Notes on Number Systems: From Everyday Examples to Generalized Radix Systems

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1 Introduction

Number systems are fundamental to both our daily lives and the inner workings of computers. We use them to count money, measure time, and perform calculations. In computing, these systems are used to represent data and instructions. In these notes, we gradually generalize the concept, starting from familiar real-life examples and moving towards a rigorous mathematical framework that covers any arbitrary radix.

2 Real-Life Examples and Scenarios

2.1 Counting and Money

In everyday life, we use the decimal (base 10) system. For instance, when counting money or measuring weight, digits from 0 to 9 are used. A price tag like $12.34 is represented as:

$$12.34 = 1 \times 10^1 + 2 \times 10^0 + 3 \times 10^{-1} + 4 \times 10^{-2}.$$

2.2 Timekeeping: Base 60

Historical systems like timekeeping use base 60. For example, one hour is divided into 60 minutes and one minute into 60 seconds. Although less common in modern arithmetic, this shows how different radices can be useful in practical applications.

2.3 Digital Computers: Binary and Beyond

Computers operate using the binary (base 2) system, where only 0 and 1 are used. Other systems such as octal (base 8) and hexadecimal (base 16) are also common as they provide a compact way to represent binary data. For example:

- Binary: 1011_2 represents a number computed as:

$$1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 8 + 0 + 2 + 1 = 11_{10}.$$

- Hexadecimal: $1A3_{16}$ is computed as:

$$1 \times 16^2 + 10 \times 16^1 + 3 \times 16^0 = 256 + 160 + 3 = 419_{10}.$$

3 Positional Numeral Systems

A positional numeral system expresses numbers as a sum of digits multiplied by powers of a base b. In general, a number N with digits $d_{n-1}, d_{n-2}, \ldots, d_0$ is given by:

$$N = d_{n-1}b^{n-1} + d_{n-2}b^{n-2} + \cdots + d_1 b^1 + d_0 b^0,$$

where each digit d_i satisfies $0 \le d_i < b$.

3.1 Examples of Popular Number Systems

- **Decimal (Base 10):** Uses digits 0–9. For example, $345_{10} = 3 \times 10^2 + 4 \times 10^1 + 5 \times 10^0$.
- **Binary (Base 2):** Uses digits 0 and 1. Example: $1101_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$.
- **Octal (Base 8):** Uses digits 0–7. Example: $157_8 = 1 \times 8^2 + 5 \times 8^1 + 7 \times 8^0$.
- **Hexadecimal (Base 16):** Uses digits 0–9 and letters A–F. Example: $1A3_{16} = 1 \times 16^2 + 10 \times 16^1 + 3 \times 16^0$.

4 Rigorous Mathematical Framework

4.1 Existence and Uniqueness Theorem

Theorem: Every nonnegative integer N has a unique representation in any base b (with $b > 1$) if we do not allow unnecessary leading zeros.

Outline of Proof:

1. **Existence:** Using the division algorithm, for any $N \geq 0$ divide N by b to obtain a quotient q_0 and remainder d_0 (with $0 \leq d_0 < b$). Repeat the process on the quotient until it is zero.

2. **Uniqueness:** Assume two different representations exist for N. Their difference yields a nonzero linear combination of powers of b, which contradicts the linear independence of these powers.

5 Representation of Real Numbers

A real number $X \geq 0$ can be written as:

$$X = I + F,$$

where I is the integer part and F is the fractional part ($0 \leq F < 1$).

5.1 Integral Part

The integral part is expressed as:

$$I = \sum_{i=0}^{n-1} d_i \, b^i.$$

5.2 Fractional Part

The fractional part is represented by an infinite series:

$$F = \sum_{i=1}^{\infty} d_{-i} \, b^{-i}.$$

Example: In base 10, the number 12.345 is:

$$12.345 = 1 \times 10^1 + 2 \times 10^0 + 3 \times 10^{-1} + 4 \times 10^{-2} + 5 \times 10^{-3}.$$

5.3 Conversion of Fractional Parts

To convert a fractional part from decimal to base b:

1. Multiply the fractional part F by b.

2. The integer part of the result becomes the first digit d_{-1}.

3. Repeat with the new fractional remainder.

Example: Converting 0.625_{10} to binary:

$$\begin{aligned} 0.625 \times 2 = 1.25 \quad &\Rightarrow \quad d_{-1} = 1, \quad \text{new fraction } 0.25, \\ 0.25 \times 2 = 0.5 \quad &\Rightarrow \quad d_{-2} = 0, \quad \text{new fraction } 0.5, \\ 0.5 \times 2 = 1.0 \quad &\Rightarrow \quad d_{-3} = 1. \end{aligned}$$

Thus, $0.625_{10} = 0.101_2$.

6 Conversion Algorithms Between Bases

6.1 From Base b to Decimal

A number N in base b is converted to decimal by:

$$N_{10} = \sum_{i=0}^{n-1} d_i \, b^i.$$

6.2 From Decimal to Base b

Use the repeated division algorithm:

1. Divide the decimal number by b.
2. Record the remainder.
3. Replace the number by the quotient and repeat until the quotient is 0.
4. The digits, read in reverse order, form the number in base b.

Example: Convert 419_{10} to hexadecimal:

$$\begin{aligned} 419 \div 16 &= 26 \quad \text{remainder 3}, \\ 26 \div 16 &= 1 \quad \text{remainder 10 (A)}, \\ 1 \div 16 &= 0 \quad \text{remainder 1}. \end{aligned}$$

Thus, $419_{10} = 1A3_{16}$.

7 Generalization to Arbitrary Radix Systems

Consider a general numeral system with an arbitrary radix r (where r is an integer greater than 1). A number X is expressed as:

$$X = \left(\sum_{i=0}^{n-1} d_i \, r^i \right) + \left(\sum_{j=1}^{\infty} d_{-j} \, r^{-j} \right),$$

with each digit d_i satisfying $0 \leq d_i < r$.

7.1 Conversion Between Two Arbitrary Bases

To convert a number from base a to base b:

1. **Convert from base a to decimal:** Compute

$$N_{10} = \sum_{i=0}^{n-1} d_i \, a^i.$$

2. **Convert from decimal to base b:** Use the repeated division (for the integer part) and repeated multiplication (for the fractional part) to obtain the digits in base b.

Example: Convert 356_7 to base 5.

1. **Base 7 to Decimal:**

$$356_7 = 3 \times 7^2 + 5 \times 7^1 + 6 \times 7^0 = 147 + 35 + 6 = 188_{10}.$$

2. **Decimal to Base 5:**

$188 \div 5 = 37$ remainder 3, $\quad 37 \div 5 = 7$ remainder 2, $\quad 7 \div 5 = 1$ remainder 2, $\quad 1 \div 5 = 0$ remainder 1.

Reading the remainders in reverse, $188_{10} = 1223_5$.