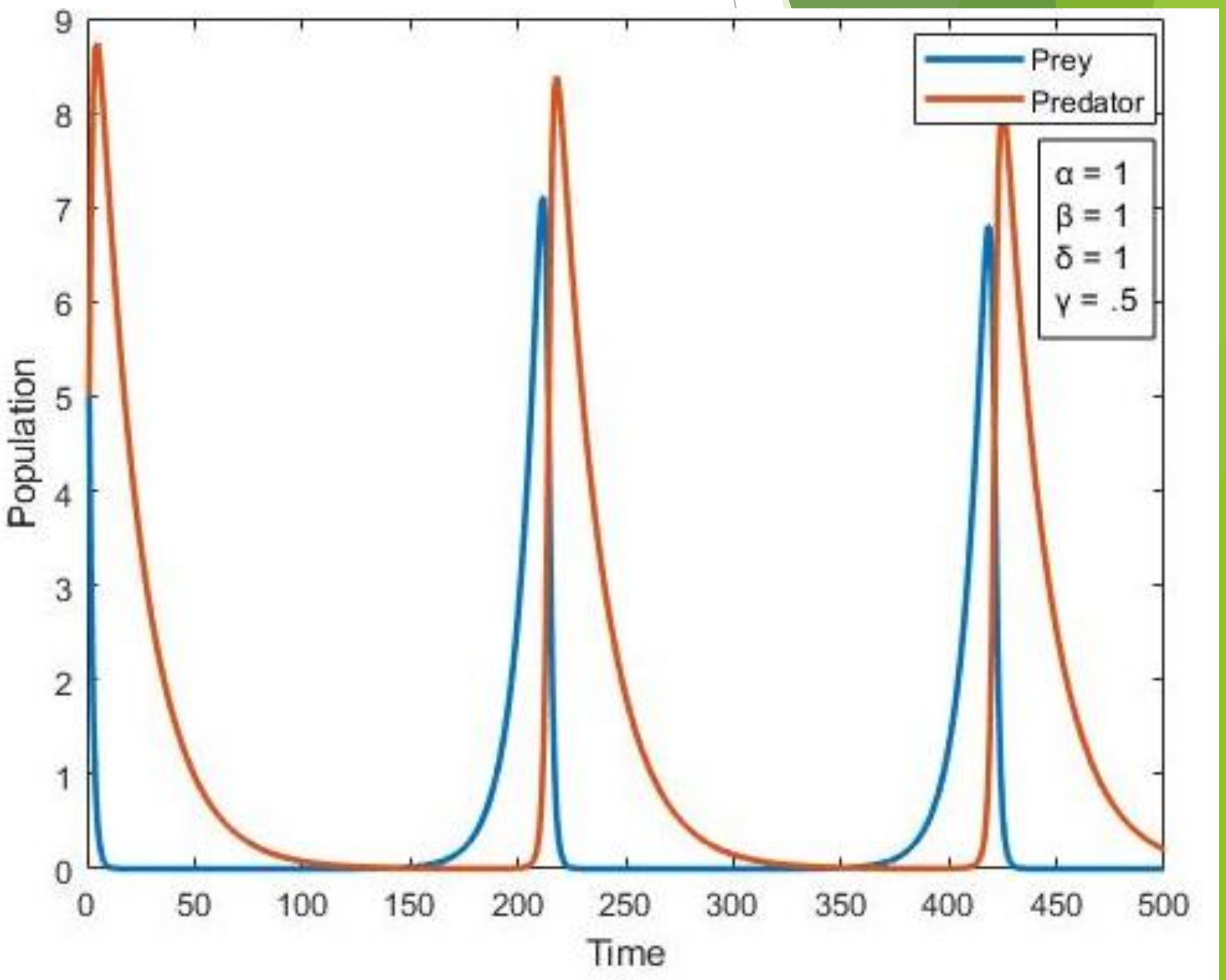




γ = predator mortality rate

$$\frac{dx}{dt} = (\alpha - \beta y)x$$
$$\frac{dy}{dt} = (\delta x - \gamma)y$$



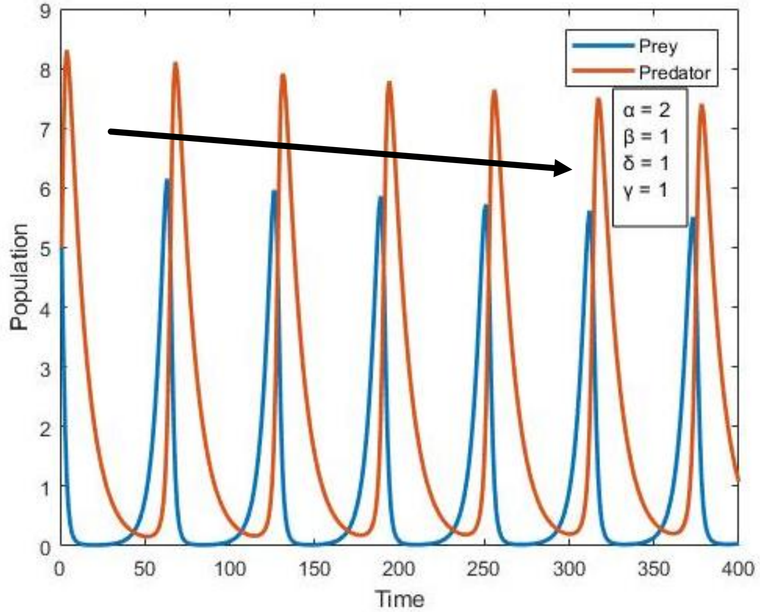


$$\frac{dx}{dt} = (\alpha - \beta y)x \qquad \frac{dy}{dt} = (\delta x - \gamma)y$$

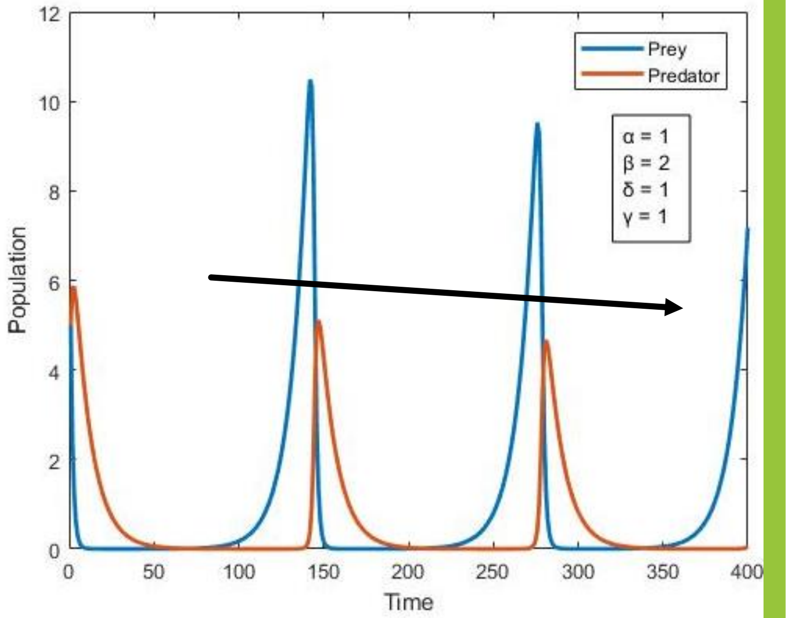
$$0 = (\alpha - \beta y)x \qquad 0 = (\delta x - \gamma)y$$

$$x = 0 \qquad y = 0$$

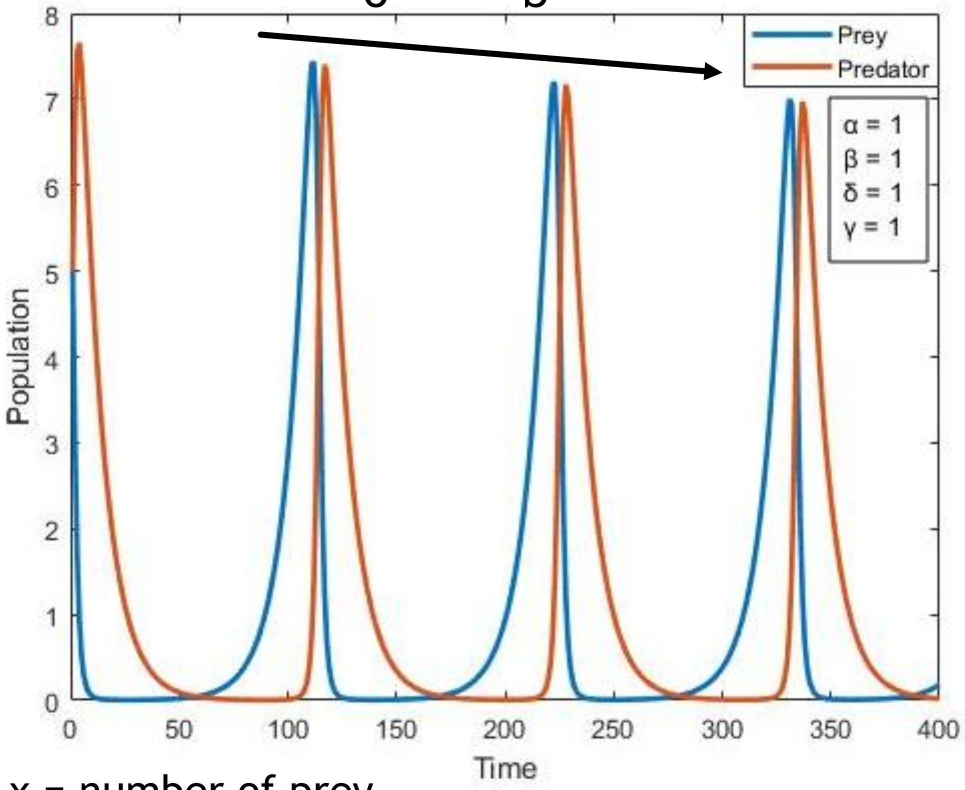
$$x = \frac{\gamma}{\delta} \qquad y = \frac{\alpha}{\beta}$$



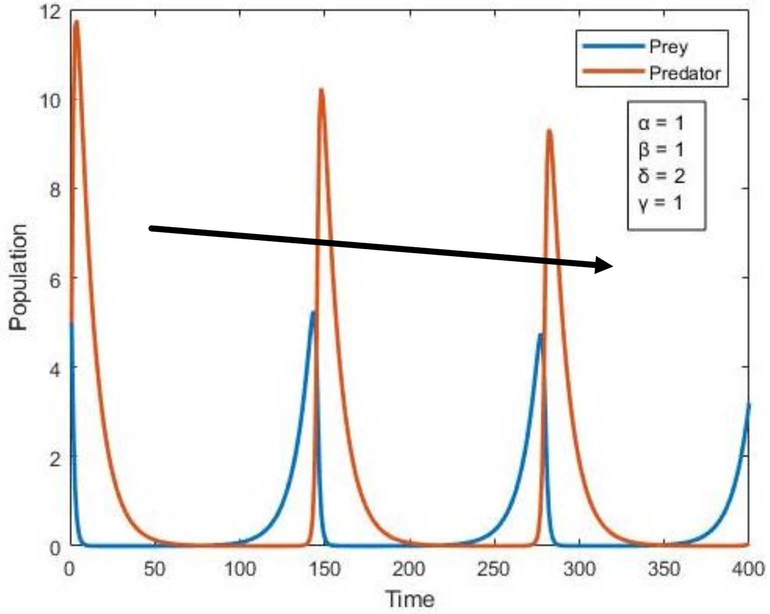
*1 prey, 2 predators



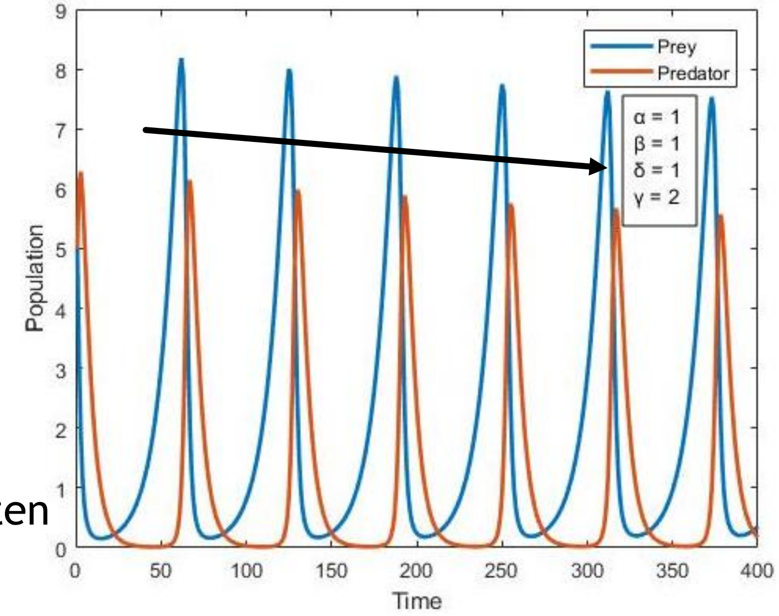
*1 prey, 1/2 predators



*1/2 prey, 1 predators

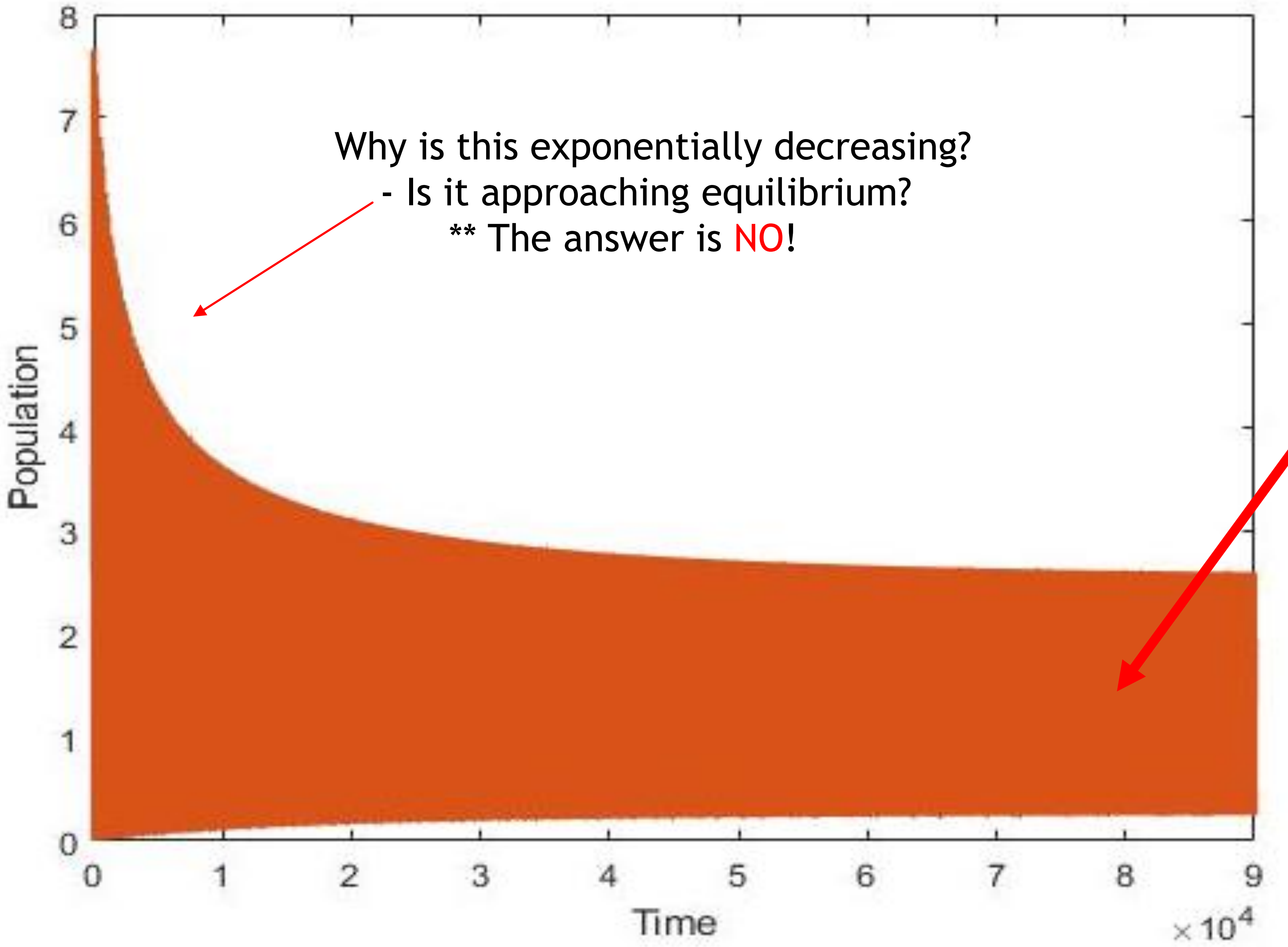


*2 prey, 1 predators

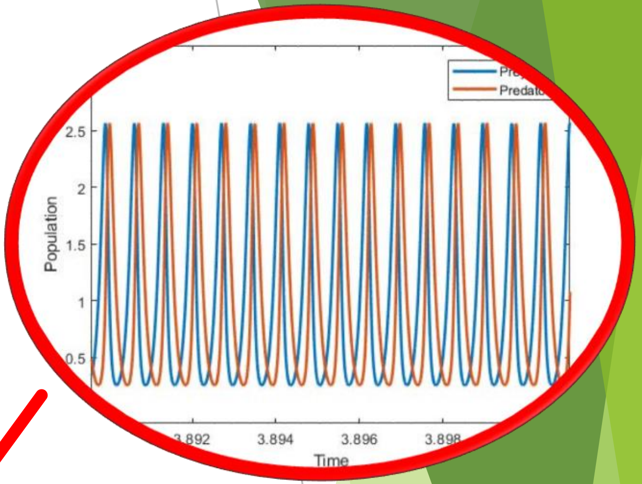


x = number of prey
y = number of predators
α = rate of prey population increase
β = predation rate coefficient
δ =reproduction rate of the predators per prey eaten
γ =predator mortality rate

What about equilibrium?



Why is this exponentially decreasing?
- Is it approaching equilibrium?
** The answer is NO!



A little interlude about Eigenvalues and Eigenvectors

- An eigenvector is a vector that when multiplied to a matrix, yields the same vector, only transformed by a factor.
- This factor is the eigenvalue

$$Ax = b$$

$$\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{2} \\ \mathbf{1} \end{bmatrix} = 4 \begin{bmatrix} \mathbf{2} \\ \mathbf{1} \end{bmatrix}}$$

Same vector, one is just multiplied by a factor of 4!

$$A\vec{v} = \lambda\ \vec{v}$$

***det(A - λI) = 0**

***then row reduce!**

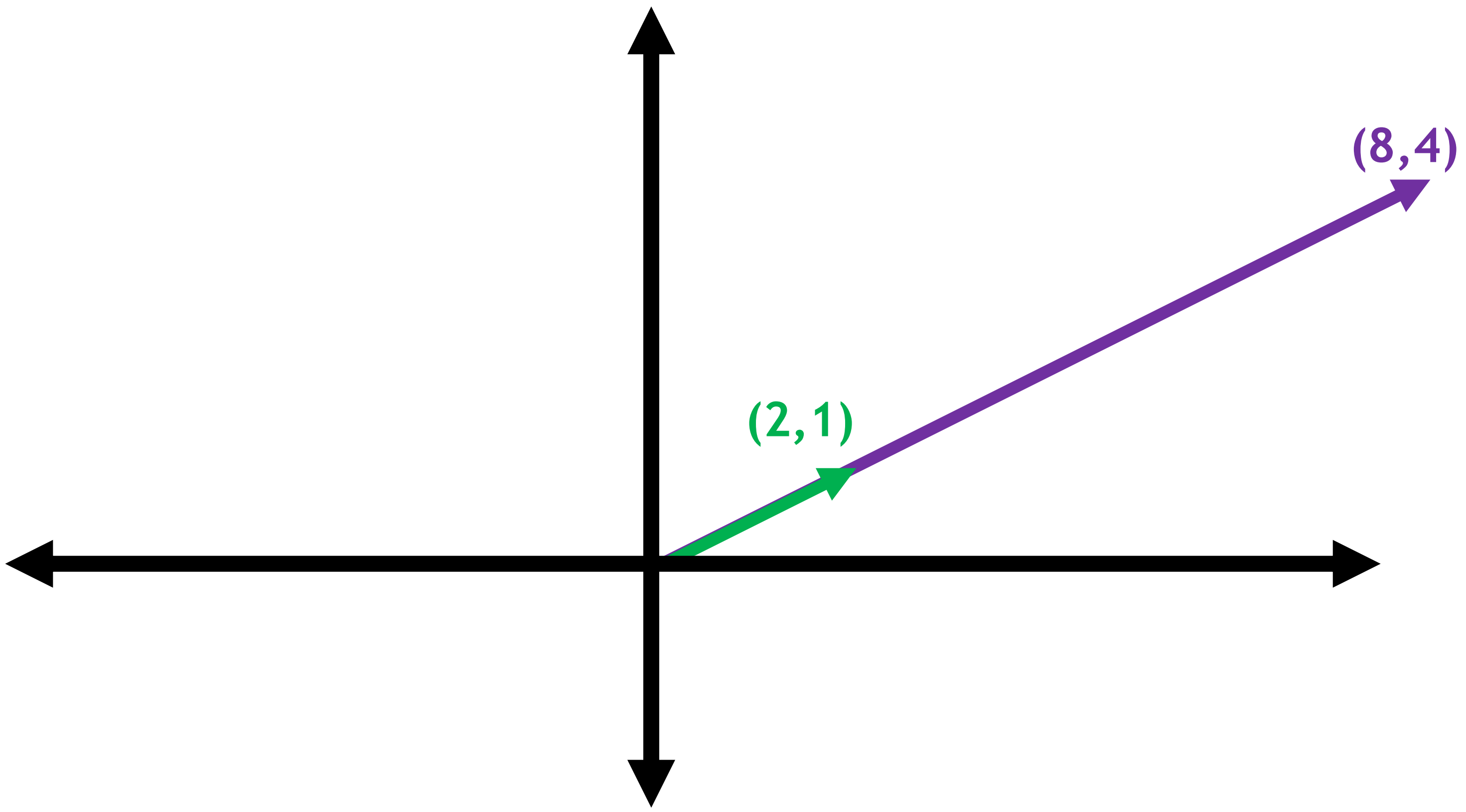
Where:

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \textit{ (eigenvector)}$$

$$\lambda = 4 \textit{ (eigenvalue)}$$

A little interlude about Eigenvalues and Eigenvectors



$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda = 4$

Back to Equilibrium

$$\frac{dx}{dt} = ax - \beta xy$$

$$\frac{dy}{dx} = -\gamma y + \delta xy$$

$$\rightarrow \begin{bmatrix} {dx}/{dt} \\ {dy}/{dt} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -\gamma \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\beta xy \\ \delta xy \end{bmatrix}$$

Since we want to find where there is no change in population:

$dx/dt = 0$ & $\boxed{(x = 0 , y = 0)}$

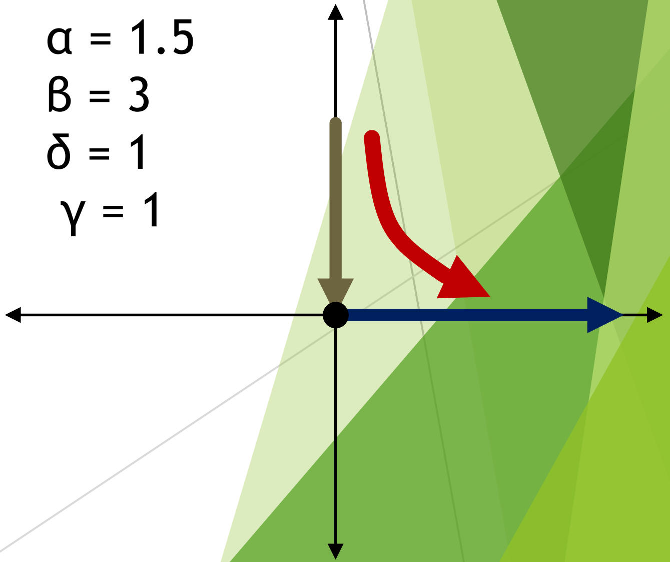
$dy/dt = 0$ $\left(x = \frac{\gamma}{\delta} , y = \frac{\alpha}{\beta} \right)$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a - \lambda & 0 \\ 0 & -\gamma - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \cancel{\begin{bmatrix} -\beta xy \\ \delta xy \end{bmatrix}}$$

$\lambda_1 = a$ $\vec{v}_1 = \begin{bmatrix} a \\ 0 \end{bmatrix}$

$\lambda_2 = -\gamma$ $\vec{v}_2 = \begin{bmatrix} 0 \\ -\gamma \end{bmatrix}$

$\alpha = 1.5$
$\beta = 3$
$\delta = 1$
$\gamma = 1$



Back to Equilibrium

$dx/dt = 0$ & $dy/dt = 0$

$(x = 0, y = 0)$

$\boxed{(x = \frac{\gamma}{\delta}, y = \frac{\alpha}{\beta})}$

Let's transform this to the point $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$:

$$\frac{dx}{dt} = (a - \beta(y - \frac{a}{\beta}))\,(x - \frac{\gamma}{\delta})$$

$$\frac{dy}{dt} = (\delta(x - \frac{\gamma}{\delta}) - \gamma)(y - \frac{a}{\beta})$$

$\rightarrow$

$$\frac{dx}{dt} = -\beta xy - \frac{\beta\gamma}{\delta}y$$

$$\frac{dy}{dt} = \delta xy + \frac{\delta a}{\beta}x$$

$$\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0-\lambda & -\frac{\beta\gamma}{\delta}\\ \frac{\delta x a}{\beta} & 0-\lambda\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + \cancel{\begin{bmatrix}-\beta xy\\ \delta xy\end{bmatrix}}$$

$$\lambda_1 = i\sqrt{a\gamma}$$

$$\lambda_2 = -i\sqrt{a\gamma}$$

Back to Equilibrium

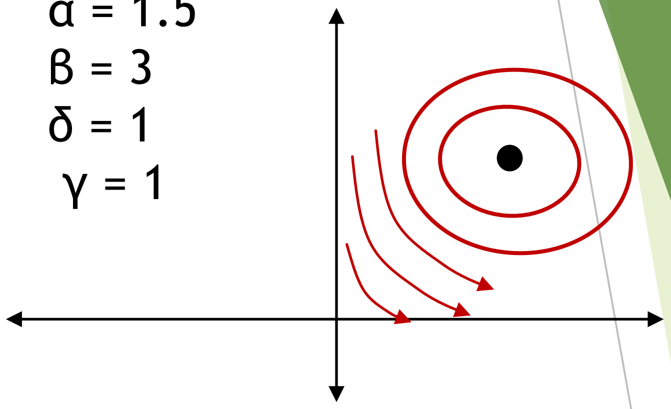
Let’s look at the behavior of the graph at this point!

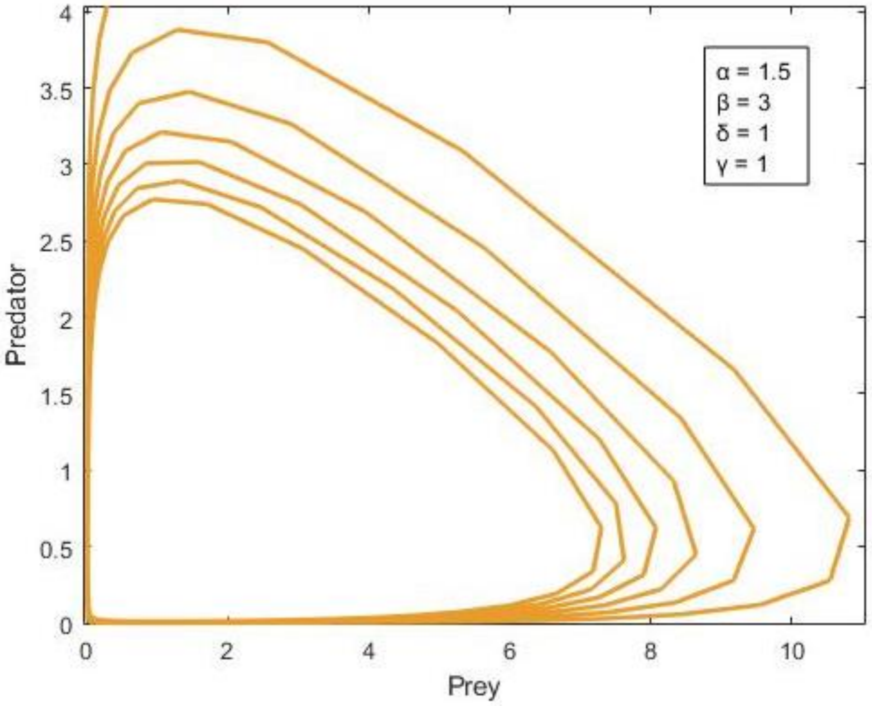
$$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{\frac{\delta a}{B}x}{-\frac{B\gamma}{\delta}y} = -\frac{a\delta^2 x}{B^2\gamma y}$$

$$\int a\delta^2 x\,dx = \int B^2\gamma y\,dy$$

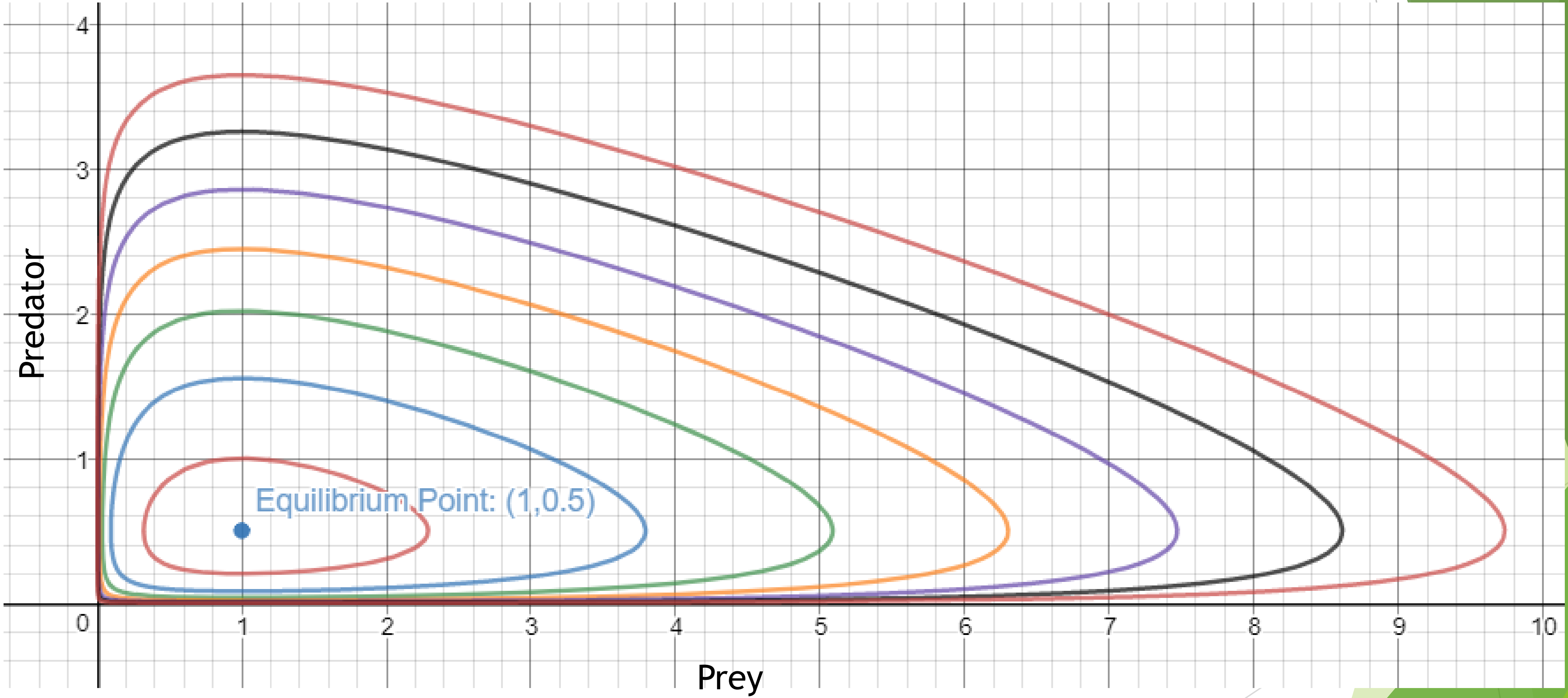
$$a\delta^2x^2 + B^2\gamma y^2 = C$$

α = 1.5
β = 3
δ = 1
γ = 1

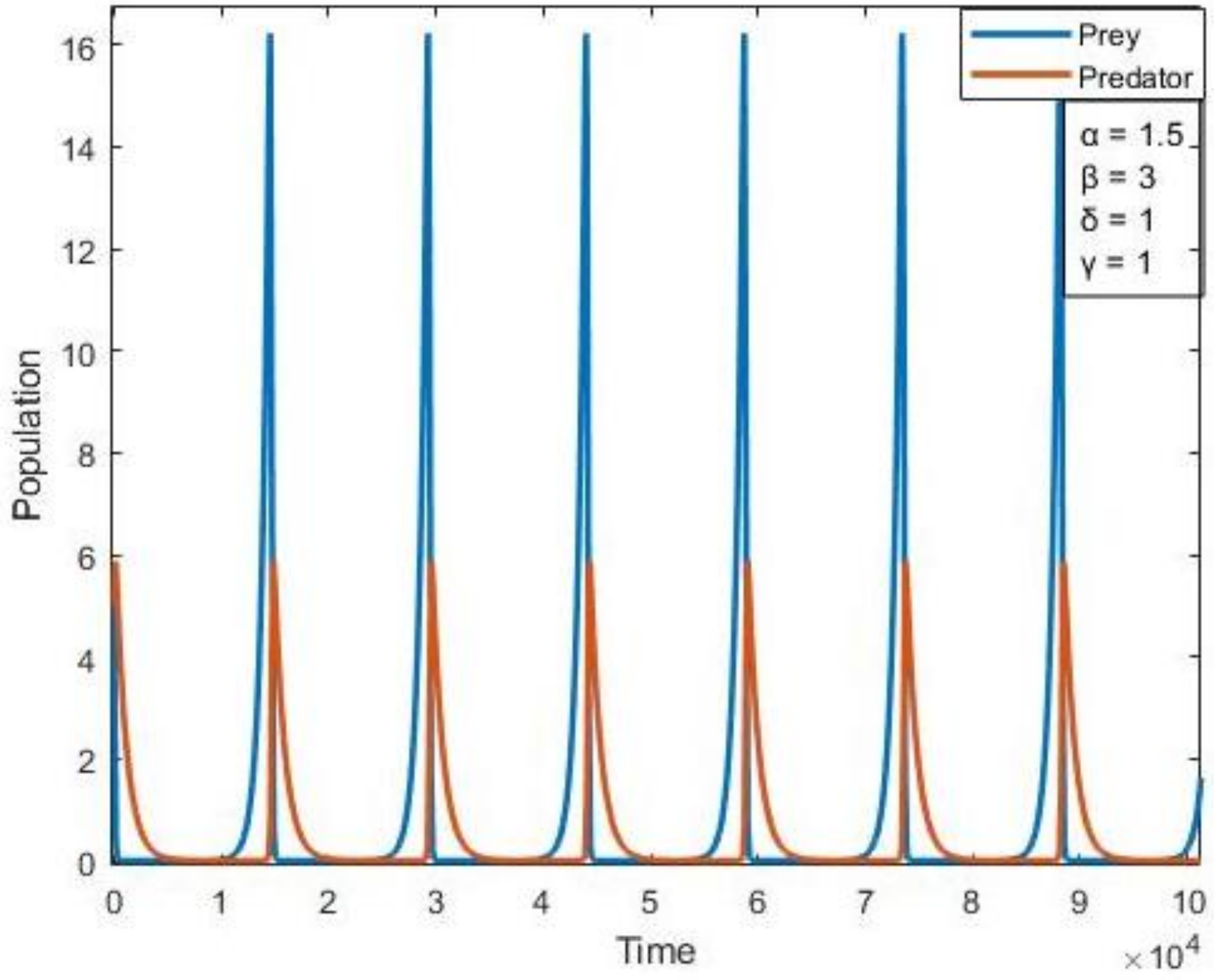




Back to Equilibrium



** Using Desmos to graph


Prey
Predator
α = 1.5
β = 3
δ = 1
γ = 1
Population
Time
×10^4
0
2
4
6
8
10
12
14
16
0
1
2
3
4
5
6
7
8
9
10

Damped Lotka – Volterra Equation

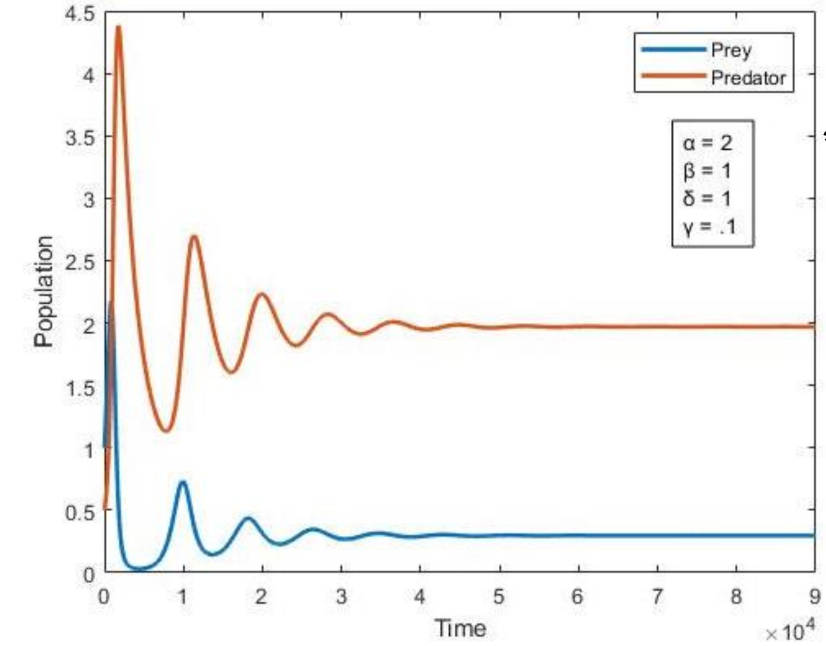
- There are more factors that affect the rate of population growth in an ecosystem.
- Factoring these other factors produces and damped version of Lotka – Volterra Equation.

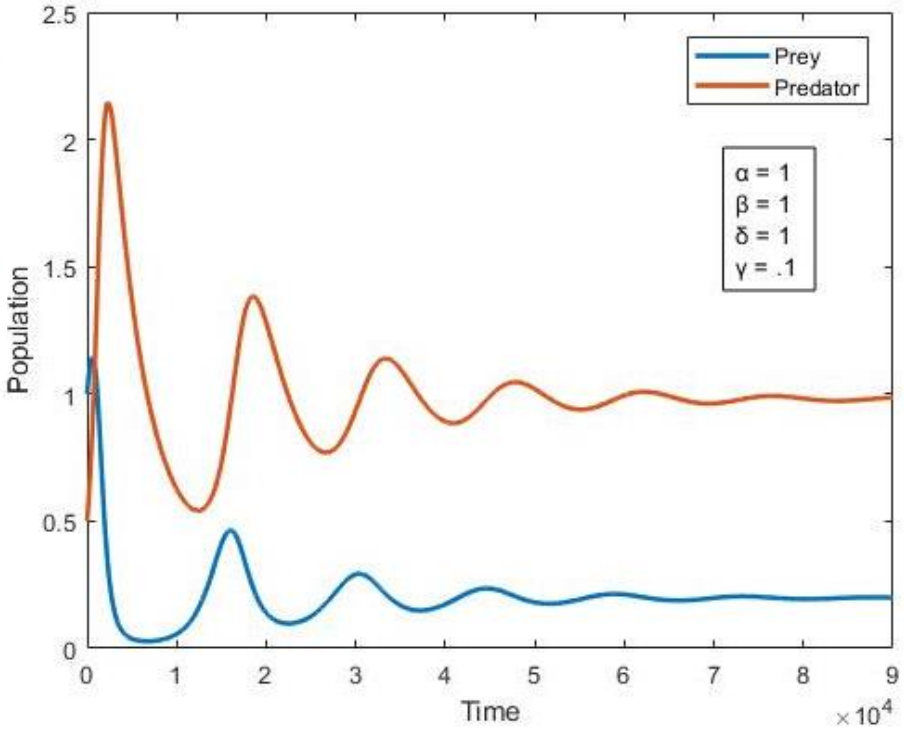
$$\frac{dx}{dt} = \alpha x - \lambda x^2 - \beta xy$$

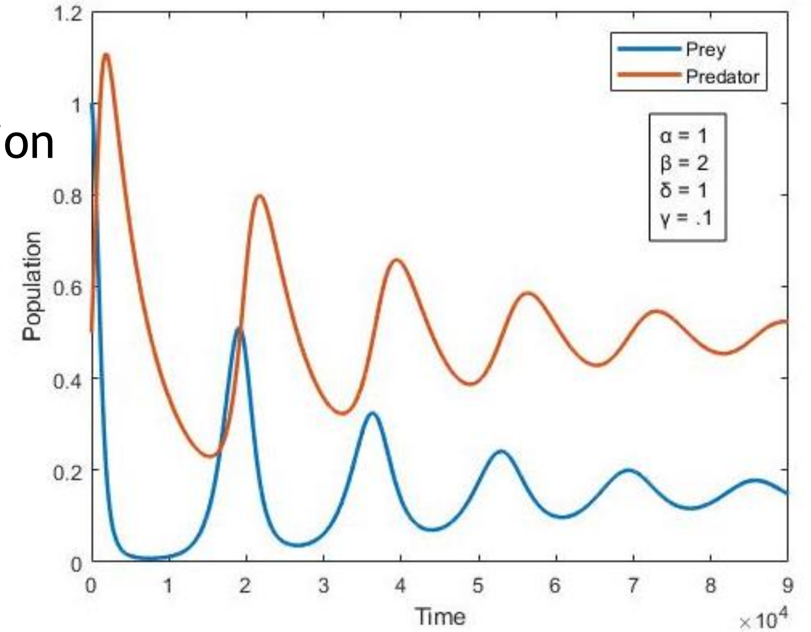
$$\frac{dy}{dt} = -\gamma y - \kappa y^2 + \delta xy,$$

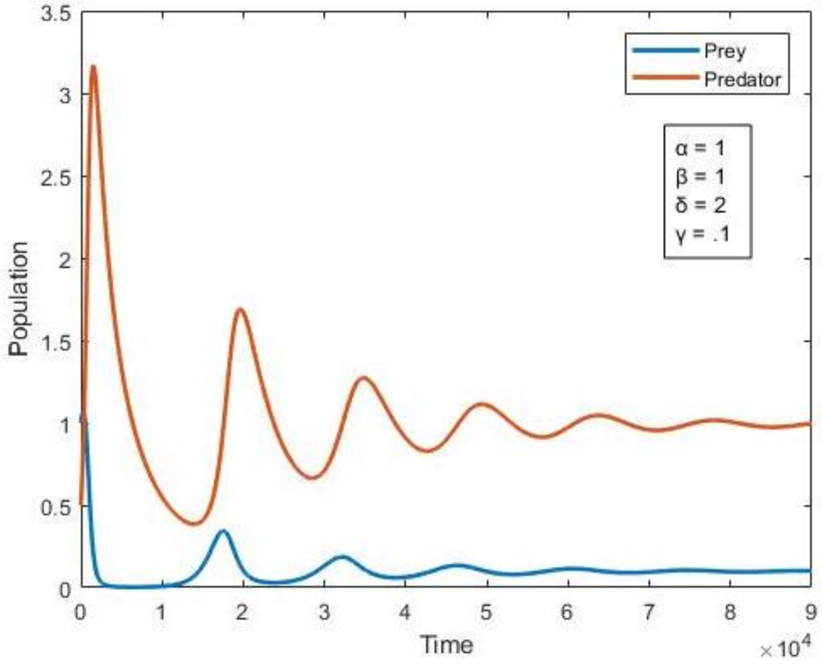
Term that introduces damping

**Graphs of damped Lotka – Volterra Equation









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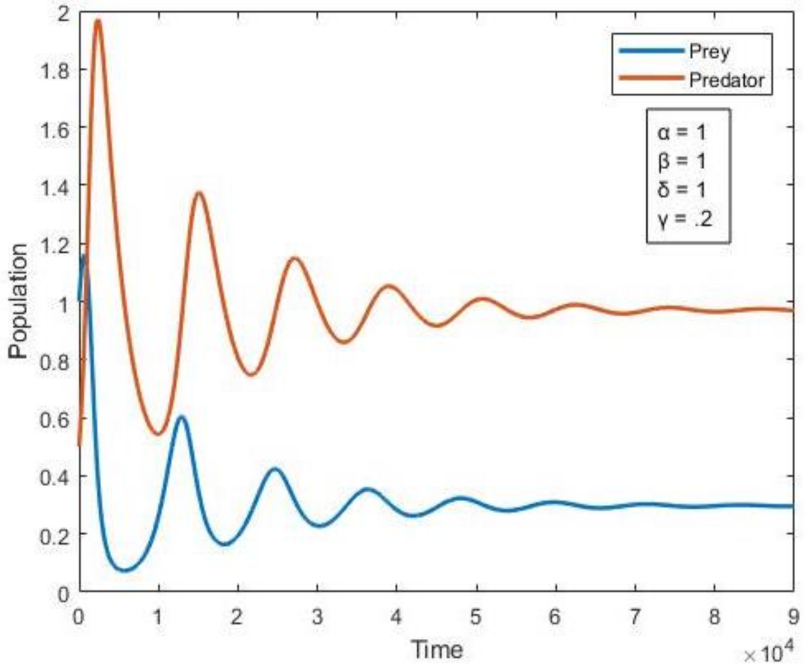

More Things to Explore!

- Competitive Lotka-Volterra equations
  - $\frac{dx_i}{dt} = rix_i\left(1 - \frac{\sum_{j=1}^{N} \alpha_{ij}x_j}{k}\right)$
- Generalized Lotka-Volterra equation
  - $\frac{dx_i}{dt} = xif_i(x)$
  - $f = r + Ax$
- Mutualism for the Lotka-Volterra equation
  - $\frac{dN}{dt} = r_1N\left(1 - \frac{N}{K_1} + \beta_{12}\frac{M}{K_1}\right)$
  - $\frac{dM}{dt} = r_2M\left(1 - \frac{M}{K_2} + \beta_{21}\frac{N}{K_2}\right)$

Questions?

References

http://www.math.ubbcluj.ro/~didactica/pdfs/2014/didmath2014-02.pdf
https://en.wikipedia.org/wiki/Lotka%E2%80%93Volterra_equations#Population_equilibrium
http://mc-stan.org/users/documentation/case-studies/lotka-volterra-predator-prey.html
https://www.math.tu-berlin.de/fileadmin/i26_fg-stannat/VorlesungsskriptSoSe2017v2.pdf
https://www.youtube.com/watch?v=pDESymjFkAU&t=322s