Solutions Manual

Third Edition

THEORY OF VIBRATION WITH APPLICATIONS

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PREFACE

This teacher's manual was assembled by the author to aid instructors teaching a course in Vibration from the author's text. There are 517 problems with discussions and solutions presented in this manual.

The author considers problem solving to be a major part of the learning process. It is only through involvement with carefully selected problems that the student will acquire a full understanding of the subject.

A variety of analytical procedures and computational methods are illustrated through these problems. The instructor will find that they emphasize and extend the scope of the text, and discussion of some of these problems in class will greatly aid the student.

The manual represents considerable effort on the part of the author, and although there is reasonable assurance of the correctness of the solutions presented, some errors could be expected. It is hoped that the intended use of the manual strictly by the instructor will be adhered to rigidly.

William T. Thomson

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1-1

$x = A \sin \omega t \qquad A = 0.20\ cm \qquad \tau = 0.15 s$

$\omega = \frac{2\pi}{\tau} = 41.89\ r/s$

$\dot{x} = \omega A \cos \omega t \qquad \dot{x}_{max} = \omega A = 8.38\ cm/s$

$\ddot{x} = -\omega^2 A \sin \omega t \qquad \ddot{x}_{max} = \omega^2 A = 350.9\ cm/s^2$

1-2

$\omega = 2\pi f = 2\pi \times 82 = 515.2\ r/s \qquad \omega^2 = 0.2655 \times 10^6$

$g = 980.4\ cm/s^2$

$x_{max} = \ddot{x}_{max} / \omega^2 = \frac{50 \times 980.4}{0.2655 \times 10^6} = 0.184\ cm$

1-3

$\omega = 2\pi f = 2\pi \times 10 = 62.83\ r/s \qquad \tau = \frac{1}{f} = 0.10\ s$

$\dot{x}_{max} = \omega A = 4.57\ m/s \qquad A = 0.07274\ m$

$= 7.274\ cm$

$\ddot{x}_{max} = \omega^2 A = 287.1\ m/s^2$

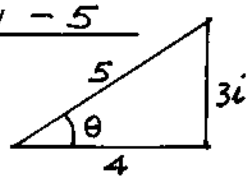
1-4

$x = A(\sin \omega_1 t + \sin \omega_2 t) = 2A \cos \frac{1}{2}(\omega_1 - \omega_2)t \cdot \sin \frac{1}{2}(\omega_1 + \omega_2)t$

let $\omega_1 = \omega, \qquad \omega_2 = \omega + \Delta\omega, \qquad \omega_1 + \omega_2 = 2\omega + \Delta\omega \cong 2\omega$

$\therefore\ x \cong 2A \cos \frac{1}{2} \Delta\omega t \cdot \sin \omega t$

1-5

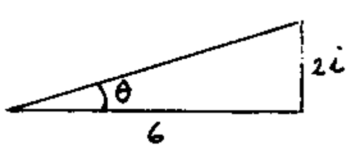


$z = 4 + 3i = 5(\cos\theta + i \sin\theta) = 5e^{i\theta}$

$\theta = \tan^{-1}\frac{3}{4} = 36°52' = 0.6435\ r$

1-6

$$\begin{array}{r} 2 + 3i \\ 4 - i \\ \hline \text{sum} = 6 + 2i \end{array}$$



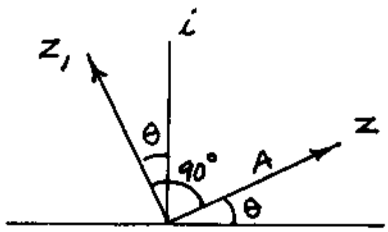
$z = Ae^{i\theta} \qquad A = \sqrt{6^2 + 2^2} = 6.325$

$\theta = \tan^{-1}\frac{2}{6} = 18°26' = 0.3217 r$

$z = 6.325\ e^{0.3217 i}$

$= 6.325\ \underline{/18°26'}$

1-7

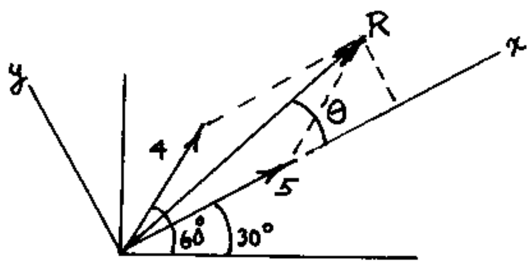


$$z = A(\cos\theta + i\sin\theta) = Ae^{i\theta}$$

$$iz = A(i\cos\theta - \sin\theta) = z_1$$

$$= A[\cos(\theta+90) + i\sin(\theta+90)]$$

1-8



$$\frac{\pi}{6} = 30^\circ$$

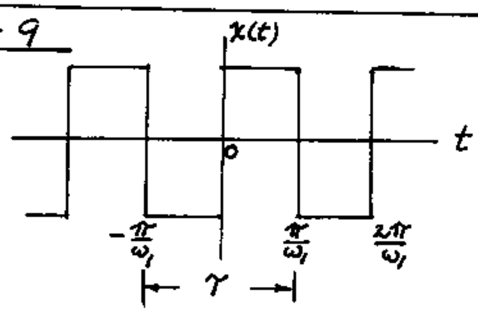
$$R_x = 5 + 4\cos 30^\circ = 5 + 3.47$$
$$= 8.47$$

$$R_y = 4\sin 30^\circ = 2.00$$

$$R = \sqrt{8.47^2 + 2.0^2} = 8.70$$

$$\theta = \tan^{-1}\frac{2}{8.70} = 12^\circ 57'$$

1-9



$x(t)$ is odd function $\therefore a_n = 0$

$$b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\sin n\omega_1 t\,dt$$

$$\tau = \frac{2\pi}{\omega_1}$$

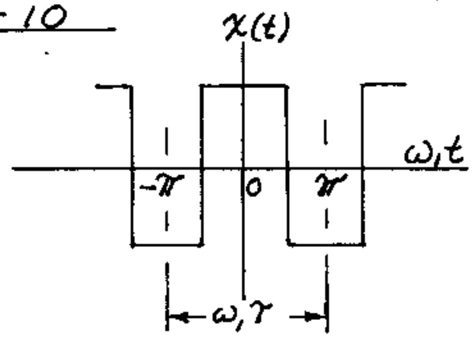
$$b_n = \frac{\omega_1}{\pi}\left[\int_{-\frac{\pi}{\omega_1}}^{0}(-1)\sin n\omega_1 t\,dt + \int_{0}^{\frac{\pi}{\omega_1}}(+1)\sin n\omega_1 t\,dt\right]$$

$$= \frac{\omega_1}{\pi}\left[\frac{\cos n\omega_1 t}{n\omega_1}\Bigg|_{-\frac{\pi}{\omega_1}}^{0} - \frac{\cos n\omega_1 t}{n\omega_1}\Bigg|_{0}^{\frac{\pi}{\omega_1}}\right] = \frac{2}{n\pi}(1 - \cos n\pi)$$

$$\therefore b_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4}{n\pi} & \text{,, } n \text{ odd} \end{cases}$$

$$x(t) = \frac{4}{\pi}\left(\sin\omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \frac{1}{5}\sin 5\omega_1 t + \cdots\right)$$

1 - 10



$x(t)$ is even function

$b_n = 0$

$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\cos n\omega_1 t \, dt$$

$$\tau = \frac{2\pi}{\omega_1} \qquad \omega_n = n\omega_1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\cos n\omega_1 t \, d(\omega_1 t) + \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2}(+1)\cos n\omega_1 t \, d(\omega_1 t) + \frac{1}{\pi}\int_{\pi/2}^{\pi}(-1)\cos n\omega_1 t \, d(\omega_1 t)$$

$$= \frac{1}{n\pi}\left\{\pm 4\right\} \quad \begin{matrix} + \text{ for } n = 1, 5, 9, \cdots \\ - \text{ for } n = 3, 7, 11, \cdots \end{matrix}$$

$$x(t) = \frac{4}{\pi}\left(\cos\omega_1 t - \frac{1}{3}\cos 3\omega_1 t + \frac{1}{5}\cos 5\omega_1 t \cdots\right)$$

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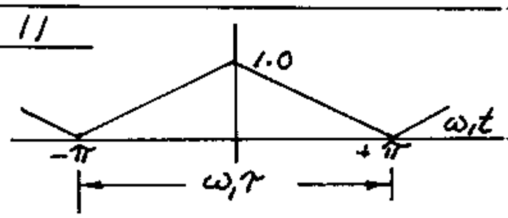
1 - 11



$x(t)$ is even function $b_n = 0$

$$x(t) = \begin{cases} \frac{1}{\pi}(t+\pi) & -\pi \le t \le 0 \\ \frac{1}{\pi}(\pi - t) & 0 \le t \le \pi \end{cases}$$

$$\frac{1}{2}a_0 = \text{average value} = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}\frac{1}{\pi}(t+\pi)\cos n\omega_1 t \, d(\omega_1 t) + \frac{1}{\pi}\int_{0}^{\pi}\frac{1}{\pi}(\pi - t)\cos n\omega_1 t \, d(\omega_1 t)$$

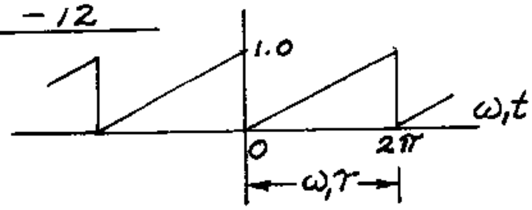
$$= \frac{2}{\pi}\int_{0}^{\pi}\frac{1}{\pi}(\pi - t)\cos n\omega_1 t \, d(\omega_1 t) \qquad \text{because of symmetry}$$

$$= \frac{2}{\pi}\left.\frac{\sin n\omega_1 t}{n}\right|_{0}^{\omega_1 t = \pi} - \frac{2}{\pi^2}\left.\left(\frac{\cos n\omega_1 t}{n^2} + \omega_1 t\frac{\sin n\omega_1 t}{n}\right)\right|_{0}^{\omega_1 t = \pi}$$

$$= \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4}{n^2\pi^2} & \text{for } n \text{ odd} \end{cases}$$

$$x(t) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos\omega_1 t + \frac{1}{3^2}\cos 3\omega_1 t + \frac{1}{5^2}\cos 5\omega_1 t + \cdots\right)$$

1 - 12



$$x = \frac{\omega_1 t}{2\pi} \qquad 0 \le \omega_1 t \le 2\pi$$

$$c_n = \frac{\omega_1}{2\pi}\int_0^{2\pi} \frac{\omega_1 t}{2\pi}\, e^{-in\omega_1 t}\, dt$$

$$c_n = \frac{1}{(2\pi)^2}\int_0^{2\pi} \omega_1 t\, e^{-in\omega_1 t}\, d(\omega_1 t) = \frac{1}{(2\pi)^2}\left[\frac{e^{-in\theta}}{(-in)^2}\left(-in\theta - 1\right)\right]_0^{2\pi} \quad \begin{matrix}\text{where}\\ \theta = \omega_1 t\end{matrix}$$

$$= \frac{1}{(2\pi)^2\, n^2}\left[-1 + (1 + i\,2\pi n)\, e^{-i2\pi n}\right] = \frac{i}{2\pi}\,\frac{1}{n}$$

$$x(t) = c_0 + \frac{i}{2\pi}\left[\left(e^{i\omega_1 t} - e^{-i\omega_1 t}\right) + \frac{1}{2}\left(e^{i2\omega_1 t} - e^{-i2\omega_1 t}\right) + \frac{1}{3}\left(e^{i3\omega_1 t} - e^{-i3\omega_1 t}\right) + \cdots\right]$$

$$= \frac{1}{2} - \frac{1}{\pi}\left[\sin\omega_1 t + \frac{1}{2}\sin 2\omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \cdots\right]$$

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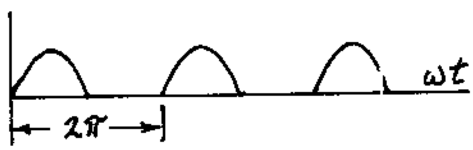
1 - 13



$$\overline{x^2} = \lim_{T\to\infty}\frac{1}{T}\int_0^{T/2} A^2\sin^2\omega t\, dt = \lim_{T\to\infty}\frac{A^2}{T}\int_0^{T/2}\frac{1}{2}(1-\cos 2\omega t)\, dt$$

$$= \lim_{T\to\infty}\frac{A^2}{2}\left(\frac{t}{2} - \frac{\sin 2\omega t}{4\omega}\right)\Bigg|_0^{T/2} = \lim_{T\to\infty}\left(\frac{A^2}{4} - \frac{\sin\omega T}{\omega T}\right) = \frac{A^2}{4}$$

$$\therefore \quad x_{RMS} = \sqrt{\overline{x^2}} = \frac{A}{2}$$

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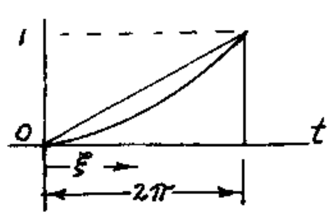
1 - 14

$$x(t) = \frac{\xi}{2\pi} \qquad x^2(t) = \frac{1}{4\pi^2}\,\xi^2 \qquad 0 \le \xi \le 2\pi$$



$$\overline{x^2} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{4\pi^2}\,\xi^2\right)d\xi = \frac{1}{2\pi}\left(\frac{1}{4\pi^2}\,\frac{\xi^3}{3}\right)\Bigg|_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(\frac{2\pi}{3}\right) = \frac{1}{3}$$

1-14 cont.

F.S. of saw tooth wave =

$$x(t) = \frac{1}{2} - \frac{1}{\pi}\left(\sin\omega_1 t + \frac{1}{2}\sin 2\omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \cdots\right)$$

$$x^2(t) = \frac{1}{4} - \frac{1}{\pi}\left(\sin\omega_1 t + \frac{1}{2}\sin 2\omega_1 t + \cdots\right)$$
$$+ \frac{1}{\pi^2}\left(\sin\omega_1 t + \frac{1}{2}\sin 2\omega_1 t + \cdots\right)^2$$

$$= \frac{1}{4} - \frac{1}{\pi}\left(\sin\omega_1 t + \frac{1}{2}\sin 2\omega_1 t + \cdots\right)$$
$$+ \frac{1}{\pi^2}\left(\sin^2\omega_1 t + \sin^2 2\omega_1 t + \cdots\right) + \text{cross products which will integrate to zero}$$

$$\frac{1}{T}\int_0^T x^2(t)\,dt = \frac{1}{4} + \frac{1}{\pi\omega_1 T}\cos\omega_1 t\Big|_0^T + \frac{1}{4\pi\omega_1 T}\cos 2\omega_1 t\Big|_0^T + \frac{1}{9\pi\omega_1 T}\cos 3\omega_1 t\Big|_0^T$$
$$+ \frac{1}{\pi^2 T}\frac{1}{2}\left(t - \frac{\sin 2\omega_1 t}{2\omega_1}\right)\Big|_0^T + \frac{1}{4\pi^2 T}\frac{1}{2}\left(t - \frac{\sin 4\omega_1 t}{4\omega_1}\right)\Big|_0^T + \cdots$$

Let $\omega_1 T = 2\pi k$ where k is an integer $\rightarrow \infty$

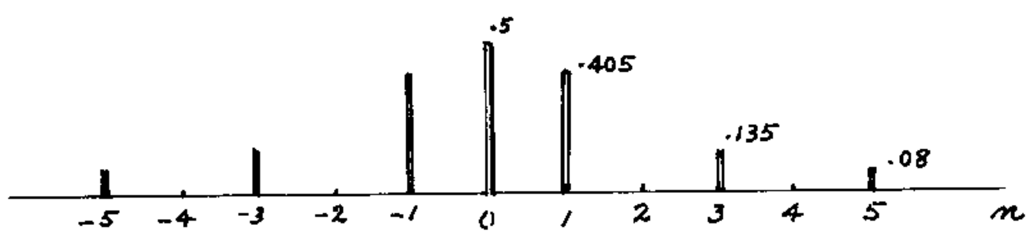
$$\overline{x^2} = \lim_{k\to\infty}\frac{1}{2\pi k}\int_0^{2\pi k} x^2(t)\,dt = \frac{1}{4} + \frac{1}{2\pi^2} + \frac{1}{2}\frac{1}{(2\pi)^2} + \frac{1}{2}\frac{1}{(3\pi)^2} + \cdots$$

$$= \frac{1}{4} + \frac{1}{2\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right) = \frac{1}{3}$$

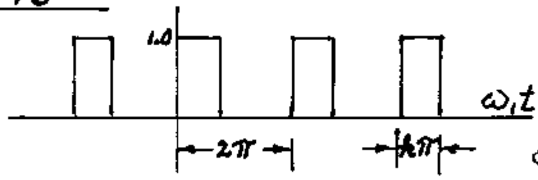
1-15 $\quad x(t) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos\omega_1 t + \frac{1}{3}\cos 3\omega_1 t + \frac{1}{5}\cos 5\omega_1 t + \cdots\right)$

Fourier Spectrum = plot of coefficients. For this case $b_n = 0$

$C_n = \sqrt{a_n^2 + b_n^2} = a_n \qquad c_0 = \frac{a_0}{2}$



1-16



$k = \frac{2}{3}$

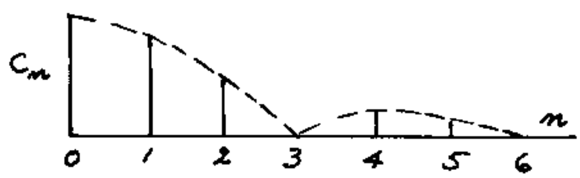
$c_0 = \frac{a_0}{2} = \text{average value} = \frac{k\pi}{2\pi} = \frac{k}{2} = \frac{1}{3}$

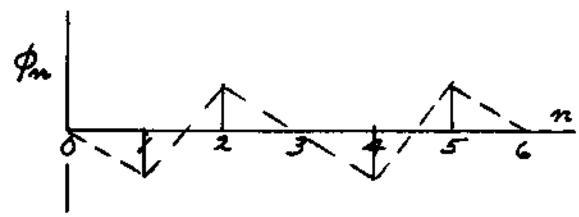
$$a_n = \frac{1}{n\pi} \sin n k \pi \quad = \frac{1}{n\pi} \sin \frac{n}{3} 2\pi$$

$$b_n = \frac{1}{n\pi} (1 - \cos n k \pi) \quad = \frac{1}{n\pi} \left(1 - \cos \frac{n}{3} 2\pi\right)$$

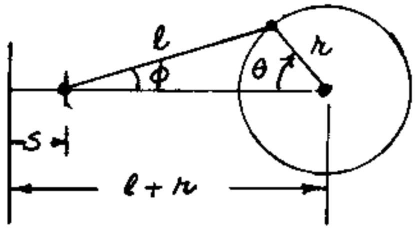
$$2 C_n = \sqrt{a_n^2 + b_n^2} \quad = \frac{\sqrt{2}}{n\pi} \sqrt{\left(1 - \cos \frac{n}{3} 2\pi\right)} \qquad \phi_n = \tan^{-1} \frac{1 - \cos \frac{n}{3} 2\pi}{\sin \frac{n}{3} 2\pi}$$

n	C_n	ϕ_n
1	.2758	−60°
2	.1379	60°
3	0	0
4	.0689	−60°
5	.0552	60°
6	0	0





1-17



$$l + r - s = r\cos\theta + l\cos\phi$$

$$l \sin\phi = r \sin\theta$$

$$\therefore \cos\phi = \left[1 - \left(\tfrac{r}{l}\right)^2 \sin^2\theta\right]^{1/2}$$

$$= 1 - \tfrac{1}{2}\left(\tfrac{r}{l}\right)^2 \sin^2\theta - \tfrac{1}{8}\left(\tfrac{r}{l}\right)^4 \sin^4\theta \cdots$$

$$S = r\left[1 - \cos\theta + \tfrac{1}{2}\left(\tfrac{r}{l}\right)\sin^2\theta + \tfrac{1}{8}\left(\tfrac{r}{l}\right)^3 \sin^4\theta + \cdots\right]$$

using $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, $\quad \sin^4\theta = \frac{1}{4}\left(\frac{3}{2} - 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right)$

$$S = r\left[1 - \cos\theta + \tfrac{1}{2}\left(\tfrac{r}{l}\right)\tfrac{1}{2}(1 - \cos 2\theta) + \cdots\right]$$

$$= r\left[1 + \tfrac{1}{4}\left(\tfrac{r}{l}\right) - \cos\theta - \tfrac{1}{4}\left(\tfrac{r}{l}\right)\cos 2\theta + \cdots\right]$$

which retains only $\left(\frac{r}{l}\right)$ to first power

Ratio of 2^{nd} harmonic / 1^{st} harmonic $= \frac{1}{4}\left(\frac{r}{l}\right) = \frac{1}{12}$

1-18

$$\overline{x^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)\,dt = \frac{A^2 k\gamma}{\gamma} = kA^2$$
$$= 0.10\,A^2$$

$$rms = \sqrt{\overline{x^2}} = 0.3162\,A$$

1-19

$$x = \left(1 - \frac{t}{\pi}\right) \qquad 0 \le t \le \pi$$

$$x^2 = 1 - \frac{2t}{\pi} + \frac{t^2}{\pi^2}$$

$$\overline{x^2} = \frac{1}{\pi}\int_0^{\pi}\left(1 - \frac{2t}{\pi} + \frac{t^2}{\pi^2}\right)dt = \frac{1}{3}$$

1-20

$$Db = 20\log_{10}\left(\frac{x_1}{x_2}\right) = 0.50$$

$$\log_{10}\left(\frac{x_1}{x_2}\right) = \frac{0.50}{20} = 0.0250$$

$$\left(\frac{x_1}{x_2}\right) = 10^{0.0250} = 1.0593$$

$$x_1 = 1.0593\,x_2 = 1.0593 \times 2.5\,mm = 2.6481$$

$$Error = 0.0593 \times 2.5\,mm = \pm\,0.148\ mm$$

1-21

$$Db = 20\log_{10}(10) = 20$$
$$Db = 20\log_{10}(50) = 33.98$$
$$Db = 20\log_{10}(100) = 40.0$$

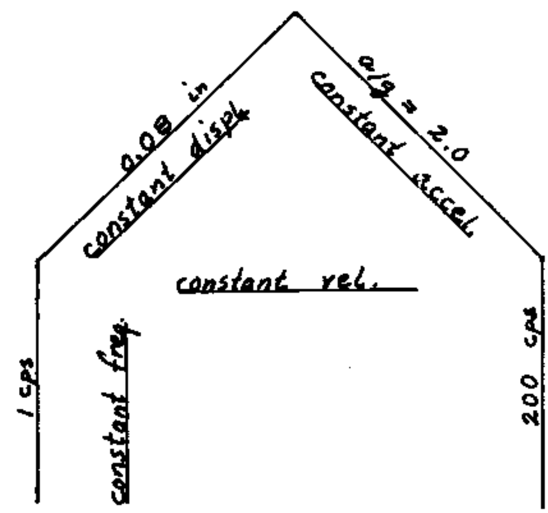
1-22

$$Db = 20\log_{10}\left(\frac{x_p}{x_{1000}}\right) = 32.$$

$$\log_{10}\left(\frac{x_p}{x_{1000}}\right) = \frac{32}{20} = 1.60$$

$$\frac{x_p}{x_{1000}} = 10^{1.60} = 39.8$$

1 - 23

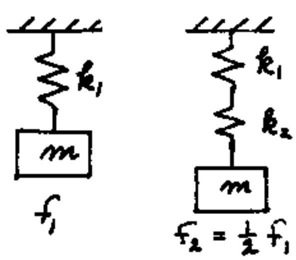


2 - 1

From Eq. 2.2-9

$$f = \frac{15.76}{\sqrt{\Delta_{mm}}} = \frac{15.76}{\sqrt{7.87}} = 5.62 \quad Hz$$

2 - 2

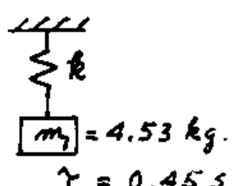


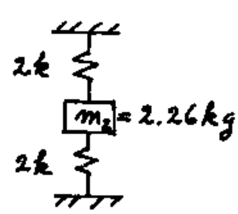
$$f_1 = \sqrt{\frac{k_1}{m}} \qquad f_2 = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}$$

$$\frac{1}{2}\sqrt{\frac{k_1}{\cancel{m}}} = \sqrt{\frac{k_1 k_2}{\cancel{m}(k_1+k_2)}}$$

$$\frac{1}{4}\cancel{k_1} = \frac{\cancel{k_1} k_2}{k_1+k_2} \qquad \therefore \quad k_2 = \frac{1}{3} k_1$$

2 - 3





$$k = \left(\frac{2\pi}{\tau_1}\right)^2 m_1 = \left(\frac{2\pi}{.45}\right)^2 4.53$$

$$= 883.5 \quad N/m$$

$$\tau_2 = 2\pi\sqrt{\frac{m_2}{4k}} = 2\pi\sqrt{\frac{2.26}{4\times 883.}}$$

$$= 0.159 \quad s$$

2 - 4

$$\frac{k}{m} = (2\pi f)^2 = \left(2\pi \frac{94}{60}\right)^2 \qquad \frac{k}{m+.453} = \left(2\pi \frac{76.7}{60}\right)^2$$

$$\frac{m+.453}{m} = \left(\frac{94}{76.7}\right)^2 \qquad \therefore \quad m = 0.9028 \text{ kg}$$

$$k = 87.48 \text{ N/m}$$

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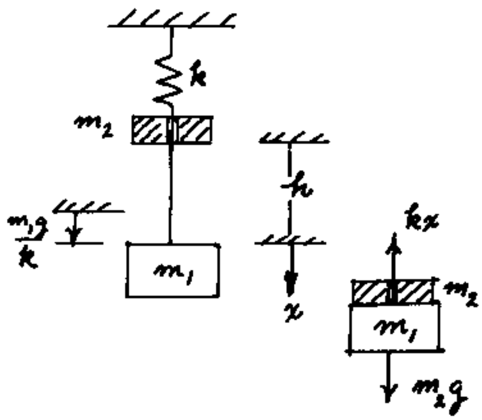
2 - 5



x measured from static equilibrium position of m_1, k

Eq. of motion after impact

$$(m_1+m_2)\ddot{x} = -kx + m_2 g$$

Gen solution:

$$x(t) = \frac{m_2 g}{k} + A\sin\omega t + B\cos\omega t$$

Initial conditions

$$x(0) = 0 = \frac{m_2 g}{k} + B \quad \therefore B = -\frac{m_2 g}{k}$$

$$\dot{x}(0) = \frac{m_2\sqrt{2gh}}{m_1+m_2} = \omega A \qquad \therefore \quad A = \frac{m_2\sqrt{2gh}}{(m_1+m_2)\omega}$$

$$\omega = \sqrt{\frac{k}{m_1+m_2}}$$

$$\therefore x(t) = \frac{m_2 g}{k} + \frac{m_2\sqrt{2gh}}{m_1+m_2}\sqrt{\frac{m_1+m_2}{k}}\sin\omega t - \frac{m_2 g}{k}\cos\omega t$$

$$= \frac{m_2 g}{k}(1-\cos\omega t) + \frac{m_2\sqrt{2gh}}{\sqrt{k(m_1+m_2)}}\sin\omega t$$

---

2 - 6 $\quad \omega_n^2 = \frac{k}{m} = 4.0 \qquad \omega_n = 2.0$

$$x = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t = 2\cos 2t - \frac{8}{2}\sin 2t$$

$$\dot{x} = -4\sin 2t - 8\cos 2t = 0 \qquad \therefore \tan 2t_p = -2$$

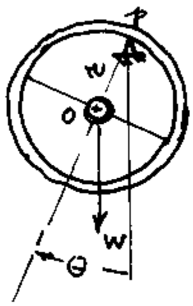
$$\therefore 2t_p = 116.57^\circ \qquad \sin 116.57^\circ = .8944 \qquad \cos 116.57 = -.4472$$

$$x_{max} = 2(-.4472) - 4(.8944) = -4.472 \text{ cm}$$

$$\ddot{x}_{max} = \omega^2 x_{max} = 4(\pm 4.472) = \pm 17.89 \text{ cm/s}^2$$

2 - 7

$$J_p\ddot{\theta} = -Wr\theta \qquad \ddot{\theta} = -\omega^2\theta$$



$$J_p = \frac{Wr}{\omega^2} = \frac{70 \times 6}{\left(\frac{2\pi}{1.22}\right)^2} = 15.83$$

$$J_o = J_p - \frac{W}{g}r^2 = 15.83 - \frac{70}{386}\times 6^2$$

$$= 9.30 \ \text{lb in sec}^2$$

---

2 - 8

$$\omega = 2\pi\,\frac{53}{60} = 5.55 \ r/s$$

$$J_p = \frac{Wr}{\omega^2} = \frac{21.35 \times .254}{5.55^2} = 0.1761$$

$$J_{cg} = J_o - \frac{Wr^2}{g} = 0.1761 - \frac{21.35 \times .254^2}{9.81} = 0.0356 \ kg\ m^2$$

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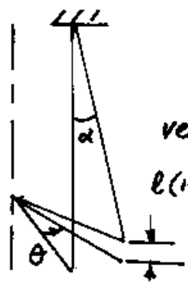
2 - 9

$$r\theta = l\alpha \qquad \text{Work done} = \text{change in KE}$$



$$\alpha = \frac{r\theta}{l} \qquad W\frac{l}{2}\,\frac{r^2}{l^2}\theta^2_{max} = \frac{1}{2}J\,\dot{\theta}^2_{max} = \frac{1}{2}J\omega^2\theta^2_{max}$$

vertical displ =

$$l(1-\cos\alpha) = l\left[1-\left(1-\tfrac{1}{2}\alpha^2+\cdots\right)\right] \qquad J = \frac{W}{g}k^2 \quad k = \text{rad. of gyr.}$$

$$= \frac{l\alpha^2}{2} = \frac{l}{2}\left(\frac{r\theta}{l}\right)^2 \qquad \frac{\cancel{W}r^2}{l} = \frac{\cancel{W}}{g}k^2\omega^2$$

$$k = \frac{r}{\omega}\sqrt{\frac{g}{l}} = \frac{.254\times 2.17}{2\pi}\sqrt{\frac{9.81}{1.829}} = .2032$$

$$k = \cancel{.4507} \ m$$

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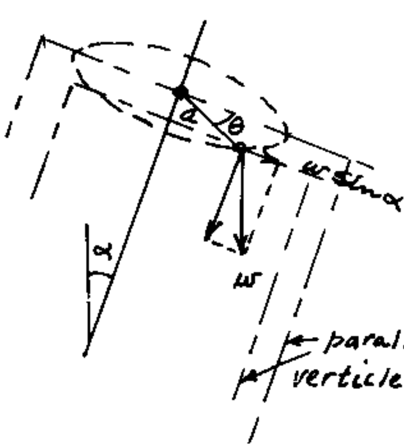
2 - 10

$$\text{Moment about shaft} = (a\sin\theta)\,w\sin\alpha$$



$$\left(J+\frac{w}{g}a^2\right)\ddot{\theta} = -(a\sin\theta)\,w\sin\alpha$$

$$\cong -(a\,w\sin\alpha)\,\theta$$

$$\therefore\ f_n = \frac{1}{2\pi}\sqrt{\frac{w\,a\sin\alpha}{J+\frac{w}{g}a^2}}$$

2-11

$$T = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} J_o \dot{\theta}^2 \qquad r\dot{\theta} = \dot{x}$$

$$= \tfrac{1}{2}\left(m + \frac{J_o}{r^2}\right)\dot{x}^2 \qquad \dot{x} = \omega x$$

$$U = \tfrac{1}{2} k x^2 \qquad \therefore\ \omega = \sqrt{\frac{k}{m + J_o/r^2}}$$

2-12

$$\tau = 2\pi\sqrt{\frac{L}{g}} \qquad L = g\left(\frac{\tau}{2\pi}\right)^2 = 9.81\left(\frac{2}{2\pi}\right)^2 = 0.994\ m$$

$$v_{max} = L(\omega\theta_o) = \frac{.003175}{.01}\ m/s \qquad \theta_o = \frac{.3175}{.994\ \pi} = .1017\ r = 5.826^\circ$$

2-13 water weighs $9802.\ N/m^3$ $\therefore\ \rho = 1.2 \times 9802. = 11762.\ N/m^3$

$$\text{buoyant force} = \pi r^2 x \cdot \rho = m\ddot{x} = \omega^2 x$$

$$\frac{1}{\omega} = \frac{\tau}{2\pi} = \sqrt{\frac{m}{\pi r^2 \rho}} \qquad m = .0372\ kg \quad r = .0032\ m$$

$$\tau = 2\pi\sqrt{\frac{.0372}{\pi \times .0032^2 \times 11762}} = 1.97\ s$$

2-14 moment about geom. center

$$-W\, 8\, \theta = J_o\ddot{\theta} = -\omega^2 J_o \theta$$

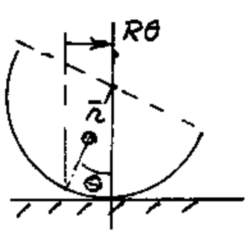
$$J_o = \frac{8W}{\omega^2} = \frac{8W\,(1.3)^2}{(2\pi)^2} = 0.3428\ W$$

2-15

$$-W h\theta = J\ddot{\theta} = -\omega^2 J\theta$$

$$\frac{1}{\omega} = \frac{\tau}{2\pi} = \sqrt{\frac{J}{Wh}} \qquad \tau = 2\pi\sqrt{\frac{J}{Wh}}$$

2-16 Displ. of cg $= (R - \bar{r})\theta$ $\qquad T_{max} = U_{max}$



$$T_{max} = \tfrac{1}{2} m (R-\bar{r})^2 \dot{\theta}^2_{max} + \tfrac{1}{2} J_{cg.}\dot{\theta}^2_{max}$$

$$= \tfrac{1}{2} m\left[(R-\bar{r})^2 + (R^2 - \bar{r}^2)\right]\omega^2\theta^2_{max}$$

$$U_{max} = mg\bar{r}\,(1 - \cos\theta_{max}) \cong mg\bar{r}\,\frac{\theta^2_{max}}{2}$$

2-16 cont $\quad T_{max} = U_{max}$

$$\omega^2 = \frac{\bar{r} g}{(R-\bar{r})^2 + (R^2 - \bar{r}^2)} = \frac{\bar{r} g}{2R(R-\bar{r})}$$

$$\tau = 2\pi\sqrt{\frac{2R(R-\bar{r})}{\bar{r} g}} \qquad \text{but } \bar{r} = \frac{2R}{\pi} \qquad \therefore \ \tau = 2\pi\sqrt{\frac{R(\pi-2)}{g}}$$

2-17

$$U = mgh(1-\cos\phi) \cong mgh\tfrac{1}{2}\phi^2 \qquad h\phi = \frac{a}{2}\theta$$

$$= mg\frac{h}{2}\left(\frac{a\theta}{2h}\right)^2 = mg\frac{a^2}{8}\frac{\theta^2}{h}$$

$$T = \frac{1}{2}\left(m\frac{L^2}{12}\right)\dot{\theta}^2 = \frac{1}{2}\left(m\frac{L^2}{12}\right)\omega^2\theta^2$$

$$T_{max} = U_{max} \qquad \therefore \ \tau = 2\pi\frac{L}{a}\sqrt{\frac{h}{3g}}$$

2-18

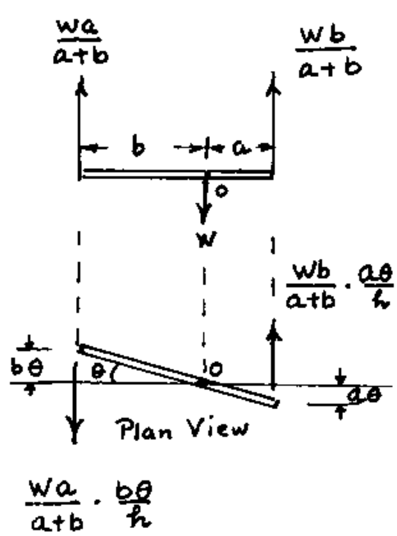
$$\tau_1 = 2\pi\sqrt{\frac{h}{g}} \qquad \text{for } \tau_2 \qquad T = \frac{1}{2}mk^2\omega^2\theta^2$$

$$U = mg\frac{L^2}{8}\frac{\theta^2}{h}$$

$$\therefore \ \tau_2 = 2\pi\sqrt{\frac{4hk^2}{gL^2}} \qquad k = \frac{\tau_2 L}{2\pi}\sqrt{\frac{g}{4h}} = \frac{\tau_2}{\tau_1}\left(\frac{L}{2}\right)$$

2-19

$$\sum M_o = J_{cg}\ddot{\theta} = -\frac{Wba}{a+b}\frac{\theta}{h}\cdot a - \frac{Wab}{a+b}\frac{\theta}{h}\cdot b$$



$$\frac{W}{g}k^2\ddot{\theta} + \frac{Wab}{h}\theta = 0$$

$$\ddot{\theta} + \left(\frac{g\,ab}{k^2 h}\right)\theta = 0$$

$$\therefore \ f = \frac{1}{2\pi}\sqrt{\frac{g\,ab}{k^2 h}}$$

$$\therefore \sum F_{horiz} = 0$$

2 - 20

J_o of wheel about torsion bar $= J_{cm} + m(24'')^2$

$= m(k^2 + 24^2) = m(9^2 + 24^2) = 657\, m$

Stiffness of torsion bar $K = \frac{GI_p}{\ell}$

$I_p = \frac{\pi D^4}{32} = \frac{\pi}{32}(1.50)^4 = 0.497 \text{ in}^4$ = polar mom. inertia of torsion bar

$G = 11.2 \times 10^6 \text{ lb/in}^2$ = shear modulus of steel

$K = \frac{(11.2\times 10^6)\times(0.497)}{50} = 0.1113 \times 10^6$ lb.in./rad.

$J_o\ddot{\theta} + K\theta = 0 \qquad f = \frac{1}{2\pi}\sqrt{\frac{K}{J_o}} = \frac{1}{2\pi}\sqrt{\frac{0.1113\times 10^6 \times 386}{38 \times 657}} = 6.60$ cps locked wheel.

With wheel free $J_o = m(24)^2 = 576\, m$

$f = \frac{1}{2\pi}\sqrt{\frac{0.1113\times 10^6 \times 386}{38 \times 576}} = 7.05$ cps.

2 - 21

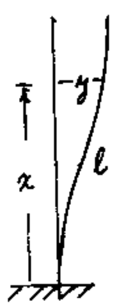
$\frac{\ell\rho}{g}\ddot{x} = -2x\rho \qquad \ddot{x} + \frac{2g}{\ell}x = 0$

$\omega^2 = \frac{2g}{\ell} \qquad \tau = 2\pi\sqrt{\frac{\ell}{2g}}$

2 - 22

$k = 2\left(\frac{12EI}{\ell^3}\right) \qquad \tau = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m\ell^3}{24EI}}$

2 - 23



$y = \frac{1}{2}y_{max}\left(1 - \cos\frac{\pi x}{\ell}\right)\sin\omega t$

$\dot{y} = \frac{1}{2}\omega y_{max}\left(1 - \cos\frac{\pi x}{\ell}\right)\cos\omega t$

$T = \frac{1}{2}\int_0^\ell m(x)\dot{y}^2\,dx = \frac{1}{2}\frac{m}{4}y_{max}^2\int_0^\ell\left(1 - \cos\frac{\pi x}{\ell}\right)^2\omega^2\,dx\,\cos^2\omega t$

$= \frac{1}{2}\cdot\frac{m}{4}y_{max}^2\,\omega^2\cos^2\omega t\int_0^\ell\left(1 - 2\cos\frac{\pi x}{\ell} + \cos^2\frac{\pi x}{\ell}\right)dx$

$= \quad '' \quad \left[x - \frac{2\ell}{\pi}\sin\frac{\pi x}{\ell} + \frac{x}{2} + \frac{1}{2}\cdot\frac{\ell}{2\pi}\sin\frac{2\pi x}{\ell}\right]_0^\ell$

2-23 Cont.

$$T = \quad '' \quad \left[\tfrac{3}{2}\ell - 0 + 0\right]$$

$$= \frac{1}{2}\left(\frac{m}{4}\cdot\frac{3\ell}{2}\right)\omega^2 y_{max}^2 \cos^2\omega t$$

$\therefore\ m_{eff} = \left(\frac{3}{8}\, m\ell\right)$ for each column, where
$m\ell$ = total mass of each column

2-24

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J\left(\frac{\dot{x}}{b}\right)^2$$ where J = moment of inertia of Linkage about pivot.

$$x = \frac{b}{a} x_m$$

$$T = \frac{1}{2} m \left(\frac{b}{a}\right)^2 \dot{x}_m^2 + \frac{1}{2} J\left(\frac{b}{a}\right)^2 \frac{1}{b^2}\dot{x}_m^2$$

$$= \frac{1}{2}\left[m\left(\frac{b}{a}\right)^2 + \frac{J}{a^2}\right]\dot{x}_m^2 \qquad \therefore\ m_{eff} = \left[m\left(\frac{b}{a}\right)^2 + \frac{J}{a^2}\right]$$

2-25

$$y = \frac{1}{2} y_0 \left[3\left(\frac{x}{\ell}\right)^2 - \left(\frac{x}{\ell}\right)^3\right]$$

$$T = \frac{1}{2} m \int_0^\ell \dot{y}^2 dx = \frac{1}{2}\frac{m}{4}\dot{y}_0^2 \int_0^\ell \left[9\left(\frac{x}{\ell}\right)^4 - 6\left(\frac{x}{\ell}\right)^5 + \left(\frac{x}{\ell}\right)^6\right] dx$$

$$= \frac{1}{2} m \dot{y}_0^2 \frac{\ell}{4}\left[\frac{9}{5} - 1 + \frac{1}{7}\right] = \frac{1}{2}\left(\frac{33}{140}\, m\ell\right)\dot{y}_0^2$$

2-26

$$T = \frac{1}{2}\left[J_0\dot{\theta}^2 + m_1(b\dot{\theta})^2\right] = \frac{1}{2}\left[J_0 + m_1 b^2\right]\dot{\theta}^2$$

$$\dot{\theta} = \dot{x}/b \qquad T = \frac{1}{2}\left[J_0/b^2 + m_1\right]\dot{x}^2 \qquad m_{eff} = J_0/b^2 + m_1$$

2-27 let θ_0 = rotation of J, T_0 = total torque

T_L = torque to Left of J

T_R = " " right " "

$$\left(\frac{1}{K_1} + \frac{1}{K_2}\right)T_L = \theta_0 \qquad \left(\frac{1}{K_2}\right)T_R = \theta_0$$

2-27 Cont.

$$T_o = T_L + T_R = \left[\frac{1}{\left(\frac{1}{K_1}+\frac{1}{K_2}\right)} + \frac{1}{\left(\frac{1}{K_2}\right)}\right]\theta_o = K\,\theta_o$$

$$\therefore\ K = \left(\frac{K_1K_2}{K_1+K_2} + K_2\right) \qquad \omega_n = \sqrt{\frac{K}{J}} = \frac{2\pi}{\tau}$$

2-28

$$T = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} J_2 \left(\frac{\dot{x}}{R_2}\right)^2 + \tfrac{1}{2} J_3 \left(\frac{R_2}{R_1}\frac{\dot{x}}{R_2}\right)^2$$

$$= \tfrac{1}{2}\left[m_1 + J_2/R_2^2 + J_3/R_1^2\right]\dot{x}^2 = \tfrac{1}{2} m_{eff}\,\dot{x}^2$$

$$U = \tfrac{1}{2} k_1 x^2 + \tfrac{1}{2} K_3 \left(\frac{x}{R_1}\right)^2 = \tfrac{1}{2}\left[k_1 + K_3/R_1^2\right]x^2 = \tfrac{1}{2} k_{eff}\, x^2$$

2-29

$$T = \tfrac{1}{2} J_1 \dot{\theta}_1^2 + \tfrac{1}{2} J_2 \left(\frac{r_1}{r_2}\dot{\theta}_1\right)^2$$

$$= \tfrac{1}{2}\left[J_1 + J_2\left(\frac{r_1}{r_2}\right)^2\right]\dot{\theta}_1^2 = \tfrac{1}{2} J_{eff}\,\dot{\theta}_1^2$$

2-30 $T = \tfrac{1}{2} J_1 \dot{\theta}_1^2 + \tfrac{1}{2}(m_0 + m_2)\dot{x}^2 + \tfrac{1}{2} J_2 \dot{\theta}_2^2$

$$\dot{\theta}_1 = \dot{x}/r_1 \qquad \dot{\theta}_2 = \dot{x}/r_2$$

$$T = \tfrac{1}{2}\left[J_1/r_1^2 + (m_0 + m_2) + J_2/r_2^2\right]\dot{x}^2$$

$$U = \tfrac{1}{2} k_1 (R\theta_1)^2 + \tfrac{1}{2} k_2 (r_2\theta_2)^2$$

$$= \tfrac{1}{2} k_1 \left(\frac{R}{r_1}x\right)^2 + \tfrac{1}{2} k_2 \left(r_2 \frac{x}{r_2}\right)^2$$

$$= \tfrac{1}{2}\left[k_1\left(\frac{R}{r_1}\right)^2 + k_2\right]x^2$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k_1\left(\frac{R}{r_1}\right)^2 + k_2}{J_1/r_1^2 + (m_1 + m_1) + J_2/r_2^2}}$$

2-31 $\quad m_{eff} = M + \frac{33}{140} ml \quad$ (see Prob. 2-25)

beam vol $= (.1016 \times .635 \times 8.89) = .5735 \ cm^3$

wt. of steel $= 0.07655 \ N/cm^3$

wt. of beam $= .5735 \times .07655 = .04390 \ N$

mass of beam $= \frac{.04390}{9.81} = .00475 \ kg = ml$

$\frac{33}{140} ml = .001055 \qquad$ beam stiffness $= \frac{3EI}{l^3} = k$

$E = 200 \times 10^9 \ N/m^2 \qquad I = \frac{bh^3}{12} = \frac{.635 \times .1016^3}{12} = .0000555 \times 10^{-8} \ m^4$

$$k = \frac{3 \times 200 \times 10^9 \times 555 \times 10^{-15}}{(.0889)^3} = 473.96 \quad N/m$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m_{eff}}} \qquad m_{eff} = M + .001055 = \frac{3EI}{l^3 4\pi^2 f^2} = \frac{473.96}{4\pi^2 \times 400}$$

$$\therefore M = 0.0289 \ kg$$

2-32

$$c_c = 2\sqrt{mk} = 2\sqrt{.907 \times 7. \times 10^2} = 50.4 \ \frac{Ns}{m}$$

2-33

$$F_d = cv \qquad c = \frac{F_d}{v} = \frac{0.50}{1.20} = 0.417 \ \frac{lb\ sec}{in}$$

$$c = 0.417 \ \frac{lb.s}{in} \times 4.448 \frac{N}{lb} \times \frac{1}{2.54} \ \frac{in}{cm} = 0.7303 \ \frac{NS}{cm} = 73.03 \frac{Ns}{m}$$

$$\zeta = \frac{c}{c_c} = \frac{73.03}{50.4} = 1.45$$

2-34

(a) $\zeta = 2. \qquad$ Eq (2.3-18) $\qquad A = -B = \frac{v_0}{2\omega_n\sqrt{\zeta^2 - 1}}$

$$\frac{x\omega_n}{v_0} = \frac{1}{3.464}\left(e^{-0.268\omega_n t} - e^{-3.732\,\omega_n t}\right)$$

(b) $\zeta = 0.50 \qquad \frac{x\omega_n}{v_0} = \frac{e^{-0.50\omega_n t}}{0.865} \sin 0.865\,\omega_n t \quad$ Eq.(2.3-16)

(c) $\zeta = 1.0 \qquad \frac{x\omega_n}{v_0} = \omega_n t \ e^{-\omega_n t}$

Eq. (2.3-19)

2-35

$$\delta = \ln \frac{x_1}{x_2} = \ln \frac{1.00}{.980} = \ln 1.020408 = 0.0202$$

$$\zeta \cong \frac{\delta}{2\pi} = .003215$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1750}{2.267}} = 27.78 \cong \omega_d$$

$$c = 2 m \omega_n \zeta = 2 \times 2.267 \times 27.78 \times .003215 = .405 \ \frac{Ns}{m}$$

2-36

(a) $$\zeta = \frac{c}{2m}\sqrt{\frac{m}{k}} = \frac{12.43}{2 \times 4.534}\sqrt{\frac{4.534}{3500}} = 0.0493$$

(b) $$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 0.3101$$

(c) $$\frac{x_m}{x_{m+1}} = e^{\delta} = (2.718)^{.3101} = 1.364$$

2-37

(a) $$\zeta = \frac{c}{2\sqrt{mk}} = \frac{70}{2\sqrt{17.5 \times 7000}} = 0.10$$

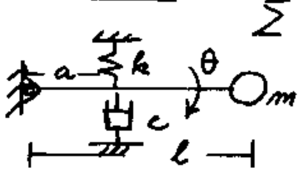
(b) $$f_d = \frac{1}{2\pi}\sqrt{1-\zeta^2}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{1-.01}\sqrt{\frac{7000}{17.5}} = 3.167 \text{ Hz}$$

(c) $$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = .6315$$

(d) $$x_m / x_{m+1} = e^{.6315} = 1.874$$

2-38

$$\sum M_o = -ac(a\dot\theta) - ak(a\theta) = m\ell^2\ddot\theta$$



$$\ddot\theta + \frac{c}{m}\left(\frac{a}{\ell}\right)^2\dot\theta + \frac{k}{m}\left(\frac{a}{\ell}\right)^2\theta = 0 \qquad \text{let } \theta = e^{st}$$

$$s_{1,2} = -\frac{c}{2m}\left(\frac{a}{\ell}\right)^2 \pm \sqrt{\left(\frac{ca^2}{2m\ell^2}\right)^2 - \frac{k}{m}\left(\frac{a}{\ell}\right)^2}$$

crit. damp. $\frac{c_c a^2}{2m\ell^2} = \frac{a}{\ell}\sqrt{\frac{k}{m}}$ $\qquad c_c = 2\frac{\ell}{a}\sqrt{km}$

$$\omega_d = \frac{a}{\ell}\sqrt{\frac{k}{m} - \left(\frac{ca}{2m\ell}\right)^2} = \frac{a}{\ell}\sqrt{\frac{k}{m}}\sqrt{1-\left(\frac{ca}{2\ell\sqrt{km}}\right)^2} = \omega_n\sqrt{1-\zeta^2}$$

$\therefore \ \omega_n = \frac{a}{\ell}\sqrt{\frac{k}{m}}, \quad \zeta = \frac{ca}{2\ell\sqrt{km}}$ $\quad$ identify from $\ddot\theta + 2\zeta\omega_n\dot\theta + \omega_n^2\theta = 0$

2-39

$$\sum M_0 = m a^2 \ddot{\theta} = -k b^2 \theta - c a^2 \dot{\theta}$$

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{k}{m}\left(\frac{b}{a}\right)^2 \theta = 0$$

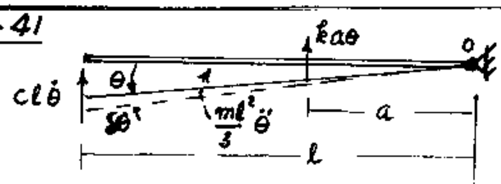
$$\therefore \quad \omega_n = \frac{b}{a}\sqrt{\frac{k}{m}} \qquad \omega_d = \sqrt{\frac{k}{m}\left(\frac{b}{a}\right)^2 - \left(\frac{c}{2m}\right)^2} \qquad c_c = \frac{2b}{a}\sqrt{km}$$

2-40

$$\delta = \ln \frac{1.0}{0.95} = \ln 1.0527 = .05129$$

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = .05129 \qquad \zeta = .00816$$

2-41



$$\delta W = -\frac{m\ell^2}{3}\ddot{\theta}\,\delta\theta - c\ell\dot{\theta}\,\ell\,\delta\theta - ka\theta\, a\,\delta\theta = 0$$

$$\ddot{\theta} + \frac{3c}{m}\dot{\theta} + \frac{3k}{m}\left(\frac{a}{\ell}\right)^2\theta = 0$$

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$$

$$\therefore \omega_n = \frac{a}{\ell}\sqrt{\frac{3k}{m}}, \qquad c_c = \frac{2a}{3\ell}\sqrt{3km}, \qquad \zeta = \frac{3}{2}\frac{c}{m\omega_n}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \frac{a}{\ell}\sqrt{\frac{3k}{m}}\sqrt{1 - \frac{9}{4}\left(\frac{c}{m\omega_n}\right)^2} = \frac{a}{\ell}\sqrt{\frac{3k}{m}}\sqrt{1 - \frac{3}{4km}\left(\frac{c\ell}{a}\right)^2}$$

2-42

$$\frac{W}{g}\ddot{x} + 2\mu A\dot{x} + kx = 0$$

$$\ddot{x} + \frac{2\mu A g}{W}\dot{x} + \frac{kg}{W}x = 0 \qquad \therefore \tau_1 = 2\pi\sqrt{\frac{W}{kg}}$$

$$f_2 = \frac{1}{\tau_2} = \frac{1}{2\pi}\sqrt{\frac{kg}{W} - \left(\frac{\mu A g}{W}\right)^2} = \frac{1}{2\pi}\sqrt{\left(\frac{2\pi}{\tau_1}\right)^2 - \left(\frac{\mu A g}{W}\right)^2}$$

square both sides

$$\left(\frac{2\pi}{\tau_2}\right)^2 - \left(\frac{2\pi}{\tau_1}\right)^2 = -\left(\frac{\mu A g}{W}\right)^2$$

$$\therefore \mu = \frac{2\pi W}{Ag}\sqrt{\frac{\tau_2^2 - \tau_1^2}{\tau_1^2\,\tau_2^2}} = \frac{2\pi W}{Ag\,\tau_1\tau_2}\sqrt{\tau_2^2 - \tau_1^2}$$

2 - 43

$$\omega_n = \sqrt{\frac{20,000 \times 32.2}{1200}} = 23.17 \text{ r/s}$$

$$\frac{1}{2} m \dot{x}_{max}^2 = \frac{1}{2} k x_{max}^2 \qquad \dot{x}_{max} = 23.17 \times 4 = 92.66 \text{ ft/s.}$$

Eq.(2.3-19) $\quad x = e^{-\omega_n t}[0 + \omega_n x(0)]t + x(0)e^{-\omega_n t}$

$$= e^{-\omega_n t}\, x(0)\,[1 + \omega_n t]$$

$$\frac{2}{12}' = e^{-\omega_n t}\, 4\,[1 - \omega_n t] \quad \text{or} \quad e^{-\omega_n t}[1 + \omega_n t] = .0417$$

solve by trial

$\omega_n t$	$e^{-\omega_n t}$	$e^{-\omega_n t}[1 + \omega_n t]$	
4.90	.00745	.0439	
4.96	.007017	.04182	← close
4.97	.006947	.04147	

$\therefore \omega_n t = 4.96 \qquad t = \frac{4.96}{23.17} = 0.214 \text{ s}$

2 - 44

$$\omega_n = \sqrt{\frac{35000}{4.53}} = 87.89 \text{ r/s}$$

$\tau = \frac{2\pi}{87.89} = .0715 \text{ s} \qquad c_c = 2\sqrt{km} = 797.04$

$$\zeta = .2197$$

$$\tau_d = \sqrt{1-\zeta^2}\,\tau = .0697$$

Eq.(2.3-16)

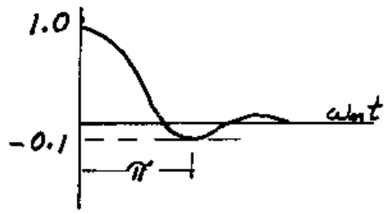
$$x = \frac{\dot{x}(0)}{\omega_n\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\sqrt{1-\zeta^2}\,\omega_n t$$

at x_{max}, $\sin\sqrt{1-\zeta^2}\,\omega_n t \cong 1.0$, also $\omega_n t = \pi/2$

$$x = \frac{15.24}{87.89 \times .9756} e^{-.2197(\pi/2)} = .1259 \text{ m}$$

$$t = \frac{1}{4}\tau_d = .0174 \text{ s.}$$

2-45



Eq.(2.3-16) for $\dot{x}(0) = 0$

$$x = x(0)\, e^{-\zeta\omega_n t} \cos\sqrt{1-\zeta^2}\,\omega_n t$$

at $\sqrt{1-\zeta^2}\,\omega_n t = \pi \qquad \cos\sqrt{1-\zeta^2}\,\omega_n t = -1$

$$-.10 = 1\, e^{-\zeta\omega_n t}(-1) \qquad \text{solve by trial}$$

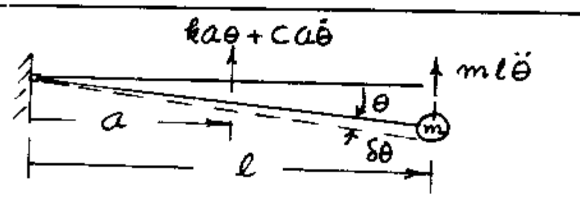
ζ	$\sqrt{1-\zeta^2}$	$\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}$	$e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$
.50	.866		.1630
.59	.8074	−2.2957	.1007 ←
.60	.800	−2.3562	.0948

$\therefore\ \zeta_1 = 0.59$

If $\zeta = \frac{1}{2}\zeta_1 = 0.295\ , \quad \sqrt{1-\zeta^2} = .9555$

$$x_{overshoot} = 1\, e^{\frac{-.295\,\pi}{.9555}} = 0.379 = 37.9\%$$

2-46 (Prob. 2-38 by V.W.)



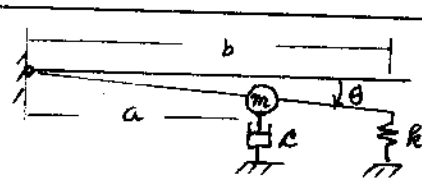
$$\delta W = -m\ell\ddot{\theta}\cdot\ell\delta\theta - (ka\theta + ca\dot{\theta})\,a\delta\theta = 0$$

$$\ddot{\theta} + \frac{ca^2}{m\ell^2}\dot{\theta} + \frac{ka^2}{m\ell^2}\theta = 0$$

$$\ddot{\theta} + 2\zeta\dot{\theta} + \omega_n^2\theta = 0 \qquad \therefore\ \omega_n = \frac{a}{\ell}\sqrt{\frac{k}{m}}, \quad \zeta = \frac{1}{2}\frac{c}{m}\left(\frac{a}{\ell}\right)^2\cdot\frac{\ell}{a}\sqrt{\frac{m}{k}} = \frac{ca}{2\ell\sqrt{km}}$$

$$\omega_d = \frac{a}{\ell}\sqrt{\frac{k}{m}}\sqrt{1 - \frac{1}{km}\left(\frac{ca}{2\ell}\right)^2}$$

2-46 (Prob. 2-39 by V.W.)



$$\delta W = -ma\ddot{\theta}\cdot a\delta\theta - ca\dot{\theta}\cdot a\delta\theta - kb\theta\cdot b\delta\theta = 0$$

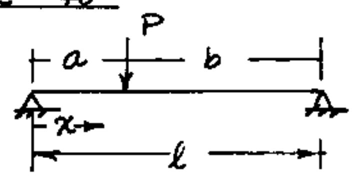
$$\therefore\ \ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{k}{m}\left(\frac{b}{a}\right)^2\theta = 0$$

2-47

$k_{eff} = (k_1 + k_2)$ in series with k_3

$$= \frac{(k_1+k_2)\,k_3}{k_1 + k_2 + k_3}$$

2-48



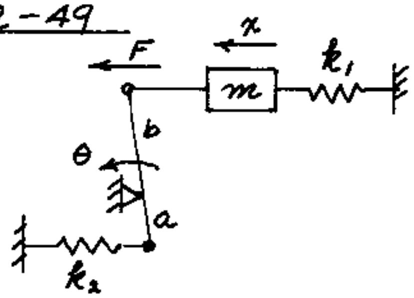
$$y(x) = \frac{Pbx}{6EI\ell}(\ell^2 - x^2 - b^2) \qquad 0 \le x \le a$$

Let $x = \frac{\ell}{3}$.

$$y\left(\tfrac{\ell}{3}\right) = \frac{P\frac{2}{3}\ell \cdot \frac{1}{3}\ell}{6EI\ell}\left(\ell^2 - \frac{\ell^2}{9} - \frac{4}{9}\ell^2\right) = \frac{P\ell^3}{EI}\cdot\frac{4}{243}$$

$$\text{Flexibility} = \frac{y}{P} = \frac{4}{243}\,\frac{\ell^3}{EI} \qquad \text{at } \frac{x}{\ell} = \frac{1}{3}$$

2-49



$$F = k_1 b\theta + \frac{a}{b}\,k_2\,a\theta$$

$$x = b\theta$$

$$\therefore F = k_1 x + \left(\frac{a}{b}\right)^2 k_2 x$$

$$k_{eff} = \frac{F}{x} = k_1 + \left(\frac{a}{b}\right)^2 k_2$$

2-50

$$k_{eff} = \frac{k_1 k_2}{k_1 + k_2}$$

2-51

Eq. (2.3-16) $\quad x(t) = e^{-\zeta\omega_n t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t + \cos\sqrt{1-\zeta^2}\,\omega_n t\right)$

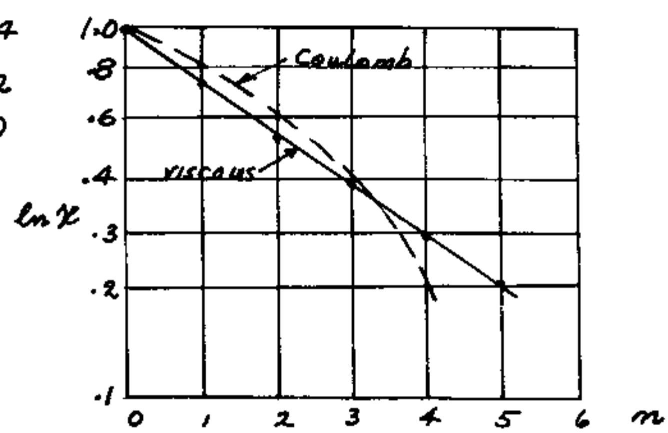
at $\omega_n t = 2\pi, 4\pi, 6\pi,$ etc. $\quad x(t) \cong e^{-\zeta\omega_n t}(0 + 1)$

n	$\omega_n t$	$e^{-.05\omega_n t}$	X_n
0	0	1.0	1.0
1	2π	.7304	.8
2	4π	.5335	.6
3	6π	.3896	.4
4	8π	.2845	.2
5	10π	.2078	0

For Coulomb friction

$$X_1 - X_2 = \frac{4F_d}{k} = \frac{4 \times .05k}{k} = .20$$

$$X_n = 1 - .2n$$



2-52

$$T = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 \dot{x}^2 + \tfrac{1}{2} I_0 \left(\frac{\dot{x}}{r}\right)^2$$

$$U = \tfrac{1}{2} k_1 x^2 + \tfrac{1}{2} k_2 \left(a\frac{x}{r} + x\right)^2$$

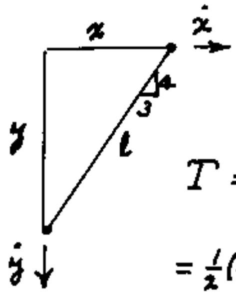
$$\frac{d}{dt}(T+U) = \left(m_1 \ddot{x} + m_2 \ddot{x} + \frac{I_0}{r^2}\ddot{x}\right)\dot{x}$$

$$+ \left(k_1 x + k_2 x + \frac{a}{r} k_2 x\right)\dot{x} = -c\,\dot{x}\,\dot{x} = \frac{dW}{dt}$$

$$\left(m_1 + m_2 + I_0/r^2\right)\ddot{x} + \left(k_1 + k_2 + \frac{a}{r}k_2\right)x + c\dot{x} = 0$$

$$c_c = 2\sqrt{k_{eff}\, m_{eff}} = 2\sqrt{\left(k_1+k_2+\frac{a}{r}k_2\right)\left(m_1+m_2+I_0/r^2\right)}$$

2-53



$$x^2 + y^2 = \ell^2$$

$$2x\,dx + 2y\,dy = 0 \qquad \therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\dot{y} = -\frac{3}{4}\dot{x}$$

$$T = \tfrac{1}{2}(m\ell)\frac{\ell^2}{3}\left(\frac{\dot{x}}{\ell}\right)^2 + \tfrac{1}{2}(m\ell)\left[\left(\frac{\dot{x}}{2}\right)^2 + \left(\frac{\dot{y}}{2}\right)^2\right] + \tfrac{1}{2}(m\ell)\frac{\ell^2}{12}\left[\frac{4\dot{x}}{5\ell} + \frac{3}{5}\frac{\dot{y}}{\ell}\right] + \tfrac{1}{2}M\dot{y}^2$$

$$= \tfrac{1}{2}(m\ell)\left\{\frac{\ell^2}{3}\left(\frac{\dot{x}}{\ell}\right)^2 + \left[\left(\frac{\dot{x}}{2}\right)^2 + \left(\frac{3\dot{x}}{8}\right)^2\right] + \frac{\ell^2}{12}\left[\frac{4}{5}\frac{\dot{x}}{\ell} + \frac{9}{20}\frac{\dot{x}}{\ell}\right]^2 + \tfrac{1}{2}M\left(\frac{3\dot{x}}{4}\right)^2\right\}$$

$$= \tfrac{1}{2}(m\ell)\left[\frac{1}{3} + \frac{1}{4} + \frac{9}{64} + \frac{1}{12}\left(\frac{16}{25} + \frac{81}{400} + \frac{72}{100}\right)\right]\dot{x}^2 + \tfrac{1}{2}M\frac{9}{16}\dot{x}^2$$

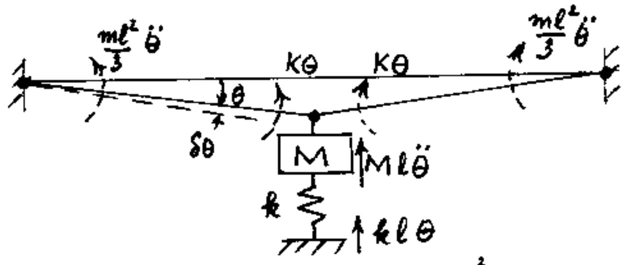
$$= \tfrac{1}{2}\left[(.854\,m\ell) + .5625\,M\right]\dot{x}^2$$

$$U = \tfrac{1}{2}k y^2 = \tfrac{1}{2}k\frac{9}{16}x^2$$

$$\frac{d}{dt}(T+U) = -c\,\frac{2}{3}\ell\,\frac{\dot{x}}{\ell} = -\frac{2}{3}c\dot{x}$$

$$\left(0.8541\,m\ell + .5625\,M\right)\ddot{x} + .5625\,kx + \frac{2}{3}c\dot{x} = 0$$

2 - 54



$$\delta W = \left[-Ml\ddot{\theta}\, l\delta\theta - kl\theta\, l\delta\theta\right] - 2K\theta\cdot\delta\theta \; - \; 2\frac{ml^2}{3}\ddot{\theta}\,\delta\theta \;=\; 0$$

$$\left(Ml^2 + \tfrac{2}{3}\, ml^2\right)\ddot{\theta} \;+\; \left(kl^2 + 2K\right)\theta \;=\; 0$$

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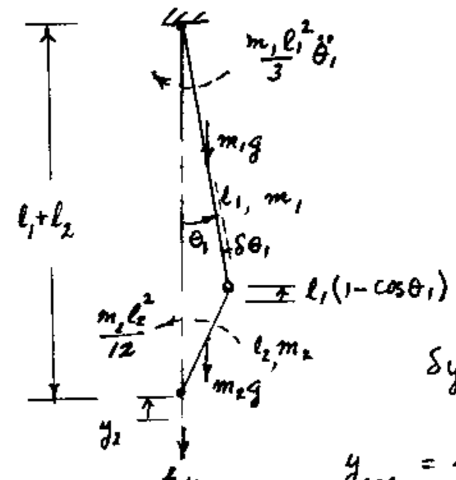
2 - 55

Accel of c.g. in vertical direction $\cong 0$
but work done by gravity is not zero



$$\delta y_{cg.1} = \frac{l_1}{2}\sin\theta_1\,\delta\theta_1 \cong \frac{l_1}{2}\,\theta_1\,\delta\theta_1$$

$$y_2 = l_1(1-\cos\theta_1) + l_2(1-\cos\theta_2)$$

$$l_1\theta_1 \cong l_2\theta_2\,, \quad \theta_2 = \frac{l_1}{l_2}\,\theta_1 \quad \therefore \; \delta\theta_2 = \frac{l_1}{l_2}\,\delta\theta_1$$

$$\delta y_2 = l_1\left(\theta_1 + \frac{l_1}{l_2}\,\theta_1\right)\delta\theta_1 \;=\; l\left(1 + \frac{l_1}{l_2}\right)\theta_1\,\delta\theta_1$$

$$y_{cg2} = l_1(1-\cos\theta_1) + \frac{l_2}{2}(1-\cos\theta_2) \quad \therefore\; \delta y_{cg2} = l_1\theta_1\,\delta\theta_1 + \frac{l_2}{2}\,\theta_2\,\delta\theta_2$$

$$= l_1\theta_1\,\delta\theta_1 + \frac{l_2}{2}\,\frac{l_1}{l_2}\,\theta_1\,\frac{l_1}{l_2}\,\delta\theta_1$$

$$= l_1\left(1 + \frac{l_1}{2l_2}\right)\theta_1\,\delta\theta_1$$

$$\delta W = -\left(\frac{m_1 l_1^2}{3}\right)\ddot{\theta}_1\,\delta\theta_1 - \left(\frac{m_2 l_2^2}{12}\,\ddot{\theta}_2\right)\delta\theta_2 - m_1 g\,\frac{l_1}{2}\,\theta_1\,\delta\theta_1 - m_2 g\, l_1\left(1 + \frac{l_1}{2l_2}\right)\theta_1\,\delta\theta_1$$

$$- k\left[l_1(1-\cos\theta_1) + l_2(1-\cos\theta_2)\right] l\left(1 + \frac{l_1}{l_2}\right)\theta_1\,\delta\theta_1 \;=\; 0$$

sub. $\ddot{\theta}_2 = \frac{l_1}{l_2}\,\ddot{\theta}_1 \quad , \quad \delta\theta_2 = \frac{l_1}{l_2}\,\delta\theta_1 \quad , \quad \theta_2 = \frac{l_1}{l_2}\,\theta_1$

$$\left[m_1\frac{l_1^2}{3} + \frac{m_2 l_2^2}{12}\left(\frac{l_1}{l_2}\right)^2\right]\ddot{\theta}_1 + \left[m_1 g\,\frac{l_1}{2} + m_2 g\, l_1\left(1 + \frac{l_1}{2l_2}\right)\right]\theta_1 \;=\; 0$$

(springforce is 2nd order infinitesmal)

3-1

$$x_{res.} = \frac{F}{c\omega_n} = \frac{F\tau}{2\pi c}$$

$$c = \frac{F\tau}{2\pi x_{res}} = \frac{24.46 \times 0.20}{2\pi \;\; 1.27\times 10^{-2}} = 61.3 \quad \frac{Ns}{m}$$

3-2

$$\frac{X_{undamped}}{X_{damped}} = \sqrt{\frac{(\omega_n^2-\omega^2)^2+(c\omega/m)^2}{(\omega_n^2-\omega^2)^2}} = R$$

$$\omega_n = \frac{2\pi}{\tau} = \frac{6.283}{.20} = 31.416 \qquad \omega = 8\pi = 25.13$$

$$\frac{c\omega}{m} = \frac{61.3 \times 8\pi}{1.95} = 790.1$$

$$R = \sqrt{\frac{(31.4^2-25.13^2)^2+(790.1)^2}{(31.4^2+25.13^2)^2}} = 2.44$$

3-3 $\quad \delta = \ln 4.2 = 1.435 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$

square & solve for ζ^2

$$2.0595(1-\zeta^2) = 39.478\,\zeta^2$$

$$\zeta^2 = \frac{2.059}{41.538} = .0496 \,, \qquad \zeta = .223$$

$$\omega_d = \frac{2\pi}{\tau_d} = \frac{2\pi}{1.80} = \omega_n\sqrt{1-\zeta^2} \qquad \omega_n = \frac{2\pi}{1.80\sqrt{1-.0496}} = 3.5806$$

$$\omega = 3 \qquad \frac{\omega}{\omega_n} = .8378$$

Eq (3.1-7) $\quad X = \frac{F_0/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} = \frac{2/525}{\sqrt{[1-.702]^2+.1396}}$

$$= \frac{2/525}{.4779} = .00797 \text{ m} = .797 \text{ cm}$$

Eq (3.1-8)

$$\phi = \tan^{-1}\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2} = \tan^{-1}\frac{.446 \times .8378}{.29801} = \tan^{-1} 1.2538 = 51.43^\circ$$

3-4 Square Eq. (3.1-7)

$$\left(\frac{X}{X_0}\right)^2 = \frac{1}{(1-r)^2+4\zeta^2 r}$$ where $X_0 = F/k$, $r = (\omega/\omega_n)^2$

$$\frac{\partial}{\partial r}\left(\frac{X}{X_0}\right)^2 = \frac{2(1-r)-4\zeta^2}{(\text{denom.})^2} = 0 \quad \therefore\ r = 1-2\zeta^2 = (\omega/\omega_n)_p^2$$

$$\left(\frac{\omega}{\omega_n}\right)_p = \sqrt{1-2\zeta^2}$$

3-5 At resonance $\frac{\omega}{\omega_n} = 1.0$, $\frac{X}{X_0} = \frac{1}{2\zeta} = \frac{.58}{X_0}$

When $\frac{\omega}{\omega_n} \neq 1.0$ $\quad\therefore\ X_0 = 1.16\,\zeta$

$$\frac{X}{X_0} = \frac{1}{\sqrt{[1-r^2]^2+[2\zeta r]^2}} = \frac{.46}{X_0}$$

Square & solve for ζ^2

$$\frac{1}{[1-.64]^2+[2\zeta]^2 .64} = \frac{.2116}{(1.16\zeta)^2}$$

$$6.359\,\zeta^2 = .1296 + 2.560\,\zeta^2$$ $\zeta^2 = .0341$, $\zeta = .1847$

3-6

Eq. 3.1-17 $$H(\omega) = \frac{1-r^2}{[1-r^2]^2+[2\zeta r]^2} - i\,\frac{2\zeta r}{[1-r^2]^2+[2\zeta r]^2}$$

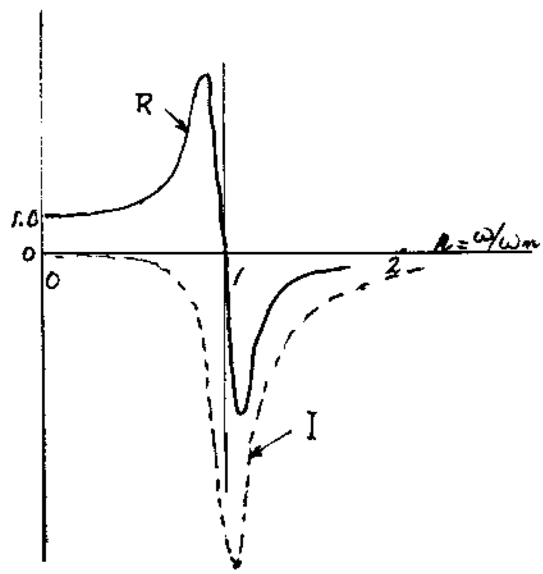
$$= R + iI$$

$\zeta = .01$

r	$1-r^2$	$(1-r^2)^2$	$2\zeta r$	$(2\zeta r)^2$	Den.	R	I
.20	.9999	.9998	.0040	.000016	.99982	1.00008	-.004001
.40	.840	.7056	.0080	.000064	.70566	1.19037	-.011337
↓							
.90	.1900	.0361	.0180	.000324	.036424	5.21634	-.494180
.94	.1164	.013549	.01880	.000353	.013902	8.37263	-1.35228
.96						12.0325	-2.94676

3 − 6 Cont.

.990	.01990	.000396	.01980	.000392	.000788	25.2525	−25.1256
.998						9.6428	−48.1658
.999						4.9578	−49.5535
1.000						0	−50.000
1.002						−9.5881	−47.988
1.020						−19.725	−9.9602
1.060	−.1236	.015277	.0212	.000449	.015726	−7.8594	−1.348



3 − 7

$$m\ddot{x}_1 = -c\dot{x}_1 + k(x_2 - x_1)$$

$$m\ddot{x}_1 + c\dot{x}_1 + kx_1 = kX_2 \sin\omega t$$

Replace excitation by $kX_2 e^{i\omega t}$, then $x = X_1 e^{i(\omega t - \phi)}$
$= X_1 e^{-i\phi} e^{i\omega t} = \bar{X}_1 e^{i\omega t}$

$$\left[(k - m\omega^2) + i\omega c\right]\bar{X}_1 \cancel{e^{i\omega t}} = kX_2 \cancel{e^{i\omega t}}$$

$$\bar{X}_1 = \frac{kX_2}{(k - m\omega^2) + i\omega c} = \frac{kX_2 e^{-i\phi}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$\therefore\ X_1 = \frac{kX_2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}, \qquad \phi = \tan^{-1}\frac{c\omega}{k - m\omega^2}$$

3 − 8

$$m\ddot{x} = c(\dot{y} - \dot{x}) + kx$$

$$m\ddot{x} + c\dot{x} + kx = c\dot{y}$$

3-8 Cont.

let $y = Y e^{i\omega t}$ $\qquad x = X e^{i(\omega t-\phi)} = X e^{-i\phi} e^{i\omega t}$

$\dot{y} = i\omega Y e^{i\omega t}$ $\qquad = \bar{X} e^{i\omega t}$

$$(k - m\omega^2 + i\omega c)\bar{X} = i\omega Y$$

$$\bar{X} = \frac{i\omega Y}{k - m\omega^2 + i\omega c} = \frac{i\omega Y e^{-i\psi}}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} = X e^{-i\phi}$$

$$X e^{-i\phi} = \frac{\omega Y}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}\, i e^{-\psi} \qquad \therefore e^{-i\phi} = i e^{-i\psi} = e^{-i(\psi - \frac{\pi}{2})}$$

$$X = \frac{\omega Y}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}} \qquad \phi = \psi - \frac{\pi}{2}, \qquad \tan\psi = \frac{c\omega}{k - m\omega^2}$$

3-9

(a) $\varphi = 90^\circ \therefore$ resonance $\qquad \omega_n = \omega = \frac{900 \times 2\pi}{60} = 2\pi f_n$

$f_n = 15$ cps $= 900$ cpm

(b) Eq. (3.2-4) at resonance

$$\zeta = \frac{em}{2MX} = \frac{2 \times .0921}{2 \times 181.4 \times 21.6 \times 10^{-3}} = 0.0118 \times 2$$

(c) At 1200 rpm $\quad \frac{\omega}{\omega_n} = \frac{1200}{900} = 1.333$

Eq. (3.2-4)

$$X = \frac{me}{M} \frac{1.333^2 \times 2}{\sqrt{[1 - 1.333^2]^2 + [2 \times .0118 \times 1.333]^2}}$$

$$= \frac{2 \times .0921}{181.4} \times \frac{1.777}{.78023} = .002314\,m = .2314\text{ cm}$$

(d)

Eq. (3.2-5) $\quad \varphi = \tan^{-1} \frac{2 \times .0235 \times 1.333}{1 - 1.333^2} = \tan^{-1}(-.0805)$

$$= 180^\circ - 4.61^\circ = 175.4^\circ$$

3 — 10

$$M\ddot{x} + c\dot{x} + kx = (me\omega^2)\sin\omega t$$

$$\text{Let } (me\omega^2)\sin\omega t = F e^{i\omega t}$$

$$\text{then } x = X e^{i(\omega t-\varphi)} = X e^{-i\varphi} e^{i\omega t} = \bar{X} e^{i\omega t}$$

$$(-\omega^2 M + ic\omega + k)\bar{X} \cancel{e^{i\omega t}} = F \cancel{e^{i\omega t}}$$

$$\bar{X} = \frac{F}{(k-\omega^2 M) + i(c\omega)}$$

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3 — 11

$$\omega = \frac{1200}{60} = 20\ rps = 20 \times 2\pi\ rad/s.$$

$$\omega_n = 18\ cps = 18 \times 2\pi\ rad/s. \quad \therefore \frac{\omega}{\omega_n} = 1.111$$

Eq. 3.2-6

$$x(t) = \underbrace{X_1 e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\,\omega_n t - \phi_1)}_{\text{transient}} + \underbrace{\frac{me\omega^2 \sin(\omega t - \varphi)}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}}_{\text{steady state}}$$

at $t=0$, $x(0)=0$

$$0 = -X_1\sin\phi_1 + \frac{(me\omega^2/k)\sin(-\varphi)}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta\,\omega/\omega_n]^2}} \qquad (1)$$

at $t=0$, $\dot{x}(0) = 0$

$$\dot{x}(t) = X_1 e^{-\zeta\omega_n t}\left[\sqrt{1-\zeta^2}\,\omega_n\cos(\sqrt{1-\zeta^2}\,\omega_n t - \phi_1) - \zeta\omega_n\sin(\sqrt{1-\zeta^2}\,\omega_n t - \phi_1)\right]$$

$$+ \frac{(me\omega^2/k)\,\omega\cos(\omega t - \varphi)}{\sqrt{[1-(\frac{\omega}{\omega_n})^2]^2 + [2\zeta\frac{\omega}{\omega_n}]^2}}$$

$$\therefore \dot{x}(0) = 0 = X_1\left[\omega_n\sqrt{1-\zeta^2}\cos\phi_1 + \zeta\omega_n\sin\phi_1\right] + \frac{(me\omega^2/k)\,\omega\cos\varphi}{\sqrt{}} \qquad (2)$$

from (1) $\quad X_1\sin\phi_1 = \dfrac{-(me\omega^2/k)\sin\varphi}{\sqrt{}}$

from (2)

$$X_1\omega_n\sqrt{1-\zeta^2}\cos\phi_1 + \zeta\omega_n\left[\frac{-(me\omega^2/k)\sin\varphi}{\sqrt{}} + \frac{(me\omega^2/k)\,\omega\cos\varphi}{\sqrt{}}\right] = 0$$

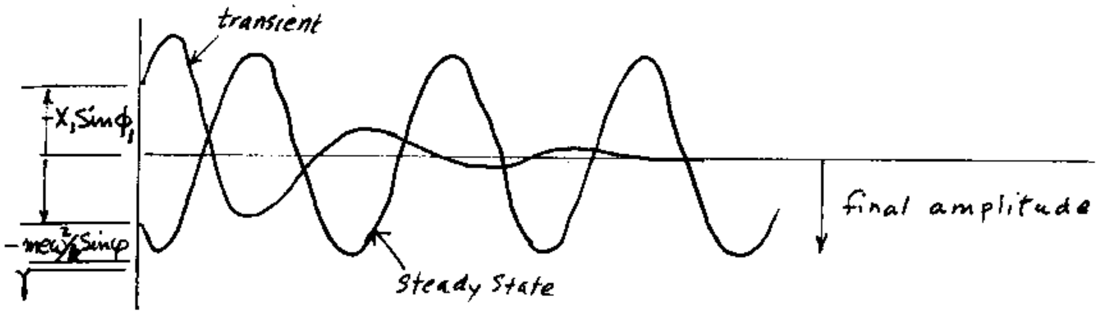
dividing

$$\therefore \tan\phi_1 = \frac{-\sqrt{1-\zeta^2}\,\sin\varphi}{\zeta\sin\varphi - \frac{\omega}{\omega_n}\cos\varphi} \qquad (3)$$

3 − 11 Cont.

at $t = \infty$, transient term $= 0$ due to $e^{-\zeta\omega_n t} \to 0$

$$\therefore \text{final ampl.} = x(\infty) = \frac{(me\omega^2/k)\sin(\omega t - \varphi)}{\sqrt{[1-(\frac{\omega}{\omega_n})^2]^2 + [2\zeta\frac{\omega}{\omega_n}]^2}}$$



phase φ can then be solved from $\tan\varphi = \dfrac{2\zeta\frac{\omega}{\omega_n}}{1-(\frac{\omega}{\omega_n})^2} = \dfrac{.2(1.111)}{1-(1.111)^2}$

$= -.94755 \quad \therefore \varphi = -43.457°$ Eq. (3.2-5)

Then ϕ_1 solved from (3) $\tan\phi_1 = -.7820$, $\phi_1 \simeq 142°$

Then solve for X_1 and sub. into Eq. (3.2-6) for build up eq.

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3 − 12 Lowest critical speed = fundamental freq. of lateral vibr.

$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ where $k = \frac{48EI}{\ell^3}$, $I = \frac{\pi d^4}{64}$

$m = m_{disk} + 0.486\, m_{shaft}$

$$k = \frac{48(29\times10^6)\pi(\frac{1}{2})^4}{(24)^3 \times 64} = \mathbf{309.}\ lb/in$$

$$.486\, m_{shaft} = \frac{.486(.283)\frac{\pi}{4}(\frac{1}{2})^2(24)}{386} = \frac{.648}{386}$$

$$m = \frac{10}{386} + \frac{.648}{386} = \frac{\mathbf{10.65}}{\mathbf{386}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{309\times386}{10.65}} = \mathbf{16.84}\ cps = 1028\ rpm$$

3-13

$w = 10 \text{ lb} = 44.48 \text{ N} \qquad m = \frac{44.48}{9.81} = 4.534 \text{ kg}$

$g = 386 \frac{in}{sec^2} = 9.81 \; m/s^2$

$$k = \frac{48EI}{l^3} = \left\{ \begin{array}{l} E = 200 \times 10^9 \; N/m^2 \\ l = 2 \times .3048 = .6096 \; m \\ d = .5 \times 2.54 \times 10^{-2} = 1.270 \times 10^{-2} \; m \\ I = \frac{\pi d^4}{64} = .1277 \times 10^{-8} \end{array} \right.$$

$$k = \frac{200 \times 10^9 \times 48 \times .1277 \times 10^{-8}}{(.6096)^3} = 54116 \; N/m$$

$$.486 \, m_{shaft} = .486 \times \frac{\pi}{4}(1.270 \times 10^{-2})^2 (.6096) \times \rho = .2938$$

$\rho = 7830 \frac{kg}{m^3}$ = density of steel

$$f = \frac{1}{2\pi}\sqrt{\frac{54116}{4.534 + .2938}} = 16.86 \text{ Hz}$$

3-14

$dia = 2.54 \text{ cm} \qquad I = \frac{\pi d^4}{64} = \frac{\pi \times 41.62}{64 \times 100^4} = 2.043 \times 10^{-8} \, m^4$

$$k = \frac{48EI}{l^3} = \frac{48(200 \times 10^9) 2.043 \times 10^{-8}}{.4064^2} = 2.922 \times 10^6 \; N/m$$

$$m = 13.6 + .486\left(\frac{\pi}{4} \times .0254^2 \times .4064\right)(7830) = 13.6 + .784$$
$$= 14.38 \text{ kg}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{2.922 \times 10^6}{14.38}} = 71.74 \text{ Hz} = 4304 \text{ rpm}$$

$$\frac{\omega}{\omega_n} = \frac{6000}{4304} = 1.394 \qquad r = \frac{e\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = -2.060 \, e$$

$me = .2879$ kg cm (given) $\qquad \therefore \quad e = \frac{.2879}{13.6} = .02117$ cm

$r = -2.060(.02117) = -.04316$ cm

$$F = m(r+e)\omega^2 = 14.38\left(\frac{-.04316 + .02117}{100}\right)(2\pi \times 100)^2 = 1273 \text{ N}$$

3-14 Cont.

for diam. = 1.905 cm

$$.486\, m_{shaft} = .486\left(\frac{\pi}{4} \times .01905^2 \times .4064\right)(7830) = .4408$$

$$m = m_{disk} + .483\, m_{shaft} = 14.04 \text{ kg.}$$

$$I = \frac{\pi}{64}\, 1.905^4 \times 100^{-4} = .6464 \times 10^{-8}$$

$$k = \frac{48\,(200 \times 10^9).6464 \times 10^{-8}}{.4064^3} = 0.9441 \times 10^6$$

$$f_n = 41.27 \text{ Hz} = 2476 \text{ rpm} \qquad \frac{\omega}{\omega_n} = \frac{6000}{2476} = 2.423$$

$$r = \frac{.02117\,(2.423)^2}{1-(2.423)^2} = -.02552 \qquad r+e = .00435$$

$$F = 14.042\left(\frac{.00435}{100}\right)\left(2\pi \times \frac{6000}{60}\right)^2 = 241.1 \text{ N}$$

3-15

$$r = r_0 + \frac{e\omega t}{2} \qquad \text{(see Ex. 3.4-1)}$$

$$.0508 = 0 + .0212\,(2\pi \times 100)\frac{t}{2}$$

$$t = \frac{.0508}{6.6602} = .0075 \text{ sec}$$

3-16

$$m\ddot{x} = -k(x-y)$$ where x = displ. of m measured from static equilib. position of m with $y = 0$

Let $y = Y \sin \frac{2\pi Vt}{L}$

then $m\ddot{x} + kx = kY \sin \frac{2\pi Vt}{L} = kY \sin \omega t$

where $\omega = \frac{2\pi V}{L}$ Sol is $x = X \sin \omega t$

$$X = \frac{Y}{1-(\omega/\omega_n)^2} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

The most unfavorable speed corresponds to $\frac{\omega}{\omega_n} = 1$

$$\therefore \quad V = \frac{L}{2\pi}\sqrt{\frac{k}{m}}$$

3 - 17   (Ref. Prob 3-16)

$$Eq. (2.2-9) \qquad f_n = \frac{15.76}{\sqrt{\Delta_{mm}}} = \frac{15.76}{\sqrt{101.6}} = 1.563 \quad Hz$$

$$\omega_n = 2\pi f_n = 9.824 \text{ r/s}$$

$$V_{crit.} = \frac{L\omega_n}{2\pi} = \frac{14.63}{2\pi} \times 9.82 = 22.87 \text{ m/s.}$$

$$V \quad 64.4 \text{ km/hr} = \frac{64400}{60^2} = 17.89 \text{ m/s}$$

$$\omega = \frac{2\pi V}{L} = \frac{2\pi \; 17.89}{14.63} = 7.683 \text{ r/s}$$

$$\left(\frac{\omega}{\omega_n}\right)^2 = \left(\frac{7.683}{9.824}\right)^2 = 0.6117 \qquad X = \frac{7.62 \text{ cm}}{1 - .6117} = 19.62 \text{ cm}$$

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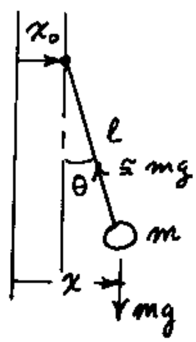
3 - 18



$$x \cong x_o + \ell\theta \qquad \theta = \frac{x - x_o}{\ell}$$

$$m\ddot{x} = -mg\theta = -\frac{mg}{\ell}(x - x_o)$$

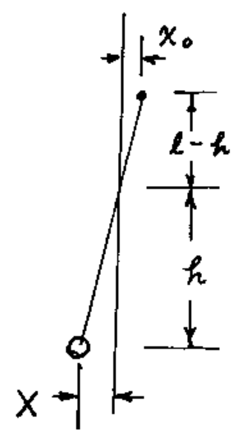
$$\ddot{x} + \frac{g}{\ell}x = \frac{g}{\ell}x_o$$

Let $x_o = X_o \sin\omega t \qquad x = X\sin\omega t$

$$\therefore X = \frac{X_o}{1 - (\omega/\omega_n)^2} \qquad \omega_n = \sqrt{\frac{g}{\ell}}$$

when $\omega = \sqrt{2}\,\omega_n \qquad X = -X_o$

$\therefore$ mode found at $\frac{\ell}{2}$



$$\frac{h}{|X|} = \frac{\ell - h}{|X_o|}$$

$$h\left|\frac{X_o}{X}\right| = \ell - h = h\left[\left(\frac{\omega}{\omega_n}\right)^2 - 1\right] \qquad \text{since } \frac{\omega}{\omega_n} > 1$$

$$h = \ell\left(\frac{\omega_n}{\omega}\right)^2$$

3-19 $\quad m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$

$y = Y\sin\omega t \qquad x = X\sin(\omega t - \psi) \qquad \dot{x} = \omega X\cos(\omega t - \psi)$

$\dot{y} = \omega Y\cos\omega t \qquad \ddot{x} = -\omega^2 X\sin(\omega t - \psi)$

Sub. into DE.

$$(k - m\omega^2)X\sin(\omega t - \psi) + c\omega X\cos(\omega t - \psi) = c\omega Y\cos\omega t + kY\sin\omega t$$

Expand $\sin(\omega t-\psi)$ & $\cos(\omega t - \psi)$ & equate coef of $\cos\omega t$ & $\sin\omega t$

coef. of $\sin\omega t \rightarrow \quad [(k - m\omega^2)\cos\psi + \omega c\sin\psi]X = kY$

" " $\cos\omega t \rightarrow \quad [(k - m\omega^2)\sin\psi - \omega c\cos\psi]X = -\omega cY$

Divide & factor $\cos\psi$ from num. & denom. to get

$$\tan\psi = \frac{mc\omega^3}{k(k - m\omega^2) + (c\omega)^2} \qquad = \text{Eq.}(3.5\text{-}9)$$

Solve for $\sin\psi$ & $\cos\psi$ & sub. into

$$\frac{X}{Y} = \frac{k}{(k - m\omega^2)\cos\psi + \omega c\sin\psi} = \sqrt{\frac{k^2 + (\omega c)^2}{(k - m\omega^2)^2 + (c\omega)^2}} = \text{Eq.}(3.5\text{-}8)$$

after much algebra

3-20

See Sec. (3.6) $\qquad f = 15.76\sqrt{\frac{1}{\Delta}\left(\frac{1}{TR} - 1\right)}$

$$\frac{1600}{60} = 15.76\sqrt{\frac{1}{\Delta_{mm}}\left(\frac{1}{.15} - 1\right)}$$

Solve for $\Delta = 2.678$ mm.

For $f = 2200$ cpm the F_{TR} is smaller

3-21 $\quad TR = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = 0.10 \qquad \therefore \left(\frac{\omega}{\omega_n}\right)^2 = 11.0$

$$\omega_n^2 = \frac{k}{m} = \frac{\omega^2}{11.0} \qquad k = \frac{m\omega^2}{11} = \frac{65}{386}\left(\frac{580 \times 2\pi}{60}\right)^2 \frac{1}{11} = 56.5 \text{ lb/in}$$

$$k_{per\ spring} = \frac{1}{3} \times 56.5 = 18.8 \text{ lb/in}$$

3 - 22

$$k = \frac{Mg}{\Delta} = \frac{453.4 \times 9.81}{.005080} = 875561 \ \frac{N}{m} = 8755.6 \ \frac{N}{cm}$$

$$\omega_n^2 = \frac{k}{M} = \frac{8755.6 \times 10^2}{453.4} = 1931.1$$

$$\left(\frac{\omega}{\omega_n}\right)^2 = \left(\frac{1200 \times 2\pi}{60}\right)^2 \frac{1}{1931.1} = 8.177$$

$$X = \frac{\frac{me}{M}\left(\frac{\omega}{\omega_n}\right)^2}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{\frac{.2303}{453.4} \times 8.177}{8.177 - 1} = 0.000579 \ m$$

$$F_{TR} = kX = 875561 \times .000679 = 506.7 \ N$$

---

3 - 23

$$\text{New } M = 453.4 + 1136. = 1589.4 \ kg$$

$$\text{New } k = 875561 \times \frac{1589.4}{453.4} = 3069.295 \times 10^3 \ \frac{N}{m}$$

$\frac{\omega}{\omega_n}$ is same

$$X = \frac{me}{M} \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{453.4}{1589.4} \times .000579 = 0.165 \times 10^{-3} \ m$$

---

3 - 24

$$M = 68 + 1200 = 1268 \ kg.$$

$$f_n = 160 \text{ cpm} \qquad \omega_n = \frac{160}{60} \times 2\pi = 16.75 \ r/s$$

$$\frac{\omega}{\omega_n} = \frac{31.4}{16.75} = 1.8746 \qquad k = \omega_n^2 M = 355951 \ \frac{N}{m}$$

$$X = \frac{100 / 0.3559 \times 10^6}{\sqrt{[1 - 1.875^2]^2 + [0.2 \times 1.875]^2}} = 110.5 \times 10^{-6} \ m$$

$$= 0.01105 \ cm$$

$$F_{TR} = kX\sqrt{1 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2} = .3559 \times 10^6 \times 110.5 \times 10^{-6} \sqrt{1 + .1406}$$

$$= 42.0 \ N$$

3-25

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{280200}{113}} = 49.796$$

$$\omega = 2\pi \times 20 = 125.6 \qquad \frac{\omega}{\omega_n} = 2.5236$$

$$\text{Accel.} = 15.24 = \omega^2 Y \quad \therefore \quad Y = .000965 \text{ cm}$$

$$\left|\frac{X}{Y}\right| = \frac{\sqrt{1+[.2\times 2.5236]^2}}{\sqrt{[1-6.368]^2 + .2547}} = 0.2078$$

$$\therefore X = .2078 \times .000965 = 0.0002005 \text{ cm}$$

$$\omega^2 X = 3.166 \text{ cm/s}^2 = \text{transmitted accel.}$$

3-26 Add mass M to instrument ∴ increase ω/ω_n

X must be reduced to $\left(\frac{2.03}{3.166}\right) .0002005 = .0001285$

$$\frac{X}{Y} = \frac{1285}{9650} = .1332 \leq \sqrt{\frac{1+(.2\frac{\omega}{\omega_n})^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(.2\frac{\omega}{\omega_n}\right)^2}}$$

Solve by trial: for $\frac{\omega}{\omega_n} = 3$, $\sqrt{} = .1321$

for $\frac{\omega}{\omega_n} = 4$ $\sqrt{} = .0893$

both of these are OK

For $\frac{\omega}{\omega_n} = 4$ $\qquad \omega_n = 31.4 = \sqrt{\frac{k}{M_i + M_o}} = \sqrt{\frac{280200}{M_i + M_o}}$

$M_i + M_o = 284. \quad \therefore \quad M_o = 171.$ kg to be added.

3-27 Compare Eq.(3.5-8) with Eq(3.6-2). They are same

$$TR = \sqrt{\frac{1+(2\zeta\frac{\omega}{\omega_n})^2}{[1-(\frac{\omega}{\omega_n})^2]^2+[2\zeta\frac{\omega}{\omega_n}]^2}}$$

To calculate $20\log|TR|$, first calculate TR with ζ fixed, varying $\frac{\omega}{\omega_n}$. Then find Db. Suggest computer program.

3-28
$$W_d = \pi c\omega X^2 = \pi\, 2\zeta\frac{\omega}{\omega_n} k\, X^2$$

$$= \frac{\pi F_0^2}{k}\,\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}$$

3-29
$$\eta = \frac{W_d}{2\pi U}, \qquad U = \tfrac{1}{2}kX_{max}^2, \qquad W_d = \pi c\omega X_{max}^2$$

$$\therefore\quad \eta = \frac{\pi c\omega X^2}{2\pi\,\frac{1}{2}kX^2} = \frac{c\omega}{k}$$

3-30
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F}{m}\sin\omega t$$

$X_{res.} = \frac{F}{2\zeta k}$ From Prob. 3-29 $\eta_{res} = \frac{c\omega}{k} = \frac{c\omega_n}{k}$ at resonan.

$$= \frac{c}{c_{cr}}\,\frac{2m\omega_n^2}{k} = 2\zeta$$

$$\therefore\quad \ddot{x} + \eta_{res.}\omega_n\dot{x} + \omega_n^2 x = \frac{F}{m}\sin\omega t$$

3-31
$$\frac{\tau_d}{\tau_n} = \frac{\omega_n}{\omega_d} = \frac{1}{\sqrt{1-\zeta^2}} \qquad \therefore\left(\frac{\tau_n}{\tau_d}\right)^2 = 1-\zeta^2$$

$\left(\frac{\tau_n}{\tau_d}\right)^2 + \zeta^2 = 1$ = circle of radius 1

3-32
$$\frac{W_d}{U} = \frac{2\zeta\pi kX^2}{\frac{1}{2}kX^2} = 4\zeta\pi$$

but $\delta \cong 2\pi\zeta$ $\quad\therefore\quad \frac{W_d}{U} \cong 2\delta$, $\qquad \zeta = \frac{c}{2m\omega_n}$

$$\delta = \frac{c\pi}{m\omega_n} = \frac{c\pi\omega_n}{k} = \frac{W_d}{2U}$$

3-33 $\quad W_d = f(X,\omega) = \pi c \omega X^2$

$$\delta = \frac{W_d}{2U} = \frac{\pi c \omega X^2}{k X^2} = \frac{\pi c \omega}{k} \quad \text{for viscous damping}$$

3-34 $\quad W_d = c_{eq.} \pi \omega X^2 = D\,4X$ for Coulomb

$$\therefore c_{eq.} = \frac{4D}{\pi \omega X} \qquad \zeta_{eq} = \frac{c_{eq}}{c_c} = \frac{4D}{\pi \omega X\, 2m\omega_n}$$

$$2\zeta_{eq}\frac{\omega}{\omega_n} = \frac{4D}{\pi \omega X\, 2m\omega_n} \cdot \frac{2\omega}{\omega_n} = \frac{4D}{\pi k X}$$

3-35

$$X = \frac{F_o/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta_{eq.}\frac{\omega}{\omega_n}\right]^2}} \qquad 2\zeta_{eq.}\frac{\omega}{\omega_n} = \frac{4D}{\pi k X}$$

square both sides and subst for $\zeta_{eq.}$

$$X^2\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[\frac{4D}{\pi k X}\right]^2\right\} = \left(F_o/k\right)^2$$

$$X^2\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 = \left(F_o/k\right)^2 - \left(\frac{4D}{\pi k}\right)^2$$

$$X = \frac{\sqrt{(F_o/k)^2 - (4D/\pi k)^2}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \qquad \therefore \text{ for } X \text{ real} \quad F_o > \frac{4D}{\pi}$$

3-36 Rewrite eq. as

$$\frac{kX}{F_o} = \frac{\sqrt{1-\left(\frac{4}{\pi}\frac{D}{F_o}\right)^2}}{1-\left(\frac{\omega}{\omega_n}\right)^2} = f\left(\frac{D}{F_o}, \frac{\omega}{\omega_n}\right)$$

3-37 $\quad J\ddot{\theta}_2 + K(\theta_2 - \theta_1) = 0 \qquad \omega_n^2 = \frac{K}{J}$

$$\ddot{\theta}_2 + \omega_n^2\theta_2 = \omega_n^2\theta_1 = \omega_n^2\,\Theta_1 \sin\omega t \qquad \text{Let } \theta_2 = \Theta_2 \sin\omega t$$

$$\therefore \Theta_2 = \frac{\omega_n^2\,\Theta_1}{\omega_n^2 - \omega^2} \quad \text{and} \quad \theta_2 - \theta_1 = (\Theta_2 - \Theta_1)\sin\omega t \qquad \begin{cases} \theta_2 = \text{outer wheel} \\ \theta_1 = \text{shaft.} \end{cases}$$

$$\text{Rel. ampl} = (\Theta_2 - \Theta_1) = \frac{\omega_n^2\,\Theta_1}{\omega_n^2 - \omega^2} - \Theta_1 = \frac{\omega^2}{\omega_n^2 - \omega^2}\,\Theta_1$$

$$= \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1-\left(\frac{\omega}{\omega_n}\right)^2}\,\Theta_1$$

$$\text{Rel to fixed ref} \quad \Theta_2 = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\,\Theta_1$$

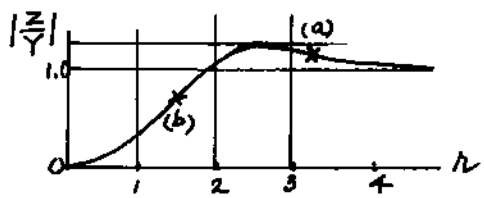
3-38 Let $r = \omega/\omega_n$

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \qquad \left(\frac{Z}{Y}\right)^2 = \frac{(r^2)^2}{(1-r^2)^2+(2\zeta r)^2}$$

$$\frac{\partial}{\partial r^2}\left(\frac{Z}{Y}\right) = 0 \quad \text{gives} \quad r_p^2 = \frac{1}{1-2\zeta^2} \quad \text{for peak ampl.}$$

for $\zeta = 0.650$, $r_p = 2.54$, $\frac{Z}{Y} = 1.012$

for (a) $\frac{Z}{Y} = 1.010$ Write above eq. as $r^4\left[1-\left(\frac{Y}{Z}\right)^2\right] - r^2(2-4\zeta^2)+1=0$



For accuracy expand $1-\left(\frac{Y}{Z}\right)^2$

$$= 1 - \frac{1}{\left(\frac{Z}{Y}\right)^2} = 1 - \frac{1}{(1+.01)^2}$$

$$= 1-(1+.01)^{-2} = 1-(1-.02+.0003-\cdots)$$

$$= 0.0197 \qquad (2-4\zeta^2) = 0.310$$

$$0.0197\, r^4 - 0.310\, r^2 + 1 = 0$$

$$\frac{\omega}{\omega_n} = r = \begin{cases} 2.125 \\ 3.350 \end{cases}$$

$\therefore$ Lowest freq. for 1% accuracy $= 3.35 \times 4.75 = \underline{\underline{15.9 \text{ cps}}}$.

(b) For 2% above plot indicates we must use $\frac{Z}{Y} = 0.98$

$$-0.0412\, r^4 - 0.310\, r^2 + 1 = 0 \quad , \quad r = 1.57$$

$$1.57 \times 4.75 = \underline{\underline{7.45 \text{ cps}}}$$

3-39
$$Z = \frac{\left(\frac{\omega}{\omega_n}\right)^2 Y}{1-\left(\frac{\omega}{\omega_n}\right)^2} \qquad Y = Z\left[\left(\frac{\omega_n}{\omega}\right)^2 - 1\right]$$

$$= 0.052\left[\left(\frac{1}{4}\right)^2-1\right] = -0.0488 \text{ cm}$$

3-40 sensitivity $= 0.096$ rms volts/cm per sec.

0.024 rms volts measured $\therefore$ cm per sec vel $= \frac{.024}{.096} = 0.250$

$\omega = 2\pi \times 30 = 188.49$ r/s.

(a) vel $= 0.250 = \omega x = 188.49\, x_{cm}$. $\qquad x = \underline{\underline{0.001326}}$ rms cm.

(b) ampl $= 0.40$ cm peak

cannot be used since 0.40 cm exceeds clearance of 0.30 cm

3-41 sensitivity = 40 mv/cm per s.

1 g accel = 9.81 m/s² = 981 cm/s² accel = $\omega^2 Y$

(a) at 10 cps $\omega = 2\pi 10 = 62.83$ r/s vel = ωY.

$$\omega Y = \frac{981}{62.83} = 15.61 \text{ cm/s.}$$

Output volts = $40 \times 15.61 = \underline{\underline{624.5 \text{ mv}}}$

(b) at 2000 cps $\omega = 12566$

$$\omega Y = \frac{981}{12566} = 0.07806 \text{ cm/s .}$$

Output volts = $40 \times .07806 = \underline{\underline{3.123 \text{ mv}}}$

3-42 For vel. pick-up $\frac{\omega}{\omega_n} >> 1 \quad \therefore \quad Z = Y$

Volts generated by instrument = $\omega Z = \omega Y$

3-43 Sensitivity = 30 mv/cm. per s.

3 mv = accuracy Limit = $30 (\omega Z)$ $\therefore$ $\omega Z = 0.10$ cm/s = Limiting vel.

$$v_{min} = 0.10 = \frac{981}{2\pi f} \quad \therefore \quad f = \underline{\underline{1561 \text{ cps}}} = \text{upper freq. Limit.}$$

At $f = 200$ $\quad v = \frac{981}{2\pi \times 200} = 0.7807$

instr. reading = $30 \times .7807 = \underline{\underline{23.42 \text{ mv}}}$.

3-44 $C = 450$ pF $\quad Q = 18$ pC/g

$C_{cable} = 5 \times 50 = 250$ pF

$$E = \frac{Q}{C} = \frac{18}{450} = 0.040 \text{ volts/g} = \underline{\underline{40 \text{ mv/g}}} \text{ open circuit}$$

With cable

$$E = 40 \times \frac{450}{450 + 250} = \underline{\underline{25.7}} \text{ mv/g}$$

3-45

$$\frac{W_d}{U} = \frac{\pi c \omega X^2}{\frac{1}{2} k X^2} = \frac{2\pi c \omega}{k}$$

$$c = \zeta c_c = \zeta 2\sqrt{km}$$

$$\frac{W_d}{U} = 2\pi \cdot 2\zeta\sqrt{km} \cdot \frac{\omega}{k} = 4\pi\zeta\frac{\omega}{\omega_n}$$

3-46

$$\frac{W_d}{U} = 4\pi\zeta\frac{\omega}{\omega_n}, \qquad \zeta \cong \frac{\delta}{2\pi}$$

$$= 2\delta\frac{\omega}{\omega_n}$$

3-47

$$\eta = \frac{W_d}{2\pi U} = \frac{1}{2\pi}\left(4\pi\zeta\frac{\omega}{\omega_n}\right) = 2\zeta\frac{\omega}{\omega_n}$$

but $2\zeta = \gamma$ at resonance

$$\therefore \eta = \gamma\frac{\omega}{\omega_n} = \gamma \quad \text{at resonance}$$

3-48

$$x = \frac{(1-r^2)}{(1-r^2)^2+(2\zeta r)^2}, \qquad y = \frac{-2\zeta r}{(1-r^2)^2+(2\zeta r)^2}$$

$$y + \frac{1}{2(2\zeta r)} = \frac{-2\zeta r}{(1-r^2)^2+(2\zeta r)^2} + \frac{1}{2(2\zeta r)} = \frac{-2(2\zeta r)^2+(1-r^2)^2+(2\zeta r)^2}{2(2\zeta r)\left[(1-r^2)^2+(2\zeta r)^2\right]}$$

$$= \frac{(1-r^2)^2-(2\zeta r)^2}{2(2\zeta r)\left[(1-r^2)^2+(2\zeta r)^2\right]}$$

$$\therefore \quad x^2 + \left(y+\frac{1}{2(2\zeta r)}\right)^2 = \frac{(1-r^2)^2}{\left[(1-r^2)^2+(2\zeta r)^2\right]} + \left\{\frac{(1-r^2)^2-(2\zeta r)^2}{2(2\zeta r)\left[(1-r^2)^2+(2\zeta r)^2\right]}\right\}^2$$

$$= \frac{1}{4(2\zeta r)^2}$$

4 - 1

Eq. (4.1-6) for impulsive response

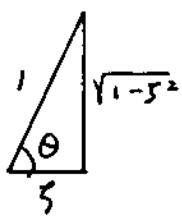
$$x = \frac{\hat{F}}{m\omega_n\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\sqrt{1-\zeta^2}\,\omega_n t$$

For max response $\frac{dx}{dt} = 0$

$$\frac{dx}{dt} = \frac{\hat{F}}{m\omega_n\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left\{-\zeta\omega_n \sin\sqrt{1-\zeta^2}\,\omega_n t + \sqrt{1-\zeta^2}\,\omega_n \cos\sqrt{1-\zeta^2}\,\omega_n t\right\} = 0$$

$$\therefore \quad \tan\sqrt{1-\zeta^2}\,\omega_n t = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

4-2



Let $\theta = \sqrt{1-\zeta^2}\,\omega_n t$

$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$ from Prob 4-1

$\sin\theta = \sqrt{1-\zeta^2}$

Eq. (4.1-6) becomes $\quad x_{peak} = \frac{\hat{F}}{m\omega_n\cancel{\sqrt{1-\zeta^2}}} e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}} \times \cancel{\sqrt{1-\zeta^2}}$

$$\frac{x_{peak}\sqrt{km}}{\hat{F}} = \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

4-3 From Ex 4.2-1

$$\frac{xk}{F_0} = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \psi\right)$$

$\tan\psi = \frac{\zeta}{\sqrt{1-\zeta^2}}$ $\qquad$ For peak response $\frac{d}{d(\omega_n t)}\left(\frac{xk}{F_0}\right) = 0$

$$\left\{-\zeta\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \psi\right) - \sqrt{1-\zeta^2}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t - \psi\right)\right) = 0$$

$$\therefore \quad \tan\left(\sqrt{1-\zeta^2}\,\omega_n t - \psi\right) = \frac{-\zeta}{\sqrt{1-\zeta^2}} \qquad \text{expand}$$

$$\frac{\tan\sqrt{1-\zeta^2}\,\omega_n t - \tan\psi}{1 + \tan\sqrt{1-\zeta^2}\,\omega_n t\cdot\tan\psi} = \frac{-\zeta}{\sqrt{1-\zeta^2}}$$

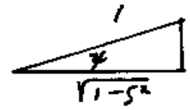
$$\tan\sqrt{1-\zeta^2}\,\omega_n t - \cancel{\frac{\zeta}{\sqrt{1-\zeta^2}}} = -\frac{\zeta}{\sqrt{1-\zeta^2}}\left[\cancel{1} + \frac{\zeta}{\sqrt{1-\zeta^2}}\tan\sqrt{1-\zeta^2}\,\omega_n t\right]$$

$$\tan\sqrt{1-\zeta^2}\,\omega_n t\left[1 + \frac{\zeta^2}{1-\zeta^2}\right] = \tan\sqrt{1-\zeta^2}\,\omega_n t = 0 \quad \therefore \quad \omega_n t = \frac{\pi}{\sqrt{1-\zeta^2}}$$

4-4 Subst. $\omega_n t = \frac{\pi}{\sqrt{1-\zeta^2}}$ into Eq. for peak $\frac{xk}{F_0}$

$$\left(\frac{xk}{F_0}\right)_{peak} = 1 - \frac{1}{\sqrt{1-\zeta^2}} \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)\cos(\pi-\psi)$$

$$= 1 + \frac{1}{\sqrt{1-\zeta^2}} \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)\cos\psi$$



$\cos\psi = \sqrt{1-\zeta^2}$

$$\left(\frac{xk}{F_0}\right)_{peak} = 1 + \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

4-5 From Ex. 4.2-1 $\quad x = \frac{F_0}{k}(1-\cos\omega_n t) \qquad t \geq 0$

For neg. step of $2F_0$ applied at $t = t_0$, we have

$$x = -\frac{2F_0}{k}\left[1 - \cos\omega_n(t-t_0)\right] \qquad t \geq t_0$$

Adding

$$x = \frac{F_0}{k}\left\{\cos\omega_n(t-t_0) - \cos\omega_n t\right\} \qquad t \geq t_0$$

4-6 With zero initial cond. Eq. 4.2-1 gives

$$x(t) = \int_0^t f(\xi)\sin\omega_n(t-\xi)\,d\xi \quad = \text{particular integral}$$

Solution for homogeneous eq. is from Eq. 2.2-6

$$x(t) = \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t + x(0)\cos\omega_n t$$

Complete sol. is sum of these solutions

4-7 From Eq. 4.2-1 response to unit step function is

$$g(t) = \int_0^t h(t-\xi)\,d\xi$$

Differentiate both sides w.r.t. time

$$\dot{g}(t) = h(t)$$

4-8 Eq. 4.2-1 $\quad x = \int_0^t f(\xi) h(t-\xi)\, d\xi$

From Prob. 4-7 $\quad \dot{g}(t-\xi) = h(t-\xi)$

$$x(t) = \int_0^t f(\xi)\, \dot{g}(t-\xi)\, d\xi \qquad \text{Integrate by parts}$$

$$x(t) = -f(\xi) g(t-\xi)\Big|_0^t + \int_0^t \dot{f}(\xi) g(t-\xi)\, d\xi$$

$$= -f(t) g(0) + f(0) g(t) + \int_0^t \dot{f}(\xi) g(t-\xi)\, d\xi \qquad \text{but } g(0) = 0$$

$$\therefore\ x(t) = f(0) g(t) + \int_0^t \dot{f}(\xi) g(t-\xi)\, d\xi$$

4-9 $\quad \bar{x}(s) = \dfrac{(ms+c)\,x(0) + m\,\dot{x}(0)}{ms^2 + cs + k}$

$$= \frac{(s + 2\zeta\omega_n)\,x(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{\dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

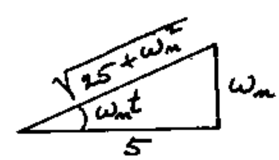
$$\mathcal{L}^{-1} \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\omega_n\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \sin\sqrt{1-\zeta^2}\,\omega_n t \qquad \text{see Append. B.}$$

$$\mathcal{L}^{-1} \frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \text{derivative of above} = e^{-\zeta\omega_n t}\left\{\cos\sqrt{1-\zeta^2}\,\omega_n t - \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

$$\therefore\ x(t) = e^{-\zeta\omega_n t}\left\{ \frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_n\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,\omega_n t + x(0) \cos\sqrt{1-\zeta^2}\,\omega_n t \right\}$$

4-10 $\quad \dot{y}(t) = 20\,u(t) - 100\,t \qquad \therefore\ \ddot{y}(t) = 20\,\delta(t) - 100$

$$\ddot{z} + \omega_n^2 z = -\ddot{y} = 100 - 20\,\delta(t)$$



$$\bar{z}(s) = \frac{100}{s(s^2+\omega_n^2)} - \frac{20}{(s^2+\omega_n^2)}$$

$$z(t) = \frac{100}{\omega_n^2}(1 - \cos\omega_n t) - \frac{20}{\omega_n} \sin\omega_n t$$

$$\dot{z}(t) = \frac{100}{\omega_n^2} \sin\omega_n t - 20 \cos\omega_n t = 0 \quad \text{for max } z$$

$$\therefore\ \tan\omega_n t = \frac{\omega_n}{5} \qquad (\text{see Fig})$$

$$z_{max} = \frac{100}{\omega_n}\left[1 - \frac{5}{\sqrt{25+\omega_n^2}}\right] - \frac{20}{\omega_n}\,\frac{\omega_n}{\sqrt{25+\omega_n^2}} \qquad \omega_n = 10$$

4-11 $\quad F_0 \sin\omega t = F_0 \sin\frac{\pi}{t_1} t$

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} \sin\frac{\pi}{t_1} t \qquad\qquad \omega_n = \frac{2\pi}{\tau}$$

Gen Sol. $\quad x(t) = A\sin\omega_n t + B\cos\omega_n t + \dfrac{F_0/m \sin\frac{\pi}{t_1} t}{1-\left(\frac{\pi}{t_1\omega_n}\right)^2}$

For $x(0) = \dot{x}(0) = 0$, $B = 0$ and $A = \dfrac{-F_0/m\left(\frac{\pi}{t_1\omega_n}\right)}{1-\left(\frac{\pi}{t_1\omega_n}\right)^2}$

With. $\dfrac{\pi}{t_1\omega_n} = \dfrac{\pi}{t_1}\dfrac{\tau}{2\pi} = \dfrac{\tau}{2t_1}$

$$x(t) = \frac{F_0}{m}\left\{\frac{-\frac{\tau}{2t_1}\sin\frac{2\pi t}{\tau}}{1-\left(\frac{\tau}{2t_1}\right)^2} + \frac{\sin\pi\frac{t}{t_1}}{1-\left(\frac{\tau}{2t_1}\right)^2}\right\}$$

$$= \frac{F_0/m}{\left(\tau/2t_1 - 2t_1/\tau\right)}\left\{\sin\frac{2\pi t}{\tau} - \frac{2t_1}{\tau}\sin\frac{\pi t}{t_1}\right\} \qquad t < t_1$$

For $t > t_1$ add same solution with t replaced by $(t-t_1)$

4-12 The triangular force can be represented by

$F_1 = \frac{2F_0}{t_1} t \qquad 0 < t < t_1/2$

$F_2 = -\frac{4F_0}{t_1}(t - t_1/2) + F_1 \qquad t_1/2 < t < t_1$

$F_3 = \frac{2F_0}{t_1}(t - t_1) + F_2 \qquad t_1 < t$

Diff. Eq. for $0 < t < t_1/2$ is

$$\ddot{x} + \omega_n x = \frac{2F_0}{mt_1} t = \left(\frac{2\omega_n^2 F_0}{k t_1}\right) t = Ct$$

$\bar{x}_1(s) = \dfrac{C}{s^2(s^2+\omega_n^2)} \quad \therefore x_1(t) = \dfrac{C}{\omega_n^3}(\omega_n t - \sin\omega_n t) = \dfrac{2F_0}{k}\left(\dfrac{t}{t_1} - \dfrac{\tau}{2\pi t_1}\sin 2\pi\dfrac{t}{\tau}\right)$

For F_2 additional excitation is $-2Ct + Ct_1 = -2C(t - t_1/2)$

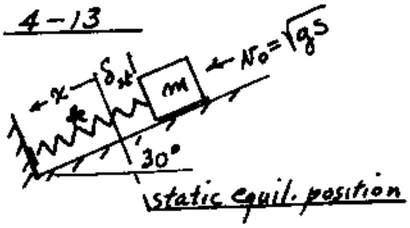
Thus $x_1(t)$ must be suplimented by $-2x_1(t - t_1/2)$

$$x_2(t) = x_1(t) - 2x_1(t - \tfrac{1}{2}t_1) \qquad \tfrac{1}{2}t_1 < t < t_1$$

For F_3 the excitation in addition to F_2 is $C(t - t_1)$

$$x_3(t) = x_2(t) + x_1(t - t_1) \qquad t_1 < t$$

4-13



$$x = x(0)\cos\omega_n t + \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t$$

$$\dot{x} = -\omega_n x(0)\sin\omega_n t + \dot{x}(0)\cos\omega_n t$$

$$k\delta_{st} = \tfrac{1}{2}mg$$

When returned to initial position

$$\left.\begin{aligned} -\delta_{st} &= -\delta_{st}\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t \\ -v_0 &= \omega_n\delta_{st}\sin\omega_n t + v_0\cos\omega_n t \end{aligned}\right\} \begin{aligned} &\div \text{ by } \cos\omega_n t \text{ and} \\ &\div \text{ the two eqs.} \end{aligned}$$

$$\frac{\delta_{st}}{v_0} = \frac{-\delta_{st} + \frac{v_0}{\omega_n}\tan\omega_n t}{\omega_n\delta_{st}\tan\omega_n t + v_0}, \qquad \tan\omega_n t = \frac{2\delta_{st}\,\omega_n v_0}{v_0^2 - \delta_{st}^2\,\omega_n^2}$$

$$\tan\omega_n t = \frac{\sqrt{\frac{m}{k}gS}}{S - \frac{gm}{4k}} \qquad \omega_n^2 = k/m \quad \delta_{st} = \frac{g}{\omega_n^2} \qquad v_0 = \sqrt{gS}$$

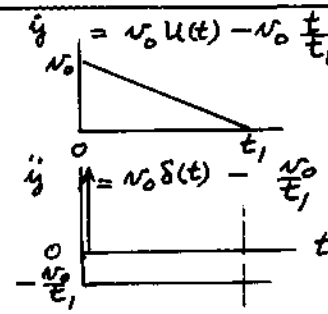
4-14

$$\Delta = \frac{38.6}{6.40} = 6.04'', \qquad \text{Max displ.} = 2\Delta = 12.08''$$

$$\omega_n = \sqrt{\frac{6.40\times 386}{38.6}} = 8.0 \text{ rad/s.} = \frac{2\pi}{\tau}$$

$$\tau = \frac{2\pi}{8} = 0.784 \text{ s.} \qquad t_{\text{max displ}} = \frac{\tau}{2} = 0.392 \text{ s.}$$

4-15



$$m\ddot{x} = -k(x-y) - f$$

Let $z = x - y$

$$\ddot{z} + \omega_n^2 z = -\ddot{y} - \frac{f}{m}$$

Gen Sol. from Eq. (4.2-5) is

$$z = \frac{-1}{\omega_n}\int_0^t \left[\ddot{y}(\xi) + \frac{f}{m}\right]\sin\omega_n(t-\xi)\,d\xi$$

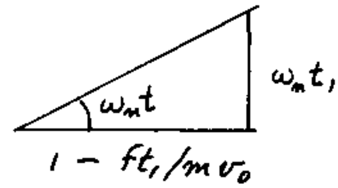
$$= -\frac{v_0}{\omega_n}\int_0^t \left[\delta(t) - \frac{1}{t_1} + \frac{f}{v_0 m}\right]\sin\omega_n(t-\xi)\,d\xi$$

$$= \frac{v_0}{\omega_n}\left\{\frac{1}{\omega_n t_1}\left(1 - \frac{ft_1}{mv_0}\right)(1-\cos\omega_n t) - \sin\omega_n t\right\}$$

4-16 Differentiate z in Prob. 4-15

$$\frac{dz}{dt} = \frac{v_0}{\omega_n}\left\{\frac{1}{t_1}\left(1-\frac{ft_1}{mv_0}\right)\sin\omega_n t - \omega_n\cos\omega_n t\right\} = 0$$

$$\therefore \tan\omega_n t = \frac{\omega_n t_1}{1-\frac{ft_1}{mv_0}}$$



$$\sin\omega_n t = \frac{\omega_n t_1}{\sqrt{(\omega_n t_1)^2 + [1-ft_1/mv_0]^2}}$$

$$\cos\omega_n t = \frac{(1-ft_1/mv_0)}{\sqrt{\text{same}}}$$ subst into eq. for Z

$$\frac{\omega_n Z_{max}}{v_0} = \frac{1}{\omega_n t_1}\left(1-\frac{ft_1}{mv_0}\right)\left\{1 - \frac{\left(1-\frac{ft_1}{mv_0}\right)}{\sqrt{(\omega_n t_1)^2+[1-ft_1/mv_0]^2}}\right\} - \frac{\omega_n t_1}{\sqrt{(\omega_n t_1)^2+[1-ft_1/mv_0]^2}}$$

$$= \frac{1}{\omega_n t_1}\left(1-\frac{ft_1}{mv_0}\right)\left\{1 - \frac{\frac{1}{\omega_n t_1}\left(1-\frac{ft_1}{mv_0}\right)}{\sqrt{1+\left[\frac{1}{\omega_n t_1}(1-ft_1/mv_0)\right]^2}}\right\} - \frac{1}{\sqrt{1+\left[\frac{1}{\omega_n t_1}(1-ft_1/mv_0)\right]^2}}$$

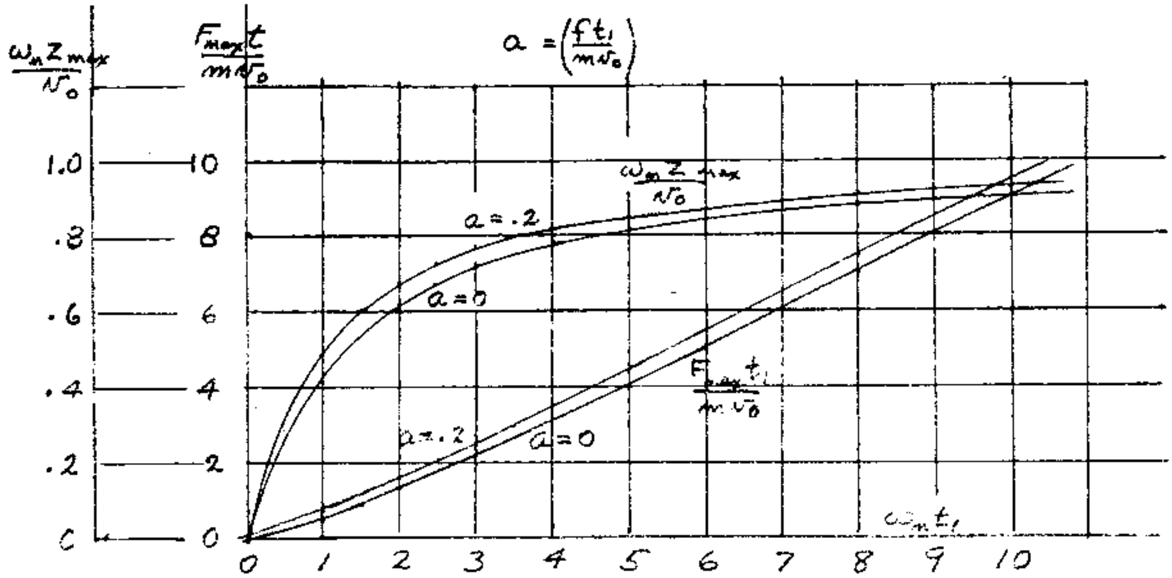
4-17 Let $a = \frac{ft_1}{mv_0}$ and $x = \omega_1 t$ $\quad y = \left|\frac{\omega_n Z_{max}}{v_0}\right|$

$$\left|\frac{\omega_n Z_{max}}{v_0}\right| = \left|\frac{1}{x}(1-a)\left\{1 - \frac{(1-a)}{\sqrt{x^2+(1-a)^2}}\right\} - \frac{x}{\sqrt{x^2+(1-a)^2}}\right|$$

$$\left|\frac{F_{max}t_1}{mv_0}\right| = a + \left|(\omega_n t_1)^2\left(\frac{\omega_n Z_{max}}{v_0}\right)\frac{1}{\omega_n t_1}\right| = a + x\left|\left(\frac{\omega_n Z_{max}}{v_0}\right)\right|$$

x	$y_{a=0}$	$\left\lvert\frac{F_m t_1}{mv_0}\right\rvert_{a=0}$	$y_{a=.20}$	$\left\lvert\frac{F_m t_1}{mv_0}\right\rvert_{a=.2}$
1	.4142	.4142	.4806	.6806
2	.6180	1.2360	.6770	1.554
3	.7208	2.1624	.7683	2.504
4	.7808	3.1232	.8198	3.479
5	.8198	4.0990	.8527	4.466
6	.8471	5.0826	.8755	5.453
8	.8828	7.0624	.9050	7.440
10	.9050	9.050	.9232	9.432

4-17 Cont:



4-18 $\quad x = \frac{F_0}{k}(1-\cos\omega_n t) \qquad 0 < t < t_0$

$$= \frac{F_0}{k}\left[\cos\omega_n(t-t_0) - \cos\omega_n t\right] \qquad t > t_0$$

For $t < t_0$ $\qquad \frac{dx}{dt} = \frac{F_0}{k}\omega_n \sin\omega_n t = 0 \qquad \begin{cases}\omega_n t_p = \pi \\ \frac{t_p}{\tau} = \frac{1}{2}\end{cases}$

$$\left(\frac{xk}{F_0}\right)_{max} = 2 \qquad \text{but } t_p = \frac{\tau}{2} < t_0$$

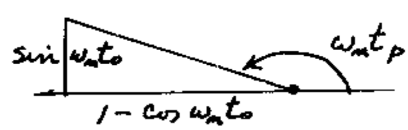
$$\therefore \frac{t_0}{\tau} > 0.50$$

For $t > t_0$

$$\frac{dx}{dt} = \frac{F_0\omega_n}{k}\left[\sin\omega_n t - \sin\omega_n(t-t_0)\right] = 0$$

$$\sin\omega_n t_p - \sin\omega_n t_p \cos\omega_n t_0 + \cos\omega_n t_p \sin\omega_n t_0 = 0$$

$$\tan\omega_n t_p = \frac{\sin\omega_n t_0}{(1-\cos\omega_n t_0)}$$



$$\sin\omega_n t_p = \frac{\sin\omega_n t_0}{\sqrt{2(1-\cos\omega_n t_0)}}$$

$$\cos\omega_n t_p = \frac{-(1-\cos\omega_n t_0)}{\sqrt{\text{same}}}$$

Subst into eq. for $t > t_0$

$$\left(\frac{kx}{F_0}\right)_{max} = \sqrt{2(1-\cos\omega_n t_0)} = 2\sin\tfrac{1}{2}\omega_n t_0 = 2\sin\frac{\pi t_0}{\tau}$$

$$\frac{t_0}{\tau} < 0.50$$

4-19 For small $\frac{t_1}{\tau}$ the sine pulse approaches an impulse

$$\hat{F} = F_0 \int_0^{t_1} \sin \frac{\pi t}{t_1} dt = \frac{2}{\pi} F_0 t_1$$

$$\text{Response} = x = \frac{\hat{F}}{m\omega_n} \sin \omega_n t = \frac{F_0}{k} \frac{4 t_1}{\tau} \sin \frac{2\pi t}{\tau}$$

$$\underline{\underline{\left(\frac{xk}{F_0}\right)_{max} = \frac{4t_1}{\tau}}} \quad \text{at } t_p = \tau/4$$

Peak response must occur at time $\gg t_1$ $\therefore \frac{t_1}{\tau} \ll \frac{1}{4}$

Let $\alpha = \frac{t_1}{\tau}$, $\xi = \frac{t}{t_1}$ for $t \geq t_1$

$$\left(\frac{xk}{F_0}\right) = \frac{1}{\left(\frac{1}{2\alpha} - 2\alpha\right)} \left[\sin 2\pi\alpha\xi + \sin 2\pi\alpha(\xi - 1)\right]$$

When $\alpha = \frac{1}{2}$ above eq. is indeterminate $= \frac{0}{0}$

$\therefore$ differentiate num. & den, w.r.t. α & divide

$$\left(\frac{xk}{F_0}\right)_{\alpha=\frac{1}{2}} = -\frac{\pi}{2} \cos \pi\xi$$

which is max when $\xi = 1 = \frac{t}{t_1}$

$\therefore t_p = t_1$

$$\text{and } \left(\frac{xk}{F_0}\right)_{\substack{\alpha=\frac{1}{2} \\ \xi=1}} = \frac{\pi}{2} = \underline{\underline{1.57}}$$

4-20 $m = \frac{16.1}{386} = 0.0417$ $\frac{t_1}{\tau} = \frac{0.40}{0.50} = 0.80$

From FIG P4-21 $\left(\frac{kx}{F_0}\right)_{max} = 1.54$

$\omega_n = \frac{2\pi}{.50} = 4\pi$ $k = m\omega_n^2 = .0417(4\pi)^2 = 6.585$ #/in

$x_{max} = 1.54 \frac{F_0}{k}$ $\hat{F} = 2.0$ lbs. $= \frac{1}{2} \times .40 \times F_0$

$\therefore F_0 = 10$

$$x_{max} = 1.54 \times \frac{10}{6.585} = \underline{\underline{2.339''}}$$

4-21 Differentiate 3rd eq. Prob. 4-12

$$\frac{dx}{dt} = \frac{2F_0}{kt_1}\left\{2\cos\frac{2\pi t_1}{\tau}\left(\frac{t}{t_1}-0.5\right) - \cos\frac{2\pi t_1}{\tau}\left(\frac{t}{t_1}-1.0\right) - \cos\frac{2\pi t}{\tau}\right\} = 0$$

or $$2\cos\frac{2\pi t_1}{\tau}\left(\frac{t_p}{t_1}-0.50\right) - \cos\frac{2\pi t_1}{\tau}\left(\frac{t_p}{t_1}-1\right) - \cos\frac{2\pi t_p}{\tau} = 0$$

which gives $t_p = t_1$

4-22 With suggested subst for $A\cos\phi$ and $A\sin\phi$

$$x = \frac{A}{k}\left\{\sin\omega_n t\cos\phi - \cos\omega_n t\sin\phi\right\} = \frac{A}{k}\sin(\omega_n t - \phi)$$

Max response occurs when $(\omega_n t - \phi) = \frac{\pi}{2}$

$$x_{max} = \frac{A}{k} \qquad \text{where} \qquad A = \omega_n\sqrt{\left[\int_0^t f(\xi)\sin\omega_n\xi\,d\xi\right]^2 + \left[\int_0^t f(\xi)\cos\omega_n\xi\,d\xi\right]^2}$$

4-23
$$F = F_0\, t/t_0 \qquad t < t_0$$
$$= 0 \qquad t > t_0$$

Eq. 4.2-1 with $h(t) = \frac{1}{m\omega_n}\sin\omega_n t$

$$x(t) = \frac{F_0}{m\omega_n}\int_0^t \frac{\xi}{t_0}\sin\omega_n(t-\xi)\,d\xi = \frac{F_0}{k}\left(\frac{t}{t_0} - \frac{\sin\omega_n t}{\omega_n t_0}\right) \quad t \quad t_0$$

For $t > t_0$ integral does not change after t_0

$$x(t) = \frac{F_0}{m\omega_n}\int_0^{t_0}\frac{\xi}{t_0}\sin\omega_n(t-\xi)\,d\xi = \frac{F_0}{k}\left[\frac{\xi}{t_0}\cos\omega_n(t-\xi) + \frac{\sin\omega_n(t-t_0)}{\omega_n t_0}\right]_0^{t_0}$$

$$= \frac{F_0}{k}\left[\cos\omega_n(t-t_0) + \frac{1}{\omega_n t_0}\left\{\sin\omega_n(t-t_0) - \sin\omega_n t\right\}\right] \quad t > t_0$$

4-24 Given vel. $= v = v_0\,(u(t) - t/t_1)$

Accel $= a = v_0\left[\delta(t) - 1/t_1\right] = \ddot{y}(t)$

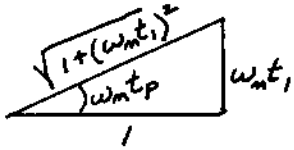
Subst into Eq. 4.2-5

$$z = -\frac{v_0}{\omega_n}\int_0^t\left[\delta(\xi) - \frac{1}{t_1}\right]\sin\omega_n(t-\xi)\,d\xi \qquad 0 < t < t_1$$

$$= \frac{v_0}{\omega_n}\left\{-\sin\omega_n t + \frac{1}{\omega_n t_1}\left(1-\cos\omega_n t\right)\right\} \qquad 0 < t < t_1$$

4-24 Cont:

$$\frac{dz}{dt} = \frac{v_0}{\omega_n}\left\{-\omega_n \cos\omega_n t + \frac{1}{t_1}\sin\omega_n t\right\} = 0$$



or $\tan\omega_n t_p = \omega_n t_1$ for z_{max}

$$\sin\omega_n t_p = \frac{\omega_n t_1}{\sqrt{1+(\omega_n t_1)^2}}$$

$$\cos\omega_n t_p = \frac{1}{\sqrt{1+(\omega_n t_1)^2}}$$

$$\therefore \frac{z_{max}\,\omega_n}{v_0} = \left\{\frac{1}{\omega_n t_1} - \frac{1}{\omega_n t_1\sqrt{1+(\omega_n t_1)^2}} - \frac{\omega_n t_1}{\sqrt{1+(\omega_n t_1)^2}}\right\}$$

4-25 If $t > t_1$ the solution is

$$z = -\frac{v_0}{\omega_n}\int_0^{t_1}\left[\delta(\xi) - \frac{1}{t_1}\right]\sin\omega_n(t-\xi)\,d\xi$$

$$= \frac{v_0}{\omega_n}\left\{-\sin\omega_n t + \frac{1}{\omega_n t_1}\left[\cos\omega_n(t-t_1) - \cos\omega_n t\right]\right\}$$

4-26 $\ddot{z} + \omega_n^2 z = -\ddot{y} = 100 - 20\,\delta(t)$

$\tau = \frac{2\pi}{\omega_n} = 0.628$ s.

choose $\Delta t = 0.05$ s

For the impulse response, it is advisable to treat it separately.

Its response is $-\frac{20}{\omega_n}\int_0^t \delta(\xi)\sin\omega_n(t-\xi)\,d\xi = -2\sin 10t$

Remaining solution can be computed numerically from D.E

(a) $\ddot{z}_i = 100 - 100\,z_i$

(b) $\ddot{z}_i\,\Delta t^2 = z_{i-1} - 2z_i + z_{i+1}$

Initial cond. $z_1 = \dot{z}_1 = 0$

Start with $\ddot{z}_1 = 100$ from (a)

From Eq(4.5-8) $z_2 = \frac{1}{2}(0.05)^2\,100 = 0.1250$

from (a) $\ddot{z}_2 = 100 - 100(0.1250)$

Find z_3 from Eq. (b) etc.

4-27 This problem is identical to Ex 4.5-2 in the text. It is suggested that the instructor change the triangular impulse duration to 0.80 s, and follow the procedure of Ex. 4.5-2.

4-28 Ex. 4A-2 is for $\dot{y} = v_0 e^{-t/t_0}$

Fig P4-28 is for $\dot{y} = 60\, e^{-0.10t}$ applies for $\omega_n t_1$ or $\omega_n t_0$

$\therefore v_0 = 60 \;, \quad t_0 = 10$

For large $\omega_n t_0$ $\quad \left(\dfrac{2z}{v_0 t_0}\right)_{max} \cong \dfrac{2}{\omega_n t_0}$ a rectangular hyperbola

at $\omega_n t_0 = 100$ Fig 4-28 gives $\left(\dfrac{2z}{v_0 t_0}\right)_{max} = 0.02$

For small $\omega_n t_0$ $\quad \left(\dfrac{2z}{v_0 t_0}\right)_{max} \cong 1.0$

4-29 $\quad m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega_n t$

With zero initial conditions, response may be evaluated from (1) (2) or (3) below

$$(1) \qquad \bar{x}(s) = \frac{\omega_n F_0/m}{(s^2+\omega_n^2)(s^2+2\zeta\omega_n + \omega_n^2)}$$

$$(2) \qquad x(t) = X_1 e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi_1\right) - \frac{F_0 \cos\omega_n t}{c\omega_n} \qquad Eq.(3.1-11)$$

$$(3) \qquad x(t) = F_0 \int_0^t \sin\omega_n (t-\tau) \cdot \frac{e^{-\zeta\omega_n t}}{m\omega_n\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,\omega_n t \qquad Eq.(4.2-2)$$

Result is :

$$x(t) = \frac{F_0}{c\omega_n}\left\{ \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \sin^{-1}\sqrt{1-\zeta^2}\right) - \cos\omega_n t \right\}$$

4-30 For small damping $\delta \cong 2\pi\zeta$

At time $t = \dfrac{1}{f_n \delta}$, $\zeta\omega_n t = \dfrac{2\pi\zeta}{\delta} \cong 1.0$ $\qquad$ Subst. into Eq. above

$$x = \frac{F_0}{c\omega_n}\left\{ e^{-1}\sin(\omega_n t + 90^\circ) - \cos\omega_n t \right\}$$

$$= \frac{F_0}{c\omega_n}\left\{ e^{-1} - 1 \right\}\cos\omega_n t = (1-e^{-1})\left(\frac{-F}{c\omega_n}\cos\omega_n t\right)$$

where steady state sol. $= \left(\dfrac{-F}{c\omega_n}\cos\omega_n t\right)$

4-31 Under harmonic force of freq. ω_n, the steady state oscillation is

$$x(t) = \frac{F_0}{c\omega_n}\cos\omega_n t \qquad x(0) = \frac{F_0}{c\omega_n}, \quad \dot{x}(0) = 0$$

Transient sol. is

$$x(t) = X_1 e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\,\omega_n t + \phi_1) \cong X_1 e^{-\zeta\omega_n t}\sin(\omega_n t + \phi_1)$$

for ζ small

$$x(0) = X_1 \sin\phi_1 = \frac{F_0}{c\omega_n}$$

$$\dot{x}(0) = X_1[\omega_n\cos\phi_1 - \zeta\omega_n\sin\phi_1] \cong X_1\omega_n\cos\phi_1 = 0$$

$$\therefore \phi_1 = 90^\circ \quad \text{and} \quad X_1 = \frac{F_0}{c\omega_n}$$

Then trans. sol. with above initial cond. is

$$x(t) = \frac{F_0}{c\omega_n} e^{-\zeta\omega_n t}\cos\omega_n t$$

At $\zeta\omega_n t = \frac{2\pi\zeta}{\delta} \cong 1.0 \qquad x(t) = e^{-1}\frac{F_0}{c\omega_n}\cos\omega_n t$

see Prob 4-30

4-32

DE $\qquad \ddot{x} = 0.25\,F(t) - 500\,x$

Let $h = H = 0.02$

$t = T(I) = H * (I-1) \qquad I = 1, 2, \dots$

$x = X(I) \qquad X(1) = 0$

$\ddot{x} = DX2(I)$

Let I go from 1 to $25 = N$

Computer program follows

4-32 Cont:

```
DIMENSION  T(28), X(28), DX2(28), F(28)
   N = 25
   H = .02
   T(1) = 0
   X(1) = 0
   DO I = 1, 25
   T(I) = H * (I-1)
   IF (I.GT.1) GO TO 2
   F(I) = 100
   DX2(I) = .25 * F(I) - 500 * X(I)
   X(I+1) = .50 * DX2(I) * H **2
   GO TO 3
2  IF (I.LT.6)  F(I) = 100
   IF (I.GE.6)  F(I) = 100 - 1000 *(T(I)-.10)
   IF (I.GT. 10)  F(I) = 0
   DX2(I) = .25 * F(I) - 500 * X(I)
   X(I+1) = 2 * X(I) - X(I-1) + DX2(I)* H **2
   IF (I = N+1)  GO TO 4
3  CONTINUE
4  WRITE (6,5)
5  FORMAT (4 IH1, TIME, FORCE, DISPL)
   WRITE (6,6) ( I(I), F(I), X(I) )
6  FORMAT (3X, F6.3, 3X, F6.3, 3X, F6.4)
   STOP
   END
```

4 - 33

We will first discuss the case where the initial conditions are not zero. The discussion on p 108 then applies.

The acceleration from the D.E. is

$$\ddot{x}_i = \frac{F_i}{m} - \omega_n^2 x_i - 2\zeta\omega_n \dot{x}_i = f(x_i, \dot{x}_i, t_i)$$

The two equations given from the Taylor series are

(a) $$x_2 = x_1 + \dot{x}_1 h + \frac{h^2}{2}\left(\frac{F_1}{m} - \omega_n^2 x_1 - 2\zeta\omega_n \dot{x}_1\right)$$

(b) $$x_1 = x_2 - \dot{x}_2 h + \frac{h^2}{2}\left(\frac{F_2}{m} - \omega_n^2 x_2 - 2\zeta\omega_n \dot{x}_2\right)$$

Since the initial conditions x_1, $\dot{x}_1$ and F_1 are assumed to be given, Eq. (a) gives x_2.
Eq. (b) is next solved for $\dot{x}_2$ which is

(c) $$\dot{x}_2 = \frac{x_2 - x_1 + \frac{h^2}{2}\left(\frac{F_2}{m} - \omega_n^2 x_2\right)}{(h + h^2\zeta\omega_n)}$$

Thus x_2 and $\dot{x}_2$ for the first interval can be calculated from Eqs. (a) and (c).

Calculations for x_3 $\dot{x}_3$, x_4 $\dot{x}_4$ etc are now made with Eq. (4.5-7') and Eq (c) generalized to index i as follows.

(d) $$x_{i+1} = 2x_i - x_{i-1} + h^2\left(\frac{F_i}{m} - \omega_n^2 x_i - 2\zeta\omega_n \dot{x}_i\right)$$

(e) $$\dot{x}_{i+1} = \frac{x_{i+1} - x_i + \frac{h^2}{2}\left(\frac{F_{i+1}}{m} - \omega_n^2 x_{i+1}\right)}{(h + h^2\zeta\omega_n)}$$

Flow diagram follows

4-33 Cont:

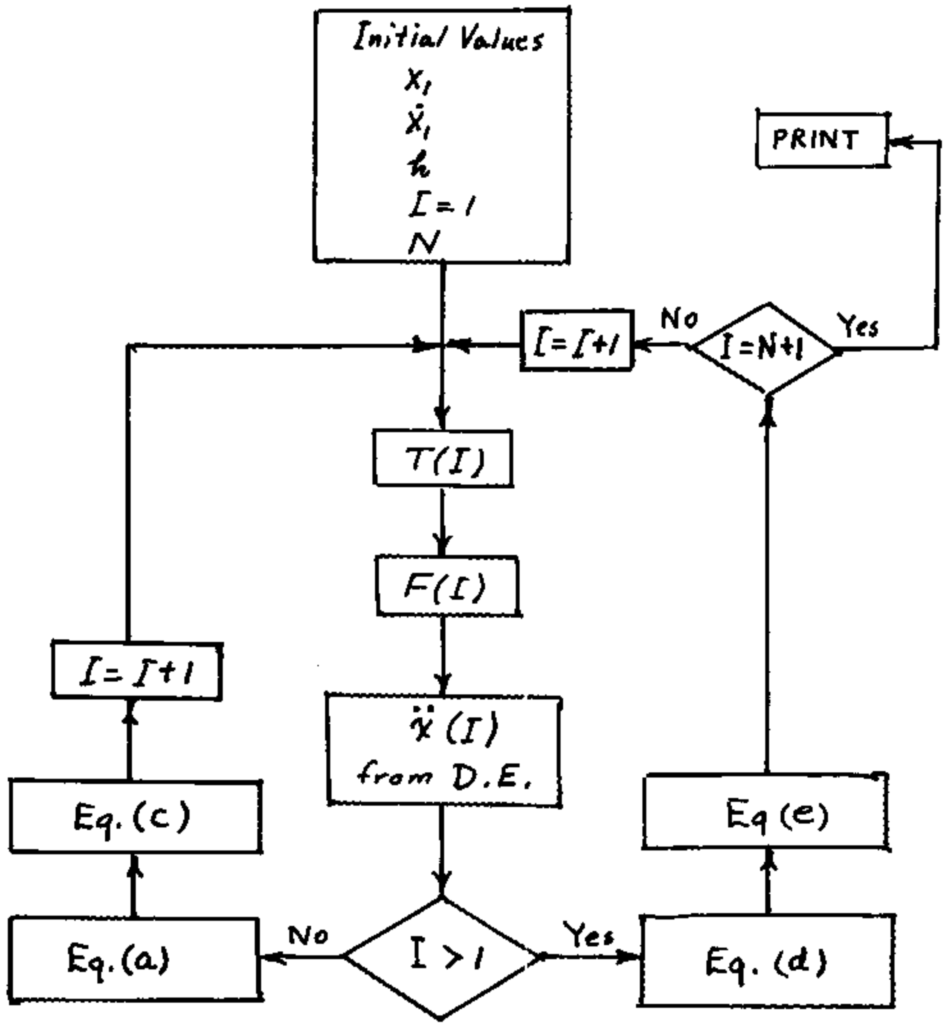


Diagram is similar to Fig. 4.5-1 with added boxes for Eq(c) and Eq.(e) for the calculation of velocities.

4-33 Cont:

When initial values are zero and force $= 0$ Eq. (a) gives $x_2 = 0$ and hence calculations can not be started. We must then use Eqs 4.5-9 and 4.5-10 and substitute $\ddot{x}_2$ from D.E. The two equations so obtained must be solved simultaneously for x_2 and $\dot{x}_2$. With these values calculations may proceed with Eqs. (d) and (e)

$$x_2 = \frac{h^2}{6}\left(\frac{F_2}{m} - \omega_n^2 x_2 - 2\zeta\omega_n \dot{x}_2\right) \qquad (4.5\text{-}10)$$

$$\dot{x}_2 = \frac{h}{2}\left(\frac{F_2}{m} - \omega_n^2 x_2 - 2\zeta\omega_n \dot{x}_2\right) \qquad (4.5\text{-}9)$$

Rearrange to

$$\left(1 + \frac{h^2}{6}\omega_n^2\right) x_2 + \left(\frac{h^2}{3}\zeta\omega_n\right)\dot{x}_2 = \frac{h^2}{6}\left(\frac{F_2}{m}\right)$$

$$\left(\frac{h}{2}\omega_n^2\right) x_2 + \left(1 + h\zeta\omega_n\right)\dot{x}_2 = \frac{h}{2}\left(\frac{F_2}{m}\right)$$

Solution:

$$x_2 = \frac{\frac{h^2}{6}\left(\frac{F_2}{m}\right)}{1 + h\omega_n\left(\frac{h}{6} + \zeta\right)} \qquad (f)$$

$$\dot{x}_2 = \frac{\frac{h}{2}\left(\frac{F_2}{m}\right)}{1 + h\omega_n\left(\frac{h}{6} + \zeta\right)} \qquad (g)$$

Previous flow diagram can now be used with Eq.(a) replaced by Eq.(f) and Eq.(c) replaced by Eq.(g). Right loop is unaltered. Calculation of F_2 is necessary prior to Eq.(a).

4 - 34

DE. for base excitation is

$$m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$$

Let $z = x - y$, then

$$\ddot{z} = -\ddot{y} - \omega_n^2 z - 2\zeta\omega_n \dot{z}$$

Thus the problem is same as that for force excitation with z replacing x and $-\ddot{y}(t)$ replacing $\left(\frac{F}{m}\right)$.

---

4 - 35 Refer to Prob. 4-33 and the given flow diagram. The exciting force $\frac{F}{m}$ is now replaced by $-\ddot{y}$

$$y = y_0 \sin \omega t$$

$$\frac{F}{m} = -\ddot{y} = \omega^2 y_0 \sin \omega t$$

The sine pulse terminates at $\omega t_p = \pi$. The time increment h should be chosen smaller than $\frac{t_p}{10}$ or $\frac{1}{10}\left(\frac{2\pi}{\omega_n}\right)$, whichever is the smallest.

A program similar to Prob. 4-32 can be written with the IF statements modified.

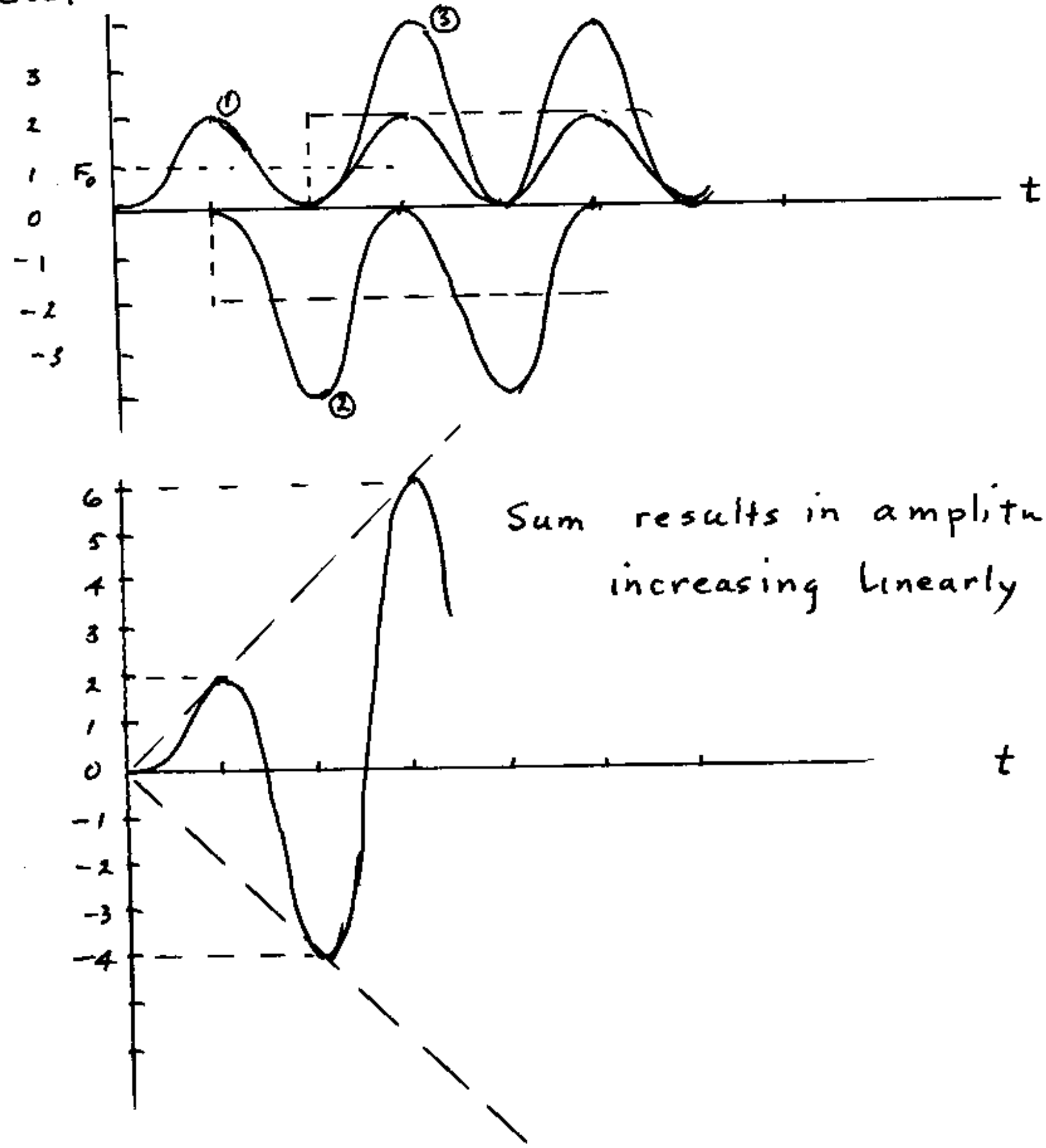
4-36 use superposition of solution to each step function

1^{st} step of F_0 results in $x(t) = \frac{F_0}{k}(1-\cos\omega_n t)$

2^{nd} step of $-2F_0$ at $t = \frac{\pi}{\omega}$ adds $-\frac{2F_0}{k}\left[1-\cos\omega_n\left(t-\frac{\pi}{\omega}\right)\right]$

3^{rd} step of $+2F_0$ at $t = \frac{2\pi}{\omega}$ adds $+\frac{2F_0}{k}\left[1-\cos\omega_n\left(t-\frac{2\pi}{\omega}\right)\right]$

etc.



Sum results in amplitudes increasing linearly

4-37

$$x_{i+1} = x_i + \dot{x}_i h + \ddot{x}_i \frac{h^2}{2} + \dddot{x}_i \frac{h^3}{6} + \ddddot{x}_i \frac{h^4}{24} + \cdots$$

$$x_{i-1} = x_i - \dot{x}_i h + \ddot{x}_i \frac{h^2}{2} - \dddot{x}_i \frac{h^3}{6} + \ddddot{x}_i \frac{h^4}{24} - \cdots$$

$$x_{i+1} + x_{i-1} = 2x_i + \ddot{x}_i h^2 + \ddddot{x}_i \frac{h^4}{12}$$

$$\ddot{x}_i = \frac{x_{i-1} - 2x_i + x_{i+1}}{h^2} - \ddddot{x}_i \frac{h^2}{12}$$

$$\text{error} = -\frac{\ddddot{x}_i}{12} h^2 = O(h^2)$$

4-38

t	$x = t^3$
.8	.5120
.9	.7290
1.0	1.0000
1.1	1.3310
1.2	1.7280

Exact value of $\dot{x}$ is

$$\dot{x} = 3t$$

$$\therefore \dot{x}(1) = 3.0000$$

Calculation of $\dot{x}$ by finite difference

using $\dot{x}_i = \frac{1}{2h}(x_{i+1} - x_{i-1})$

$h = 0.10$ $\quad \dot{x}_1 = \frac{1}{2 \times .10}(1.3310 - .7290) = 3.010$

$$\therefore \text{Error} = .01 = .1^2 = h^2$$
$$= O(h^2)$$

$h = 0.20$ $\quad \dot{x}_1 = \frac{1}{2 \times .20}(1.7280 - .5120) = 3.04$

$$\therefore \text{Error} = .04 = .2^2 = h^2$$
$$= O(h^2)$$

4-39

Refer to Prob. 4-38 but use

$$\dot{x}_i = \frac{1}{h}(x_i - x_{i-1})$$

$h = .10$ $\quad \dot{x}_1 = \frac{1}{.10}(1.0 - .7290) = 2.71$

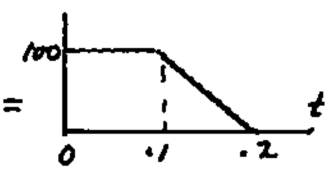
$\therefore$ Error $= .290 = 2.9\,h = O(h)$

$h = .20$ $\quad \dot{x}_1 = \frac{1}{.20}(1.0 - .5120) = 2.44$

$\therefore$ Error $= .560 = 2.8\,h = O(h)$

4-40

D.E. $\quad 4\ddot{x} + 2000\,x = F(t)$

$\therefore\ m = 4, \quad k = 2000 \qquad F(t) =$ 

$\omega_n = \sqrt{\frac{k}{m}} = 22.36$

1^{st} sol. due to step function (see Fig 4.5-4)

$$x_1 = \frac{F}{k}(1 - \cos\omega_n t) = \frac{100}{2000}(1 - \cos 22.36\,t)$$

$0 \le t \le 0.10$

2^{nd} sol. due to ramp function of

$1000\,(t - .10)$ at $t \ge .10$

DE $\quad \ddot{x} + 500\,x = 250\,t'$ $\quad$ where $t' = t - .10$

Lapl. Trans $\quad x(s) = \dfrac{250}{s^2(s^2 + 22.36^2)}$

$$x_2 = \frac{250}{22.36^2}\left[22.36\,t' - \sin 22.36\,t'\right]$$

$$= \frac{1}{2}t' - .02236 \sin 22.36\,t'$$

4 - 40

$\therefore$ add to first sol x_1, the second sol. which is $x_2 = \frac{1}{2}(t-.10) - .02236 \sin 22.36(t-.10)$

at $t \geq .10$

similarly the third sol. is same as x_2 with $(t-.20)$

---

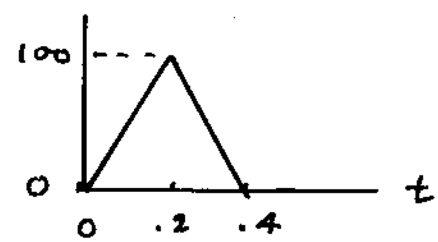
4 - 41

DE to be solved is

$$\ddot{x} + 16\pi^2 x = 2F(t)$$

Let $y = \dot{x}$, then

$$\dot{y} = f(x,t) = 2F(t) - 16\pi^2 x$$

where $F(t) =$ 

Follow Calc. procedure of Example 4.6-1

$\omega_n = 4\pi = 12.56 = \frac{2\pi}{\tau}$ $\qquad \tau = .500$

$\therefore$ suggest $h = .02$

Runge-Kutta Program

```
      PROBLEM 4-41 THOMSON
      DIMENSION T(50),T1(50),T2(50),T3(50),T4(50),X(50),X1(50),X2(50),
     1X3(50),X4(50),Y(50),Y1(50),Y2(50),Y3(50),Y4(50),F(50),F1(50),
     1F2(50),F3(50),F4(50)
      N=49
      DH=0.02
      X(1)=0.0
      Y(1)=0.0
      T(1)=0.0
      PRINT5
    5 FORMAT(20X,'J',5X,'TIME',9X,'DISPL',5X,'ACCELERATION',11X,'F(J)',
     110X,'FORCE')
      DO 10 J=1,N
      F(J)=FXY(T(J),X(J),Y(J),DF)
      PRINT8, J,T(J),X(J),Y(J),F(J),DF
    8 FORMAT(18X,I3,2X,F7.3,2X,E12.3,5X,E12.3,3X,E12.3,7X,F8.3)
      T1(J)=T(J)
      X1(J)=X(J)
      Y1(J)=Y(J)
      F1(J)=FXY(T1(J),X1(J),Y1(J),DF)
      T2(J)=T(J)+DH/2.
      X2(J)=X(J)+Y1(J)*DH/2.
      Y2(J)=Y(J)+F1(J)*DH/2.
      F2(J)=FXY(T2(J),X2(J),Y2(J),DF)
      T3(J)=T(J)+DH/2.
      X3(J)=X(J)+Y2(J)*DH/2.
      Y3(J)=Y(J)+F2(J)*DH/2.
      F3(J)=FXY(T3(J),X3(J),Y3(J),DF)
      T4(J)=T(J)+DH
      X4(J)=X(J)+Y3(J)*DH
      Y4(J)=Y(J)+F3(J)*DH
      F4(J)=FXY(T4(J),X4(J),Y4(J),DF)
      X(J+1)=X(J)+DH/6.*(Y1(J)+2.*Y2(J)+2.*Y3(J)+Y4(J))
      Y(J+1)=Y(J)+DH/6.*(F1(J)+2.*F2(J)+2.*F3(J)+F4(J))
      T(J+1)=T(J)+DH
   10 CONTINUE
      STOP
      END


      FUNCTION FXY(T,X,Y,DF)
      IF(T.GT.0.4) GO TO 50
      IF(T.GT.0.2) GO TO 49
      DF=500.*T
      GO TO 51
   49 DF=200.-500.*T
      GO TO 51
   50 DF=0.0
   51 FXY=2.*DF-16.*3.1415**2*X
      RETURN
      END
```

4-41 Cont.

TIME	DISPL	ACCELERATION	F(J)	J	FORCE
0.000	0.000E 00	0.000E 00	0.000E 00	1	0.000
0.020	0.133E-02	0.199E 00	0.198E 02	2	10.000
0.040	0.105E-01	0.783E 00	0.383E 02	3	20.000
0.060	0.350E-01	0.172E 01	0.545E 02	4	30.000
0.080	0.811E-01	0.294E 01	0.672E 02	5	40.000
0.100	0.154E 00	0.438E 01	0.757E 02	6	50.000
0.120	0.257E 00	0.593E 01	0.794E 02	7	60.000
0.140	0.392E 00	0.752E 01	0.782E 02	8	70.000
0.160	0.557E 00	0.903E 01	0.720E 02	9	80.000
0.180	0.752E 00	0.104E 02	0.613E 02	10	90.000
0.200	0.970E 00	0.115E 02	0.468E 02	11	100.000
0.220	0.120E 01	0.118E 02	-0.103E 02	12	90.000
0.240	0.144E 01	0.110E 02	-0.667E 02	13	80.000
0.260	0.164E 01	0.918E 01	-0.119E 03	14	70.000
0.280	0.180E 01	0.634E 01	-0.164E 03	15	60.000
0.300	0.189E 01	0.271E 01	-0.198E 03	16	50.000
0.320	0.190E 01	-0.150E 01	-0.220E 03	17	40.000
0.340	0.183E 01	-0.601E 01	-0.228E 03	18	30.000
0.360	0.166E 01	-0.105E 02	-0.222E 03	19	20.000
0.380	0.141E 01	-0.148E 02	-0.202E 03	20	10.000
0.400	0.107E 01	-0.185E 02	-0.169E 03	21	0.000
0.420	0.671E 00	-0.213E 02	-0.106E 03	22	0.000
0.440	0.229E 00	-0.227E 02	-0.361E 02	23	0.000
0.460	-0.228E 00	-0.227E 02	0.360E 02	24	0.000
0.480	-0.671E 00	-0.213E 02	0.106E 03	25	0.000
0.500	-0.107E 01	-0.185E 02	0.169E 03	26	0.000
0.520	-0.140E 01	-0.146E 02	0.222E 03	27	0.000
0.540	-0.165E 01	-0.976E 01	0.260E 03	28	0.000
0.560	-0.179E 01	-0.430E 01	0.283E 03	29	0.000
0.580	-0.182E 01	0.143E 01	0.287E 03	30	0.000
0.600	-0.173E 01	0.707E 01	0.274E 03	31	0.000
0.620	-0.154E 01	0.123E 02	0.243E 03	32	0.000
0.640	-0.125E 01	0.167E 02	0.197E 03	33	0.000
0.660	-0.879E 00	0.201E 02	0.139E 03	34	0.000
0.680	-0.454E 00	0.222E 02	0.717E 02	35	0.000
0.700	-0.712E-03	0.229E 02	0.112E 00	36	0.000
0.720	0.453E 00	0.222E 02	-0.715E 02	37	0.000
0.740	0.878E 00	0.201E 02	-0.139E 03	38	0.000
0.760	0.125E 01	0.167E 02	-0.197E 03	39	0.000
0.780	0.154E 01	0.123E 02	-0.243E 03	40	0.000
0.800	0.173E 01	0.709E 01	-0.274E 03	41	0.000
0.820	0.182E 01	0.145E 01	-0.287E 03	42	0.000
0.840	0.179E 01	-0.428E 01	-0.283E 03	43	0.000
0.860	0.165E 01	-0.974E 01	-0.261E 03	44	0.000
0.880	0.141E 01	-0.146E 02	-0.222E 03	45	0.000
0.900	0.107E 01	-0.185E 02	-0.169E 03	46	0.000
0.920	0.672E 00	-0.213E 02	-0.106E 03	47	0.000
0.940	0.230E 00	-0.227E 02	-0.362E 02	48	0.000
0.960	-0.227E 00	-0.227E 02	0.359E 02	49	0.000

4 - 42

The stiffness of the crane boom is represented by k_c, measured from the extended straight Line.

$$\frac{W}{g}\ddot{y} = k_c(x-y) - W$$

$$\ddot{y} + \left(\frac{k_c g}{W}\right)y = \left(\frac{k_c g}{W}\right)x - g \quad , \quad \omega^2 = \left(\frac{k_c g}{W}\right)$$

$$\ddot{y} + \omega^2 y = \omega^2 Vt - g \quad , \quad x = Vt$$

By L.T.

$$s^2\bar{y}(s) - s\,y(0) - \dot{y}(0) + \omega^2\bar{y}(s) = \frac{\omega^2 V}{s^2} - \frac{g}{s}$$

$$\bar{y}(s) = \frac{s\,y(0)}{s^2+\omega^2} + \frac{\dot{y}(0)}{s^2+\omega^2} + \frac{\omega^2 V}{s^2(s^2+\omega^2)} - \frac{g}{s(s^2+\omega^2)}$$

Inverse L.T.

$$y(t) = y(0)\cos\omega t + \frac{\dot{y}(0)}{\omega}\sin\omega t + \frac{V}{\omega}(\omega t - \sin\omega t) - \frac{g}{\omega^2}(1-\cos\omega t)$$

5-1

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$
$$m\ddot{x}_2 = -k(x_2 - x_1)$$

origin of x_1 & x_2 = equilib position

$$\left.\begin{aligned}(2 - \tfrac{m\omega^2}{k})X_1 &= X_2 \\ X_1 &= (1 - \tfrac{m\omega^2}{k})X_2\end{aligned}\right\} \text{let } \lambda = \frac{m\omega^2}{k}$$

characteristic eq. $\lambda^2 - 3\lambda + 1 = 0$

$$\lambda = \begin{cases} 0.382 = \dfrac{m\omega_1^2}{k} \\ 2.618 = \dfrac{m\omega_2^2}{k} \end{cases} \qquad \begin{aligned}(X_1/X_2)_1 &= 1 - \lambda_1 = 0.614 \\ (X_1/X_2)_2 &= 1 - \lambda_2 = -1.618\end{aligned}$$

5-2 & 3

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k+nk) & -nk \\ -nk & (k+nk) \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

Let $\lambda = m\omega^2/k$

$$\begin{vmatrix} (1+n-\lambda) & -n \\ -n & (1+n-\lambda) \end{vmatrix} = 0 \qquad \begin{aligned}&\lambda^2 - 2(1+n)\lambda + (1+2n) = 0 \\ &\lambda = (1+n) \pm n, \quad X_1/X_2 = \frac{1+n-\lambda}{n}\end{aligned}$$

For $n = 1$.

$$\lambda_1 = 1 = \frac{m\omega_1^2}{k} \qquad (X_1/X_2)_1 = 1$$
$$\lambda_2 = 3 = \frac{m\omega_2^2}{k} \qquad (X_1/X_2)_2 = -1$$

5-4

$$m\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

$\lambda = \dfrac{m\omega^2}{k}$

$$\begin{vmatrix} (2-3\lambda) & -1 \\ -1 & (4-\lambda) \end{vmatrix} = 0 \qquad \lambda^2 - \frac{14}{3}\lambda + \frac{7}{3} = 0$$

$$\lambda = \frac{14}{6} \pm \sqrt{\left(\frac{14}{6}\right)^2 - \frac{7}{3}} = \begin{cases} 0.570 = \dfrac{m\omega_1^2}{k} \\ 4.096 = \dfrac{m\omega_2^2}{k} \end{cases}$$

$$\frac{X_1}{X_2} = (4 - \lambda) \qquad \left(\frac{X_1}{X_2}\right)_1 = 3.43 \qquad \left(\frac{X_1}{X_2}\right)_2 = -0.096$$

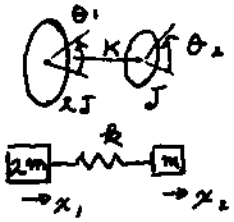
5-5

$$J_2\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + K\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix} \qquad \lambda = \frac{\omega^2 J_2}{K}$$

$$\begin{vmatrix} (2-2\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0 \qquad \begin{aligned}2(1-\lambda)^2 - 1 &= 0 \\ (1-\lambda) &= \pm 1/\sqrt{2}\end{aligned} \qquad \lambda = 1 \mp 1/\sqrt{2}$$

$$\lambda_{1,2} = \begin{cases} .293 = J_2\dfrac{\omega_1^2}{K} \\ 1.707 = J_2\dfrac{\omega_2^2}{K} \end{cases} \quad \left(\frac{\theta_1}{\theta_2}\right)_1 = 0.707 \qquad \left(\frac{\theta_1}{\theta_2}\right)_2 = -0.707$$

5-6



for $k_1 = 0$

$$\begin{vmatrix} (1-2\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0 \qquad \lambda(2\lambda - 3) = 0$$

$\therefore \lambda = 0, \quad \lambda = 1.5$

↑free rotation

$$\frac{\theta_1}{\theta_2} = -\frac{1}{2}$$

In general

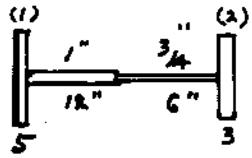
$$J_1\ddot{\theta}_1 - K_2(\theta_2 - \theta_1) = 0$$
$$J_2\ddot{\theta}_2 + K_2(\theta_2 - \theta_1) = 0$$
$$(\ddot{\theta}_2 - \ddot{\theta}_1) + \left(\frac{K_2}{J_2} + \frac{K_2}{J_1}\right)(\theta_2 - \theta_1) = 0$$

Let $\phi = \pm(\theta_2 - \theta_1)$

$$\ddot{\phi} + K_2\left(\frac{J_1 + J_2}{J_1 J_2}\right)\phi = 0$$

$$\omega_n = \sqrt{K_2\left(\frac{J_1 + J_2}{J_1 J_2}\right)}$$

5-7



$$K_1 = \frac{GI_p}{l} = \frac{(11.5\times10^6)}{12''}\frac{\pi 1^4}{32} = 0.0941\times10^6$$

$$K_2 = \left(\frac{11.5\times10^6}{6''}\right)\frac{\pi\left(\frac{3}{4}\right)^4}{32} = 0.0595\times10^6$$

$$K_{eff} = \frac{K_1K_2}{K_1+K_2} = 0.0365\times10^6$$

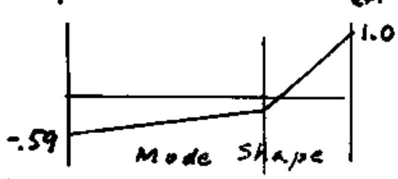
$$\omega_n = \sqrt{K_{eff}\left(\frac{J_1+J_2}{J_1J_2}\right)} = \sqrt{\frac{8}{15}\times .0365\times10^6} = 139.4 \text{ rad/s.}$$

characteristic eq.

$$\begin{vmatrix} \left(1 - \frac{\omega^2 J_1}{K_{eff}}\right) & -1 \\ -1 & \left(1 - \frac{\omega^2 J_2}{K_{eff}}\right) \end{vmatrix} = 0 \qquad \left(\frac{\theta_1}{\theta_2}\right) = \left(1 - \frac{\omega^2 J_2}{K_{eff}}\right) = 1 - 139.4^2\frac{3}{.0365\times10^6}$$

$$= 1 - 1.5979 = -0.5979$$

Twist in each shaft $\propto$ to $\frac{1}{K_i}$

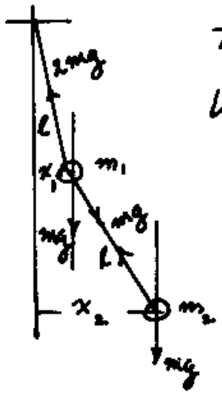


5-8 Total stiffness between cars = 16,000 #/in. System has node at middle of spring ∴ oscillates as two single deg. freed. system with each spring of $2k$ = 32,000 #/in and mass m

$$\omega_n = \sqrt{\frac{2k}{m}} = \sqrt{k\left(\frac{m_1+m_2}{m_1m_2}\right)} = \sqrt{\frac{2\times16{,}000\times386}{50{,}000}} = 15.72 \text{ rad/s.}$$

5-9

For small amplitudes angles are $\frac{x_1}{\ell}$ and $\frac{x_2 - x_1}{\ell}$.
Tensions are approx $2mg$ and mg.



Use ΣF_x

$$m\ddot{x}_1 = -2mg\left(\frac{x_1}{\ell}\right) + mg\left(\frac{x_2 - x_1}{\ell}\right)$$

$$m\ddot{x}_2 = -mg\left(\frac{x_2 - x_1}{\ell}\right) \qquad \text{Let } \lambda = \frac{\omega^2 \ell}{g}$$

$$(3-\lambda)X_1 - X_2 = 0$$

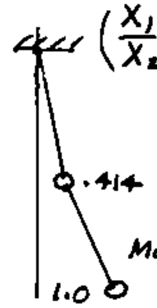
$$-X_1 + (1-\lambda)X_2 = 0$$

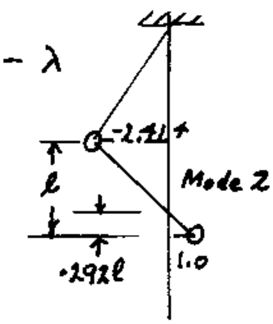
$$\lambda^2 - 4\lambda + 2 = 0, \quad \lambda = 2 \pm \sqrt{2} = \begin{cases} 0.586 \\ 3.414 \end{cases}$$

$$\omega_n = \sqrt{(2 \pm \sqrt{2})g/\ell} = \begin{cases} 0.764\sqrt{\frac{g}{\ell}} \\ 1.850\sqrt{\frac{g}{\ell}} \end{cases} \qquad \left(\frac{X_1}{X_2}\right) = 1 - \lambda$$

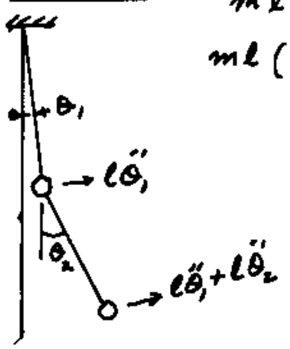
$$\left(\frac{X_1}{X_2}\right)_1 = 1 - \lambda_1 = 0.414$$

$$\left(\frac{X_1}{X_2}\right)_2 = -2.414$$





5-10



$$m\ell\ddot{\theta}_1 = -2mg\,\theta_1 + mg\,\theta_2$$

$$m\ell(\ddot{\theta}_1 + \ddot{\theta}_2) = -mg\,\theta_2 \qquad \lambda = \frac{\omega^2 \ell}{g}$$

$$(2-\lambda)\theta_1 - \theta_2 = 0$$

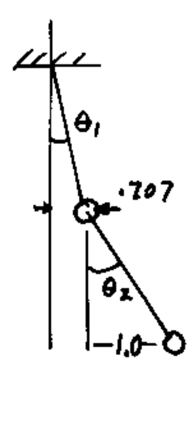
$$-\lambda\theta_1 + (1-\lambda)\theta_2 = 0$$

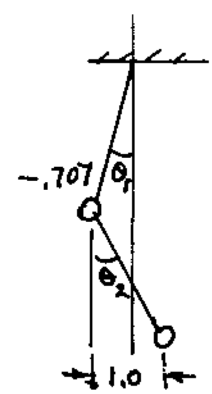
$$\lambda^2 - 4\lambda + 2 = 0 \qquad \text{Same as Prob 5-9}$$

$$\frac{\theta_1}{\theta_2} = \frac{1-\lambda}{\lambda}$$

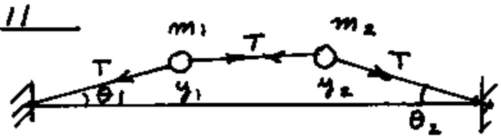
$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.707 \\ 1.00 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} -0.707 \\ 1.00 \end{Bmatrix}$$





5-11



$\theta_1 \cong y_1/\ell \qquad \theta_2 \cong y_2/\ell$

$$m_1 \ddot{y}_1 = -T\frac{y_1}{\ell} + T\left(\frac{y_2-y_1}{\ell}\right)$$

$$m_2 \ddot{y}_2 = -T\left(\frac{y_2-y_1}{\ell}\right) - T\frac{y_2}{\ell}$$

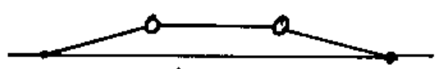
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \frac{T}{\ell}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

5-12 Let $\lambda = \omega^2 \dfrac{m\ell}{T}$ in Prob 5-11

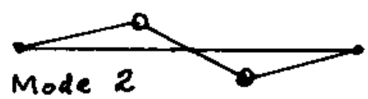
$$\begin{bmatrix} (2-\lambda) & -1 \\ -1 & (2-\lambda) \end{bmatrix}\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix} \qquad \lambda^2 - 4\lambda + 3 = 0$$

$\lambda_1 = 1 \qquad \omega_1 = \sqrt{T/m\ell} \qquad \left(\dfrac{Y_1}{Y_2}\right)_1 = 1$

$\lambda_2 = 3 \qquad \omega_2 = \sqrt{3T/m\ell} \qquad \left(\dfrac{Y_1}{Y_2}\right)_2 = -1$



Mode 1



Mode 2

5-13

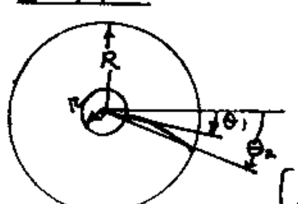
$$\begin{vmatrix} (2-2\lambda) & -1 \\ -1 & (2-\lambda) \end{vmatrix} = 0 \qquad 2\lambda^2 - 6\lambda + 3 = 0$$

$$\lambda = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 1.5}$$

$\lambda_1 = 0.634 \qquad \omega_1 = 0.796\sqrt{T/m\ell} \qquad \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1.00 \\ 0.732 \end{Bmatrix}$

$\lambda_2 = 2.366 \qquad \omega_2 = 1.538\sqrt{T/m\ell} \qquad \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1.00 \\ -2.732 \end{Bmatrix}$

5-14



$$\left.\begin{aligned} J_1\ddot{\theta}_1 &= -K_1\theta_1 + K_2(\theta_2-\theta_1) \\ J_2\ddot{\theta}_2 &= -K_2(\theta_2-\theta_1) \end{aligned}\right\} \quad K_2 = 4k_2R^2$$

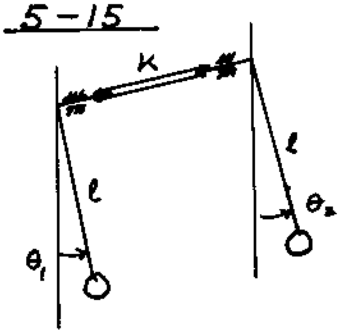
Defl. of spring $=$ $R(\theta_2-\theta_1) = \dfrac{\text{Force}}{k_2} = \dfrac{F}{k_2}$

Torque $= RF = k_2R^2(\theta_2-\theta_1)$

$$\begin{bmatrix} \left(\dfrac{K_1+K_2}{J_1} - \omega^2\right) & -\dfrac{K_2}{J_1} \\ -\dfrac{K_2}{J_2} & \left(\dfrac{K_2}{J_2} - \omega^2\right) \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

$$\omega^4 - \left[\frac{K_1}{J_1} + \frac{K_2}{J_1} + \frac{K_2}{J_2}\right]\omega^2 + \frac{K_1}{J_1}\frac{K_2}{J_2} = 0$$

5-15



$$m\ell^2\ddot{\theta}_1 = -mg\ell\theta_1 + K(\theta_2-\theta_1)$$

$$m\ell^2\ddot{\theta}_2 = -mg\ell\theta_2 - K(\theta_2-\theta_1)$$

$$\begin{bmatrix} \left(\frac{g}{\ell}+\frac{K}{m\ell^2}-\omega^2\right) & -\frac{K}{m\ell^2} \\ -\frac{K}{m\ell^2} & \left(\frac{g}{\ell}+\frac{K}{m\ell^2}-\omega^2\right)\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}0\end{Bmatrix}$$

$$\omega^4 - 2\left(\frac{g}{\ell}+\frac{K}{m\ell^2}\right)\omega^2 + \frac{g}{\ell}\left(\frac{g}{\ell}+2\frac{K}{m\ell^2}\right) = 0$$

$$\omega^2 = \frac{g}{\ell} + \frac{K}{m\ell^2}(1\pm 1) \qquad \frac{\theta_1}{\theta_2} = \frac{\frac{g}{\ell}+\frac{K}{m\ell^2}-\omega^2)}{K/m\ell^2}$$

$$\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix}_1 = \begin{Bmatrix}1\\ 1\end{Bmatrix} \qquad \begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}1\\ -1\end{Bmatrix}$$

For $\begin{Bmatrix}\ell = 19.3'' \\ mg = 3.86\ lb \\ K = 2.0\ lb.in/rad\end{Bmatrix}$ $\quad \omega_1 = \sqrt{\frac{g}{\ell}} = 4.4721$

$$\omega_2 = \sqrt{\frac{g}{\ell}+\frac{2K}{m\ell^2}} = \sqrt{20+1.0739} = 4.5906$$

For $\theta_1(0)=0,\ \theta_2(0)=\theta_0,\quad \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$

$$\theta_1 = \frac{\theta_0}{2}\cos\omega_1 t - \frac{\theta_0}{2}\cos\omega_2 t = \theta_0 \sin\tfrac{1}{2}(\omega_2-\omega_1)t\cdot\sin\tfrac{1}{2}(\omega_1+\omega_2)t$$

$$\theta_2 = \frac{\theta_0}{2}\cos\omega_1 t + \frac{\theta_0}{2}\cos\omega_2 t = \theta_0 \cos\tfrac{1}{2}(\omega_2-\omega_1)t\cdot\cos\tfrac{1}{2}(\omega_1+\omega_2)t$$

$$\left.\begin{aligned}\theta_1 &= \theta_0 \sin 0.0593t\cdot\sin 4.5314t\\ \theta_2 &= \theta_0\cos 0.0593t\cdot\cos 4.5314t\end{aligned}\right\}$$

Beat period $T_b = \frac{\pi}{.0593} = 52.978$ sec.

5-16 From Prob. 5-4 $\quad \lambda = \begin{cases}0.5691\\ 4.0972\end{cases}$ $\quad (X_1/X_2)_1 = 3.4309$, $\ (X_1/X_2)_2 = -0.0972$

For $x_1(0)=A,\ x_2(0)=0$

$\dot{x}_1(0) = \dot{x}_2(0) = 0$

$$x_1 = 3.4309C_1\cos\omega_1 t - 0.0972C_2\cos\omega_2 t$$

$$x_2 = 1.0000C_1\cos\omega_1 t + 1.0000C_2\cos\omega_2 t$$

i.e. Mode 1 is multiplied by C_1 and mode 2 by C_2

At $t=0$

$$A = 3.4309C_1 - 0.0972C_2$$

$$0 = C_1 + C_2 \qquad \therefore\ C_1 = -C_2$$

5-16 Cont.

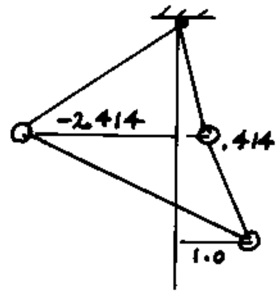
$$A = -3.4309\,C_2 - 0.0972\,C_2 = -3.5281\,C_2$$

$$\therefore C_2 = -0.2834\,A\,, \qquad C_1 = 0.2834\,A$$

$$x_1 = 0.9724\,A\cos\omega_1 t + 0.0276\,A\cos\omega_2 t$$

$$x_2 = 0.2834\,A\cos\omega_1 t - 0.2834\,A\cos\omega_2 t$$

5-17



Add A times mode 1 and B times mode 2

$$x_1 = 0.414\,A\cos\omega_1 t - 2.414\,B\cos\omega_2 t$$

$$x_2 = 1.00\,A\cos\omega_1 t + 1.00\,B\cos\omega_2 t$$

At $t = 0$ $\quad x_1(0) = x_2(0) = X$

$$X = 0.414\,A - 2.414\,B$$

$$X = 1\,A + 1\,B \qquad \therefore A = X - B$$

$$X = 0.414(X - B) - 2.414\,B \qquad \therefore B = -0.2072\,X$$

$$A = 1.2072\,X$$

$$x_1 = X\left\{0.4998\cos\omega_1 t + 0.5002\cos\omega_2 t\right\}$$

$$x_2 = X\left\{1.2072\cos\omega_1 t - 0.2072\cos\omega_2 t\right\}$$

5-18 From Prob 5-1 $(X_1/X_2)_1 = 0.614$, $(X_1/X_2)_2 = -1.618$

$x_1(0) = \dot{x}_1(0) = x_2(0) = 0\,, \quad \dot{x}_2(0) = V$

The 4 initial conditions require 4 arbitrary constants. We choose A times mode 1, B times mode 2 and phase angles φ_1 and φ_2

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = A\begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix}\cos(\omega_1 t + \varphi_1) + B\begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix}\cos(\omega_2 t + \varphi_2)$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = -\omega_1 A\begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix}\sin(\omega_1 t + \varphi_1) - \omega_2 B\begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix}\sin(\omega_2 t + \varphi_2)$$

For $t = 0$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = A\begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix}\cos\varphi_1 + B\begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix}\cos\varphi_2$$

5-18 Cont.

$$\begin{Bmatrix} 0 \\ V \end{Bmatrix} = -\omega_1 A \begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix} \sin\varphi_1 - \omega_2 B \begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix} \sin\varphi_2$$

From 1st two eqs. $\varphi_1 = \varphi_2 = 90°$

From 3rd eq. $\omega_1 A(0.614) = \omega_2 B(1.618)$

From 4th eq. $V = \frac{-1.618}{0.614}\omega_2 B - \omega_2 B = -3.635\omega_2 B$

$$\therefore\ B = \frac{-V}{3.635\omega_2} = -.2751\frac{V}{\omega_2}$$

$$A = \frac{1.618}{.614}\frac{\omega_2}{\omega_1}\left(-.2751\frac{V}{\omega_2}\right) = -0.7249\frac{V}{\omega_1}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0.7249\frac{V}{\omega_1}\begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix}\sin\omega_1 t + \frac{V}{3.635\omega_2}\begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix}\sin\omega_2 t$$

where: $\omega_1 = 0.618\sqrt{\frac{k}{m}}$, $\omega_2 = 1.618\sqrt{\frac{k}{m}}$, $\varphi_1 = \varphi_2 = 90°$

5-19 Using same gen. eqs. of Prob. 5-18

$$\begin{Bmatrix} 0 \\ 1.0 \end{Bmatrix} = A\begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix}\cos\varphi_1 + B\begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix}\cos\varphi_2$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = -\omega_1 A\begin{Bmatrix} 0.614 \\ 1.00 \end{Bmatrix}\sin\varphi_1 - \omega_2 B\begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix}\sin\varphi_2$$

The second set of eqs. require $\varphi_1 = \varphi_2 = 0$

$\therefore\ 0.614\,A = 1.618\,B$ $\quad B = .3795\,A$

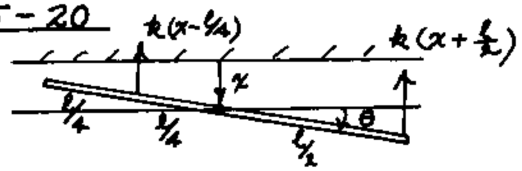
$1.0 = A + .3795\,A$ $\quad \therefore\ A = .7249$

$B = .2751$

Subst. into gen. eqs.

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0.4451 \\ 0.7249 \end{Bmatrix}\cos\omega_1 t + \begin{Bmatrix} -0.4451 \\ 0.2751 \end{Bmatrix}\cos\omega_2 t$$

5-20



$$m\ddot{x} = -k(x+\tfrac{l}{2}\theta) - k(x-\tfrac{l}{4}\theta)$$
$$J\ddot{\theta} = -k(x+\tfrac{l}{2}\theta)\tfrac{l}{2} + k(x-\tfrac{l}{4}\theta)\tfrac{l}{4}$$

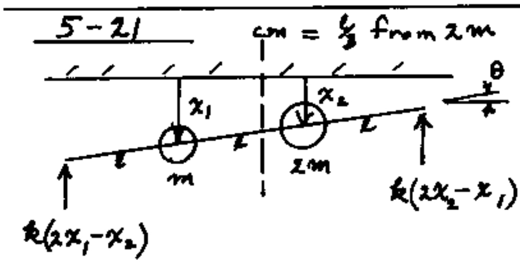
$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 2k & \frac{kl}{4} \\ \frac{kl}{4} & k\frac{5l}{16} \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

$$\begin{vmatrix} (2-\frac{\omega^2 m}{k}) & \frac{l}{4} \\ \frac{1}{4} & (\frac{5}{16}-\frac{\omega^2 J}{kl}) \end{vmatrix} = 0$$

$$\omega^4\left(\frac{m}{k}\frac{J}{kl}\right) - \omega^2\left(\frac{5}{16}\frac{m}{k} + \frac{2J}{kl}\right) + \frac{9}{16} = 0$$

$$\frac{x}{\theta} = -4\left(\frac{5}{16} - \frac{\omega^2 J}{kl}\right)$$

5-21



$$m\ddot{x}_1 + 2m\ddot{x}_2 = -k(2x_1-x_2) - k(2x_2-x_1)$$
$$J_{cm} = m(\tfrac{2}{3}l)^2 + 2m(\tfrac{l}{3})^2 = \tfrac{2}{3}ml^2$$
$$\theta = \frac{x_1-x_2}{l}$$

Moment eq. $\frac{2}{3}ml^2\left(\frac{\ddot{x}_1-\ddot{x}_2}{l}\right) = k(2x_2-x_1)\frac{4l}{3} - k(2x_1-x_2)\frac{5l}{3}$

$$m\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k\begin{bmatrix} 1 & 1 \\ 14 & -13 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix} \qquad \text{Let } \lambda = \frac{\omega^2 m}{k}$$

$$\begin{vmatrix} (1-\lambda) & (1-2\lambda) \\ (14-2\lambda) & -(13-2\lambda) \end{vmatrix} = 0$$

$$2\lambda^2 - 15\lambda + 9 = 0$$
$$\lambda_1 = 0.658 \qquad \omega_1 = 0.811\sqrt{\frac{k}{m}}$$
$$\lambda_2 = 6.842 \qquad \omega_2 = 2.62\sqrt{\frac{k}{m}}$$

$$\frac{X_1}{X_2} = \frac{(1-2\lambda)}{-(1-\lambda)} \qquad \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.921 \\ 1.00 \end{Bmatrix} \qquad \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2 = \begin{Bmatrix} -2.17 \\ 1.00 \end{Bmatrix}$$

5-22

$$2m(\ddot{x}_1 - l\ddot{\theta}) + m\ddot{x}_1 = -k(x_1+l\theta) - k(x_1-2l\theta)$$
$$J_{cm}\ddot{\theta} = k(x_1-2l\theta)\tfrac{4l}{3} - k(x_1-l\theta)\tfrac{5l}{3}$$

Both static + dynamic coupling present.

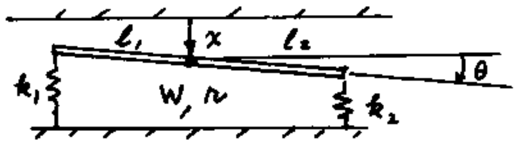
5-23 From Prob 5-9

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \frac{g}{\ell}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \qquad \therefore \text{ Static coupling}$$

From Prob 5-10

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \frac{g}{\ell}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0 \qquad \therefore \text{ Static \& dynamic coupling}$$

5-24



Let $a = \frac{1}{m}(k_1 + k_2)$

$b = \frac{1}{m}(k_2\ell_2 - k_1\ell_1)$

$c = \frac{1}{mr^2}(k_1\ell_1^2 + k_2\ell_2^2)$

r = rad. of gyr. about cm.

$$\ddot{x} + ax + b\theta = 0$$

$$\ddot{\theta} + c\theta + \frac{b}{r^2}x = 0$$

$$\omega^4 - (a+c)\omega^2 + \left(ac - \frac{b^2}{r^2}\right) = 0$$

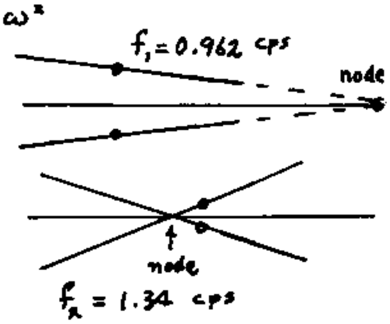
$$\frac{X}{\theta} = \frac{-b}{a - \omega^2}$$

For data given

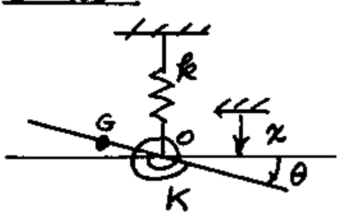
$a = 40.5, \quad b = 42.8, \quad c = 65.6$

$\omega_1^2 = 36.57 \qquad \left(\frac{X}{\theta}\right)_1 = -10.9$ ft/rad

$\omega_2^2 = 69.53 \qquad \left(\frac{X}{\theta}\right)_2 = 1.47$ ft/rad.



5-25



$$m(\ddot{x} - e\ddot{\theta}) + kx = 0$$

$$\Sigma M_G = J_G\ddot{\theta} + K\theta + kex = 0$$

$$J_o = J_G + me^2$$

$$(J_o - me^2)\ddot{\theta} + K\theta + kex = 0$$

5-26

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}\sin\omega t$$

$$\begin{bmatrix} 0.01 & 0 \\ 0 & 0.005 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 30 & -10 \\ -10 & 10 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}\sin\omega t$$

5-26 Cont.

$$\begin{bmatrix} (30-0.01\omega^2) & -10 \\ -10 & (10-0.005\omega^2) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

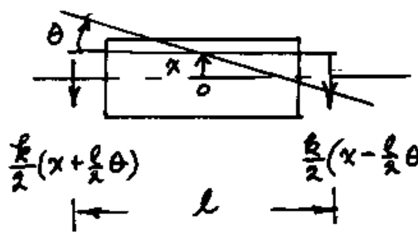
$$\omega^4 - 5000\,\omega^2 + 4\times 10^6 = 0$$

$$\omega^2 = 2500 \pm \sqrt{2500^2 - 4\times 10^6} = 2500 \pm \sqrt{2.25\times 10^6}$$

$$\omega_1 = 31.6 \text{ rad/s} \qquad \left(\frac{X_1}{X_2}\right)_1 = \frac{1}{2}$$

$$\omega_2 = 63.3 \text{ rad/s} \qquad \left(\frac{X_1}{X_2}\right)_2 = -1.0$$

5-27

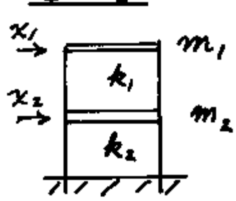


$$\Sigma F = -kx + mr\omega^2\cos\omega t = (M+m)\ddot{x} \cong M\ddot{x}$$

$$\Sigma M_o = -\frac{k}{4}l^2\theta + mr\omega^2 b\cos\omega t = J_o\ddot{\theta}$$

$$m \ll M$$

5-28



$$m_1\ddot{x}_1 + k_1(x_1-x_2) = 0$$

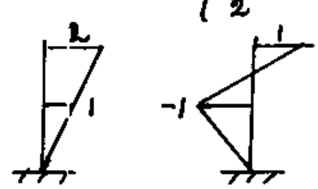
$$m_2\ddot{x}_2 - k_1(x_1-x_2) + k_2x_2 = 0$$

Let $\lambda = \frac{\omega^2 m_1}{k_1}$, $m_2 = 2m_1$, $k_2 = 2k_1$

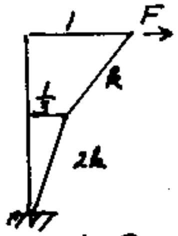
$$\left.\begin{aligned} (1-\lambda)x_1 - x_2 &= 0 \\ -x_1 + (3-2\lambda)x_2 &= 0 \end{aligned}\right\} \qquad \lambda^2 - \frac{5}{2}\lambda + 1 = 0$$

$$\lambda = \begin{cases} \frac{1}{2} \\ 2 \end{cases} \qquad \omega_1 = \sqrt{\frac{k_1}{2m_1}} \qquad \omega_2 = \sqrt{\frac{2k_1}{m_1}}$$

$$\frac{X_1}{X_2} = \frac{1}{1-\lambda} = \begin{cases} 2 \\ -1 \end{cases}$$



5-29



Initial Displ.

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} A\cos\omega_1 t + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} B\cos\omega_2 t$$

Satisfies $\dot{x}_1(0) = \dot{x}_2(0) = 0$

$$\begin{Bmatrix} 1 \\ \frac{1}{3} \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} A + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} B \qquad \therefore A = \frac{4}{9}, \quad B = -\frac{1}{9}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 8/9 \\ 4/9 \end{Bmatrix}\cos\omega_1 t + \begin{Bmatrix} 1/9 \\ -1/9 \end{Bmatrix}\cos\omega_2 t.$$

5-30 From Prob 5-29

$$x_1 = \frac{8}{9}\cos\omega_1 t + \frac{1}{9}\cos\omega_2 t \qquad \omega_1 = \sqrt{\frac{k_1}{2m_1}}$$

$$x_2 = \frac{4}{9}\cos\omega_1 t - \frac{1}{9}\cos\omega_2 t \qquad \omega_2 = \sqrt{\frac{2k_1}{m_1}} = 2\omega_1$$

$$\left.\begin{array}{l} \text{shear in } 1^{st} \text{ story} = k_2 x_2 = 2k_1 x_2 \\ \text{" " } 2^{nd} \text{ " } = k_1(x_1 - x_2) \end{array}\right\} \text{Ratio} = \frac{2x_{2\,max}}{(x_1 - x_2)_{max}}$$

$$\frac{\partial x_2}{\partial t} = -\frac{4}{9}\omega_1 \sin\omega_1 t + \frac{1}{9}\omega_2 \sin\omega_2 t = 0$$

$$= -\frac{\omega_1}{9}\left\{4\sin\omega_1 t - 2\sin 2\omega_1 t\right\} = 0$$

$$= \left\{2\sin\omega_1 t - 2\sin\omega_1 t\cos\omega_1 t\right\} = 0$$

$$\therefore \cos\omega_1 t = 1, \quad \text{and} \quad \omega_1 t = 0,\ 360^\circ \quad \therefore x_{2\,max} = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$$

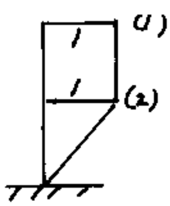
$$\frac{\partial(x_1 - x_2)}{\partial t} = -\frac{4}{9}\omega_1 \sin\omega_1 t - \frac{4}{9}\omega_1 \sin 2\omega_1 t = 0$$

$$\text{or} \quad \sin\omega_1 t(1 + 2\cos\omega_1 t) = 0, \quad \cos\omega_1 t = -\frac{1}{2}, \quad \omega_1 t = 120^\circ$$

$$\therefore (x_1 - x_2)_{max} = \frac{4}{9}\left(-\frac{1}{2}\right) + \frac{2}{9}\left(-\frac{1}{2}\right) = \frac{1}{3}$$

$$\text{Ratio of shears} = \frac{2\left(\frac{1}{3}\right)}{\left(\frac{1}{3}\right)} = 2$$

5-31 See Prob. 5-29

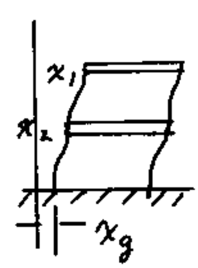


$$\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} A + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} B$$

$$A = \frac{2}{3}, \quad B = \frac{1}{3}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 4/3 \\ 2/3 \end{Bmatrix}\cos\omega_1 t + \begin{Bmatrix} -1/3 \\ 1/3 \end{Bmatrix}\cos\omega_2 t$$

5-32 see Prob. 5-28 with $\lambda = \frac{\omega^2 m_1}{k_1}$



$$\begin{bmatrix} (1-\lambda) & -1 \\ -1 & (3-2\lambda) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2X_g \end{Bmatrix}$$

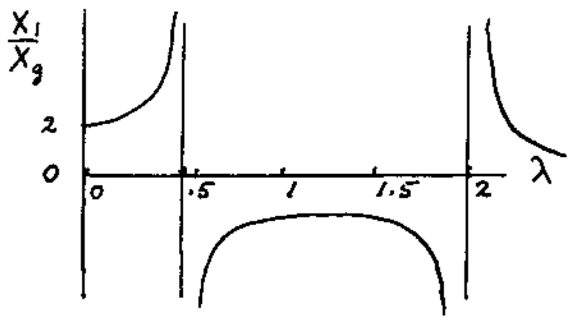
5-32 Cont. $\lambda_1 = \frac{1}{2}$ $\lambda_2 = 2$

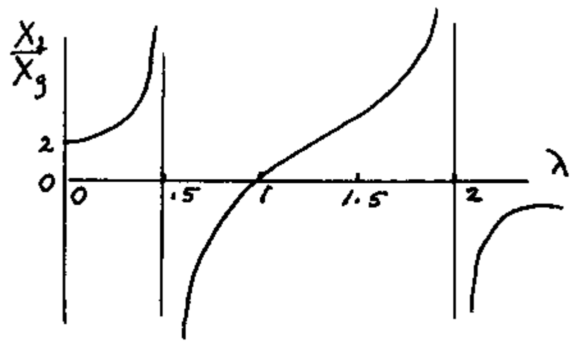
$\therefore$ Denominator $= (\lambda - \frac{1}{2})(\lambda - 2)$ use Cramer's rule

$$X_1 = \frac{\begin{vmatrix} 0 & -1 \\ 2 & (3-2\lambda) \end{vmatrix} X_g}{(\lambda-\frac{1}{2})(\lambda-2)} \qquad X_2 = \frac{\begin{vmatrix} (1-\lambda) & 0 \\ -1 & 2 \end{vmatrix} X_g}{(\lambda-\frac{1}{2})(\lambda-2)}$$

Response $= x_1 = X_1 \sin\omega t$ $\qquad x_2 = X_2 \sin\omega t$

$$\frac{x_1}{X_g} = \frac{2\sin\omega t}{(\lambda-\frac{1}{2})(\lambda-2)} \qquad \frac{x_2}{X_g} = \frac{2(1-\lambda)\sin\omega t}{(\lambda-\frac{1}{2})(\lambda-2)}$$





5-33 $\qquad M(\ddot{y}_0 - l_0\ddot{\theta}) = K_h(y_G - y_0)$

$$M\rho_0^2\ddot{\theta} = K_R(y_G - y_0)l_0 - K_R\theta + Mgl_0\theta$$

ρ_c = rad. of gyration of bldg about its c.m.

5-34 $\qquad$ Let $\omega_h^2 = \frac{K_h}{M}$ $\qquad \omega_r^2 = \frac{K_r}{M\rho_c^2}$

$$\lambda = \frac{\omega}{\omega_h}, \quad \left(\frac{\rho_c}{l_0}\right)^2 = \frac{1}{3}, \quad \left(\frac{\omega_r}{\omega_h}\right)^2 = 4$$

$$\begin{bmatrix} (1-\lambda^2) & \lambda^2 \\ 1 & \frac{1}{3}(4-\lambda^2) \end{bmatrix} \begin{Bmatrix} Y_0 \\ l_0\theta_0 \end{Bmatrix} = Y_G e^{i\phi} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Characteristic eq.

$$\lambda^4 - 8\lambda^2 + 4 = 0$$

$$\lambda^2 = 4 \pm \sqrt{12} = \begin{cases} 0.54 \\ 7.46 \end{cases}$$

$$\lambda = \frac{\omega}{\omega_h} = \begin{cases} 0.734 \\ 2.73 \end{cases}$$

Ampl. ratio

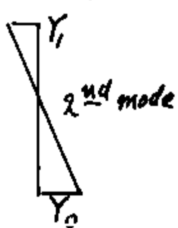
1st mode $\quad \frac{Y_0}{l_0\theta_0} = \frac{\lambda^2 - 4}{3} = -1.15$

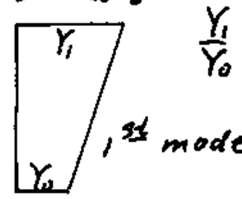
Let $Y_1 = Y_0 - 2l_0\theta_0$ = displ of top, then $\frac{Y_1}{l_0\theta_0} = \frac{Y_0}{l_0\theta_0} - 2 = -1.15 - 2 = -3.15$

$$\frac{Y_1}{Y_0} = \frac{-3.15}{-1.15} = 2.73$$

2nd mode

$$\left.\begin{aligned} \frac{Y_0}{l_0\theta_0} &= 1.15 \\ \frac{Y_1}{l_0\theta_0} &= -.85 \end{aligned}\right\} \frac{Y_1}{Y_0} = -.74$$





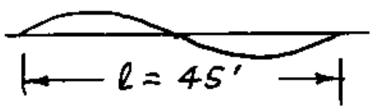
5-35 From Prob. 5-34 $\text{Det.} = \lambda^4 - 8\lambda^2 + 4$

For Det = 0, $\lambda = \dfrac{\omega}{\omega_h} = \begin{cases} 0.732 & 1^{st}\text{ mode} \\ 2.732 & 2^{nd}\text{ mode} \end{cases}$

Using Cramer's rule

$$\left.\begin{aligned} \frac{Y_0}{Y_G} &= \frac{\begin{vmatrix} 1 & \lambda^2 \\ 1 & \frac{1}{3}(4-\lambda^2) \end{vmatrix}}{\lambda^4 - 8\lambda^2 + 4} = \frac{4}{3}\,\frac{(1-\lambda^2)}{\lambda^4 - 8\lambda^2 + 4} \\ \frac{l_0\theta_0}{Y_G} &= \frac{\begin{vmatrix} (1-\lambda^2) & 1 \\ 1 & 1 \end{vmatrix}}{\lambda^4 - 8\lambda^2 + 4} = \frac{-\lambda^2}{\lambda^4 - 8\lambda^2 + 4} \end{aligned}\right\} \text{Plot for various values of } \lambda \text{ to Check Fig P5-35}$$

5-36



Unfavorable speed is found from $v\tau = l$

$$v = \frac{l}{\tau} = l f_n$$

From Prob 5-24

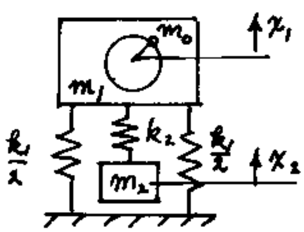
$f_1 = 0.962$ predominately bouncing

$f_2 = 1.327$ " pitching

$v_1 = 45 \times 0.962 = 43.29$ ft/s $= 29.5$ mph

$v_2 = 45 \times 1.327 = 59.72$ ft/s $= 40.7$ mph

5-37



$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = m_0 e\omega^2 \sin\omega t$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$$

$\dfrac{k_2}{m_2}$ should aqual exciting freq ω^2

$$k_2 = \omega^2 m_2 = \left(\frac{2\pi\,1800}{60}\right)^2\left(\frac{50}{386}\right) = 4600 \text{ lb/in}$$

The absorber will then make $x_1 = 0$ and the absorber force will be aqual and opposite to the exciting force.

$$k_2 X_2 = m_0 e\omega^2$$

$$X_2 = \frac{m_0 e\,\omega^2}{k_2} = \frac{m_0 e}{m_2} = \frac{2}{50} = 0.04''$$

5-38

$$m_1\ddot{x}_1 + c(\dot{x}_1-\dot{x}_2) + k_1x_1 + k_2(x_1-x_2) = m_0 e\omega^2(e^{i\omega t})$$

$$m_2\ddot{x}_2 + c(\dot{x}_2-\dot{x}_1) + k_2(x_2-x_1) = 0$$

Let $x_1 = X_1 e^{i\omega t}$, $x_2 = X_2 e^{i\omega t}$ where X_1 & X_2 are complex

$$\left(\frac{k_1+k_2}{m_1} - \omega^2 + i\frac{\omega c}{m_1}\right)X_1 - \left(\frac{k_2}{m_1} + \frac{i\omega c}{m_1}\right)X_2 = \frac{(m_0e)\omega^2}{m_1}$$

$$-\left(\frac{k_2}{m_2} + \frac{i\omega c}{m_2}\right)X_1 + \left(\frac{k_2}{m_2} - \omega^2 + \frac{i\omega c}{m_2}\right)X_2 = 0$$

$$X_1 = \frac{(m_0e)\omega^2(k_2 - m_2\omega^2 + i\omega c)}{[(k_1+k_2) - m_1\omega^2 + i\omega c][k_2 - m_2\omega^2 + i\omega c] - (k_2 + i\omega c)^2}$$

$$\frac{X_2}{X_1} = \frac{(k_2 - i\omega c)}{(k_1 - m_2\omega^2 + i\omega c)}$$

5-39

$$I\ddot{\theta}_1 + 4ka^2(\theta_1-\theta_2) = M_0\sin\omega t$$

$$I_d\ddot{\theta}_2 + 4ka^2(\theta_2-\theta_1) = 0$$

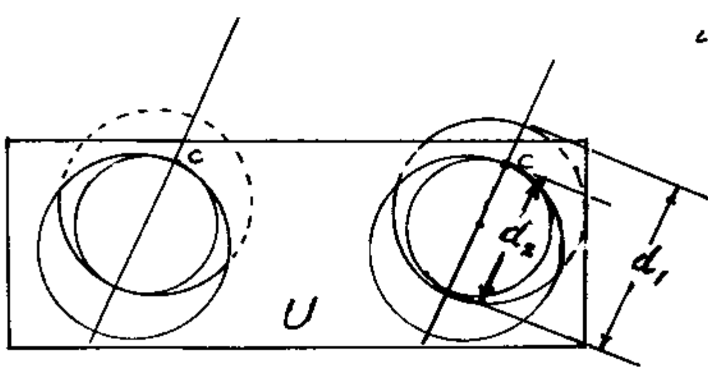
Let $\lambda = \frac{\omega^2 I}{ka^2}$ and $n = \frac{I_d}{I}$

$$\begin{bmatrix} (4-\lambda) & -1 \\ -1 & (4-n\lambda) \end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix} M_0 \\ 0\end{Bmatrix}\sin\omega t$$

$$\theta_1 = \frac{(4-n\lambda)M_0\sin\omega t}{n\lambda^2 - 4(1+n)\lambda + 15} \qquad \therefore\ \theta_1 = 0 \text{ when } \lambda = \frac{4}{n}$$

5-40

Point of contact c of U with small circle moves in circle of radius $r = d_1 - d_2$



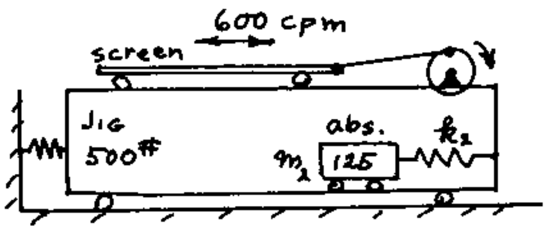
5-41 From Eq. 5.6-5 the natural freq. is

$$\omega_n = n\sqrt{\frac{R}{r}} = 4\,(\text{rot. speed}) = 4n$$

$$\therefore \frac{r}{R} = \frac{1}{16} \qquad r = d_1 - d_2 = \frac{3}{4}'' - d_2$$

$$\therefore r = \frac{R}{16} = \frac{4''}{16} = \frac{1}{4} = \frac{3}{4} - d_2 \qquad d_2 = \frac{1}{2}''$$

5-42



$f_1 = 400$ cpm.

Excit. $\omega = \frac{2\pi\,600}{60} = 20\pi$ rad/s.

Nat. freq. of absorber must equal the excitation freq.

$$\omega_{22}^2 = \frac{k_2}{m_2} = \frac{386\,k_2}{125} = (20\pi)^2$$

$$\therefore k_2 = 1278.\ \#/\text{in}$$

Nat. freq. of system is found from the denominator of Eq 5.5-1 which can be reduced to

$$\left(\frac{\omega}{\omega_{22}}\right)^4 - \left[1 + \left(\frac{\omega_{11}}{\omega_{22}}\right)^2\left\{1 + \mu\left(\frac{\omega_{22}}{\omega_{11}}\right)^2\right\}\right]\left(\frac{\omega}{\omega_{22}}\right)^2 + \left(\frac{\omega_{11}}{\omega_{22}}\right)^2 = 0$$

$$\mu = \frac{m_2}{m_1} = \frac{125}{500} = 0.25\,, \qquad \left(\frac{\omega_{11}}{\omega_{22}}\right)^2 = \left(\frac{400}{600}\right)^2 = \frac{1}{2.25}\,, \qquad \text{Let } \lambda = \frac{\omega}{\omega_{22}}$$

$$\lambda^4 - 1.695\,\lambda^2 + \frac{1}{2.25} = 0\,, \qquad \lambda^2 = \left(\frac{\omega}{\omega_{22}}\right)^2 = 0.845 \pm .164$$

5-43 Refer to FIG 5.5-3

With trial weight of 2 lb tuned to 232 rpm, the two nat. freqs. are

$$\left(\frac{\omega}{\omega_{22}}\right) = \frac{198}{232} = 0.854 \quad \text{and} \quad \left(\frac{\omega}{\omega_{22}}\right) = \frac{272}{232} = 1.17$$

These numbers establish the mass ratio from FIG 5.5-3 to be $\mu \cong 0.10$

To move nat. freqs. outside specified freqs. of

$$\left(\frac{\omega}{\omega_{22}}\right) = \frac{160}{232} = 0.69 \quad \text{and} \quad \left(\frac{\omega}{\omega_{22}}\right) = \frac{320}{232} = 1.38$$

FIG 5.5-3 shows $\mu \geq 0.57$

5-43 Cont.

Since $\mu_1 = \frac{(m_2)_1}{m_1} = \frac{2}{m_1} = 0.10 \qquad \therefore\ m_1 = 20$

$$\mu_2 = \frac{(m_2)_2}{m_1} = 0.57\ , \quad (m_2)_2 = .57 \times 20 = 11.4\ \ lb.$$

The stiffness should be $k_2 = m_2\omega^2 = \frac{11.4}{386}\left(\frac{2\pi\, 232}{60}\right)^2 = 17.9$ #/in

---

5-44 Assume linear velocity distribution of fluid between disk and case. The torque is

$$T = \mu(\text{velocity gradient})(\text{radius})(\text{area})$$

$$= 2\int_{R_o}^{R} 2\pi\mu\left(\frac{\omega r}{h}\right) r^2\, dr + 2\pi\mu\left(\frac{\omega R}{h}\right) R^2 b$$

$$= 2\pi\frac{\mu\omega R^3}{h}\left[\frac{1}{2}\left(R - \frac{R_o^4}{R^3}\right) + b\right]$$

5-45 Optimum damping given by Eq. 5.7-6

$$\zeta_o = \frac{\mu}{\sqrt{2(1+\mu)(2+\mu)}} = \frac{.25}{\sqrt{2(1.25)(2.25)}} = 0.1054$$

The freq. at which the damper is most effective (with peak ampl.) is given by Eq. 5.7-7

$$\frac{\omega}{\omega_n} = \sqrt{\frac{2}{2+\mu}} = \sqrt{\frac{2}{2.25}} = 0.943$$

5-46 The peak amplitude for any μ and ζ can be found from Fig 5.7-4. It is seen that the optimum (lowest point on curve) for $\mu = .25$ is $\zeta \cong .105$ as computed in Prob. 5-45. Thus

$$\frac{\theta_{max}\ \zeta = .10}{\theta_{max}\ \zeta = .105} \cong 1.0$$

5-47 In Fig 5.7-3 all curves pass through a common point p. This point is at the frequency for optimum damping. Thus by equating $\left|\frac{K\theta_0}{M_0}\right|^2$ for $\zeta = 0$ and $\zeta = \infty$ Eq. 5.7-7 is obtained

$$\frac{\mu^2(\omega/\omega_m)^2}{\mu^2(\omega/\omega_m)^2(1-\omega^2/\omega_m^2)} = \frac{4}{4\left[\mu\left(\frac{\omega}{\omega_m}\right)^2-(1-\omega^2/\omega_m^2)\right]^2}$$

or. $$\frac{1}{1-(\omega/\omega_m)^2} = \frac{1}{(\omega/\omega_m)^2(1+\mu)-1}$$

Thus $$\omega/\omega_m = \sqrt{\frac{2}{2+\mu}}$$

Eq. 5.7-6 is found by differentiating $\left|\frac{K\theta_0}{M_0}\right|^2$ w.r.t. $\left(\frac{\omega}{\omega_m}\right)^2$, equating it to zero and substituting Eq. 5.7-7

Let $r^2 = \frac{2}{2+\mu} = \left(\frac{\omega}{\omega_m}\right)^2$ $\qquad (1-r^2) = \frac{\mu}{2+\mu} = r^2(1+\mu)-1$

$q = \frac{4\zeta^2}{\mu^2}$ $\qquad$ Rewrite Eq. 5.7-5

$$\left|\frac{K\theta_0}{M_0}\right|^2 = \frac{r^2+q}{r^2(1-r^2)^2+q\left[r^2(1+\mu)-1\right]^2}$$

$$\frac{\partial|\ |^2}{\partial r^2} = r^2(1-r^2)^2+q\left[r^2(1+\mu)-1\right]^2-(r^2+q)\Big\{(1-r^2)^2-2r^2(1-r^2)$$
$$+2q\left[r^2(1+\mu)-1\right](1+\mu)\Big\} = 0$$

$$= r^2(1-r^2)^2+q(1-r^2)^2-(r^2+q)\Big\{\quad \text{same} \quad\Big\} = 0$$

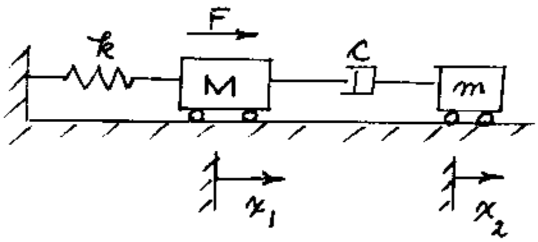
$$= (r^2+q)\left[\cancel{\left(\frac{\mu}{2+\mu}\right)^2}-\cancel{\left(\frac{\mu}{2+\mu}\right)^2}+2r^2\left(\frac{\mu}{2+\mu}\right)-2q\left(\frac{\mu}{2+\mu}\right)(1+\mu)\right] = 0$$

$$\therefore\ r^2-q(1+\mu) = 0 \quad \text{and} \quad q = \frac{r^2}{1+\mu} = \frac{4\zeta^2}{\mu^2}$$

$$\zeta^2 = \frac{\mu^2}{4}\,\frac{1}{1+\mu}\,\frac{2}{2+\mu} = \frac{\mu^2}{2(1+\mu)(2+\mu)}$$

$$\zeta_{opt} = \frac{\mu}{\sqrt{2(1+\mu)(2+\mu)}}$$

5 - 48



$$M\ddot{x}_1 = -kx_1 - c(\dot{x}_1 - \dot{x}_2) + Fe^{i\omega t}$$

$$m\ddot{x}_2 = c(\dot{x}_1 - \dot{x}_2) \qquad \text{Let } x_1 = X_1 e^{i\omega t}, \quad x_2 = X_2 e^{i\omega t}$$

$$\left[\left(\frac{k}{M} - \omega^2\right) + i\left(\frac{c\omega}{M}\right)\right]X_1 - i\left[\frac{c\omega}{M}\right]X_2 = \frac{F}{M}$$

$$\left[-\omega^2 + i\frac{c\omega}{m}\right]X_2 = i\left(\frac{c\omega}{m}\right)X_1 \qquad \text{eliminate } X_2$$

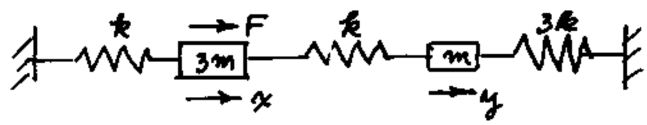
$$\left\{\left[\left(\frac{k}{M} - \omega^2\right) + i\left(\frac{c\omega}{M}\right)\right] - i\left(\frac{c\omega}{M}\right)\frac{i\left(\frac{c\omega}{m}\right)}{\left(-\omega^2 + \frac{ic\omega}{m}\right)}\right\}X_1 = \frac{F}{M}$$

$$\left\{\left[(k - M\omega^2) + ic\omega\right]\left(\frac{-m\omega^2 + ic\omega}{\cancel{m}}\right) + \frac{(c\omega)^2}{\cancel{m}}\right\}X_1 =$$

$$F\left(\frac{-m\omega^2 + ic\omega}{\cancel{m}}\right)$$

$$\therefore \frac{X_1}{F} = \frac{(\omega^2 m - ic\omega)}{m\omega^2(k - M\omega^2) + ic\omega\left[m\omega^2 - (k - M\omega^2)\right]}$$

5-49



$$\left.\begin{aligned}\ddot{x} &= -\frac{2}{3}\frac{k}{m}x + \frac{1}{3}\frac{k}{m}y + \frac{F}{3m} \\ \ddot{y} &= \frac{k}{m}x - 4\frac{k}{m}y\end{aligned}\right\} \quad \begin{aligned}\omega_1 &= 0.751\sqrt{\frac{k}{m}} \\ \omega_2 &= 2.04\sqrt{\frac{k}{m}}\end{aligned}$$

Let $k = m = 1$, then $\tau_1 = 8.35$ sec, $\tau_2 = 3.07$ sec.

Let $\Delta t = 0.20$

$$x_{i+1} = \ddot{x}_i\,\Delta t^2 + 2x_i - x_{i-1}$$
$$y_{i+1} = \ddot{y}_i\,\Delta t^2 + 2y_i - y_{i-1}$$

Initial Cond. $(I=1)$

$$x(1) = y(1) = \ddot{y}(1) = 0, \qquad \ddot{x}_1(1) = \frac{100}{3}$$

From Eq. (4.5-8) $\quad x(2) = \frac{1}{2}(0.20)^2\left(\frac{100}{3}\right) = 0.666$

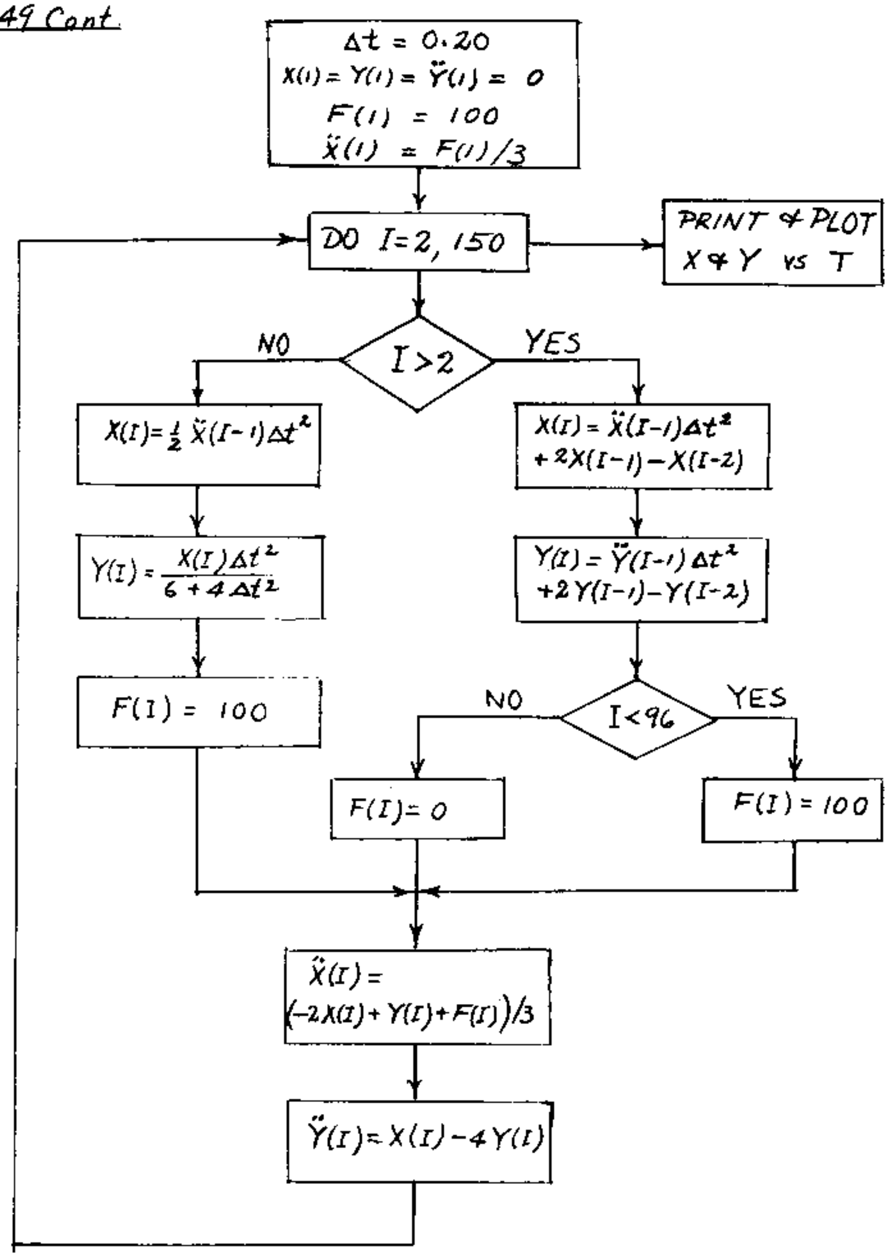
From Eq (4.5-10) and DE for $\ddot{y}$

$$y(2) = \frac{1}{6}\Delta t^2\,\ddot{y}(2) = \frac{\Delta t^2}{6}\left[x(2) - 4y(2)\right]$$

$$y(2)\left[1 + \frac{2}{3}\Delta t^2\right] = \frac{1}{6}\Delta t^2 \times 0.666 \qquad \therefore\ y(2) = 0.00432$$

Flow diagram, computer program and plot of results follow.

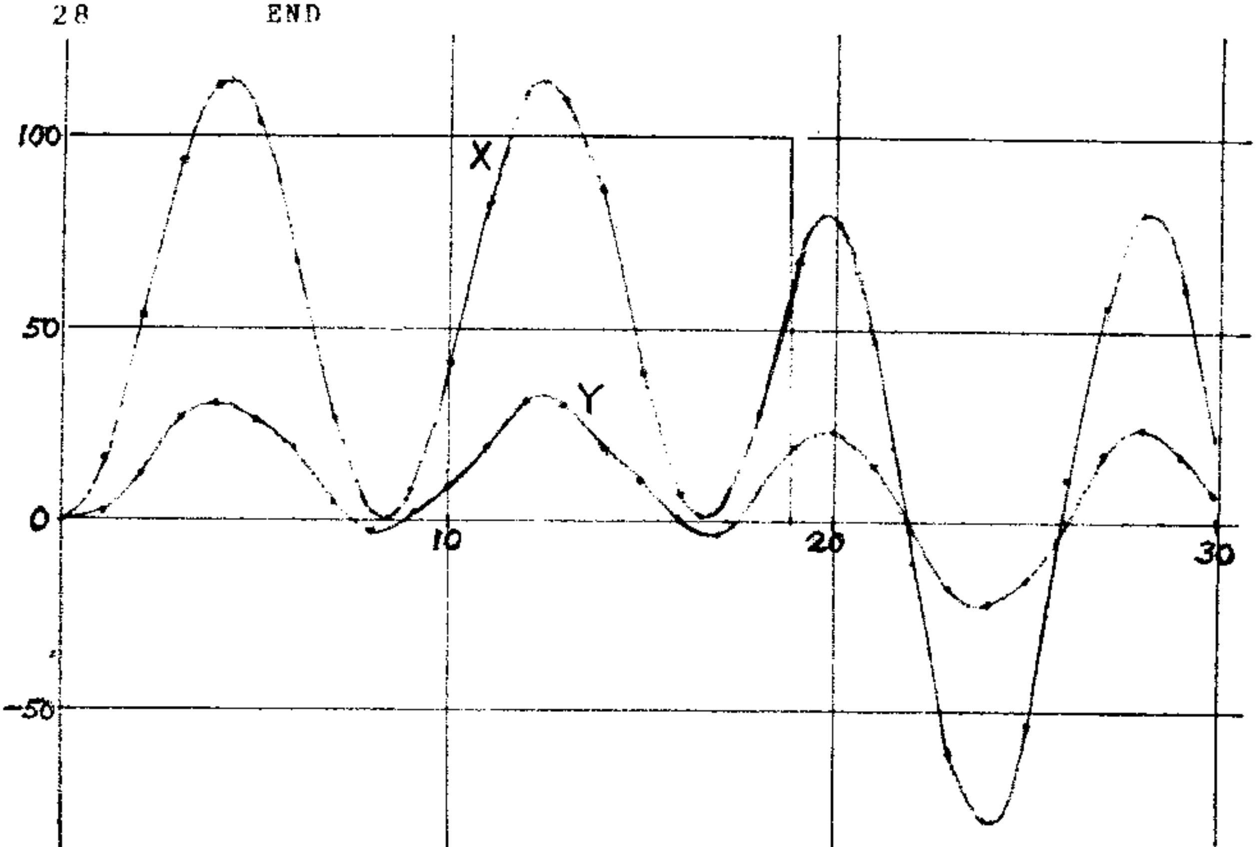
5-49 Cont.



Flow Diagram Prob 5-49

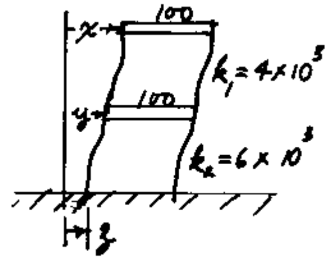
Computer Program – Prob 5–49

```
C       PROBLEM 5-49 THOMSON
        DIMENSION X(155),Y(155),DX(155),DY(155),T(155)
        DT=0.20
        F=100.
        X(1)=0.0
        Y(1)=0.0
        DY(1)=0.0
        DX(1)=F/3.
        T(1)=0.0
        I=2
      5 IF(I.GT.2) GOTO 10
        X(I)=0.5*DX(I-1)*DT**2
        Y(I)=X(I)*DT**2/(6.+4.*DT**2)
        GO TO 30
     10 X(I)=DX(I-1)*DT**2+2.*X(I-1)-X(I-2)
        Y(I)=DY(I-1)*DT**2+2.*Y(I-1)-Y(I-2)
        IF(I.GT.95) GOTO 20
        F=100.
        GO TO 30
     20 F=0.0
     30 DX(I)=(-2.*X(I)+Y(I)+F)/3.
        DY(I)=X(I)-4.*Y(I)
        T(I)=T(I-1)+DT
        I=I+1
        IF(I.LE.151) GOTO 5
        DO 50 I=1,150
     50 PRINT,T(I),X(I),Y(I)
        STOP
        END
```



5-50

$$100\ddot{x} = -4\times10^3(x-y)$$
$$100\ddot{y} = 4\times10^3(x-y) - 6\times10^3(y-z)$$



$$\therefore\ \ddot{x} = -40x + 40y$$
$$\ddot{y} = 40x - 100y + 60z \qquad z = 10\sin\pi t$$

Use Eq. 4.5-10 to start computation

$$x(2) = \tfrac{1}{6}\ddot{x}(2)h^2 = \tfrac{h^2}{6}\left[-40x(2) + 40y(2)\right]$$

$$y(2) = \tfrac{h^2}{6}\ddot{y}(2) = \tfrac{h^2}{6}\left[40x(2) - 100y(2) + 60z(2)\right]$$

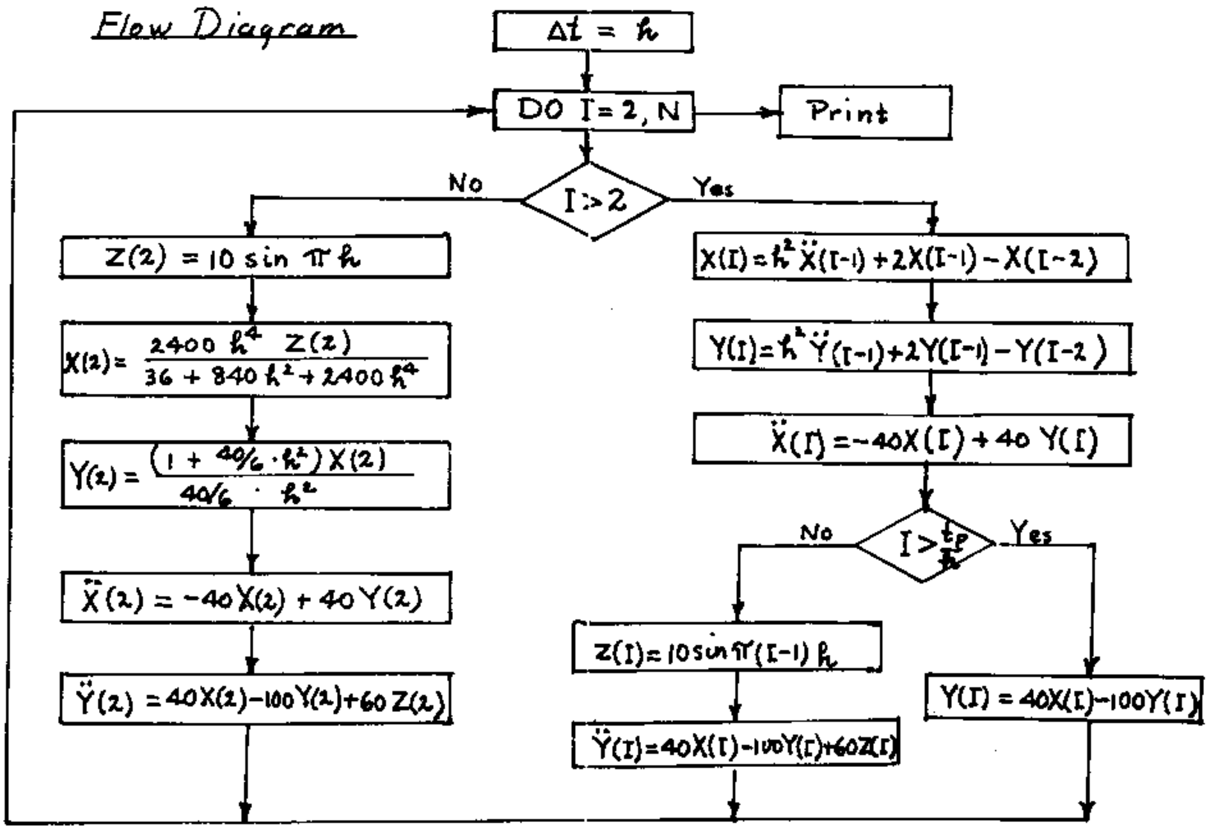
$$x(2)\left[1 + \tfrac{40}{6}h^2\right] = y(2)\,\tfrac{40}{6}h^2$$

$$y(2)\left[1 + \tfrac{100}{6}h^2\right] = x(2)\tfrac{40}{6}h^2 + 10h^2z(2)$$

$$\therefore\ x(2) = \frac{40}{6}h^2\,\frac{1}{1+\frac{40}{6}h^2}\cdot\frac{1}{1+\frac{100}{6}h^2}\cdot\left[x(2)\tfrac{40}{6}h^2 + 10h^2z(2)\right]$$

$$\left.\begin{aligned} x(2) &= 2400h^4z(2)/(36 + 840h^2 + 2400h^4) \\ y(2) &= \left(1 + \tfrac{40}{6}h^2\right)x(2)\Big/\tfrac{40}{6}h^2 \end{aligned}\right\}\ \text{starting equations}$$

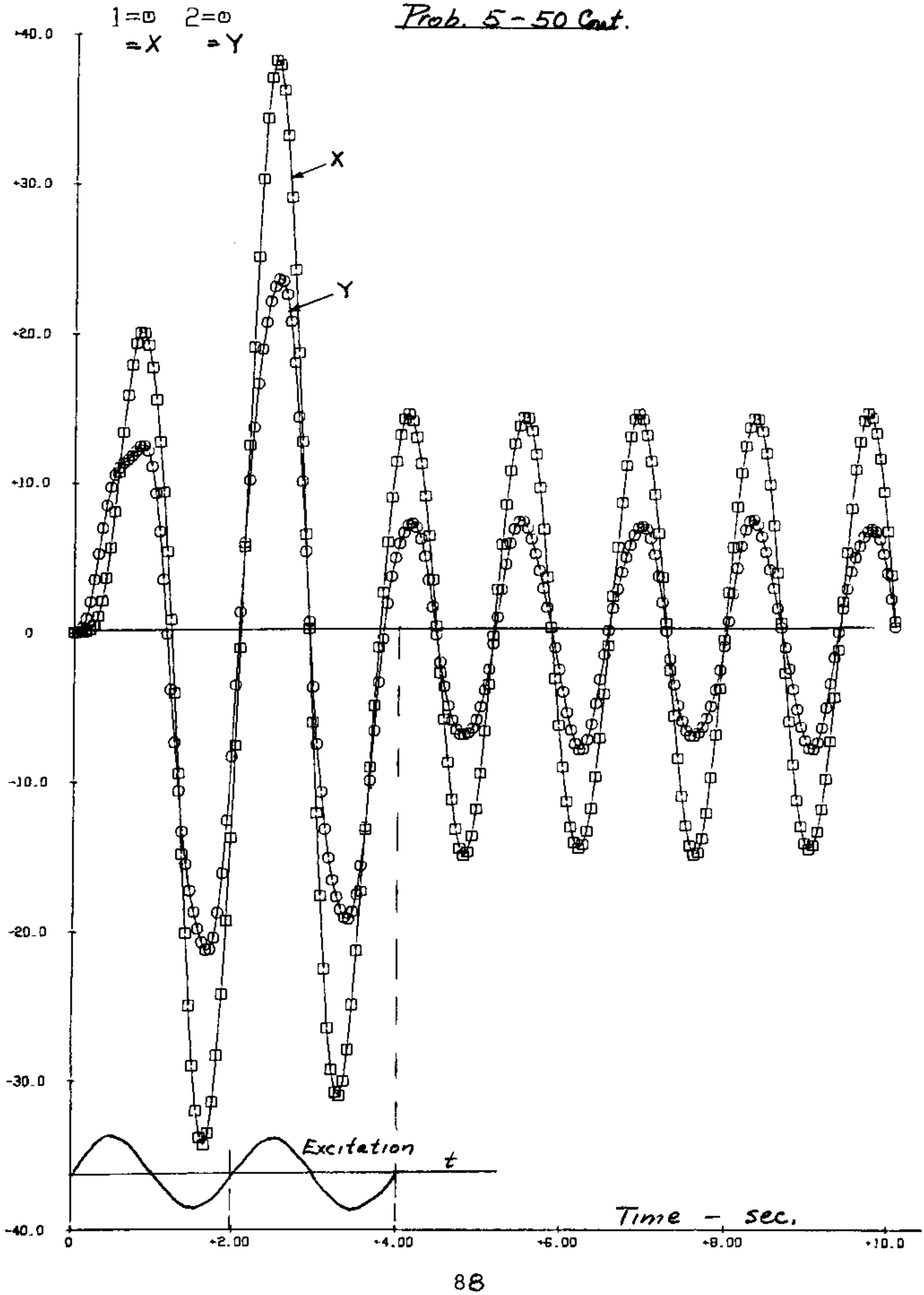
Flow Diagram



Computer Program

```
      PROBLEM 5-50 THOMSON
      DIMENSION X(220),Y(220),DX(220),DY(220),T(220),Z(220)
      DT=0.05
      F=100.
      X(1)=0.0
      Y(1)=0.0
      DY(1)=0.0
      DX(1)=0.
      T(1)=0.0
      I=2
      Z(1)=0.0
    5 T(I)=T(I-1)+DT
      IF(T(I).GT.4.) GO TO 10
      Z(I)=10.*SIN(3.14*T(I))
      GO TO 15
   10 Z(I)=0.0
   15 IF(I.GT.2) GOTO 20
      X(I)=(400.*6.*DT**4*Z(I))/(36.+840.*DT**2+2400.*DT**4)
      Y(I)=(1+40./6.*DT**2)*X(I)/(40./6.*DT**2)
      GO TO 50
   20 X(I)=DX(I-1)*DT**2+2.*X(I-1)-X(I-2)
      Y(I)=DY(I-1)*DT**2+2.*Y(I-1)-Y(I-2)
   50 DX(I)=-40.*X(I)+40.*Y(I)
      DY(I)=40.*X(I)-100.*Y(I)+60.*Z(I)
      I=I+1
      IF(I.LE.202) GOTO 5
      DO 70 I=1,202
      PRINT60,T(I),Z(I),X(I),Y(I)
   60 FORMAT(10X,4F12.4)
   70 CONTINUE
      CALL EZPLOT(T,X,202)
      CALL EZPLOT(T,Y,-202)
      CALL FINISH
      STOP
      END
```

Prob. 5-50 Cont.



5-51

$$\left.\begin{aligned} J_1 \ddot{\theta}_1 &= K_1(\theta_2 - \theta_1) \\ J_2 \ddot{\theta}_2 &= -K_1(\theta_2 - \theta_1) + K_2(\theta_3 - \theta_2) \\ J_3 \ddot{\theta}_3 &= -K_2(\theta_3 - \theta_3) \end{aligned}\right\}$$

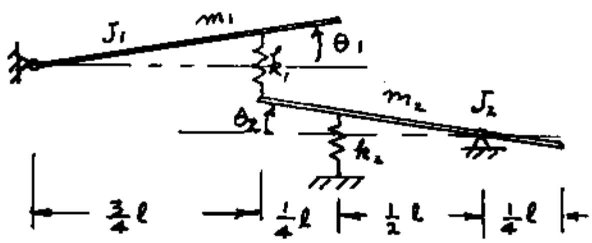
If $\phi = (\theta_2 - \theta_1)$, $\psi = (\theta_3 - \theta_2)$ the DEs can be written as

$$\left.\begin{aligned} J_2 \ddot{\phi} &= -k_1\left(1 + \frac{J_2}{J_1}\right)\phi + K_2 \psi \\ J_3 \ddot{\psi} &= K_1 \frac{J_3}{J_2} \phi - K_2\left(1 + \frac{J_3}{J_2}\right)\psi \end{aligned}\right\} \text{ DEs in two coordinates}$$

Freq. Eq. from either set of equations is

$$\omega^4 - \left[\frac{K_1}{J_1} + \frac{K_3}{J_2}\left(1 + \frac{K_1}{K_2} + \frac{J_2}{J_3}\right)\right]\omega^2 + \frac{K_1}{J_1}\frac{K_2}{J_2}\left(\frac{J_1 + J_2 + J_3}{J_3}\right) = 0$$

5-52



$$J_1 \ddot{\theta}_1 = -\tfrac{3}{4}\ell k_1(\theta_1 - \theta_2)\tfrac{3}{4}\ell$$

$$J_2 \ddot{\theta}_2 = \tfrac{3}{4}\ell k_1(\theta_1 - \theta_2)\tfrac{3}{4}\ell - \tfrac{1}{2}\ell k_2 \theta_2 \tfrac{1}{2}\ell$$

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \ell^2 k_1 \begin{bmatrix} \frac{9}{16} & -\frac{9}{16} \\ -\frac{9}{16} & \left(\frac{9}{16} + \frac{1}{4}\frac{k_2}{k_1}\right) \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

$$J_1 = m_1 \ell^2 \frac{1}{3} \qquad J_2 = m_2 \ell^2\left(\frac{1}{12} + \frac{1}{16}\right)$$

5-53

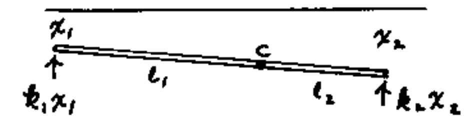
$$\begin{bmatrix} \left[K_1\left(1 + \frac{J_2}{J_1}\right) - \omega^2 J_2\right] & -K_2 \\ -K_1 \frac{J_3}{J_2} & \left[K_2\left(1 + \frac{J_3}{J_2}\right) - \omega^2 J_3\right] \end{bmatrix}\begin{Bmatrix} \phi \\ \psi \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

$$\frac{\phi}{\psi} = \frac{K_2}{K_1\left(1 + \frac{J_2}{J_1}\right) - \omega^2 J_2} = \frac{\theta_2 - \theta_1}{\theta_3 - \theta_2} = \frac{\frac{\theta_2}{\theta_1} - 1}{\frac{\theta_3}{\theta_1} - \frac{\theta_2}{\theta_1}}$$

<u>Assign numerical</u> values for K_s & J_s and prove

$$J_1 \theta_1^{(i)} \theta_1^{(j)} + J_2 \theta_2^{(i)} \theta_2^{(j)} + J_3 \theta_3^{(i)} \theta_3^{(j)} = 0$$

5-54



$$\text{Displ. of } c = x_1 + \frac{\ell_1}{\ell_1+\ell_2}(x_2 - x_1)$$

$$= \frac{\ell_2 x_1 + \ell_1 x_2}{\ell_1 + \ell_2}$$

$$T = \frac{1}{2} m \left(\frac{\ell_2 \dot{x}_1 + \ell_1 \dot{x}_2}{\ell_1 + \ell_2} \right)^2 + \frac{1}{2} J_c \left(\frac{\dot{x}_2 - \dot{x}_1}{\ell_1 + \ell_2} \right)^2$$

$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2$$ $\therefore$ Dynamic coupling exists

---

5-55

Equations to be solved are:

$$\ddot{x} = -720 x + 360 y$$ (see p 132)

$$\ddot{y} = 1440 (x - y) + 160$$

$$(s^2 + 720)\bar{x}(s) - 360\, \bar{y}(s) = 0$$

$$-1440\, \bar{x}(s) + (s^2 + 1440)\, \bar{y}(s) = \frac{160}{s}$$

from 1st eq. $\bar{x}(s) = \left(\dfrac{360}{s^2 + 720} \right) \bar{y}(s)$

subst. into 2nd eq. $-1440\left(\dfrac{360}{s^2+720}\right)\bar{y}(s) + (s^2 + 1440)\bar{y}(s) = \dfrac{160}{s}$

x by $(s^2 + 720)$

$$\left[-518,400 + s^4 + 2160 s^2 + 1,036,800 \right] \bar{y}(s) = \frac{160(s^2 + 720)}{s}$$

$$\left[s^4 + 2160 s^2 + 518,400 \right] \bar{y}(s) = \frac{160}{s}(s^2 + 720)$$

$$\bar{y}(s) = \frac{160(s^2 + 720)}{s(s^4 + 2160 s^2 + 518,400)}$$

Roots: $s^2 = -1080 \pm \sqrt{1,166,400 - 518,400}$

$$= -1080 \pm \sqrt{648,000}$$

$$= -1080 \pm 804.9845$$

also $s = 0$ is a root.

5-55 Cont.

$$\therefore \quad s_1^2 = -275.0155 \qquad s_1 = \pm i\, 16.5836$$

$$s_2^2 = -1884.9845 \qquad s_2 = \pm i\, 43.4164$$

Rewrite $\bar{y}(s)$ as

$$\bar{y}(s) = \frac{160\,(s^2+720)}{s\,(s^2+275.0155)(s^2+1884.9845)}$$

$$= \frac{160\,(s^2+720)}{s(s+i\,16.5836)(s-i\,16.5836)(s+i\,43.4164)(s-i\,43.4164)}$$

$$= \frac{C_1}{(s+i\,16.5836)} + \frac{C_2}{(s-i\,16.5836)} + \frac{C_3}{(s+i\,43.4164)}$$

$$+ \frac{C_4}{(s-i\,43.4164)} + \frac{C_5}{s}$$

$$C_1 = \lim_{s\to s_{1-}} \left[\frac{160\,(s^2+720)\cancel{(s+i\,16.5836)}}{s\,\cancel{(s+i\,16.5836)}(s-i\,16.58)(s^2+1884.9845)}\right]$$

$$= \frac{160\,(-275.0155+720)}{(-i\,16.5836)(-i\,33.1672)(-275.0155+1884.9845)} = -.0804007$$

$$C_2 = \lim_{s\to s_{1-}} \left[\frac{160\,(s^2+720)\cancel{(s-i\,16.5836)}}{s(s+i\,16.5836)\cancel{(s-i\,16.5836)}(s^2+1884.98)}\right] = -.0804007$$

$$C_3 = \lim_{s\to s_{2-}} \left[\frac{160(s^2+720)\,(s+i\,43.4164)}{s\,(s^2+275.0155)(s+i\,43.4164)\,(s-i\,43.4164)}\right] = -.03071$$

$$C_4 = C_3$$

$$C_5 = \frac{160\,(720)}{(275.0155)(1884.9845)} = 0.2222$$

$$\bar{y}(s) = \frac{.2222}{s} - \frac{.0804007}{(s+i\,16.5836)} - \frac{.0804007}{(s-i\,16.5836)}$$

$$- \frac{.03071}{(s+i\,43.4164)} - \frac{.03071}{(s-i\,43.4164)}$$

Inverse

$$y(t) = 0.2222 - .0804007\left(e^{-i16.5836} + e^{i16.5836}\right)$$

$$- .03071\left(e^{-i43.4164} + e^{i43.4164}\right)$$

$$= 0.2222 - .1608\cos 16.58t - .06142\cos 43.41t$$

$$= .1608(1-\cos 16.58t) + .06142(1-\cos 43.41t) \text{ meters}$$

move dec. pt. 2 places to left for centimeters

Solution for $x(t)$

$$\bar{x}(s) = \frac{360}{(s^2+720)}\,\bar{y}(s) = \frac{360}{\cancel{(s^2+720)}} \cdot \frac{160\cancel{(s^2+720)}}{s(s^2+275.0155)(s^2+1884.9845)}$$

$\therefore$ roots are same. Solve as above with partial fractions, the equation

$$\bar{x}(s) = \frac{57600}{s(s^2+275.0155)(s^2+1884.9845)}$$

Ans.

$$x(t) = .1301(1-\cos 16.58t) - .0190(1-\cos 43.41t) \text{ meters.}$$

5-56 Examine for example the subsidiary sol. for Prob 5-55

$$\bar{y}(s) = \sum \frac{c_i}{s-s_i} \quad \text{where } s_i \text{ are roots}$$

$\therefore$ Results are sums of normal modes

5 - 57

$$m\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\\ddot{x}_2\end{Bmatrix} + k\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\x_2\end{Bmatrix} = \begin{Bmatrix}F_0\\0\end{Bmatrix}\sin\omega t$$

$$ms^2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}\bar{x}_1(s)\\\bar{x}_2(s)\end{Bmatrix} + k\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{Bmatrix}\bar{x}_1(s)\\\bar{x}_2(s)\end{Bmatrix} = \begin{Bmatrix}\frac{F_0\omega}{s^2+\omega^2} + msx_1(0) + m\dot{x}_1(0)\\ msx_2(0) + m\dot{x}_2(0)\end{Bmatrix}$$

$$\begin{bmatrix}(ms^2+2k) & -k\\-k & (ms^2+2k)\end{bmatrix}\begin{Bmatrix}\bar{x}_1(s)\\\bar{x}_2(s)\end{Bmatrix} = \begin{Bmatrix} \text{''} \end{Bmatrix}$$

Det. of matrix $(ms^2+2k)^2 - k^2 = 0$, $s^2 = -\frac{2k}{m} \pm \frac{k}{m}$

$$\therefore \left(s^2 + \frac{k}{m}\right)\left(s^2 + \frac{3k}{m}\right)m$$

$$\bar{x}_1(s) = \frac{\begin{vmatrix}\left[\frac{F_0\omega}{s^2+\omega^2} + msx_1(0) + m\dot{x}(0)\right] & -k\\ \left[msx_2(0) + m\dot{x}_2(0)\right] & (ms^2+2k)\end{vmatrix}}{\left(s^2+\frac{k}{m}\right)\left(s^2+\frac{3k}{m}\right)m}$$

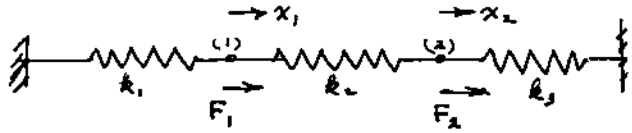
$$\bar{x}_2(s) = \frac{\begin{vmatrix}(ms^2+2k) & \left[\frac{F_0\omega}{s^2+\omega^2} + msx_1(0) + m\dot{x}_1(0)\right]\\ -k & \left[msx_2(0) + m\dot{x}_2(0)\right]\end{vmatrix}}{\left(s^2+\frac{k}{m}\right)\left(s^2+\frac{3k}{m}\right)m}$$

For steady state vibration $x(0)$, $\dot{x}(0) = 0$ and $s = i\omega$

$$x_1(t) = \frac{\left(2\frac{k}{m} - \omega^2\right)F_0\sin\omega t}{\left(\frac{k}{m} - \omega^2\right)\left(3\frac{k}{m} - \omega^2\right)}$$

$$x_2(t) = \frac{k F_0 \sin\omega t}{\left(\frac{k}{m} - \omega^2\right)\left(3\frac{k}{m} - \omega^2\right)m}$$

6-1



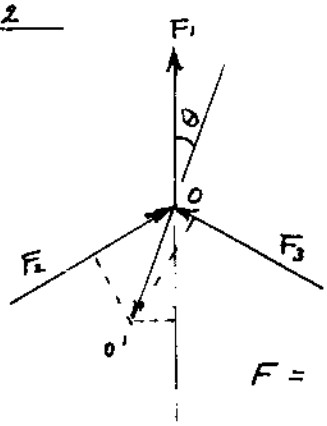
$$F_1 = -k_1 x_1 - k_2(x_1 - x_2)$$

$$F_2 = -k_3 x_2 - k_2(x_2 - x_1)$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} K \end{bmatrix}^{-1} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{-1}{k_1k_2 + k_1k_3 + k_2k_3} \begin{bmatrix} (k_2+k_3) & k_2 \\ k_2 & (k_1+k_2) \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

6-2



Let o move to o' with δ small

Then $F_1 = \delta k \cos\theta$

$F_2 = \delta k \cos(60-\theta)$

$F_3 = \delta k \cos(60+\theta)$

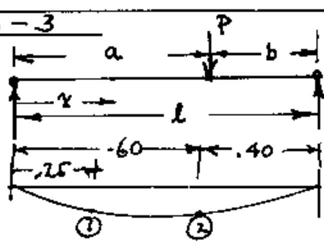
Force along δ is

$$F = F_1 \cos\theta + F_2 \cos(60-\theta) + F_3 \cos(60+\theta)$$

$$= k\delta\left[\cos^2\theta + \cos^2(60-\theta) + \cos^2(60+\theta)\right] = 1.5\,k\delta$$

$\therefore \dfrac{\delta}{F} = \dfrac{1}{1.5k}$ and independent of θ

6-3



$$y = \frac{Pbx}{6EI\ell}\left(\ell^2 - x^2 - b^2\right) \qquad x \le a$$

Position ① $= 0.25\ell$, position ② $= 0.60\ell$

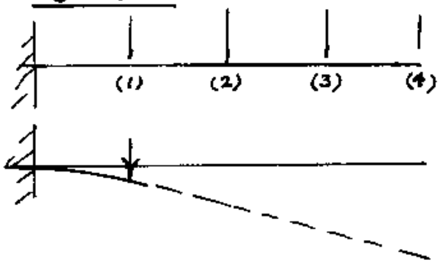
$$a_{11} = \frac{(.75)(.25)\ell^3}{6\,EI}\left(1 - .25^2 - .75^2\right) = 0.0114\,\frac{\ell^3}{EI}$$

$$a_{12} = a_{21} = \frac{(0.25)(.40)\ell^3}{6\,EI}\left(1 - .40^2 - .25^2\right) = 0.0130\,\frac{\ell^3}{EI}$$

$$a_{22} = \frac{(.40)(.60)\ell^3}{6\,EI}\left(1 - .60^2 - .40^2\right) = 0.0192\,\frac{\ell^3}{EI}$$

$$a = \frac{\ell^3}{EI}\begin{bmatrix} 0.0114 & 0.0130 \\ 0.0130 & 0.0192 \end{bmatrix}$$

6 - 4



$$a_{11} = \frac{\left(\frac{\ell}{4}\right)^3}{3EI} = \frac{\ell^3}{192EI}$$

$$\Theta_{11} = \frac{\left(\frac{\ell}{4}\right)^2}{2EI} = \frac{\ell^2}{32EI}$$

$$a_{21} = a_{11} + \Theta_{11}\frac{\ell}{4} = \frac{\ell^3}{192EI} + \frac{\ell^3}{128EI} = \frac{2.50\,\ell^3}{192EI}$$

$$a_{31} = a_{21} + \frac{\ell^3}{128EI} = \frac{4\,\ell^3}{192EI}$$

$$a_{41} = a_{31} + \frac{\ell^3}{128EI} = \frac{5.50\,\ell^3}{192EI}$$

Place unit load at ② and repeat

$$a_{22} = \frac{\left(\frac{\ell}{2}\right)^3}{3EI} = \frac{\ell^3}{24EI}$$

$$\Theta_{22} = \frac{\left(\frac{\ell}{2}\right)^2}{2EI} = \frac{\ell^2}{8EI}$$

$$a_{32} = a_{22} + \Theta_{22}\frac{\ell}{4} = \left(\frac{1}{24} + \frac{1}{32}\right)\frac{\ell^3}{EI} = \frac{7}{96}\frac{\ell^3}{EI}$$

$$a_{42} = \left(\frac{7}{96} + \frac{1}{32}\right)\frac{\ell^3}{EI} = \frac{10}{96}\frac{\ell^3}{EI}$$

$$a_{33} = \frac{\left(\frac{3}{4}\ell\right)^3}{3EI} = \frac{9}{64}\frac{\ell^3}{EI}, \qquad \Theta_{33} = \frac{\left(\frac{3}{4}\ell\right)^2}{2EI} = \frac{9}{32}\frac{\ell^2}{EI}$$

$$a_{43} = a_{33} + \Theta_{33}\frac{\ell}{4} = \left(\frac{9}{64} + \frac{9}{128}\right)\frac{\ell^3}{EI} = \frac{27}{128}\frac{\ell^3}{EI}$$

$$a_{44} = \frac{\ell^3}{3EI}$$

$$a = \frac{\ell^3}{EI}\begin{bmatrix} \frac{1}{192} & \frac{2.5}{192} & \frac{4}{192} & \frac{5.5}{192} \\ \frac{2.5}{192} & \frac{1}{24} & \frac{7}{96} & \frac{10}{96} \\ \frac{4.0}{192} & \frac{7}{96} & \frac{9}{64} & \frac{27}{128} \\ \frac{5.5}{192} & \frac{10}{96} & \frac{27}{128} & \frac{1}{3} \end{bmatrix} = \frac{\ell^3}{192EI}\begin{bmatrix} 1 & 2.5 & 4.0 & 5.5 \\ 2.5 & 8.0 & 14.0 & 20.0 \\ 4.0 & 14.0 & 27.0 & 40.5 \\ 5.5 & 20.0 & 40.5 & 64.0 \end{bmatrix}$$

Computer Program for Matrix Inversion

```
C       PROBLEM 6-4 THOMSON
        REAL A(4,4),AINV(4,4),WKAREA(4)
        DO 10 I=1,4
        READ,(A(I,J),J=1,4)
     10 CONTINUE
        DO 16 I=1,4
        PRINT15,(A(I,J),J=1,4)
     15 FORMAT(10X,4F12.4)
     16 CONTINUE
        N=4
        IA=4
        IDGT=0
        CALL LINV1F (A,N,IA,AINV,IDGT,WKAREA,IER)
        DO 26 I=1,4
        PRINT25,(AINV(I,J),J=1,4)
     25 FORMAT(' ',10X,4F12.4)
     26 CONTINUE
        STOP
        END
```

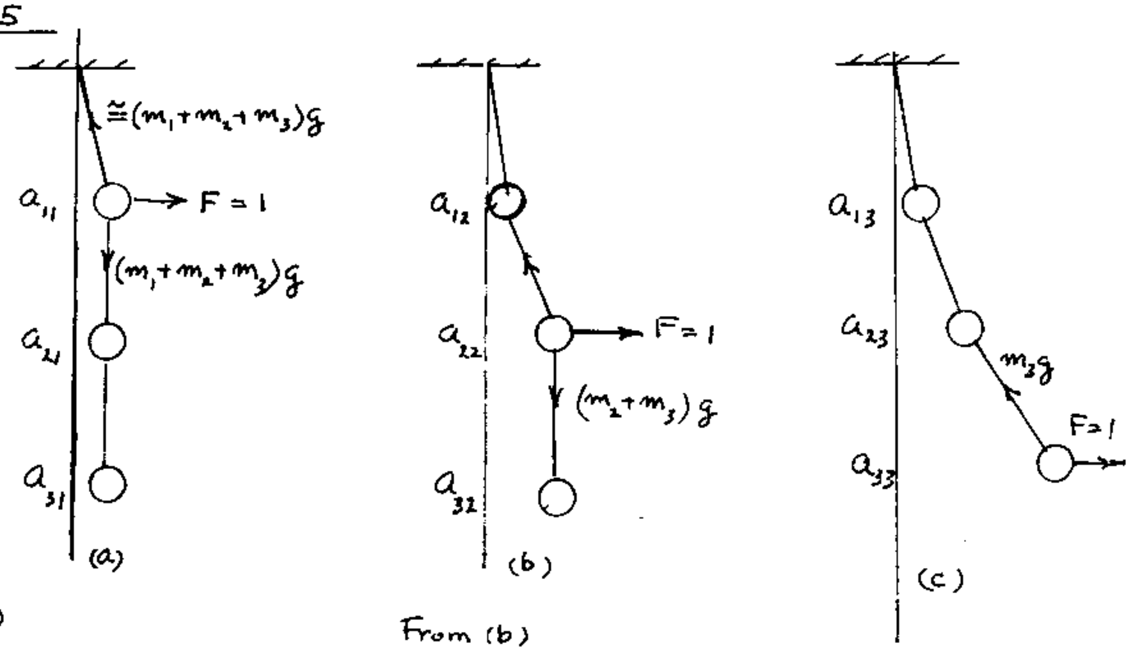
Original Matrix

1.0000	2.5000	4.0000	5.5000
2.5000	8.0000	14.0000	20.0000
4.0000	14.0000	27.0000	40.5000
5.5000	20.0000	40.5000	64.0000

Inverted Matrix

6.2680	-3.9381	1.4945	-0.2474
-3.9381	4.7835	-3.1959	0.8660
1.4945	-3.1959	3.2990	-1.2165
-0.2474	0.8660	-1.2165	0.5361

6-5



From (a)

$$\frac{a_{11}}{\ell_1}(m_1+m_2+m_3)g = 1$$

$$a_{11} = a_{21} = a_{31} = \frac{1}{m_1+m_2+m_3}\cdot\frac{\ell_1}{g}$$

From (b)

$$\left(\frac{a_{22}-a_{12}}{\ell_2}\right)(m_2+m_3)g = 1$$

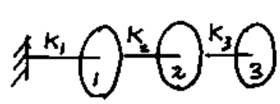
$$a_{22} = \frac{1}{m_2+m_3}\frac{\ell_2}{g} + \frac{1}{m_1+m_2+m_3}\frac{\ell_1}{g} = a_{32}$$

From (c)

$$\left(\frac{a_{33}-a_{23}}{\ell_3}\right)m_3 g = 1 \qquad a_{33} = \frac{1}{m_3}\frac{\ell_3}{g} + \frac{1}{m_2+m_3}\frac{\ell_2}{g} + \frac{1}{m_1+m_2+m_3}\frac{\ell_1}{g}$$

$$a_{ij} = a_{ji}$$

6-6



Give each disk a unit rotation holding other disks with zero rotation. Torque required is then:

$\theta_1 = 1.0 \quad T_1 = (K_1+K_2)\,, \quad T_0 = -K_1\,, \quad T_2 = -K_2\,, \quad T_3 = 0$

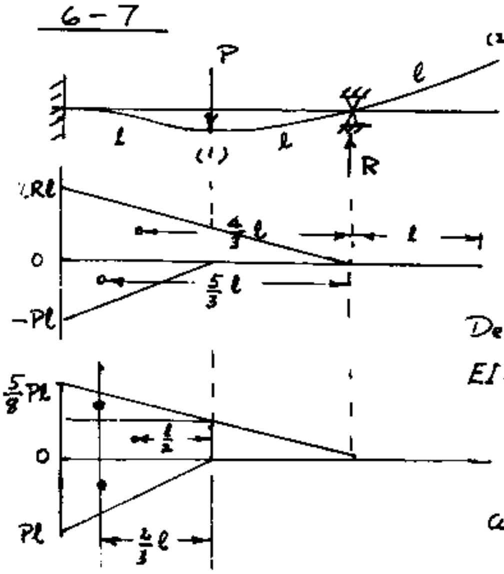
$\theta_2 = 1.0 \quad T_1 = -K_2\,, \quad T_2 = (K_2+K_3)\,, \quad T_3 = -K_3$

$\theta_3 = 1.0 \quad T_1 = 0\,, \quad T_2 = -K_3\,, \quad T_3 = K_3$

$$[K] = \begin{bmatrix} (K_1+K_2) & -K_2 & 0 \\ -K_2 & (K_2+K_3) & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix} \qquad [a] = \begin{bmatrix} 1/K_1 & 1/K_1 & 1/K_1 \\ 1/K_1 & (1/K_1+1/K_2) & (1/K_1+1/K_2) \\ 1/K_1 & (1/K_1+1/K_2) & (1/K_1+1/K_2+1/K_3) \end{bmatrix}$$

For flexibility apply unit torque to each disk in turn and measure rotation.

6-7



Defl. at R = 0

$$\frac{1}{2}(2Rl\times 2l)\frac{4}{3}l - \frac{1}{2}(Pl\times l)\frac{5}{3}l = 0$$

$$\therefore\ R = \frac{5}{16}P$$

$$2Rl = \frac{5}{8}Pl$$

Defl. at (1) = moment of area about (1)

$$EIy_1 = \left(\frac{5}{16}Pl\times l\right)\frac{l}{2} + \frac{1}{2}\left(\frac{5}{16}Pl^2\right)\frac{2}{3}l - \frac{1}{2}\left(Pl^2\right)\frac{2}{3}l$$

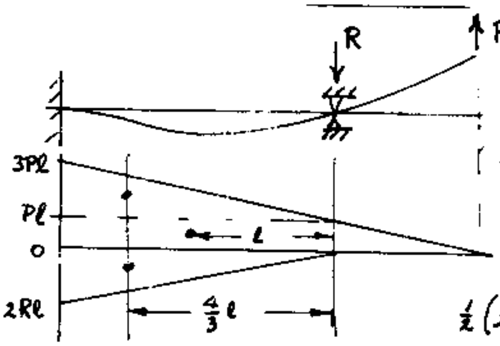
$$= Pl^3\left(\frac{5}{32} + \frac{5}{48} - \frac{1}{3}\right) = -\frac{3.5}{48}Pl^3$$

$$a_{11} = \frac{3.50}{48}\frac{l^3}{EI} = \frac{7}{96}\frac{l^3}{EI}$$

Defl. at (2) = moment of area about (2)

$$EIy_2 = \frac{1}{2}\left[\frac{5}{8}Pl\times 2l\right]\frac{7}{3}l - \frac{1}{2}\left[Pl^2\right]\frac{8}{3}l = Pl^3\left(\frac{35}{24} - \frac{8}{6}\right) = \frac{1}{8}Pl^3$$

$$\therefore\ a_{21} = a_{12} = \frac{1}{8}\frac{l^3}{EI}$$



Defl at R = 0

$$(2Pl^2)\,l + \frac{1}{2}(2Pl\times 2l)\frac{4}{3}l - \frac{1}{2}(2Rl\times 2l)\frac{4}{3}l = 0$$

$$Pl^3\left(2 + \frac{8}{3}\right) = Rl^3\left(\frac{8}{3}\right) \quad \therefore\ R = \frac{7}{4}P$$

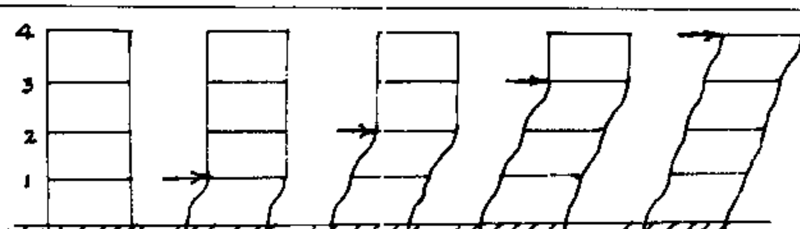
For a_{22}

$$\frac{1}{2}(3Pl\times 3l)2l - \frac{1}{2}\left(\frac{7}{2}Pl\times 2l\right)\left(l + \frac{4}{3}l\right) = EIy_2$$

$$Pl^3(9 - 8.1666) = \left(\frac{54}{6} - \frac{49}{6}\right)Pl^3 = \frac{5}{6}Pl^3 \qquad \therefore\ a_{22} = \frac{5}{6}\frac{l^3}{EI}$$

$$[a] = \frac{l^3}{EI}\begin{bmatrix} 7/96 & 1/8 \\ 1/8 & 5/6 \end{bmatrix}$$

6-8



$a_{11} = \frac{1}{12}\frac{l^3}{EI}$ $\quad a_{12} = \frac{1}{12}\frac{l^3}{EI}$ $\quad a_{13} = \frac{1}{12}\frac{l^3}{EI}$ $\quad a_{14} = \frac{1}{12}\frac{l^3}{EI}$

$a_{21} = \frac{1}{12}$ " $\quad a_{22} = \frac{2}{12}$ " $\quad a_{23} = \frac{2}{12}$ " $\quad a_{24} = \frac{2}{12}$ "

$a_{31} = \frac{1}{12}$ " $\quad a_{32} = \frac{2}{12}$ " $\quad a_{33} = \frac{3}{12}$ " $\quad a_{34} = \frac{3}{12}$ "

$a_{41} = \frac{1}{12}$ " $\quad a_{42} = \frac{2}{12}$ " $\quad a_{43} = \frac{3}{12}$ " $\quad a_{44} = \frac{4}{12}$ "

$$[a] = \frac{l^3}{12EI}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

6-8 Cont.

$$[k] = [a]^{-1}$$

$$D = \det.[a] = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 2(12-9) - 1(12-9) + 1(8-6) - 1(0) \quad - 2(8-6) + 2(4-3)$$
$$- 2(4-3) + 0 + 0 - 0 + 0 - 0 = 1$$

$$C_{11} = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{vmatrix} \quad C_{12} = -\begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{vmatrix} \quad C_{13} = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 0 \quad C_{14} = 0$$

$$C_{21} = -\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{vmatrix} \quad C_{22} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{vmatrix} \quad C_{23} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} \quad C_{24} = 0$$

$$C_{31} = 0 \quad C_{32} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{vmatrix} \quad C_{33} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{vmatrix} \quad C_{34} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

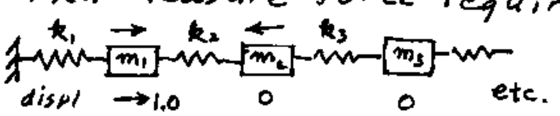
$$C_{41} = 0 \quad C_{42} = 0 \quad C_{43} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} \quad C_{44} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$$C_{11}' = C_{11} = 2, \quad C_{12}' = -1 \quad C_{21}' = -1, \quad C_{22}' = 2 \quad C_{23}' = -1$$

$$C_{32}' = -1 \quad C_{33}' = 2 \quad C_{34}' = -1 \quad C_{43}' = -1 \quad C_{44}' = 1$$

$$\therefore [k] = \frac{12EI}{l^3}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

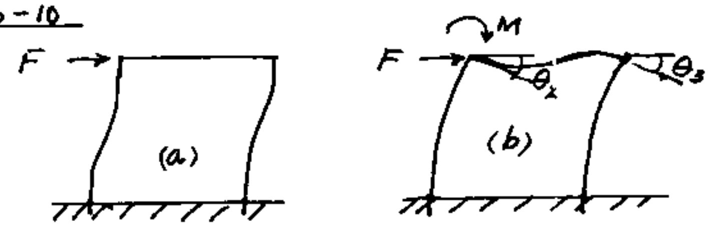
6-9 Give each mass a unit displ. keeping all others at zero. Then measure force required. ie for $x_1 = 1.0$



etc.

$$[k] = \begin{bmatrix} (k_1+k_2) & -k_2 & 0 & 0 & 0 & \cdots \\ -k_2 & (k_2+k_3) & -k_3 & 0 & 0 & \cdots \\ 0 & -k_3 & (k_3+k_4) & -k_4 & 0 & \cdots \\ 0 & 0 & -k_4 & (k_4+k_5) & -k_5 & \cdots \\ \vdots & & & & & \end{bmatrix}$$

6-10



(a) $k = 2 \times \dfrac{12EI}{\ell^3} = 24\dfrac{EI}{\ell^3}$

(b) $F = \dfrac{EI}{\ell^3}\left[24 - 6\theta_2\ell - 6\theta_3\ell\right]$ $\theta_2 = \theta_3 = \theta$

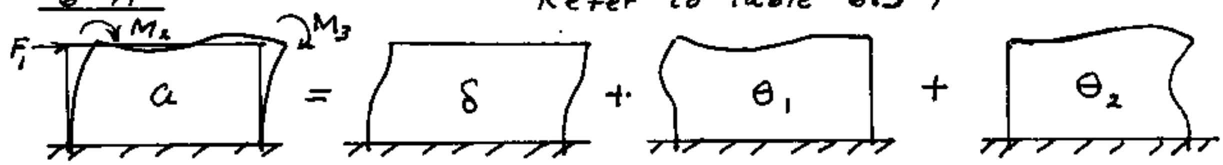
$M = \dfrac{EI}{\ell^2}\left[-6 + 8\theta_2\ell + 2\theta_3\ell\right] = 0$ (see Prob 6-11)

$-6 + 10\theta\ell = 0 \quad \therefore \theta = \dfrac{6}{10\ell}$

$F = \dfrac{EI}{\ell^3}\left[24 - 12 \times \dfrac{6}{10}\right] = 16.8\dfrac{EI}{\ell^3}$

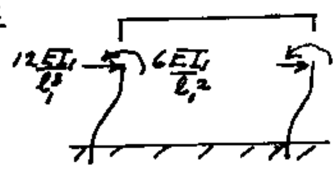
Ratio $\dfrac{(a)}{(b)} = \dfrac{24}{16.8}$

6-11 Refer to Table 6.3-1

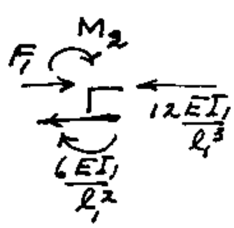


a is superposition of $\delta + \theta_1 + \theta_2$

δ mode

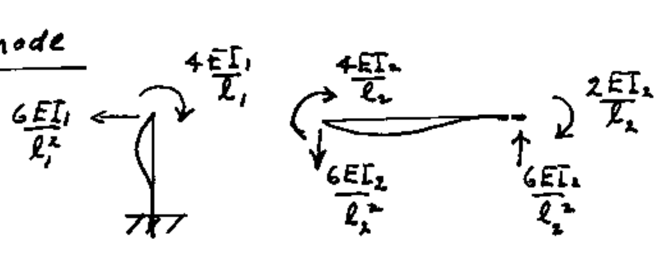


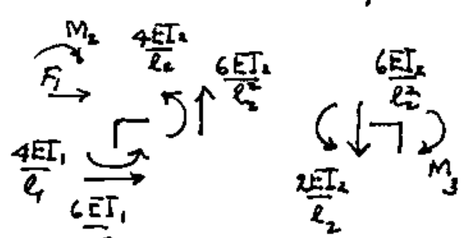
FBD of corner



$\therefore F_1 = 24\dfrac{EI_1}{\ell_1^3}\delta \qquad M_2 = M_3 = -6\dfrac{EI_1}{\ell_1^2}\delta$

θ_1 mode



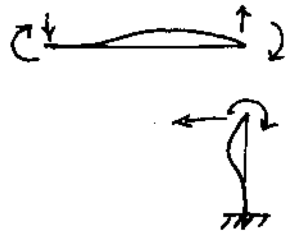


FBD of corners

$F_1 = -\dfrac{6EI_1}{\ell_1^2}\theta_1, \qquad M_2 = \left(\dfrac{4EI_1}{\ell_1} + \dfrac{4EI_2}{\ell_2}\right)\theta_1, \qquad M_3 = \left(\dfrac{2EI_2}{\ell_2}\right)\theta_1$

6-11 Cont.

θ_2 mode

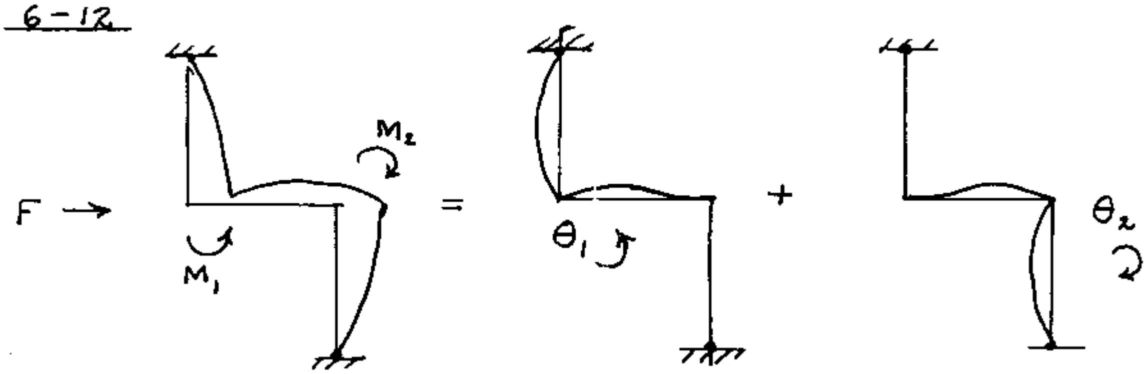


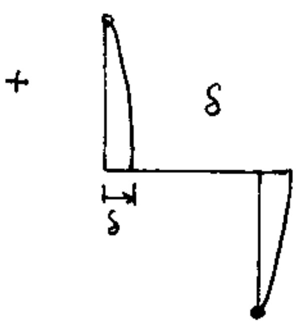
Same as mode θ_1, but shifted to right corner

Sum results for δ, θ_1 & θ_2

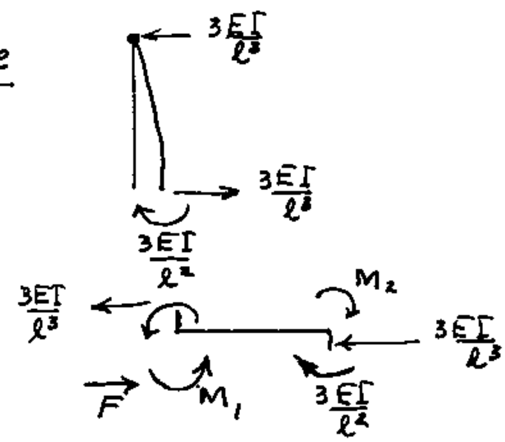
$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{24EI_1}{\ell_1^3} & -6\frac{EI_1}{\ell_1^2} & -6\frac{EI_1}{\ell_1^2} \\ -6\frac{EI_1}{\ell_1^2} & \left(\frac{4EI_1}{\ell_1}+\frac{4EI_2}{\ell_2}\right) & \frac{2EI_2}{\ell_2} \\ -6\frac{EI_1}{\ell_1^2} & \frac{2EI_2}{\ell_2} & \left(\frac{4EI_1}{\ell_1}+\frac{4EI_2}{\ell_2}\right) \end{bmatrix} \begin{Bmatrix} \delta \\ \theta_1 \\ \theta_2 \end{Bmatrix}$$

6-12





δ-mode



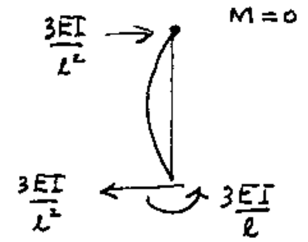
$$F = 6\frac{EI}{\ell^3}\delta$$

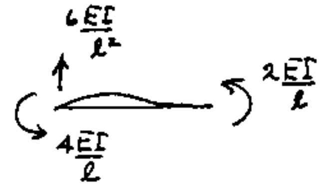
$$M_1 = -3\frac{EI}{\ell^2}\delta$$

$$M_2 = -3\frac{EI}{\ell^2}\delta$$

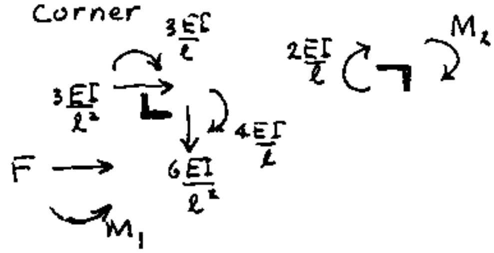
from corners

θ_1-mode





Corner



$$F = -\frac{3EI}{\ell^2}\theta_1$$

$$M_1 = \left(\frac{3EI}{\ell} + \frac{4EI}{\ell}\right)(\theta_1)$$

$$M_2 = -\frac{2EI}{\ell}(\theta_1)$$

θ_2-mode this mode is same as above moved to right corner with θ_1 replaced by $+\theta_2$

Add results

$$\begin{Bmatrix} F \\ M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{6EI}{\ell^3} & -\frac{3EI}{\ell^2} & -\frac{3EI}{\ell^2} \\ -\frac{3EI}{\ell^2} & \left(\frac{3EI}{\ell} + \frac{4EI}{\ell}\right) & -\frac{2EI}{\ell} \\ -\frac{3EI}{\ell^2} & -\frac{2EI}{\ell} & \left(\frac{3EI}{\ell} + \frac{4EI}{\ell}\right) \end{bmatrix} \begin{Bmatrix} \delta \\ \theta_1 \\ \theta_2 \end{Bmatrix}$$

Since no moments were applied at corners $M_1 = M_2 = 0$

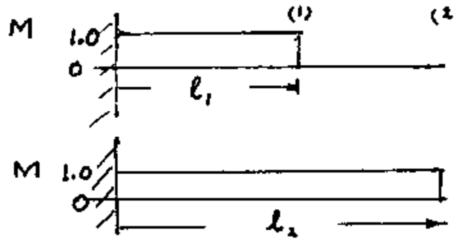
$$\left.\begin{aligned} 0 &= -\frac{3EI}{\ell^2}\delta + \frac{7EI}{\ell}\theta_1 - \frac{2EI}{\ell}\theta_2 \\ 0 &= -\frac{3EI}{\ell^2}\delta - \frac{2EI}{\ell}\theta_1 + \frac{7EI}{\ell}\theta_2 \end{aligned}\right\}$$

Subtr. we find $\theta_1 = \theta_2 = \theta$ as expected by symmetry

$$M_1 = -\frac{3EI}{\ell^2}\delta + \frac{5EI}{\ell}\theta = 0 \qquad \therefore\ \theta = \frac{3\delta}{5\ell}$$

$$F = \frac{6EI}{\ell^3}\delta - \frac{6EI}{\ell^2}\left(\frac{3\delta}{5\ell}\right) = \underline{\underline{2.40\,\frac{EI}{\ell^3}\,\delta}}$$

6 – 13 Moment Diagram



$$EI\,\theta_{21} = EI\,\theta_{12} = M_1\ell_1 = M_2\ell_2$$

$$M_1 = M_2 = 1$$

$$\therefore\ \theta_{12} = \theta_{21}$$

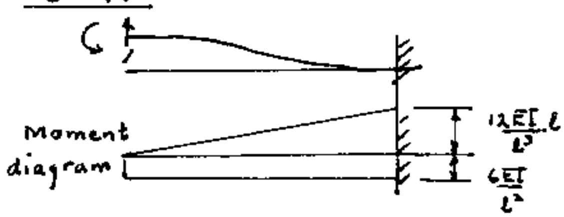
By Work done

M_1 1st followed by M_2 $\quad W = \frac{1}{2}M_1\theta_{11} + \frac{1}{2}M_2\theta_{22} + M_1\theta_{12}$

M_2 1st " " M_1 $\quad W = \frac{1}{2}M_2\theta_{22} + \frac{1}{2}M_1\theta_{11} + M_2\theta_{21}$

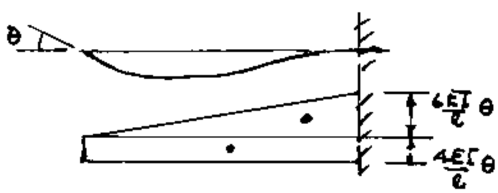
---

6 – 14



Difference in slope $= 0$

$$\left(\frac{1}{2}\cdot\frac{12EI}{\ell^3}\cdot\ell\right) - \left(\frac{6EI}{\ell^2}\cdot\ell\right) = 0$$

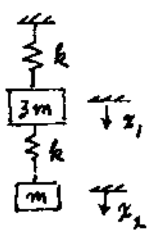


Difference in defl. $= 0$

$$\left(\frac{1}{2}\ \frac{6EI}{\ell}\right)\frac{2}{3}\ell - \left(\frac{4EI}{\ell}\cdot\ell\right)\frac{\ell}{2} = 0$$

---

6 – 15



$$m\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix} + k\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} = \begin{Bmatrix}0\end{Bmatrix} \qquad \text{Let } \lambda = \frac{m\omega^2}{k}$$

$$\begin{vmatrix}(2-3\lambda) & -1\\ -1 & (1-\lambda)\end{vmatrix} = 0 \qquad \lambda^2 - \frac{5}{3}\lambda + \frac{1}{3} = 0$$

$$\text{Adj. matrix} = \begin{bmatrix}(1-\lambda) & 1\\ 1 & (2-3\lambda)\end{bmatrix}$$

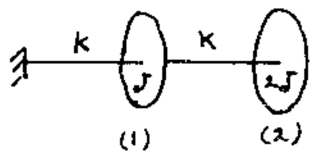
$$\lambda = 0.8333 \pm \sqrt{.6944 - .3333}$$

$$= 0.8333 \pm 0.6010 = \begin{cases}0.2323\\ 1.4343\end{cases}$$

Subst. λ_1 into either column for 1st mode $= \begin{Bmatrix}.7677\\ 1.000\end{Bmatrix}$

Subst λ_2 into " " " 2nd mode $= \begin{Bmatrix}-.4343\\ 1.000\end{Bmatrix}$

6-16



$$\begin{bmatrix} J & 0 \\ 0 & 2J \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \{0\}$$

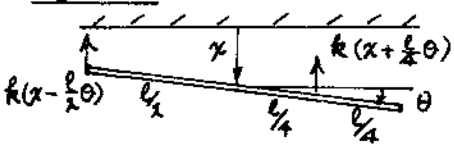
Let $\lambda = \frac{\omega^2 J}{K}$

$$\begin{vmatrix} (2-\lambda) & -1 \\ -1 & (1-2\lambda) \end{vmatrix} = 0 \qquad 2\lambda^2 - 5\lambda + 1 = 0$$

$$\lambda = \frac{5}{4} \pm \sqrt{\frac{25}{16} - \frac{8}{16}} = \begin{cases} 0.2192 \\ 2.2808 \end{cases}$$

$$\text{Adj. matrix} = \begin{bmatrix} (1-2\lambda) & 1 \\ 1 & (2-\lambda) \end{bmatrix} \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1.000 \\ 1.781 \end{Bmatrix} \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1.00 \\ -.2808 \end{Bmatrix}$$

6-17



$$m\frac{\ell^2}{12}\ddot{\theta} = \frac{\ell}{2}k(x - \tfrac{\ell}{2}\theta) - \frac{\ell}{4}k(x + \tfrac{\ell}{4}\theta) = \frac{k\ell}{4}x - \frac{5\ell^2}{16}k\theta$$

$$m\ddot{x} = -k(x - \tfrac{\ell}{2}\theta) - k(x + \tfrac{\ell}{4}\theta) = -2kx + \frac{k\ell}{4}\theta$$

$$m\begin{bmatrix} \frac{\ell^2}{12} & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{x} \end{Bmatrix} + k\begin{bmatrix} \frac{5}{16}\ell^2 & -\frac{\ell}{4} \\ -\frac{\ell}{4} & 2 \end{bmatrix}\begin{Bmatrix} \theta \\ x \end{Bmatrix} = \{0\} \qquad \text{Let } \lambda = \frac{\omega^2 m}{k}$$

$$\begin{vmatrix} (\frac{5}{16}\ell^2 - \frac{\ell^2}{12}\lambda) & -\frac{\ell}{4} \\ -\frac{\ell}{4} & (2-\lambda) \end{vmatrix} = 0 \qquad \lambda^2 - \frac{23}{4}\lambda + \frac{9\times 12}{16} = 0$$

$$\lambda = \frac{23}{8} \pm \sqrt{\frac{97}{16}}$$

$$\lambda = \frac{1}{8}(23 \pm 9.849) = \begin{cases} 1.644 \\ 4.106 \end{cases} \qquad \frac{\ell\theta}{x} = 4(2-\lambda) = \begin{cases} 1.424 \\ -8.424 \end{cases}$$

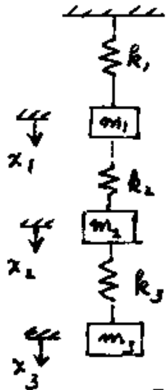
$$\begin{Bmatrix} \theta \\ x \end{Bmatrix}_1 = \begin{Bmatrix} 1.424/\ell \\ 1.000 \end{Bmatrix} \qquad \begin{Bmatrix} \theta \\ x \end{Bmatrix}_2 = \begin{Bmatrix} -8.424/\ell \\ 1.000 \end{Bmatrix}$$

$$P = \begin{bmatrix} 1.424/\ell & -8.424/\ell \\ 1.000 & 1.000 \end{bmatrix}$$ To show that $P'KP$ = diagonal

$$\begin{bmatrix} 1.424/\ell & 1.0 \\ -8.424/\ell & 1.0 \end{bmatrix}\begin{bmatrix} \frac{5}{16}\ell^2 & -\frac{\ell}{4} \\ -\frac{\ell}{4} & 2 \end{bmatrix}\begin{bmatrix} 1.424/\ell & -8.424/\ell \\ 1.000 & 1.000 \end{bmatrix} = \begin{bmatrix} 1.424/\ell & 1.0 \\ -8.424/\ell & 1.0 \end{bmatrix}\begin{bmatrix} .195\ell & -2.8825 \\ 1.644 & 4.106 \end{bmatrix}$$

$$= \begin{bmatrix} 1.9217 & 0.0013 \\ 0.0013 & 28.388 \end{bmatrix} \cong \begin{bmatrix} 1.9217 & 0 \\ 0 & 28.388 \end{bmatrix}$$

6-18



Place unit load at 1.

$$a_{11} = \frac{1}{k_1} = a_{21} = a_{31} = a_{12} = a_{13}$$

Place unit load at 2

$$a_{22} = \left(\frac{1}{k_1} + \frac{1}{k_2}\right) = a_{32} = a_{23}$$

Place unit load at 3

$$a_{33} = \left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right)$$

Eq. of motion:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \omega^2 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

6-19

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix} \qquad \text{Let } \lambda = \frac{m\omega^2}{k}$$

$$\begin{vmatrix} (2-\lambda) & -1 \\ -1 & (2-2\lambda) \end{vmatrix} = 0 \qquad \lambda^2 - 3\lambda + \frac{3}{2} = 0$$

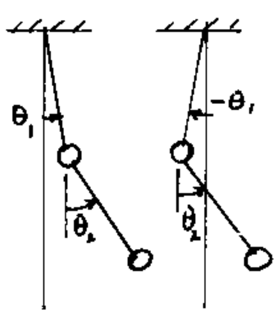
$$\lambda = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{3}{2}} = 1.5 \pm 0.866 = \begin{cases} 0.634 \\ 2.366 \end{cases}$$

$$\frac{x_1}{x_2} = \frac{2(1-\lambda)}{1} = \begin{cases} 0.732 \\ -2.732 \end{cases} \qquad \therefore P = \begin{bmatrix} 0.732 & -2.732 \\ 1.000 & 1.000 \end{bmatrix}$$

$$P'MP\,\ddot{y} + P'KP\,y = 0$$

$$\begin{bmatrix} 2.535 & 0 \\ 0 & 9.48 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} 1.606 & 0 \\ 0 & 22.33 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix} \quad \text{uncoupled.}$$

6-20



$$\frac{l}{g} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0$$

$\therefore$ dynamically coupled.

From Prob. 5-10

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.707 \\ 1.00 \end{Bmatrix} \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} -0.707 \\ 1.00 \end{Bmatrix}$$

6-20 Cont.

Gen. mass

mode 1 $\quad (.707 \;\; 1.0)\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} .707 \\ 1.0 \end{Bmatrix} = 3.414$

mode 2 $\quad (-.707 \;\; 1.0)\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} = 0.586$

$$P = \begin{bmatrix} .707 & -.707 \\ 1.0 & 1.0 \end{bmatrix} \qquad \tilde{P} = \begin{bmatrix} .207 & -1.207 \\ .293 & 1.707 \end{bmatrix}$$

$$\tilde{P}' M \tilde{P} = \begin{bmatrix} .207 & .293 \\ -1.207 & 1.707 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} .207 & -1.207 \\ .293 & 1.707 \end{bmatrix} = \begin{bmatrix} 0.293 & 0 \\ 0 & 1.707 \end{bmatrix}$$

$$\tilde{P}' K \tilde{P} = \begin{bmatrix} & " & \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} & " & \end{bmatrix} = \begin{bmatrix} .1714 & 0 \\ 0 & 5.82 \end{bmatrix}$$

Normalized eq. of motion becomes

$$\begin{bmatrix} .293 & 0 \\ 0 & 1.707 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} .1714 & 0 \\ 0 & 5.82 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix} \quad \therefore \text{ uncoupled}$$

6-21 $\qquad$ See Prob. 6-11

$$\begin{bmatrix} (m_1+m_2) & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} \frac{24EI_1}{\ell_1^3} & -6\frac{EI_1}{\ell_1^2} & -6\frac{EI_1}{\ell_1^2} \\ -6\frac{EI_1}{\ell_1^2} & \left(\frac{4EI_1}{\ell_1}+\frac{4EI_2}{\ell_2}\right) & 2\frac{EI_2}{\ell_2} \\ -6\frac{EI_1}{\ell_1^2} & 2\frac{EI_2}{\ell_2} & \left(\frac{4EI_1}{\ell_1}+\frac{4EI_2}{\ell_2}\right) \end{bmatrix}\begin{Bmatrix} x \\ \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

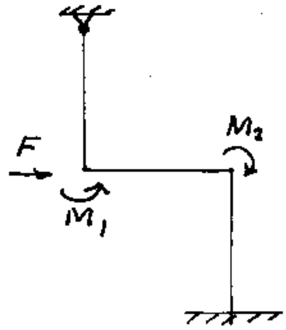
Assign numbers before solving.

6-22 $\qquad$ See Prob 6-12

$$\begin{bmatrix} (m_1+m_2) & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} \frac{6EI}{\ell^3} & -3\frac{EI}{\ell^2} & -3\frac{EI}{\ell^2} \\ -3\frac{EI}{\ell^2} & 7\frac{EI}{\ell} & -2\frac{EI}{\ell} \\ -3\frac{EI}{\ell^2} & -2\frac{EI}{\ell} & 7\frac{EI}{\ell} \end{bmatrix}\begin{Bmatrix} x \\ \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

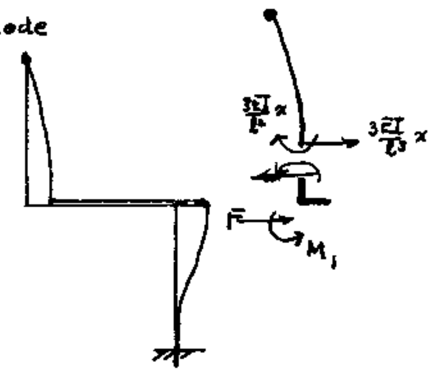
Assign numbers before solving

6-23

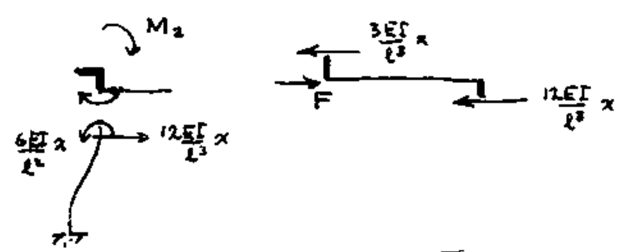


The problem is the superposition of the three modes below. Use Table 6.3-1 and examine the free-body-diagrams of members and corners.

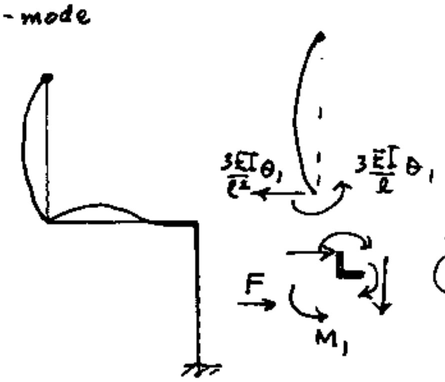
x-mode



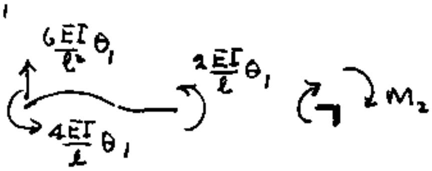
$$F = \frac{15EI}{\ell^3}x \qquad M_1 = -\frac{3EI}{\ell^2}x, \qquad M_2 = \frac{6EI}{\ell^2}x$$



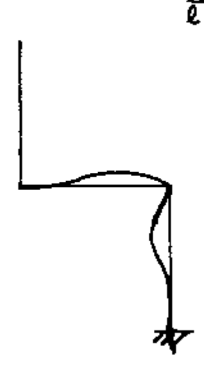
θ_1-mode

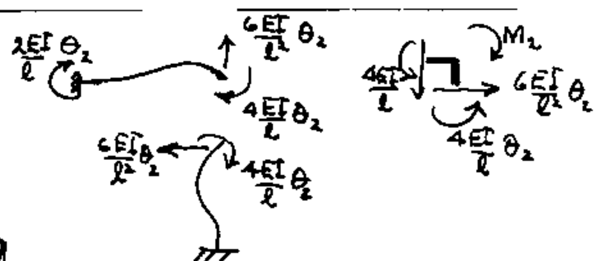


$$F = -\frac{3EI}{\ell^2}\theta_1, \qquad M_1 = \frac{7EI}{\ell}\theta_1, \qquad M_2 = -\frac{2EI}{\ell}\theta_1$$



θ_2-mode





$$\begin{Bmatrix} F \\ M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{15EI}{\ell^3} & -\frac{3EI}{\ell^2} & \frac{6EI}{\ell^2} \\ -\frac{3EI}{\ell^2} & \frac{7EI}{\ell} & -\frac{2EI}{\ell} \\ \frac{6EI}{\ell^2} & -\frac{2EI}{\ell} & \frac{8EI}{\ell} \end{bmatrix} \begin{Bmatrix} x \\ \theta_1 \\ \theta_2 \end{Bmatrix}$$

6-24

$$m\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\\ddot{x}_2\end{Bmatrix} + c\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}\dot{x}_1\\\dot{x}_2\end{Bmatrix} + k\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\x_2\end{Bmatrix} = \begin{Bmatrix}F_0\\0\end{Bmatrix}\sin\omega t$$

$\therefore$ Damping is not proportional

6-25

char. eq. $\lambda = \frac{\omega^2 m}{k}$

$$\begin{vmatrix}(2-\lambda) & -1\\-1 & (2-\lambda)\end{vmatrix} = 0 \qquad \lambda = \begin{cases}1\\3\end{cases} \qquad \begin{Bmatrix}x_1\\x_2\end{Bmatrix}_1 = \begin{Bmatrix}1\\1\end{Bmatrix}$$

$$\begin{Bmatrix}x_1\\x_2\end{Bmatrix}_2 = \begin{Bmatrix}-1\\1\end{Bmatrix}$$

$$\therefore\ P = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} \qquad \tilde{P} = \frac{1}{\sqrt{2m}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

Let $x = \tilde{P}y$ in eq. Prob. 6-24 and premult. by $\tilde{P}'$

$$\tilde{P}'M\tilde{P}\ddot{y} + \tilde{P}'C\tilde{P}\dot{y} + \tilde{P}'k\tilde{P}y = \tilde{P}'F$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{y}_1\\\ddot{y}_2\end{Bmatrix} + \frac{c}{2m}\begin{bmatrix}1 & -1\\-1 & 5\end{bmatrix}\begin{Bmatrix}\dot{y}_1\\\dot{y}_2\end{Bmatrix} + \frac{k}{m}\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{Bmatrix}y_1\\y_2\end{Bmatrix} = \begin{Bmatrix}F_0\\-F_0\end{Bmatrix}\frac{\sin\omega t}{\sqrt{2m}}$$

$\therefore$ coupled only by damping

$$\ddot{y}_1 + \frac{c}{2m}(\dot{y}_1 - \dot{y}_2) + \frac{k}{m}y_1 = \frac{F_0}{\sqrt{2m}}\sin\omega t$$

$$\ddot{y}_2 + \frac{c}{2m}(-\dot{y}_1 + 5\dot{y}_2) + 3\frac{k}{m}y_2 = \frac{-F_0}{\sqrt{2m}}\sin\omega t$$

Steady State solution

$$\left(\frac{k}{m} - \omega^2 + i\frac{c}{2m}\omega\right)Y_1 - i\left(\frac{c}{2m}\omega\right)Y_2 = \frac{F_0}{\sqrt{2m}}$$

$$-i\left(\frac{c}{2m}\omega\right)Y_1 + \left(\frac{3k}{m} - \omega^2 + i\frac{5c}{2m}\omega\right)Y_2 = \frac{-F_0}{\sqrt{2m}}$$

$$Y_1 = \frac{\begin{vmatrix}1 & i\omega\frac{c}{2m}\\-1 & \left(\frac{3k}{m} - \omega^2 + i\omega\frac{5c}{2m}\right)\end{vmatrix}\frac{F_0}{\sqrt{2m}}}{\left(\frac{k}{m} - \omega^2 + i\frac{c}{2m}\omega\right)\left(\frac{3k}{m} - \omega^2 + i\omega\frac{5c}{2m}\right) + \left(\frac{\omega c}{2m}\right)^2}$$

6-26

Let. $\omega_0^2 = \dfrac{k}{m}$, $\alpha = \dfrac{k_1}{c}$, $\beta = \dfrac{k_1}{m}$

The equations are rewritten as

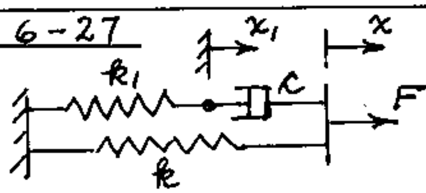
$$\ddot{x} = -\omega_0^2 x - \beta x_1 + \frac{F}{m}$$

$$\dot{x}_1 = \dot{x} - \alpha x_1$$

Let $x_1 = z_1$, $\dot{x}_1 = \dot{z}_1$, $x = z_2$, $\dot{x} = z_3 = \dot{z}_2$, then

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{Bmatrix} = \begin{bmatrix} -\alpha & 0 & 1 \\ 0 & 0 & 1 \\ -\beta & -\omega_0^2 & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \frac{F}{m} \end{Bmatrix}$$

6-27



$$F = kx + c(\dot{x} - \dot{x}_1) \qquad (a)$$

$$k_1 x_1 = c(\dot{x} - \dot{x}_1) \qquad (b)$$

Assume F to be harmonic; then from (b)

$$x_1 = \frac{i\omega c}{k_1 + i\omega c}\, x = \frac{i(\omega c/k_1)}{1 + i(\omega c/k_1)}\, x$$

Subst. into (a)

$$F = kx + i\omega c\left[1 - \frac{i(\omega c/k_1)}{1 + i(\omega c/k_1)}\right] x$$

$$= \frac{\left[k\left(1 + \frac{i\omega c}{k_1}\right) + i\omega c\right]}{\left(1 + \frac{i\omega c}{k_1}\right)} \cdot \frac{\left(1 - \frac{i\omega c}{k_1}\right)}{\left(1 - \frac{i\omega c}{k_1}\right)}\, x$$

$$= \left\{ \frac{k + (k + k_1)\left(\frac{\omega c}{k_1}\right)^2}{1 + \left(\frac{\omega c}{k_1}\right)^2} + \frac{i\omega c}{1 + \left(\frac{\omega c}{k_1}\right)^2} \right\} x$$

$$= \left\{ k_{eq} + i\omega\, c_{eq} \right\} x$$

6 - 28 Refer to Prob. 6-16

$$X_1 = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1.00 \\ 1.781 \end{Bmatrix} \qquad X_2 = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1.00 \\ -0.2808 \end{Bmatrix}$$

$$X_1' K X_2 = (1.00 \quad 1.781)\, k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1.00 \\ -0.2808 \end{Bmatrix}$$

$$= (1.00 \quad 1.781) \begin{Bmatrix} 2.2808 \\ -1.2808 \end{Bmatrix} = 2.2808 - 2.2811$$

$$= -0.0003 \cong 0$$

---

6 - 29

ϕ = normal mode

$$k\phi_s = \omega_s^2 M \phi_s \qquad \times \text{ by } KM^{-1}$$

$$KM^{-1}K\phi_s = \omega_s^2 KM^{-1}M\phi_s = \omega_s^2 K\phi_s$$

$$\underline{\underline{\phi_n'[KM^{-1}K]\phi_s}} = \omega_s^2[\phi_n' K \phi_s] = \underline{\underline{0}} \quad \text{for } n \neq s$$

Repeat $KM^{-1}K\phi_s = \omega_s^2 K\phi_s$ $\times$ by KM^{-1}

$$KM^{-1}KM^{-1}K\phi_s = \omega_s^2 [KM^{-1}K]\phi_s \qquad \times \text{ by } \phi_n'$$

$$\underline{\underline{\phi_n'[KM^{-1}]^2 K\phi_s}} = \omega_s^2 \phi_n' [KM^{-1}K]\phi_s = \underline{\underline{0}} \quad n \neq s$$

Repeat to obtain $\phi_n'[KM^{-1}]^h K\phi_s = 0$

---

6 - 30

$$K\phi_s = \omega_s^2 M\phi_s, \qquad M\underline{K^{-1}K}\phi_s = \omega_s^2 MK^{-1}M\phi_s$$

$$\therefore \; M\phi_s = \omega_s^2 MK^{-1}M\phi_s \qquad \text{ie } K^{-1}K = 1$$

$$\phi_n' M\phi_s = \omega_s^2 \phi_n'[MK^{-1}M]\phi_s \underset{n \neq s}{=} 0 \qquad \text{ie } \phi_n' M \phi_s \underset{n \neq s}{=} 0$$

Repeat

$$MK^{-1}M\phi_s = \omega_s^2 [MK^{-1}]^2 M\phi_s$$

$$\phi_n' MK^{-1}M\phi_s = \underline{\underline{\omega_s^2 \phi_n' [MK^{-1}]^2 M\phi_s}} = \underline{\underline{0}} \qquad n \neq s$$

Repeat h times.

6-31 Refer to Ex. 6.9-1 for ϕ_i & ω_i

For second mode

$$m_{22}\ddot{q}_2 + c_{22}\dot{q}_2 + k_{22} q_2 = -\ddot{u}_0(t)\sum_{i=1}^{10} m_i \phi_2(x_i)$$

$$m_{22} = \sum_{i=1}^{10} m_i \phi_2^2(x_i) = 5.5235\, m$$

$$c_{22} = 2\zeta_2\,\omega_2\,m_{22} = 2\zeta_2\left(0.4451\sqrt{\tfrac{k}{m}}\right)m_{22} = 0.8902\,\zeta_2\sqrt{\tfrac{k}{m}}\; m_{22}$$

$$k_{22} = \omega_2^2 m_{22} = \left(0.1981\,\tfrac{k}{m}\right) m_{22}$$

$$\sum_{i=1}^{10} m_i\phi_2(x_i) = -2.2470\, m$$

$$\therefore \quad \ddot{q}_2 + 0.8902\,\zeta_2\sqrt{\tfrac{k}{m}}\,\dot{q}_2 + 0.1981\,\tfrac{k}{m}\,q_2 = \frac{2.247}{5.5235}\,\ddot{u}_0(t)$$

$$= 0.4068\,\ddot{u}_0(t)$$

For 3rd mode

$$m_{33} = 8.5957\, m, \quad c_{33}/m_{33} = 2\zeta_3\left(.7307\sqrt{\tfrac{k}{m}}\right) = 1.4614\zeta_2\sqrt{\tfrac{k}{m}}$$

$$k_{33} = 0.5339\,\tfrac{k}{m}, \quad \sum_{i=1}^{10} m_i\phi_3(x_i) = 2.8095\, m$$

$$\ddot{q}_3 + 1.4614\,\zeta_3\sqrt{\tfrac{k}{m}}\,\dot{q}_3 + 0.5339\,\tfrac{k}{m}\,q_3 = -0.3268\,\ddot{u}_0(t)$$

6-32 From Ex. 6.9-1

$$\ddot{q}_1 + 0.299\,\zeta_1\sqrt{\tfrac{k}{m}}\,\dot{q}_1 + 0.02235\,\tfrac{k}{m}\,q_1 = -1.2672\,\ddot{u}_0(t)$$

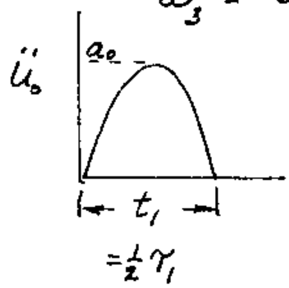
$$\omega_1^2 = 0.02235\,\tfrac{k}{m}, \quad \omega_1 = 0.1495\sqrt{\tfrac{k}{m}} = \frac{2\pi}{\tau_1}$$

$$\therefore \tau_1 = \frac{2\pi}{.1495}\sqrt{\tfrac{m}{k}} = 42.028\sqrt{\tfrac{m}{k}}$$

From Ex. 6.9-1

$$\omega_2 = \frac{2\pi}{\tau_2} = 0.4451\sqrt{\frac{k}{m}}\,, \qquad \tau_2 = 14.1168\sqrt{\frac{m}{k}}$$

$$\omega_3 = 0.7307\sqrt{\frac{k}{m}}\,, \qquad \tau_3 = 8.5989\sqrt{\frac{m}{k}}$$



See FIG 4.4-3 for shock spectrum

$$\frac{t_1}{\tau_1} = 0.50 \qquad \left(\frac{xk}{F_o}\right)_{max} = 1.5$$

$$\frac{t_1}{\tau_1}\frac{\tau_1}{\tau_2} = \frac{t_1}{\tau_2} = .5\,\frac{42.028}{14.116} = 1.4886 \quad \therefore \left(\frac{xk}{F_o}\right)_{max} = 1.5$$

$$\frac{t_1}{\tau_3} = .50\,\frac{42.028}{8.5989} = 2.4438 \quad \therefore \left(\frac{xk}{F_o}\right)_{max} = 1.13$$

The right side of the DEs are:

$$-\ddot{u}_o(t)\,\frac{\sum m\phi_1}{\sum m\phi_1^2} = -1.2672\,\ddot{u}_o = \frac{F_o}{m} \quad \text{by comparison}$$

$$\text{with } \ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = \frac{F_o}{m}$$

$$\therefore F_o = -1.2672\, m\, a_o \qquad \therefore \text{ in place of } \left(\frac{xk}{F_o}\right)_{max}$$

we use $\left(\frac{qk}{F_o}\right)_{max} = \frac{qk}{-1.2672\, m a_o}$ for mode 1. $= 1.5$

$$\therefore (q_1)_{max} = -1.5 \times 1.2672\,\frac{m}{k}a_o = \underline{\underline{-1.9008\,\frac{m a_o}{k}}}$$

Similarly for 2nd mode & 3rd mode

$$(q_2)_{max} = 1.5 \times .4068\,\frac{m a_o}{k} = \underline{\underline{0.6102\,\frac{m a_o}{k}}}$$

$$(q_3)_{max} = 1.13 \times (-.3268)\,\frac{m a_o}{k} = \underline{\underline{-0.3693\,\frac{m a_o}{k}}}$$

6-32 Cont:

$$x(t) = \phi_1 q_1 + \phi_2 q_2 + \phi_3 q_3 + \cdots$$

but at 10^{th} floor $\phi_i = 1.0$

$$\therefore x(t) = q_1 + q_2 + q_3$$

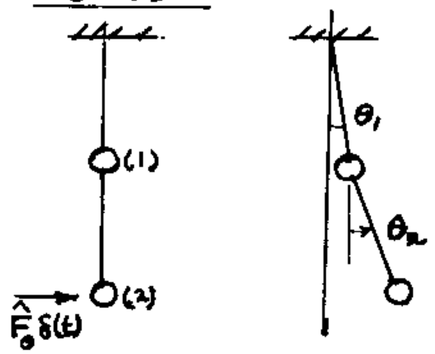
From Eq. (6.9-6)

$$|x(10)|_{max} = (q_1)_{max} + \sqrt{(q_2)_{max}^2 + (q_3)_{max}^2}$$

$$= 1.90 + \sqrt{.610^2 + .369^2}$$

$$= 1.90 + 0.711 = \underline{\underline{2.61}} \; \frac{ma_0}{k}$$

6-33



Impulse = change in momentum

At $t=0$ mass at (2) will acquire a velocity of $\frac{\hat{F}_0}{m} = v(0) = \ell\dot{\theta}_2(0)$

$$\therefore \dot{\theta}_2(0) = \frac{\hat{F}}{m\ell} \qquad \dot{\theta}_1(0) = 0$$

$$\theta = \phi_1 q_1 + \phi_2 q_2$$

$$\therefore \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} .707 \\ 1.00 \end{Bmatrix} q_1 + \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} q_2$$

$$= \begin{Bmatrix} .707 \\ 1.00 \end{Bmatrix} A \sin .764\sqrt{\frac{g}{\ell}}\,t + \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} B \sin 1.85\sqrt{\frac{g}{\ell}}\,t$$

$$\sqrt{\frac{\ell}{g}} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = .764 \begin{Bmatrix} .707 \\ 1.00 \end{Bmatrix} A \cos .764\sqrt{\frac{g}{\ell}}\,t + 1.85 \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} B \cos 1.85\sqrt{\frac{g}{\ell}}\,t$$

At $t = 0$

$$\sqrt{\frac{\ell}{g}} \begin{Bmatrix} 0 \\ \frac{\hat{F}}{m\ell} \end{Bmatrix} = .764 \begin{Bmatrix} .707 \\ 1.00 \end{Bmatrix} A + 1.85 \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} B$$

6-33 Cont

$$\therefore \quad 0 = (.764)\cancel{(.707)}A - (1.85)\cancel{(.707)}B$$

$$B = 0.413\,A$$

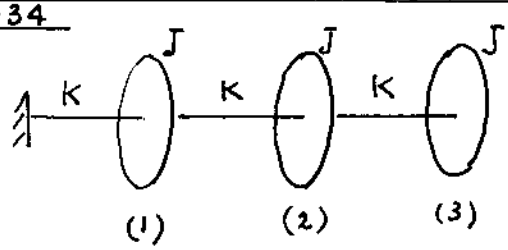
$$\sqrt{\frac{\ell}{g}}\,\frac{\hat{F}_o}{m\ell} = .764\,A + 1.85\,(.413\,A) = 1.528\,A$$

$$\therefore \quad A = 0.6544\sqrt{\frac{\ell}{g}}\,\frac{\hat{F}}{m\ell}$$

$$B = 0.2703\sqrt{\frac{\ell}{g}}\,\frac{\hat{F}}{m\ell}$$

$$\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \sqrt{\frac{\ell}{g}}\,\frac{\hat{F}}{m\ell}\left[(0.6544)\begin{Bmatrix}.707\\ 1.00\end{Bmatrix}\sin .764\sqrt{\frac{g}{\ell}}\,t + (.2703)\begin{Bmatrix}-.707\\ 1.00\end{Bmatrix}\sin 1.85\sqrt{\frac{g}{\ell}}\,t\right]$$

6-34



$$\lambda = \frac{J\omega^2}{K}$$

$$\phi = \begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix}$$

$$J\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{\theta}_3\end{Bmatrix} + K\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ M_o u(t)\end{Bmatrix}$$

Let $\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} = P\begin{Bmatrix}q_1\\ q_2\\ q_3\end{Bmatrix}$ & premultiply by P'

$$P'JP\ddot{q} + P'KPq = P'\{M\}$$

$$m_{ii}\ddot{q}_i + k_{ii}q_i = M_o u(t)\,\theta_{3\,i}$$

$$\ddot{q}_i + \omega_i^2 q_i = \frac{M_o(\theta_3)_i}{m_{ii}}\,u(t)$$

$$m_{11} = \phi_1' J \phi_1 = J(.328^2 + .591^2 + .737^2) = 1.00\,J \qquad \omega_1^2 = .198\,\frac{K}{J}$$

$$m_{22} = J(.737^2 + .328^2 + .591^2) = 1.00\,J \qquad \omega_2^2 = 1.555\,\frac{K}{J}$$

$$m_{33} = 1.00\,J \qquad \omega_3^2 = 3.247\,\frac{K}{J}$$

Right side of eq $\frac{M_o(\theta_3)_i u(t)}{m_{ii}} =$

$$\frac{.737}{J}M_o u(t) \qquad -\frac{.591}{J}M_o u(t) \qquad \text{and} \qquad \frac{.328}{J}M_o u(t)$$

6-34 Cont. $\therefore$ Modal eqs. are

$$\left.\begin{aligned}\ddot{q}_1 + \left(.198\frac{K}{J}\right)q_1 &= \frac{.737}{J}M_o\,\mathcal{U}(t)\\ \ddot{q}_2 + \left(1.555\frac{K}{J}\right)q_2 &= -\frac{.591}{J}M_o\,\mathcal{U}(t)\\ \ddot{q}_3 + \left(3.247\frac{K}{J}\right)q_3 &= \frac{.328}{J}M_o\,\mathcal{U}(t)\end{aligned}\right\}$$

For $\mathcal{U}(t)$ the max response $= 2\times\frac{F_o}{k}$ if DE = $\ddot{x}+\omega_n^2 x = \frac{F_o}{m}$

Sol: $x = \frac{F_o}{k}(1-\cos\omega_n t)$

$$\therefore\ q_1(t) = \frac{.737 M_o}{\cancel{J}\left(.198\frac{K}{\cancel{J}}\right)}\left(1-\cos\sqrt{.198\frac{K}{J}}\,t\right) = 3.72\frac{M_o}{K}(1-\cos\omega_1 t)$$

$$q_2(t) = \frac{-.591}{1.555}\frac{M_o}{K}(1-\cos\omega_2 t) = -0.380\frac{M_o}{K}(1-\cos\omega_2 t)$$

$$q_3(t) = \frac{.328}{3.247}\frac{M_o}{K}(1-\cos\omega_3 t) = 0.101\frac{M_o}{K}(1-\cos\omega_3 t)$$

$$\theta_3(t) = \phi_1(\theta_3)q_1(t) + \phi_2(\theta_3)q_2(t) + \phi_3(\theta_3)q_3(t)$$

$$= 0.737\,q_1(t) - .591\,q_2(t) + .328\,q_3(t)$$ = superposition of above q_i

By Shock Spectrum technique

$$\left|\theta_3(t)\right|_{max} = .737\,q_{1\,max} + \sqrt{(.591\,q_{2\,max})^2 + (.328\,q_{3\,max})^2}$$

$$= .737\left(2\times 3.72\frac{M_o}{K}\right) + \sqrt{\left(.591\times 2\times .380\frac{M_o}{K}\right)^2 + \left(.328\times 2\times .101\frac{M}{K}\right)^2}$$

$$= (5.483 + .454)\frac{M_o}{K} = \underline{\underline{5.937\frac{M_o}{K}}}$$

6-35

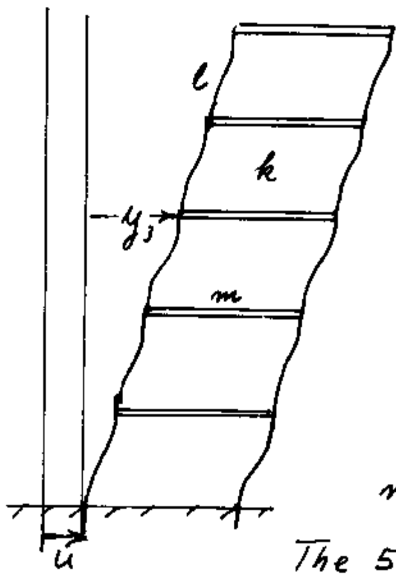
$$y(t) = \phi_1 q_1 + \phi_2 q_2 + \phi_3 q_3 + \phi_4 q_4 + \phi_5 q_5$$

1st Two Normal Modes obtained from Computer

$$\frac{m\omega_1^2}{k} = 0.08101 \qquad \begin{Bmatrix} .1699\\ .3260\\ .4557\\ .5485\\ .5969\end{Bmatrix} = \phi_1, \qquad \frac{m\omega_2^2}{k} = 0.6903 \qquad \begin{Bmatrix} .4557\\ .5969\\ .3260\\ -.1699\\ -.5485\end{Bmatrix} = \phi_2$$

$$\tau_1 = 22.08\sqrt{\frac{m}{k}} \qquad \tau_2 = 7.563\sqrt{\frac{m}{k}}$$

$$\frac{\sum m\phi_1}{\sum m\phi_1^2} = 2.097 \qquad \frac{\sum m\phi_2}{\sum m\phi_2^2} = 0.6602$$



For $K_R = \infty$, $\theta = 0$ and we have only translation of ground plus elastic translation of each floor. The more general case of $\theta \neq 0$ should be deferred until Ch 8, with Lagrange's Eq.

Write force eq. for one mass (say 3rd floor)

$$m(\ddot{u} + \ddot{y}_3) = -k(y_3 - y_2) + k(y_4 - y_3)$$

The 5 equations like the above can be written in matrix form

$$m\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \ddot{y}_5 \end{Bmatrix} + k\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{Bmatrix} = -m\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\ddot{u}$$

or

$$M\ddot{y} + Ky = -M\ddot{u}$$

Let $y = \phi_1 q_1 + \phi_2 q_2$ & decouple by $P'KP$

or $\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \left[\{\phi_1\}\{\phi_2\}\right]\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = Pq$

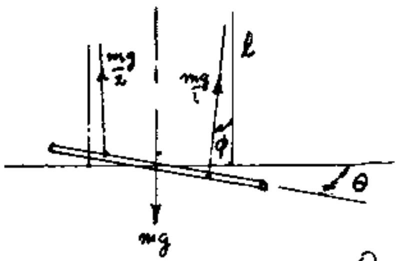
$$P'MP\ddot{q} + P'KPq = -P'M\ddot{u}$$

The modal eqs. become

$$\ddot{q}_1 + \omega_1^2 q_1 = -\frac{\sum m\phi_1}{\sum m\phi_1^2}\ddot{u}$$

$$\ddot{q}_2 + \omega_2^2 q_2 = -\frac{\sum m\phi_2}{\sum m\phi_2^2}\ddot{u}$$

6 - 36



$\phi \cong \frac{a}{2}\frac{\theta}{\ell}$ Torsional Oscil.

$$J\ddot{\theta} = -\cancel{2}\frac{mg}{\cancel{2}}\frac{a\theta}{2\ell}\frac{a}{2} = -mg\frac{a^2}{4\ell}\theta$$

$$\therefore \omega_T^2 = \frac{mga^2}{4\ell J} = \frac{\cancel{m}ga^2}{4\ell\frac{\cancel{m}L^2}{12}} = 3\frac{g}{\ell}\left(\frac{a}{L}\right)^2$$

Out-of plane oscillation is that of simple pendulum with $\omega^2 = \frac{g}{\ell}$. For $\omega^2 = \omega_T^2$

$$\frac{g}{\ell} = 3\frac{g}{\ell}\left(\frac{a}{L}\right)^2 \qquad \therefore \frac{a}{L} = \frac{1}{\sqrt{3}}$$

With small eccentricity to excite torsional oscil, there will be beating (see Prob. 1-4)

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6 - 37

Modal damping given as $\zeta_1 = .05$, $\zeta_2 = .13$

From Eq. (6.8-9)

$\omega_1 = .445\sqrt{\frac{k}{m}}$
$\omega_2 = 1.247\sqrt{\frac{k}{m}}$
$\omega_3 = 1.802\sqrt{\frac{k}{m}}$

$$2\zeta_1\omega_1 = \alpha + \beta\omega_1^2$$
$$2\zeta_2\omega_2 = \alpha + \beta\omega_2^2$$

$$2(\zeta_1\omega_1 - \zeta_2\omega_2) = \beta(\omega_1^2 - \omega_2^2)$$

$$\therefore \beta = \frac{2(\zeta_2\omega_2 - \zeta_1\omega_1)}{\omega_2^2 - \omega_1^2}, \qquad \alpha = \frac{2\omega_1\omega_2(\zeta_1\omega_2 - \zeta_2\omega_1)}{\omega_2^2 - \omega_1^2}$$

$$\therefore \beta = \frac{2(.05\times.445 - .13\times1.247)\sqrt{\frac{k}{m}}}{(.198 - 1.555)\frac{k}{m}} = \underline{\underline{0.2061\sqrt{\frac{m}{k}}}}$$

$$\alpha = \frac{2\times.445\times1.247(.05\times1.247 - .13\times.445)\sqrt{\frac{k}{m}}}{1.357} = \underline{\underline{0.00370\sqrt{\frac{k}{m}}}}$$

For 3nd mode

$$2\zeta_3\omega_3 = \alpha + \beta\omega_3^2 \qquad \zeta_3 = \frac{\alpha + \beta\omega_3^2}{2\omega_3}$$

$$\zeta_3 = \frac{.00370 + .2061\times3.247}{2\times1.802} = \underline{\underline{0.1867}}$$

6-39 See Ex. 6.5-2

$$C_i = \frac{X_i' M u(0)}{X_i' M X_i}$$

From data given $X_i' M X_i = 1.0\ m$

$$\therefore\ C_1 = \frac{(.737\ \ .591\ \ .328)\ \cancel{m}}{1.0\ \cancel{m}} \begin{Bmatrix} .520 \\ -.100 \\ .205 \end{Bmatrix} = .3914$$

$$C_2 = (-.591\ \ .328\ \ .737) \begin{Bmatrix} .520 \\ -.100 \\ .205 \end{Bmatrix} = -.1890 = -.4829\ C_1$$

$$C_3 = (.328\ \ -.737\ \ .591) \begin{Bmatrix} .520 \\ -.100 \\ .205 \end{Bmatrix} = .3654 = .9336\ C_1$$

6-38

$$X_i' M X_j = (-\ -\ -)_i\, m[I] \begin{Bmatrix} - \\ - \\ - \end{Bmatrix}_j = m(-\ -\ -)_i \begin{Bmatrix} - \\ - \\ - \end{Bmatrix}_j$$

With the modes given $X_i' M X_j = \begin{cases} 0 & \text{for } j \neq i \\ 1 & \text{for } j = i \end{cases}$

6-40

$$X(t) = \sum_i X_i \left(A_i \sin\omega_i t + B_i \cos\omega_i t \right)$$

$$\dot{X}(t) = \sum_i \omega_i X_i \left(A_i \cos\omega_i t - B_i \sin\omega_i t \right)$$

$$\dot{X}(0) = \sum_i \omega_i X_i A_i$$

$$X_j' M \dot{X}(0) = \sum_i \omega_i X_j' M X_i A_i = \omega_j X_j' M X_j A_j$$

$$\therefore\ A_j = \frac{X_j' M \dot{X}(0)}{\omega_j X_j' M X_j} \qquad B_j = \frac{X_j' M X(0)}{X_j' M X_j}$$

6 - 41 The bending of the shaft is measured from the slope of the bearing, which is β. Thus y and θ in Example 6.1-4 must first be replaced by η and $(\theta-\beta)$ respectively.

$$\begin{Bmatrix} \eta \\ \theta-\beta \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} P \\ M \end{Bmatrix}$$

From geometry Fig P6-41

$$y = \frac{P}{k} + \beta \ell + \eta$$

$$M_B = K\beta = P\ell + M \qquad \therefore \beta = \frac{P\ell}{K} + \frac{M}{K}$$

$$\text{and } \eta = y - \frac{P}{k} - \beta\ell = y - \frac{P}{k} - \frac{P\ell^2}{K} - \frac{M\ell}{K}$$

$$\therefore \quad y - \frac{P}{k} - \frac{P\ell^2}{K} - \frac{M\ell}{K} = a_{11} P + a_{12} M$$

$$\theta - \frac{P\ell}{K} - \frac{M}{K} = a_{21} P + a_{22} M$$

$$\begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{bmatrix} (a_{11} + \frac{1}{k} + \frac{\ell^2}{K}) & (a_{12} + \frac{\ell}{K}) \\ (a_{21} + \frac{\ell}{K}) & (a_{22} + \frac{1}{K}) \end{bmatrix} \begin{Bmatrix} P \\ M \end{Bmatrix} = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix} \begin{Bmatrix} P \\ M \end{Bmatrix}$$

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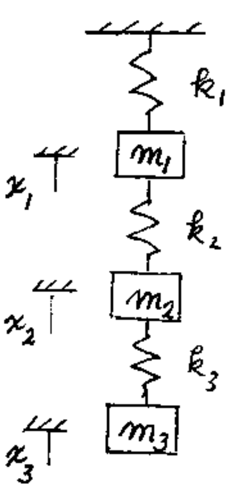
6 - 42



$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$M\ddot{X} + KX = 0$$

$$\left[-\omega^2 I + M^{-1}K\right]X = 0$$

$$\left[A - \lambda I\right]X = 0$$

where $A = M^{-1}K$

6-42 Cont.

$$M^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix}$$

$$A = M^{-1}K = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix} \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{m_1}(k_1+k_2) & -\frac{1}{m_1}k_2 & 0 \\ -\frac{1}{m_2}k_2 & \frac{1}{m_2}(k_2+k_3) & -\frac{1}{m_2}k_3 \\ 0 & -\frac{1}{m_3}k_3 & \frac{1}{m_3}k_3 \end{bmatrix} \quad \therefore \text{ not symmetric}$$

Next Follow Footnote p165

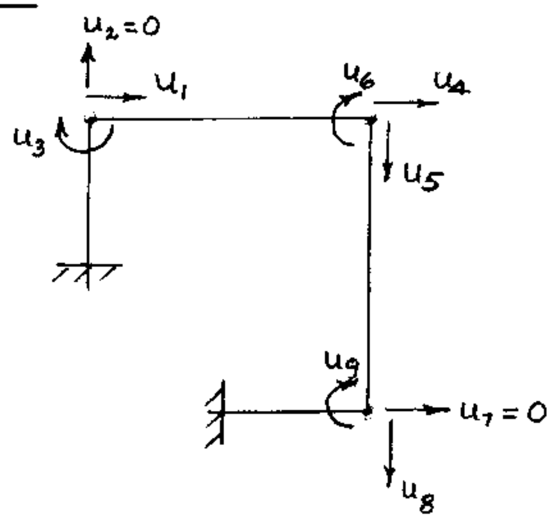
$$Q = M^{\frac{1}{2}} = \begin{bmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{bmatrix} \qquad Q^{-1} = \begin{bmatrix} \frac{1}{\sqrt{m_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{m_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{m_3}} \end{bmatrix} = Q^{-T}$$

$$\text{New } A = Q^{-T} K Q^{-1} = \begin{bmatrix} \frac{1}{\sqrt{m_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{m_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{m_3}} \end{bmatrix} \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{m_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{m_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{m_3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{m_1}(k_1+k_2) & -\frac{1}{\sqrt{m_1 m_2}}k_2 & 0 \\ -\frac{1}{\sqrt{m_1 m_2}}k_2 & \frac{1}{m_2}(k_2+k_3) & -\frac{1}{\sqrt{m_2 m_3}}k_3 \\ 0 & -\frac{1}{\sqrt{m_2 m_3}}k_3 & \frac{1}{m_3}k_3 \end{bmatrix} \quad \therefore \text{ symmetric \& standard form.}$$

This procedure is valid for M a full symmetric matrix such as those encountered in the finite element formulation.

7-1



Disregard

$u_2 = u_7 = 0$

Constraints are

$u_1 = u_4$

$u_5 = u_8$

There are 7 coordinates and 2 constraint eqs.

$\therefore$ DOF $= 5$

7-2

7 Coordinates are $u_1 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \quad u_8 \quad u_9$

4 of the above coordinates are associated with the two constraint equations, $u_1 \; u_4 \; u_5 \; u_8$.

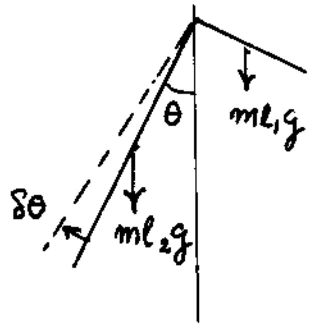
$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_4 \\ u_5 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

Above can be written as $\begin{Bmatrix} u_1 \\ u_5 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_8 \end{Bmatrix}$ where u_4 & u_8 are chosen as gen. coords.

gen. coords.

$$\therefore \; \underbrace{\begin{Bmatrix} u_1 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_8 \\ u_9 \end{Bmatrix}}_{\text{all coords.}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_6 \\ u_8 \\ u_9 \end{Bmatrix}$$

7-3



$$-m\ell_2 g\,\frac{1}{2}\ell_2\left[\cos\theta-\cos(\theta+\delta\theta)\right]+m\ell_1 g\,\frac{\ell_1}{2}\left[\sin(\theta+\delta\theta)-\sin\theta\right]=0$$

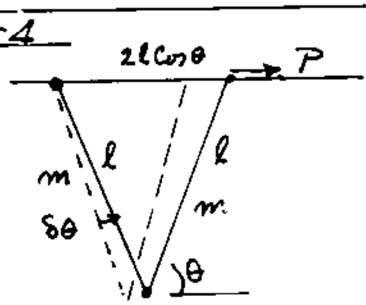
$$\ell_2^2\left[\cos\theta-(\cos\theta\cos\delta\theta-\sin\theta\sin\delta\theta\right]+\ell_1^2\left[\sin\theta\cos\delta\theta+\cos\theta\sin\delta\theta\right]=0$$

$$\cos\delta\theta\cong 1\,,\qquad \sin\delta\theta\cong\delta\theta$$

$$\ell_2^2\left[\sin\theta\cdot\delta\theta\right]+\ell_1^2\left[\cos\theta\cdot\delta\theta\right]=0$$

$$\therefore\quad \tan\theta=\left(\frac{\ell_1}{\ell_2}\right)^2$$

7-4



$$U=2mg\,\frac{\ell}{2}\sin\theta$$

$$\delta U=mg\ell\cos\theta\,\delta\theta$$

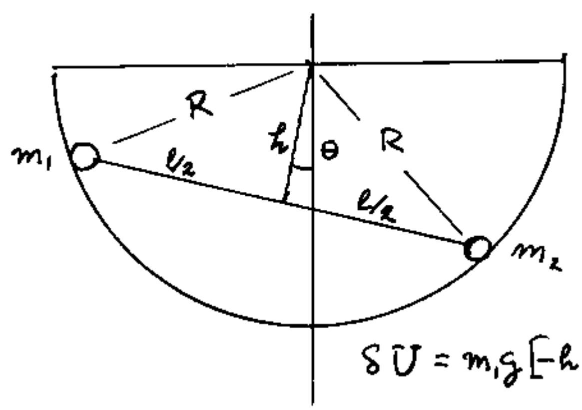
$$x=2\ell\cos\theta$$

$$\delta x=2\ell\sin\theta\,\delta\theta$$

$$\delta W=mg\ell\cos\theta\,\delta\theta-P2\ell\sin\theta\,\delta\theta=0$$

$$\therefore\ \tan\theta=\frac{mg}{2P}$$

7-5



$$h=\sqrt{R^2-\left(\frac{\ell}{2}\right)^2}$$

$$2h=\sqrt{(2R)^2-\ell^2}$$

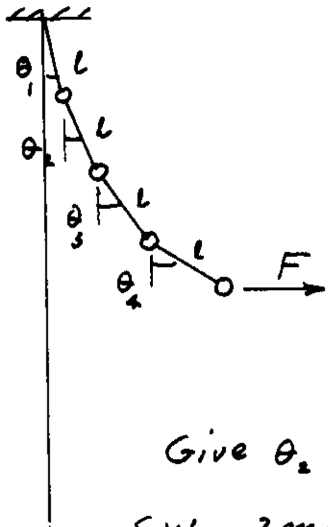
$$U=m_1 g\left[h\cos\theta-\frac{\ell}{2}\sin\theta\right]+m_2 g\left[h\cos\theta+\frac{\ell}{2}\sin\theta\right]$$

$$\delta U=m_1 g\left[-h\sin\theta\,\delta\theta-\frac{\ell}{2}\cos\theta\,\delta\theta\right]+m_2 g\left[-h\sin\theta\,\delta\theta+\frac{\ell}{2}\cos\theta\,\delta\theta\right]=0$$

$$-(m_1+m_2)h\sin\theta+(m_2-m_1)\frac{\ell}{2}\cos\theta=0$$

$$\tan\theta=\frac{(m_2-m_1)\,\ell/2}{(m_1+m_2)\,h}=\left(\frac{m_2-m_1}{m_2+m_1}\right)\frac{\ell}{\sqrt{(2R)^2+\ell^2}}$$

7-6



Top mass position $x = l\sin\theta_1$
$y = l\cos\theta_1$

Give θ_1 virtual displ $\delta\theta_1$

$$\delta x = l\cos\theta_1\,\delta\theta_1$$
$$\delta y = -l\sin\theta_1\,\delta\theta_1$$

$$\delta W = 4mg(-l\sin\theta_1\,\delta\theta_1) + Fl\cos\theta_1\,\delta\theta_1 = 0$$

$$\therefore \tan\theta_1 = \frac{F}{4mg}$$

Give θ_2 virtual displ. $\delta\theta_2$

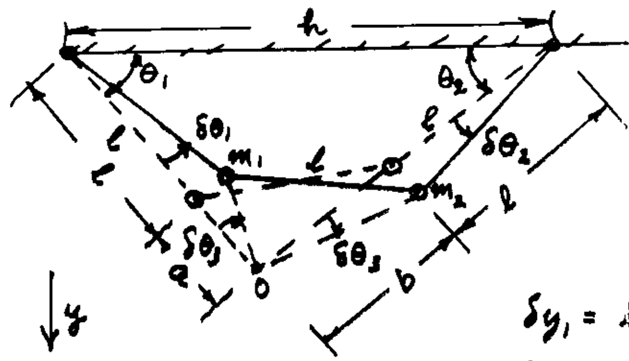
$$\delta W = 3mg(-l\sin\theta_2\,\delta\theta_2) + Fl\cos\theta_2\,\delta\theta_2 = 0$$

$$\therefore \tan\theta_2 = \frac{F}{3mg}$$

etc. $\tan\theta_3 = \dfrac{F}{2mg}$, $\tan\theta_4 = \dfrac{F}{mg}$

7-8

dotted lines show original position in equilibrium



$$\delta W = m_1 g\,\delta y_1 - m_2 g\,\delta y_2 = 0$$

The middle length rotates $\delta\theta_3$ about 0.

$$l\,\delta\theta_1 = a\,\delta\theta_3$$
$$l\,\delta\theta_2 = b\,\delta\theta_3$$

$$\delta y_1 = l\,\delta\theta_1\cos\theta_1 = a\,\delta\theta_3\cos\theta_1$$
$$\delta y_2 = l\,\delta\theta_2\cos\theta_2 = b\,\delta\theta_3\cos\theta_2$$

$$\therefore \delta W = (m_1 g\,a\cos\theta_1 - m_2 g\,b\cos\theta_2)\,\delta\theta_3 = 0$$

$$\therefore\; m_1 a\cos\theta_1 - m_2 b\cos\theta_2 = 0$$

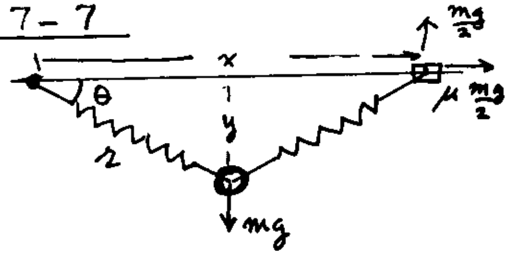
Geometric Eqs.

$$(l+a)\cos\theta_1 + (l+b)\cos\theta_2 = h$$
$$l\sin\theta_1 - l\sin\theta_2 = b\sin\theta_2 - a\sin\theta_1$$
$$l^2 = (a\cos\theta_1 + b\cos\theta_2)^2 + (b\sin\theta_2 - a\sin\theta_1)^2$$

4 eqs
4 unknowns
a, b, θ_1, θ_2

$$\sin(\theta_1+\theta_2) = \frac{h}{l}\,\frac{(m_1\cos\theta_1\sin\theta_2 - m_2\cos\theta_2\sin\theta_1)}{(m_1\cos\theta_1 - m_2\cos\theta_2)}$$

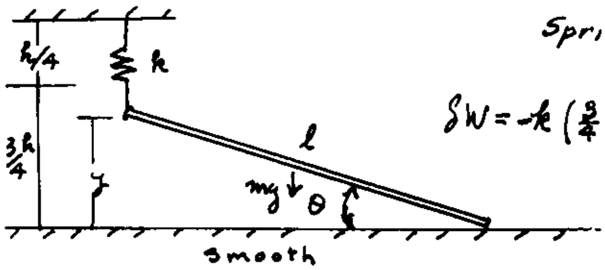
7-7



$$y = r \sin\theta$$
$$\delta y = r \cos\theta \, \delta\theta$$
$$x = 2r\cos\theta$$
$$\delta x = -2r\sin\theta \, \delta\theta$$

$$\delta W = mg\,\delta y + \mu \frac{mg}{2}\,\delta x = 0 \qquad \therefore \tan\theta = \frac{1}{\mu}$$

7-9



Stretch of spring $= \frac{3}{4}h - y$

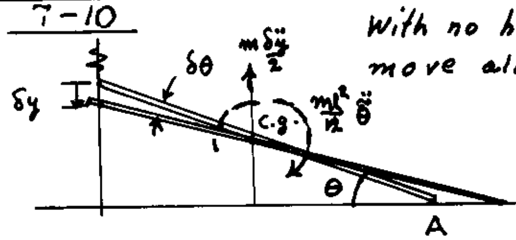
Spring force $= k\left(\frac{3}{4}h - y\right)$

$$\delta W = -k\left(\frac{3}{4}h - y\right)\cancel{\delta y} + mg\frac{\cancel{\delta y}}{2} = 0$$

$$\therefore \quad y = \frac{3}{4}h - \frac{mg}{2k}$$

$$\sin\theta = \frac{y}{l} = \frac{1}{l}\left(\frac{3h}{4} - \frac{mg}{2k}\right)$$

7-10



With no horizontal force on the bar, the c.g. must move along the vertical line.

Let $\delta\theta = \tilde{\theta} \quad \therefore \theta = \theta_0 - \tilde{\theta}$

$$\sin(\theta_0 - \tilde{\theta}) \cong \sin\theta_0 - \tilde{\theta}\cos\theta_0$$
$$\cos(\theta_0 - \tilde{\theta}) \cong \cos\theta_0 + \tilde{\theta}\sin\theta_0$$

Forces are balanced at θ_0. $\quad \delta y = l\sin\theta_0 - l\sin(\theta_0 - \tilde{\theta}) = (l\cos\theta_0)\tilde{\theta}$

spring force due to $\tilde{\theta}$ is $\quad -k\delta y = -(k l \cos\theta_0)\tilde{\theta}$

$$\delta W = (-k l\cos\theta_0)\tilde{\theta}\cdot(l\cos\theta_0\cancel{\tilde{\theta}}) - \left(\frac{m}{2} l\cos\theta_0\ddot{\tilde{\theta}}\right)\left(\frac{l\cos\theta_0}{2}\cancel{\tilde{\theta}}\right) - \left(\frac{ml^2}{12}\ddot{\tilde{\theta}}\right)\cancel{\tilde{\theta}} = 0$$

$$\left(\frac{ml^2}{12} + \frac{ml^2}{4}\cos^2\theta_0\right)\ddot{\tilde{\theta}} + (k l^2\cos^2\theta_0)\tilde{\theta} = 0$$

For small θ_0 above eq. reduces to

$$\left(\frac{ml^2}{3}\right)\ddot{\tilde{\theta}} + (kl^2)\tilde{\theta} = 0$$

7-11

$$\left[(m\ell_1)\frac{\ell_1^2}{3} + (m\ell_2)\frac{\ell_2^2}{3}\right]\ddot{\theta} = -(m\ell_2 g)\frac{\ell_2}{2}\sin\theta + (m\ell_1 g)\frac{\ell_1}{2}\cos\theta$$

Let $\theta = \theta_0 + \theta_\sim$

$$\sin\theta \cong \sin\theta_0 + \theta_\sim \cos\theta_0$$

$$\cos\theta \cong \cos\theta_0 - \theta_\sim \sin\theta_0$$

but $-(m\ell_2 g)\frac{\ell_2}{2}\sin\theta_0 + (m\ell_1 g)\frac{\ell_1}{2}\cos\theta_0 = 0$

$$\therefore \left(\frac{\cancel{m}\ell_1^3}{3} + \frac{\cancel{m}\ell_2^3}{3}\right)\ddot{\theta}_\sim = -\frac{\cancel{m}g}{2}\left(\ell_2^2\cos\theta_0 + \ell_1^2\sin\theta_0\right)\theta_\sim$$

$$\ddot{\theta}_\sim + \frac{3}{2}g\left(\frac{\ell_2^2\cos\theta_0 + \ell_1^2\sin\theta_0}{\ell_1^3 + \ell_2^3}\right)\theta_\sim = 0$$

where $\tan\theta_0 = \left(\frac{\ell_1}{\ell_2}\right)^2$

7-12

$$T = \frac{1}{2}(m_1 + m_2)R^2\dot{\theta}^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) = (m_1 + m_2)R^2\ddot{\theta} \qquad h = \sqrt{R^2 + \left(\frac{\ell}{2}\right)^2}$$

$$U = -(m_1 + m_2)gh\cos\theta - (m_2 - m_1)g\frac{\ell}{2}\sin\theta$$

$$\frac{\partial U}{\partial\theta} = (m_1 + m_2)gh\sin\theta - (m_2 - m_1)g\frac{\ell}{2}\cos\theta$$

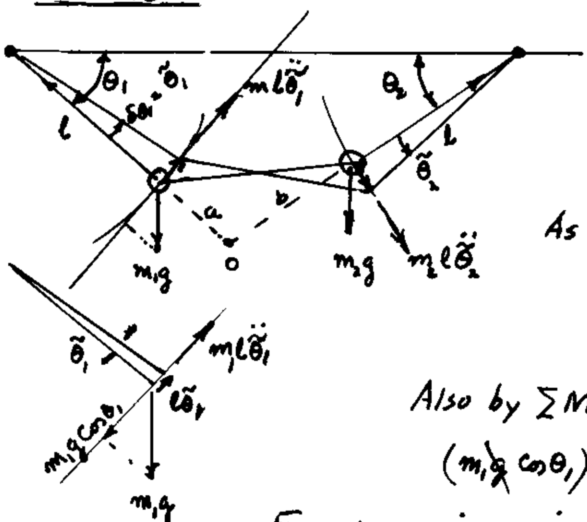
Lagranges Eq.

$$(m_1 + m_2)R^2\ddot{\theta} + (m_1 + m_2)gh\sin\theta - (m_2 - m_1)g\frac{\ell}{2}\cos\theta = 0$$

Let $\theta = \theta_0 + \theta_\sim$

$$(m_1 + m_2)R^2\ddot{\theta}_\sim + \left[(m_1 + m_2)gh\cos\theta_0 + (m_2 - m_1)g\frac{\ell}{2}\sin\theta_0\right]\theta_\sim = 0$$

7 – 13



Motion of each mass is along circle of radius ℓ. Virtual displ $= \ell\tilde{\theta}_1$ and $\ell\tilde{\theta}_2$ where $\theta_1 = \bar{\theta}_1 + \tilde{\theta}_1$ and $\theta_2 = \bar{\theta}_2 + \tilde{\theta}_2$

As in Prob 7-8

$$\delta\theta_1 = \tilde{\theta}_1 = \frac{a}{\ell}\delta\theta_3 = \frac{a}{\ell}\tilde{\theta}_3$$

$$\delta\theta_2 = \tilde{\theta}_2 = \frac{b}{\ell}\delta\theta_3 = \frac{b}{\ell}\tilde{\theta}_3 = \frac{b}{a}\tilde{\theta}_1$$

Also by $\sum M_o$

$$(m_1\cancel{g}\cos\theta_1)a = (m_2\cancel{g}\cos\theta_2)b \qquad \text{same eq. as Prob 7-8}$$

For dynamics include inertia forces $m_1\ell\ddot{\tilde{\theta}}_1$ & $m_2\ell\ddot{\tilde{\theta}}_2$ in tangential direction $\ell\tilde{\theta}_1$ and $\ell\tilde{\theta}_2$. Then work done by virtual displ. $\ell\tilde{\theta}_1$ and $\ell\tilde{\theta}_2$ is

$$\delta W = (m_1\ell\ddot{\tilde{\theta}}_1 - m_1 g\cos\theta_1)\cancel{\ell}\tilde{\theta}_1 + (m_2\ell\ddot{\tilde{\theta}}_2 + m_2 g\cos\theta_2)\cancel{\ell}\tilde{\theta}_2 = 0$$

For small $\tilde{\theta}$ with $\theta = \bar{\theta} \pm \tilde{\theta}$

$$\cos\theta_1 = \cos\bar{\theta}_1 - \tilde{\theta}_1\sin\bar{\theta}_1$$

$$\cos\theta_2 = \cos\bar{\theta}_2 + \tilde{\theta}_2\sin\bar{\theta}_2$$

$$\tilde{\theta}_2 = \frac{b}{a}\tilde{\theta}_1 = \frac{m_1}{m_2}\frac{\cos\bar{\theta}_1}{\cos\bar{\theta}_2}\tilde{\theta}_1$$

$$+\left[m_1\ell + m_2\ell\left(\frac{b}{a}\right)^2\right]\ddot{\tilde{\theta}}_1\tilde{\theta}_1 - g\left\{m_1(\cancel{\cos\bar{\theta}_1} - \tilde{\theta}_1\sin\bar{\theta}_1)\tilde{\theta}_1 - m_2(\cancel{\cos\bar{\theta}_2} + \frac{b}{a}\tilde{\theta}_1\sin\bar{\theta}_2)\frac{b}{a}\tilde{\theta}_1\right\} = 0$$

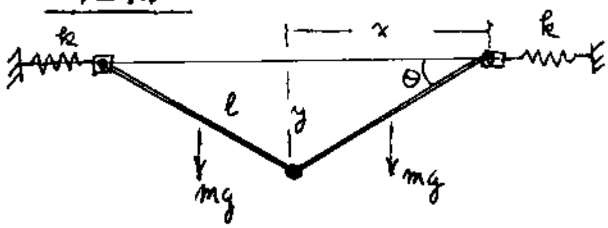
since $m_1\cos\bar{\theta}_1\cdot a - m_2\cos\bar{\theta}_2\cdot b = 0$ from Prob. 8-6 & above

$$\left[m_1\ell + m_2\ell\left(\frac{b}{a}\right)^2\right]\ddot{\tilde{\theta}}_1\cancel{\tilde{\theta}_1} + g\left\{m_1\sin\bar{\theta}_1 + m_2\left(\frac{m_1\cos\bar{\theta}_1}{m_2\cos\bar{\theta}_2}\right)^2\right\}\tilde{\theta}_1\cancel{\tilde{\theta}_1} = 0$$

$$\left[m_1 + m_2\left(\frac{m_1\cos\bar{\theta}_1}{m_2\cos\bar{\theta}_2}\right)^2\right]\ddot{\tilde{\theta}}_1 + \frac{g}{\ell}\left[m_1\sin\bar{\theta}_1 + m_2\left(\frac{m_1}{m_2}\frac{\cos\bar{\theta}_1}{\cos\bar{\theta}_2}\right)^2\right]\tilde{\theta}_1 = 0$$

$$\therefore \omega_n^2 = \frac{g}{\ell}\cdot\frac{m_1\sin\bar{\theta}_1 + m_2\left(\frac{m_1\cos\bar{\theta}_1}{m_2\cos\bar{\theta}_2}\right)^2}{m_1 + m_2\left(\frac{m_1\cos\bar{\theta}_1}{m_2\cos\bar{\theta}_2}\right)^2}$$

7–14



Let θ_o = equilib angle

Spring force = F_{so} at $\theta = \theta_o$

$x = l\cos\theta \qquad \delta x = -l\sin\theta_o\,\delta\theta$

$y = l\sin\theta \qquad \delta y = l\cos\theta_o\,\delta\theta$

$$\delta W = 2mg\,\frac{\delta y}{2} + 2F_{so}\,\delta x = 0$$

$$(mg\cos\theta_o - 2F_{so}\sin\theta_o)\,l\,\delta\theta = 0$$

If $F_s = 0$ at $\theta = 0$ then $F_{so} = k(l-x) = kl(1-\cos\theta_o)$

$$\tan\theta_o = \frac{mg}{2kl(1-\cos\theta_o)}$$

solve by trial for given value of $\dfrac{mg}{2kl}$

7–15

$$T = \frac{1}{2}m_o\left[(r\dot\theta)^2 + \dot r^2\right] + \frac{1}{2}\left(m\frac{l^2}{3}\right)\dot\theta^2$$

$$U = \frac{1}{2}k(r-r_o)^2 - m_o g\,r\cos\theta - mg\frac{l}{2}\cos\theta$$

$$\frac{\partial T}{\partial\dot\theta} = m_o r^2\dot\theta + m\frac{l^2}{3}\dot\theta \qquad \frac{\partial T}{\partial\theta} = 0$$

$$\frac{\partial U}{\partial\theta} = m_o g\,r\sin\theta + mg\frac{l}{2}\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial U}{\partial\theta} = 0$$

$$m_o\left[r\ddot\theta + 2\dot r\dot\theta\right]r + \frac{ml^2}{3}\ddot\theta + \left(m_o g\,r + mg\frac{l}{2}\right)\sin\theta = 0$$

$$\frac{\partial T}{\partial\dot r} = m_o\dot r \qquad \frac{\partial T}{\partial r} = m_o r\dot\theta^2$$

$$\frac{\partial U}{\partial r} = -m_o g\cos\theta + k(r-r_o)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot r}\right) - \frac{\partial T}{\partial r} + \frac{\partial U}{\partial r} = 0$$

$$m_o\ddot r - m_o r\dot\theta^2 + k(r-r_o) - m_o g\cos\theta = 0$$

7-16

$$T = \frac{1}{2}\int_0^\ell m\,\dot{y}^2\,dx$$

$$U = \frac{1}{2}k\,y^2(\ell) + \frac{1}{2}K\,y'^2(0) + \frac{1}{2}\int_0^\ell EI\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$y = \frac{x}{\ell}q_1 + q_2\sin\frac{\pi x}{\ell}$$

$$\dot{y}^2 = \left(\frac{x}{\ell}\right)^2\dot{q}_1^2 + 2\left(\frac{x}{\ell}\right)\dot{q}_1\dot{q}_2\sin\frac{\pi x}{\ell} + \dot{q}_2^2\sin^2\frac{\pi x}{\ell}$$

$$y' = \frac{1}{\ell}q_1 + q_2\frac{\pi}{\ell}\cos\frac{\pi x}{\ell}$$

$$y'^2 = \frac{1}{\ell^2}q_1^2 + \frac{2}{\ell}q_1 q_2\frac{\pi}{\ell}\cos\frac{\pi x}{\ell} + q_2^2\left(\frac{\pi}{\ell}\right)^2\cos^2\frac{\pi x}{\ell}$$

$$y'' = -q_2\left(\frac{\pi}{\ell}\right)^2\sin\frac{\pi x}{\ell}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} = \ddot{q}_1\int_0^\ell m\left(\frac{x}{\ell}\right)^2 dx + \ddot{q}_2\int_0^\ell \frac{x}{\ell}\sin\frac{\pi x}{\ell}\,dx = \frac{m\ell}{3}\ddot{q}_1 + \frac{\ell}{\pi}\ddot{q}_2$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} = \ddot{q}_1\int_0^\ell \frac{x}{\ell}\sin\frac{\pi x}{\ell}\,dx + \ddot{q}_2\int_0^\ell \sin^2\frac{\pi x}{\ell}\,dx = \frac{\ell}{\pi}\ddot{q}_1 + \frac{\ell}{2}\ddot{q}_2$$

$$\frac{\partial U}{\partial q_1} = k q_1 + \frac{K}{\ell^2}q_1 + \frac{K\pi}{\ell^2}q_2$$

$$\frac{\partial U}{\partial q_2} = \frac{K\pi}{\ell^2}q_1 + K\left(\frac{\pi}{\ell}\right)^2 q_2 + EI\left(\frac{\pi}{\ell}\right)^4\frac{\ell}{2}q_2$$

$$m\ell\begin{bmatrix}\frac{1}{3} & \frac{1}{2\pi}\\ \frac{1}{2\pi} & \frac{1}{2}\end{bmatrix}\begin{Bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{Bmatrix} + \frac{EI}{\ell^3}\begin{bmatrix}\left(k + \frac{K}{\ell^2}\right)\frac{\ell^3}{EI} & -\frac{\pi K}{\ell^2}\frac{\ell^3}{EI}\\ -\frac{\pi K}{\ell^2}\frac{\ell^3}{EI} & \left(\frac{\pi^4}{2} + \frac{\pi^2 K}{\ell^2}\frac{\ell^3}{EI}\right)\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix} = \{0\}$$

7-17

$$T = \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\dot{\theta}_1^{\,2} + \frac{1}{2}\left(\frac{m\ell^2}{12}\right)\dot{\theta}_2^{\,2} + \frac{1}{2}m\left(\ell\dot{\theta}_1 + \frac{\ell}{2}\dot{\theta}_2\right)^2$$

$$U = \frac{1}{2}k\left(\frac{\ell}{2}\theta_1\right)^2 + mg\frac{\ell}{2}(1-\cos\theta_1) + mg\left[\ell(1-\cos\theta_1) + \frac{\ell}{2}(1-\cos\theta_2)\right]$$

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}_1} = \left(\frac{m\ell^2}{3}\right)\ddot{\theta}_1 + m\left(\ell\ddot{\theta}_1 + \frac{\ell}{2}\ddot{\theta}_2\right)\ell$$

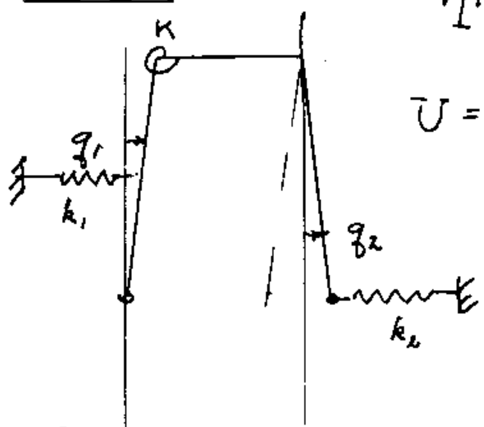
$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}_2} = \left(\frac{m\ell^2}{12}\right)\ddot{\theta}_2 + m\left(\ell\ddot{\theta}_1 + \frac{\ell}{2}\ddot{\theta}_2\right)\frac{\ell}{2}$$

$$\frac{\partial U}{\partial\theta_1} = k\left(\frac{\ell}{2}\theta_1\right) + mg\frac{\ell}{2}\sin\theta_1 + mg\,\ell\sin\theta_1$$

$$\frac{\partial U}{\partial\theta_2} = mg\frac{\ell}{2}\sin\theta_2$$

$$\begin{bmatrix} \left(\frac{m\ell^2}{3} + m\ell^2\right) & m\frac{\ell^2}{2} \\ m\frac{\ell^2}{2} & \left(\frac{m\ell^2}{12} + \frac{m\ell^2}{4}\right) \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} \left(k\frac{\ell}{2} + \frac{3}{2}mg\ell\right) & 0 \\ 0 & mg\frac{\ell}{2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \{0\}$$

7-18



$$T = \tfrac{1}{2}(m_1+m_2)(2\ell\dot{q}_1)^2 + \tfrac{1}{2}J_1\dot{q}_1^{\,2} + \tfrac{1}{2}J_2\dot{q}_2^{\,2}$$

$$U = \tfrac{1}{2}k_1(\ell q_1)^2 + \tfrac{1}{2}Kq_1^2 + \tfrac{1}{2}k_2(2\ell)^2(q_1+q_2)^2$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} = (m_1+m_2)(4\ell^2)\ddot{q}_1 + J_1\ddot{q}_1$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} = J_2\ddot{q}_2$$

$$\frac{\partial U}{\partial q_1} = \ell^2 k_1 q_1 + Kq_1 + k_2(4\ell^2)(q_1+q_2)$$

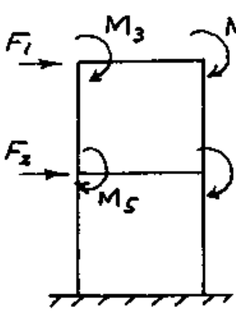
$$\frac{\partial U}{\partial q_2} = 4\ell^2 k_2(q_1+q_2)$$

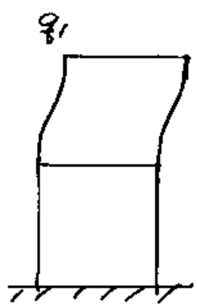
$$\left[(m_1+m_2)4\ell^2 + J_1\right]\ddot{q}_1 + \left[\ell^2 k_1 + K + 4\ell^2 k_2\right]q_1 + 4\ell^2 k_2 q_2 = 0$$

$$J_2\ddot{q}_2 + 4\ell^2 k_2(q_1+q_2) = 0$$

7-19

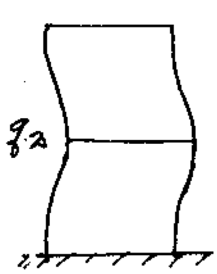
Refer to Fig 7.1-4. Assume $\ell_2 = \ell_1 = \ell$

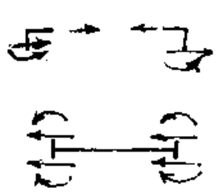




$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 24 & 0 & 0 & 0 & 0 & 0 \\ -24 & \vdots & \vdots & \vdots & \vdots & \vdots \\ -6\ell & \vdots & \vdots & \vdots & \vdots & \vdots \\ -6\ell & \vdots & \vdots & \vdots & \vdots & \vdots \\ -6\ell & \vdots & \vdots & \vdots & \vdots & \vdots \\ -6\ell & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}\begin{Bmatrix} q_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

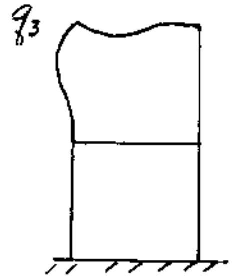
Examine FBD of each of the four corners for above
Refer to table 7.3-1





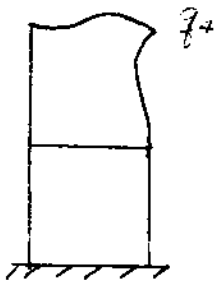
$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 0 & -24 & 0 & 0 & 0 & 0 \\ \vdots & 48 & \vdots & \vdots & \vdots & \vdots \\ \vdots & 6\ell & \vdots & \vdots & \vdots & \vdots \\ \vdots & 6\ell & \vdots & \vdots & \vdots & \vdots \\ \vdots & 0 & \vdots & \vdots & \vdots & \vdots \\ \vdots & 0 & \vdots & \vdots & \vdots & \vdots \end{bmatrix}\begin{Bmatrix} 0 \\ q_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

7-19 Cont.

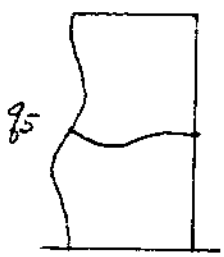


Examine FBD of each corner

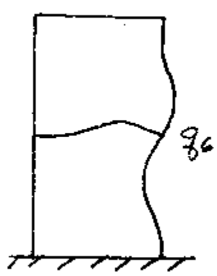
$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & -6\ell & 0 & 0 & 0 \\ \vdots & \vdots & 6\ell & \vdots & \vdots & \vdots \\ \vdots & \vdots & 8\ell^2 & \vdots & \vdots & \vdots \\ \vdots & \vdots & 2\ell^2 & \vdots & \vdots & \vdots \\ \vdots & \vdots & 2\ell^2 & \vdots & \vdots & \vdots \\ \vdots & \vdots & 0 & \vdots & \vdots & \vdots \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ q_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$



$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & 0 & -6\ell & 0 & 0 \\ \vdots & \vdots & \vdots & 6\ell & \vdots & \vdots \\ \vdots & \vdots & \vdots & 2\ell^2 & \vdots & \vdots \\ \vdots & \vdots & \vdots & 8\ell^2 & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & 2\ell^2 & \vdots & \vdots \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ q_4 \\ 0 \\ 0 \end{Bmatrix}$$



$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & 0 & 0 & -6\ell & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & 2\ell^2 & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & 12\ell^2 & \vdots \\ \vdots & \vdots & \vdots & \vdots & 12\ell^2 & \vdots \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ q_5 \\ 0 \end{Bmatrix}$$



$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -6\ell \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 2\ell^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 2\ell^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 12\ell^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ q_6 \end{Bmatrix}$$

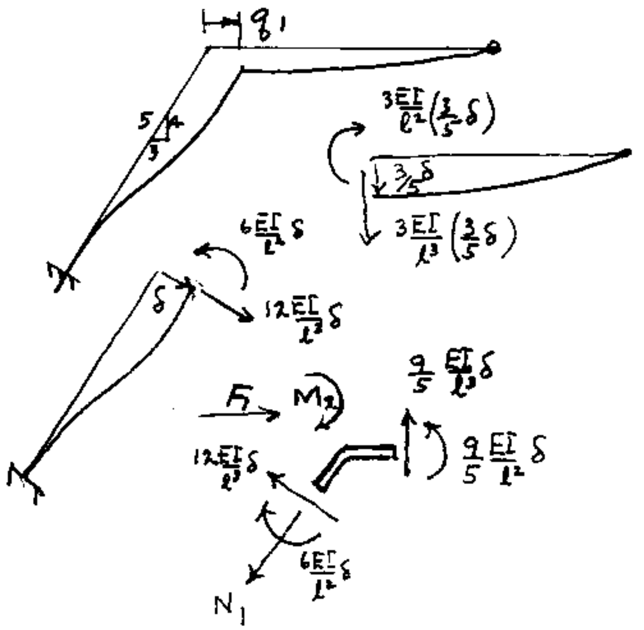
$$[k] = \frac{EI}{\ell^3}\left[\begin{array}{cc|cccc} 24 & -24 & -6\ell & -6\ell & -6\ell & -6\ell \\ -24 & 48 & 6\ell & 6\ell & 0 & 0 \\ \hline -6\ell & 6\ell & 8\ell^2 & 2\ell^2 & 2\ell^2 & 0 \\ -6\ell & 6\ell & 2\ell^2 & 8\ell^2 & 0 & 2\ell^2 \\ -6\ell & 0 & 2\ell^2 & 0 & 12\ell^2 & 2\ell^2 \\ -6\ell & 0 & 0 & 2\ell^2 & 2\ell^2 & 12\ell^2 \end{array}\right]$$

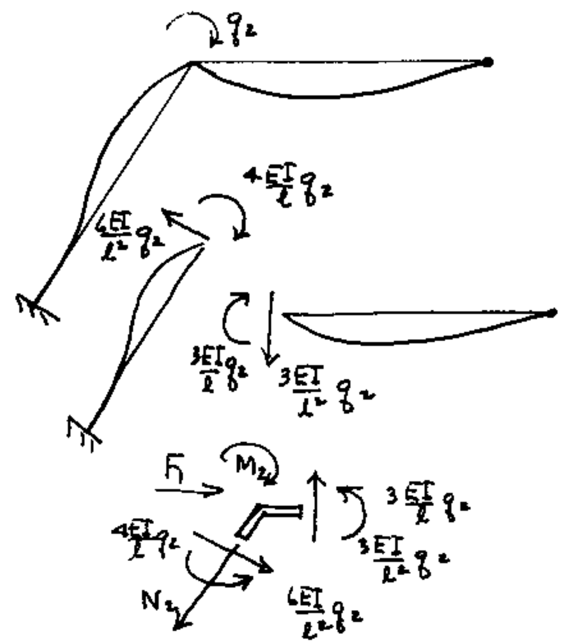
7-19 Cont.

$$[m] = \begin{bmatrix} 2m & 0 & 0 & 0 & 0 & 0 \\ 0 & 2m & 0 & 0 & 0 & 0 \\ 0 & 0 & J & 0 & 0 & 0 \\ 0 & 0 & 0 & J & 0 & 0 \\ 0 & 0 & 0 & 0 & J & 0 \\ 0 & 0 & 0 & 0 & 0 & J \end{bmatrix}$$

$$[m]\{\ddot{q}\} + [k]\{q\} = 0$$

7-20





$$\Sigma F_y = -\frac{4}{5}N_1 + \left(\frac{9}{5} + \frac{3}{5}\times 12\right)\frac{EI}{\ell^3}\delta = 0$$

$$\therefore N_1 = \frac{45}{4}\frac{EI}{\ell^3}\delta$$

$$\Sigma F_x = -F_1 + \left[\frac{3}{5}\left(\frac{45}{4}\right) + \frac{4}{5}\times 12\right]\frac{EI}{\ell^3}\delta = 0$$

$$F_1 \simeq 16.35\frac{EI}{\ell^3}\delta$$

$$\Sigma M \qquad M_2 = \left(\frac{9}{5} - 6\right)\frac{EI}{\ell^2}\delta = -4.20\frac{EI}{\ell^2}\delta$$

but $\delta = \frac{5}{4}q_1$

$$\therefore F_1 = 20.44\frac{EI}{\ell^3}q_1$$

$$M_2 = -5.25\frac{EI}{\ell^2}q_1$$

$$\Sigma F_y = 3\frac{EI}{\ell^2}q_2 - \frac{3}{5}\left(6\frac{EI}{\ell^2}\right)q_2 - \frac{4}{5}N_2 = 0$$

$$\therefore N_2 = -.75\frac{EI}{\ell^2}q_2$$

$\Sigma F_x = 0$ gives

$$F_1 = \left[\frac{3}{5}(-.75) - 6\times\frac{4}{5}\right]\frac{EI}{\ell^2}q_2$$

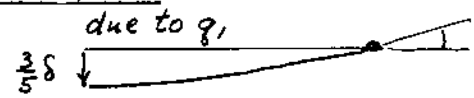
$$F_1 = -5.25\frac{EI}{\ell^2}q_2$$

$$M_2 = 7\frac{EI}{\ell}q_2$$

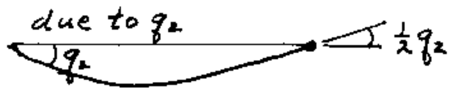
7-20 Cont.

$$\begin{Bmatrix} F_1 \\ M_2 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 20.44 & -5.25\ell \\ -5.25\ell & 7.0\,\ell^2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

7-21 From Table 6.3-1 rotation of right end is



$$\frac{3}{2}\left(\frac{3}{5}\frac{\delta}{\ell}\right) = \frac{3}{2}\left(\frac{3}{5}\times\frac{5}{4}\frac{q_1}{\ell}\right) = \frac{9}{8}\,\frac{q_1}{\ell}$$



$$T = \frac{1}{2}(m_1+m_2)\dot{q}_1^{\,2} + \frac{1}{2}J_1\dot{q}_2^{\,2} + \frac{1}{2}J_2\left(\frac{9}{8}\frac{\dot{q}_1}{\ell} + \frac{1}{2}\dot{q}_2\right)^2$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} = (m_1+m_2)\ddot{q}_1 + J_2\left(\frac{9}{8\ell}\ddot{q}_1 + \frac{1}{2}\ddot{q}_2\right)\frac{9}{8\ell}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} = J_1\ddot{q}_2 + J_2\left(\frac{9}{8\ell}\ddot{q}_1 + \frac{1}{2}\ddot{q}_2\right)\frac{1}{2}$$

$$U = \frac{1}{2}(q_1\ q_2)\Big[\ k\ \Big]\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} + \frac{1}{2}k_0 q_1^{\,2} + \frac{1}{2}K_0\left(\frac{9}{8\ell}q_1 + \frac{1}{2}q_2\right)^2$$

$$\frac{\partial U}{\partial q_{1,2}} = \begin{bmatrix} \left\{20.44\frac{EI}{\ell^3} + k_0 + \left(\frac{9}{8\ell}\right)^2 K_0\right\} & \left\{-5.25\frac{EI}{\ell^2} + \frac{1}{2}\times\frac{9}{8\ell}K_0\right\} \\ \left\{-5.25\frac{EI}{\ell^2} + \frac{1}{2}\times\frac{9}{8\ell}K_0\right\} & \left\{7.0\frac{EI}{\ell} + \frac{1}{4}K_0\right\} \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

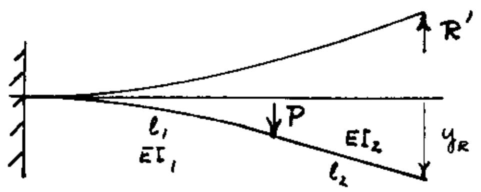
Eq. of motion

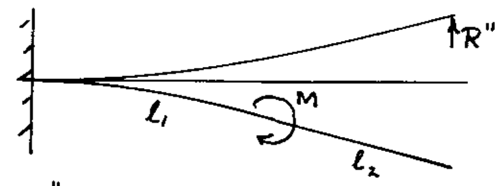
$$\begin{bmatrix} \left\{m_1+m_2+\left(\frac{9}{8\ell}\right)^2 J_2\right\} & \left\{\frac{1}{2}\times\frac{9}{8\ell}J_2\right\} \\ \left\{\frac{1}{2}\times\frac{9}{8\ell}J_2\right\} & \left\{J_1 + \frac{1}{4}J_2\right\} \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix}$$

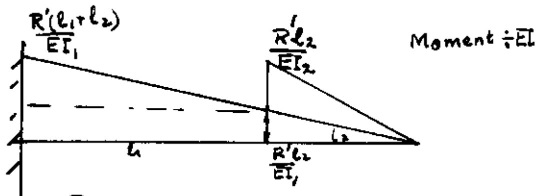
$$+\begin{bmatrix} \left\{20.44\frac{EI}{\ell^3} + k_0 + \left(\frac{9}{8\ell}\right)^2 K_0\right\} & \left\{-5.25\frac{EI}{\ell^2} + \frac{9}{16\ell}K_0\right\} \\ \left\{-5.25\frac{EI}{\ell^2} + \frac{9}{16\ell}K_0\right\} & \left\{7.0\frac{EI}{\ell} + \frac{1}{4}K_0\right\} \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \Big\{\,0\,\Big\}$$

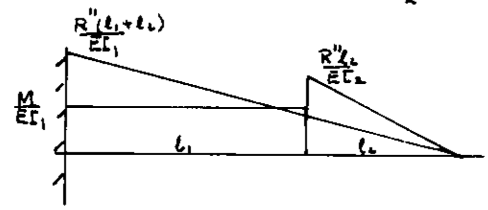
Assign numbers for m_1, m_2 J_1, J_2, k_0, K_0 & ℓ for normal mode determination.

7-22









Due to P

$$y_P = \frac{Pl_1^3}{3EI_1} \qquad y'_P = \frac{Pl_1^2}{2EI_1}$$

$$\therefore\ y_R = \frac{Pl_1^3}{3EI_1} + \left(\frac{Pl_1^2}{2EI_1}\right)l_2$$

Due to R'

$$y_R = \left(\frac{1}{2}\,\frac{R'l_2}{EI_2}\,l_2\right)\frac{2}{3}l_2 + \left(\frac{R'l_2}{EI_1}\cdot l_1\right)\left(l_2 + \frac{1}{2}l_1\right)$$

$$+\left(\frac{1}{2}\,\frac{R'l_1}{EI_1}\cdot l_1\right)\left(l_2 + \frac{2}{3}l_1\right)$$

Equate y_R due to P and due to R' (use $\frac{EI_1}{EI_2}=2$)

$$\frac{P}{\cancel{EI_1}}\left[\frac{l_1^3}{3} + \frac{l_1^2 l_2}{2}\right] = \frac{R'}{\cancel{EI_1}}\left[\frac{2}{3}l_2^3 + l_1 l_2^2 + l_2 l_1^2 + \frac{l_1^3}{3}\right]$$

$$= R'[b]$$

Due to M

$$y_M = \left(\frac{M}{EI_1}\cdot l_1\right)\frac{l_1}{2}$$

$$y'_M = \frac{M}{EI_1}l_1$$

$$y_R = \frac{M}{EI_1}\,\frac{l_1^2}{2} + \frac{Ml_1 l_2}{EI_1}$$

Due to R''

y_R same as that for R' with R' replaced by R''

Equate y_R due to M and due to R''

$$\frac{M}{\cancel{EI_1}}\left(\frac{l_1^2}{2} + l_1 l_2\right) = \frac{R''}{\cancel{EI_1}}\left[\frac{2}{3}l_2^3 + l_1 l_2^2 + l_2 l_1^2 + \frac{l_1^3}{3}\right]$$

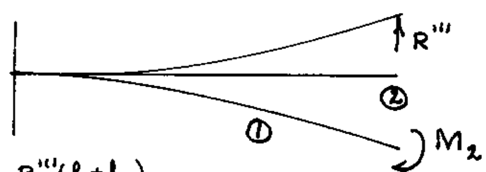
$$= R''[b]$$

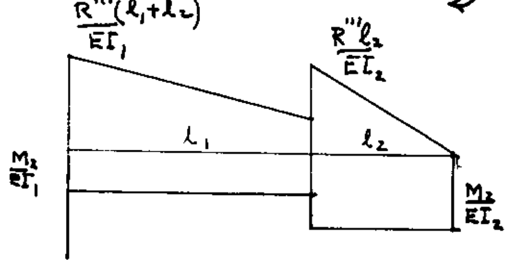
$$M_1 = -M - Pl_1 + R(l_1 + l_2) \qquad \text{where } R = R' + R''$$

$$R = \frac{P\left(\frac{1}{3}l_1^3 + \frac{1}{2}l_1^2 l_2\right) + M\left(\frac{1}{2}l_1^2 + l_1 l_2\right)}{\frac{2}{3}l_2^3 + l_1 l_2^2 + l_2 l_1^2 + \frac{1}{3}l_1^3}$$

7-23

Need stiffness matrix for M, P and M_2 at end of beam





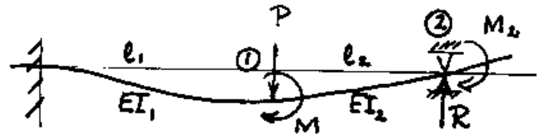
$$y_R = \frac{M_2}{EI_1}l_1\left(l_2 + \frac{l_1}{2}\right) + \frac{M_2}{EI_2}\,\frac{l_2^2}{2}$$

$$y_R = \frac{R'''}{EI_1}\left[\frac{2}{3}l_2^3 + l_1 l_2^2 + l_2 l_1^2 + \frac{1}{3}l_1^3\right]$$

Equate

$$\frac{M_2}{\cancel{EI_1}}\left[l_1 l_2 + \frac{l_1^2}{2} + 2\cdot\frac{l_2^2}{2}\right] = \frac{R'''}{\cancel{EI_1}}[b]$$

7-23 Cont:



Determine y_1 θ_1 y_2 θ_2

Set. $y_2 = 0$ to find R

Determine flexibility matrix & invert.
$EI_1 = 2\ EI_2$

$$y_1 = \frac{P\ell_1^3}{3EI_1} + \frac{M\ell_1^2}{2EI_1} + \frac{M_2\ell_1^2}{2EI_1} - \frac{1}{2}\left[\frac{R(\ell_1+\ell_2)\ell_1}{EI_1} - \frac{R\ell_1\ell_2}{EI_1}\right]\frac{2}{3}\ell_1 - \frac{R\ell_2}{EI_1}\frac{\ell_1^2}{2}$$

$$\theta_1 = \frac{P\ell_1^2}{2EI_1} + \frac{M\ell_1}{EI_1} + \frac{M_2\ell_1}{EI_1} - \frac{1}{2}\left[\frac{R(\ell_1+\ell_2)}{EI_1} + \frac{R\ell_2}{EI_1}\right]\ell_1$$

$$y_2 = \frac{1}{2}\frac{P\ell_1^2}{EI_1}\left(\frac{2}{3}\ell_1+\ell_2\right) + \frac{M\ell_1}{EI_1}\left(\frac{1}{2}\ell_1+\ell_2\right) + \frac{M_2\ell_1}{EI_1}\left(\frac{\ell_1}{2}+\ell_2\right) + \frac{M_2}{EI_2}\frac{\ell_2^2}{2}$$
$$-\frac{1}{2}\frac{R(\ell_1+\ell_2)}{EI_1}(\ell_1+\ell_2)\frac{2}{3}(\ell_1+\ell_2) - \frac{1}{2}\left[\frac{R\ell_2^2}{EI_2} - \frac{R\ell_2\ell_2}{EI_1}\right]\frac{2}{3}\ell_2$$

$$\theta_2 = \frac{P\ell_1^2}{2EI_1} + \frac{M\ell_1}{EI_1} + \frac{M_2\ell_1}{EI_1} + \frac{M_2\ell_2}{EI_2} - \frac{1}{2}\frac{R(\ell_1+\ell_2)^2}{EI_1} - \frac{1}{2}\left(\frac{R\ell_2}{EI_2} - \frac{R\ell_2}{EI_1}\right)\ell_2$$

To simplify algebra let $\ell_1 = \ell_2 = \ell$ & $EI_1 = 2EI_2$, result is

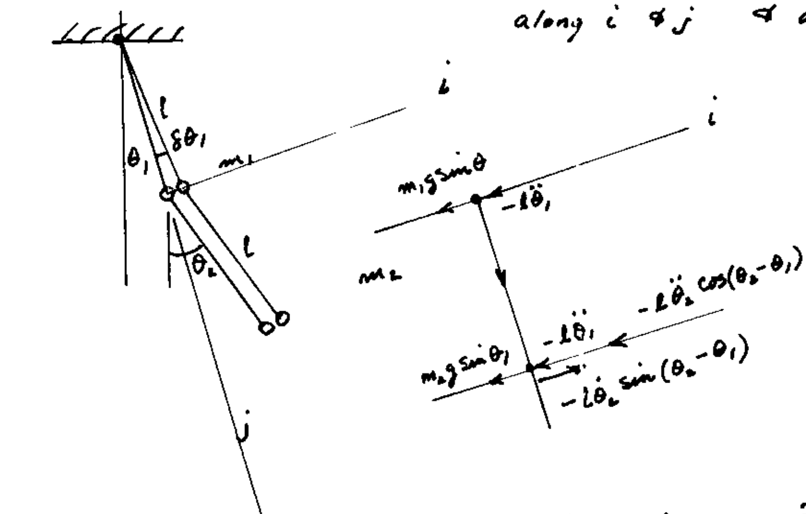
$$\begin{Bmatrix} y_1 \\ \theta_1 \\ \theta_2 \\ y_2=0 \end{Bmatrix} = \frac{1}{EI_1}\begin{bmatrix} \frac{\ell^3}{3} & \frac{\ell^2}{2} & \frac{\ell^2}{2} & -\frac{5}{6}\ell^3 \\ \frac{\ell^2}{2} & \ell & \ell & -\frac{3}{2}\ell^2 \\ \frac{\ell^2}{2} & \ell & 3\ell & -\frac{5}{2}\ell^2 \\ \frac{5}{6}\ell^3 & \frac{3}{2}\ell^2 & \frac{5}{2}\ell^2 & -3\ell^3 \end{bmatrix}\begin{Bmatrix} P \\ M \\ M_2 \\ R \end{Bmatrix}$$

from $y_2=0$ $\quad R = \frac{5}{18}P + \frac{1}{2}\frac{M}{\ell} + \frac{5}{6}\frac{M_2}{\ell}$ $\quad$ Subst. into above to obtain

$$\begin{Bmatrix} y_1 \\ \theta_1 \\ \theta_2 \end{Bmatrix} = \frac{1}{EI_1}\begin{bmatrix} 0.1018\ell^3 & 0.0833\ell^2 & -0.1944\ell^2 \\ 0.0833\ell^2 & 0.25\ell & -0.25\ell \\ -0.1944\ell^2 & -0.25\ell & 0.0916\ell \end{bmatrix}\begin{Bmatrix} P \\ M \\ M_2 \end{Bmatrix}$$

invert by computer program to obtain stiffness matrix (See Prob. 6-4)
need numeric value of ℓ.

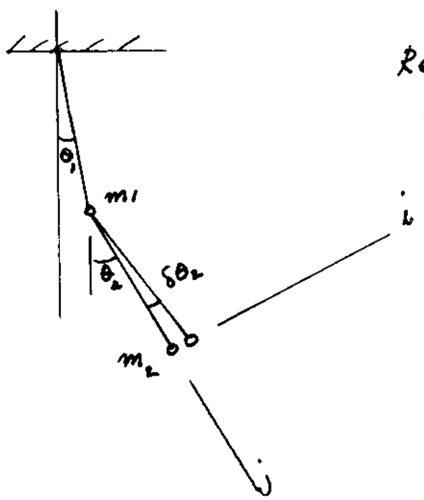
Resolve all forces including inertia forces along i & j & dot product with $l\delta\theta_1\, i$



$$\Sigma(F - m\ddot{r})\cdot l\delta\theta_1\, i = -(m_1+m_2)g\sin\theta_1 - (m_1+m_2)l\ddot{\theta}_1 - m_2 l\ddot{\theta}_2\cos(\theta_2-\theta_1) + m_2 l\dot{\theta}_2^2\sin(\theta_2-\theta_1) = 0$$

or $$(m_1+m_2)l\ddot{\theta}_1 + m_2 l\ddot{\theta}_2\cos(\theta_2-\theta_1) - m_2 l\dot{\theta}_2\sin(\theta_2-\theta_1) + (m_1+m_2)g\sin\theta_1 = 0$$

Resolve all forces along new i j $\perp$ & $\parallel$ to l_2 and dot with $l\delta\theta_2\, i$



$$m_2 l\ddot{\theta}_2 + m_2 l\ddot{\theta}_1\cos(\theta_2-\theta_1) + m_2 l\dot{\theta}_1^2\sin(\theta_2-\theta_1) + m_2 g\sin\theta_2 = 0$$

Try checking these equations from Lagrange's eqs.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}_i}\right) - \frac{\partial T}{\partial\theta_i} + \frac{\partial U}{\partial\theta_i} = 0$$

8-1

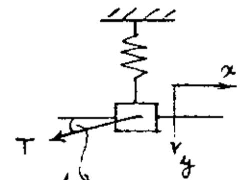
$$c = \sqrt{\frac{T}{\rho}} = \sqrt{\frac{444}{.372}} = 34.55 \ \text{m/s}$$

8-2 see Sec. 8-1

$$f_n = \frac{n}{2l}\sqrt{\frac{T}{\rho}}, \qquad n = 1, 2, 3 \ ---$$

8-3

Gen sol. $y(x,t) = \left(A \sin\frac{\omega x}{c} + B\cos\frac{\omega x}{c}\right)\sin\omega t$



At $x=0$ $\quad y = 0 \quad \therefore B = 0$

At $x=l$ $\quad y = y(l,t)$ of spring mass

slope $\frac{dy}{dx}(l,t)$ is negative

Vertical force $= -T\frac{dy}{dx}(l,t)$

$$= \left(-TA\frac{\omega}{c}\cos\frac{\omega l}{c}\right)\sin\omega t$$

$$= m\ddot{y} + ky$$

$$\therefore y(l) = \frac{-TA\frac{\omega}{c}\cos\frac{\omega l}{c}}{k - m\omega^2} = \frac{-TA\frac{\omega}{c}\cos\frac{\omega l}{c}}{m\omega_n^2\left(1-\frac{\omega^2}{\omega_n^2}\right)}, \qquad \omega_n = \sqrt{\frac{k}{m}}$$

or $$\tan\frac{\omega l}{c} = -\left(\frac{T}{kl}\right)\frac{\left(\frac{\omega l}{c}\right)}{\left[1-\left(\frac{\omega l}{c}\right)^2\left(\frac{mc^2}{kl^2}\right)\right]}$$

8-4

$$y_1 = a\cos kx\sin\omega t, \qquad y_2 = a\cos\left(\frac{k}{x}+\frac{\pi}{2}\right)\sin\left(\omega t+\frac{\pi}{2}\right)$$

$$= -a\sin kx\cos\omega t$$

$$y = y_1 + y_2 = a\left[\cos kx\sin\omega t - \sin kx\cos\omega t\right]$$

$$= a\sin(\omega t - kx) = a\sin k\left(\frac{\omega}{k}t - x\right) \quad \therefore c = \frac{\omega}{k}$$

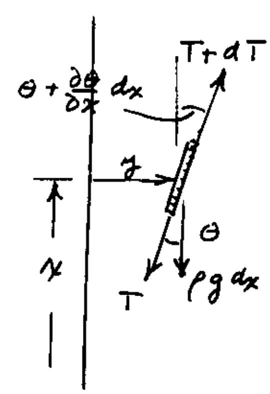
8-5

$$c = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{200\times10^9}{7810}} = 5060 \ \text{m/s} = 16{,}600 \ \text{ft/s}$$

8-6 $dT \cong \rho g\,dx$

$$(T+dT)\left(\theta+\frac{\partial\theta}{\partial x}dx\right) - T\theta = \rho\,dx\,\ddot{y}$$

$$\therefore T\frac{\partial\theta}{\partial x} + \rho g\,\theta = \rho\ddot{y} \qquad \theta = \frac{\partial y}{\partial x}, \quad T = \rho g x$$



$$\rho g x\frac{\partial^2 y}{\partial x^2} + \rho g\frac{\partial y}{\partial x} = \rho\ddot{y}$$

$$\ddot{y} = g\left(x\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x}\right)$$

8 - 7

$$y = Y(x)\cos\omega t$$
$$\ddot{y} = -\omega^2 Y(x)\cos\omega t$$

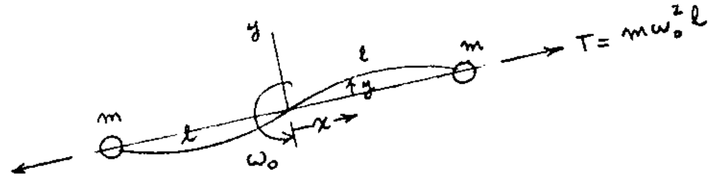
$$-\omega^2 Y(x) = g\left(x\frac{d^2Y(x)}{dx^2} + \frac{dY(x)}{dx}\right)$$

Let $z^2 = \frac{4\omega^2}{g}x$ $\qquad dx = \frac{g}{4\omega^2}2z\,dz, \quad (dx)^2 = \left(\frac{g2z}{4\omega^2}\right)^2 dz^2$

$$-\omega^2 Y = g\left\{\frac{g}{4\omega^2}z^2\frac{d^2Y}{\frac{gz^2}{4\omega^2}dz^2} + \frac{2\omega^2}{gz}\frac{dY}{dz}\right\}$$

$$\therefore \quad \frac{d^2Y}{dz^2} + \frac{1}{z}\frac{dY}{dz} + Y = 0 \qquad \text{Bessel's D.E}$$

8 - 8



Assume mode shape as shown. Accel. at y is $\ddot{y} - y\omega_0^2$ in lateral direction

$$T\frac{\partial^2 y}{dx^2} = \rho(\ddot{y} - y\omega_0^2) \qquad \text{Let } y = Y(x)e^{i\omega t}$$

$$\frac{d^2Y}{dx^2} + \left[\left(\frac{\omega}{c}\right)^2 + \left(\frac{\omega_0}{c}\right)^2\right]Y(x) = 0 \qquad c = \sqrt{\frac{T}{\rho}}$$

$Y(0) = 0 \quad \therefore \quad B = 0$ and $Y(x) = A\sin\Omega x \neq 0$

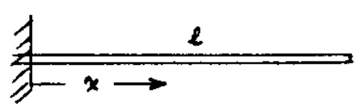
where $\Omega = \sqrt{\left(\frac{\omega}{c}\right)^2 + \left(\frac{\omega_0}{c}\right)^2}$

$Y(l) = 0 \quad \therefore \quad \sin\Omega l = 0$

$\sqrt{\left(\frac{\omega}{c}\right)^2 + \left(\frac{\omega_0}{c}\right)^2}\; l = \pi \qquad \omega^2 = \left(\frac{\pi c}{l}\right)^2 - \omega_0^2$

$$\therefore \quad \underline{\underline{\omega^2 = \left(\frac{\pi}{l}\right)^2\left(\frac{m\omega_0^2 l}{\rho}\right) - \omega_0^2}}$$

8 - 9



$$u = \left(A\sin\frac{\omega x}{c} + B\cos\frac{\omega x}{c}\right)\sin\omega t$$

$u(0) = 0 \quad \therefore B = 0$

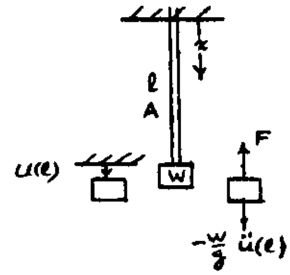
$$\sigma = E\left(\frac{du}{dx}\right)_{x=l} = 0 \quad \therefore \quad \frac{\omega}{c}\cdot\cos\frac{\omega l}{c} = 0$$

$$\frac{\omega l}{c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \cdots = (n+\tfrac{1}{2})\pi \qquad n = 0, 1, 2 \cdots$$

$$f_n = \frac{\omega_n}{2\pi} = (n+\tfrac{1}{2})\frac{c}{2l}, \quad n = 0, 1, 2, \cdots$$

8-10

$$u = C_1 \sin\frac{\omega x}{c}\sin\omega t$$



$$F = A\sigma = AE\frac{\partial u}{\partial x} = AEC_1\frac{\omega}{c}\cos\frac{\omega x}{c}\sin\omega t$$

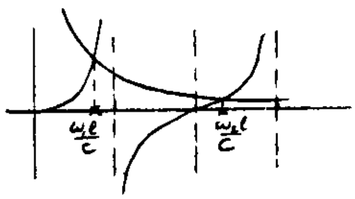
$$= -\frac{W}{g}\ddot{u}(l)$$

$$\therefore\ AE\frac{\omega}{c}\cos\frac{\omega l}{c} = \omega^2\frac{W}{g}\sin\frac{\omega l}{c}$$

$$\frac{\omega l}{c}\tan\frac{\omega l}{c} = \frac{A\rho l}{W} = \frac{W_{rod}}{W_{end}}$$

8-11

From Prob. 8-10 $\quad \tan\frac{\omega l}{c} = \dfrac{\left(\frac{m_{rod}}{M}\right)}{\frac{\omega l}{c}}$



Let freq. eq. be $\omega_1 = m_1\sqrt{\frac{Eg}{\rho_w}}$

where m_1 depends on end cond.

ρ_w = weight density

ρ = mass "

$$\omega_1 = m_1\sqrt{\frac{EA}{l}\frac{gl^2}{\rho_w Al}} = m_1 l\sqrt{\frac{EA}{l}\frac{g}{\rho_w Al}} = m_1 l\sqrt{k/m_{rod}}$$

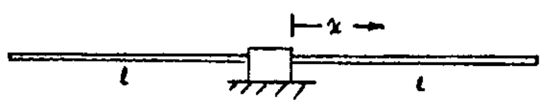
Let $r = \frac{m_{rod}}{M}$, then $\omega_1 = m_1 l\sqrt{\frac{k}{rM}}$

Approx sol.

$$\omega_{1\,appr.} \doteq \sqrt{\frac{AE/l}{M+\frac{1}{3}m_{rod}}} = \sqrt{\frac{k}{M+\frac{r}{3}M}}$$

$$\frac{\omega_{appr.}}{\omega_1} = \sqrt{\frac{k}{M(1+\frac{1}{3}r)}}\sqrt{\frac{rM}{k}}\left(\frac{1}{m_1 l}\right) = \frac{1}{\beta_1}\sqrt{\frac{3r}{3+r}}$$

8-12



$$u(0) = 0 \quad \therefore\ u(x) = A\sin\omega\frac{x}{l}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=l} = 0 \quad \therefore\ \cos\frac{\omega l}{c} = 0 \qquad \frac{\omega l}{c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \cdots$$

$$\omega_1 = \frac{\pi}{2}\frac{c}{l} = \frac{\pi}{2l}\sqrt{\frac{Eg}{\rho_w}} = 2\pi(20{,}000)$$

$$l = \frac{1}{4\times 20{,}000}\sqrt{\frac{Eg}{\rho_w}} = \frac{10^3}{80{,}000}\sqrt{\frac{30\times 386}{0.31}} = 7.64''$$

Note $\rho = \frac{\rho_w}{g}$ = mass density

8 - 13

$$m\frac{\partial^2 u}{\partial t^2} = AE\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial u}{\partial t} + \frac{p_0}{l}\,p(x)\,f(t)$$

$u = \sum_i \phi_i(x)\,q_i(t)$ subst into eq. above

$$m\sum_i \phi_i \ddot{q}_i = AE\sum_i \phi_i'' q_i - \alpha\sum_i \phi_i \dot{q}_i + \frac{p_0}{l}\,p(x)\,f(t)$$

mult. by $\phi_j\,dx$ and integrate over $x = 0$ to l

$$m\int_0^l \phi_j \sum_i \phi_i \ddot{q}_i\,dx = AE\int_0^l \phi_j \sum_i \phi_i'' q_i\,dx - \alpha\int_0^l \phi_j \sum_i \phi_i \dot{q}_i\,dx + \frac{p_0}{l}\int_0^l p(x)\phi_j\,dx\,f(t)$$

since ϕ_j and ϕ_i are orthogonal

$$\ddot{q}_j\int_0^l m\,\phi_j^2\,dx = AE\,q_j\int_0^l \phi_j\phi_j''\,dx - \alpha\,\dot{q}_j\int_0^l \phi_j^2\,dx + \frac{p_0}{l}f(t)\int_0^l p(x)\phi_j\,dx$$

$$\ddot{q}_j + 2\zeta\omega_j\dot{q}_j + \omega_j^2 q_j = \frac{p_0}{ml}f(t)\int_0^l p(x)\,\phi_j\,dx$$

$$\text{with}\quad b_j = \frac{1}{l}\int_0^l p(x)\,\phi_j\,dx$$

$$\ddot{q}_j + 2\zeta\omega_j\dot{q}_j + \omega_j^2 q_j = \frac{p_0}{m}\,b_j\,f(t)$$

From Eq 4.2-1

$$q_j = \frac{p_0}{m}\,b_j\int_0^l f(t-\tau)\,e^{-\zeta\omega_j\tau}\sin\omega_j\sqrt{1-\zeta^2}\,\tau\,d\tau$$

$$u = \sum_j \phi_j\,q_j$$

---

8 - 14 From Eq. 8.3-3 $c = \sqrt{\dfrac{gG}{\rho_w}}$

for steel $G = 12\times10^6$ psi

$\rho_w = 0.282$ lb/in^3

$$c = 10^3\sqrt{\frac{12\times386}{0.282}} = 128{,}000 \text{ in/sec} = 10{,}660 \text{ ft/sec}$$

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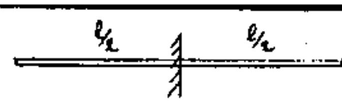
8 - 15



$$\theta = A\sin\frac{\omega x}{c}$$

$$\left(\frac{\partial\theta}{\partial x}\right)_{x=\frac{l}{2}} = 0 \qquad \therefore\quad \cos\frac{\omega l}{2c} = 0$$

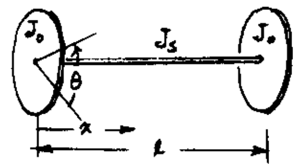
$$\frac{\omega l}{2c} = \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2}\ \cdots$$

$$\omega_n = \frac{\pi c}{l},\ \frac{3\pi c}{l},\ \cdots = (2n-1)\frac{c}{l} \qquad n = 1, 2, 3\ \cdots$$

8-16

$$J_0 \ddot{\theta}_{x=0} = G I_p \left(\frac{d\theta}{dx}\right)_{x=0}$$

$$J_0 \ddot{\theta}_{x=l} = -G I_p \left(\frac{d\theta}{dx}\right)_{x=l}$$



$$\theta = \left(A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c}\right) \sin \omega t$$

$$-\omega^2 J_0 B = G I_p \frac{\omega}{c} A$$

$$-\omega^2 J_0 \left[A \sin \frac{\omega l}{c} + B \cos \frac{\omega l}{c}\right] = -G I_p \frac{\omega}{c}\left[A \cos \frac{\omega l}{c} - B \sin \frac{\omega l}{c}\right]$$

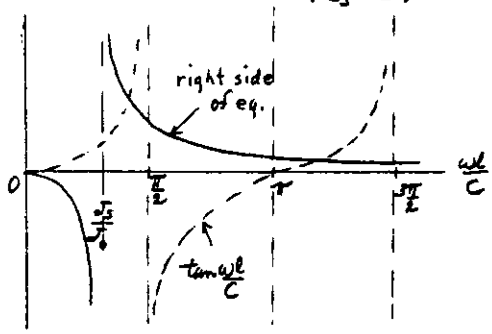
$$\therefore \quad B = -\frac{G I_p}{\omega c J_0} A$$

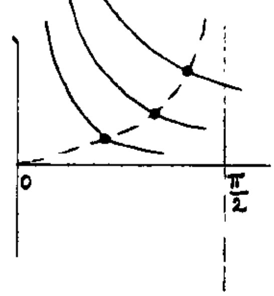
$$\sin \frac{\omega l}{c} - \frac{G I_p}{\omega c J_0} \cos \frac{\omega l}{c} = \frac{G I_p}{\omega c J_0}\left(\cos \frac{\omega l}{c} + \frac{G I_p}{\omega c J_0} \sin \frac{\omega l}{c}\right)$$

$$\frac{G I_p}{\omega c J_0} = \frac{G I_p}{\omega c J_0} \frac{\rho l}{g} \frac{g}{\rho l} = \frac{G g}{\rho} \frac{J_s}{J_0} \frac{1}{\omega l c} = \frac{J_s}{J_0} \frac{c}{\omega l}$$

$$\therefore \left\{\tan \frac{\omega l}{c}\right\}\left\{1 + \left(\frac{J_s}{J_0} \frac{c}{\omega l}\right)^2\right\} = 2 \frac{J_s}{J_0} \frac{c}{\omega l}$$

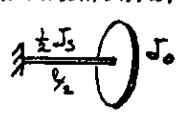
$$\tan \frac{\omega l}{c} = \frac{2\left(\frac{J_s}{J_0} \frac{\omega l}{c}\right)}{\left(\frac{J_0}{J_s} \frac{\omega l}{c}\right)^2 - 1} \qquad \frac{J_s}{J_0} < 1$$





When ends are free ($J_0 = 0$), $\frac{\omega l}{c} = \frac{\pi}{2}$. The effect of J_0 is to Lower nat. freq. If J_0/J_s is very large the right side of eq

$\cong \dfrac{2}{\left(\frac{J_0}{J_s} \frac{\omega l}{c}\right)}$ Fundamental freq. has node at center



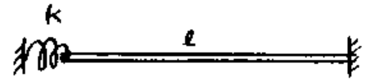
$$\omega_1 \cong \sqrt{\frac{2GI_p/l}{J_0 + \frac{1}{6} J_s}} = \sqrt{\frac{\frac{2Gg}{l} \, J_s/l^2}{J_0 + \frac{1}{6} J_s}} = \frac{c}{l}\sqrt{\frac{2 J_s}{J_0 + \frac{1}{6} J_s}} = \frac{c}{l}\sqrt{\frac{2}{\frac{J_0}{J_s} + \frac{1}{6}}}$$

When $\frac{J_0}{J_s} = 0$ $\quad \frac{\omega_1 l}{c} = \sqrt{3}$

When $\frac{J_0}{J_s} = 5$, exact eq $\tan \frac{\omega l}{c} = \dfrac{10 \frac{\omega l}{c}}{25\left(\frac{\omega l}{c}\right)^2 - 1}$ has root $\frac{\omega_1 l}{c} = 0.622$

Approx eq. $\frac{\omega_1 l}{c} = \sqrt{\frac{2}{5 + \frac{1}{6}}}$ gives 0.62

8 - 17



$$\theta = \left(A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c}\right) \sin \omega t$$

$$\theta(\ell) = 0 \quad \therefore \quad A \sin \frac{\omega \ell}{c} + B \cos \frac{\omega \ell}{c} = 0 , \qquad B = -A \tan \frac{\omega \ell}{c}$$

$$\theta = A\left(\sin \frac{\omega x}{c} - \tan \frac{\omega \ell}{c} \cos \frac{\omega x}{c}\right) \sin \omega t$$

$$\text{Torque at } x = 0 \;=\; K\,\theta(0) = KA\left(-\tan \frac{\omega \ell}{c}\right)$$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = A \, \frac{\omega}{c}(1)$$

$$GI_p \left(\frac{d\theta}{dx}\right)_{x=0} = A \, \frac{\omega}{c} \, GI_p = -KA \tan \frac{\omega \ell}{c}$$

$$\therefore \tan \frac{\omega \ell}{c} = -\frac{\omega I_p G}{Kc} = -\frac{I_p G}{K\ell}\left(\frac{\omega \ell}{c}\right)$$

8 - 18

$$y = A \cosh \beta x + B \sinh \beta x + C \cos \beta x + D \sin \beta x$$

At $x = 0$ and ℓ, $\frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = 0$ so the 4 eqs become

$$A + 0 - C + 0 = 0$$
$$0 + B + 0 - D = 0$$

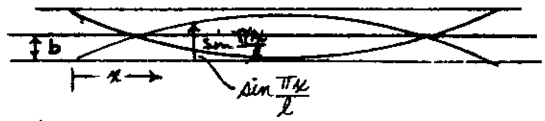
$$A \cosh \beta \ell + B \sinh \beta \ell - C \cos \beta \ell - D \sin \beta \ell = 0$$
$$A \sinh \beta \ell + B \cosh \beta \ell + C \sin \beta \ell - D \cos \beta \ell = 0$$

$$\therefore C = A \quad \text{and} \quad D = B$$

$$-\frac{A}{B} = \frac{(\sinh \beta \ell - \sin \beta \ell)}{(\cosh \beta \ell - \cos \beta \ell)} = \frac{(\cosh \beta \ell - \cos \beta \ell)}{(\sinh \beta \ell + \sin \beta \ell)}$$

$\therefore$ freq. eq. becomes $\cosh \beta \ell \cos \beta \ell = 1.0$

8 — 19



Momentum =

$$m\int_0^l \left(\sin\frac{\pi x}{l} - b\right) dx = 0 \qquad \text{gives} \quad b = \frac{2}{\pi}$$

$$T = \frac{1}{2} m\omega^2 \int_0^l \left(\sin\frac{\pi x}{l} - b\right)^2 dx = \frac{1}{2} m\omega^2\left[\frac{l}{2} - \frac{8}{\pi^2}l + \frac{4}{\pi^2}l\right] = \frac{1}{2} m\omega^2 l\left(\frac{1}{2} - \frac{4}{\pi^2}\right)$$

$$U = \frac{1}{2}EI\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{1}{2}EI\left(\frac{\pi}{l}\right)^4\int_0^l \frac{1}{2}\left(1 - \cos\frac{2\pi x}{l}\right)dx = \frac{1}{4}EI\left(\frac{\pi}{l}\right)^4\frac{l}{2}$$

Equate T & U $\qquad \omega_1^2 = \frac{\pi^6}{\pi^2 - 8}\left(\frac{EI}{ml^4}\right) = 512\left(\frac{EI}{ml^4}\right)$

$\omega_1 = 24.6\sqrt{\frac{EI}{ml^4}}$ $\qquad$ node at $\left(\sin\frac{\pi x}{l} - \frac{2}{\pi}\right) = 0$ or $\frac{x}{l} = 0.22$

---

8 — 20

$$2\pi f_1 = 22.4\sqrt{\frac{EI}{ml^4}} = 2\pi\ 1690$$

$$m = 2\times 2\times 1\times\frac{153}{1732}\times\frac{1}{386} = 916\times 10^{-6}$$

$$I = \frac{2\times 2^3}{12} = \frac{4}{3} \qquad \frac{EI}{ml^4} = \left(\frac{2\pi\times 1690}{22.4}\right) = 224{,}000$$

$$E = \frac{224{,}000\times 916\times 10^{-6}\times 12^4}{4/3} = 3{,}480{,}000 \ \ \text{lb/in}^2$$

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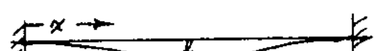
8 — 21



Start with Eq B.4-12, At $x = 0$ and $x = l$, $y = \frac{dy}{dx} = 0$

$$\left.\begin{array}{l} A + 0 + C + 0 = 0 \\ 0 + B + 0 + D = 0 \end{array}\right\} \quad \begin{array}{l} C = -A \\ D = -B \end{array}$$

$$A(\cosh\beta l - \cos\beta l) + B(\sinh\beta l - \sin\beta l) = 0$$

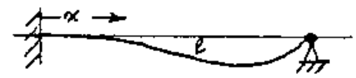
$$A(\sinh\beta l + \sin\beta l) + B(\cosh\beta l - \cos\beta l) = 0$$

$$-\frac{A}{B} = \frac{\sinh\beta l - \sin\beta l}{\cosh\beta l - \cos\beta l} = \frac{\cosh\beta l - \cos\beta l}{\sinh\beta l + \sin\beta l}$$

or $\qquad \cosh\beta l\cdot\cos\beta l = 1$

8-22

Start with Eq. 8.4-12



At $x=0$, $y = \frac{dy}{dx} = 0$ gives $C = -A$

$D = -B$

At $x = \ell$ $y = \frac{d^2y}{dx^2} = 0$

$$A(\cosh\beta\ell - \cos\beta\ell) + B(\sinh\beta\ell - \sin\beta\ell) = 0$$

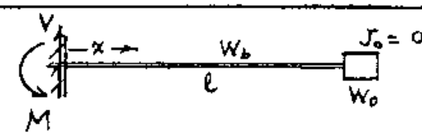
$$A(\cosh\beta\ell + \cos\beta\ell) + B(\sinh\beta\ell + \sin\beta\ell) = 0$$

$$-\frac{A}{B} = \frac{\sinh\beta\ell - \sin\beta\ell}{\cosh\beta\ell - \cos\beta\ell} = \frac{\sinh\beta\ell + \sin\beta\ell}{\cosh\beta\ell + \cos\beta\ell}$$

$$\therefore \cosh\beta\ell\sin\beta\ell - \sinh\beta\ell\cos\beta\ell = 0$$

or $\tanh\beta\ell = \tan\beta\ell$

8-23



At $x=0$, $y = \frac{dy}{dx} = 0$

$\therefore$ $C = -A$, $D = -B$

At $x = \ell$ $-EI\frac{d^3y}{dx^3} = -V = \frac{W_0}{g}\ddot{y}(\ell)$

$$-\beta^3\left[A(\sinh\beta\ell - \sin\beta\ell) + B(\cosh\beta\ell + \cos\beta\ell)\right]$$
$$= -\frac{W_0}{g}\frac{\omega^2}{EI}\left[A(\cosh\beta\ell - \cos\beta\ell) + B(\sinh\beta\ell - \sin\beta\ell)\right.$$

At $x = \ell$ $-M = -EI\frac{d^2y}{dx^2} = J_0\left(\frac{dy}{dx}\right) = 0$

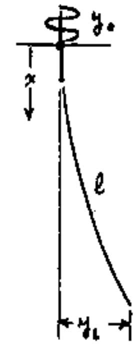
$$\beta^2\left[A(\cosh\beta\ell + \cos\beta\ell) + B(\sinh\beta\ell + \sin\beta\ell)\right] = 0$$

$$-\frac{A}{B} = \frac{(\cosh\beta\ell + \cos\beta\ell) - \frac{W_0\omega^2}{\beta^3 gEI}(\sinh\beta\ell - \sin\beta\ell)}{(\sinh\beta\ell - \sin\beta\ell) - \frac{W_0\omega^2}{\beta^3 gEI}(\cosh\beta\ell - \cos\beta\ell)} = \frac{(\sinh\beta\ell + \sin\beta\ell)}{(\cosh\beta\ell + \cos\beta\ell)}$$

$$\beta^2 = \omega\sqrt{\frac{w\ell}{gEI\ell}} = \omega\sqrt{\frac{W_b}{gEI\ell}}$$

$$\therefore (1 - \cosh\beta\ell\cos\beta\ell) = \frac{W_0}{W_b}\beta\ell(-\cosh\beta\ell\sin\beta\ell + \sinh\beta\ell\cos\beta\ell)$$

8 - 24



At $x=0$, $y=y_0$ $\therefore$ $y_0 = A + C$

At $x=0$ $\frac{d^2y}{dx^2}=0$ $\therefore$ $C = A$

At $x=\ell$ $\frac{d^2y}{dx^2}=0$

$$\beta^2\left[A(\cosh\beta\ell - \cos\beta\ell) + B\sinh\beta\ell - D\sin\beta\ell\right] = 0$$

At $x=\ell$ $\frac{d^3y}{dx^3}=0$

$$\beta^3\left[A(\sinh\beta\ell + \sin\beta\ell) + B\cosh\beta\ell - D\cos\beta\ell\right] = 0$$

At $x=\ell$, $y = y_\ell$

$$y_\ell = A(\cosh\beta\ell + \cos\beta\ell) + B\sinh\beta\ell + D\sin\beta\ell$$

$$\frac{y_0}{y_\ell} = \frac{2A}{A(Ch+C)+BSh+Ds} = \frac{2A}{A(Ch+c)+BSh+A(Ch-c)+BSh}$$

where $Ch = \cosh\beta\ell$, $C = \cos\beta\ell$, $Sh = \sinh\beta\ell$, $s = \sin\beta\ell$

Also $y_0 = 2A$

$$\frac{y_0}{y_\ell} = \frac{\sinh\beta\ell\cos\beta\ell - \cosh\beta\ell\sin\beta\ell}{\sinh\beta\ell - \sin\beta\ell}$$

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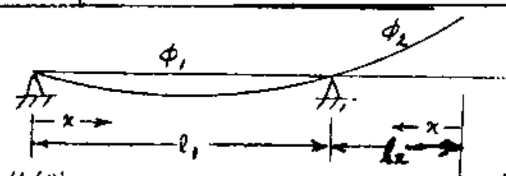
8 - 25



Boundary & intermediate cond.	$y_1(0)=0$	$y_1(\ell)=0$	$y_2''(0)=0$
	$\frac{d^2y_1(0)}{dx^2}=0$	$y_1''(\ell_1)=y_2''(\ell_2)$	$y_2'''(0)=0$

$$\left.\begin{aligned} y_1(0) &= A_1 + C_1 = 0 \quad \therefore C_1 = -A_1 \\ y_1''(0) &= A_1 - C_1 = 0 \quad \therefore C_1 = +A_1 \end{aligned}\right\} \therefore A_1 = C_1 = 0$$

$$y_1 = B_1\sinh\beta x + D_1\sin\beta x$$

$$y_1(\ell_1) = B_1\sinh\beta\ell + D_1\sin\beta\ell = 0 \quad \therefore B_1 = -D_1\frac{\sin\beta\ell_1}{\sinh\beta\ell_1}$$

$$y_1 = D_1\left(\sin\beta x - \frac{\sin\beta\ell_1}{\sinh\beta\ell_1}\sinh\beta x\right) = \phi_1(x)$$

$$y_2''(0) = A_2 - C_2 = 0 \quad \therefore C_2 = A_2$$

$$y_2'''(0) = B_2 - D_2 = 0 \quad \therefore D_2 = B_2$$

$$y_2(\ell_2) = A_2(\cosh\beta\ell_2 + \cos\beta\ell_2) + B_2(\sinh\beta\ell_2 + \sin\beta\ell_2)$$

$$\therefore B_2 = -A\,\frac{\cosh\beta\ell_2 + \cos\beta\ell_2}{\sinh\beta\ell_2 + \sin\beta\ell_2}$$

$$y_2 = A_2\left\{(\cosh\beta x + \cos\beta x) - \left(\frac{\cosh\beta\ell_2 + \cos\beta\ell_2}{\sinh\beta\ell_2 + \sin\beta\ell_2}\right)(\sinh\beta x + \sin\beta x)\right\} = \phi_2(x)$$

9-1 $\quad m\ddot{x} + kx = P_0$

$$x = \frac{P_0}{k}(1-\cos\omega t)\ , \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\therefore\ x_{max} = 2\frac{P_0}{k}$$

9-2

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \frac{P_0}{m}$$

$$q_i = \frac{P_0}{m}\left[1 - \frac{e^{-\zeta_i\omega_i t}}{\sqrt{1-\zeta_i^2}}\cos(\sqrt{1-\zeta_i}\ \omega_i t - \psi_i)\right]$$

$$\tan\psi_i = \frac{\zeta_i}{\sqrt{1-\zeta_i^2}} \qquad \text{For small damping } \psi_i \cong 0$$

$$\sqrt{1-\zeta_i^2} \cong 1$$

$$\therefore\ q_i \cong \frac{P_0}{k}\left[1 - e^{-\zeta_i\omega_i t}\cos\omega_i t\right] = \frac{P_0}{k}D_i(t)$$

9-3 $\quad \ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \dfrac{f(t)}{M}\displaystyle\int_0^l \frac{P_0}{l}\phi_i(x)\,dx$

where $f(x,t) = \dfrac{P_0}{l}f(t)$ and P_0 = total force

Mode participation factor $= \dfrac{1}{l}\displaystyle\int_0^l \phi_i(x)\,dx$

9-4 $\quad$ Generalized force $= \dfrac{P_0}{M_i}\left\{\dfrac{1}{l}\displaystyle\int_0^l p(x)\phi_i(x)\,dx\right\}f(t)$

(See Sec. 9.1)

$$K_i = \frac{1}{l}\int_0^l p(x)\phi_i(x)\,dx\ , \qquad p(x) = l\,\delta(x-a)$$

$$\therefore\ K_i = \frac{1}{l}\int_0^l l\,\delta(x-a)\phi_i(x)\,dx = \phi_i(a)$$

$$q_i = \frac{P_0 K_i}{M_i\omega_i^2}D_i(t)\ , \qquad y(x,t) = \sum_i q_i\phi_i = \sum_i \frac{P_0\phi_i(a)\phi_i(x)}{M_i\omega_i^2}D_i(t)$$

$$\omega_i^2 = (\beta_i l)^2\frac{EI}{M_i l^3} \qquad \therefore\ y(x,t) = \frac{P_0 l^3}{EI}\sum_i \frac{\phi_i(a)\phi_i(x)}{(\beta_i l)^4}D_i(t)$$

9-5

$$K_i = \lim_{\epsilon \to 0} \frac{1}{\ell}\int_0^{\ell} \ell^2 \left[\frac{\delta(x-a-\epsilon)-\delta(x-a)}{\epsilon}\right]\phi_i(x)\,dx$$

$$= \lim_{\epsilon\to 0} \ell\left[\frac{\phi_i(a+\epsilon)-\phi_i(a)}{\epsilon}\right] = \ell \left.\frac{d\phi_i(x)}{dx}\right|_{x=a}$$

$$= \beta_i \ell \cdot \frac{1}{\beta_i}\left.\frac{d\phi_i(x)}{dx}\right|_{x=a} = \beta_i \ell\ \phi_i'(a)$$

see Append. D for ϕ'

9-6

$$K_i = \frac{1}{\ell}\int_0^{\ell} \ell\ \delta(x-\tfrac{\ell}{2})\phi_i(x)\,dx = \phi_i(\tfrac{\ell}{2})$$

$\phi_n(x) = \sqrt{2}\ \sin n\pi\frac{x}{\ell}$ when normalized to total mass M

$\phi_n(\frac{\ell}{2}) = \sqrt{2}\ \sin\frac{n\pi}{2}$, $n = 1, 3, 5, \cdots$

$\omega_n^2 = (\beta_n\ell)^4 \frac{EI}{M\ell^3}$, $\beta_n = \frac{n\pi}{\ell}$, $n = 1, 3, 5, \cdots$

$$y(x,t) = \frac{2P_0\ell^3}{EI}\sum_n \left\{\frac{\sin\frac{n\pi}{2}\cdot \sin n\pi\frac{x}{\ell}}{(n\pi)^4} D_n(t)\right\} \quad n = 1, 3, 5\cdots$$

9-7 From Prob. 9-5 $K_n = \beta\ell\ \phi_n'(a)$, $\beta_n = \frac{n\pi}{\ell}$

$\phi_n(x) = \sqrt{2}\ \sin n\pi\frac{x}{\ell}$ $\frac{d\phi_n}{dx} = \sqrt{2}\ \frac{n\pi}{\ell}\cos n\pi\frac{x}{\ell}$

$\phi_n'(\frac{\ell}{2}) = \frac{1}{\beta_n}\frac{d\phi_n(\frac{\ell}{2})}{dx} = \frac{\sqrt{2}}{\ell}\cos\frac{n\pi}{2}$, $n = 2, 4, 6, \cdots$

$$\ddot{q}_n + \omega_n^2 q_n = \frac{\mathcal{M}_0}{M}\left[\beta_n \ell\ \phi_n'(a)\right] f(t)$$

$$q_n = \frac{\mathcal{M}_0}{M\omega_n^2}\beta_n\ell\ \phi_n'(\tfrac{\ell}{2}) D_n(t)$$

$$y(x,t) = \frac{\mathcal{M}_0}{M}\sum_n \frac{\beta_n\ell}{\omega_n^2}\phi_n'(\tfrac{\ell}{2})\phi_n(x)D_n(t) = \frac{\mathcal{M}_0\ell^3}{EI}\sum_n \frac{\phi_n'(\frac{\ell}{2})\phi_n(x)}{(\beta_n\ell)^3}D_n(t)$$

$$= \frac{2\mathcal{M}_0\ell^2}{EI}\sum_{n=2,4,\cdots}\frac{\cos\frac{n\pi}{2}\sin\frac{n\pi x}{\ell}}{(n\pi)^3}D_n(t)$$

9 - 8 Only even modes, $n = 2, 6, \cdots$ will give $K_n \neq 0$

$$\phi_n = \sqrt{2} \sin \frac{n\pi x}{\ell}$$

$$K_n = \frac{1}{\ell}\int_0^{\ell/2} \sqrt{2} \sin\frac{n\pi x}{\ell} dx - \frac{1}{\ell}\int_{\ell/2}^{\ell} \sqrt{2} \sin\frac{n\pi x}{\ell} dx = \frac{2\sqrt{2}}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)$$

$$D_n = \omega_n \int_0^t \sin \omega_n (t-\xi)\, d\xi = 1 - \cos\omega_n t$$

$$y(x,t) = \sum_n \frac{p_0 \ell K_n \phi_n}{\omega_n^2 M} D_n = \sum_{n=2,6,\cdots} \frac{p_0\ell}{\omega_n^2 M}\frac{2\sqrt{2}}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)\sqrt{2}\sin\frac{n\pi x}{\ell}(1-\cos\omega_n t)$$

$$= \frac{2p_0\ell}{\pi M}\sum_{n=2,6,\cdots} 2\left(1-\cos\frac{n\pi}{2}\right)\sin\frac{n\pi x}{\ell}(1-\cos\omega_n t)$$

1st mode, $n = 2$

$$y_1(x,t) = \frac{4p_0\ell}{\pi M\omega_2^2}\sin\frac{2\pi x}{\ell}(1-\cos\omega_2 t)$$

2nd mode, $n = 6$

$$y_2(x,t) = \frac{4p_0\ell}{\pi M\omega_6^2}\sin\frac{6\pi x}{\ell}(1-\cos\omega_6 t)$$

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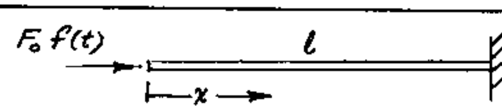
9 - 9



Normal mode $u(x) = \mathcal{A}\sin\frac{\omega x}{c} + \mathcal{B}\cos\frac{\omega x}{c}$

$$\frac{du}{dx} = \frac{\omega}{c}\left\{\mathcal{A}\cos\frac{\omega x}{c} - \mathcal{B}\sin\frac{\omega x}{c}\right\}$$

Boundary Cond.

$x = 0$, $\frac{du}{dx} = 0$ $\therefore \mathcal{A} = 0$

$x = \ell$, $u = 0$ $\therefore \mathcal{B}\cos\frac{\omega\ell}{c} = 0$

$$\frac{\omega\ell}{c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \cdots$$

$$u(x) = \mathcal{B}\cos\frac{\omega x}{c}$$

$$\int_0^\ell \frac{M}{\ell}\mathcal{B}^2\cos^2\frac{\omega x}{c}dx = M = \frac{\beta^2}{\ell}\int_0^\ell \frac{1}{2}\left(1+\cos\frac{2\omega x}{c}\right)dx = \frac{\beta^2}{2}$$

$$\therefore \beta = \sqrt{2} \qquad \phi_n(x) = \sqrt{2}\cos\frac{n\pi}{2}\frac{x}{\ell}, \quad n = 1, 3, 5, \cdots$$

9 - 9 Cont:

$$\ddot{q}_n + \omega_n^2 q_n = \frac{f(t)}{M}\int_0^\ell F_0\,\delta(x)\sqrt{2}\cos\frac{n\pi x}{2\ell}\,dx = \frac{F_0\sqrt{2}}{M}f(t)$$

$$q_n = \frac{F_0 K_n}{\omega_n^2 M} D_n(t) = \frac{\sqrt{2}F_0}{\left(\frac{n\pi}{2}\frac{c}{\ell}\right)^2 M} D_n(t) = \frac{\sqrt{2}F_0 \cancel{M}\ell^{\cancel{2}} D_n(t)}{\left(\frac{n\pi}{2}\right)^2 \cancel{M} E\cancel{\ell} A}$$

$$u(x,t) = \sum_n \phi_n(x) q_n(t) = \frac{2F_0\ell}{EA}\left\{\frac{\cos\frac{\pi x}{2\ell}}{\left(\frac{\pi}{2}\right)^2}\cdot D_1(t) + \frac{\cos\frac{3\pi x}{2\ell}}{\left(\frac{3\pi}{2}\right)^2}\cdot D_3(t) + \cdots\right\}$$

---

9 - 10

$$K_n = \frac{1}{\ell}\int_0^\ell \ell\,\delta\left(x-\frac{\ell}{3}\right)\sqrt{2}\cos\frac{n\pi}{2}\frac{x}{\ell}\,dx = \sqrt{2}\cos\frac{n\pi}{6}$$

$n = 1, 3, 5, \cdots$

n					
1	$\cos\frac{\pi}{6}$	$= \cos 30°$	$= .866$		
3	$\cos\frac{3\pi}{6}$	$= \cos 90°$	$= 0$	∴	mode absent
5	$\cos\frac{5\pi}{6}$	$= \cos 150°$	$= -.866$		
7	$\cos\frac{7\pi}{6}$	$= \cos 210°$	$= -.866$		
9	$\cos\frac{9\pi}{6}$	$= \cos 270°$	$= 0$	∴	mode absent
11	$\cos\frac{11\pi}{6}$	$= \cos 330°$	$= .866$		

∴ modes present are $1, 5, 7, 11, 13, \cdots$

---

9 - 11

$$K_n = \sqrt{2}\cos\frac{n\pi}{6}$$

$$q_n = \frac{F_0 K_n}{\omega_n^2 M} D_n(t) \qquad \omega_n^2 = \left(\frac{n\pi}{2}\frac{c}{\ell}\right)^2 = \left(\frac{n\pi}{2}\right)^2\frac{EA}{M\ell}$$

$$u(x,t) = \frac{2F_0\ell}{AE}\left\{\frac{0.866}{\left(\frac{\pi}{2}\right)^2}\cos\frac{\pi x}{2\ell}\cdot D_1(t) - \frac{0.866}{\left(\frac{5\pi}{2}\right)^2}\cos\frac{5\pi x}{2\ell}\cdot D_5(t)\right.$$

$$\left. - \frac{0.866}{\left(\frac{7\pi}{2}\right)^2}\cos\frac{7\pi x}{2\ell}\cdot D_7(t) + \frac{0.866}{\left(\frac{11\pi}{2}\right)^2}\cos\frac{11\pi x}{2\ell}\cdot D_{11}(t) + \cdots\right\}$$

9-12 $\quad y(x,t) = q_1 \sin\frac{\pi x}{\ell} + q_2$

$$T = \frac{1}{2}\int_0^\ell m\dot{y}^2 dm = \frac{1}{2}\int_0^\ell (\varphi_1 \dot{q}_1 + \varphi_2 \dot{q}_2)^2 dx$$

$$= \frac{1}{2} m\ell \left[\frac{1}{2}\dot{q}_1^2 + \frac{4}{\pi}\dot{q}_1\dot{q}_2 + \dot{q}_2^2 \right]$$

$$U = \frac{1}{2}\left(\frac{k}{2}\right)y^2(0) + \frac{1}{2}\left(\frac{k}{2}\right)y^2(\ell) + \frac{1}{2}\int_0^\ell EI\, y''^2 dx$$

$$= \frac{1}{2}kq_2^2 + \frac{1}{2}EI\,q_1^2\int_0^\ell \left(\frac{\pi}{\ell}\right)^4 \sin^2\frac{\pi x}{\ell}\,dx$$

$$= \frac{1}{2}kq_2^2 + \frac{1}{2}EI\,q_1^2\left(\frac{\pi}{\ell}\right)^4\frac{\ell}{2} \qquad \text{sub. into } \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} = 0$$

$$\ddot{q}_1 + \frac{4}{\pi}\ddot{q}_2 + \pi^4\frac{EI}{m\ell^4}q_1 = 0, \quad \text{and} \quad \frac{2}{\pi}\ddot{q}_1 + \ddot{q}_2 + \frac{k}{m\ell}q_2 = 0$$

rewrite

$$\ddot{q}_1 + \frac{4}{\pi}\ddot{q}_2 + \omega_{11}^2 q_1 = 0 \quad \text{and} \quad \frac{2}{\pi}\ddot{q}_1 + \ddot{q}_2 + \omega_{22}^2 q_2 = 0$$

$$\left.\begin{aligned} (\omega_{11}^2 - \omega^2)q_1 - \frac{4}{\pi}\omega^2 q_2 &= 0 \\ -\frac{2}{\pi}\omega^2 q_1 + (\omega_{22}^2 - \omega^2)q_2 &= 0 \end{aligned}\right\} \quad \omega^4 - \left(\frac{\pi^2}{\pi^2-8}\right)(\omega_{11}^2 + \omega_{22}^2)\omega^2 + \frac{\pi^2\omega_{11}^2\omega_{22}^2}{\pi^2-8} = 0$$

$$\omega^2 = \frac{\omega_{22}^2}{2}\left(\frac{\pi^2}{\pi^2-8}\right)\left[(1+R) \pm \sqrt{(1-R)^2 + \frac{32}{\pi^2}R}\right] \quad \text{where } R = \left(\frac{\omega_{11}}{\omega_{22}}\right)^2$$

Assume $y = (b + \sin\frac{\pi x}{\ell})q$

$$T = \frac{1}{2}m\dot{q}^2\int_0^\ell \left(b^2 + 2b\sin\frac{\pi x}{\ell} + \sin^2\frac{\pi x}{\ell}\right)dx = \frac{1}{2}m\dot{q}^2\left(b^2\ell + 4b\frac{\ell}{\pi} + \frac{\ell}{2}\right)$$

$$U = \left[\frac{1}{2}kb^2 + \frac{1}{2}EI\left(\frac{\pi}{\ell}\right)^4\frac{\ell}{2}\right]q^2 \qquad T = U \text{ gives}$$

$$\omega^2 = \frac{\frac{k}{M}b^2 + \frac{1}{2}\omega_{11}^2}{b^2 + \frac{4b}{\pi} + \frac{1}{2}}$$

Momentum = 0 gives

$$kb - m\omega^2\int_0^\ell \left(\sin\frac{\pi x}{\ell} + b\right)dx = 0 \quad \therefore\ kb - m\omega^2\left(-2\frac{\ell}{\pi} + b\ell\right) = 0$$

$$b = \frac{2}{\pi}\left(\frac{\omega^2}{\omega_{22}^2 - \omega^2}\right) = q_2/q_1 \quad \text{or} \quad \omega^2 = \frac{\omega_{22}^2 b}{b + \frac{2}{\pi}}$$

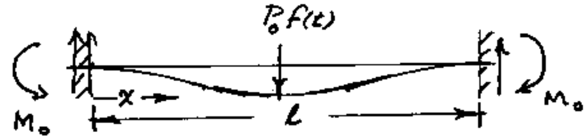
Equating ω^2

$$\frac{\omega_{22}^2 b}{b + \frac{2}{\pi}} = \frac{\omega_{22}^2 b^2 + \frac{1}{2}\omega_{11}^2}{b^2 + \frac{4b}{\pi} + \frac{1}{2}} \quad \therefore\ b^2 + \frac{\pi}{4}(1-R)b + \frac{1}{2}R = 0$$

$$b = \frac{\pi}{8}\left[(R-1) \pm \sqrt{(R-1)^2 + \frac{32}{\pi^2}R}\right] \qquad R = \left(\frac{\omega_{11}}{\omega_{22}}\right)^2$$

9 - 13

$y = \sum_n \phi_n q_n$



Generalized force

$$= P_0 f(t) \int_0^\ell \phi_n(x)\delta(x-\tfrac{\ell}{2})\,dx = P_0 \phi_n(\tfrac{\ell}{2}) f(t)$$

$$y(\tfrac{\ell}{2}, t) = \sum_n \phi_n(\tfrac{\ell}{2}) q_n(t) \quad \text{where } q_n = \text{solution of}$$

$$\ddot{q}_n + \omega_n^2 q_n = \frac{P_0}{M}\phi_n(\tfrac{\ell}{2}) f(t)$$

Solution

$$q_n(t) = q_n(0)\cos\omega_n t + \frac{1}{\omega_n}\dot{q}_n(0)\sin\omega_n t + \frac{P_0\phi_n(\frac{\ell}{2})}{M\omega_n^2}\,\omega_n\int_0^t f(\xi)\sin\omega_n(t-\xi)\,d\xi$$

From Appendix D

$\phi_1(\frac{\ell}{2}) = 1.583$,	$\beta_1\ell = 4.73$	$\phi_1''(0) = 2.0$
$\phi_2(\frac{\ell}{2}) = 0$	$\beta_2\ell = 7.85$	$\phi_2''(0) = 2.0$
$\phi_3(\frac{\ell}{2}) = -1.372$	$\beta_3\ell = 10.99$	$\phi_3''(0) = 2.0$

$$M_0 = EI\left(\frac{d^2y}{dx^2}\right)_0 = EI\sum_n q_n(t)\left(\frac{d^2\phi_n}{dx^2}\right)_{x=0} = EI\sum_n q_n(t)\beta_n^2\phi_n''(0)$$

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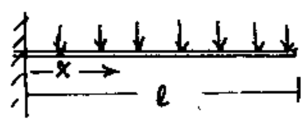
9 - 14



$$K_i = \frac{1}{\ell}\int_0^\ell \phi_i(x)\,dx$$

The above integral for K_i can be evaluated from tables in App. D. Add column of $\phi_i(x)$ and mult. by interval $\frac{dx}{\ell} = 0.04$. 1st mode column sum is 20.57 ∴ $20.57 \times 0.04 = 0.822$. Exact value from integrating Eq. 8.4-12 with proper boundary conditions is 0.783. i.e. use $y(x) = (\cosh\beta x - \cos\beta x) - \left(\frac{\sinh\beta\ell - \sin\beta\ell}{\cosh\beta\ell + \cos\beta\ell}\right)(\sinh\beta x - \sin\beta x)$

<u>Note:</u> The number 20.57 is too large since it was computed with the largest deflection of $2.0 \times .04 + 1.89 \times .04 + \cdots$ etc. If the sum is started with $1.89 \times .04 + \cdots$ etc, then a figure of 18.57 is obtained. Averaging $\frac{1}{2}(18.57 + 20.57) \times .04$ we obtain the exact value 0.783.

Letting $\alpha_n = \frac{\sinh\beta\ell - \sin\beta\ell}{\cosh\beta\ell + \cos\beta\ell}$, the integral can be shown to be

$$\int_0^\ell \phi_n(x)\frac{dx}{\ell} = \frac{2\alpha_n}{\beta_n\ell}$$

9-15 From Eq. 9.3-8 $\cancel{\bar{q}} M(\omega_1^2-\omega^2) = -k\phi_1^2(\frac{l}{3})\cancel{\bar{q}}$

$$1-\left(\frac{\omega}{\omega_1}\right)^2 = -\frac{k}{M}\phi_1^2(\tfrac{l}{3})\frac{1}{\omega_1^2} \qquad \left(\frac{\omega}{\omega_1}\right)^2 = 1+\frac{k}{M}\phi_1^2(\tfrac{l}{3})\frac{1}{\omega_1^2}$$

For simply supported beam $\phi_1(x) = \sqrt{2}\sin\frac{\pi x}{l}$

$$\therefore \phi_1^2(\tfrac{l}{3}) = 2\sin^2\frac{\pi}{3} = 1.50 \qquad \omega_1^2 = \pi^4\frac{EI}{Ml^3} \text{ for unconstrained beam.}$$

$$\therefore \left(\frac{\omega}{\omega_1}\right)^2 = 1+1.5\left(\frac{k}{M}\right)\left(\frac{Ml^3}{\pi^4 EI}\right)$$

9-16 From Eq. 9.3-8

$$\bar{q}_1 = \frac{1}{M(\omega_1^2-\omega^2)}\left\{-k\phi_1(\tfrac{l}{3})\left[\bar{q}_1\phi_1(\tfrac{l}{3})+\bar{q}_2\phi_2(\tfrac{l}{3})\right]\right\}$$

$$\bar{q}_2 = \frac{1}{M(\omega_2^2-\omega^2)}\left\{-k\phi_2(\tfrac{l}{3})\left[\bar{q}_1\phi_1(\tfrac{l}{3})+\bar{q}_2\phi_2(\tfrac{l}{3})\right]\right\}$$

Freq. Eq. becomes

$$\begin{vmatrix} \left[(\omega_1^2-\omega^2)+\frac{k}{M}\phi_1^2(\tfrac{l}{3})\right] & \frac{k}{M}\phi_1(\tfrac{l}{3})\phi_2(\tfrac{l}{3}) \\ \frac{k}{M}\phi_1(\tfrac{l}{3})\phi_2(\tfrac{l}{3}) & \left[(\omega_2^2-\omega^2)+\frac{k}{M}\phi_2^2(\tfrac{l}{3})\right] \end{vmatrix} = 0$$

9-17 We use here a somewhat different procedure than that of the text in that the influence coefs. are found from the beam eq. $F(a,t) = -k\,y(\frac{l}{3})$

$$\alpha(a,x) = \alpha(\tfrac{l}{3},\tfrac{l}{3}) = \frac{(\frac{l}{3})(\frac{2l}{3})}{6EIl}\left(l^2-\frac{4l^2}{9}-\frac{l^2}{9}\right) = \frac{4}{243}\frac{l^3}{EI}$$

From Eq. 9.4-4

$$y(\tfrac{l}{3}) = -k\,y(\tfrac{l}{3})\frac{4}{243}\frac{l^3}{EI}+\left(\frac{\omega}{\omega_1}\right)^2 q_1\phi_1(\tfrac{l}{3})+\left(\frac{\omega}{\omega_2}\right)^2 q_2(\tfrac{l}{3})+\cdots$$

Using 1st mode only

$$y(\tfrac{l}{3})\left[1+\frac{4}{243}\frac{kl^3}{EI}\right] = \left(\frac{\omega}{\omega_1}\right)^2 q_1\phi_1(\tfrac{l}{3})$$

but $y(\frac{l}{3}) = \phi_1(\frac{l}{3})q_1$ + higher modes which are neglected.

$$\therefore \left(\frac{\omega}{\omega_1}\right)^2 = 1+\frac{4}{243}\frac{kl^3}{EI}$$

Comparison with Prob. 9-15

$\frac{1.5}{\pi^4} = 0.0154$

$\frac{4}{243} = 0.0165$

9 - 18 From Eq. 9.4-4 with 1 mode

$$\phi_1(a)\ddot{\bar{q}}_1(t) = F(a,t)\,\alpha(a,a) - \left(\frac{\omega}{\omega_1}\right)^2 \bar{q}_1(t)\,\phi_1(a)$$

$$\phi_1(a)\bar{q}_1(t) = -k\,\phi_1(a)\,\bar{q}_1(t)\,\frac{\phi_1^2(a)}{M\omega_1^2} - \left(\frac{\omega}{\omega_1}\right)^2 \bar{q}_1(t)\,\phi_1(a)$$

$$\therefore \quad \left(\frac{\omega}{\omega_1}\right)^2 = 1 + \frac{k}{M\omega_1^2}\phi_1^2(a)$$

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9 - 19

$$\bar{q}_1(\omega_1^2 - \omega^2) = -\frac{K}{M}\varphi_1'(0)\left[\bar{q}_1\varphi_1'(0) + \bar{q}_2\varphi_2'(0)\right]$$

$$\bar{q}_2(\omega_2^2 - \omega^2) = -\frac{K}{M}\varphi_2'(0)\left[\bar{q}_1\varphi_1'(0) + \bar{q}_2\varphi_2'(0)\right]$$

Freq. eq. becomes

$$\begin{vmatrix} \left[(\omega_1^2-\omega^2) + \frac{K}{M}\varphi_1'^2(0)\right] & \frac{K}{M}\varphi_1'(0)\varphi_2'(0) \\ \frac{K}{M}\varphi_1'(0)\varphi_2'(0) & \left[(\omega_2^2-\omega^2)+\frac{K}{M}\varphi_2'(0)\right] \end{vmatrix} = 0$$

$\omega_2^2 = 16\,\omega_1^2$ for simply supported beam, Let $\lambda = \left(\frac{\omega}{\omega_1}\right)^2$

$$\left[(1-\lambda)+\frac{K}{M\omega_1^2}\varphi_1'^2(0)\right]\left[(16-\lambda)+\frac{K}{M\omega_1^2}\varphi_2'^2(0)\right] - \left(\frac{K}{M\omega_1^2}\right)^2\left[\varphi_1'(0)\,\varphi_2'(0)\right]^2 = 0$$

$$\lambda^2 - \left\{17 + \frac{K}{M\omega_1^2}\left[\varphi_1'^2(0)+\varphi_2'^2(0)\right]\right\}\lambda + \left\{16 + \frac{K}{M\omega_1^2}\left[\varphi_2'^2(0) + 16\,\varphi_1'^2(0)\right]\right\} = 0$$

$\varphi_1(x) = \sqrt{2}\sin\frac{\pi x}{\ell}$ $\qquad \varphi_1'(0) = \sqrt{2}\,\frac{\pi}{\ell}$ $\qquad \varphi_1'^2(0) = 2\left(\frac{\pi}{\ell}\right)^2$

$\varphi_2(x) = \sqrt{2}\sin\frac{2\pi x}{\ell}$ $\qquad \varphi_2'(0) = 2\sqrt{2}\,\frac{\pi}{\ell}$ $\qquad \varphi_2'^2(0) = 8\left(\frac{\pi}{\ell}\right)^2$

If $K = 0$

$$\lambda^2 - 17\lambda + 16 = (\lambda-1)(\lambda-16) = 0$$

$$\therefore \begin{cases}\lambda_1 = 1 \\ \lambda_2 = 16\end{cases}$$

If $K \neq 0$ Let $\alpha = \frac{K}{M\omega_1^2}$

$$\lambda^2 - \left[17 + 10\left(\frac{\pi}{\ell}\right)^2\alpha\right]\lambda + \left[16 + 40\left(\frac{\pi}{\ell}\right)^2\alpha\right] = 0$$

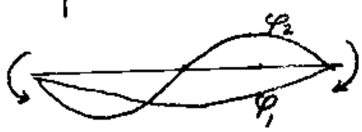
9 - 20

$$\bar{q}_1(\omega_1^2-\omega^2) = -\frac{K}{M}\varphi_1'(0)[\bar{q}_1\varphi_1'(0)+\bar{q}_2\varphi_2'(0)] - \frac{K}{M}\varphi_1'(\ell)[\bar{q}_1\varphi_1'(\ell)+\bar{q}_2\varphi_2'(\ell)]$$

$$\bar{q}_2(\omega_2^2-\omega^2) = -\frac{K}{M}\varphi_2'(0)[\bar{q}_1\varphi_1'(0)+\bar{q}_2\varphi_2'(0)] - \frac{K}{M}\varphi_2'(\ell)[\bar{q}_1\varphi_1'(\ell)+\bar{q}_2\varphi_2'(\ell)]$$

Freq. eq.

$$\begin{vmatrix} (\omega_1^2-\omega^2)+\frac{K}{M}[\varphi_1'^2(0)+\varphi_1'^2(\ell)] & \frac{K}{M}[\varphi_1'(0)\varphi_2'(0)+\varphi_1'(\ell)\varphi_2'(\ell)] \\ \frac{K}{M}[\varphi_1'(0)\varphi_2'(0)+\varphi_1'(\ell)\varphi_2'(\ell)] & (\omega_1^2-\omega^2)+\frac{K}{M}[\varphi_2'^2(0)+\varphi_2'^2(\ell)] \end{vmatrix} = 0$$



since $\varphi_1'(0) = -\varphi_1'(\ell)$

$\varphi_1'^2(0) = \varphi_1'^2(\ell)$

$\varphi_2'^2(0) = \varphi_2'^2(\ell)$

$\varphi_1'(0)\varphi_2'(0) = 4\left(\frac{\pi}{\ell}\right)^2$ and $\varphi_1'(\ell)\varphi_2'(\ell) = -4\left(\frac{\pi}{\ell}\right)^2$ off diagonal terms are zero

$$[(1-\lambda)+\alpha\, 2\varphi_1'^2(0)][(16-\lambda)+\alpha\, 2\varphi_2'^2(0)] = 0$$

$$\lambda^2 - \left[17+2\alpha\, 10\left(\frac{\pi}{\ell}\right)^2\right]\lambda + \left[16+2\alpha\, 40\left(\frac{\pi}{\ell}\right)^2\right] = 0$$

$\therefore$ α of Prob 9-19 is changed to 2α in Prob 9-20

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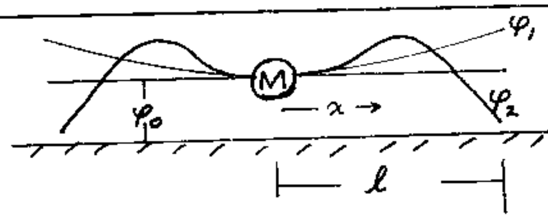
9 - 21



$\varphi_0 = 1$

$\varphi_1(x)$ and $\varphi_2(x)$ etc are cantilever modes

$$y(x,t) = \varphi_0 q_0(t) + \sum_{n=1}^{\infty}\varphi_n(x)q_n(t)$$

For free-free natural modes, momentum must be zero.

$$\therefore 2\int_0^\ell m\,y(x,t)\,dx + M_0\,y(0,t) = 0$$

$$2\sum_{n=1}^{\infty} q_n(t)\int_0^\ell m\,\varphi_n(x)\,dx + (M_0+2m)\varphi_0 q_0 = 0$$

If only 1 mode + translation is used $(M+2m)q_0 + 2q_1 m\ell(.783) = 0$

$$q_0 = -\left(\frac{2\times .783\, m\ell}{M_0+2m}\right)q_1$$

$$T = \frac{1}{2}\int_0^\ell 2m\,\dot{y}^2(x,t)\,dx + \frac{1}{2}M_0\,\dot{y}^2(0,t)$$

$$= \int_0^\ell m[\varphi_0\dot{q}_0+\varphi_1\dot{q}_1]^2\,dm + \frac{1}{2}M_0\varphi_0^2\dot{q}_0^2$$

9 - 21 Cont:

$$T = \int_0^\ell m\varphi_0^2 \dot{q}_0^2 dm + \int_0^\ell 2m\varphi_0\varphi_1 \dot{q}_0 \dot{q}_1 dx + \int_0^\ell m\varphi_1^2 \dot{q}_1^2 dx + \tfrac{1}{2} M_0 \varphi_0^2 \dot{q}_0^2$$

$$= \tfrac{1}{2}\dot{q}_0^2 \left\{ M_0 + 2m\ell \right\} + \left\{ 2m\ell \cdot .783 \right\} \dot{q}_0 \dot{q}_1 + m\ell \dot{q}_1^2$$

$$U = \tfrac{1}{2}\int_0^\ell 2EI\, y''^2(x,t)\,dx = \tfrac{1}{2}\left\{ 2\omega_1^2 m\ell\, q_1^2 \right\} \qquad \text{see Eq. 9.1-7}$$

Lagranges Eqs.
$$\ddot{q}_0 (M_0 + 2m\ell) + 2\times .783\, m\ell\, \ddot{q}_1 = 0$$
$$\ddot{q}_1 2m\ell + 2\times .783\, m\ell\, \ddot{q}_0 + 2\omega_1^2 m\ell\, q_1 = 0$$

Freq. Eq
$$\begin{vmatrix} -(M_0 + 2m\ell)\omega^2 & -2\times .783\, m\ell\omega^2 \\ -2\times .783\, m\ell\omega^2 & 2(\omega_1^2 - \omega^2)m\ell \end{vmatrix} = 0$$

Let $2m\ell = M$

$$-(M_0 + M)\omega^2(\omega_1^2 - \omega^2)M - (.783)^2 M^2 \omega^4 = 0$$

$$\frac{\omega}{\omega_1} = \sqrt{\frac{M_0 M + M^2}{M_0 M + .387 M^2}}$$ where ω_1 = 1st nat. freq. of cantilever beam of length ℓ and mass $\frac{1}{2}M$

If $M_0 \to 0$, then $\omega = \omega_1 \dfrac{1}{\sqrt{.387}} = 1.61\,\omega_1$

Since $\omega_1 = 3.52\sqrt{\dfrac{EI}{m\ell^4}}$ and for free-free beam of length 2ℓ has nat. freq $\omega_{1_{ff}} = 22.4\sqrt{\dfrac{EI}{m(2\ell)^4}} = 5.57\sqrt{\dfrac{EI}{m\ell^4}}$

$\dfrac{5.57}{3.52} = 1.58$. Since the airplane becomes a free-free beam of length 2ℓ as $M_0 \to 0$, the above result is approximately correct.

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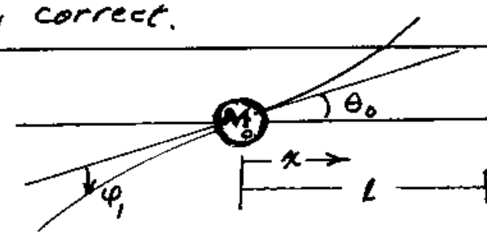
9 - 22



$$y(x,t) = q_0 \frac{x}{\ell} + \varphi_1 q_1$$

$$\theta_0 = \left(\frac{dy}{dx}\right)_{x=0} = \frac{q_0}{\ell}$$

$$T = \tfrac{1}{2} I_0 \left(\frac{\dot{q}_0}{\ell}\right)^2 + \tfrac{1}{2}\int_0^\ell 2m\left[\dot{q}_0 \frac{x}{\ell} + \varphi_1 \dot{q}_1\right]^2 dx$$

$$= \tfrac{1}{2} I_0 \left(\frac{\dot{q}_0}{\ell}\right)^2 + \tfrac{1}{3} m\ell \dot{q}_0^2 + 2m\dot{q}_0 \dot{q}_1 \int_0^\ell \frac{x}{\ell}\varphi_1(x)\,dx + m\dot{q}_1^2 \int_0^\ell \varphi_1^2\, dx$$

9 - 22 Cont:

$$U = \frac{1}{2}\left\{2\omega_1^2(m\ell)q_1^2\right\} \qquad \text{see Eq. 9.1-7}$$

Lagranges Eq. & characteristic eq.

$$\begin{vmatrix} -\left(\frac{\bar{I}_0}{\ell^2}+\frac{2m\ell}{3}\right)\omega^2 & -\frac{2m\omega^2}{\ell}\int_0^\ell x\varphi_1\,dx \\ -\frac{2m\omega^2}{\ell}\int_0^\ell x\varphi_1\,dx & 2m\left(\ell\omega^2-\omega^2\int_0^\ell\varphi_1^2dx\right)\end{vmatrix}=0$$

For cantilever $\int_0^\ell \varphi_1^2 dx = \ell$, $\int_0^\ell x\varphi_1 dx = \frac{2}{\beta_1^2} = \frac{2\ell}{3.516} = \frac{\ell^2}{1.758}$

$$\omega^4\left[2m\ell\left(\frac{I_0}{\ell^2}+\frac{2}{3}m\ell\right)-\frac{(2m\ell)^2}{3.08}\right]=\omega^2\omega_1^2\,2m\ell\left(\frac{I_0}{\ell^2}+\frac{2}{3}m\ell\right)$$

$$\omega = \omega_1\sqrt{\frac{(I_0+\frac{2}{3}m\ell^3)}{(I_0+\frac{2}{3}m\ell^3)-\frac{2m\ell^3}{3.090}}}$$

To check for case $I_0=0$, the results should be the 2^{nd} mode of a hinged-free beam of length ℓ or 2^{nd} mode of free-free beam of length 2ℓ.

For $I_0=0$, $\quad \omega = \omega_1\sqrt{\frac{3.09}{.090}} = \omega_1\sqrt{34.5} = 5.85\,\omega_1$

$$= 20.6\sqrt{\frac{EI}{m\ell^4}}$$ * However correct result should be $15.4\sqrt{\frac{EI}{m\ell^4}}$

*Difficulty due to difference of two numbers in denominator which is small. Needs 2^{nd} mode φ_2 for better results.

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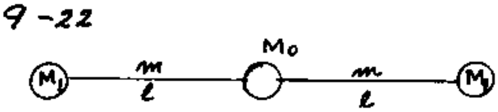
9 - 23 See Prob. 9-21 and 9-22



T has added term $2(\frac{1}{2})M_1\dot{y}^2(\ell,t)$

$$= M_1[\varphi_0\dot{q}_0+\varphi_1(\ell)\dot{q}_1]^2 = M_1\varphi_0^2\dot{q}_0^2+2M_1\varphi_0\varphi_1(\ell)\dot{q}_0\dot{q}_1+M_1\varphi_1^2(\ell)\dot{q}_1^2$$

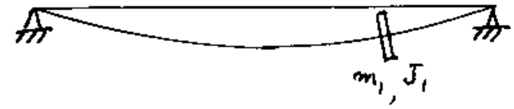
Lagrange's Eq. same as Prob 9-21 $+\,2M_1[\ddot{q}_0+\varphi_1(\ell)\ddot{q}_1]+2M_1[\varphi_1(\ell)\ddot{q}_0+\varphi_1^2(\ell)\ddot{q}_1]$

New freq. eq.

$$\begin{vmatrix} -(M_0+2m\ell+2M_1)\omega^2 & -(2\times.783m\ell+2M_1\varphi_1(\ell))\omega^2 \\ -(2\times.783m\ell+2M_1\varphi_1(\ell))\omega^2 & [2m\ell\omega_1^2-\omega^2(2m\ell+2M_1\varphi_1^2(\ell))]\end{vmatrix}=0$$

9 - 24

The additional mass changes only $T' = \frac{1}{2}m_1\dot{y}^2 + \frac{1}{2}J_1\dot{y}'^2 + \frac{1}{2}\int \dot{y}^2 dm$



$y(x,t) = \varphi_1(x)\, q_1(t)$

$T' = \frac{1}{2} m_1 \varphi_1^2(a)\dot{q}_1^2 + \frac{1}{2} J_1 \varphi_1'^2(a)\dot{q}_1^2 + \frac{1}{2}\int \varphi_1^2(x)\,dm \cdot \dot{q}_1^2$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} = m_1\varphi_1^2(a)\ddot{q}_1 + J_1\varphi_1'^2(a)\ddot{q}_1 + \ddot{q}_1\int_0^\ell \varphi_1^2(x)\,dx$$

$$= \left[M_1 + m_1\varphi_1^2(a) + J_1\varphi_1'^2(a)\right]\ddot{q}_1$$

$$= M_1\left[1 + \frac{m_1}{M_1}\varphi_1^2(a) + \frac{J_1}{M_1}\varphi_1'^2(a)\right]\ddot{q}_1 = M'\ddot{q}_1$$

$$M'\ddot{q}_1 + C_1\dot{q}_1 + K_1 q_1 = \int_0^\ell p(x,t)\,\varphi_1(x)\,dx$$

$$\therefore \frac{C_1}{M_1} = 2\zeta'\omega_1 = \frac{C_1}{M_1\left[1 + \frac{m_1}{M_1}\varphi_1^2(a) + \frac{J_1}{M_1}\varphi_1'^2(a)\right]}$$

$$\frac{K_1}{M'} = \omega'^2 = \frac{K}{\text{same denominator}} \qquad \text{etc.}$$

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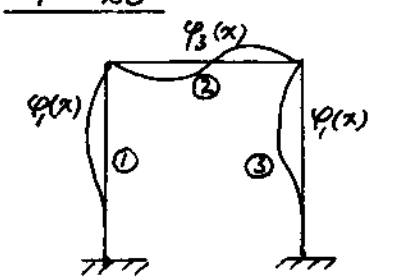
9 - 25



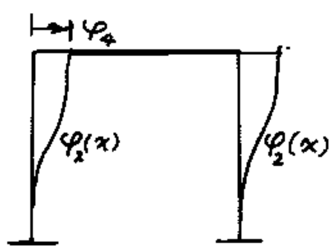
The modes are available in App. D.

Member ① $w_1(x) = \varphi_1 p_1 + \varphi_2 p_2$

Member ② $w_2(x) = \varphi_3 p_3$

$u_2(x) = 1\, p_4$

Member ③ $w_3(x) = \varphi_1 p_5 + \varphi_2 p_6$



So there are 6 coordinates $p_1 \cdots p_6$

Boundary conditions (4 eqs.)

$w_1(\ell) = u_2(0) \qquad \varphi_1(\ell)p_1 + \varphi_2(\ell)p_2 = p_4$

$w_1'(\ell) = w_2'(0) \qquad \varphi_1'(\ell)p_1 + \varphi_2'(\ell)p_2 = \varphi_3'(0)p_3$

$w_2'(\ell) = w_3'(\ell) \qquad \varphi_3'(\ell)p_3 = \varphi_1'(\ell)p_5 + \varphi_2'(\ell)p_6$

$u_2(\ell) = w_3(\ell) \qquad p_4 = \varphi_1'(\ell)p_5 + \varphi_2'(\ell)p_6$

9-25 Cont: Gen. mass & gen. stiffness from T & U

$$T = \frac{1}{2}\int_0^{l_1} \dot{w}_1^2\, m\, dx + \frac{1}{2}\int_0^{l_2} [\dot{w}_2^2 + \dot{u}_2^2]\, m\, dx + \frac{1}{2}\int_0^{l_1} \dot{w}_3^2\, m\, dx$$

$$U = \frac{1}{2} EI\int_0^{l_1} w_1''^2\, dx + \frac{1}{2} EI\int_0^{l_2} w_2''^2\, dx + \frac{1}{2} EI\int_0^{l_1} w_3''^2\, dx$$

Mass Matrix
$$\left[\begin{array}{cc|cc|cc} m_{11} & m_{12} & & & & \\ m_{21} & m_{22} & & & & \\ \hline & & m_{33} & m_{34} & & \\ & & m_{43} & m_{44} & & \\ \hline & & & & m_{55} & m_{56} \\ & & & & m_{65} & m_{66} \end{array}\right]$$

Stiffness Matrix
$$\left[\begin{array}{cc|cc|cc} k_{11} & k_{12} & & & & \\ k_{21} & k_{22} & & & & \\ \hline & & k_{33} & k_{34} & & \\ & & k_{43} & k_{44} & & \\ \hline & & & & k_{55} & k_{56} \\ & & & & k_{65} & k_{66} \end{array}\right]$$

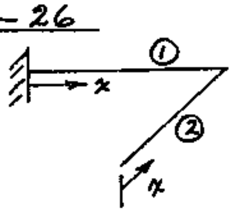
Constraint Eqs.

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{bmatrix} \\ & 2\times 6 \\ & \text{Matrix} \\ \\ \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

where q_1 & q_2 may be any of the p_s

System reduces to a 2×2 matrix equation.

9-26



Let $w_1(x) = \phi_1 p_1 + \phi_2 p_2 + \phi_3 p_3$

$\theta_1(x) = \phi_4 p_4$

$w_2(x) = \phi_5 p_5 + \phi_6 p_6 + \phi_7 p_7$

$\theta_2(x) = \phi_8 p_8 + \phi_9 p_9$

Subst. into

$$T = \frac{1}{2}\int \dot{w}^2 dm + \frac{1}{2}\int \frac{J}{A}\dot{\theta}^2 dm$$

$$U = \frac{1}{2}\int EI\left(\frac{d^2w}{dx^2}\right)^2 dx + \frac{1}{2}\int C\left(\frac{d\theta}{dx}\right)^2 dx$$

to establish m_{ij} and k_{ij}

9 - 26 Cont: Constraint eq. at junction of 1 and 2

(1)	$w_1(\ell) = w_2(\ell)$	defl 1 = defl 2
(2)	$\theta_1(\ell) = -w_2'(\ell)$	twist 1 = bending slope 2
(3)	$C\,\theta_1'(\ell) = EI\,w_2''(\ell)$	torque 1 = bend. moment 2
(4)	$w_1'''(\ell) = -w_2'''(\ell)$	shear 1 = shear 2
(5)	$EI\,w_1''(\ell) = -C\,\theta_2'(\ell)$	bend. mom. 1 = torque at 2
(6)	$w_1'(\ell) = \theta_2(\ell)$	bend slope 1 = twist 2

Mass Matrix

$$\left[\begin{array}{ccc|c|ccc|cc}
m_{11} & m_{12} & m_{13} & & & & & & \\
m_{21} & m_{22} & m_{23} & & & \text{all } 0 & & & \\
m_{31} & m_{32} & m_{33} & & & & & & \\
\hline
 & & & m_{44} & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & & m_{55} & m_{56} & m_{57} & 0 & 0 \\
 & & & & m_{65} & m_{66} & m_{67} & 0 & 0 \\
 & \text{all } 0 & & & m_{75} & m_{76} & m_{77} & 0 & 0 \\
\hline
 & & & & & & & m_{88} & m_{89} \\
 & & & & & & & m_{98} & m_{99}
\end{array}\right] = 9 \times 9$$

stiffness Matrix

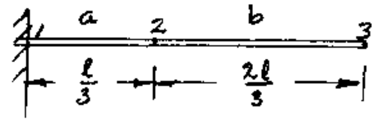
$$\left[\begin{array}{ccc|c|ccc|cc}
k_{11} & k_{12} & k_{13} & & & & & & \\
k_{21} & k_{22} & k_{23} & & & & & & \\
k_{31} & k_{32} & k_{33} & & & & & & \\
\hline
 & & & k_{44} & & & & & \\
\hline
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & k_{77} & & \\
\hline
 & & & & & & & & \\
 & & & & & & & & k_{99}
\end{array}\right] = 9 \times 9$$

Constraint Matrix $= 9 \times 3$ $\left[\begin{array}{c|c|c} \, & \, & \, \\ \, & \, & \, \\ \, & \, & \, \end{array}\right]$

3 gen. coords, any 3 of the p_s $\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$

Result is $\begin{bmatrix} M \\ 3\times 3 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} K \\ 3\times 3 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \{0\}$

10 − 1 (see Ex. 10.1−1)



$M_a = \frac{1}{3} M \qquad k_a = \frac{EA}{\ell/3} = \frac{6}{2}\frac{EA}{\ell}$

$M_b = \frac{2}{3} M \qquad k_b = \frac{EA}{2\ell/3} = \frac{3}{2}\frac{EA}{\ell}$

with $u_1 = 0$

$$\text{Mass Matrix} = \frac{1}{6}\begin{bmatrix} 2(M_a + M_b) & M_b \\ M_b & 2M_b \end{bmatrix} \qquad \text{Ex. 10.1-1}$$

$$\text{Stiffness Matrix} = \begin{bmatrix} (k_a + k_b) & -k_b \\ -k_b & k_b \end{bmatrix}$$

Diff. Eq. of Motion with $\ddot{u} = -\omega^2 u$

$$\left\{ \frac{-\omega^2 M}{3 \times 6}\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} + \frac{EA}{2\ell}\begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \right\} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } \lambda = \frac{\omega^2 M}{18} \cdot \frac{2\ell}{EA} = \frac{\omega^2 M\ell}{9EA}$$

Characteristic Eq.

$$\begin{vmatrix} (9-6\lambda) & -(3+2\lambda) \\ -(3+2\lambda) & (3-4\lambda) \end{vmatrix} = 0 \quad \longrightarrow \quad \lambda^2 - 3.30\lambda + .90 = 0$$

$$\lambda = \begin{cases} .30 \\ 3.00 \end{cases}$$

$$\omega^2 = 9 \times \begin{cases} .3 \\ 3.00 \end{cases} \qquad \begin{aligned} \omega_1 &= 1.643\sqrt{EA/M\ell} \\ \omega_2 &= 5.196\sqrt{EA/M\ell} \end{aligned} \qquad \text{Exact} \rightarrow \begin{cases} 1.5708 \\ 4.7124 \end{cases}$$

∴ 1^{st} mode is 4.6% high

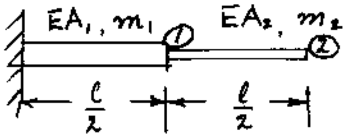
2^{nd} mode is 10.3% high

With sta. at $\frac{\ell}{2}$, $\lambda^2 - \frac{10}{7}\lambda + \frac{1}{7} = 0$

$\omega_1 = 1.6115\sqrt{\frac{EA}{M\ell}} = 1.025 \times \text{Exact} = 2.6\%$ high

$\omega_2 = 5.6293\sqrt{\frac{EA}{M\ell}} = 1.1946 \times$ " $= 19.5\%$ high

10 − 2



$EA_1 = 2\,EA_2$

$m_1 = 2\,m_2$

$k_2 = \frac{2EA_2}{\ell}$

$$\left[-\frac{\omega^2}{6}\begin{bmatrix}2(M_a+M_b) & M_b\\ M_b & 2M_b\end{bmatrix} + \begin{bmatrix}(k_a+k_b) & -k_b\\ -k_b & k_b\end{bmatrix}\right]\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

$$\left[-\frac{\omega^2 M_2}{6}\begin{bmatrix}2(2+1) & 1\\ 1 & 2\end{bmatrix} + \frac{2EA_2}{\ell}\begin{bmatrix}(2+1) & -1\\ -1 & 1\end{bmatrix}\right]\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

Let $\lambda = \frac{\omega^2 M_2}{6}\left(\frac{\ell}{2EA}\right) = \left(\frac{\omega^2 M_2 \ell}{12EA_2}\right) \quad \therefore \quad \left|-\lambda\begin{bmatrix}6 & 1\\ 1 & 2\end{bmatrix} + \begin{bmatrix}3 & -1\\ -1 & 1\end{bmatrix}\right| = 0$

$$\begin{vmatrix}(3-6\lambda) & -(1+\lambda)\\ -(1+\lambda) & (1-2\lambda)\end{vmatrix} = 0 \qquad \lambda^2 - 1.2727\lambda + .1818 = 0$$

$$\lambda_{1,2} = \begin{cases} .1640\\ 1.1088\end{cases}$$

$\omega_1 = 1.4029\sqrt{\frac{EA_2}{M_2\ell}}$

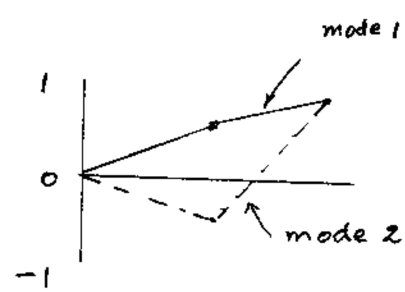
$\omega_2 = 3.6477\sqrt{\frac{EA_2}{M_2\ell}}$

Mode shapes $\qquad (3-6\lambda)u_1 = (1+\lambda)u_2$

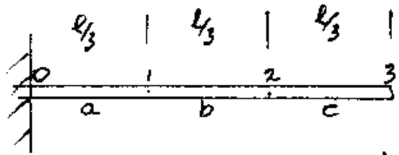
or $(1+\lambda)u_1 = (1-2\lambda)u_2$

$\left(\frac{u_1}{u_2}\right)_1 = \frac{(1-2\lambda)}{(1+\lambda)} = .5773$

$\left(\frac{u_1}{u_2}\right)_2 = -.5258$



10 - 3



$$u = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$M_a = M_b = M_c = \frac{M}{3}$$

$$k_a = k_b = k_c = \frac{EA_a}{\ell/3} = \frac{3EA}{\ell}$$

Mass matrix

$$\frac{M_a}{6}\begin{bmatrix} 2 & 1 & & \\ 1 & 2+2 & 1 & \\ & 1 & 2+2 & 1 \\ & & 1 & 2 \end{bmatrix}$$

Stiffness matrix

$$k_a\begin{bmatrix} 1 & -1 & & \\ -1 & 1+1 & -1 & \\ & -1 & 1+1 & -1 \\ & & -1 & 1 \end{bmatrix}$$

$$\frac{M_a}{6}\begin{bmatrix} \cancel{2} & \cancel{1} & \cancel{0} & \cancel{0} \\ \cancel{1} & 4 & 1 & 0 \\ \cancel{0} & 1 & 4 & 1 \\ \cancel{0} & 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \cancel{\ddot{u}_0} \\ \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + k_a\begin{bmatrix} \cancel{1} & \cancel{-1} & \cancel{0} & \cancel{0} \\ \cancel{-1} & 2 & -1 & 0 \\ \cancel{0} & -1 & 2 & -1 \\ \cancel{0} & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \cancel{u_0} \\ u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$u_0 = 0 \therefore$ cross out as above

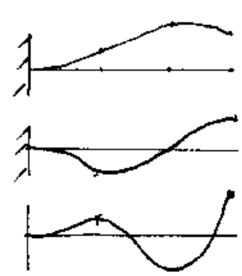
$$\frac{M}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \frac{3EA}{\ell}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Computer solution: $\lambda = \dfrac{M\ell\omega^2}{54EA}$

$\omega_1 = 1.522\sqrt{\dfrac{EA}{M\ell}}$	$\left(\sqrt{\dfrac{EA}{M\ell}} = \dfrac{1}{\ell}\sqrt{\dfrac{E}{\rho}}\right)$	Exact 1.571	$\dfrac{1}{\ell}\sqrt{\dfrac{E}{\rho}}$	$\therefore$ 3% low
$\omega_2 = 5.1545$ "		4.712	"	9% high
$\omega_3 = 9.4329$ "		7.85	"	20% high

Mode Shapes

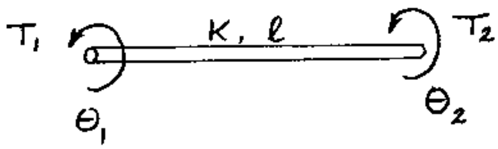
$$\begin{Bmatrix} 0 \\ .415 \\ .728 \\ .544 \end{Bmatrix}_1 \quad \begin{Bmatrix} 0 \\ -.811 \\ .0268 \\ .583 \end{Bmatrix}_2 \quad \begin{Bmatrix} 0 \\ .410 \\ -.684 \\ .602 \end{Bmatrix}_3$$



10 – 4

With linear twist, the problem is identical to that of the longitudinal vibration.

Torsional element of length ℓ



Mass $= \rho A \ell$ Stiffness $= \dfrac{I_p G}{\ell}$ of length ℓ

$J_p = \dfrac{MR^2}{2}$ = mass polar moment of inertia

Mass matrix $= \dfrac{\rho A \ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ Stiffness $= \dfrac{I_p G}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

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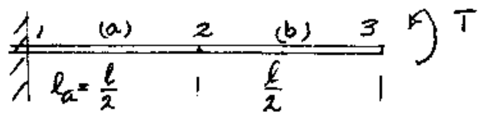
10 – 5



Torsion

$M_a \dfrac{R^2}{2} = J_a$ Let J = mass polar moment of inertia for entire rod

Then $J_a = \frac{1}{2} J$, $K_a = \dfrac{I_p G}{\ell_a} = 2\dfrac{I_p G}{\ell}$

$$\frac{J_a}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + K_a\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Units: J = mass in², I_p = in⁴, G = lb/in², ℓ = in.

Let $\lambda = \dfrac{\omega^2 J_a}{6 K_a}$ $J_a = J/2$, $K_a = 2K$

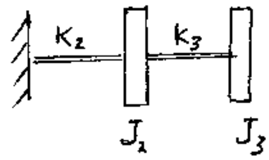
Same prob. as longitudinal vibr.

$$\omega = \begin{cases} 1.6114\sqrt{\dfrac{K}{J}} \\ 5.629 \quad " \end{cases}$$

where $K = \dfrac{I_p G}{\ell}$, $J = M\dfrac{R^2}{2}$

$$\frac{K}{J} = \frac{2 I_p G}{M \ell R^2} = \frac{GA}{M\ell}$$

10 - 6 Lumped mass torsional system



$$\begin{bmatrix} J_2 & 0 \\ 0 & J_3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} (K_2+K_3) & -K_3 \\ -K_3 & K_3 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} [(K_2+K_3) - \omega^2 J_2] & -K_3 \\ -K_3 & [K_3 - \omega^2 J_3] \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \therefore \text{ same form}$$

The problem here is assigning proper values of J_2 and J_3, to be equivalent to distributed system.

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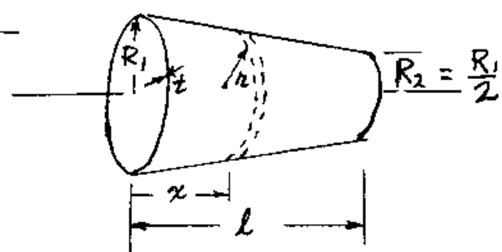
10 - 7

Longitudinal Vibration



$$\xi = x/l$$

$$\varphi_1 = (1-\xi)$$

$$\varphi_2 = \xi$$

$$u = u_1(1-\xi) + u_2\xi$$

$$m_{ij} = \int \varphi_i \varphi_j \, dm \qquad r = R_1(1 - \tfrac{1}{2}\xi)$$

$$dm = \rho\, 2\pi r t \cdot dx \quad , \quad \rho = \text{mass density}$$

$$A = 2\pi r t$$

$$\therefore m_{11} = \rho 2\pi t R_1 \int_0^1 (1-\tfrac{1}{2}\xi)(1-\xi)^2 l \, d\xi = m_1 \int_0^1 \left(1 - \tfrac{5}{2}\xi + 2\xi^2 - \tfrac{1}{2}\xi^3\right) d\xi$$

where $m_1 = \rho 2\pi t R_1 l = \rho A_1 l$

$$m_{11} = m_1\left(1 - \tfrac{5}{4} + \tfrac{2}{3} - \tfrac{1}{8}\right) = \frac{m_1}{24}(24 - 30 + 16 - 3) = \frac{7}{24} m_1$$

$$A_1 = 2\pi t R_1$$

10-7 Cont.

$$m_{12} = m_1 \int_0^1 (1-\tfrac{1}{2}\xi)(1-\xi)\xi \, d\xi = \frac{3}{24} m_1$$

$$m_{22} = m_1 \int_0^1 (1-\tfrac{1}{2}\xi)\xi^2 \, d\xi = \frac{5}{24} m_1$$

$$\therefore \quad m = \frac{m_1}{24}\begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}$$

$$k_{ij} = \int EA \varphi_i' \varphi_j' \, \ell \, d\xi \qquad \varphi_1' = \frac{d\varphi_1}{dx} = -\frac{1}{\ell}, \qquad \varphi_2' = \frac{1}{\ell}$$

$$k_{11} = E \frac{2\pi t R_1}{\ell^2}\int_0^1 (1-\tfrac{1}{2}\xi)\ell \, d\xi = \frac{EA_1}{\ell}\left(1-\tfrac{1}{4}\right) = \frac{3}{4}\frac{EA_1}{\ell}$$

where $A_1 = 2\pi t R_1$

$$k_{12} = -\frac{3}{4}\frac{EA_1}{\ell} = k_{21} \quad , \qquad k_{22} = \frac{3}{4}\frac{EA_1}{\ell}$$

$$\therefore \quad k = \frac{3}{4}\frac{EA_1}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Eq. of motion

$$\frac{\rho A_1 \ell}{24}\begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{3}{4}\frac{EA_1}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\lambda = \dfrac{\omega^2 \rho \ell^2}{18E}$

$$\begin{vmatrix} (1-7\lambda) & -(1+3\lambda) \\ -(1+3\lambda) & (1-5\lambda) \end{vmatrix} = 0 \quad \therefore \quad \lambda(26\lambda - 6) = 0$$

$$\lambda_0 = 0$$

$$\lambda_1 = \frac{6}{26} = \frac{\omega^2 \rho \ell^2}{18E} = \frac{\omega^2 (\rho A_1 \ell)\ell}{18EA_1} = \frac{\omega^2 m_1 \ell}{18EA_1}$$

10-7 Cont.

$$\omega_1^2 = \left(\frac{6}{26}\times 18\right)\left(\frac{E}{\rho\ell^2}\right) = 4.1538\left(\frac{E}{\rho\ell^2}\right)$$

$$\omega_1 = 2.038\sqrt{\frac{EA_1}{m_1\ell}} = \underline{\underline{2.038\sqrt{\frac{E}{\rho\ell^2}}}}$$ for 1-element solution for Longit. vib.

where A_1 = cross section area at ①

$M_1 = m_1\ell$ = mass of entire uniform cylinder of radius R_1 and length ℓ.

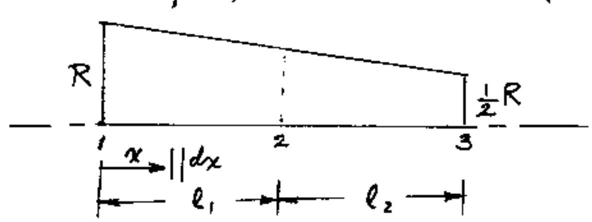
If we have a uniform cylinderical shell of Radius R_1 and length ℓ, with one end fixed with other end free, treated like a helical spring with effective mass $\frac{m_1}{3}$ at free end, the natural frequency will be

$$\omega_1 = \sqrt{\frac{EA_1}{\left(\frac{m_1}{3}\right)\ell}} = 1.732\sqrt{\frac{EA_1}{m_1\ell}}$$

10-8 Same conical tube as in 10-7 treated as two elements of equal length, 1-2 and 2-3.

$$r_{1-2} = R\left(1 - \frac{1}{4}\frac{x}{\ell_1}\right)$$

$$\xi = \frac{x}{\ell_1}$$



$$m_{ij} = \int \varphi_i\varphi_j\,dm = \int \varphi_i\varphi_j\,2\pi\rho t\,r_{1-2}\,dx = 2\pi\rho t R\int_0^1 (1-\tfrac{1}{4}\xi)\varphi_i\varphi_j\ell_1 d\xi$$

$$m_{11} = 2\pi\rho t R\ell_1\int_0^1 (1-\tfrac{1}{4}\xi)(1-\xi)^2 d\xi = 2\pi\rho t R\ell_1(.3125)$$

$$m_{12} = \quad '' \quad \int_0^1 (1-\tfrac{1}{4}\xi)(1-\xi)\xi\,d\xi = \quad '' \quad (.1458)$$

$$m_{22} = \quad '' \quad \int_0^1 (1-\tfrac{1}{4}\xi)\xi^2 d\xi = \quad '' \quad (.2708)$$

10-8 Cont.

$$m = 2\pi\rho t R \ell_1 \begin{bmatrix} .3125 & .1458 \\ .1458 & .2708 \end{bmatrix}$$

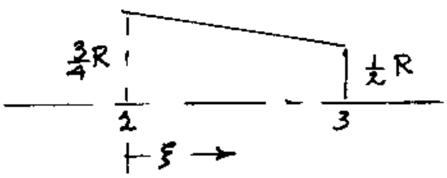
$$k_{ij} = \int_0^1 EA\,\varphi_i'\varphi_j'\,\ell_1\,d\xi \qquad \therefore k_{11} = \int_0^1 2\pi RE\left(1-\tfrac{1}{4}\xi\right)\left(-\tfrac{1}{\ell_1}\right)^2 \ell_1\,d\xi$$

$$= \frac{2\pi RE}{\ell_1}(.8750)$$

$$k_{12} = -k_{11}, \qquad k_{22} = k_{11}$$

$$k = \frac{2\pi RE}{\ell_1}\begin{bmatrix} .8750 & -.8750 \\ -.8750 & .8750 \end{bmatrix}$$

Element 2-3



$$r = \tfrac{3}{4}R\left(1-\tfrac{1}{3}\xi\right)$$

$$A = 2\pi t r = 2\pi t R\,\tfrac{3}{4}\left(1-\tfrac{1}{3}\xi\right)$$

$$dm = 2\pi\rho t\,r\,dx = 2\pi\rho t R\left(1-\tfrac{1}{3}\xi\right)\ell_1\,d\xi$$

$$m_{11} = \tfrac{3}{2}\pi\rho t R\ell_1 \int_0^1 \left(1-\tfrac{1}{3}\xi\right)(1-\xi)^2 d\xi = \tfrac{3}{2}\pi\rho t R\ell_1\,(.3056)$$

$$m_{12} = \tfrac{3}{2}\pi\rho t R\ell_1\,(.1389), \qquad m_{22} = \tfrac{3}{2}\pi\rho t R\ell_1\,(.2500)$$

$$m = \tfrac{3}{2}\pi\rho t R\ell_1 \begin{bmatrix} .3056 & .1389 \\ .1389 & .250 \end{bmatrix} = 2\pi\rho t R\ell_1 \begin{bmatrix} .2292 & .1042 \\ .1042 & .1875 \end{bmatrix}$$

$$k_{11} = \tfrac{3}{2}\pi\rho\frac{tR}{\ell_1}\int_0^1 \left(1-\tfrac{1}{3}\xi\right)d\xi = \tfrac{3}{2}\pi\rho\frac{tR}{\ell_1}\left(\tfrac{5}{6}\right)$$

$$k_{12} = -k_{11}, \qquad k_{22} = k_{11}$$

$$\therefore k = \tfrac{3}{2}\pi \times \tfrac{5}{6}\,\rho\frac{tR}{\ell_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

10-8 Cont. Assemble element matrices

$$m = 2\pi\rho t R \ell_1 \begin{bmatrix} .3125 & .1458 & 0 \\ .1458 & (.2708+.2292) & .1042 \\ 0 & .1042 & .1875 \end{bmatrix}$$

$$k = \frac{\pi R E t}{\ell_1} \begin{bmatrix} 1.75 & -1.75 & \\ -1.75 & (1.75+1.25) & -1.25 \\ & -1.25 & 1.25 \end{bmatrix} \qquad \lambda = \frac{\omega^2 \rho \ell_1}{E}$$

with $u_1 = 0$, eqs. of motion leads to chara. eq.

$$\left| -\lambda \begin{bmatrix} .5000 & .1042 \\ .1042 & .1875 \end{bmatrix} + \begin{bmatrix} 3.00 & -1.25 \\ -1.25 & 1.25 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} (3.00 - .500\lambda) & -(1.25 + .1042\lambda) \\ -(1.25 + .1042\lambda) & (1.25 - .1875\lambda) \end{vmatrix} = 0$$

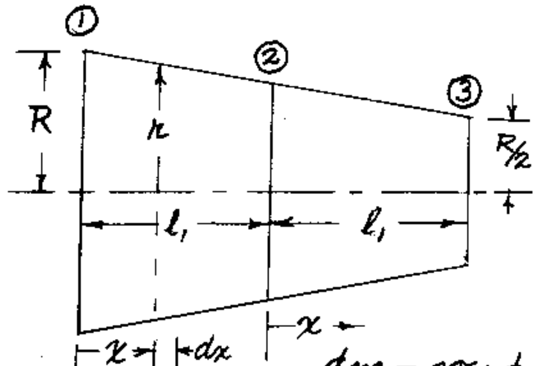
$$\lambda^2 - 17.46\lambda + 26.380 = 0$$

$$\lambda_1 = 1.6708 \qquad \omega_1 = 1.2926\sqrt{\frac{E}{\rho \ell_1^2}} \qquad \ell_1 = \frac{\ell}{2}$$

$$\lambda_2 = 15.7892 \qquad \omega_2 = 3.9736\sqrt{\frac{E}{\rho \ell_1^2}}$$

$$\therefore \left.\begin{aligned} \omega_1 &= 2.585\sqrt{\frac{E}{\rho \ell^2}} \\ \omega_2 &= 7.947\sqrt{\frac{E}{\rho \ell^2}} \end{aligned}\right\} \text{longitudinal vibr (2 elements)}$$

10 – 9(a) Torsional Vibration of Conical Shell.



$r_{1-2} = R(1-\frac{1}{4}\xi)$

$\xi = \frac{x}{l_1}$, $dx = l_1 d\xi$

$r_{2-3} = \frac{3}{4}R(1-\frac{1}{3}\xi)$

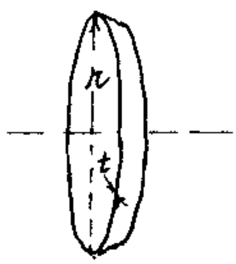
$\varphi_1 = (1-\xi)$

$\varphi_2 = \xi$

$\varphi_1' = -\frac{1}{l_1}$

$\varphi_2' = \frac{1}{l_1}$

$dm = 2\pi r t \rho dx$



$dJ_{1-2} = r^2 dm = 2\pi\rho t R^3(1-\frac{1}{4}\xi)^3 l_1 d\xi$

$I_p = 2\pi r t \cdot r^2$ = 2nd moment (polar) of cross-section area

Stiffness of element 1-2.

$$k_{ij} = G\int_0^1 I_p \varphi_i' \varphi_j' dx = G\cdot 2\pi t\, R^3 \int_0^1 (1-\tfrac{1}{4}\xi)^3 \varphi_i' \varphi_j' l_1 d\xi$$

$$k_{11} = G\cdot 2\pi t\, R^3 \int_0^1 (1-\tfrac{1}{4}\xi)^3 \left(\tfrac{1}{l_1^2}\right) l_1 d\xi = \frac{G2\pi t R^3}{l_1}\int_0^1 (1-\tfrac{3}{4}\xi+\tfrac{3}{16}\xi^2-\tfrac{1}{64}\xi^3)d\xi$$

$$= \frac{G2\pi t R^3}{l_1}\left(1-\tfrac{3}{8}+\tfrac{1}{16}-\tfrac{1}{256}\right) = \frac{G2\pi t R^3}{l_1}(.6836)$$

$k_{12} = k_{21} = -k_{11}$, $\quad k_{22} = k_{11}$

$$\therefore \quad K_{1-2} = \frac{G2\pi t R^3}{l_1}(.6836)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stiffness of element 2-3.

$$k_{11} = \frac{G2\pi t}{l_1}(.75R)^3 \int_0^1 (1-\tfrac{1}{3}\xi)^3 d\xi = \frac{G2\pi t R^3}{l_1}(.4219)\int_0^1 (1-\xi+\tfrac{1}{3}\xi^2-\tfrac{1}{27}\xi^3)d\xi$$

$$= \frac{G2\pi t R^3}{l_1}(.4219)(.6019) = \frac{G2\pi t R^3}{l_1}(.2539)$$

$$K_{2-3} = \frac{G2\pi t R^3}{l_1}(.2539)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{G2\pi t R^3}{l_1}(.6836)\begin{bmatrix} .3714 & -.3714 \\ -.3714 & .3714 \end{bmatrix}$$

10 – 9 (a) Cont.

$$\text{Combined stiffness} = \frac{G2\pi tR^3}{\ell/2}(.6836)\begin{bmatrix} 1 & -1 & \\ -1 & 1.3714 & -.3714 \\ & -.3714 & .3714 \end{bmatrix}$$

For $\theta_1 = 0$

$$K\theta = (1.3672)\frac{G2\pi tR^3}{\ell}\begin{bmatrix} 1.3714 & -.3714 \\ -.3714 & .3714 \end{bmatrix}\begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix}$$

Mass matrix, element 1-2

$$J_{ij} = \int_0^1 \varphi_i \varphi_j \, dJ = 2\pi\rho t R^3 \int_0^1 (1-\tfrac{1}{4}\xi)^3 \varphi_i \varphi_j \, \ell_1 \, d\xi$$

$$J_{11} = 2\pi\rho t R^3 \int_0^1 (1-\tfrac{1}{4}\xi)^3 (1-\xi)^2 \ell_1 \, d\xi$$

$$= 2\pi\rho t R^3 \ell_1 \int_0^1 (1 - \tfrac{11}{4}\xi + \tfrac{43}{16}\xi^2 - \tfrac{73}{64}\xi^3 + \tfrac{7}{32}\xi^4 - \tfrac{1}{64}\xi^5)\, d\xi$$

$$= \quad '' \quad (1 - \tfrac{11}{8} + \tfrac{43}{48} - \tfrac{73}{256} + \tfrac{7}{160} - \tfrac{1}{384})$$

$$= 2\pi\rho t R^3 \ell_1 (.2768)$$

$$J_{12} = 2\pi\rho t R^3 \ell_1 \int_0^1 (1-\tfrac{1}{4}\xi)^3 (1-\xi)\xi \, d\xi$$

$$'' \quad \int_0^1 (\xi - \tfrac{7}{4}\xi^2 + \tfrac{15}{16}\xi^3 - \tfrac{13}{64}\xi^4 + \tfrac{1}{64}\xi^5)\, d\xi$$

$$= 2\pi\rho t R^3 \ell_1 \; (.1130)$$

$$J_{22} = 2\pi\rho t R^3 \ell_1 \int_0^1 (1-\tfrac{1}{4}\xi)^3 \xi^2 d\xi = \int_0^1 (\xi^2 - \tfrac{3}{4}\xi^3 + \tfrac{3}{16}\xi^4 - \tfrac{1}{64}\xi^5) d\xi \, (\quad)$$

$$= 2\pi\rho t R^3 \ell_1 (\tfrac{1}{3} - \tfrac{3}{16} + \tfrac{3}{80} - \tfrac{1}{384}) = 2\pi\rho t R^3 \ell_1 (.1804)$$

10 – 9(a) Cont.

$$J_{1-2} = 2\pi\rho t R^3 \ell_1 \begin{bmatrix} .2768 & .1130 \\ .1130 & .1804 \end{bmatrix} \qquad (\ell_1 = \tfrac{\ell}{2})$$

Mass matrix, element 2 – 3 $\qquad r_{2-3} = \frac{3}{4}R(1 - \frac{1}{3}\xi)$

$$dJ_{2-3} = 2\pi r t \rho \cdot r^2 = 2\pi\rho t (\tfrac{3}{4}R)^3 (1 - \tfrac{1}{3}\xi)^3$$

$$J_{22} = 2\pi\rho t R^3 (.4219) \int_0^1 (1 - \tfrac{1}{3}\xi)^3 (1-\xi)^2 \ell_1 \, d\xi$$

$$\text{''} \quad \ell_1 \int_0^1 (1 - 3\xi + \tfrac{10}{3}\xi^2 - 1.7036\xi^3 + .40737\xi^4 - .037043\xi^5) d\xi$$

$$= 2\pi\rho t R^3 \ell_1 (.4219)(.2605) = 2\pi\rho t R^3 \ell_1 (.1099)$$

$$J_{23} = \text{same term} \times \ell_1 \int_0^1 (1 - \tfrac{1}{3}\xi)^3 (1-\xi)\xi \, d\xi$$

$$= \quad \text{''} \quad \int_0^1 (\xi - 2\xi^2 + \tfrac{4}{3}\xi^3 - \tfrac{10}{27}\xi^4 + \tfrac{1}{27}\xi^5) d\xi$$

$$= 2\pi\rho t R^3 \ell_1 (.4219)(.0988) = 2\pi\rho t R^3 \ell_1 (.0417)$$

$$J_{33} = \text{same term} \times \ell_1 \int_0 (1 - \tfrac{1}{3}\xi)^3 \xi^2 d\xi = 2\pi\rho t R^3 \ell_1 (.4219)(.1438)$$

$$= 2\pi\rho t R^3 \ell_1 (.0607)$$

Combined mass term with $\theta_1 = 0$

$$2\pi\rho t R^3 \ell_1 \begin{bmatrix} (.1099 + .1804) & .0417 \\ .0417 & .0607 \end{bmatrix} \begin{Bmatrix} \ddot\theta_2 \\ \ddot\theta_3 \end{Bmatrix} \qquad (\ell_1 = \tfrac{\ell}{2})$$

$$J = \pi\rho t R^3 \ell \begin{bmatrix} .2903 & .0417 \\ .0417 & .0607 \end{bmatrix}$$

10-9 (a) Cont.

Eq. of motion: Let $\lambda = \dfrac{\omega^2 \cancel{\pi} \rho \cancel{t} \cancel{R^3} \ell}{1.3672\, G\, 2 \cancel{\pi} \cancel{t} \cancel{R^3}/\ell}$

$$\lambda = \frac{\omega^2 \rho \ell^2}{2.734\, G}$$

$$\left[-\lambda \begin{bmatrix} .2903 & .0417 \\ .0417 & .0607 \end{bmatrix} + \begin{bmatrix} 1.3714 & -.3714 \\ -.3714 & .3714 \end{bmatrix} \right] \begin{Bmatrix} \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} (1.3714 - .2903\lambda) & -(.3714 + .0417\lambda) \\ -(.3714 - .0417\lambda) & (.3714 - .0607\lambda) \end{vmatrix} = 0$$

$$\lambda^2 - 10.069\lambda + 23.358 = 0$$

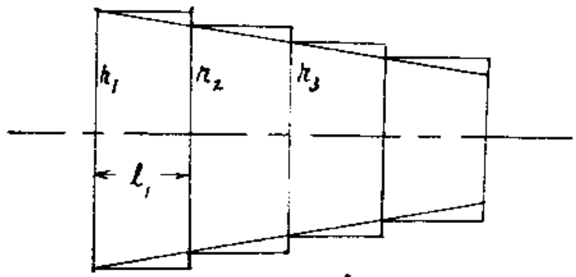
$\lambda_1 = 3.624$ $\qquad$ $\lambda_2 = 6.438$

$\omega_1^2 = 9.908 \dfrac{G}{\rho \ell^2}$ $\qquad$ $\omega_2^2 = 17.603 \dfrac{G}{\rho \ell^2}$

$\underline{\underline{\omega_1 = 3.147 \sqrt{\dfrac{G}{\rho \ell^2}}}}$ $\qquad$ $\underline{\underline{\omega_2 = 4.196 \sqrt{\dfrac{G}{\rho \ell^2}}}}$

Results unchecked.

10-9 (b)



Torsion problem.

Assume short elements ∴ constant radius for each section.

Element stiffness:

$$A_i = 2\pi r_i t\,, \quad I_{p_i} = A r_i^2\,, \quad GI_{p_i} = GA_i r^2$$

$$K_i = \frac{GI_{p_i}}{l_i} = G\frac{2\pi t\, r_i^3}{l_i}$$

Assume straight line variation in twist ∴ φ_{1_i}, φ_{2_i} applies to each element $\varphi_{1_i} = (1-\xi)$, $\varphi_{2_i} = \xi$ where $\xi = \frac{x}{l_i}$. Then $K_i = \frac{GI_{p_i}}{l_i}\int_0^1 \varphi_{1_i}'\varphi_{2_i}' l_i\, d\xi$

$$\int_0^1 \varphi_i'\varphi_j' l_i\, d\xi = \pm 1$$

$$K \text{ for element } = G\left(\frac{2\pi t\, r_i^3}{l_i}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad l_i = \frac{l}{n}$$

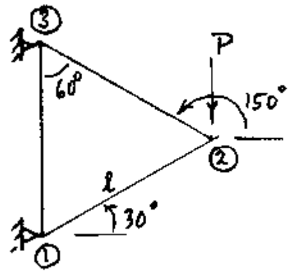
Element mass:

$$J \text{ for element } = \frac{2\pi t \rho l_i\, r_i^3}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

N section eq.

$$\rho\frac{2\pi t\, l}{6n}\begin{bmatrix} 2r_1^3 & r_1^3 & & & \\ r_1^3 & 2(r_1^3+r_2^3) & r_2^3 & & \\ & r_2^3 & 2(r_2^3+r_3^3) & r_3^3 & \\ & & r_3^3 & 2(r_3^3+r_4^3) & \\ & & & & \text{etc.} \end{bmatrix}\begin{Bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \\ \ddot\theta_3 \\ \ddot\theta_4 \\ \vdots \end{Bmatrix} + \frac{G\, 2\pi t n}{l}\begin{bmatrix} r_1^3 & -r_1^3 & & \\ -r_1^3 & (r_1^3+r_2^3) & -r_2^3 & \\ & -r_2^3 & (r_2^3+r_3^3) & \\ & & & \text{etc} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \end{Bmatrix}$$

10 − 10



Element 1−2 $\quad \alpha = 30^\circ \quad c = .866, \quad s = .500$

$c^2 = .750 \quad s^2 = .250 \quad cs = .433$

$$\bar{k}_{12} = \frac{EA}{l}\begin{bmatrix} \cdots & \begin{matrix} .750 & .433 \\ .433 & .250 \end{matrix} \end{bmatrix} \begin{Bmatrix} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}$$

Element 2−3, $\quad \alpha = 150^\circ \quad c = -.866 \quad s = .50$

$$\bar{k}_{23} = \frac{EA}{l}\begin{bmatrix} .750 & -.433 & -.750 & .433 \\ -.433 & .250 & .433 & -.250 \\ -.750 & .433 & .750 & -.433 \\ .433 & -.250 & -.433 & .250 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \end{Bmatrix}$$

Assemble

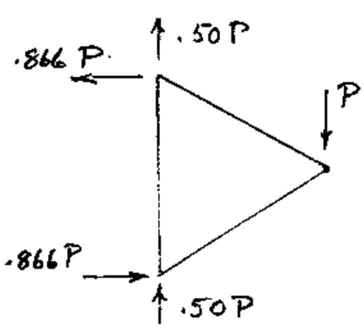
$$\begin{Bmatrix} F_{2x} = 0 \\ F_{2y} = -P \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \frac{EA}{l}\begin{bmatrix} 1.50 & 0 & -.75 & .433 \\ 0 & .50 & .433 & -.25 \\ -.75 & .433 & .75 & -.433 \\ .433 & -.25 & -.433 & .25 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 = 0 \\ \bar{v}_3 = 0 \end{Bmatrix}$$

$$F_{2x} = 0 = \frac{EA}{l}\, 1.50\, \bar{u}_2 \qquad \therefore\ \bar{u}_2 = 0$$

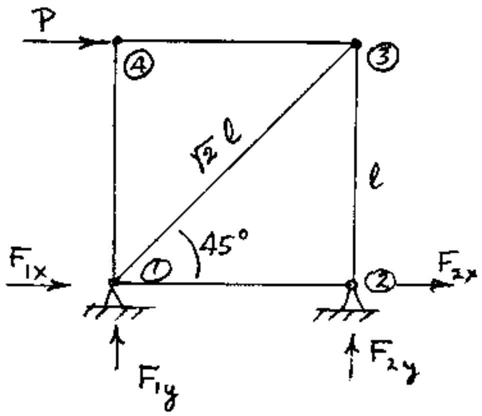
$$F_{2y} = -P = \frac{EA}{l}\, .50\, \bar{v}_2 \qquad \therefore\ \bar{v}_2 = -\frac{2Pl}{EA}$$

$$F_{3x} = \frac{EA}{l}\, .433\, \bar{v}_2 = \frac{EA}{l}\, .433\left(\frac{-2Pl}{EA}\right) = -.866P$$

$$F_{3y} = -.25\, \bar{v}_2 \frac{EA}{l} = -.25 \frac{EA}{l}\left(-\frac{2Pl}{EA}\right) = 0.50P$$



10 − 11



$$\bar{k}_{12} = \frac{EA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \bar{m}_{12} = \frac{m\ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Element 1−2, $\alpha = 0$, $s = 0$, $c = 1$

$$\bar{k}_{1-2} = \frac{EA}{\ell}\begin{array}{c} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}\begin{array}{c} 1 \\ \\ 2 \end{array}$$

Element 2−3 $\alpha = 90°$, $s = 1$, $c = 0$

$$\bar{k}_{2-3} = \frac{EA}{\ell}\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right]\begin{array}{c} 2 \\ \\ 3 \end{array}$$

Element 3−4 $\alpha = 180°$ $s = 0$, $c = -1$

$$\bar{k}_{3-4} = \frac{EA}{\ell}\left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{c} 3 \\ \\ 4 \end{array}$$

Element 4−1 $\alpha = 270°$ $s = -1$, $c = 0$

$$\bar{k}_{41} = \frac{EA}{\ell}\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right]\begin{array}{c} 4 \\ \\ 1 \end{array}$$

Element 1 − 3 $\alpha = 45°$, $s = \frac{\sqrt{2}}{2}$, $c = \frac{\sqrt{2}}{2}$ Length $= \sqrt{2}\,\ell$

$$\bar{k}_{13} = \frac{EA}{\sqrt{2}\,\ell}\left[\begin{array}{cc|cc} .5 & .5 & -.5 & -.5 \\ .5 & .5 & -.5 & -.5 \\ \hline -.5 & -.5 & .5 & .5 \\ -.5 & -.5 & .5 & .5 \end{array}\right]\begin{array}{c} 1 \\ \\ 3 \end{array}$$

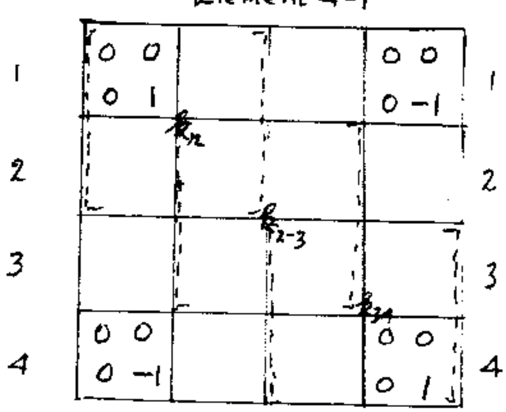
10 - 11 Cont.

$$\begin{Bmatrix} \bar{F}_{1x} \\ \bar{F}_{1y} \\ \bar{F}_{2x} \\ \bar{F}_{2y} \\ \bar{F}_{3x} \\ \bar{F}_{3y} \\ \bar{F}_{4x} \\ \bar{F}_{4y} \end{Bmatrix} = \left[\; K \; \right] \begin{Bmatrix} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \bar{u}_2 = 0 \\ \bar{v}_2 = 0 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

Since all displacements for 1 and 2 are zero, we need only the lower part of this matrix or the lower right section of K. However for instructional purposes we will fill in the elements into the 8×8 space for K.

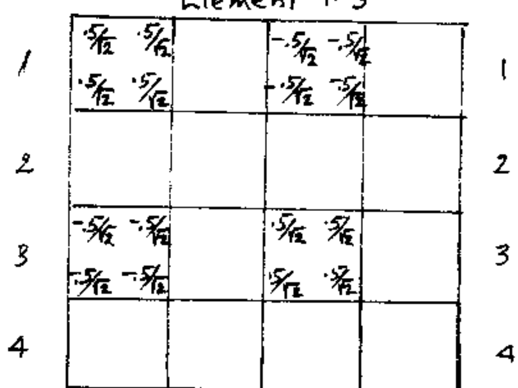
There is no problem with placement of $\bar{k}_{12}$, $\bar{k}_{23}$ and $\bar{k}_{34}$. Element $\bar{k}_{4-1}$ and $\bar{k}_{1-3}$ must be placed as follows:

Element 4-1



dotted squares indicate placement of k_{12}, k_{23}, k_{34}

Element 1-3



superimpose this onto diagram on left.

Considering only those terms in lower right quarter, we have

$$\begin{Bmatrix} \bar{F}_{3x} = 0 \\ \bar{F}_{3y} = 0 \\ \bar{F}_{4x} = P \\ \bar{F}_{4y} = 0 \end{Bmatrix} = \frac{EA}{\ell} \begin{bmatrix} (1 + .5/\sqrt{2}) & .5/\sqrt{2} & -1 & 0 \\ .5/\sqrt{2} & (1 + .5/\sqrt{2}) & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

10 - 11 Cont. The last matrix give

(1) $$0 = \frac{EA}{\ell}\left[\left(1+\frac{.5}{\sqrt{2}}\right)\bar{u}_3 + \frac{.5}{\sqrt{2}}\bar{v}_3 - \bar{u}_4\right]$$

(2) $$0 = \frac{EA}{\ell}\left[\frac{.5}{\sqrt{2}}\bar{u}_3 + \left(1+\frac{.5}{\sqrt{2}}\right)\bar{v}_3\right] \rightarrow \bar{u}_3 = -3.828\,\bar{v}_3$$

(3) $$P = \frac{EA}{\ell}\left[-\bar{u}_3 + \bar{u}_4\right]$$

(4) $$0 = \frac{EA}{\ell}\left[\bar{v}_4\right] \quad \therefore \bar{v}_4 = 0$$

add (1) and (3) to eliminate $\bar{u}_4$ & subst (2) into it.

$$\frac{P\ell}{EA} = \frac{.5}{\sqrt{2}}\bar{u}_3 + \frac{.5}{\sqrt{2}}\bar{v}_3 \rightarrow \bar{u}_3 + \bar{v}_3 = \frac{1}{.3536}\frac{P\ell}{AE}$$

$$\bar{u}_3 - \frac{1}{3.828}\bar{u}_3 = \frac{P\ell}{.3536\,AE} \quad \therefore \bar{u}_3 = 3.828\frac{P\ell}{AE}$$

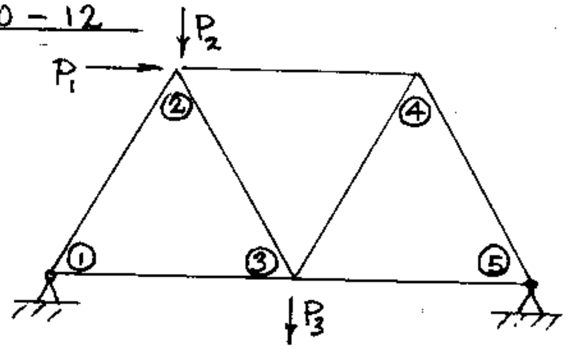
$$\bar{v}_3 = -\frac{P\ell}{AE}$$

$$\bar{u}_4 = 4.828\frac{P\ell}{AE}$$

Complete Equation

$$\begin{Bmatrix} \bar{F}_{1x} \\ \bar{F}_{1y} \\ \bar{F}_{2x} \\ \bar{F}_{2y} \\ \bar{F}_{3x} \\ \bar{F}_{3y} \\ \bar{F}_{4x} \\ \bar{F}_{4y} \end{Bmatrix} = \frac{EA}{\ell}\left[\begin{array}{cc|cc|cc|cc} (1+.5/\sqrt{2}) & .5/\sqrt{2} & -1 & 0 & -.5/\sqrt{2} & -.5/\sqrt{2} & 0 & 0 \\ .5/\sqrt{2} & (1+.5/\sqrt{2}) & 0 & 0 & -.5/\sqrt{2} & -.5/\sqrt{2} & 0 & -1 \\ \hline -1 & 0 & 1 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 1 & 0 & -1 & & \\ \hline -.5/\sqrt{2} & -.5/\sqrt{2} & 0 & 0 & (1+.5/\sqrt{2}) & .5/\sqrt{2} & -1 & 0 \\ -.5/\sqrt{2} & -.5/\sqrt{2} & 0 & -1 & .5/\sqrt{2} & (1+.5/\sqrt{2}) & 0 & 0 \\ \hline 0 & 0 & & & -1 & 0 & 1 & 0 \\ 0 & -1 & & & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \bar{u}_2 = 0 \\ \bar{v}_2 = 0 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

10 - 12



Elements 1-2, 3-4, $\alpha = 60°$ $c = .5$, $s = .866$, $cs = .433$

$c^2 = .250$ $s^2 = .750$

$$\bar{k} = \frac{EA}{\ell}\left[\begin{array}{cc|cc} .250 & .433 & -.250 & -.433 \\ .433 & .750 & -.433 & -.750 \\ \hline & & .250 & .433 \\ & & & .750 \end{array}\right]$$

Elements 2-3, 4-5 $\alpha = 300°$ $c = .5$ $s = -.866$ $cs = -.433$

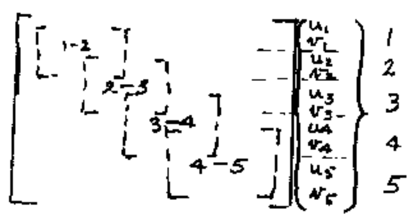
$c^2 = .250$ $s^2 = .750$

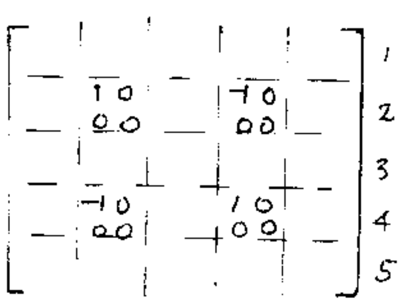
$$\bar{k} = \frac{EA}{\ell}\left[\begin{array}{cc|cc} .250 & -.433 & -.250 & .433 \\ & .750 & .433 & -.750 \\ & & .250 & -.433 \\ & & & .750 \end{array}\right]$$

Element 1-3, 2-4, 3-5 $\alpha = 0$ $c = 1$ $s = 0$ $cs = 0$

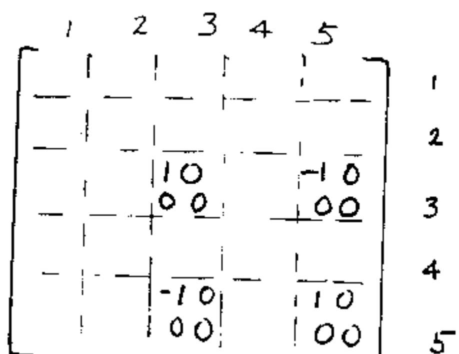
$c^2 = 1$ $s^2 = 0$

$$\bar{k} = \frac{EA}{\ell}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



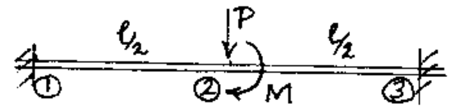


Element 2-4



Element 3-5

10 − 13



From Example 10.5−1 and noting that the ends 1 and 3 are fixed, the stiffness matrix and displacement vectors are:

$$\begin{Bmatrix} F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix} = \frac{8EI}{\ell^3}\left[\begin{array}{cc|cc} 24 & 0 & -12 & 3\ell \\ 0 & 2\ell^2 & -3\ell & .5\ell^2 \\ \hline -12 & -3\ell & 12 & -3\ell \\ 3\ell & .5\ell^2 & -3\ell & \ell^2 \end{array}\right]\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ -M \end{Bmatrix} = \frac{8EI}{\ell^3}\begin{bmatrix} 24 & 0 \\ 0 & 2\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

$$\therefore \quad \bar{v}_2 = -\frac{P\ell^3}{192\,EI} \qquad \theta_2 = -\frac{M\ell}{16\,EI}$$

$$M_3 = \frac{8EI}{\ell^3}\left[3\ell\,\bar{v}_2 + .5\ell^2\theta_2\right] = \frac{8EI}{\ell^3}\left[3\ell\left(\frac{-P\ell^3}{192EI}\right) + .5\ell^2\left(\frac{-M\ell}{16EI}\right)\right]$$

---

10 − 14 Refer to Prob 10.5−1 with $v_3 = \theta_3 = 0$

$$\left[-\frac{\omega^2 m\ell}{840}\begin{bmatrix} 312 & 0 \\ 0 & 2\ell^2 \end{bmatrix} + \frac{8EI}{\ell^3}\begin{bmatrix} 24 & 0 \\ 0 & 2\ell^2 \end{bmatrix}\right]\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\lambda = \left(\frac{\omega^2 m\ell}{840}\right)\left(\frac{\ell^3}{8EI}\right) = \frac{m\ell^4\omega^2}{6720\,EI}$

$$\begin{vmatrix} (24 - 312\lambda) & 0 \\ 0 & 2\ell^2(1-\lambda) \end{vmatrix} = 0$$

$$\lambda_1 = \frac{24}{312} = \frac{m\ell^4\omega_1^2}{6720\,EI}$$

$$\lambda_2 = 1 = \frac{m\ell^4\omega_2^2}{6720\,EI}$$

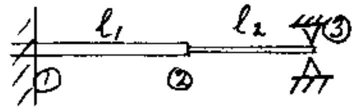
$\therefore$

$\omega_1 = 22.74\sqrt{\frac{EI}{m\ell^4}}$ ⟶ 1.6 % high

$\omega_2 = 81.98\sqrt{\frac{EI}{m\ell^4}}$ ⟶ 33 % high.

Exact values $\begin{cases} \omega_1 = 22.37\sqrt{\ } \\ \omega_2 = 61.67\sqrt{\ } \end{cases}$

10 - 15



Element 1-2

$$\bar{k} = \frac{EI_1}{\ell_1^3}\begin{bmatrix} R & 0 & 0 \\ 0 & 12 & -6\ell_1 \\ 0 & -6\ell_1 & 4\ell_1^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix} \qquad \bar{m} = \frac{m_1\ell_1}{420}\begin{bmatrix} 156 & -22\ell_1 \\ -22\ell_1 & 4\ell_1^2 \end{bmatrix}\begin{Bmatrix} \ddot{v}_2 \\ \ddot{\theta}_2 \end{Bmatrix}$$

Element 2-3

$$\bar{k} = \frac{EI_2}{\ell_2^3}\begin{bmatrix} R & 0 & 0 & -R & 0 & 0 \\ 0 & 12 & 6\ell_2 & 0 & -12 & 6\ell_2 \\ 0 & 6\ell_2 & 4\ell_2^2 & 0 & -6\ell_2 & 2\ell_2^2 \\ -R & 0 & 0 & R & 0 & 0 \\ 0 & -12 & -6\ell_2 & 0 & 12 & -6\ell_2 \\ 0 & 6\ell_2 & 2\ell_2^2 & 0 & -6\ell_2 & 4\ell_2^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 = 0 \\ \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 \\ 0 \\ \theta_3 \end{Bmatrix}$$

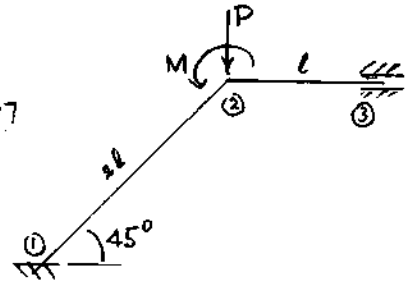
$$\bar{m} = \frac{m_2\ell_2}{420}\begin{bmatrix} N & 0 & 0 & \frac{1}{2}N & 0 & 0 \\ 0 & 156 & 22\ell_2 & 0 & 54 & -13\ell_2 \\ 0 & 22\ell_2 & 4\ell_2^2 & 0 & 13\ell_2 & -3\ell_2^2 \\ \frac{1}{2}N & 0 & 0 & N & 0 & 0 \\ 0 & 54 & 13\ell_2 & 0 & 156 & -22\ell_2 \\ 0 & -13\ell_2 & -3\ell_2^2 & 0 & -22\ell_2 & 4\ell_2^2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_2 = 0 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \\ \ddot{u}_3 = 0 \\ \ddot{v}_3 = 0 \\ \theta_3 \end{Bmatrix}$$

Free vibr. eq.

$$\frac{1}{420}\begin{bmatrix} 156(m_1\ell_1 + m_2\ell_2) & 22(m_2\ell_2^2 - m_1\ell_1^2) & -13\, m_2\ell_2^2 \\ 22(m_2\ell_2^2 - m_1\ell_1^2) & 4(m_2\ell_2^3 + m_1\ell_1^3) & -3\, m_2\ell_2^3 \\ -13\, m_2\ell_2^3 & -3\, m_2\ell_2^3 & 4\, m_2\ell_2^3 \end{bmatrix}\begin{Bmatrix} \ddot{v}_2 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix}$$

$$+E\begin{bmatrix} 12\left(\frac{I_2}{\ell_2^3} + \frac{I_1}{\ell_1^3}\right) & 6\left(\frac{I_2}{\ell_2^2} - \frac{I_1}{\ell_1^2}\right) & 6\,\frac{I_2}{\ell_2^2} \\ 6\left(\frac{I_2}{\ell_2^2} - \frac{I_1}{\ell_1^2}\right) & 4\left(\frac{I_2}{\ell_2} + \frac{I_1}{\ell_1}\right) & 2\,\frac{I_2}{\ell_2} \\ 6\,\frac{I_2}{\ell_2^2} & 2\,\frac{I_2}{\ell_2} & 4\,\frac{I_2}{\ell_2} \end{bmatrix}\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

10 − 16



Element 1-2 $\quad \alpha = 45^\circ \quad C = .707 \quad S = .707$

$$c^2 = s^2 = cs = .50$$

$$\ell_{12} = 2\ell$$

$$\bar{k} = \left(\frac{EI}{8\ell^3}\right)\begin{bmatrix} (.5R_{1\text{-}2}+6) & .50(R_{1\text{-}2}-12) & 12\ell\cdot.707 \\ .50(R_{1\text{-}2}-12) & .5\,(R_{1\text{-}2}+12) & -12\ell\cdot.707 \\ 12\ell\cdot.707 & -12\ell\cdot.707 & 16\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

All $R_{1\text{-}2} = \frac{A}{I}(2\ell)^2 = \frac{4A\ell^2}{I}$

Element 2-3 $\quad \alpha = 0 \quad C = 1\;, \; S = 0\;, \; \ell_{23} = \ell$

$$\bar{k} = \left(\frac{EI}{\ell^3}\right)\begin{bmatrix} R_{2\text{-}3} & 0 & 0 \\ 0 & 12 & 6\ell \\ 0 & 6\ell & 4\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

All $R_{2\text{-}3} = \frac{A\ell^2}{I} = \frac{1}{4}R_{1\text{-}2}$

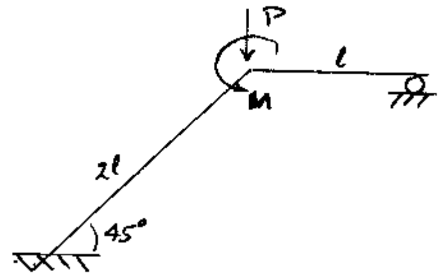
Assembled $\quad$ with all $R_s = R_{12} = R$

$$\bar{k} = \left(\frac{EI}{8\ell^3}\right)\begin{bmatrix} (.5R+2R+6) & .5(R-12) & .707\times12\ell \\ .5\,(R+12) & .5(R+12)+96 & -.707\times12\ell+48\ell \\ 12\times.707\,\ell & -.707\times12\ell+48\ell & 16\ell^2+32\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ -P \\ M \end{Bmatrix} = \left(\frac{EI}{8\ell^3}\right)\begin{bmatrix} (2.5R+6) & .5\,(R-12) & 8.484\ell \\ .5\,(R-12) & (.5R+102) & 39.52\ell \\ 8.484\ell & 39.52\ell & 48\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

$$R = \frac{4A\ell^2}{I}$$

10 – 17



Element 1 – 2

$$\bar{k} = \left(\frac{EI}{8l^3}\right)\begin{bmatrix} (.5R+6) & .5(R-12) & 8.484\,l \\ .5(R-12) & .5(R+12) & -8.484\,l \\ 8.484\,l & -8.484\,l & 16\,l^2 \end{bmatrix}$$

Element 2 – 3

$$\bar{k} = \left(\frac{EI}{l^3}\right)\begin{bmatrix} R_{2\text{-}3} & 0 & 0 \\ 0 & 3 & 3l \\ 0 & 3l & 3l^2 \end{bmatrix} = \left(\frac{EI}{8l^3}\right)\begin{bmatrix} 2R & 0 & 0 \\ 0 & 24 & 24l \\ 0 & 24l & 24l^2 \end{bmatrix}$$

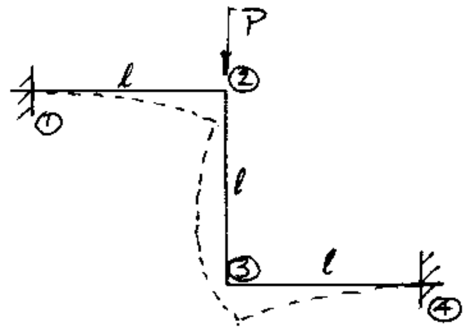
Assembled

$$\begin{Bmatrix} 0 \\ -P \\ M \end{Bmatrix} = \left(\frac{EI}{8l^3}\right)\begin{bmatrix} (2.5R+6) & .5(R-12) & 8.484\,l \\ .5(R-12) & (.5R+30) & 15.52\,l \\ 8.484\,l & 15.52\,l & 40\,l^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

From 1st of above equation express $\bar{u}_2$ in terms of $\bar{v}_2$ and θ_2 and write

$$\begin{Bmatrix} -P \\ M \end{Bmatrix} = \left(\frac{EI}{8l^3}\right)\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

10 - 18(a)



$$\begin{Bmatrix} \bar{F}_{1x} \\ \vdots \\ \bar{M}_4 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} & & \\ & 12\times 12 & \\ & & \end{bmatrix}\begin{Bmatrix} \bar{u}_1 \\ \vdots \\ \theta_4 \end{Bmatrix}$$

Elements 1-2 and 3-4 $\alpha = 0$ $C = 1,$ $S = 0$

$$\bar{k}_{1-2} = \bar{k}_{3-4} = \frac{EI}{\ell^3}\left[\begin{array}{ccc|ccc} R & 0 & 0 & -R & 0 & 0 \\ 0 & 12 & 6\ell & 0 & -12 & 6\ell \\ 0 & 6\ell & 4\ell^2 & 0 & -6\ell & 2\ell^2 \\ \hline -R & 0 & 0 & R & 0 & 0 \\ 0 & -12 & -6\ell & 0 & 12 & -6\ell \\ 0 & 6\ell & 2\ell^2 & 0 & -6\ell & 4\ell^2 \end{array}\right] \quad \text{vectors}\begin{Bmatrix} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \theta_1 = 0 \\ \hline \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix} \text{or} \begin{Bmatrix} \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \\ \hline \bar{u}_4 = 0 \\ \bar{v}_4 = 0 \\ \theta_4 = 0 \end{Bmatrix}$$

Element 2-3 $\alpha = 270^\circ$ or -90° $C = 0,$ $S = -1$

$$\bar{k}_{2-3} = \frac{EI}{\ell^3}\left[\begin{array}{ccc|ccc} 12 & 0 & 6\ell & -12 & 0 & 6\ell \\ 0 & R & 0 & 0 & -R & 0 \\ 6\ell & 0 & 4\ell^2 & -6\ell & 0 & 2\ell^2 \\ \hline -12 & 0 & -6\ell & 12 & 0 & -6\ell \\ 0 & -R & 0 & 0 & R & 0 \\ 6\ell & 0 & 2\ell^2 & -6\ell & 0 & 4\ell^2 \end{array}\right] \quad \text{vector}\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \\ \hline \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$$

Only displacements involving 2 and 3 need to be considered. Thus we need only to add lower right quarter of $\bar{k}_{1-2}$ to upper left quarter of $\bar{k}_{2-3}$, and upper left qtr. of $\bar{k}_{3-4}$ to lower right qtr. of $\bar{k}_{2-3}$

$$\begin{Bmatrix} \bar{F}_{2x} = 0 \\ -P \\ \bar{M}_2 = 0 \\ \hline \bar{F}_{3x} = 0 \\ \bar{F}_{3y} = 0 \\ \bar{M}_3 = 0 \end{Bmatrix} = \frac{EI}{\ell^3}\left[\begin{array}{ccc|ccc} \cancel{(R+12)} & \cancel{0} & \cancel{6\ell} & \cancel{-12} & \cancel{0} & \cancel{6\ell} \\ 0 & (R+12) & -6\ell & 0 & -R & 0 \\ 6\ell & -6\ell & 8\ell^2 & -6\ell & 0 & 2\ell^2 \\ \hline \cancel{-12} & \cancel{0} & \cancel{-6\ell} & \cancel{(R+12)} & \cancel{0} & \cancel{-6\ell} \\ 0 & -R & 0 & 0 & (R+12) & 6\ell \\ 6\ell & 0 & 2\ell^2 & -6\ell & 6\ell & 8\ell^2 \end{array}\right]\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \\ \hline \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$$

symmetric matrix

10 – 18(a) Cont.

For no axial extension $\bar{u}_2 = -\bar{u}_3 = 0$

this eliminates 1^{st} and 4^{th} columns

Then $F_{2x} = F_{3x} = 0$ and the 1^{st} and 4^{th} rows requires that $\theta_3 = -\theta_2$ which eliminates the 1^{st} and 4^{th} rows, leaving the 4×4 matrix

$$\begin{Bmatrix} -P \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{\ell^3} \left[\begin{array}{cc|cc} (R+12) & -6\ell & -R & 0 \\ -6\ell & 8\ell^2 & 0 & 2\ell^2 \\ \hline -R & 0 & (R+12) & 6\ell \\ 0 & 2\ell^2 & 6\ell & 8\ell^2 \end{array}\right] \begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$$

symmetric

Since $\bar{v}_3 = \bar{v}_2$ add 3^{rd} column to 1^{st} column.
Also since $\theta_3 = -\theta_2$, subtract 4^{th} col. from 2^{nd} col,
This also reduces the matrix to a 2×2

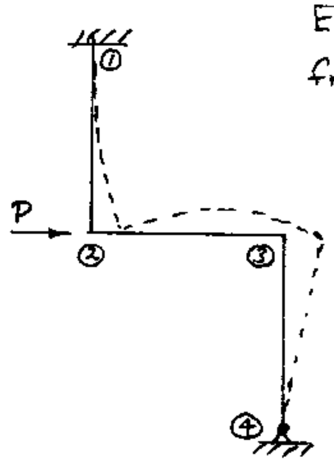
$$\begin{Bmatrix} -P \\ 0 \end{Bmatrix} = \frac{EI}{\ell^3} \begin{bmatrix} 12 & -6\ell \\ -6\ell & 6\ell^2 \end{bmatrix} \begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

Solving the two equations

$$\ell\theta_2 = \bar{v}_2$$

$$\boxed{\bar{v}_2 = \frac{-P\ell^3}{6EI} \qquad \theta_2 = -\frac{P\ell^2}{6EI}}$$

10 − 18 (b)



Elements 1-2 and 2-3 are already available from Prob 10−18(a) by interchanging 1-2 and 2-3

The new element 3−4 is obtained from Eq.(10.8 − 3) with $\alpha = -90^\circ$ $C=0$, $S=-1$

Element 3−4

$$\frac{EI}{\ell^3}\left[\begin{array}{ccc|ccc} 3 & 0 & 3\ell & -3 & 0 & 3\ell \\ 0 & R & 0 & 0 & -R & 0 \\ 3\ell & 0 & 3\ell^2 & 0 & 0 & 0 \\ \hline -3 & 0 & -3\ell & 3 & 0 & -3\ell \\ 0 & -R & 0 & 0 & R & 0 \\ 0 & 0 & 0 & -3\ell & 0 & 3\ell^2 \end{array}\right]$$

$$\begin{Bmatrix} \bar{F}_{1x} \\ \bar{F}_{1y} \\ \bar{M}_1 \\ \hline \bar{F}_{2x}=P \\ \bar{F}_{2y}=0 \\ \bar{M}_2=0 \\ \hline \bar{F}_{3x}=0 \\ \bar{F}_{3y}=0 \\ \bar{M}_3=0 \\ \hline \bar{F}_{4x} \\ \bar{F}_{4y} \\ \bar{M}_4 \end{Bmatrix} = \frac{EI}{\ell^3}\left[\begin{array}{ccc|ccc|ccc|ccc} & & & -12 & 0 & 6\ell & & & & & & \\ & & & 0 & -R & 0 & & & & & & \\ & & & -6\ell & 0 & 2\ell^2 & & & & & & \\ \hline & & & (12+R) & 0 & -6\ell & -R & 0 & 0 & & & \\ & & & 0 & (12+R) & 6\ell & 0 & -12 & 6\ell & & & \\ & & & -6\ell & 6\ell & 8\ell^2 & 0 & -6\ell & 2\ell^2 & & & \\ \hline & & & -R & 0 & 0 & (R+3) & 0 & 3\ell & -3 & 0 & 3\ell \\ & & & 0 & -12 & -6\ell & 0 & (R+12) & -6\ell & 0 & -R & 0 \\ & & & 0 & 6\ell & 2\ell^2 & 3\ell & -6\ell & 7\ell^2 & 0 & 0 & 0 \\ \hline & & & & & & -3 & 0 & -3\ell & 3 & 0 & -3\ell \\ & & & & & & 0 & -R & 0 & 0 & R & 0 \\ & & & & & & 0 & 0 & 0 & -3\ell & 0 & 3\ell^2 \end{array}\right] \begin{Bmatrix} 0 \\ \hline \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \\ \hline \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \\ \hline 0 \end{Bmatrix}$$

We need only the central part of the matrix

10 – 18 (b) Cont. $\quad \bar{v}_2 = \bar{v}_3 = 0 \quad \therefore$ eliminate cols 2 & 5

$$\begin{Bmatrix} \bar{F}_{2x}=P \\ \bar{M}_2 \\ \bar{F}_{3x} \\ \bar{M}_3 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} (R+12) & -6\ell & -R & 0 \\ -6\ell & 8\ell^2 & 0 & 2\ell^2 \\ -R & 0 & (R+3) & 3\ell \\ 0 & 2\ell^2 & 3\ell & 7\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \theta_2 \\ \bar{u}_3 \\ \theta_3 \end{Bmatrix}$$

$\bar{u}_2 = \bar{u}_3 \quad \therefore$ add col 3 to 1

$$\begin{Bmatrix} \frac{P\ell^3}{EI} \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 12 & -6\ell & 0 \\ -6\ell & 8\ell^2 & 2\ell^2 \\ 3 & 0 & 3\ell \\ 3\ell & 2\ell^2 & 7\ell^2 \end{bmatrix}\begin{Bmatrix} u_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

Eq. no.

$$\frac{P\ell^3}{EI} = 12\bar{u}_2 - 6\ell\theta_2 \quad (1)$$
$$-6\ell\bar{u}_2 + 8\ell^2\theta_2 + 2\ell^2\theta_3 = 0 \quad (2)$$
$$3\bar{u}_2 + 3\ell\theta_3 = 0 \quad (3)$$
$$3\ell\bar{u}_2 + 2\ell^2\theta_2 + 7\ell^2\theta_3 = 0 \quad (4)$$

$2\times(4) + (2) = \quad 0\,u_2 + 12\ell^2\theta_2 + 16\ell^2\theta_3 = 0 \quad \therefore \quad \theta_2 = -\frac{4}{3}\theta_3$

(3) gives $\quad \bar{u}_2 = -\ell\theta_3 \quad \therefore = \frac{3}{4}\ell\theta_2$

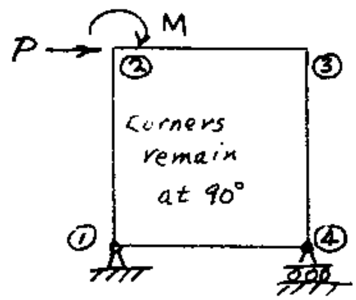
Subst $\theta_2 = -\frac{4}{3}\theta_3$ into (1)

$$\frac{P\ell^3}{EI} = 12u_2 - 6\ell\left(-\frac{4}{3}\theta_3\right) = 12\bar{u}_2 + 8\ell\theta_3 = 12\bar{u}_2 - 8\bar{u}_2 = 4\bar{u}_2$$

$$\therefore \quad \bar{u}_2 = \frac{P\ell^3}{4EI}, \qquad \theta_3 = -\frac{P\ell^2}{4EI}, \qquad \theta_2 = \frac{P\ell^2}{3EI}$$

Comparing with Prob(10–18(a)) the deflection under P is larger with pinned end, and the two rotations are dissimular and also larger. Conclusion: Pinned end leads to more flexible structure.

10 - 18 (c)



Element

Element		
1-2	$c = 0$,	$s = 1$
2-3	$c = 1$,	$s = 0$
3-4	$c = 0$,	$s = -1$
4-1	$c = -1$	$s = 0$

Element stiffness

$$①\text{-}② \begin{bmatrix} 12 & 0 & -6\ell & -12 & 0 & -6\ell \\ 0 & R & 0 & 0 & -R & 0 \\ -6\ell & 0 & 4\ell^2 & 6\ell & 0 & 2\ell^2 \\ -12 & 0 & 6\ell & 12 & 0 & 6\ell \\ 0 & -R & 0 & 0 & R & 0 \\ -6\ell & 0 & 2\ell^2 & 6\ell & 0 & 4\ell^2 \end{bmatrix} \qquad ②\text{-}③ \begin{bmatrix} R & 0 & 0 & -R & 0 & 0 \\ 0 & 12 & 6\ell & 0 & -12 & 6\ell \\ 0 & 6\ell & 4\ell^2 & 0 & -6\ell & 2\ell^2 \\ -R & 0 & 0 & R & 0 & 0 \\ 0 & -12 & -6\ell & 0 & 12 & -6\ell \\ 0 & 6\ell & 2\ell^2 & 0 & -6\ell & 4\ell^2 \end{bmatrix} = ①\text{-}④$$

$$③\text{-}④ \begin{bmatrix} 12 & 0 & 6\ell & -12 & 0 & 6\ell \\ 0 & R & 0 & 0 & -R & 0 \\ 6\ell & 0 & 4\ell^2 & -6\ell & 0 & 2\ell^2 \\ -12 & 0 & -6\ell & 12 & 0 & -6\ell \\ 0 & -R & 0 & 0 & R & 0 \\ 6\ell & 0 & 2\ell^2 & -6\ell & 0 & 4\ell^2 \end{bmatrix} \qquad ④\text{-}① \begin{bmatrix} R & 0 & 0 & -R & 0 & 0 \\ 0 & 12 & -6\ell & 0 & -12 & -6\ell \\ 0 & -6\ell & 4\ell^2 & 0 & 6\ell & 2\ell^2 \\ -R & 0 & 0 & R & 0 & 0 \\ 0 & -12 & 6\ell & 0 & 12 & 6\ell \\ 0 & -6\ell & 2\ell^2 & 0 & 6\ell & 4\ell^2 \end{bmatrix}$$

Displacements.

$\bar{u}_1 = \bar{v}_1 = 0$

$\bar{v}_2 = \bar{v}_3 = 0$

$\bar{u}_2 = \bar{u}_3$

$\bar{v}_4 = 0$

$\bar{u}_4 \cong 0$

Coordinates not zero

$\bar{u}_2,\ \bar{u}_3,\ \theta_1,\ \theta_2,\ \theta_3,\ \theta_4$

none of the θ_s need to be equal, however we can assume that $\bar{u}_2 = \bar{u}_3$

This leaves 5 unknowns.

10 – 18 (c) Cont.

Assembled matrix (symmetric)

$$
\begin{array}{c}
\\ (1) \\ \\ \\ (2) \\ \\ \\ (3) \\ \\ \\ (4) \\ \\
\end{array}
\left[\begin{array}{ccc|ccc|ccc|ccc}
(12+R) & 0 & -6\ell & -12 & 0 & -6\ell & & & & -R & 0 & 0 \\
0 & (12+R) & 6\ell & 0 & -R & 0 & & & & 0 & -12 & 6\ell \\
-6\ell & 6\ell & 8\ell^2 & 6\ell & 0 & 2\ell^2 & & & & 0 & -6\ell & 2\ell^2 \\
\hline
-12 & 0 & 6\ell & (12+R) & 0 & 6\ell & -R & 0 & 0 & & & \\
0 & -R & 0 & 0 & (12+R) & 6\ell & 0 & -12 & 6\ell & & & \\
-6\ell & 0 & 2\ell^2 & 6\ell & 6\ell & 8\ell^2 & 0 & -6\ell & 2\ell^2 & & & \\
\hline
 & & & -R & 0 & 0 & (12+R) & 0 & 6\ell & -12 & 0 & 6\ell \\
 & & & 0 & -12 & -6\ell & 0 & (12+R) & -6\ell & 0 & -R & 0 \\
 & & & 0 & 6\ell & 2\ell^2 & 6\ell & -6\ell & 8\ell^2 & -6\ell & 0 & 2\ell^2 \\
\hline
-R & 0 & 0 & & & & -12 & 0 & -6\ell & (12+R) & 0 & -6\ell \\
0 & -12 & -6\ell & & & & 0 & -R & 0 & 0 & (12+R) & -6\ell \\
0 & 6\ell & 2\ell^2 & & & & 6\ell & 0 & 2\ell^2 & -6\ell & -6\ell & 8\ell^2
\end{array}\right]
\begin{array}{c}
0 \\ 0 \\ \theta_1 \\ u_2 \\ 0 \\ \theta_2 \\ \bar{u}_3 = \bar{u}_2 \\ 0 \\ \theta_3 \\ 0 \\ 0 \\ \theta_4
\end{array}
\quad
\begin{array}{c}
\bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \theta_1 \\ \bar{u}_2 \\ \bar{v}_2 = 0 \\ \theta_2 \\ \bar{u}_3 \\ \bar{v}_3 = 0 \\ \theta_3 \\ \bar{u}_4 = 0 \\ \bar{v}_4 = 0 \\ \theta_4
\end{array}
$$

$\bar{u}_3 = \bar{u}_2$ $\therefore$ add col. 7 to 4 and rewrite

$$
\begin{Bmatrix}
\bar{F}_{1x} \\ \bar{F}_{1y} \\ 0 \\ P \\ 0 \\ -M \\ 0 \\ 0 \\ 0 \\ \bar{F}_{4x} \\ \bar{F}_{4y} \\ 0
\end{Bmatrix}
= \frac{EI}{\ell^3}
\begin{bmatrix}
-6\ell & -12 & -6\ell & 0 & 0 \\
6\ell & 0 & 0 & 0 & 6\ell \\
8\ell^2 & 6\ell & 2\ell^2 & 0 & 2\ell^2 \\
6\ell & 12 & 6\ell & 0 & 0 \\
0 & 0 & 6\ell & 6\ell & 0 \\
2\ell^2 & 6\ell & 8\ell^2 & 2\ell^2 & 0 \\
0 & 12 & 0 & 6\ell & 6\ell \\
0 & 0 & -6\ell & -6\ell & 0 \\
0 & 6\ell & 2\ell^2 & 8\ell^2 & 2\ell^2 \\
0 & -12 & 0 & -6\ell & -6\ell \\
-6\ell & 0 & 0 & 0 & -6\ell \\
2\ell^2 & 6\ell & 0 & 2\ell^2 & 8\ell^2
\end{bmatrix}
\begin{Bmatrix}
\theta_1 \\ \bar{u}_2 \\ \theta_2 \\ \theta_3 \\ \theta_4
\end{Bmatrix}
$$

12x 5 matrix

5th and 8th equations are identical — both give $\theta_2 = -\theta_3$
(remove 5th & 8th eqs. after subtracting 4th col from 3rd col.)

10 – 18 (c) Cont   Rewrite eq with col 4 subtracted from col 3 and 5th & 8th eqs. removed

$$\begin{Bmatrix} \bar{F}_{1x} \\ \bar{F}_{1y} \\ 0 \\ P \\ -M \\ 0 \\ 0 \\ \bar{F}_{4x} \\ \bar{F}_{4y} \\ 0 \end{Bmatrix} = \frac{EI}{\ell^3} \begin{bmatrix} -6\ell & -12 & -6\ell & 0 \\ 6\ell & 0 & 0 & 6\ell \\ 8\ell^2 & 6\ell & 2\ell^2 & 2\ell^2 \\ 6\ell & 12 & 6\ell & 0 \\ 2\ell^2 & 6\ell & 6\ell^2 & 0 \\ 0 & 12 & -6\ell & 6\ell \\ 0 & 6\ell & -6\ell^2 & 2\ell^2 \\ 0 & -12 & 6\ell & -6\ell \\ -6\ell & 0 & 0 & -6\ell \\ 2\ell^2 & 6\ell & -2\ell^2 & 8\ell^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \bar{u}_2 \\ \theta_2 \\ \theta_4 \end{Bmatrix}$$

10 x 4 matrix

Eqs. for $\bar{F}_{1y}$ and $\bar{F}_{4y}$ are equal but of opposite signs

$\therefore \quad \bar{F}_{1y} = -\bar{F}_{4y}$ (remove these two eqs.)

6^{th} eq and $\bar{F}_{4x}$ are of opposite signs but 6^{th} eq $= 0$

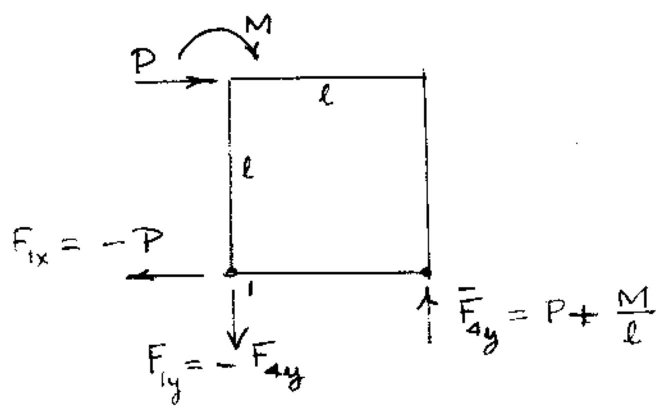
$\therefore \quad \bar{F}_{4x} = 0$ (remove these two eqs)

Eqs for $\bar{F}_{1x}$ and P are of opposite signs

$\therefore \quad F_{1x} = -P$

Moment about ① gives

$F_{4y}\,\ell = P\ell + M$



The above is just for checking, since these results can be obtained from statics.

10 - 18 (c) Cont.

Rewrite unused eqs from 10x 4 matrix

$$\frac{P\ell^3}{EI} = 6\ell\theta_1 + 12\bar{u}_2 + 6\ell\theta_2 \qquad (1)$$

$$-\frac{M\ell^2}{EI} = 2\ell\theta_1 + 6\bar{u}_2 + 6\ell\theta_2 \qquad (2)$$

$$0 = 8\ell^2\theta_1 + 6\ell\bar{u}_2 + 2\ell^2\theta_2 + 2\ell^2\theta_4 \qquad (3)$$

$$0 = 2\ell^2\theta_1 + 6\ell\bar{u}_2 - 2\ell^2\theta_2 + 8\ell^2\theta_4 \qquad (4)$$

Eliminate θ_4 betw. (3) & (4)

$$0 = 30\ell^2\theta_1 + 18\ell\bar{u}_2 + 10\ell^2\theta_2 \qquad (5)$$

Eqs. (1) (2) & (5) has 3 unknowns with 3 eqs.

In (5) solve for $\ell\theta_1$ in terms of $\bar{u}_2$ and θ_2

$$\ell\theta_1 = -(.60\,\bar{u}_2 + .3333\,\ell\theta_2) \quad \text{subst. into(1) and(2)}$$

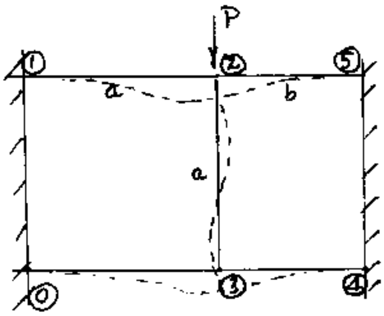
$$\frac{P\ell^3}{EI} = (12 - 3.60)\bar{u}_2 + (6\ell - 2\ell)\theta_2 = 8.40\bar{u}_2 + 4\,\ell\theta_2$$

$$-\frac{M\ell^2}{EI} = (6 - 1.20)\bar{u}_2 + (6\ell - .666\ell)\theta_2 = 4.80\,\bar{u}_2 + 5.333\ell\theta_2$$

Eliminate θ_2 for eq. for $\bar{u}_2$, then eliminate $\bar{u}_2$

$$\boxed{\begin{aligned} \bar{u}_2 &= \frac{1}{4.80}\left(\frac{P\ell^3}{EI} + .75\,\frac{M\ell^2}{EI}\right) \\ \theta_2 &= \frac{-1}{5.333\,\ell}\left(\frac{P\ell^3}{EI} + 1.75\,\frac{M\ell^2}{EI}\right) \end{aligned}}$$

10 - 18(d)



Let $a = l_1 = l$

$b = l_2 = \frac{b}{a} l_1$

$\bar{u}_2 = \bar{u}_3 = 0$

$\bar{v}_2 = \bar{v}_3$

$$\begin{Bmatrix} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \theta_1 = 0 \\ \hline \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \\ \hline \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \\ \hline \bar{u}_4 = 0 \\ \bar{v}_4 = 0 \\ \theta_4 = 0 \end{Bmatrix}$$

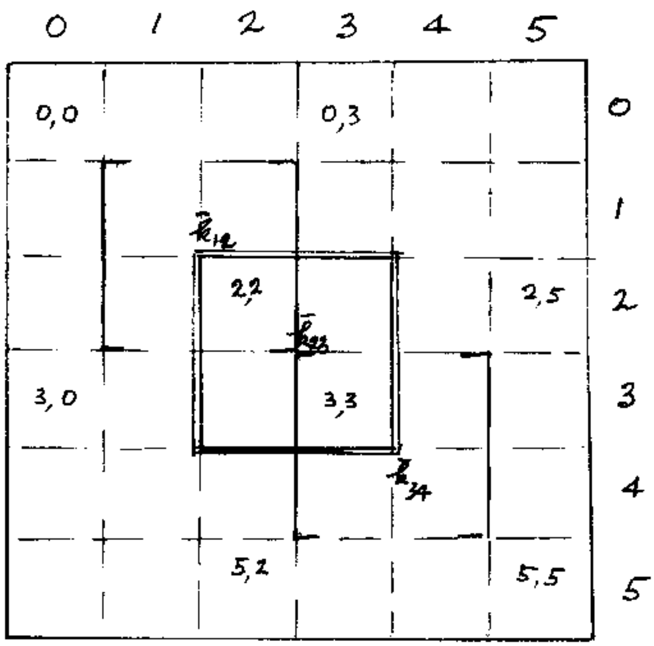
$$\bar{k}_{1-2} = \frac{EI}{l_1^3}\left[\begin{array}{ccc|ccc} & & & -R & 0 & 0 \\ & & & 0 & -12 & 6l \\ & & & 0 & -6l & 2l^2 \\ \hline - & - & & R & 0 & 0 \\ & & & 0 & 12 & -6l \\ & & & 0 & -6l & 4l^2 \end{array}\right] = \bar{k}_{0-3}$$

$$\bar{k}_{2-3} = \frac{EI}{l_1^3}\left[\begin{array}{ccc|ccc} 12 & 0 & 6l & -12 & 0 & 6l \\ 0 & R & 0 & 0 & -R & 0 \\ 6l & 0 & 4l^2 & -6l & 0 & 2l^2 \\ \hline -12 & 0 & -6l & 12 & 0 & -6l \\ 0 & -R & 0 & 0 & R & 0 \\ 6l & 0 & 2l^2 & -6l & 0 & 4l^2 \end{array}\right]$$

$$\bar{k}_{3-4} = \frac{EI}{l_2^3}\left[\begin{array}{ccc|ccc} R & 0 & 0 & & & \\ 0 & 12 & 6l_2 & & & \\ 0 & 6l_2 & 4l_2^2 & - & - & - \\ \hline -R & 0 & 0 & & & \\ 0 & -12 & -6l_2 & & & \\ 0 & 6l_2 & 2l_2^2 & & & \end{array}\right] = \frac{EI}{l^3}\left[\begin{array}{ccc|ccc} R(\frac{a}{b})^3 & 0 & 0 & & & \\ 0 & 12(\frac{a}{b})^3 & 6l(\frac{a}{b})^2 & & & \\ 0 & 6l(\frac{a}{b})^2 & 4l^2(\frac{a}{b}) & - & - & - \\ \hline -R(\frac{a}{b})^3 & 0 & 0 & & & \\ 0 & -12(\frac{a}{b})^3 & -6l(\frac{a}{b})^2 & & & \\ 0 & 6l(\frac{a}{b})^2 & 2l^2(\frac{a}{b}) & & & \end{array}\right] \begin{array}{l} = \bar{k}_{2-5} \\ = \bar{k}_{3-4} \end{array}$$

For the assembly we can first consider a matrix space of 6 x 6 each of which is composed of a 3x3 elements. ie. $\bar{k}_{2-3}$ has 4 sub-matrices, each corner of which has a 3x3 matrix. Since ①, ⓪, ④ and ⑤ all have zero deflections, we will be concerned only with the displacements of ② and ③ which occupies the central portion of the 6x6 space.

10 – 18 (d) Cont.



Sub matrices of $\bar{k}_{2-5}$

2,2 2,5

5,2 5,5

shown in 6 x 6 space

Sub matrices of $\bar{k}_{0-3}$

0,0 0,3

3,0 3,3

shown in 6 x 6 space

The matrix of interest which is indicated by the square in double lines also involves the lower right quarter of $\bar{k}_{12}$, all of $\bar{k}_{23}$ and upper left quarter of $\bar{k}_{3-4}$

$$\frac{EI}{\ell^3}\left[\begin{array}{ccc|ccc}
\begin{array}{c}R+12\\+R(\frac{a}{b})^3\end{array} & \begin{array}{c}0+0\\0\end{array} & \begin{array}{c}0+6\ell\\0\end{array} & -12 & 0 & 6\ell \\
\begin{array}{c}0+0\\0\end{array} & \begin{array}{c}12+R\\12(\frac{a}{b})^3\end{array} & \begin{array}{c}-6\ell+0\\6\ell(\frac{a}{b})^2\end{array} & 0 & -R & 0 \\
\begin{array}{c}0+6\ell\\0\end{array} & \begin{array}{c}-6\ell+0\\6\ell(\frac{a}{b})^2\end{array} & \begin{array}{c}4\ell^2+4\ell^2\\4\ell^2(\frac{a}{b})\end{array} & -6\ell & 0 & 2\ell^2 \\
\hline
-12 & 0 & -6\ell & \begin{array}{c}R+12\\R(\frac{a}{b})^3\end{array} & \begin{array}{c}0+0\\0\end{array} & \begin{array}{c}0-6\ell\\0\end{array} \\
0 & -R & 0 & \begin{array}{c}0+0\\0\end{array} & \begin{array}{c}12+R\\12(\frac{a}{b})^3\end{array} & \begin{array}{c}-6\ell+0\\6\ell(\frac{a}{b})^2\end{array} \\
6\ell & 0 & 2\ell^2 & 0-6\ell & \begin{array}{c}-6\ell+0\\6\ell(\frac{a}{b})^2\end{array} & \begin{array}{c}4\ell^2+4\ell^2\\4\ell^2(\frac{a}{b})\end{array}
\end{array}\right]
\begin{Bmatrix}\bar{u}_2\\ \bar{v}_2\\ \theta_2\\ \bar{u}_3\\ \bar{v}_3\\ \theta_3\end{Bmatrix}$$

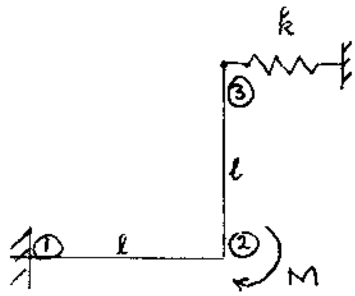
10 - 18 (d) Cont

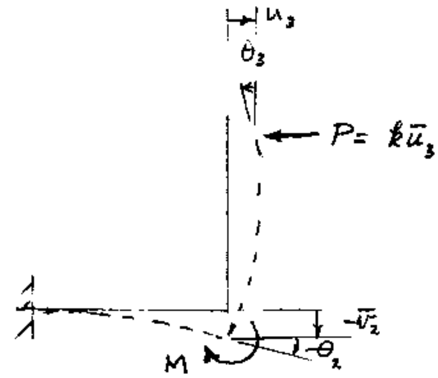
Note that $\begin{cases} \bar{u}_2 = \bar{u}_3 = 0 \\ \bar{v}_2 = \bar{v}_3 \\ \theta_2 = \theta_3 \end{cases}$ cross out cols. 1 and 4; add col. 5 to 2; add col. 6 to 3

Resulting equations

$$\begin{Bmatrix} \bar{F}_{2y} = -P \\ \bar{M}_2 = 0 \end{Bmatrix} = \frac{EI}{a^3} \begin{bmatrix} (12)\left[1+\left(\frac{a}{b}\right)^3\right] & 6\ell\left[\left(\frac{a}{b}\right)^2-1\right] \\ 6\ell\left[\left(\frac{a}{b}\right)^2-1\right] & 4\ell^2\left[2.5+\left(\frac{a}{b}\right)\right] \end{bmatrix} \begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

10-18 (e)





$$\begin{Bmatrix} \bar{F}_{1x} \\ \bar{F}_{1y} \\ \bar{M}_1 \\ \bar{F}_{2x} \\ \bar{F}_{2y} \\ \bar{M}_2 \\ \bar{F}_{3x} \\ \bar{F}_{3y} \\ \bar{M}_3 \end{Bmatrix} = \frac{EI}{\ell^3} \begin{bmatrix} R & 0 & 0 & -R & 0 & 0 & & & \\ 0 & 12 & 6\ell & 0 & -12 & 6\ell & & & \\ 0 & 6\ell & 4\ell^2 & 0 & -6\ell & 2\ell^2 & & & \\ -R & 0 & 0 & R+12 & 0+0 & 0-6\ell & -12 & 0 & -6\ell \\ 0 & -12 & -6\ell & 0+0 & 12+R & -6\ell+0 & 0 & -R & 0 \\ 0 & 6\ell & 2\ell^2 & 0-6\ell & -6\ell+0 & 4\ell^2+4\ell^2 & 6\ell & 0 & 2\ell^2 \\ & & & -12 & 0 & 6\ell & \left(12+\frac{k\ell^3}{EI}\right) & 0 & 6\ell \\ & & & 0 & -R & 0 & 0 & R & 0 \\ & & & -6\ell & 0 & 2\ell^2 & 6\ell & 0 & 4\ell^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \theta_1 = 0 \\ \bar{u}_2 = 0 \\ \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$$

$\bar{v}_3 = \bar{v}_2$ $\therefore$ add col. 8 to col 5

$$\begin{Bmatrix} 0 \\ -M \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{\ell^3} \begin{bmatrix} 12 & -6\ell & 0 & 0 \\ -6\ell & 8\ell^2 & 6\ell & 2\ell^2 \\ 0 & 6\ell & \left(12+\frac{k\ell^3}{EI}\right) & 6\ell \\ 0 & 2\ell^2 & 6\ell & 4\ell^2 \end{bmatrix} \begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 \\ \theta_3 \end{Bmatrix}$$ Sym. 4 x 4

4 eqs, 4 unknowns

10 – 18 (e) Cont.

$$① \quad 0 = 12\bar{v}_2 - 6\ell\theta_2$$

$$② \quad -\frac{M\ell^3}{EI} = -6\ell\bar{v}_2 + 8\ell^2\theta_2 + 6\ell\bar{u}_3 + 2\ell^2\theta_3$$

$$③ \quad 0 = 6\ell\theta_2 + \left(12 + \frac{k\ell^3}{EI}\right)\bar{u}_3 + 6\ell\theta_3$$

$$④ \quad 0 = 2\ell^2\theta_2 + 6\ell\bar{u}_3 + 4\ell^2\theta_3$$

From ① $\underline{\bar{v}_2 = \frac{\ell}{2}\theta_2}$ sub into ②

$$②' \quad -\frac{M\ell^3}{EI} = 5\ell^2\theta_2 + 6\ell\bar{u}_3 + 2\ell^2\theta_3$$

From ④ $\underline{\theta_2 = -\frac{3}{\ell}\bar{u}_3 - 2\theta_3}$

Sub. into ③ $$0 = -6\ell\left(\frac{3}{\ell}\bar{u}_3 + 2\theta_3\right) + \left(12 + \frac{k\ell^3}{EI}\right)\bar{u}_3 + 6\ell\theta_3$$

$$= \left(-18 + 12 + \frac{k\ell^3}{EI}\right)\bar{u}_3 - 12\ell\theta_3 + 6\ell\theta_3$$

$$\therefore \quad \underline{6\ell\theta_3 = \left(-6 + \frac{k\ell^3}{EI}\right)\bar{u}_3}$$

$$-\frac{M\ell^3}{EI} = 5\ell^2\left(-\frac{3}{\ell}\bar{u}_3 - 2\theta_3\right) + 6\ell\bar{u}_3 + 2\ell^2\theta_3 = -9\ell\bar{u}_3 - 8\ell^2\theta_3$$

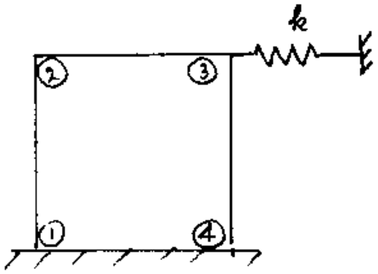
Sub for θ_3 in above eq.

$$-\frac{M\ell^3}{EI} = -9\ell\bar{u}_3 - 8\ell^2\left(\frac{-6 + \frac{k\ell^3}{EI}}{6\ell}\right)\bar{u}_3 = \left(-\ell - \frac{4}{3}\frac{k\ell^3}{EI}\right)\bar{u}_3$$

$$\therefore \quad \boxed{\bar{u}_3 = \frac{M\ell^2}{\left(1 + \frac{4}{3}\frac{k\ell^3}{EI}\right)EI}} \qquad \boxed{\theta_3 = \frac{1}{6\ell}\frac{\left(-6 + \frac{k\ell^3}{EI}\right)}{\left(1 + \frac{4}{3}\frac{k\ell^3}{EI}\right)}\frac{M\ell^2}{EI}}$$

$$\boxed{\theta_2 = -\frac{3}{\ell}\bar{u}_3 - 2\theta_3} \qquad \boxed{\bar{v}_2 = -\frac{3}{2}\bar{u}_3 - \ell\theta_2}$$

10 – 19 (a)



Stiffness matrix

$$\frac{EI}{l^3}\begin{bmatrix} (12+R) & 0 & 6l & -R & 0 & 0 \\ 0 & (12+R) & 6l & 0 & -12 & 6l \\ 6l & 6l & 8l^2 & 0 & -6l & 2l^2 \\ -R & 0 & 0 & (12+R+\frac{kl^3}{EI}) & 0 & 6l \\ 0 & -12 & -6l & 0 & (12+R) & -6l \\ 0 & 6l & 2l^2 & 6l & -6l & 8l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 = 0 \\ \theta_2 \\ \bar{u}_3 \\ \bar{v}_3 = 0 \\ \theta_3 \end{Bmatrix} \qquad \bar{u}_3 = \bar{u}_2$$

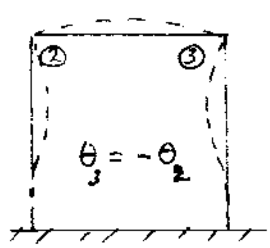
Upper left qtr = lower right qtr of elem. 1-2 plus upper left qtr of elem. 2-3.

Lower right qtr = upper left qtr of elem. 3-4 plus Lower right qtr of elem. 2-3.

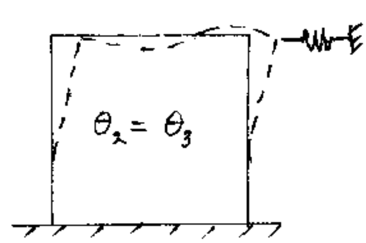
mass matrix $$\frac{ml}{420}\begin{bmatrix} (156+N) & 0 & 22l & \frac{1}{2}N & 0 & 0 \\ 0 & (156+N) & 22l & 0 & 54 & -13l \\ 22l & 22l & 8l^2 & 0 & 13l & -3l^2 \\ \frac{1}{2}N & 0 & 0 & (156+N) & 0 & 22l \\ 0 & 54 & -13l & 0 & (156+N) & -22l \\ 0 & -13l & -3l^2 & 22l & -22l & 8l^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_2 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \\ \ddot{u}_3 \\ \ddot{v}_3 \\ \ddot{\theta}_3 \end{Bmatrix}$$

Assumptions

$\bar{u}_2 = \bar{u}_3$, $\bar{v}_2 = \bar{v}_3 = 0$, two modes $\begin{cases} \theta_3 = -\theta_2 & \rightarrow \text{spring not involved} \\ \theta_3 = \theta_2 \end{cases}$



spring not in action



10 – 19 (a) Cont

General eq. of motion

$$\frac{m\ell}{420}\begin{bmatrix}(156+N) & 22\ell & \frac{1}{2}N & 0\\ 22\ell & 8\ell^2 & 0 & -3\ell^2\\ \frac{1}{2}N & 0 & (156+N) & 22\ell\\ 0 & -3\ell^2 & 22\ell & 8\ell^2\end{bmatrix}\begin{Bmatrix}\ddot{\bar{u}}_2\\ \ddot{\theta}_2\\ \ddot{\bar{u}}_3\\ \ddot{\theta}_3\end{Bmatrix} + \frac{EI}{\ell^3}\begin{vmatrix}(12+R) & 6\ell & -R & 0\\ 6\ell & 8\ell^2 & 0 & 2\ell^2\\ -R & 0 & (12+R+\frac{k\ell^3}{EI}) & 6\ell\\ 0 & 2\ell^2 & 6\ell & 8\ell^2\end{vmatrix}\begin{Bmatrix}\bar{u}_2\\ \theta_2\\ \bar{u}_3\\ \theta_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}$$

Symmetric mode $\bar{u}_2 = \bar{u}_3 = 0$ $\therefore$ cross out col 1, col 3

$\theta_3 = -\theta_2$ row 1, row 3

$$\frac{m\ell}{420}\begin{bmatrix}8\ell^2 & -3\ell^2\\ -3\ell & 8\ell^2\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_2\\ \ddot{\theta}_3\end{Bmatrix} + \frac{EI}{\ell^3}\begin{bmatrix}8\ell^2 & 2\ell^2\\ 2\ell^2 & 8\ell^2\end{bmatrix}\begin{Bmatrix}\theta_2\\ \theta_3\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

$$\left[-\omega^2\frac{m\ell}{420}(11) + \frac{EI}{\ell^3}(6)\right]\ell^2\theta_2 = 0$$

$$\omega^2 = \frac{6\times 420}{11}\frac{EI}{m\ell^4} \qquad \underline{\underline{\omega = 15.14\sqrt{\frac{EI}{m\ell^4}}}}$$

Antisym. mode $\bar{u}_2 = \bar{u}_3 \neq 0$, $\theta_2 = \theta_3$

$$\left[-\frac{\omega^2 m\ell}{420}\begin{bmatrix}(156+\frac{3}{2}N) & 22\ell\\ 22\ell & 5\ell^2\end{bmatrix} + \frac{EI}{\ell^3}\begin{bmatrix}(12+\frac{k\ell^3}{EI}) & 6\ell\\ 6\ell & 10\ell^2\end{bmatrix}\right]\begin{Bmatrix}\bar{u}\\ \theta\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

with $\lambda = \left(\frac{\omega^2 m\ell^4}{420EI}\right)$ $N = 140$ Characteristic eq. =

$$\begin{vmatrix}(12+\frac{k\ell^3}{EI}-366\lambda) & (6-22\lambda)\ell\\ (6-22\lambda)\ell & (10-5\lambda)\ell^2\end{vmatrix} = 0$$

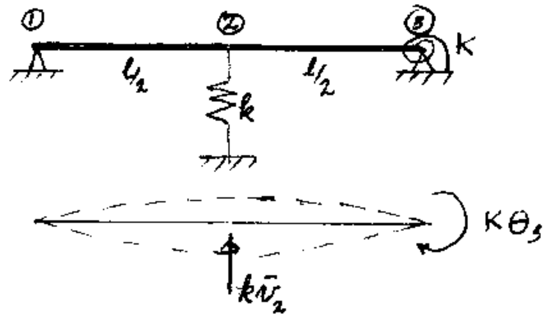
$$\lambda^2 - (2.6657 + .003715\frac{k\ell^3}{EI})\lambda + .062407 = 0$$

$\therefore$ must decide on numerical value of $\frac{k\ell^3}{EI}$ before solving

If $\frac{k\ell^3}{EI} = 0$, then $\lambda_1 = .0236$ $\omega_1 = 3.149\sqrt{EI/m\ell^4}$

$\lambda_2 = 2.642$ $\omega_2 = 33.311\sqrt{EI/m\ell^4}$ (Note numerical error on p279 text)

10 − 19 (b)



$\bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0$

Element Stiffness 1−2 and 2−3 Eq. (10.2−1) with $l = \frac{\ell}{2}$

$$\frac{8EI}{\ell^3}\begin{bmatrix} 12 & 3\ell & -12 & 3\ell \\ 3\ell & \ell^2 & -3\ell & \frac{1}{2}\ell^2 \\ -12 & -3\ell & \left(12+\frac{k\ell^3}{8EI}\right) & -3\ell \\ 3\ell & \frac{1}{2}\ell^2 & -3\ell & \left(\ell^2+\frac{K\ell^3}{8EI}\right) \end{bmatrix} \quad \text{displ.} = \begin{Bmatrix} \bar{v}_1 \\ \theta_1 \\ \bar{v}_2 \\ \theta_3 \end{Bmatrix} \text{ or } \begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$$

The term $\frac{k\ell^3}{8EI}$ is used only in one of the $\bar{v}_2$ elements

Element Mass 1−2 and 2−3 Eq. (10.2−10) with $\ell = \frac{\ell}{2}$

$$\frac{m\ell}{840}\begin{bmatrix} 156 & 11\ell & 54 & -6.5\ell \\ 11\ell & \ell^2 & 6.5\ell & -.75\ell^2 \\ 54 & 6.5\ell & 156 & -11\ell \\ -6.5\ell & -.75\ell^2 & -11\ell & \ell^2 \end{bmatrix}$$

Assembled

$$\frac{8EI}{\ell^3}\begin{bmatrix} 12 & 3\ell & -12 & 3\ell & & \\ 3\ell & \ell^2 & -3\ell & .5\ell^2 & & \\ -12 & -3\ell & \left(24+\frac{k\ell^3}{8EI}\right) & 0 & -12 & 3\ell \\ 3\ell & .5\ell^2 & 0 & 2\ell^2 & -3\ell & .5\ell^2 \\ & & -12 & -3\ell & 12 & -3\ell \\ & & 3\ell & .5\ell^2 & -3\ell & \left(\ell^2+\frac{K\ell^3}{8EI}\right) \end{bmatrix} \quad \text{displ} = \begin{Bmatrix} \bar{v}_1 = 0 \\ \theta_1 \\ \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 = 0 \\ \theta_3 \end{Bmatrix}$$

∴ delete cols 1 and 5 and rows 1 and 5

10-19(b) Cont.

Assembled.

$$
\frac{m\ell}{840}\begin{bmatrix}
156 & 11\ell & 54 & -6.5\ell & & \\
11\ell & \ell^2 & 6.5\ell & -.75\ell^2 & & \\
54 & 6.5\ell & 312 & 0 & 54 & -6.5\ell \\
-6.5\ell & -.75\ell^2 & 0 & 2\ell^2 & 6.5\ell & -.75\ell^2 \\
 & & 54 & 6.5\ell & 156 & -11\ell \\
 & & -6.5\ell & -.75\ell^2 & -11\ell & \ell^2
\end{bmatrix}
$$

Eq. of motion is a 4x4

$$
\left[-\frac{\omega^2 m\ell}{840}\begin{bmatrix}
\ell^2 & 6.5\ell & -.75\ell^2 & 0 \\
6.5\ell & 312 & 0 & -6.5\ell \\
-.75\ell^2 & 0 & 2\ell^2 & -.75\ell^2 \\
0 & -6.5\ell & -.75\ell^2 & \ell^2
\end{bmatrix}
+ \frac{8EI}{\ell^3}\begin{bmatrix}
\ell^2 & -3\ell & .5\ell^2 & 0 \\
-3\ell & \left(24+\frac{k\ell^3}{8EI}\right) & 0 & 3\ell \\
.5\ell^2 & 0 & 2\ell^2 & .5\ell^2 \\
0 & 3\ell & .5\ell^2 & \left(\ell^2+\frac{K\ell^3}{8EI}\right)
\end{bmatrix}\right]
\begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}
= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

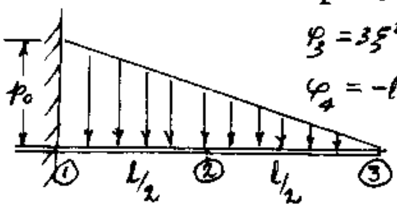
10-20 (a)

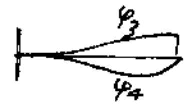
$\varphi_1 = 1 - 3\xi^2 + 2\xi^3$

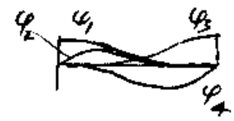
$\varphi_2 = \ell\xi - 2\ell\xi^2 + \ell\xi^3$

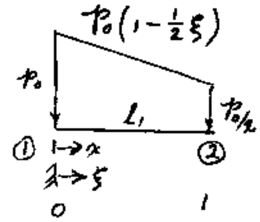
$\varphi_3 = 3\xi^2 - 2\xi^3$

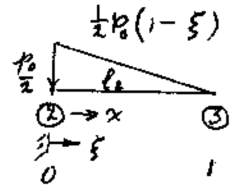
$\varphi_4 = -\ell\xi^2 + \ell\xi^3$











Element ①-② See Eq.(10.7-4)

since $\bar{v}_1 = 0$ only φ_3 & φ_4 are needed.

$$\bar{F}_2 = \int_0^1 p_0(1-\tfrac{1}{2}\xi)(3\xi^2-2\xi^3)\ell_1 d\xi$$

$$M_2 = \int_0^1 p_0(1-\tfrac{1}{2}\xi)(-\ell\xi^2+\ell\xi^3)\ell_1 d\xi$$

Integrated results.

$\bar{F}_2 = .3250\, p_0\ell_1$

$M_2 = -.0583\, p_0\ell_1^2$

Element ②-③ See Eq.(10.7-4)

all 4 φ_s are necessary

$$\bar{F}_3 = \int_0^1 \frac{p_0}{2}(1-\xi)(3\xi^2-2\xi^3)\ell_2 d\xi$$

$$M_3 = \int_0^1 \frac{p_0}{2}(1-\xi)(-\ell_2\xi^2+\ell\xi^3)\ell_2 d\xi$$

$$\bar{F}_2 = -\int_0^1 \frac{p_0}{2}(1-\xi)(1-3\xi^2+2\xi^3)\ell_2 d\xi$$

$$M_2 = -\int_0^1 \frac{p_0}{2}(1-\xi)(\ell_2\xi-2\ell\xi^2+\ell\xi^3)\ell_2 d\xi$$

Integrated results

$\bar{F}_2 = .25\, p_0\ell_2$

$M_2 = -.850\, p_0\ell_2^2$

$\bar{F}_3 = .0750\, p_0\ell_2$

$M_3 = .0583\, p_0\ell_2^2$

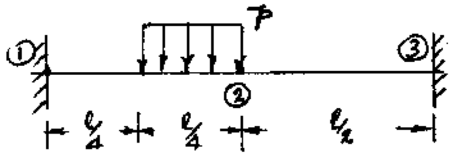
Assembly of both sections

$$\begin{Bmatrix} \bar{F}_2 \\ M_2 \\ \bar{F}_3 \\ M_3 \end{Bmatrix} = -p_0 \begin{Bmatrix} .325\,\ell_1 + .250\,\ell_2 \\ -.0583\,\ell_1^2 - .850\,\ell_2^2 \\ .0750\,\ell_2 \\ .0583\,\ell_2^2 \end{Bmatrix}$$

with $\ell_1 = \ell_2 = \ell/2$

$$\begin{Bmatrix} \bar{F}_2 \\ M_2 \\ \bar{F}_3 \\ M_3 \end{Bmatrix} = -p_0\ell \begin{Bmatrix} .2875 \\ -.2271\,\ell \\ .0375 \\ .0146\,\ell \end{Bmatrix}$$

10-20(b)



$$\bar{F}_2 = -p\int_{1/2}^{1} \varphi_3 \, l \, d\xi = -p\int_{1/2}^{1} (3\xi^2 - 2\xi^3) l_1 \, d\xi = -p l_1 \left[3\left(\frac{\xi^3}{3}\right) - 2\left(\frac{\xi^4}{4}\right)\right]_{1/2}^{1}$$

$$= -p\left[.4063\right] l_1 = -pl\left[.2031\right] \qquad l_1 = \frac{l}{2}$$

$$M_2 = -p\int_{1/2}^{1} (-l_1\xi^2 + l_1\xi^3) l_1 \, d\xi = .0286 \, pl^2$$

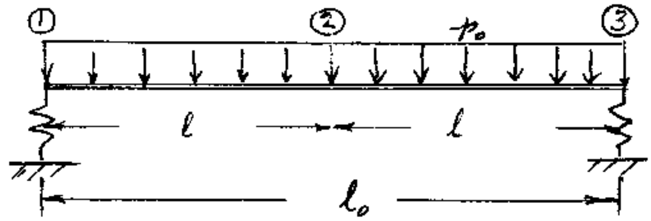
with $\bar{v}_1 = \bar{v}_3 = \theta_1 = \theta_3 = 0$ (see Ex. 10.6-1)

$$\begin{Bmatrix} -P_2 \\ M_2 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 192 & \\ & 16 l^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix} = -pl\begin{Bmatrix} .2031 \\ .0286 l \end{Bmatrix}$$

$$\therefore, \bar{v}_2 = -\frac{.2031}{192}\frac{pl^4}{EI}$$

$$\theta_2 = -\frac{.0286}{16}\frac{pl^3}{EI}$$

10 - 21



Element $\begin{pmatrix}1-2\\2-3\end{pmatrix}$ Eq.(10.2-1) for $\bar{k}$ and Eq.(10.2-10) for $\bar{m}$

$$\bar{k} = \frac{EI}{l^3}\left[\begin{array}{cc|cc} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ \hline -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{array}\right] \qquad \bar{m} = \frac{ml}{420}\left[\begin{array}{cc|cc} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ \hline 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{array}\right]$$

Assembled

$$\bar{k} = \frac{EI}{l^3}\left[\begin{array}{cc|cc|cc} 12 & 6l & -12 & 6l & & \\ 6l & 4l^2 & -6l & 2l^2 & & \\ \hline -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ \hline & & -12 & -6l & 12 & -6l \\ & & 6l & 2l^2 & -6l & 4l^2 \end{array}\right]\begin{Bmatrix} \bar{v}_1 \\ \theta_1 \\ \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$$

must replace l by $\frac{l_o}{2}$

$$\bar{m} = \frac{ml}{420}\left[\begin{array}{cc|cc|cc} 156 & 22l & 54 & -13l & & \\ 22l & 4l^2 & 13l & -3l^2 & & \\ \hline 54 & 13l & 312 & 0 & 54 & -13l \\ -13l & -3l^2 & 0 & 8l^2 & 13l & -3l^2 \\ \hline & & 54 & 13l & 156 & -22l \\ & & -13l & -3l^2 & -22l & 4l^2 \end{array}\right]$$

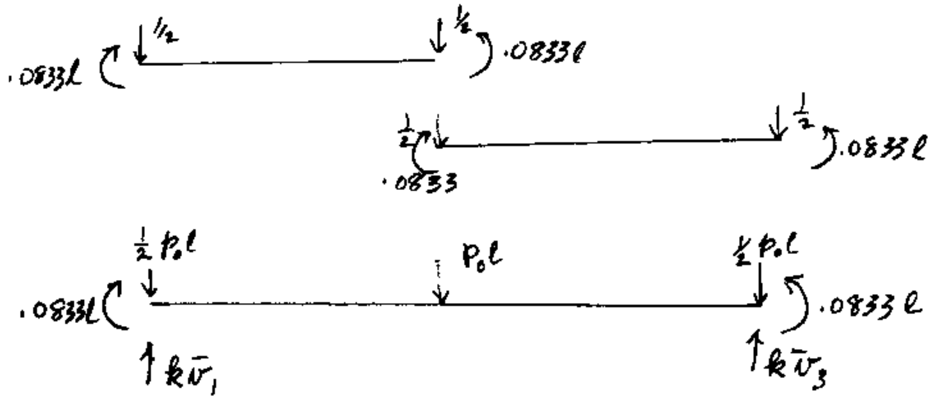
For loads, must determine generalized forces & moments at stations

$$\bar{F}_1 = -p_0 f(t) \int_0^1 \varphi_1 \, l \, d\xi = -p_0 f(t) \left[1 - 1 + \tfrac{1}{2} \right] l = -\tfrac{1}{2} \, p_0 \, l \, f(t)$$

$$\bar{M}_1 = -p_0 f(t) \, l \int_0^1 \varphi_2 \, l \, d\xi = -p_0 \, l^2 f(t) \left[\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4} \right] = -.0833 \, p_0 \, l^2 f(t)$$

$$\bar{F}_2 = -p_0 f(t) \int_0^1 \varphi_3 \, l \, d\xi = -p_0 \, l \, f(t) \left[1 - \tfrac{2}{4} \right] = -\tfrac{1}{2} \, p_0 \, l \, f(t)$$

$$\bar{M}_2 = -p_0 f(t) \, l \int_0^1 \varphi_4 \, l \, d\xi = -p_0 \, l^2 f(t) \left[-\tfrac{1}{3} + \tfrac{1}{4} \right] = .0833 \, p_0 \, l^2 f(t)$$



$$\begin{Bmatrix} \bar{F}_1 - k\bar{v}_1 \\ \bar{M}_1 \\ \bar{F}_2 \\ \bar{M}_2 \\ \bar{F}_3 - k\bar{v}_3 \\ M_3 \end{Bmatrix} = \begin{Bmatrix} -.50 - k\bar{v}_1 \\ -.0833\,l \\ -1.0 \\ 0 \\ -.50 - k\bar{v}_3 \\ .0833\,l \end{Bmatrix} p_0 \, l \, f(t)$$

$-k\bar{v}_1$ and $-k\bar{v}_3$ must now be shifted to stiffness matrix as $+k\bar{v}_1$ and $+k\bar{v}_3$

10-21 Cont

General equation of motion 6×6

$$\frac{m\ell}{420}\begin{bmatrix} 156 & 22\ell & 54 & -13\ell & 0 & 0 \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 & 0 & 0 \\ 54 & 13\ell & 312 & 0 & 54 & -13\ell \\ -13\ell & -3\ell & 0 & 8\ell^2 & 13\ell & -3\ell^2 \\ 0 & 0 & 54 & 13\ell & 156 & -22\ell \\ 0 & 0 & -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix}\begin{Bmatrix} \ddot{\bar{v}}_1 \\ \ddot{\theta}_1 \\ \ddot{\bar{v}}_2 \\ \ddot{\theta}_2 \\ \ddot{\bar{v}}_3 \\ \ddot{\theta}_3 \end{Bmatrix} + \frac{EI}{\ell^3}\begin{bmatrix} (12+\frac{k\ell^3}{EI}) & 6\ell & -12 & 6\ell & 0 & 0 \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 & 0 & 0 \\ -12 & -6\ell & 24 & 0 & -12 & 6\ell \\ 6\ell & 2\ell^2 & 0 & 8\ell^2 & -6\ell & 2\ell^2 \\ 0 & 0 & -12 & -6\ell & (12+\frac{k\ell^3}{EI}) & -6\ell \\ 0 & 0 & 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_1 \\ \theta_1 \\ \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -.50 \\ -.0833\ell \\ -1.0 \\ 0 \\ -.50 \\ .0833\ell \end{Bmatrix} p_0\ell f(t)$$

10-22

Symmetric mode (free-free vibration)

$$\bar{v}_1 = \bar{v}_3 , \qquad \theta_2 = 0 \qquad \theta_1 = -\theta_3$$

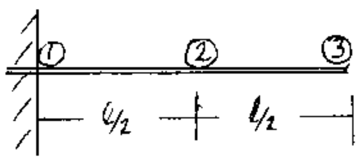
Due to symmetry only $\bar{v}_1$ θ_1 and $\bar{v}_2$ are necessary

From above equation we have

$$\frac{m\ell}{420}\begin{bmatrix} 156 & 22\ell & 54 \\ 22\ell & 4\ell^2 & 13\ell \\ 54 & 13\ell & 312 \end{bmatrix}\begin{Bmatrix} \ddot{\bar{v}}_1 \\ \ddot{\theta}_1 \\ \ddot{\bar{v}}_2 \end{Bmatrix} + \frac{EI}{\ell^3}\begin{bmatrix} (12+\frac{k\ell^3}{EI}) & 6\ell & -12 \\ 6\ell & 4\ell^2 & -6\ell \\ -12 & -6\ell & 24 \end{bmatrix}\begin{Bmatrix} \bar{v}_1 \\ \theta_1 \\ \bar{v}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

10-23 Problem deals with uniform cantilever equation in Ex. 10.5-1 whose equation is

$$\left[-\lambda\begin{bmatrix}312 & 0 & 54 & -6.5\ell\\ 0 & 2\ell^2 & 6.5\ell & -.75\ell^2\\ 54 & 6.5\ell & 156 & -11\ell\\ -6.5\ell & -.75\ell^2 & -11\ell & \ell^2\end{bmatrix} + \begin{bmatrix}24 & 0 & -12 & 3\ell\\ 0 & 2\ell^2 & -3\ell & .5\ell^2\\ -12 & -3\ell & 12 & -3\ell\\ 3\ell & .5\ell & -3\ell & \ell^2\end{bmatrix}\right]\begin{Bmatrix}\bar{v}_2\\ \theta_2\\ \bar{v}_3\\ \theta_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}$$



Numerical value of ℓ is needed

Let $\ell = 20''$

$$\lambda = \left(\frac{\omega^2 m\ell}{840}\right)\left(\frac{\ell^3}{8EI}\right) = \frac{\omega^2 m\ell^3}{6720\,EI}$$

Then

$$M = \begin{bmatrix}320 & 0 & 54 & -130\\ 0 & 800 & 130 & -300\\ 54 & 130 & 156 & -220\\ -130 & -300 & -220 & 400\end{bmatrix} \qquad K = \begin{bmatrix}24 & 0 & -12 & 60\\ 0 & 800 & -60 & 200\\ -12 & -60 & 12 & -60\\ 60 & 200 & -60 & 400\end{bmatrix}$$

Computation was carried by two different computer programs. For the program CAL (which is somewhat similar to NROOT in that only M and K needed to be inputted) the eigenvalues and eigenvectors were printed out. The mode shapes are for the **original** displacement X (see computational notes)

10-23 Cont.

Computer program CAL.

```
LOAD M R=4 C=4
ARRAY NAME = M        NUMBER OF ROWS =   4 NUMBER OF COLUMNS =   4
P M

COL#  =            1            2            3            4
ROW   1    320.00000       .00000     54.00000   -130.00000
ROW   2       .00000    800.00000    130.00000   -300.00000
ROW   3     54.00000    130.00000    156.00000   -220.00000
ROW   4   -130.00000   -300.00000   -220.00000    400.00000
LOAD K R=4 C=4
ARRAY NAME = K        NUMBER OF ROWS =   4 NUMBER OF COLUMNS =   4
P K

COL#  =            1            2            3            4
ROW   1     24.00000       .00000    -12.00000     60.00000
ROW   2       .00000    800.00000    -60.00000    200.00000
ROW   3    -12.00000    -60.00000     12.00000    -60.00000
ROW   4     60.00000    200.00000    -60.00000    400.00000
JACOBI K V M E
P V
```

Eigenvectors

$\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix} =$

```
COL#  =            1            2            3            4
ROW   1       .02341      -.04958      -.00769       .03200
ROW   2       .00401       .00154       .02962       .03379
ROW   3       .06891       .06961      -.07766       .12991
ROW   4       .00474       .01663      -.03745       .12536
P E

COL#  =            1
ROW   1       .00183
ROW   2       .07203
ROW   3       .84016
ROW   4      7.02150
RETURN
```

Eigen values $= \lambda$

$$\omega^2 = 6720.\frac{EI}{M\ell^3}\,\lambda$$

$$\omega = \left.\begin{matrix} 3.507 \\ 22.001 \\ 75.139 \\ 217.220 \end{matrix}\right\} \times \sqrt{\frac{EI}{M\ell^3}}$$

Computational Notes

The matrix equation for the normal mode vibration is generally written as

$$\left[-M\lambda + K\right] X = 0 \tag{1}$$

where M and K are both symmetric matrices and λ is related to the natural frequencies by $\lambda = \omega^2$. Premultiplying the above equation by M^{-1}, we have another form of the equation

$$\left[-\lambda I + A\right] X = 0 \tag{2}$$

where

$$A = M^{-1}K \tag{3}$$

is called the dynamic matrix. In general $M^{-1}K$ is not symmetric.

It is desirable to introduce a coordinate transformation to rewrite the above equation in the standard form inoorporated into most computer programs. This standard form is

$$\left[-\lambda I + \tilde{A}\right] U = 0 \tag{4}$$

where I is a unit matrix and $\tilde{A}$ is a symmetric matrix.

For this transformation, let

$$X = Q^{-1}U \tag{5}$$

and substitute into Eq. (1).

$$\left[-\lambda M Q^{-1} + K Q^{-1}\right] U = 0$$

Premultiply by the transpose of Q which we will designate by Q^{-T}

$$\left[-\lambda Q^{-T} M Q^{-1} + Q^{-T} K Q^{-1}\right] U = 0$$

If we now decompose M as

$$M = Q^{T} Q \tag{6}$$

the first term of the last equation becomes a unit matrix.

i.e. $$Q^{-T} Q^{T} \cdot Q Q^{-1} = I$$

We note here that

$$M = Q^{T} Q = M^{\frac{1}{2}} M^{\frac{1}{2}}$$

and since M is symmetric, we obtain

$$Q = Q^{T} = M^{\frac{1}{2}} \tag{7}$$

and

$$\tilde{A} = Q^{-T} K Q^{-1} = M^{-\frac{1}{2}} K M^{-\frac{1}{2}} \tag{8}$$

which is symmetric. Thus the standard form of the computer equation for the eigenvalues and eigenvectors is achieved.

The eigenvalues, λ, of Eq.(4) will be identical to those of Eqs.(1) or (2). However, the eigenvectors of the computer equation, Eq.(4), will now be U instead of X. To obtain the eigenvactors of the original equation, the transformation equation, Eq.(5), must be used.

For lumped mass systems, where the coordinates are chosen at each mass, the mass matrix is diagonal and $Q = M^{1/2}$ is easily determined. However when M is a full matrix, as in most finite element formulations, the determination of $M^{1/2}$ is not a simple task. For such cases, $M^{1/2}$ is determined from the equation

$$M^{\frac{1}{2}} = Y \Lambda^{\frac{1}{2}} Y^{T} \tag{9}$$

where Λ is the diagonal matrix of the eigenvalues of M and Y is the orthonormal matrix of the eigenvectors of M,

i.e. $Y^{T} Y = I$. (M must here be a positive definite real symmetric matrix, which the mass matrix satisfies. See Meirovitch,L. Analytical Methods in Vibration, p 76, The Macmillan Co., N.Y., 1967)

Ref: Meirovitch,L. Computational Methods in Structural Dynamics, p61, Sijthoff & Noordhoff, The Netherlands, 1980.

10-23 Cont. (2^{nd} Procedure) The steps in Computational Notes were carried out on MacPlus with BASIC programs in Math Package and Scientific Analysis Program SAP

Computation for $M^{1/2} = Y\Lambda^{1/2}Y^T$

INPUT MATRIX M IN SYMMETRIC FORM

$$\begin{bmatrix} 320 & & & \\ 0 & 800 & & \\ 54 & 130 & 156 & \\ -130 & -300 & -220 & 400 \end{bmatrix} = M$$

EIGENVALUES OF M (by SYMEIG)

1026.2458250538
427.64188324099
201.17548671333
20.936804991822

$$\Lambda = \begin{bmatrix} 1026.24 & & & \\ & 427.64 & & \\ & & 201.17 & \\ & & & 20.93 \end{bmatrix}$$

CORRESPONDING EIGENVECTORS (COLS) OF M

$$\begin{bmatrix} -.11251123964196 & .69783094122845 & -.69987273057571 & .10271981023802 \\ -.81659856817195 & -.46966360076447 & -.32448266951864 & .08540420223171 \\ -.25654886082296 & .28577999506147 & .44493141685562 & .80904172372534 \\ .50466888295643 & -.45910551510988 & -.45489030817742 & .57236898247361 \end{bmatrix} = Y$$

$$(-.112511 \quad -.816598 \quad -.256548 \quad .5046688)\begin{Bmatrix} -.112511 \\ -.816598 \\ -.256548 \\ .5046688 \end{Bmatrix}$$

$$= .01266 + .666832 + .065817 + .25491 = 1.0000$$

The Ys are orthonormal as shown above checking

$$\Lambda^{1/2} = \begin{bmatrix} 32.051 & 0 & 0 & 0 \\ 0 & 20.679 & 0 & 0 \\ 0 & 0 & 14.184 & 0 \\ 0 & 0 & 0 & 4.576 \end{bmatrix}$$

Computation for $Y\Lambda^{1/2}Y^{P} = M^{1/2}$

THE PRODUCT MATRIX [C] = [A] X [B] IS : -

-3.605737	14.42981	-9.925963	.4699552
-26.17285	-9.7121	-4.602424	.3907904
-7.900571	5.90799	6.310462	3.701984
16.17293	-9.49373	-6.450883	2.618845

$\} \; Y\Lambda^{1/2}$

THE PRODUCT MATRIX [C] = [A] X [B] IS : -

17.46832	-.5734422	1.011564	-3.660839
-.5723705	27.45703	2.207047	-6.432595
.9751865	1.94517	9.515408	-7.450424
-3.661047	-6.428366	-7.611399	16.95033

$\} \; Y\Lambda^{1/2}Y^{T} = M^{1/2} = Q$

The inverse of the input matrix is : -

0.0608	0.0050	0.0068	0.0180
0.0050	0.0405	0.0050	0.0186
0.0070	0.0063	0.1635	0.0758
0.0181	0.0193	0.0768	0.1040

$= Q^{-1}$

Multiplication for $Q^{-T} K Q^{-1}$
by SAP (Matrix)

THE PRODUCT MATRIX [C] = [A] X [B] IS : -

2.4612	7.2	-2.0316	11.468
1.2024	35.882	-3.5724	15.742
2.8092	9.550001	-3.0276	22.318
5.7624	31.132	-6.6624	41.852

$Q^{-T}K$

THE PRODUCT MATRIX [C] = [A] X [B] IS : -

.3789906	.5124393	.601312	1.216898
.5124393	1.740547	.8124846	2.055429
.601312	.8124846	1.285862	2.319776
1.216898	2.055429	2.319776	4.530376

$Q^{-T}KQ^{-1} = \tilde{A}$

INPUT MATRIX $\tilde{A}$ IN SYMMETRIC FORM

Eigen values & vectors by Math Package (SYMEIG)

```
.37899
.51243   1.7405
.601312   .81248   1.2858
1.2168   2.0554   2.3197   4.5303
```

EIGENVALUES OF $\tilde{A}$

```
7.0278455556106
.83378223012746
.072052235090057
.0019099791723867
```

$\lambda = \frac{\omega^2 M\ell^3}{6720 EI}$

$\omega^2 = \frac{6720.\, EI}{M\ell^3}\lambda$	ω computed		ω exact
47226.	217.315		
5603.	74.853	$\times\sqrt{\frac{EI}{M\ell^3}}$	$\omega_3 = 61.7$
484.	22.000		$\omega_2 = 22.03$
12.83	3.5825		$\omega_1 = 3.516$

CORRESPONDING EIGENVECTORS (COLS) OF $\tilde{A}$ $= U$

```
.21289484519686   .092870186818959 -.856536608263   .46086435313892
.39308961378661 -.88527738654973   .11908614674759   .21813526548855
.40095055035245   .4060306420499   .49588307863158   .65458211571615
.79962178346634   .20687724616463 -.079142281272342 -.55816064689359
```

must change to $X = Q^{-1} U$

use SAP(MAT)

Original eigenvectors X

$$X = \bar{Q}^{-1}\, U$$

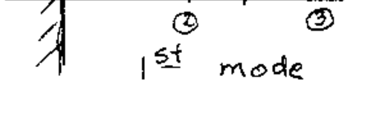
```
THE PRODUCT MATRIX [C] = [A] X [B] IS : -
3.202216E-02   7.699439E-03 -4.953256E-02   2.351194E-02
3.385756E-02 -2.951012E-02   1.54474E-03   4.028889E-03
.1301223   7.712928E-02   6.982172E-02   .0693064
.1253841   3.728332E-02   1.664509E-02   4.773013E-03
```

$= \begin{matrix} \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 \\ \theta_3 \end{matrix}$

4^{th} mode 3^{rd} mode 2^{nd} mode 1^{st} mode

Checking mode shapes (for 1^{st} mode)

$$\left(\frac{\bar{v}_2}{\bar{v}_3}\right)_1 = \frac{.0235}{.0693} = .339 \qquad \left(\frac{\theta_2}{\theta_3}\right) = \frac{.004029}{.004773} = .844$$



1^{st} mode

11-1

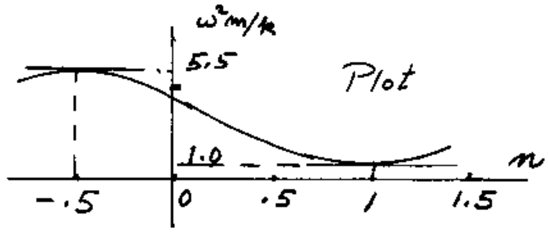
$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 \qquad U = \frac{1}{2} k x_1^2 + \frac{1}{2}(3k)(x_2 - x_1)^2 + \frac{1}{2}(2k)x_2^2$

Let $\dot{x} = \omega x$ & equate

$$\omega^2 m x_1^2 + 2\omega^2 m x_2^2 = k x_1^2 + 3k(x_2 - x_1)^2 + 2k x_2^2$$

Let $x_2/x_1 = n$, then

$$\frac{\omega^2 m}{k} = \frac{1 + 3(n-1)^2 + 2n^2}{1 + 2n^2}$$

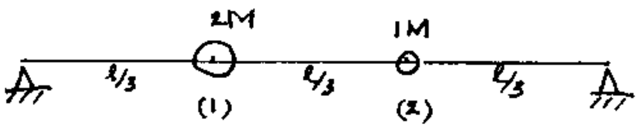


Can be checked by

$$\frac{\partial}{\partial n}\left(\frac{\omega^2 m}{k}\right) = 0 \longrightarrow \text{gives} \quad n = -.5 \quad \text{and} \quad n = 1.0$$

Roots are $\frac{\omega^2 m}{k} = 1.0$ and $\frac{\omega^2 m}{k} = 5.50$

11-2



$y(x) = \frac{Wbx}{6lEI}(l^2 - x^2 - b^2)$

$0 < x < (l - b)$

Due to 2M

$$y_1 = \frac{(2Mg)\frac{2}{3}l\,\frac{1}{3}l}{6lEI}\left[l^2 - \left(\frac{l}{3}\right)^2 - \left(\frac{2l}{3}\right)^2\right] = \frac{16}{486}\left(\frac{Mgl^3}{EI}\right)$$

$$y_2 = \frac{(2Mg)\frac{1}{3}l\,\frac{1}{3}l}{6l\,EI}\left[l^2 - \left(\frac{l}{3}\right)^2 - \left(\frac{l}{3}\right)^2\right] = \frac{14}{486}\left(\frac{Mgl^3}{EI}\right)$$

Due to 1M

$$y_1 = \frac{Mg\,\frac{1}{3}l\,\frac{1}{3}l}{6l\,EI}\left[l^2 - \left(\frac{l}{3}\right)^2 - \left(\frac{l}{3}\right)^2\right] = \frac{7}{486}\left(\frac{Mgl^3}{EI}\right)$$

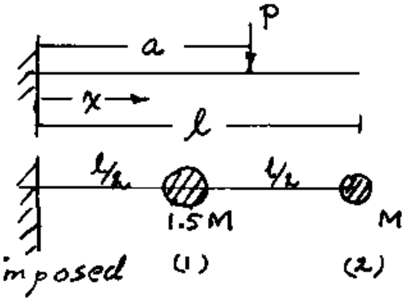
$$y_2 = \frac{Mg\,\frac{1}{3}l\,\frac{2}{3}l}{6l\,EI}\left[l^2 - \left(\frac{2}{3}l\right)^2 - \left(\frac{l}{3}\right)^2\right] = \frac{8}{486}\left(\frac{Mgl^3}{EI}\right)$$

Add $\quad y_1 = \frac{23}{486}\left(\frac{Mgl^3}{EI}\right) \qquad y_2 = \frac{22}{486}\left(\frac{Mgl^3}{EI}\right)$

$$\omega_1^2 = \frac{g\left[(2M)\frac{23}{486} + (M)\frac{22}{486}\right]}{\left[2M\left(\frac{23}{486}\right)^2 + M\left(\frac{22}{486}\right)^2\right]}\,\frac{1}{\frac{Mgl^3}{EI}} = 21.43\,\frac{EI}{Ml^3}$$

$$\omega_1 = 4.63\sqrt{\frac{EI}{Ml^3}} \quad \text{rad/s.}$$

11-3



$$y(x) = \frac{Px^2}{6EI}(3a - x) \qquad 0 \le x \le a$$

$$= \frac{Pa^2}{6EI}(3x - a) \qquad a \le x \le \ell$$

Due to 1.5M + M with results superimposed

$$y_1 = \frac{1.5Mg\left(\frac{\ell}{2}\right)^2}{3EI} + \frac{Mg\left(\frac{\ell}{2}\right)^2}{6EI}\left(3\ell - \frac{\ell}{2}\right) = \frac{16}{96}\,\frac{Mg\ell^3}{EI}$$

$$y_2 = \frac{1.5Mg\left(\frac{\ell}{2}\right)^2}{6EI}\left(3\ell - \frac{\ell}{2}\right) + \frac{Mg\ell^3}{3EI} = \frac{47}{96}\,\frac{Mg\ell^3}{EI}$$

$$\omega^2 = \frac{\cancel{g}\left[1.5\left(\frac{16}{96}\right) + \left(\frac{47}{96}\right)\right]}{\left[1.5\left(\frac{16}{96}\right)^2 + \left(\frac{47}{96}\right)^2\right]}\,\frac{EI}{M\cancel{g}\ell^3} = 2.628\,\frac{EI}{M\ell^3}$$

$$\therefore \omega_1 = 1.621\sqrt{\frac{EI}{M\ell^3}}$$

11-4

$$a_{11} = \frac{1}{24}\frac{\ell^3}{EI} \qquad a_{21} = a_{12} = \frac{10}{96}\,\frac{\ell^3}{EI}$$

$$a_{22} = \frac{\ell^3}{3EI} \qquad \therefore [a] = \frac{\ell^3}{96EI}\begin{bmatrix} 4 & 10 \\ 10 & 32 \end{bmatrix}$$

$$[k] = [a]^{-1} = \frac{96EI}{\ell^3}\,\frac{1}{28}\begin{bmatrix} 32 & -10 \\ -10 & 4 \end{bmatrix} = 13.7143\,\frac{EI}{\ell^3}\begin{bmatrix} 8 & -2.5 \\ -2.5 & 1 \end{bmatrix}$$

Set up eq. of motion using flexibility

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{\omega^2 M\ell^3}{96EI}\begin{bmatrix} 4 & 10 \\ 10 & 32 \end{bmatrix}\begin{bmatrix} 1.5 & 0 \\ 0 & 1.0 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$= \frac{\omega^2 M\ell^3}{96EI}\begin{bmatrix} 6 & 10 \\ 15 & 32 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \qquad \text{Let } \lambda = \frac{\omega^2 M\ell^3}{96EI}$$

$$\begin{bmatrix} (1 - 6\lambda) & -10\lambda \\ -15\lambda & (1 - 32\lambda) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

11-4 Cont.

$$42\lambda^2 - 38\lambda + 1 = 0$$

$$\lambda^2 - 0.9048\lambda + 0.0238 = 0$$

$$\lambda = 0.4524 \pm \sqrt{.2048 - .0238} = 0.4524 \pm 0.4253$$

$$\lambda = \frac{\omega^2 M\ell^3}{96EI} = \begin{cases} 0.0271 \\ 0.8771 \end{cases} \qquad \omega^2 = \begin{cases} 2.6016 \ \frac{EI}{M\ell^3} \\ 84.2576 \ \frac{EI}{M\ell^3} \end{cases}$$

$$\omega = \begin{cases} 1.6129 \sqrt{\frac{EI}{M\ell^3}} \\ 9.1792 \sqrt{\frac{EI}{M\ell^3}} \end{cases} \qquad \left(\frac{X_1}{X_2}\right)_1 = \frac{10\lambda}{1-6\lambda} = 0.3237$$

To verify $\omega^2 = \dfrac{X'KX}{X'MX}$

$$\omega^2 = \frac{(.3237 \quad 1.0)\begin{bmatrix} 8 & -2.5 \\ -2.5 & 1 \end{bmatrix}\begin{Bmatrix} .3237 \\ 1.0 \end{Bmatrix} 13.7143\frac{EI}{\ell^3}}{(.3237 \quad 1.0)\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} .3237 \\ 1.0 \end{Bmatrix}} = 2.605\frac{EI}{M\ell^3}$$

$$\therefore \omega_1 = 1.614\sqrt{\frac{EI}{M\ell^3}}$$

11-5

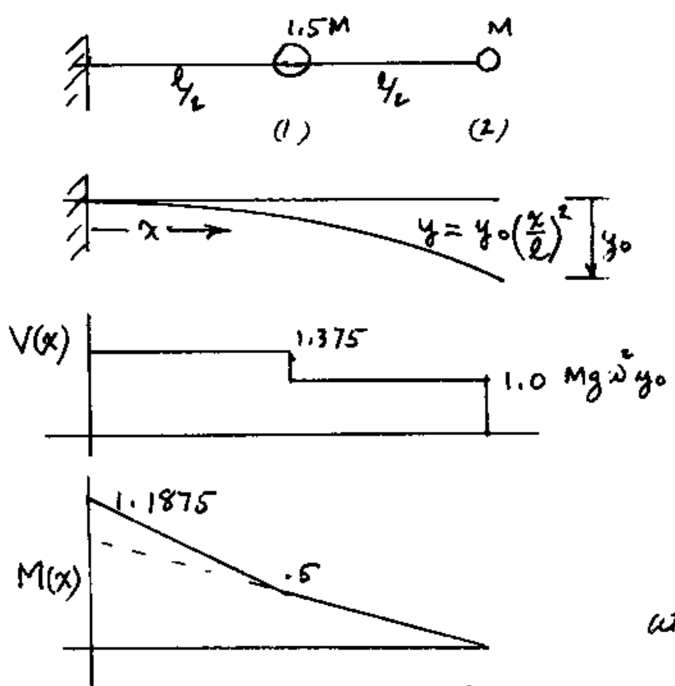
$$\omega^2 = \frac{X'MX}{X'M'aMX} =$$

$$= \frac{(.3237 \quad 1.0)\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} .3237 \\ 1.0 \end{Bmatrix}}{(.3237 \quad 1.0)\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 10 \\ 10 & 32 \end{bmatrix}\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} .3237 \\ 1.0 \end{Bmatrix}} \quad \frac{96EI}{M\ell^3}$$

$$= \frac{1.1572}{42.654} \cdot \frac{96EI}{M\ell^3} = 2.6045\frac{EI}{M\ell^3}$$

$$\omega_1 = 1.6138\sqrt{\frac{EI}{M\ell^3}}$$

11-6



$\dot{y} = \omega y = \omega y_0 \left(\frac{x}{l}\right)^2$

$\ddot{y} = -\omega^2 y = -\omega^2 y_0 \left(\frac{x}{l}\right)^2$

$-m\ddot{y} = \omega^2 y_0 \left(\frac{x}{l}\right)^2 m$

For lumped mass, dynamic loads are

$\omega^2 M y_0 g$ at (2)

and $\omega^2 1.5 M y_0 g \left[\frac{1}{4}\right] + \omega^2 M y_0 g [1]$

at (1) $= \omega^2 M g y_0 \, 1.375$

$$\therefore V(x) = Mg\omega^2 y_0 \qquad \frac{l}{2} \le x \le l$$

$$= 1.375\, Mg\omega^2 y_0 \qquad 0 \le x \le \frac{l}{2}$$

$$M(x) = \int_x^l V(\xi)\,d\xi = Mg\omega^2 y_0\, \xi \Big|_x^l = Mg\omega^2 y_0 (l - x) \qquad \frac{l}{2} \le x \le l$$

$$= Mg\omega^2 y_0 \left[\frac{l}{2} + 1.375\left(\frac{l}{2} - x\right)\right] \qquad 0 \le x \le \frac{l}{2}$$

$$U = \frac{1}{2}\int \frac{M^2}{EI}dx = \frac{(Mg\omega^2 y_0)^2}{2EI}\left[\int_{l/2}^{l} (l-x)^2 dx + \int_0^{l/2} \left\{\frac{l}{2} + 1.375\left(\frac{l}{2} - x\right)\right\}^2 dx\right]$$

$$= \frac{1}{2}\frac{(Mg\omega^2 y_0)^2 l^3}{EI}\left[\frac{1}{24} + \frac{9.0156}{24}\right] = \frac{1}{2}\frac{M^2 g^2 \omega^4 y_0^2}{EI}(0.4173)$$

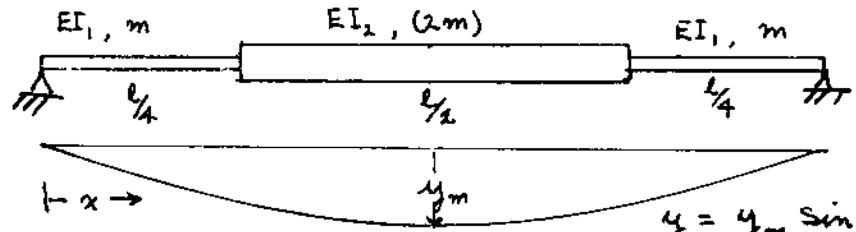
$$T = \frac{1}{2} M\omega^2 y_0^2 + \frac{1}{2}(1.5M)\omega^2 y_0^2 \left(\frac{1}{2}\right)^4 = \frac{1}{2} M\omega^2 y_0^2 (1.0938)$$

11-6 Cont $U = T$ gives

$$\omega^2 = \frac{1.0938}{0.4173}\frac{EI}{M\ell^3} = 2.619\frac{EI}{M\ell^3}$$

$$\omega_1 = 1.618\sqrt{\frac{EI}{M\ell^3}} \qquad \text{Exact val} = 1.6129\sqrt{\frac{EI}{M\ell^3}}$$

11-7



$$y = y_m \sin\frac{\pi x}{\ell}$$

$$\frac{d^2y}{dx^2} = -y_m\left(\frac{\pi}{\ell}\right)^2 \sin\frac{\pi x}{\ell}$$

$$U = \frac{1}{2}\int EI\left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{1}{2}y_m^2\left(\frac{\pi}{\ell}\right)^4\left[EI_1\, 2\int_0^{\ell/4}\sin^2\frac{\pi x}{\ell}\,dx + EI_2\, 2\int_{\ell/4}^{\ell/2}\sin^2\frac{\pi x}{\ell}\,dx\right]$$

$$= y_m^2\left(\frac{\pi}{\ell}\right)^4\left[EI_1\,\frac{1}{2}\left(x - \frac{\ell}{2\pi}\sin\frac{2\pi x}{\ell}\right)_0^{\ell/4} + EI_2\,\frac{1}{2}\left(x - \frac{\ell}{2\pi}\sin\frac{2\pi x}{\ell}\right)_{\ell/4}^{\ell/2}\right]$$

$$= y_m^2\left(\frac{\pi}{\ell}\right)^4 \ell\left[.0454\,EI_1 + .2046\,EI_2\right]$$

$$T = \frac{1}{2}\int m\dot{y}^2 dx = \frac{1}{2}\omega^2 y_m^2\left[m\, 2\int_0^{\ell/4}\sin^2\frac{\pi x}{\ell}\,dx + (2m)\, m\int_{\ell/4}\sin^2\frac{\pi x}{\ell}\,dx\right]$$

$$= \omega^2 y_m^2\, m\ell\left[.0454 + .4092\right] = 0.4546\,\omega^2 y_m^2\, m\ell$$

Equating

$$\omega_1^2 = \frac{\pi^4}{m\ell^4}\left[0.100\,EI_1 + 0.450\,EI_2\right]$$

$$\omega_1 = \pi^2\sqrt{\frac{0.100\,EI_1 + 0.450\,EI_2}{m\ell^4}}$$

11-7 Cont:

(a) if $EI_2 = EI_1$ & $m_1 = m$, $m_2 = 2m$

$$\omega_1 = .7416\,\pi^2 \sqrt{\frac{EI_1}{ml^4}} = 7.32 \sqrt{\frac{EI_1}{ml^4}}$$

(b) if $EI_2 = 4EI_1$

$$\omega_1 = 1.3785\,\pi^2 \sqrt{\frac{EI_1}{ml^4}}$$

11-8
$$y = y_0 \frac{x}{l}\left(1 - \frac{x}{l}\right) \qquad \frac{dy}{dx} = \frac{y_0}{l}\left(1 - 2\frac{x}{l}\right)$$

$$\frac{d^2y}{dx^2} = -y_0 \frac{2}{l^2}$$

$$U = \frac{1}{2}\,\frac{4y_0^2}{l^4}\left[2EI_1\int_0^{l/4} dx + 2EI_2\int_{l/4}^{l/2} dx\right] = \frac{1}{2}\left(\frac{4y_0^2}{l^4}\right)\frac{l}{2}\left[EI_1 + EI_2\right]$$

$$= \frac{y_0^2}{l^3}\left[EI_1 + EI_2\right]$$

$$T = \cancel{\tfrac{1}{2}}\,\omega^2 y_0^2\left[m\,\cancel{2}\int_0^{l/4}\left(\frac{x}{l}\right)^2\left(1 - \frac{x}{l}\right)^2 dx + (2m)\cancel{2}\int_{l/4}^{l/2}\left(\frac{x}{l}\right)^2\left(1 - \frac{x}{l}\right)^2 dx\right]$$

$$= \omega^2 y_0^2\left[ml\left\{\frac{1}{3}\left(\frac{x}{l}\right)^3 - \frac{1}{2}\left(\frac{x}{l}\right)^4 + \frac{1}{5}\left(\frac{x}{l}\right)^5\right\}_0^{l/4} + 2ml\left\{\quad \text{same} \quad\right\}_{l/4}^{l/2}\right]$$

$$= \omega^2 y_0^2\, ml\left[(.008659) + (.0264)\right] = .03506\,\omega^2 y_0^2\, ml$$

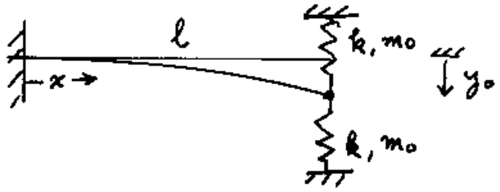
Equating $T = U$

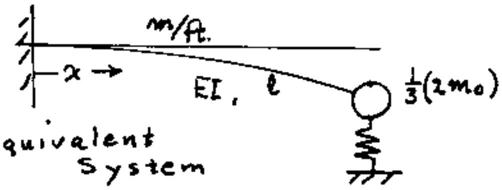
$$\omega_1^2 = 28.52\left(\frac{EI_1 + EI_2}{ml^4}\right)$$

For $EI_2 = EI_1$ $\qquad \omega_1 = \sqrt{57.04}\sqrt{\frac{EI_1}{ml^4}} = 7.55\sqrt{\frac{EI_1}{ml^4}}$

for Sine Curve Prob. 11-7 $\qquad \omega_1 = 7.32\sqrt{\frac{EI_1}{ml^4}}$

11-9





Equivalent System

$$T_m = \frac{1}{2}\left(\frac{33\,m\ell}{140}\right)\omega^2 y_o^2 + \frac{1}{2}\left(\frac{2}{3}m_o\right)\omega^2 y_o^2$$

$$U_m = \frac{1}{2}\left(\frac{3EI}{\ell^3}\right)y_o^2 + \frac{1}{2}(2k)\,y_o^2$$

Equate

$$\omega_1^2 = \frac{\frac{3EI}{\ell^3} + 2k}{\left(\frac{33\,m\ell}{140} + \frac{2}{3}m_o\right)}$$

11-10

$$y_m = \sin\frac{\pi x}{\ell} + b$$

$$U = \frac{1}{2}EI\int_0^\ell (y'')^2\,dx + \frac{1}{2}\left(\frac{k}{2}\right)y_{x=0}^2 + \frac{1}{2}\left(\frac{k}{2}\right)y_{x=\ell}^2$$

$$U_{max} = \frac{1}{2}EI\int_0^\ell \left(\frac{\pi}{\ell}\right)^4 \sin^2\frac{\pi x}{\ell}\,dx + \frac{1}{2}kb^2$$

$$T_{max} = \frac{1}{2}\int_0^\ell \frac{M}{\ell}\dot{y}^2\,dx = \frac{1}{2}\omega^2\frac{M}{\ell}\int_0^\ell\left(\sin^2\frac{\pi x}{\ell} + 2b\sin\frac{\pi x}{\ell} + b^2\right)dx$$

$$= \frac{M\omega^2}{2}\left(\frac{1}{2} + \frac{4b}{\pi} + b^2\right)$$

Equating $U_{max} = T_{max}$

$$\omega_1^2 = \frac{2k}{M}\left(\frac{\frac{K}{k}\frac{\pi^4}{4} + \frac{b^2}{2}}{\frac{1}{2} + \frac{4b}{\pi} + b^2}\right)$$

where $K = EI/\ell^3$

$\frac{\partial\omega_1^2}{\partial b} = 0$ gives

$$b = -\frac{\pi}{4}\left(\frac{1}{2} - \frac{K\pi^4}{2k}\right) \pm \sqrt{\left[\frac{\pi}{4}\left(\frac{1}{2} - \frac{K\pi^4}{2k}\right)\right]^2 + \frac{\pi^4 K}{2k}}$$

11-11

$$y = y_m\left[3\left(\frac{x}{\ell}\right) - 4\left(\frac{x}{\ell}\right)^3\right] \qquad 0 \le x \le \frac{\ell}{2}$$

$$m(x) = m_0\left(\frac{x}{\ell}\right)\left(1 - \frac{x}{\ell}\right)$$

$$T_m' = \frac{1}{2} \times 2\int_0^{\ell/2} m(x)\,\omega^2 y^2\,dx = m_0\,\omega^2 y_m^2 \int_0^{\ell/2} \frac{x}{\ell}\left[9\left(\frac{x}{\ell}\right)^2 - 24\left(\frac{x}{\ell}\right)^4 + 16\left(\frac{x}{\ell}\right)^6\right] dx$$

$$= m_0\omega^2 y_m^2 \ell\left[\frac{9}{4}\left(\frac{1}{2}\right)^4 - \frac{24}{6}\left(\frac{1}{2}\right)^6 + \frac{16}{8}\left(\frac{1}{2}\right)^8 - \frac{9}{5}\left(\frac{1}{2}\right)^5 + \frac{24}{7}\left(\frac{1}{2}\right)^7 - \frac{16}{9}\left(\frac{1}{2}\right)^9\right]$$

$$= 0.0529\; m_0\ell\;\omega^2 y_m^2$$

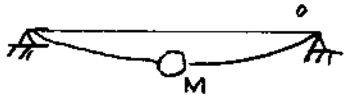
$$U_m = \frac{1}{2} \times 2\int_0^{\ell/2} EI\left(\frac{d^2y}{dx^2}\right)^2 dx = y_m^2\; EI\int_0^{\ell/2} \frac{24^2}{\ell^6}\,x^2\,dx = y_m^2\; EI\,\frac{24^2}{\ell^6}\,\frac{1}{3}\left(\frac{\ell}{2}\right)^3$$

Equating

$$\omega_1^2 = 453\,\frac{EI}{(m_0\ell)\,\ell^3} \qquad \omega_1 = 21.30\sqrt{\frac{EI}{(m_0\ell)\,\ell^3}}$$

To check, first find total mass

$$\int m\,dx = 2m_0\int_0^{\ell/2}\left(\frac{x}{\ell} - \frac{x^3}{\ell^3}\right)dx = 2m_0\left[\frac{x^2}{2\ell} - \frac{x^4}{4\ell^3}\right]_0^{\ell/2} = 0.2188\; m_0\ell = M_T$$



For massless beam with M at midspan $k = \frac{48EI}{\ell^3}$

$$T' = \frac{1}{2}\, m_{eff}\,\cancel{\omega^2}\,\cancel{y_m^2} = 0.0529\; m_0\ell\;\cancel{\omega^2}\,\cancel{y_m^2} \text{ from above}$$

$$\therefore m_{eff} = .1058\; m_0\ell = N\;M_T = N\;.2188\; m_0\ell$$

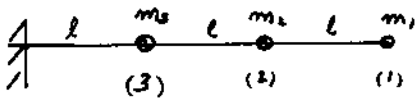
$$\therefore N = .4835 \qquad \therefore m_{eff} = .4835\;(\text{total mass})$$

$$\omega^2 = \frac{k}{m_{eff}} = \frac{48EI}{.4835\,M_T\,\ell^3} = 99.28\;\frac{EI}{M_T\,\ell^3}$$

$$\omega = 9.96\sqrt{\frac{EI}{M_T\,\ell^3}}$$ $\therefore$ results appear to be reasonable

11-12

From Eq. 11.2-3 $\quad \dfrac{1}{\omega_1^2} \cong a_{11} m_1 + a_{22} m_2 + a_{33} m_3$



$$a_{11} = \frac{(3l)^3}{3EI} = \frac{27 l^3}{3EI}$$

$$a_{22} = \frac{(2l)^3}{3EI} = \frac{8 l^3}{3EI}$$

$$a_{33} = \frac{l^3}{3EI}$$

$$\therefore \frac{1}{\omega_1^2} \cong (27 m_1 + 8 m_2 + m_3)\frac{l^3}{3EI} \qquad \omega_1 \cong \sqrt{\frac{3EI}{l^3}\left(\frac{1}{27m_1 + 8m_2 + m_3}\right)}$$

11-13

Flex. $\quad a_{xb} = \dfrac{bx}{6EIl}\left(l^2 - x^2 - b^2\right)$

$$a_{11} = \frac{\frac{3}{4}l \cdot \frac{1}{4}l}{6EI\,l}\left(l^2 - \frac{l^2}{16} - \frac{9}{16}l^2\right) = \frac{3}{16^2}\frac{l^3}{EI} = a_{33}$$

$$a_{22} = \frac{\frac{1}{2}l \cdot \frac{1}{2}l}{6EIl}\left(l^2 - \frac{l^2}{4} - \frac{l^2}{4}\right) = \frac{1}{48}\frac{l^3}{EI} = \frac{16}{3} \cdot \frac{l^3}{16^2 EI}$$

$$\frac{1}{\omega_1^2} \cong a_{11}\frac{W_1}{g} + a_{22}\frac{W_2}{g} + a_{33}\frac{W_3}{g}$$

$$= \frac{l^3}{16^2 EIg}\left(3W_1 + \frac{16}{3}W_2 + 3W_3\right) = \frac{l^3 W}{16^2 EI}\left(3 + \frac{16}{3}\times 4 + 6\right)$$

$$\therefore \omega_1 \cong 16\sqrt{\frac{3}{91}\frac{EIg}{Wl^3}} = 2.905\sqrt{\frac{EIg}{Wl^3}}$$

11-14

Use Eq.(b) Ex 11.2-1 with $a_{22} = \dfrac{0.78}{1000}$

$$\frac{1}{\omega_1^2} \cong \frac{1}{\omega_{11}^2} + \frac{0.78}{1000}\,\frac{320}{386} = \left(\frac{60}{2\pi\times 622}\right)^2 + 0.6466\times 10^{-3}$$

$$= (0.2357 + 0.6466)\times 10^{-3}$$

$$= 0.8823\times 10^{-3}$$

$$\omega_1 \cong \sqrt{\frac{1}{0.8823\times 10^{-3}}} = 33.66 \text{ rad/sec} = 5.358 \text{ Hz.}$$

11-15

$$\frac{1}{(\omega_1)_1^2} - \frac{1}{\omega_{11}^2} = a_{22}(m_2)_1$$

$$\frac{1}{(\omega_1)_2^2} - \frac{1}{\omega_{11}^2} = a_{22}(m_2)_2 \qquad \text{divide to eliminate } a_{22}$$

$$\frac{\omega_{11}^2 - (\omega_1)_1^2}{(\omega_1)_1^2 \cancel{\omega_{11}^2}} \; \frac{(\omega_1)_2^2 \cancel{\omega_{11}^2}}{\omega_{11}^2 - (\omega_1)_2^2} = \frac{(m_2)_1}{(m_2)_2}$$

$$(\omega_1)_1 = 435 \times 2\pi \qquad (m_2)_1 = 5.44$$

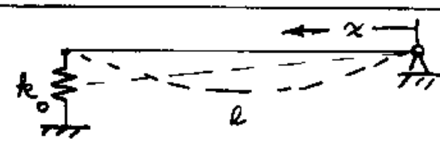
$$(\omega_1)_2 = 398 \times 2\pi \qquad (m_2)_2 = 5.44 + 4.52 = 9.96$$

$$\left(\frac{398}{435}\right)^2 \left[\frac{f_{11}^2 - 435^2}{f_{11}^2 - 398^2}\right] = \frac{5.44}{9.96} \qquad \therefore f_{11}^2 = 245219$$

$$f_{11} = 495.2 \text{ cps}$$

11-16

$$\phi_1 = \frac{x}{l}, \qquad \phi_2 = \sin\frac{\pi x}{l}$$



$$m_{11} = \int m\,\phi_1\phi_1\,dx = \frac{m}{l^2}\int_0^l x^2\,dx = \frac{ml}{3}$$

$$m_{12} = \int_0^l m\,\frac{x}{l}\cdot\sin\frac{\pi x}{l}\,dx = \frac{ml}{\pi^2}\int_0^{\frac{\pi x}{l}=\pi}\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi x}{l}\right)d\left(\frac{\pi x}{l}\right)$$

$$= \frac{ml}{\pi^2}\left[\sin\frac{\pi x}{l} - \left(\frac{\pi x}{l}\right)\cos\frac{\pi x}{l}\right]_0^{\frac{\pi x}{l}=\pi} = \frac{ml}{\pi}$$

$$m_{22} = m\int_0^l \sin^2\frac{\pi x}{l}\,dx = m\int_0^l \frac{1}{2}\left[1 - \cos\frac{2\pi x}{l}\right]dx = \frac{ml}{2}$$

11-16 Cont.

$$EI\int_0^\ell \phi_1''\phi_1''\,dx = 0 \qquad EI\int_0^\ell \phi_1''\phi_2''\,dx = 0$$

$$EI\int_0^\ell \phi_2''\phi_2''\,dx = EI\left(\frac{\pi}{\ell}\right)^4\int_0^\ell \sin^2\frac{\pi x}{\ell}\,dx = \left(\frac{\pi}{\ell}\right)^4 EI\,\frac{\ell}{2}$$

$$u = C_1\,\phi_1(x) + C_2\,\phi_2(x)$$

$$U = \frac{1}{2}\int_0^\ell EI\,(u'')^2\,dx + \frac{1}{2}k_0\,u^2(\ell) \qquad \text{where } u^2(\ell) = C_1^2$$

$$= \frac{1}{2}k_0 C_1^2 + \frac{1}{2}\left(\frac{\pi}{\ell}\right)^4 EI\frac{\ell}{2}\;C_2^2$$

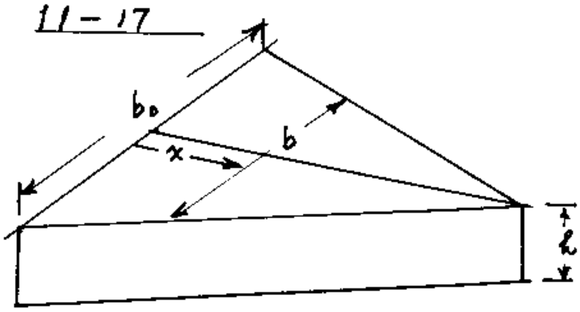
$$\frac{\partial U}{\partial C_1} = k_0 C_1 \qquad \frac{\partial U}{\partial C_2} = \left(\frac{\pi}{\ell}\right)^4 EI\frac{\ell}{2}\;C_2$$

$\therefore$ Eq. 11.3-7 becomes

$$\begin{bmatrix} \left(k_0 - \omega^2\frac{m\ell}{3}\right) & -\omega^2\frac{m\ell}{\pi} \\ -\omega^2\frac{m\ell}{\pi} & \left\{\left(\frac{\pi}{\ell}\right)^4 EI\frac{\ell}{2} - \omega^2\frac{m\ell}{2}\right\} \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \{0\}$$

$$\left[\left(\frac{1}{6}-\frac{1}{\pi^2}\right)(m\ell)^2\right]\omega^4 - m\ell\left[\frac{\pi^4 EI}{6\,\ell^3} + \frac{k_0}{2}\right]\omega^2 + \frac{k_0}{2}\pi^4\frac{EI}{\ell^3} = 0$$

11-17



$$m(x) = m_0\left(1 - \frac{x}{l}\right)$$

$$b(x) = b_0\left(1 - \frac{x}{l}\right)$$

$$I(x) = \frac{b h^3}{12} = \frac{h^3}{12} b_0\left(1 - \frac{x}{l}\right)$$

$$\therefore\ EI = EI_0\left(1 - \frac{x}{l}\right)$$

$$y = C_1 x^2 + C_2 x^3 = C_1\phi_1 + C_2\phi_2$$

$$\phi_1 = x^2 \qquad \phi_2 = x^3 \qquad k_{ij} = \int_0^l EI\,\phi_i''\phi_j''\,dx$$

$$\phi_1'' = 2 \qquad \phi_2'' = 6x \qquad m_{ij} = \int m(x)\,\phi_i\phi_j\,dx$$

$$k_{11} = 4EI_0\int_0^l \left(1 - \frac{x}{l}\right)dx = 4EI_0\left(l - \frac{l}{2}\right) = 2EI_0 l$$

$$k_{12} = 12EI_0\int_0^l x\left(1 - \frac{x}{l}\right)dx = 12EI_0\left(\frac{l^2}{2} - \frac{l^3}{3l}\right) = 2EI_0 l^2.$$

$$k_{22} = 36EI_0\int_0^l x^2\left(1 - \frac{x}{l}\right)dx = 36\left(\frac{l^3}{3} - \frac{l^4}{4l}\right) = 3EI_0 l^3$$

$$m_{11} = m_0\int_0^l x^4\left(1 - \frac{x}{l}\right)dx = m_0\left(\frac{l^5}{5} - \frac{l^6}{6l}\right) = \frac{1}{30}m_0 l^5$$

$$m_{12} = m_0\int_0^l x^5\left(1 - \frac{x}{l}\right)dx = m_0\left(\frac{l^6}{6} - \frac{l^6}{7}\right) = \frac{1}{42}m_0 l^6$$

$$m_{22} = m_0\int_0^l x^6\left(1 - \frac{x}{l}\right)dx = m_0\left(\frac{l^7}{7} - \frac{l^7}{8}\right) = \frac{1}{56}m_0 l^7$$

11-17 Cont. Subst. into Eq. 9.3-7

$$\begin{bmatrix} \left(2EI_0 l - \omega^2 \frac{ml^5}{30}\right) & \left(2EI_0 l^2 - \omega^2 \frac{m_0 l^6}{42}\right) \\ \left(2EI_0 l^2 - \omega^2 \frac{m_0 l^6}{42}\right) & \left(3EI_0 l^3 - \omega^2 \frac{m_0 l^7}{56}\right) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

Freq. Eq. from determinant of above eq = 0

$$6(EI_0)^2 l^4 - \omega^2\left[\frac{m_0 l^5}{30} 3EI_0 l^3 + \frac{m_0 l^7}{56} 2EI_0 l\right] + \omega^4\left(\frac{m_0 l^5}{30} \cdot \frac{m_0 l^7}{56}\right)$$

$$-4(EI_0)^2 l^4 + \left[4EI_0 l^2 \frac{ml^6}{42}\right]\omega^2 - \omega^4\left(\frac{m_0^2 l^{12}}{42^2}\right) = 0$$

$$\omega^4\left[m_0^2 l^{12}\left(\frac{1}{30 \times 56} - \frac{1}{42^2}\right)\right] - \omega^2\left[m_0 EI_0 l^8\left(\frac{3}{30} + \frac{2}{56} - \frac{4}{42}\right)\right] + 2(EI_0)^2 l^4 = 0$$

$$\left(28.345 \times 10^{-6} m_0^2 l^{12}\right)\omega^4 - \left(0.0405 m_0 EI_0 l^3\right)\omega^2 + 2(EI_0)^2 l^4 = 0$$

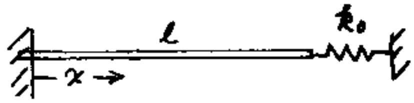
$$\omega^4 - 1429.3\left(\frac{EI_0}{m_0 l^4}\right)\omega^2 + 70559.\left(\frac{EI_0}{m_0 l^4}\right)^2 = 0$$

$$\omega^2 = \begin{Bmatrix} 51.20 \\ 1378 \end{Bmatrix} \times \frac{EI_0}{m_0 l^4}$$

$$\omega_1 = 7.155 \sqrt{\frac{EI_0}{m_0 l^4}}$$

$$\omega_2 = 37.12 \sqrt{\frac{EI_0}{m_0 l^4}}$$

11-18



Normal modes of fixed-free uniform rods are

$$\phi_1 = \sin\frac{\pi}{2}\frac{x}{\ell} \qquad \phi_2 = \sin\frac{3\pi}{2}\frac{x}{\ell} \qquad \text{See Eq. 8.2-8}$$

$$\therefore \quad u(x) = C_1 \sin\frac{\pi}{2}\frac{x}{\ell} + C_2 \sin\frac{3\pi}{2}\frac{x}{\ell}$$

$$\frac{\partial u}{\partial x} = C_1 \frac{\pi}{2\ell}\cos\frac{\pi}{2}\frac{x}{\ell} + C_2 \frac{3\pi}{2}\cos\frac{3\pi}{2}\frac{x}{\ell}$$

$$U = \frac{1}{2}AE\int_0^\ell \left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{1}{2}k_0 u^2(\ell)$$

$$= \frac{1}{2}AE\int_0^\ell C_1^2\left(\frac{\pi}{2\ell}\right)^2\cos^2\frac{\pi}{2}\frac{x}{\ell}\,dx + AE\,C_1C_2\int_0^\ell\left(\frac{\pi}{2\ell}\right)\left(\frac{3\pi}{2\ell}\right)\cos\frac{\pi}{2}\frac{x}{\ell}\cos\frac{3\pi}{2}\frac{x}{\ell}\,dx$$

$$+ \frac{1}{2}AE\int_0^\ell C_2^2\left(\frac{3\pi}{2\ell}\right)^2\cos^2\frac{3\pi}{2\ell}x\,dx + \frac{1}{2}k_0(C_1 - C_2)^2$$

$$\frac{\partial U}{\partial C_1} = \left[AE\left(\frac{\pi}{2\ell}\right)^2\int_0^\ell \cos^2\frac{\pi}{2}\frac{x}{\ell}\,dx + k_0\right]C_1 - k_0 C_2$$

$$+ AE\,C_2\int_0^\ell \cancelto{0}{\left(\frac{\pi}{2\ell}\right)\left(\frac{3\pi}{2\ell}\right)\cos\frac{\pi}{2}\frac{x}{\ell}\cos\frac{3\pi}{2}\frac{x}{\ell}\,dx}$$

$$\frac{\partial U}{\partial C_2} = \left[AE\left(\frac{3\pi}{2\ell}\right)^2\int_0^\ell \cos^2\frac{3\pi}{2\ell}x\,dx\right]C_2 - k_0C_1 + k_0C_2$$

$$\therefore \quad k_{11} = AE\left(\frac{\pi}{2\ell}\right)^2\int_0^\ell \cos^2\frac{\pi}{2}\frac{x}{\ell}\,dx + k_0$$

$$k_{12} = k_{21} = -k_0$$

$$k_{22} = AE\left(\frac{3\pi}{2\ell}\right)^2\int_0^\ell \cos^2\frac{3\pi}{2}\frac{x}{\ell}\,dx + k_0$$

11-18 Cont:

$$\text{Since} \int_0^{\ell} \cos^2 n\theta \, d\theta = \frac{\ell}{2} \qquad n = \frac{\pi}{2\ell},\ \frac{3\pi}{2\ell},$$

$$k_{11} = AE\left(\frac{\pi}{2\ell}\right)^2 \frac{\ell}{2} + k_0 \qquad k_{12} = -k_0$$

$$k_{22} = AE\left(\frac{3\pi}{2\ell}\right)^2 \frac{\ell}{2} + k_0$$

$$m_{ij} = \int_0^{\ell} m\,\phi_i \phi_j \, dx = \begin{cases} m_{11} = m\frac{\ell}{2} \\ m_{12} = 0 \\ m_{22} = \frac{m\ell}{2} \end{cases}$$

Subst into Eq. 11.3-7

$$\begin{bmatrix} \left\{AE\left(\frac{\pi}{2\ell}\right)^2 \frac{\ell}{2} + k_0 - \omega^2 \frac{m\ell}{2}\right\} & -k_0 \\ -k_0 & \left\{AE\left(\frac{3\pi}{2\ell}\right)^2 \frac{\ell}{2} + k_0 - \omega^2 \frac{m\ell}{2}\right\} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

Freq. Eq.

$$\left(\frac{m\ell}{2}\omega^2\right)^2 - \left(\frac{m\ell}{2}\omega^2\right)\left[AE\left(\frac{\pi}{2\ell}\right)^2 \frac{\ell}{2} + k_0 + AE\left(\frac{3\pi}{2\ell}\right)^2 \frac{\ell}{2} + k_0\right]$$

$$+ \left[AE\left(\frac{\pi}{2\ell}\right)^2 \frac{\ell}{2} + k_0\right]\left[AE\left(\frac{3\pi}{2\ell}\right)^2 \frac{\ell}{2} + k_0\right] - k_0^2 = 0$$

Reduces to

$$\omega^4 - \omega^2\left[10\left(\frac{\pi}{2\ell}\right)^2 \frac{AE}{m} + \frac{4k_0^2}{m\ell}\right] + \left[9\left(\frac{\pi}{2\ell}\right)^4\left(\frac{AE}{m}\right)^2 + 20\left(\frac{\pi}{2\ell}\right)^2 \frac{AE}{m}\frac{k_0}{m\ell}\right] = 0$$

11-19 Refer to Prob 11-18

With $k_o = 0$

$$k_{11} = AE\left(\frac{\pi}{2\ell}\right)^2\frac{\ell}{2} \qquad k_{12} = 0$$

$$k_{22} = AE\left(\frac{3\pi}{2\ell}\right)^2\frac{\ell}{2}$$

$$\text{Additional } KE = \frac{1}{2}m_o\dot{u}^2_{(\ell)} = \frac{1}{2}m_o(\dot{C}_1 - \dot{C}_2)^2$$

$$= \frac{1}{2}m_o(\dot{C}_1^2 - 2\dot{C}_1\dot{C}_2 + \dot{C}_2^2)$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{C}_1} = m_o\ddot{C}_1 - m_o\ddot{C}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{C}_2}\right) = -m_o\ddot{C}_1 + m_o\ddot{C}_2$$

$$\therefore\ m_{11} = \frac{m\ell}{2} + m_o$$

$$m_{12} = m_{21} = -m_o$$

$$m_{22} = \frac{m\ell}{2} + m_o$$

Eq. 11.3-7 becomes

$$\begin{bmatrix} \left\{AE\left(\frac{\pi}{2\ell}\right)^2\frac{\ell}{2} - \omega^2\left(\frac{m\ell}{2}+m_o\right)\right\} & \omega^2 m_o \\ \omega^2 m_o & \left\{AE\left(\frac{3\pi}{2\ell}\right)^2\frac{\ell}{2} - \omega^2\left(\frac{m\ell}{2}+m_o\right)\right\} \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \{0\}$$

11-20

$$m(x) = m_0 \frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)$$

$$\phi_1 = \sin\frac{\pi x}{\ell} \qquad \phi_2 = \sin\frac{2\pi x}{\ell}$$

$$y = \phi_1 q_1 + \phi_2 q_2$$

$$k_{11} = EI\int_0^\ell \phi_1''\phi_1''dx = EI\left(\frac{\pi}{\ell}\right)^4\int_0^\ell \sin^2\frac{\pi x}{\ell}dx = EI\left(\frac{\pi}{\ell}\right)^4\frac{\ell}{2}$$

$$k_{12} = k_{21} = EI\int_0^\ell \phi_1''\phi_2'' = 0$$

$$k_{22} = EI\int_0^\ell \phi_2''\phi_2''dx = EI\left(\frac{2\pi}{\ell}\right)^4\int_0^\ell \sin^2\frac{2\pi x}{\ell}dx = EI\left(\frac{2\pi}{\ell}\right)^4\frac{\ell}{2}$$

$$m_{11} = \int_0^\ell m(x)\phi_1\phi_1 dx = m_0\int_0^\ell \frac{x}{\ell}\left(1-\frac{x}{\ell}\right)\sin^2\frac{\pi x}{\ell}dx$$

$$= m_0\int_0^\ell \frac{x}{\ell}\,\frac{1}{2}\left(1-\cos\frac{2\pi x}{\ell}\right)dx - m_0\int_0^\ell\left(\frac{x}{\ell}\right)^2\frac{1}{2}\left(1-\cos\frac{2\pi x}{\ell}\right)dx$$

$$= m_0\left[\frac{\ell}{4} - \frac{\ell}{2\pi^2}\left(\frac{\cos\left(\frac{2\pi x}{\ell}\right)}{\left(\frac{2\pi}{\ell}\right)^2} - \left(\frac{\pi x}{\ell}\right)\frac{\sin\left(\frac{2\pi x}{\ell}\right)}{\left(\frac{2\pi}{\ell}\right)}\right)\Bigg|_0^\ell\,\right]$$

$$- m_0\left[\frac{\ell}{6} - \frac{\ell}{16\pi^3}\int_0^\ell\left(\frac{2\pi x}{\ell}\right)^2\cos\frac{2\pi x}{\ell}\,d\left(\frac{2\pi x}{\ell}\right)\right]$$

$$= \frac{m_0\ell}{12} + \frac{m_0\ell}{16\pi^3}\left[2\left(\frac{2\pi x}{\ell}\right)\cos\frac{2\pi x}{\ell} + \left\{\left(\frac{2\pi x}{\ell}\right)^2 - 2\right\}\sin\frac{2\pi x}{\ell}\right]_0^\ell$$

$$= \frac{m_0\ell}{12} + \frac{m_0\ell}{4\pi^2} = 0.10866\,m_0\ell$$

11-20 Cont:

$$m_{22} = m_o\int_0^\ell \left(\frac{x}{\ell}\right)\left(1-\frac{x}{\ell}\right)\sin^2\frac{2\pi x}{\ell}\,dx = m_o\int_0^\ell \frac{x}{\ell}\left(1-\frac{x}{\ell}\right)\frac{1}{2}\left(1-\cos\frac{4\pi x}{\ell}\right)dx$$

$$= \frac{m_o\ell}{12} - \frac{m_o}{2}\int_0^\ell \frac{x}{\ell}\left(1-\frac{x}{\ell}\right)\cos\frac{4\pi x}{\ell}\,dx = \frac{m_o\ell}{12} + \frac{m_o\ell}{(4\pi)^2}$$

$$= 0.08966\, m_o\ell$$

$$m_{12} = m_o\int_0^\ell \left(\frac{x}{\ell}\right)\left(1-\frac{x}{\ell}\right)\sin\frac{\pi x}{\ell}\sin\frac{2\pi x}{\ell}\,dx = 0 \quad \text{by inspection}$$

ie $\frac{x}{\ell}\left(1-\frac{x}{\ell}\right)$ = symmetric function

$\sin\frac{\pi x}{\ell}\sin\frac{2\pi x}{\ell}$ = unsymmetric function

Eq. (11.3-7) becomes

$$\begin{bmatrix} \left[EI\left(\frac{\pi}{\ell}\right)^4\frac{\ell}{2} - 0.10866\, m_o\ell\,\omega^2\right] & [\quad 0 \quad] \\ [\quad 0 \quad] & \left[EI\left(\frac{2\pi}{\ell}\right)^4\frac{\ell}{2} - 0.08966\, m_o\ell\,\omega^2\right] \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = 0$$

$\therefore$ C_1 & C_2 are independent & Rayleigh-Ritz method fails.

However from above

$$\left[EI\left(\frac{\pi}{\ell}\right)^4\frac{\ell}{2} - 0.10866\, m_o\ell\,\omega^2\right] = 0 \qquad \left[EI\left(\frac{2\pi}{\ell}\right)^4\frac{\ell}{2} - 0.08966\, m_o\ell\,\omega^2\right] = 0$$

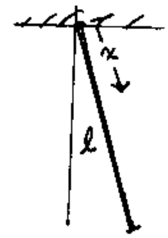
reduces to two Rayleigh's method.

$$\omega_1^2 = \frac{\pi^4}{2\times .10866}\left(\frac{EI}{m_o\ell^4}\right) = 448\,\frac{EI}{m_o\ell^4} \qquad \omega_1 = 21.2\sqrt{\frac{EI}{m_o\ell^4}}$$

$$\omega_2^2 = \frac{8\pi^4}{.08966}\left(\frac{EI}{m_o\ell^4}\right) = 8691\,\frac{EI}{m_o\ell^4} \qquad \omega_2 = 93\sqrt{\frac{EI}{m_o\ell^4}}$$

1st mode is sine $\frac{\pi x}{\ell}$, 2nd mode is $\sin\frac{2\pi x}{\ell}$. Compare with Prob 11-11.

11-21



$$\phi_1 = \frac{x}{\ell} \qquad \phi_1'' = 0$$

$$\phi_2 = \sin\frac{\pi x}{\ell} \qquad \phi_2'' = -\left(\frac{\pi}{\ell}\right)^2 \sin\frac{\pi x}{\ell}$$

$$\phi_3 = \sin\frac{2\pi x}{\ell} \qquad \phi_3'' = -\left(\frac{2\pi}{\ell}\right)^2 \sin\frac{2\pi x}{\ell}$$

$$k_{11} = EI\int_0^\ell \phi_1''\phi_1''\,dx = 0 \qquad k_{12} = k_{13} = 0$$

$$k_{22} = EI\left(\frac{\pi}{\ell}\right)^4 \int_0^\ell \sin^2\frac{\pi x}{\ell}\,dx = EI\left(\frac{\pi}{\ell}\right)^4 \frac{\ell}{2}$$

$$k_{33} = EI\left(\frac{2\pi}{\ell}\right)^4 \frac{\ell}{2}$$

$$k_{23} = EI\left(\frac{\pi}{\ell}\right)^2\left(\frac{2\pi}{\ell}\right)^2 \int_0^\ell \sin\frac{\pi x}{\ell}\cdot\sin\frac{2\pi x}{\ell}\,dx = 0$$

$$m_{11} = m_0\int_0^\ell \phi_1^2\,dx = m_0\int_0^\ell \left(\frac{x}{\ell}\right)^2 dx = \frac{m_0\ell}{3}$$

$$m_{22} = m_0\int_0^\ell \sin^2\frac{\pi x}{\ell}\,dx = \frac{m_0\ell}{2}$$

$$m_{33} = m_0\int_0^\ell \sin^2\frac{2\pi x}{\ell}\,dx = \frac{m_0\ell}{2}$$

$$m_{12} = m_0\int_0^\ell \frac{x}{\ell}\sin\frac{\pi x}{\ell}\,dx = \frac{m_0}{\ell}\left[\frac{\sin\frac{\pi x}{\ell}}{\left(\frac{\pi}{\ell}\right)^2} - x\frac{\cos\frac{\pi x}{\ell}}{\left(\frac{\pi}{\ell}\right)}\right]_0^\ell$$

$$= \frac{m_0\ell}{\pi} = m_{21}$$

$$m_{13} = m_0\int_0^\ell \frac{x}{\ell}\sin\frac{2\pi x}{\ell}\,dx = -\frac{m_0\ell}{2\pi}$$

11-21 Cont

$$m_{23} = m_0 \int_0^{\ell} \sin\frac{\pi x}{\ell} \sin\frac{2\pi x}{\ell}\, dx = 0$$

$$\begin{bmatrix} \left(0 - \omega^2 \frac{m_0 \ell}{3}\right) & \left(0 - \omega^2 \frac{m_0 \ell}{\pi}\right) & \left(0 + \omega^2 \frac{m_0 \ell}{2\pi}\right) \\ \left(0 - \omega^2 \frac{m_0 \ell}{\pi}\right) & \left(EI\left(\frac{\pi}{\ell}\right)^4 \frac{\ell}{2} - \omega^2 \frac{m_0 \ell}{2}\right) & (0 - 0) \\ \left(0 - \omega^2 \frac{m_0 \ell}{2\pi}\right) & (0 - 0) & \left(EI\left(\frac{2\pi}{\ell}\right)^4 \frac{\ell}{2} - \omega^2 \frac{m_0 \ell}{2}\right) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \{0\}$$

Freq. Eq.

$$-\omega^2 \frac{m_0 \ell}{3} \left[EI\left(\frac{\pi}{\ell}\right)^4 \frac{\ell}{2} - \omega^2 \frac{m_0 \ell}{2}\right]\left[EI\left(\frac{2\pi}{\ell}\right)^4 \frac{\ell}{2} - \omega^2 \frac{m_0 \ell}{2}\right]$$

$$+ \omega^2 \frac{m_0 \ell}{\pi} \left[-\left(\omega^2 \frac{m_0 \ell}{\pi}\right)\left\{EI\left(\frac{2\pi}{\ell}\right)^4 \frac{\ell}{2} - \omega^2 \frac{m_0 \ell}{2}\right\}\right]$$

$$- \omega^2 \frac{m_0 \ell}{2\pi} \left[-\left(\omega^2 \frac{m_0 \ell}{2\pi}\right)\left\{EI\left(\frac{\pi}{\ell}\right)^4 \frac{\ell}{2} - \omega^2 \frac{m_0 \ell}{2}\right\}\right] = 0$$

$$-\left(\omega^2 m_0 \ell\right)^3 \left[\frac{1}{12} - \frac{1}{2\pi^2} + \frac{1}{8\pi^2}\right] + \left(\omega^2 m_0 \ell\right)^2 EI \left[\frac{\ell}{12}\left(\frac{\pi}{\ell}\right)^4 + \frac{\ell}{12}\left(\frac{2\pi}{\ell}\right)^4 - \frac{\ell}{2\pi^2}\left(\frac{2\pi}{\ell}\right)^4 + \frac{\ell}{8\pi^2}\left(\frac{\pi}{\ell}\right)^4\right]$$

$$-\left(\omega^2 m_0 \ell\right)\left(EI\right)^2 \frac{\ell^2}{12} \left(\frac{\pi}{\ell}\right)^4 \left(\frac{2\pi}{\ell}\right)^4 = 0$$

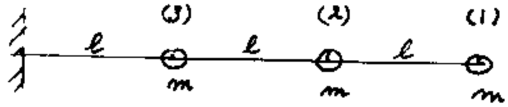
$$-0.045338\left(\omega^2\right)^3 + 60.273\left(\omega^2\right)^2 \left(\frac{EI}{m_0 \ell^4}\right) - 12651.37\, \omega^2 \left(\frac{EI}{m_0 \ell^4}\right)^2 = 0$$

$$\left(\omega^2\right)^2 - \left(1329.4146 \frac{EI}{m_0 \ell^4}\right)\omega^2 + 279045.61 \left(\frac{EI}{m_0 \ell^4}\right)^2 = 0$$

$$\omega^2 = \left(664.70 \pm 403.47\right)\left(\frac{EI}{m_0 \ell^4}\right) = \begin{Bmatrix} 261.23 \\ 1068.17 \end{Bmatrix} \frac{EI}{m_0 \ell^4}$$

$$\omega = \begin{Bmatrix} 0.0 \\ 16.16 \\ 32.68 \end{Bmatrix} \sqrt{\frac{EI}{m_0 \ell^4}} \qquad \text{Exact Sol} \quad \omega = \begin{Bmatrix} 0 \\ 15.4 \\ 50. \end{Bmatrix} \sqrt{\frac{EI}{m_0 \ell^4}}$$

11-22



From Ex. 6.1-2

$$[a] = \frac{\ell^3}{3EI}\begin{bmatrix} 27 & 14 & 4 \\ 14 & 8 & 2.5 \\ 4 & 2.5 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \lambda \begin{bmatrix} 27 & 14 & 4 \\ 14 & 8 & 2.5 \\ 4 & 2.5 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad \text{where } \lambda = \frac{m\omega^2\ell^3}{3EI}$$

Start iteration with

$$\begin{Bmatrix} 1 \\ .3 \\ .1 \end{Bmatrix} \rightarrow \begin{Bmatrix} 31.60 \\ 16.65 \\ .75 \end{Bmatrix} = 31.6\begin{Bmatrix} 1.00 \\ .5269 \\ .0237 \end{Bmatrix} \rightarrow \begin{Bmatrix} 34.47 \\ 18.27 \\ 5.409 \end{Bmatrix} = 34.47\begin{Bmatrix} 1.00 \\ .530 \\ .1569 \end{Bmatrix}$$

$$\rightarrow \begin{Bmatrix} 35.047 \\ 18.632 \\ 5.4819 \end{Bmatrix} = 35.047\begin{Bmatrix} 1.00 \\ .5316 \\ .1564 \end{Bmatrix} \rightarrow \begin{Bmatrix} 35.068 \\ 18.6438 \\ 5.4854 \end{Bmatrix} = 35.068\begin{Bmatrix} 1.00 \\ .5316 \\ .1564 \end{Bmatrix}$$

$$\therefore \quad 35.068\,\frac{m\omega_1^2\ell^3}{3EI} = 1 \qquad \omega_1^2 = .0855\,\frac{EI}{m\ell^3}$$

$$\omega_1 = 0.2925\sqrt{\frac{EI}{m\ell^3}}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1.00 \\ .5316 \\ .1564 \end{Bmatrix}$$

2^{nd} mode

$$[S] = \begin{bmatrix} 0 & -\frac{x_2}{x_1} & -\frac{x_3}{x_1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -.5316 & -.1564 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 27 & 14 & 4 \\ 14 & 8 & 2.5 \\ 4 & 2.5 & 1 \end{bmatrix} \begin{bmatrix} 0 & -.5316 & -.1564 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -.3532 & -.2228 \\ 0 & .5576 & .3184 \\ 0 & .3736 & .3744 \end{bmatrix}$$

Start iteration with

$$\begin{Bmatrix} 1 \\ 0 \\ -.5 \end{Bmatrix} \longrightarrow \begin{Bmatrix} .1114 \\ -.1592 \\ -.1868 \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ -1.429 \\ -1.677 \end{Bmatrix} \xrightarrow{2^{nd}\text{ iter}} \begin{Bmatrix} .8784 \\ -1.3308 \\ -1.1617 \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ -1.5150 \\ -1.3225 \end{Bmatrix}$$

$$\xrightarrow{3^{rd}\text{ iter}} \begin{Bmatrix} .8298 \\ -1.2658 \\ -1.0611 \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ -1.5254 \\ -1.2787 \end{Bmatrix} \longrightarrow \begin{Bmatrix} .8237 \\ -1.2577 \\ -1.0486 \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ -1.5269 \\ -1.2731 \end{Bmatrix}$$

$$\longrightarrow \begin{Bmatrix} .8229 \\ -1.2568 \\ -1.0471 \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ -1.5273 \\ -1.2725 \end{Bmatrix} \longrightarrow \begin{Bmatrix} .8229 \\ -1.2568 \\ -1.0470 \end{Bmatrix} = .8229 \begin{Bmatrix} 1.00 \\ -1.5273 \\ -1.2723 \end{Bmatrix}$$

$$\therefore \quad \omega_2^2 = \frac{3}{.8229} \frac{EI}{m\ell^3} = 3.6456 \left(\frac{EI}{m\ell^3} \right)$$

$$\omega_2 = 1.9094 \sqrt{\frac{EI}{m\ell^3}} \qquad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1.00 \\ -1.5273 \\ -1.2723 \end{Bmatrix}$$

11-22 Cont. $\quad 3^{rd}$ mode

$$C_1 = \bar{x}_1 + .5316\,\bar{x}_2 + .1564\,\bar{x}_3 = 0$$

$$C_2 = \bar{x}_1 - 1.5273\,\bar{x}_2 - 1.2723\,\bar{x}_3 = 0$$

Subtr. $\quad \bar{x}_2 = -1.1207\,\bar{x}_3 \quad$ subst. back into C_1

to get $\quad \bar{x}_1 = .4394\,\bar{x}_3 \quad \therefore \quad \bar{x}_3 = 2.2758\,\bar{x}_1$

$$\bar{x}_2 = -2.5505\,\bar{x}_1$$

$$[S] = \begin{bmatrix} 0 & 0 & 1.00 \\ 0 & 0 & -2.5505 \\ 0 & 0 & 2.2758 \end{bmatrix} \qquad \text{Mult. by } [a]$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \lambda \begin{bmatrix} 0 & 0 & .3962 \\ 0 & 0 & -.7145 \\ 0 & 0 & -.1005 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

start iteration with

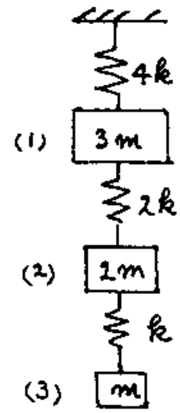
$$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \rightarrow \begin{Bmatrix} .3962 \\ -.7145 \\ -.1005 \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ -1.8034 \\ -.2537 \end{Bmatrix} \xrightarrow{2^{nd}\text{ iter.}} \begin{Bmatrix} .1005 \\ -.1812 \\ .0255 \end{Bmatrix} = .1005 \begin{Bmatrix} 1.00 \\ -1.8034 \\ -.2537 \end{Bmatrix}$$

$$\therefore \quad \frac{.1005\, m\omega^2 \ell^3}{3EI} = 1 \qquad \omega_3^2 = 29.851 \frac{EI}{m\ell^3}$$

$$\omega_3 = 5.4636 \sqrt{\frac{EI}{m\ell^3}}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}^{(3)} = \begin{Bmatrix} 1.00 \\ -1.803 \\ -.2537 \end{Bmatrix}$$

11-23



$$a_{11} = \frac{1}{4k} = a_{21} = a_{12} = a_{31} = a_{13}$$

$$a_{22} = \frac{4k+2k}{4k \cdot 2k} = \frac{3}{4k} = a_{32} = a_{23}$$

$$a_{33} = \frac{1}{4k} + \frac{1}{2k} + \frac{1}{k} = \frac{7}{4k}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{4k} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \lambda [a][m]\{x\}$$

$$= \lambda \begin{bmatrix} 3 & 2 & 1 \\ 3 & 6 & 3 \\ 3 & 6 & 7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Start iteration with $\begin{Bmatrix} 1 \\ 3 \\ 7 \end{Bmatrix}$

$$\begin{Bmatrix} 1 \\ 3 \\ 7 \end{Bmatrix} \to \begin{Bmatrix} 16 \\ 48 \\ 70 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3.0 \\ 4.375 \end{Bmatrix} = \begin{Bmatrix} .229 \\ .686 \\ 1.00 \end{Bmatrix}$$

normalized to $x_3 = 1.0$

2^{nd} iter.

$$\to \begin{Bmatrix} 3.059 \\ 7.803 \\ 11.803 \end{Bmatrix} = \begin{Bmatrix} .259 \\ .661 \\ 1.00 \end{Bmatrix} \xrightarrow{3^{rd}\text{ iter}} \begin{Bmatrix} 3.099 \\ 7.743 \\ 11.743 \end{Bmatrix} = \begin{Bmatrix} .264 \\ .659 \\ 1.00 \end{Bmatrix} \to$$

$$\xrightarrow{4^{th}} \begin{Bmatrix} 3.110 \\ 7.746 \\ 11.746 \end{Bmatrix} = \begin{Bmatrix} .265 \\ .659 \\ 1.00 \end{Bmatrix} \xrightarrow{5th} \begin{Bmatrix} 3.113 \\ 7.749 \\ 11.749 \end{Bmatrix} = \begin{Bmatrix} .265 \\ .660 \\ 1.00 \end{Bmatrix} \times 11.749$$

$$11.749 \frac{m\omega^2}{4k} = 1.0 \qquad \omega_1^2 = 0.341 \frac{k}{m}$$

$$\omega_1 = 0.584 \sqrt{\frac{k}{m}}$$

11-23 Cont $\quad$ 2^{nd} mode

$$[S] = \begin{bmatrix} 0 & -\left(\frac{2}{3}\right)\left(\frac{.660}{.265}\right) & -\left(\frac{1}{3}\right)\left(\frac{1.00}{.265}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1.660 & -1.258 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\{x\} = \lambda [a][m][S]\{x\}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \lambda \begin{bmatrix} 0 & -2.980 & -2.774 \\ 0 & 1.020 & -.774 \\ 0 & 1.020 & 3.226 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \qquad \text{start with } \begin{Bmatrix} -.2 \\ .6 \\ 1.0 \end{Bmatrix}$$

$$\rightarrow \begin{Bmatrix} -4.562 \\ -.162 \\ 3.838 \end{Bmatrix} = \begin{Bmatrix} -1.189 \\ -.042 \\ 1.00 \end{Bmatrix} \xrightarrow{2^{nd}} \begin{Bmatrix} -2.649 \\ -.817 \\ 3.183 \end{Bmatrix} = \begin{Bmatrix} -.832 \\ -.257 \\ 1.00 \end{Bmatrix} \xrightarrow{3^{rd}}$$

took 13 iteration to stabilize

$$\xrightarrow{13th} \begin{Bmatrix} -1.466 \\ -1.222 \\ 2.778 \end{Bmatrix} = 2.778 \begin{Bmatrix} -.528 \\ -.440 \\ 1.00 \end{Bmatrix} \qquad \omega_2^2 = 1.440\ k/m$$

$$\omega_2 = 1.200\sqrt{\frac{k}{m}}$$

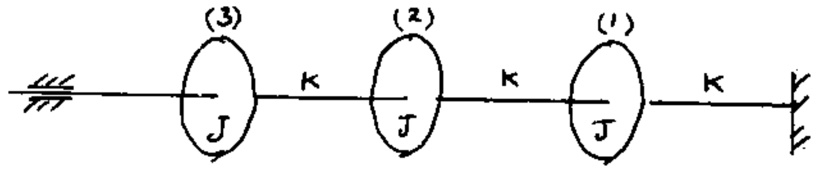
— 3^{rd} mode —

$$[S] = \begin{bmatrix} 0 & 0 & 1.581 \\ 0 & 0 & -1.710 \\ 0 & 0 & 1.00 \end{bmatrix} \qquad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \lambda \begin{bmatrix} 0 & 0 & 2.323 \\ 0 & 0 & -2.517 \\ 0 & 0 & 1.483 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$1.483\ \frac{m\omega^2}{4k} = 1 \qquad \omega_2^2 = 2.697\ \frac{k}{m}$$

$$\omega_3 = 1.642\sqrt{\frac{k}{m}}$$

11-24



$$a_{11} = a_{21} = a_{12} = a_{31} = a_{13} = \frac{1}{K}$$

$$a_{22} = a_{32} = a_{23} = \frac{1}{K} + \frac{1}{K} = \frac{2}{K} \qquad a_{33} = \frac{3}{K}$$

$$[a] = \frac{1}{K}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{\omega^2 J}{K}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

start with

$$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \rightarrow \begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix} = \begin{Bmatrix} .50 \\ .833 \\ 1.00 \end{Bmatrix} \rightarrow \begin{Bmatrix} 2.337 \\ 4.1666 \\ 5.1666 \end{Bmatrix} = \begin{Bmatrix} .4516 \\ .8065 \\ 1.000 \end{Bmatrix} \rightarrow \begin{Bmatrix} 2.2581 \\ 4.0645 \\ 5.0646 \end{Bmatrix} = \begin{Bmatrix} .4458 \\ .8025 \\ 1.000 \end{Bmatrix}$$

$$\rightarrow \begin{Bmatrix} 2.2483 \\ 4.0508 \\ 5.0508 \end{Bmatrix} = \begin{Bmatrix} .4451 \\ .8020 \\ 1.000 \end{Bmatrix} \rightarrow \begin{Bmatrix} 2.2471 \\ 4.0491 \\ 5.0491 \end{Bmatrix} = \begin{Bmatrix} .4450 \\ .8019 \\ 1.000 \end{Bmatrix} \rightarrow \begin{Bmatrix} 2.2469 \\ 4.0488 \\ 5.0488 \end{Bmatrix} = \begin{Bmatrix} .4450 \\ .8019 \\ 1.000 \end{Bmatrix}$$

$$\omega_1 = \sqrt{\frac{1}{5.0488}\frac{K}{J}} = 0.4450\sqrt{\frac{K}{J}}$$

2nd mode

$$\bar{\theta}_1 = -\frac{.8019}{.4450}\bar{\theta}_2 - \frac{1.00}{.4450}\bar{\theta}_3 = -1.8020\,\bar{\theta}_2 - 2.2472\,\bar{\theta}_3$$

$$[S] = \begin{bmatrix} 0 & -1.8020 & -2.2472 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

11-24 Cont.

$$\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} = \frac{\omega^2 J}{K}\begin{bmatrix}0 & -.8020 & -1.2472\\ 0 & .1980 & -.2472\\ 0 & .1980 & .7528\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} \text{ converges to } \begin{Bmatrix}-.8019\\ -.3571\\ .6429\end{Bmatrix} = \begin{Bmatrix}-1.2473\\ -.5559\\ 1.000\end{Bmatrix}$$

$$\omega_2 = \sqrt{\frac{1}{.6429}\frac{k}{J}} = 1.247\sqrt{\frac{k}{J}}$$

3rd mode

$C_1 = 0$ gives $\quad \bar{\theta}_1 = -1.802\,\bar{\theta}_2 - 2.2472\,\bar{\theta}_3$

$C_2 = 0$ " $\quad \bar{\theta}_1 = -.4451\,\bar{\theta}_2 + .8018\,\bar{\theta}_3$

$$0 = -1.3569\,\bar{\theta}_2 - 3.0490\,\bar{\theta}_3$$

$$\bar{\theta}_2 = -2.247\,\bar{\theta}_3$$

$$\bar{\theta}_1 = 1.8020\,\bar{\theta}_3$$

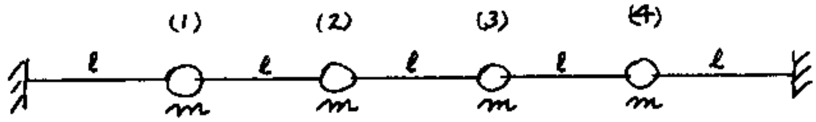
$$[S]_2 = \begin{bmatrix}0 & 0 & 1.8020\\ 0 & 0 & -2.247\\ 0 & 0 & 1.00\end{bmatrix}$$

$$\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} = \frac{\omega^2 J}{K}\begin{bmatrix}0 & -.8020 & -1.2472\\ 0 & .1980 & -.2472\\ 0 & .1980 & .7528\end{bmatrix}\begin{bmatrix}0 & 0 & 1.8020\\ 0 & 0 & -2.247\\ 0 & 0 & 1.00\end{bmatrix}$$

$$= \frac{\omega^2 J}{K}\begin{bmatrix}0 & 0 & .5557\\ 0 & 0 & -.6921\\ 0 & 0 & .3079\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} \text{ converges to } \begin{Bmatrix}.5557\\ .6921\\ .3079\end{Bmatrix} = \begin{Bmatrix}1.803\\ -2.248\\ 1.000\end{Bmatrix}$$

$$\omega_3 = \sqrt{\frac{1}{.3079}\frac{k}{J}} = 1.802\sqrt{\frac{k}{J}}$$

11-25



T = tension

From unit defl. at each pt. & $\sum F$

$$a_{11} = \frac{4}{5}\frac{\ell}{T} \qquad a_{21} = a_{12} = \frac{3}{5}\frac{\ell}{T} \qquad a_{31} = a_{13} = \frac{2}{5}\frac{\ell}{T}$$

$$a_{41} = \frac{1}{5}\frac{\ell}{T} \qquad a_{22} = \frac{6}{5}\frac{\ell}{T} \qquad a_{32} = \frac{4}{5}\frac{\ell}{T} \qquad a_{42} = \frac{2}{5}\frac{\ell}{T}$$

$$a = \frac{\ell}{5T}\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \qquad m = m\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \frac{\omega^2 m\ell}{5T}\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

start with $\begin{Bmatrix} .5 \\ 1.0 \\ 1.0 \\ .5 \end{Bmatrix}$

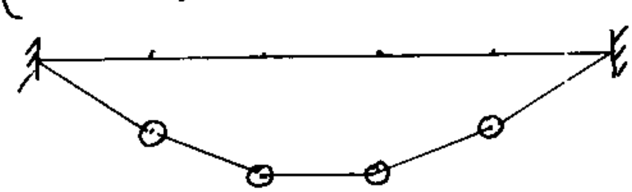
$$\rightarrow \begin{Bmatrix} 7.5 \\ 12.50 \\ 12.50 \\ 7.5 \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.666 \\ 1.666 \\ 1.00 \end{Bmatrix} \rightarrow \begin{Bmatrix} 13.333 \\ 21.666 \\ 21.666 \\ 13.333 \end{Bmatrix} = \begin{Bmatrix} 1.000 \\ 1.625 \\ 1.625 \\ 1.000 \end{Bmatrix} \rightarrow \text{etc}$$

converges to

$$\begin{Bmatrix} 13.095 \\ 21.190 \\ 21.190 \\ 13.095 \end{Bmatrix} = 13.095\begin{Bmatrix} 1.000 \\ 1.618 \\ 1.618 \\ 1.000 \end{Bmatrix}$$

$$\omega_1^2 = \frac{5}{13.09}\,\frac{T}{m\ell}$$

$$\omega_1 = 0.618\sqrt{\frac{T}{m\ell}}$$



symmetric

0 - nodes

11-25 Cont:

$2^{\underline{nd}}$ mode

$$[S]_1 = \begin{bmatrix} 0 & -1.618 & -1.618 & -1.000 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Y = \lambda [a][S]_1 Y \quad , \qquad \lambda = \frac{\omega^2 m \ell}{5T}$$

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \lambda \begin{bmatrix} 0 & -3.4720 & -4.4720 & -3.000 \\ 0 & 1.1460 & -0.8540 & -1.000 \\ 0 & 0.7640 & 2.7640 & 1.000 \\ 0 & 0.3820 & 1.3820 & 3.000 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$2^{\underline{nd}}$ mode will be unsymmetric ∴ start with

$$\begin{Bmatrix} 1.0 \\ .6 \\ -.6 \\ -1.0 \end{Bmatrix} \rightarrow \begin{Bmatrix} 3.6000 \\ 2.2000 \\ -2.2000 \\ -3.6000 \end{Bmatrix} = \begin{Bmatrix} -1.000 \\ -.6111 \\ .6111 \\ 1.000 \end{Bmatrix} \rightarrow \begin{Bmatrix} -3.6111 \\ -2.2222 \\ 2.2222 \\ 3.6111 \end{Bmatrix} = \begin{Bmatrix} -1.000 \\ -.6154 \\ .6154 \\ 1.000 \end{Bmatrix}$$

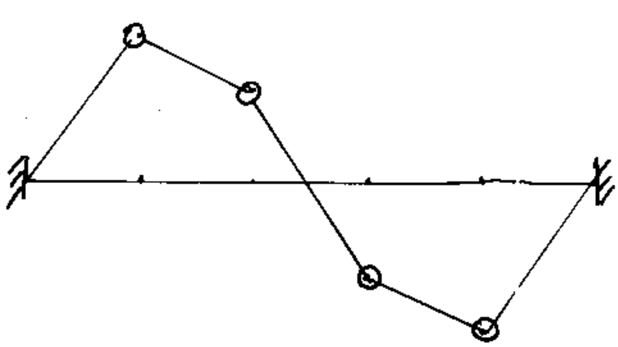
etc. converges after 4 more iterations to

$$\begin{Bmatrix} -3.6180 \\ -2.2360 \\ 2.2360 \\ 3.3180 \end{Bmatrix} = 3.3180 \begin{Bmatrix} -1.000 \\ -.6180 \\ .6180 \\ 1.000 \end{Bmatrix}$$

$$\omega_2^2 = \frac{5}{3.318} \frac{T}{m\ell}$$

$$= 1.3820 \frac{T}{m\ell}$$

$$\omega_1 = 1.1756 \sqrt{\frac{T}{m\ell}}$$



antisymmetric

1 node

11-25 Cont: 3rd mode

$$C_1 = 0 = 1.0\,\bar{y}_1 + 1.618\,\bar{y}_2 + 1.618\,\bar{y}_3 + 1.00\,\bar{y}_4$$

$$C_2 = 0 = -1.0\,\bar{y}_1 - .618\,\bar{y}_2 + .618\,\bar{y}_3 + 1.00\,\bar{y}_4$$

adding $\quad 1.00\,\bar{y}_2 + 2.236\,\bar{y}_3 + 2.0\,\bar{y}_4 = 0$

$$\bar{y}_2 = -2.236\,\bar{y}_3 - 2.00\,\bar{y}_4$$

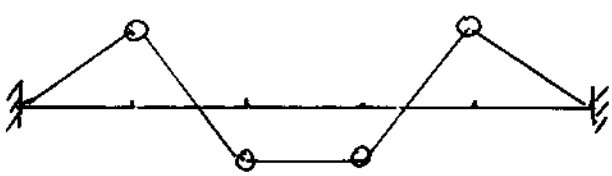
$$[S]_2 = \begin{bmatrix} 0 & -.618 & .618 & 1.00 \\ 0 & 0 & -2.236 & -2.0 \\ 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix} \qquad Y = \lambda[a][S]_1[S]_2 Y$$

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \lambda \begin{bmatrix} 0 & 0 & 3.2914 & 3.9440 \\ 0 & 0 & -3.4165 & -3.2920 \\ 0 & 0 & 1.0557 & -.5280 \\ 0 & 0 & .5278 & 2.2360 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

3rd mode must have 2 nodes & be symmetric, start with

$$\begin{Bmatrix} 1.0 \\ -.6 \\ -.6 \\ 1.0 \end{Bmatrix} \longrightarrow \text{converges to} \quad \begin{Bmatrix} 1.9098 \\ -1.1806 \\ -1.1806 \\ 1.9098 \end{Bmatrix} = \begin{Bmatrix} 1.0000 \\ -.6182 \\ -.6182 \\ 1.000 \end{Bmatrix}$$

$$\omega_3^2 = \frac{5}{1.9098}\frac{T}{m\ell} = 2.618\,\frac{T}{m\ell} \qquad \omega_3 = 1.618\sqrt{\frac{T}{m\ell}}$$



symmetric
2 nodes

11-25 Cont: 4^{th} mode

$$C_1 = 0 = 1.0\,\bar{y}_1 + 1.618\,\bar{y}_2 + 1.618\,\bar{y}_3 + 1.00\,\bar{y}_4$$

$$C_2 = 0 = -1.00\,\bar{y}_1 - .618\,\bar{y}_2 + .618\,\bar{y}_3 + 1.00\,\bar{y}_4$$

$$C_3 = 0 = 1.00\,\bar{y}_1 - .618\,\bar{y}_2 - .618\,\bar{y}_3 + 1.00\,\bar{y}_4$$

solving $\quad \bar{y}_2 = 1.618\,\bar{y}_4 \qquad \bar{y}_3 = -1.618\,\bar{y}_4$

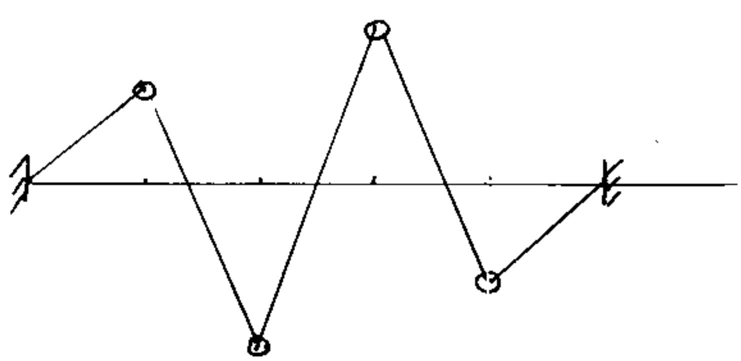
$\bar{y}_1 = 0.618\,\bar{y}_3$

$$[S]_3 = \begin{bmatrix} 0 & 0 & .618 & 0 \\ 0 & 0 & 0 & 1.618 \\ 0 & 0 & 0 & -1.618 \\ 0 & 0 & 0 & 1.0 \end{bmatrix} \qquad Y = \lambda\,[a_m][S]_1[S]_2[S]_3\,Y$$

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \lambda \begin{bmatrix} 0 & 0 & 0 & -1.3815 \\ 0 & 0 & 0 & 2.2357 \\ 0 & 0 & 0 & -2.2361 \\ 0 & 0 & 0 & 1.3820 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

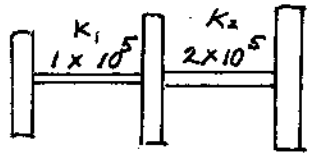
$$\omega_4^2 = \frac{5}{1.382}\,\frac{T}{m\ell} = 3.6178\,\frac{T}{m\ell}$$

$$\omega_4 = 1.902\sqrt{\frac{T}{m\ell}}$$



3 nodes
antisymmetric

12 – 1   Using the HP-25 programmable calculator the following simple program consisting of 36 program steps is written



$J_1 = 5$ $J_2 = 11$ $J_3 = 22$

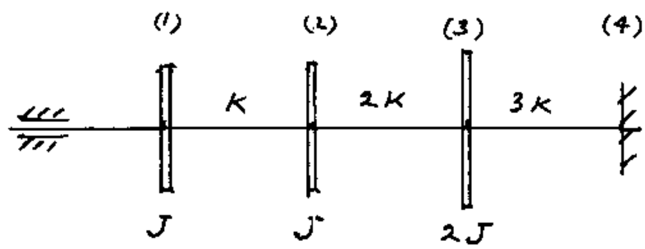
	Program Mode	Program Gives	Run Mode	Disply
00	$\boxed{f}\ \boxed{\text{PRGM}}$		$\boxed{f}\ \boxed{\text{PRGM}}$	
02	1 $\boxed{\uparrow}$	θ_1	ω $\boxed{g}$ 2	
04	5 $\boxed{\uparrow}$	J_1	$\boxed{\text{STO}}$ 0	
06	$\boxed{\text{RCL}}$ 0 $\boxed{\times}$	$\omega^2 J_1$	$\boxed{\text{R/S}}$	T_3
07	$\boxed{\text{STO}}$ 1	$\omega^2 J_1$		
12	1 $\boxed{\text{EEX}}$ 5 $\boxed{\div}$ $\boxed{-}$	$\theta_2 = 1 - \frac{\omega^2 J_1}{K_1}$	$\boxed{\text{RCL}}$ 2	θ_2
13	$\boxed{\text{STO}}$ 2	θ_2	$\boxed{\text{RCL}}$ 4	θ_3
14	11 $\boxed{\times}$	$J_2\theta_2$	$\boxed{\text{RCL}}$ 0	ω^2
18	$\boxed{\text{RCL}}$ 0 $\boxed{\times}$	$\omega^2 J_2 \theta_2$	$\boxed{f}\ \boxed{\sqrt{\ }}$	ω
20	$\boxed{\text{RCL}}$ 1 $\boxed{+}$	$T_2 = \omega^2 J_1 + \omega^2 J_2 \theta_2$		
21	$\boxed{\text{STO}}$ 3	T_2		
25	2 $\boxed{\text{EEX}}$ 5 $\boxed{\div}$	T_2/K_2		
28	$\boxed{\text{RCL}}$ 2 $\boxed{-}$ $\boxed{\text{CHS}}$	$\theta_3 = \theta_2 - \frac{T_2}{K_2}$		
29	$\boxed{\text{STO}}$ 4	θ_3		
32	22 $\boxed{\times}$	$J_3\theta_3$		
34	$\boxed{\text{RCL}}$ 0 $\boxed{\times}$	$\omega^2 J_3 \theta_3$		
36	$\boxed{\text{RCL}}$ 3 $\boxed{+}$	$T_3 = T_2 + \omega^2 J_3 \theta_3$	Newton meters	

The first column indicates the number of steps (49 are allowed). After the program is keyed in, switch to the Run Mode. Choosing any ω, ω^2 is stored in space 0 and by pressing $\boxed{\text{R/S}}$ the calculations in the program mode is executed automatically with T_3 displayed. $\boxed{\text{RCL}}$ 2 and $\boxed{\text{RCL}}$ 4 then displays θ_2 and θ_3 respectively. To repeat for any other frequency it is only necessary to input the new ω in the run mode and press $\boxed{\text{R/S}}$.

12-1 Cont. The following ω_s were keyed in to obtain T_3 and θ_2 θ_3 at natural frequencies

ω	$T_3 \times 10^{-3}$	θ_1	θ_2	θ_3
			Mode Shape	
20	14.66			
40	52.32 ($T_3 = 52.32\times10^3$ Nm)			
60	95.43			
80	119.38			
100	99.50			
120	20.75			
123.666	−.00118	1.00	0.2353	−0.3449
140.	−109.65			
160	−246.96			
180	−290.74			
200	−64.00			
202	−16.78			
202.6584	.00134	1.00	−1.0535	0.2995
203	8.95			
205	64.89			
220	710.55 Nm			

12-2



$J = 1\ kg.m^2$

$K = 0.2\times10^6\ \frac{Nm}{r.}$

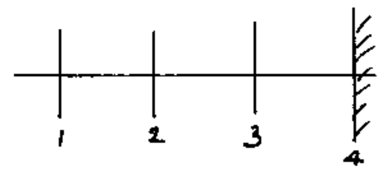
This problem may be solved by digital computer using Eqs. (12.2-1) & (12.2-2), however for these simple systems, the hand calculator can be easily programmed, as in Prob. 12-1. Only slight changes are necessary.

1st mode was also obtained by matrix iteration as $\omega_1 = 281.3$

Eq. for iteration →
$$\begin{Bmatrix}\theta_1\\\theta_2\\\theta_3\end{Bmatrix} = \frac{J\omega^2}{6K}\begin{bmatrix}11&5&2\\5&5&2\\2&2&2\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}\begin{Bmatrix}\theta_1\\\theta_2\\\theta_3\end{Bmatrix}$$

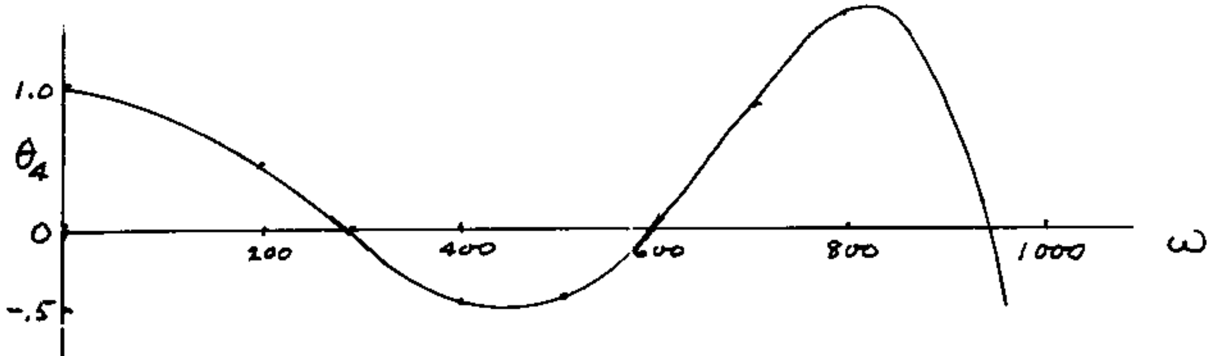
12-2 Cont: Use same computer program HP-25 as in Prob. 1-1 with following changes.

		Run Mode	Display	
00				
02				
04	1 $\boxed{\uparrow}$	Same	$\boxed{R/S}$	θ_4
06			$\boxed{RCL}$ 2	θ_2
07			$\boxed{RCL}$ 4	θ_3
12	2 $\boxed{EEX}$ 5 $\boxed{\div}$ $\boxed{-}$			
13				
14	1 $\boxed{\times}$			
18				
20				
21				
25	4 $\boxed{EEX}$ 5 $\boxed{\div}$			
28				
29				
32	2 $\boxed{\times}$			
34				
36				
40	6 $\boxed{EEX}$ 5 $\boxed{\div}$			
44	$\boxed{RCL}$ 4 $\boxed{-}$ $\boxed{CHS}$			



Results:

ω	θ_1	θ_2	θ_3	θ_4	
20	1.00	.9980	.9960	.9933	
200				.417	
281.24	1.00	.6045	.2872	−.00000812	1st Mode
400				−.4507	
500				−.432	
589.368	1.00	−.7368	−.9654	−.0000082	2nd Mode
700.				.936	
800				1.597	
900				.8954	
934.6382	1.00	−3.3677	1.8031	.0000067	3rd Mode



12-3

```
C       PROBLEM 10.3 THOMSON
        DIMENSION RJ(4),RK(4),W(60),DE(60,4),T(60,4),TF(60)
        M=4
        L=60
        READ5,(RJ(J),J=1,M)
      5 FORMAT(4F10.3)
        READ6,(RK(J),J=1,M)
      6 FORMAT(4E10.3)
        DO 20 I=1,L
        DE(I,1)=1.
        W(I)=(I-1)*10
        T(I,1)=W(I)**2*RJ(1)*DE(I,1)
        DO 10 J=2,M
        DE(I,J)=DE(I,J-1)-T(I,J-1)/RK(J-1)
        T(I,J)=T(I,J-1)+W(I)**2*RJ(J)*DE(I,J)
     10 CONTINUE
     20 CONTINUE
        DO 25 J=1,M
        PRINT24,J,RJ(J),RK(J)
     24 FORMAT(20X,I3,5X,F8.3,5X,E8.3)
     25 CONTINUE
        DO 40 I=1,L
        TF(I)=T(I,M)
        PRINT30,W(I),DE(I,M),T(I,M)
     30 FORMAT(10X,F8.2,10X,F12.4,10X,F12.4)
     40 CONTINUE
        CALL EZPLOT(W,TF,L)
        CALL FINISH
        STOP
        END
```

n	J	K
1	1.130	.169E+06
2	1.130	.169E+06
3	1.130	.226E+06
4	2.260	.0

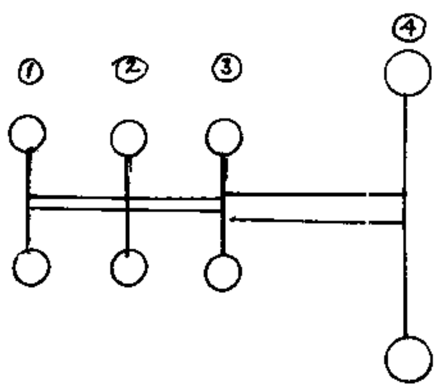
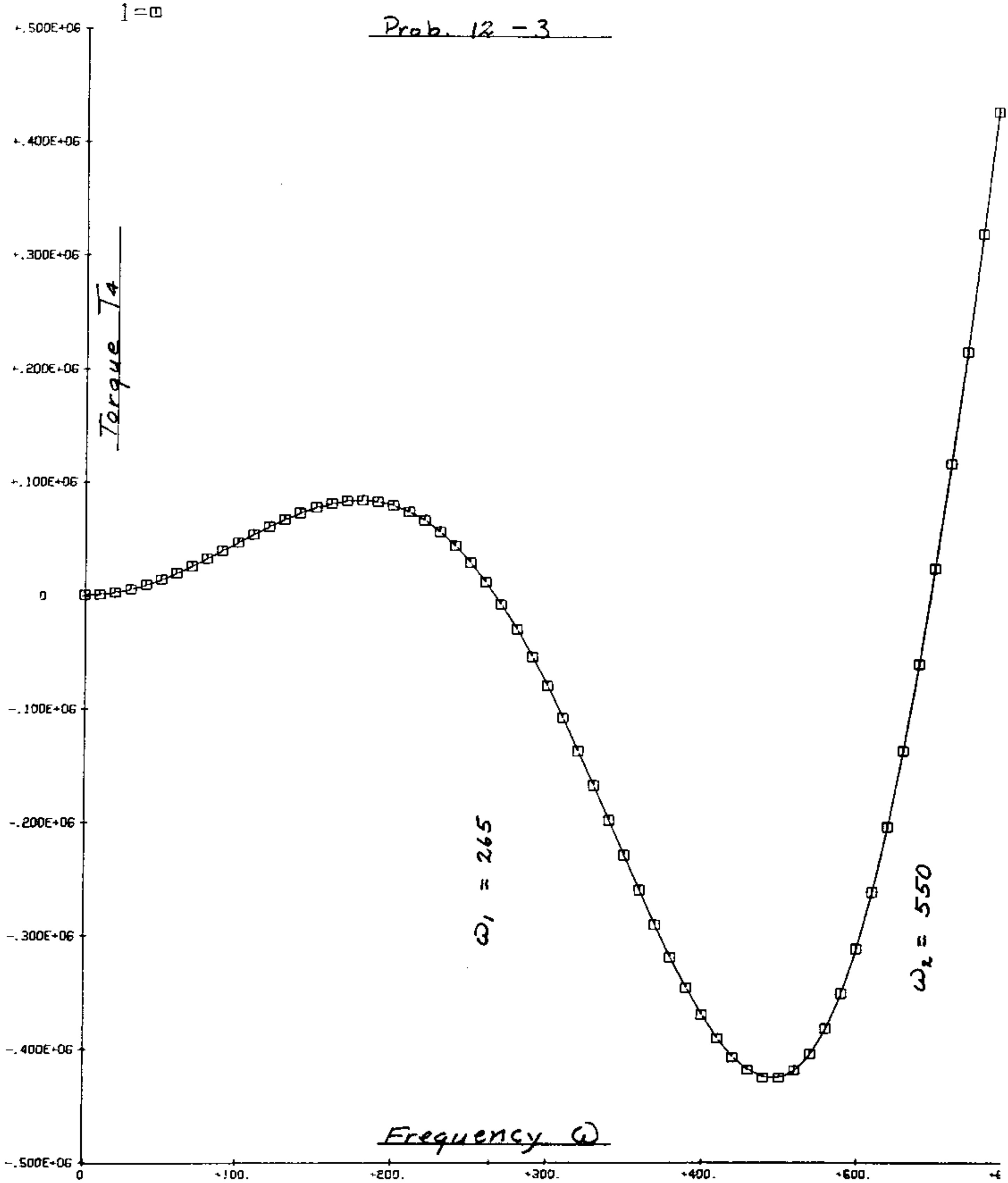
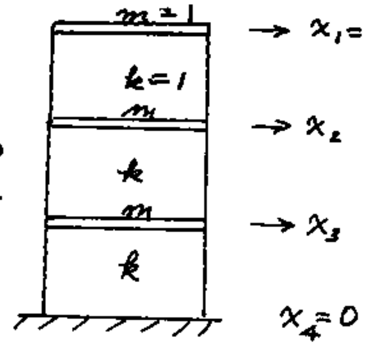


FIG. P 12-3

12-3 ω	θ_4	T_4
0.0	1.0000	0.0
10.00	0.9965	563.9058
20.00	0.9860	2242.5156
30.00	0.9686	4996.6680
40.00	0.9444	8761.6445
50.00	0.9135	13447.9648
60.00	0.8761	18942.5312
70.00	0.8325	25110.0469
80.00	0.7829	31794.7812
90.00	0.7276	38822.5586
100.00	0.6670	46003.1094
110.00	0.6015	53132.5898
120.00	0.5315	59996.4297
130.00	0.4574	66372.3125
140.00	0.3797	72033.5000
150.00	0.2990	76752.1875
160.00	0.2157	80303.0000
170.00	0.1304	82466.8750
180.00	0.0438	83034.6875
190.00	-0.0436	81811.2500
200.00	-0.1312	78618.8750
210.00	-0.2183	73301.9375
220.00	-0.3042	65729.9375
230.00	-0.3884	55801.7617
240.00	-0.4701	43449.1133
250.00	-0.5488	28640.1875
260.00	-0.6236	11382.6250
270.00	-0.6941	-8273.0625
280.00	-0.7596	-30231.3125
290.00	-0.8194	-54348.7500
300.00	-0.8730	-80432.5000
310.00	-0.9197	-108238.687
320.00	-0.9590	-137471.437
330.00	-0.9905	-167782.687
340.00	-1.0137	-198772.437
350.00	-1.0280	-229990.312
360.00	-1.0332	-260936.125
370.00	-1.0290	-291063.687
380.00	-1.0150	-319784.562
390.00	-0.9911	-346471.562
400.00	-0.9571	-370465.062
410.00	-0.9131	-391080.000
420.00	-0.8591	-407613.687
430.00	-0.7952	-419355.500
440.00	-0.7216	-425595.937
450.00	-0.6387	-425642.500
460.00	-0.5470	-418828.125
470.00	-0.4471	-404532.187
480.00	-0.3395	-382193.375
490.00	-0.2253	-351330.875
500.00	-0.1054	-311564.437
510.00	0.0192	-262637.062
520.00	0.1470	-204442.125
530.00	0.2767	-137046.375
540.00	0.4066	-60724.3125
550.00	0.5349	24013.4375
560.00	0.6595	116383.625
570.00	0.7782	215294.125


1=□
Prob. 12 -3
Torque T_4
+.500E+06
+.400E+06
+.300E+06
+.200E+06
+.100E+06
0
-.100E+06
-.200E+06
-.300E+06
-.400E+06
-.500E+06
0
+100.
+200.
+300.
+400.
+600.
$\omega_1 = 265$
$\omega_2 = 550$
Frequency ω

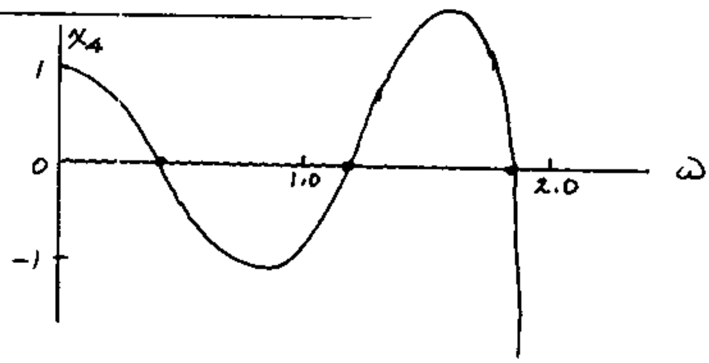
12 - 4 Programmed for HP-25



Program Mode			Run Mode	
00	$\boxed{f}$ $\boxed{PRGM}$		$\boxed{f}$ $\boxed{PRGM}$	
02	1 $\boxed{\uparrow}$	$x_1=1$, $m_1=1$	ω $\boxed{\uparrow}$	$=\omega$
04	$\boxed{RCL}$ 0 $\boxed{-}$	$1-\omega^2 m_1 = x_2$	$\boxed{g}$ 2	$=\omega^2$
05	$\boxed{STO}$ 1	x_2	$\boxed{STO}$ 0	$=\omega^2$
07	1 $\boxed{+}$	$(1+x_2)$	$\boxed{R/S}$	$=x_4$
09	$\boxed{RCL}$ 0 $\boxed{\times}$	$\omega^2(1+x^2)=F_2$	$\boxed{RCL}$ 2	$=x_3$
12	$\boxed{RCL}$ 1 $\boxed{-}$ $\boxed{CHS}$	$x_3=x_2-F_2$	$\boxed{RCL}$ 1	$=x_2$
13	$\boxed{STO}$ 2	x_3		
15	$\boxed{RCL}$ 1 $\boxed{+}$	x_2+x_3		
17	1 $\boxed{+}$	$1+x_2+x_3$		
19	$\boxed{RCL}$ 0 $\boxed{\times}$	$\omega^2(1+x_2+x_3)=F_3$		
22	$\boxed{RCL}$ 2 $\boxed{-}$ $\boxed{CHS}$	$=x_4$		

Freq. Plot. Scan.

	ω	x_4
	.2	.768
$\omega_1 \rightarrow$	.6	-.5587
	.8	-1.054
	1.0	-1.000
$\omega_2 \rightarrow$	1.5	1.422
	1.7	1.283
$\omega_3 \rightarrow$	1.9	-2.545



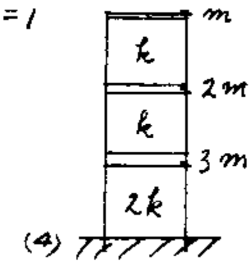
ω	x_4	x_3	x_2	x_1	
$\omega_1=$ .44504	.0000068	.4450	.8019	1.00	$\therefore\ \omega_1 = .44504\sqrt{\frac{k}{m}}$
$\omega_2=$ 1.247	.0001	-1.247	-.555	1.00	$\omega_2 = 1.247\sqrt{\frac{k}{m}}$
$\omega_3=$ 1.802	-.0012	1.8027	-2.247	1.00	$\omega_3 = 1.802\sqrt{\frac{k}{m}}$

Mode Shapes

12 - 5

$F_1 = m\omega^2 1$ $\qquad x_2 = 1 - \frac{m\omega^2}{k}$

$x_1 = 1$



$F_2 = 2m\omega^2 x_2$

$F_1 + F_2 = m\omega^2 + 2m\omega^2 x_2 \qquad x_3 = x_2 - \frac{F_1 + F_2}{k}$

$F_3 = 3m\omega^2 x_3$

$F_1 + F_2 + F_3 = m\omega^2 + 2m\omega^2 x_2 + 3m\omega^2 x_3$

$$x_4 = x_3 - \frac{F_1 + F_2 + F_3}{2k}$$

$\boxed{Prog}$ $\boxed{f}$ $\boxed{PRGM}$

1 $\boxed{\uparrow}$

$\boxed{RCL}$ 0 $\boxed{-}$ $\boxed{STO}$ 1 $= x_2$

2 $\boxed{\times}$ 1 $\boxed{+}$ $\boxed{STO}$ 2

$\boxed{RCL}$ 0 $\boxed{\times}$ $\boxed{STO}$ 3

$\boxed{RCL}$ 1 $\boxed{RCL}$ 3 $\boxed{-}$ $= x_3$

$\boxed{STO}$ 4 $= x_3$

3 $\boxed{\times}$ $\boxed{RCL}$ 2 $\boxed{+}$

$\boxed{RCL}$ 0 $\boxed{\times}$ 2 $\boxed{\div}$ $\boxed{CHS}$

$\boxed{RCL}$ 4 $\boxed{+}$ $= x_4$

$\boxed{STO}$ 5 $= x_4$

Run Mode

$\boxed{f}$ $\boxed{PRGM}$

ω $\boxed{\uparrow}$

$\boxed{g}$ 2 $= \omega^2$

$\boxed{STO}$ 0 $= \omega^2$

$\boxed{R/S}$ $= x_4$

$\boxed{RCL}$ 1 $= x_2$

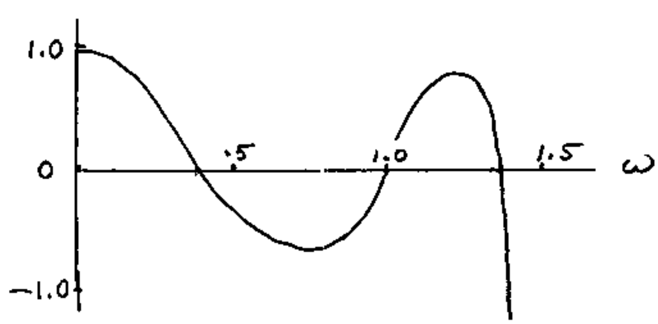
$\boxed{RCL}$ 4 $= x_3$

Result only of nat. freq.

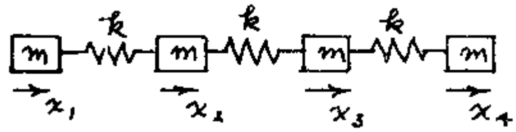
n	ω_n	x_4	x_3	x_2	x_1
1	.4385	−.0004	.3330	.8164	1.00
2	1.000	.000	−1.00	.000	1.00
3	1.3478	.0003	.3336	−.8166	1.00

$\therefore \omega_1 = .4385\sqrt{\frac{k}{m}}$

etc.



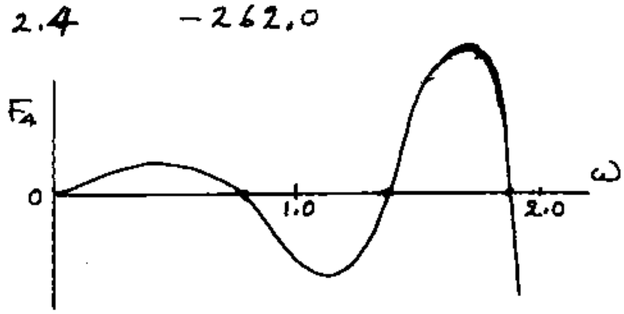
12-7 Same program as Prob 10-4 but extended to $F_4 = 0$



Program identical through 22

22	- - - - - - -	
23	$\boxed{\text{STO}}$ 3	x_4
25	$\boxed{\text{RCL}}$ 2 $\boxed{+}$	$x_3 + x_4$
27	$\boxed{\text{RCL}}$ 1 $\boxed{+}$	$x_2 + x_3 + x_4$
29	1 $\boxed{+}$	$1 + x_2 + x_3 + x_4$
31	$\boxed{\text{RCL}}$ 0 $\boxed{\times}$	$\omega^2(1 + x_2 + x_3 + x_4) = F_4$

ω	F_4	x_4	x_3	x_2	x_1
			Mode Shape		
.01	.0004				
.2	.144				
.4	.408				
.6	.407				
.76537	-.000011	-1.00	-.4142	.4142	1.00
.8	-.131				
1.0	-1.000				
1.4	-1.567				
1.4142	-.0002	1.00	-1.00	-1.00	1.00
1.6	2.417				
1.8	1.858				
1.848	.0	-1.00	2.418	-2.418	1.00
2.0	-16.00				
2.4	-262.0				

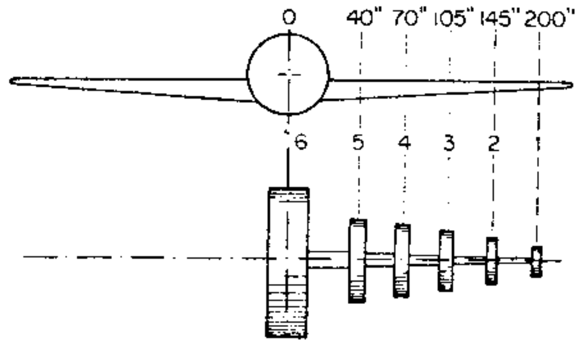


$\omega_1 = .76537 \sqrt{\frac{k}{m}}$

$\omega_2 = 1.4142 \sqrt{\frac{k}{m}}$

$\omega_3 = 1.848 \sqrt{\frac{k}{m}}$

n	J lb. in. sec.^{2}	K lb. in./rad.
1	50	15×10^6
2	138	30
3	145	22
4	181	36
5	260	120
6	$\frac{1}{2} \times 140{,}000$	



Sta.	J	K
1	50.000	0.150E+08
2	138.000	0.300E+08
3	145.000	0.220E+08
4	181.000	0.360E+08
5	260.000	0.120E+09
6	70000.000	0.0

```
      PROBLEM 10.9 THOMSON,ANTISYMMETRIC
      DIMENSION RJ(15),RK(15),W(200),DE(200,15),T(200,15),TF(200)
      M=6
      L=100
      READ5,(RJ(J),J=1,M)
    5 FORMAT(6F10.3)
      READ6,(RK(J),J=1,M)
    6 FORMAT(6E10.3)
      DO 20 I=1,L
      DE(I,1)=1.
      W(I)=(I-1)*10
      T(I,1)=W(I)**2*RJ(1)*DE(I,1)
      DO 10 J=2,M
      DE(I,J)=DE(I,J-1)-T(I,J-1)/RK(J-1)
      T(I,J)=T(I,J-1)+W(I)**2*RJ(J)*DE(I,J)
   10 CONTINUE
   20 CONTINUE
      DO 25 J=1,M
      PRINT24,J,RJ(J),RK(J)
   24 FORMAT(20X,I3,5X,F10.3,5X,E10.3)
   25 CONTINUE
      DO 40 I=1,L
      TF(I)=T(I,M)
      PRINT30,W(I),DE(I,M),T(I,M)
   30 FORMAT(10X,F8.2,10X,F12.4,10X,E12.4)
   40 CONTINUE
      CALL EZPLOT(W,TF,L)
      CALL FINISH
      STOP
      END
```

12-B Cont.

ω	Θ_6	T_6
0.0	1.0000	0.0
10.00	0.9955	0.7045E+07
20.00	0.9819	0.2780E+08
30.00	0.9594	0.6113E+08
40.00	0.9283	0.1052E+09
50.00	0.8889	0.1574E+09
60.00	0.8416	0.2147E+09
70.00	0.7870	0.2733E+09
80.00	0.7256	0.3294E+09
90.00	0.6581	0.3784E+09
100.00	0.5852	0.4158E+09
110.00	0.5078	0.4372E+09
120.00	0.4267	0.4381E+09
130.00	0.3428	0.4143E+09
140.00	0.2570	0.3621E+09
150.00	0.1702	0.2781E+09
160.00 ω_1 Antisym.	0.0834	0.1600E+09
170.00 ω_1 Symm.	-0.0023	0.5965E+07
180.00	-0.0862	-0.1848E+09
190.00	-0.1672	-0.4122E+09
200.00	-0.2446	-0.6748E+09
210.00	-0.3173	-0.9702E+09
220.00	-0.3846	-0.1295E+10
230.00	-0.4459	-0.1644E+10
240.00	-0.5004	-0.2012E+10
250.00	-0.5475	-0.2392E+10
260.00	-0.5869	-0.2775E+10
270.00	-0.6180	-0.3154E+10
280.00	-0.6406	-0.3519E+10
290.00	-0.6546	-0.3859E+10
300.00	-0.6598	-0.4166E+10
310.00	-0.6564	-0.4427E+10
320.00	-0.6445	-0.4634E+10
330.00	-0.6244	-0.4777E+10
340.00	-0.5966	-0.4848E+10
350.00	-0.5615	-0.4837E+10
360.00	-0.5197	-0.4740E+10
370.00	-0.4721	-0.4552E+10
380.00	-0.4195	-0.4270E+10
390.00	-0.3628	-0.3894E+10
400.00	-0.3030	-0.3426E+10
410.00	-0.2411	-0.2871E+10
420.00	-0.1783	-0.2236E+10
430.00	-0.1157	-0.1532E+10
440.00	-0.0545	-0.7718E+09
450.00	0.0042	0.2731E+08
460.00	0.0592	0.8467E+09
470.00	0.1094	0.1665E+10
480.00	0.1539	0.2457E+10
490.00	0.1916	0.3199E+10
500.00	0.2218	0.3863E+10

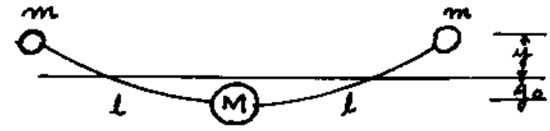
12-8 Cont.

ω	Θ_6	T_6
510.00	0.2436	0.4422E+10
520.00	0.2566	0.4848E+10
530.00	0.2603	0.5115E+10
540.00	0.2546	0.5198E+10
550.00	0.2394	0.5075E+10
560.00	0.2149	0.4729E+10
570.00	0.1817	0.4147E+10
580.00	0.1403	0.3323E+10
590.00	0.0918	0.2258E+10
600.00	0.0372	0.9615E+09
610.00	-0.0219	-0.5465E+09
620.00	-0.0840	-0.2237E+10
630.00	-0.1472	-0.4068E+10
640.00	-0.2095	-0.5989E+10
650.00	-0.2688	-0.7935E+10
660.00	-0.3227	-0.9833E+10
670.00	-0.3690	-0.1160E+11
680.00	-0.4056	-0.1314E+11
690.00	-0.4302	-0.1436E+11
700.00	-0.4411	-0.1516E+11
710.00	-0.4366	-0.1546E+11
720.00	-0.4159	-0.1515E+11
730.00	-0.3784	-0.1419E+11
740.00	-0.3247	-0.1253E+11
750.00	-0.2563	-0.1018E+11
760.00	-0.1759	-0.7202E+10
770.00	-0.0878	-0.3728E+10
780.00	0.0010	0.1259E+08
790.00	0.0849	0.3671E+10
800.00	0.1506	0.6753E+10
810.00	0.1853	0.8583E+10
820.00	0.1722	0.8265E+10
830.00	0.0905	0.4643E+10
840.00	-0.0848	-0.3753E+10
850.00	-0.3835	-0.1876E+11
860.00	-0.8410	-0.4266E+11

Comments: System was cut at centerline using $\frac{1}{2}$ the J of the fuselage. Antisymmetric modes are given for $\Theta_6 = 0$ and Symmetric modes for $T_6 = 0$. Note that they are very close to each other because of the very large J_6

12 - 9

From conservation of momentum



$$M y_o = 2m y \qquad \frac{M}{m} = n$$

$$(y + y_o) = \frac{P\ell^3}{3EI} = \frac{(m\omega^2 y)\ell^3}{3EI} = y + \frac{2m}{M} y = y\left(1 + \frac{2}{n}\right)$$

$$\therefore \ \omega^2 = \frac{3EI}{M\ell^3}(n+2) = \frac{6EI}{M\ell^3}\left(1 + \frac{n}{2}\right) \qquad \omega = \sqrt{\frac{6EI}{M\ell^3}\left(1+\frac{n}{2}\right)}$$

---

12 - 10 Using Eq. 12.9 - 5

$$\begin{Bmatrix} -V_3 \\ M_3 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & m\omega^2 \\ \ell & 1 & 0 & m\omega^2\ell \\ 3\alpha\ell & 6\alpha & 1 & m\omega^2 3\alpha\ell \\ \alpha\ell^2 & 3\alpha\ell & \ell & (1+m\omega^2\alpha\ell^2) \end{bmatrix}^2 \begin{Bmatrix} -0 \\ 0 \\ 1 \\ \theta_1 \end{Bmatrix} \qquad \text{where } \alpha = \frac{\ell}{6EI}$$

Calculation can be limited to last 2 rows of last two columns.

$$\left.\begin{aligned} u_{33} + u_{34}\theta_1 &= 0 \\ u_{43} + u_{44}\theta_1 &= 0 \end{aligned}\right\} \text{ or } \quad \underbrace{u_{33}u_{44} - u_{34}u_{43} = 0}_{\text{freq. eq.}}$$

$$\text{For } A^2 = AA = U, \qquad u_{ij} = \sum_k a_{ik} a_{kj}$$

$$\therefore \ u_{33} = \sum_k a_{3k} a_{k3} = 1 + m\omega^2 3\alpha\ell^2$$

$$u_{44} = \sum_k a_{4k} a_{k4} = m\omega^2\alpha\ell^2 + m\omega^2 3\alpha\ell^2 + m\omega^2 3\alpha\ell^2 + (1+m\omega^2\alpha\ell^2)^2$$
$$= 1 + 9m\omega^2\alpha\ell^2 + (m\omega^2\alpha\ell^2)^2$$

$$u_{34} = \sum_k a_{3k} a_{k4} = m\omega^2 3\alpha\ell + m\omega^2 6\alpha\ell + m\omega^2 3\alpha\ell + m\omega^2 3\alpha\ell(1+m\omega^2\alpha\ell^2)$$
$$= 15 m\omega^2\alpha\ell + 3(m\omega^2\alpha\ell)^2\ell$$

$$u_{43} = \ell + \ell(1+m\omega^2\alpha\ell^2) = 2\ell + m\omega^2\alpha\ell^3$$

12-10 Cont: subst. into freq. eq.

$$(1+m\omega^2 3\alpha\ell^2)\left[1+9\omega^2 m\alpha\ell^2+(m\omega^2\alpha\ell^2)^2\right]$$
$$+\ (2\ell+m\omega^2\alpha\ell^3)\left[-15m\omega^2\alpha\ell-3\ell(m\omega^2\alpha\ell)^2\right] = 0$$

or $$1-18(m\omega^2\alpha\ell^2)+7(m\omega^2\alpha\ell^2)^2 = 0$$

let $\beta = m\omega^2\alpha\ell^2$ then $\beta^2-\frac{18}{7}\beta+\frac{1}{7} = 0$

$$\beta = 1.2857 \pm \sqrt{1.6531-.1429} = 1.2857 \pm 1.2289$$

$$\beta = \begin{cases} .0568 \\ 2.5615 \end{cases} \qquad \sqrt{\beta} = \begin{cases} .2384 \\ 1.5857 \end{cases} \qquad \omega = \left\{ \begin{matrix} .584 \\ 3.884 \end{matrix} \right\} \sqrt{\frac{EI}{m\ell^3}}$$

12-11

Fortran H program given here can be used for other 3 mass cantilever system.

If all m_s are equal & all ℓ_s are equal, the results for $m=1$, $\ell=1$, and $EI = 1\times10^6$ can be used for other values of m and ℓ and EI by evaluating β_1 and β_2.

SYSTEM/370 FORTRAN H EXTENDED (ENHANCED)

```
      PROBLEM 10.12 THOMSON
      DIMENSION ZV(2,4),ZM(2,4),ZD(2,4),ZY(2,4),WM(3),WL(3),WEI(3),
     1DE(200),DY(200),W(200)
      M=4
      L=200
      N=M-1
      READ5,(WM(J),J=1,N)
    5 FORMAT(3F10.3)
      READ6,(WL(J),J=1,N)
    6 FORMAT(3F10.3)
      READ7,(WEI(J),J=1,N)
    7 FORMAT(3E10.3)
      ZD(1,1)=0.0
      ZD(2,1)=1.0
      ZY(1,1)=1.0
      ZY(2,1)=0.0
      DO 60 K=1,L
      W(K)=(K-1)*10.
      DO 50 I=1,2
      ZV(I,1)=0.0
      ZM(I,1)=0.0
      DO 40 J=2,M
      ZV(I,J)=W(K)**2*WM(J-1)*ZY(I,J-1)+ZV(I,J-1)
      ZM(I,J)=W(K)**2*WM(J-1)*WL(J-1)*ZY(I,J-1)+ZM(I,J-1)+WL(J-1)*
     1ZV(I,J-1)
      ZD(I,J)=W(K)**2*WM(J-1)*WL(J-1)**2/(2.*WEI(J-1))*ZY(I,J-1)+
     1ZD(I,J-1)+WL(J-1)/WEI(J-1)*ZM(I,J-1)+WL(J-1)**2/(2.*WEI(J-1))*
     1ZV(I,J-1)
      ZY(I,J)=(1.+W(K)**2*WM(J-1)*WL(J-1)**3/(6.*WEI(J-1)))*ZY(I,J-1)+
     1WL(J-1)*ZD(I,J-1)+WL(J-1)**2/(2.*WEI(J-1))*ZM(I,J-1)+WL(J-1)**3/
     1(6.*WEI(J-1))*ZV(I,J-1)
   40 CONTINUE
   50 CONTINUE
      DE(K)=-ZD(1,M)/ZD(2,M)
      DY(K)=ZY(1,M)+ZY(2,M)*DE(K)
   60 CONTINUE
      DO 70 J=1,N
      PRINT65,WM(J),WL(J),WEI(J)
   65 FORMAT(10X,F10.3,5X,F10.3,5X,E10.3)
   70 CONTINUE
      DO 80 K=1,L
      PRINT75,W(K),DY(K)
   75 FORMAT(20X,F8.2,5X,F12.4)
   80 CONTINUE
      CALL EZPLOT(W,DY,L)
      CALL FINISH
      STOP
      END
```

m	ℓ	EI
1.000	1.000	0.100E+07
1.000	1.000	0.100E+07
1.000	1.000	0.100E+07

ω	y_4	ω	y_4
150.00	0.6955	1760.00	-0.4357
160.00	0.6456	1770.00	-0.4084
170.00	0.6038	1780.00	-0.3808
180.00	0.5605	1790.00	-0.3532
190.00	0.5156	1800.00	-0.3255
200.00	0.4694	1810.00	-0.2977
210.00	0.4220	1820.00	-0.2698
220.00	0.3734	1830.00	-0.2418
230.00	0.3239	1840.00	-0.2137
240.00	0.2735	1850.00	-0.1855
250.00	0.2224	1860.00	-0.1572
260.00	0.1707	1870.00	-0.1289
270.00	0.1185	1880.00	-0.1005
280.00	0.0660	1890.00	-0.0720
290.00 $\leftarrow \omega_1$	0.0131	1900.00	-0.0434
300.00	-0.0398	1910.00 $\leftarrow \omega_2$	-0.0148
310.00	-0.0929	1920.00	0.0139
320.00	-0.1458	1930.00	0.0427
330.00	-0.1987	1940.00	0.0715
340.00	-0.2513	1950.00	0.1004
350.00	-0.3036	1960.00	0.1293
360.00	-0.3556	1970.00	0.1583
370.00	-0.4071	1980.00	0.1873
380.00	-0.4581	1990.00	0.2164

other data deleted.

Data obtained for $m = 1$, $\ell = 1$, and $EI = 1 \times 10^6$

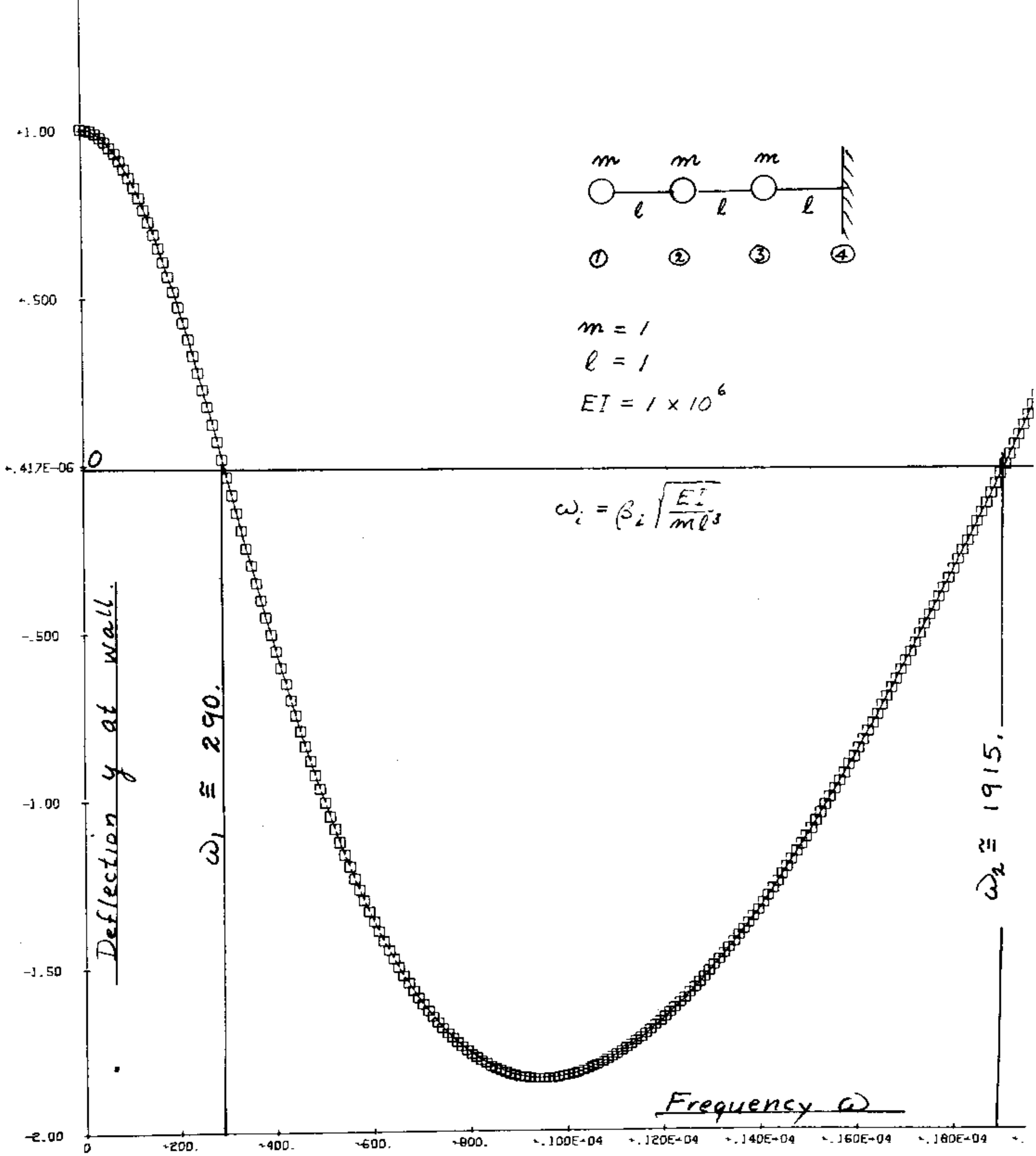
For other values of m, ℓ, and EI, first evaluate β from $\omega_i = \beta_i \sqrt{\frac{EI}{m\ell^3}}$. This β can then be used for other m, ℓ, & EI.

i.e. for 1st mode $\omega_1 = \beta_1 \sqrt{\frac{10^6}{1}}$ $\therefore$ $\beta_1 = \omega_1 \times 10^{-3}$

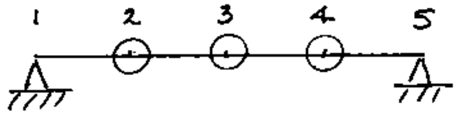
$\cong 0.290$ From Prob 11-22 $\omega_1 = 0.2925 \sqrt{\frac{EI}{m\ell^3}}$

$\omega_2 = 1.909 \sqrt{\frac{EI}{m\ell^3}}$ β_2 from computer sol. $= 1.915$

Prob. 12-11 Cont:



12 - 12



stations must be numbered as 1 to 5 with

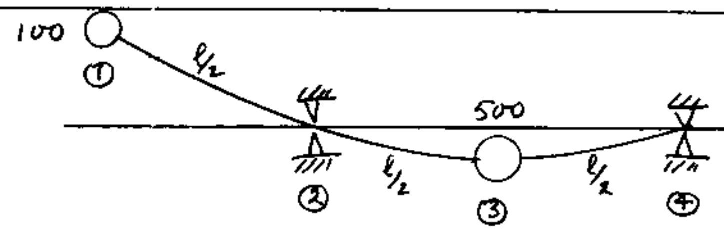
$m_1 = m_5 = 0$

$$\begin{Bmatrix} -V \\ 0 \\ \theta \\ 0 \end{Bmatrix}_5 = \left[\begin{array}{cc|cc} - & - & - & - \\ u_{21} & - & u_{23} & - \\ - & - & - & - \\ u_{41} & - & u_{43} & - \end{array}\right] \begin{Bmatrix} -V \\ 0 \\ \theta \\ 0 \end{Bmatrix}_1$$

$$\therefore \quad \begin{vmatrix} u_{21} & u_{23} \\ u_{41} & u_{43} \end{vmatrix} = 0 \quad \text{or} \quad u_{43} = \frac{u_{41}\, u_{23}}{u_{21}}$$

---

12 - 13



$m_2 = m_4 = 0$

$y_1 = 1.0$

$y_2 = y_4 = 0$

Eq. 1, 1-5

$$\begin{Bmatrix} -V \\ M \\ \theta \\ 0 \end{Bmatrix}_2 = \begin{bmatrix} \text{sec } 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta_1 \\ 1 \end{Bmatrix}_1 \qquad \begin{Bmatrix} -V \\ M \\ \theta \\ y \end{Bmatrix}_3 = \begin{bmatrix} \text{sec. } 2 \end{bmatrix} \begin{Bmatrix} -V \\ M \\ \theta \\ 0 \end{Bmatrix}_2 \quad \text{etc}$$

this is constraint eq.

$$\begin{Bmatrix} -V \\ 0 \\ \theta \\ 0 \end{Bmatrix}_4 = \begin{bmatrix} 0\ 0 & \\ 0\ \text{sec } 1 & \\ 0\ 0 & \\ 0\ 0 & \end{bmatrix} \begin{bmatrix} & 0 \\ \text{sec } 2 & 0 \\ & 0 \\ & 0 \end{bmatrix} \begin{bmatrix} \text{sec } 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta_1 \\ 1 \end{Bmatrix}_1 = \begin{bmatrix} - & - & - & - \\ - & - & u_{23} & u_{24} \\ - & - & - & - \\ - & - & u_{43} & u_{44} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta_1 \\ 1 \end{Bmatrix}$$

$$\begin{vmatrix} u_{23} & u_{24} \\ u_{43} & u_{44} \end{vmatrix} = 0$$

12-14 Eqs. 12.4-1 to 6 can be arranged in the following matrix form

$$\begin{Bmatrix} -V \\ M \\ \theta \\ y \\ \hdashline T \\ \varphi \end{Bmatrix}_{i+1} = \left[\begin{array}{cccc:cc} 1 & 0 & 0 & m\omega^2 & 0 & m\omega^2 c \\ \ell & 1 & 0 & m\omega^2\ell & 0 & m\omega^2 c\ell \\ \frac{\ell^2}{2EI} & \frac{\ell}{EI} & 1 & \frac{m\omega^2\ell^2}{2EI} & 0 & \frac{m\omega^2 c\ell^2}{2EI} \\ \frac{\ell^2}{6EI} & \frac{\ell^2}{2EI} & \ell & \left(1+\frac{m\omega^2\ell^3}{6EI}\right) & 0 & \frac{m\omega^2 c\ell^3}{6EI} \\ \hdashline 0 & 0 & 0 & m\omega^2 c & 1 & J\omega^2 \\ 0 & 0 & 0 & m\omega^2 c h & h & (1+J\omega^2 h) \end{array}\right]_i \begin{Bmatrix} -V \\ M \\ \theta \\ y \\ \hdashline T \\ \varphi \end{Bmatrix}_i$$

$$\begin{Bmatrix} V \\ \hdashline W \end{Bmatrix}_3 = \left[\begin{array}{c:c} A & 2B \\ \hdashline 2C & D \end{array}\right]\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right]\begin{Bmatrix} V \\ \hdashline W \end{Bmatrix}_1$$

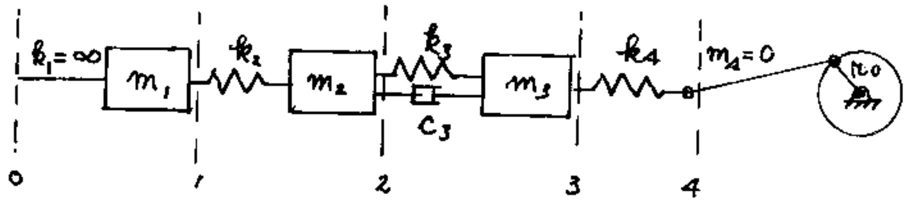
section 2 differs from Sec 1 by $2c_i$ in place of c_i.

$$= \left[\begin{array}{c:c} (A^2+2BC) & (AB+2BD) \\ \hdashline (2AC+CD) & (2BC+D^2) \end{array}\right]\begin{Bmatrix} V \\ \hdashline W \end{Bmatrix}_1$$

$$\begin{Bmatrix} -V_3 \\ M_3 \\ 0 \\ 0 \\ T_3 \\ 0 \end{Bmatrix} = \begin{bmatrix} & u_{ij} & \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \theta_1 \\ 1 \\ 0 \\ \varphi_1 \end{Bmatrix}$$

$$\therefore \begin{bmatrix} u_{33} & u_{34} & u_{36} \\ u_{43} & u_{44} & u_{46} \\ u_{63} & u_{64} & u_{66} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ 1 \\ \varphi_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Set determinant to zero for nat. freqs.



$x_0=1$

$F_0=0$

$x_4 = r_0$

$$\begin{Bmatrix} x \\ F \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ -m_1\omega^2 \end{Bmatrix} = \begin{bmatrix} 1 & \frac{1}{\infty} \\ -m_1\omega^2 & 1-\frac{m_1\omega^2}{\infty} \end{bmatrix}_1 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}_0 = \begin{Bmatrix} 1 \\ -m_1\omega^2 \end{Bmatrix}$$

$$\begin{Bmatrix} x \\ F \end{Bmatrix}_2 = \begin{bmatrix} 1 & \frac{1}{k_2} \\ -m_2\omega^2 & \left(1-\frac{m_2\omega^2}{k_2}\right) \end{bmatrix}_2 \begin{Bmatrix} x \\ F \end{Bmatrix}_1$$

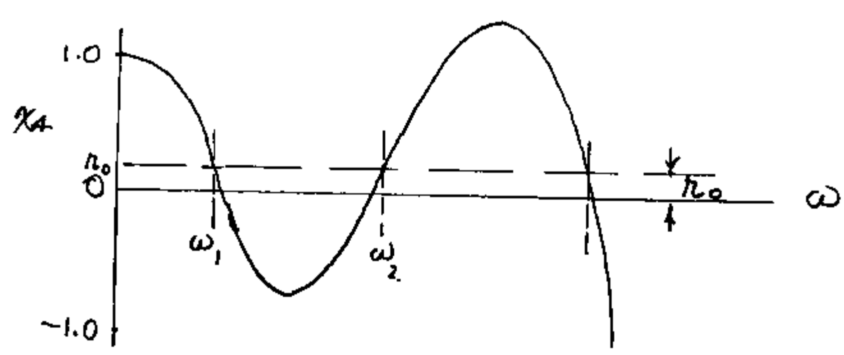
$$\begin{Bmatrix} x \\ F \end{Bmatrix}_3 = \begin{bmatrix} 1 & \frac{1}{k_3+i\omega c_3} \\ -m_3\omega^2 & \left(1-\frac{m_3\omega^2}{k_3+i\omega c_3}\right) \end{bmatrix}_3 \begin{Bmatrix} x \\ F \end{Bmatrix}_2$$

$$\begin{Bmatrix} x \\ F \end{Bmatrix}_4 = \begin{bmatrix} 1 & \frac{1}{k_4} \\ 0 & 0 \end{bmatrix}_4 \begin{Bmatrix} x \\ F \end{Bmatrix}_3$$

Boundry Cond.

$$\therefore\ x_4 = x_3 + \frac{F_3}{k_4} = r_0$$

<u>Problem is identical in calculation to torsional system and same program as Prob 12-10 can be used.</u>



$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_0^R = \begin{Bmatrix}1\\-\omega^2 J_1\end{Bmatrix}$$

$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_m^R = \begin{bmatrix}1 & 0\\-\omega^2 J & 1\end{bmatrix}_m \begin{bmatrix}1 & \frac{1}{K+i\omega g}\\0 & 1\end{bmatrix}_m \begin{Bmatrix}\theta\\T\end{Bmatrix}_{m-1}^R = \begin{bmatrix}1 & \frac{1}{K+i\omega g}\\-\omega^2 J & \left(1-\frac{\omega^2 J}{K+i\omega g}\right)\end{bmatrix}_m \begin{Bmatrix}\theta\\T\end{Bmatrix}_{m-1}^R$$

Elements of above transfer matrix are:

$$\frac{1}{K+i\omega g} = \left[\frac{K}{K^2+(\omega g)^2}\right] - i\left[\frac{\omega g}{K^2+(\omega g)^2}\right]$$

$$1-\frac{\omega^2 J}{K+i\omega g} = \left[1-\frac{\omega^2 J K}{K^2+(\omega g)^2}\right] + i\left[\frac{\omega^2 J\omega g}{K^2+(\omega g)^2}\right]$$

Sample calc. for $\omega^2 = 0.5\times10^5$

Renumber J_s from J_0 to J_3

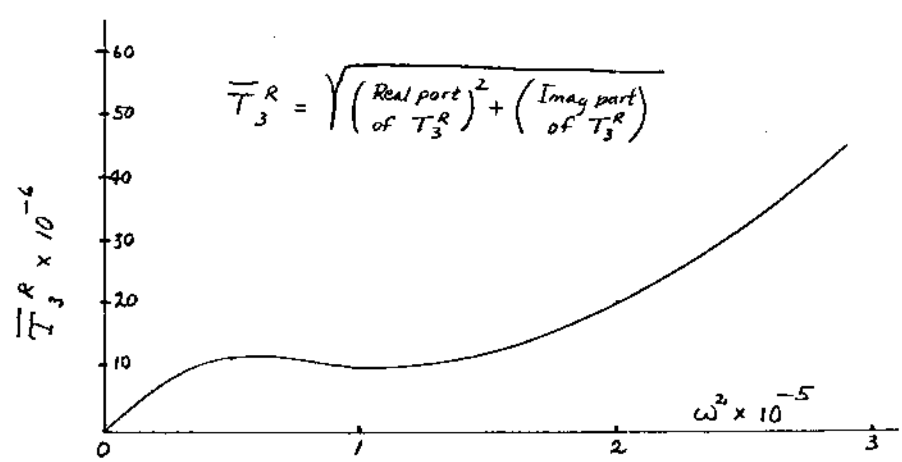
$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_0^R = \begin{Bmatrix}1.0\\-1\times10^6\end{Bmatrix}$$

$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_1^R = \begin{bmatrix}1 & -i(.446\times10^{-6})\\-5\times10^6 & (1+i\,2.23)\end{bmatrix}\begin{Bmatrix}1.0\\-1\times10^6\end{Bmatrix} = \begin{Bmatrix}1 + .446\,i\\(-6-i\,2.23)\times10^6\end{Bmatrix}$$

$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_2^R = \begin{bmatrix}1 & .2\times10^{-6}\\-.5\times10^6 & .90\end{bmatrix}\begin{Bmatrix}1+.446\,i\\(-6-i\,2.23)10^6\end{Bmatrix} = \begin{Bmatrix}-.20\\(-5.9-i\,2.23)\,10^6\end{Bmatrix}$$

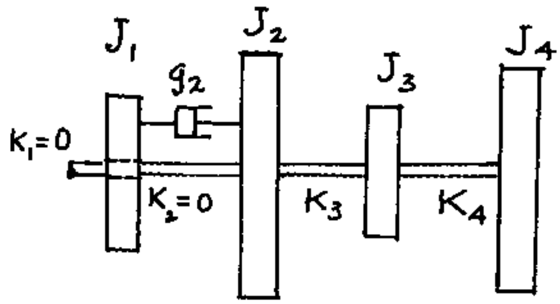
$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_3^R = \begin{bmatrix}1 & .10\times10^{-6}\\-1.5\times10^6 & .85\end{bmatrix}\begin{Bmatrix}-.20\\(-5.9-i\,2.23)10^6\end{Bmatrix} = \begin{Bmatrix}-.79-i\,.223\\(-4.72-i\,1.9)\,10^6\end{Bmatrix}$$

Repeat for other frequencies — should be programmed for digital computer.



12-16 Cont:

Eqs. 12.6-3 & 12.6-4



$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_n^R = \begin{bmatrix} 1 & \dfrac{1}{K+ig} \\ (i\omega c-\omega^2 J) & 1+\dfrac{i\omega c-\omega^2 J}{K+i\omega g} \end{bmatrix}_n \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{n-1}^R$$

n	J	K	c	g
1	20.000	0.0000E 00	0.0000E 00	0.0000E 00
2	100.000	0.0000E 00	0.0000E 00	0.1000E 05
3	10.000	0.5000E 07	0.0000E 00	0.0000E 00
4	30.000	0.1000E 08	0.0000E 00	0.0000E 00

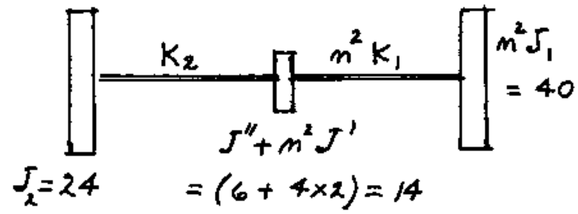
```
      PROBLEM 10.21 THOMSON
      DIMENSION RJ(4),RK(4),RC(4),RG(4),W(60)
      COMPLEX DE(60,4),T(60,4),CRJ(4),CRK(4),CMPLX
      M=4
      L=60
      READ,(RJ(J),J=1,M)
      READ,(RK(J),J=1,M)
      READ,(RC(J),J=1,M)
      READ,(RG(J),J=1,M)
      DO 20 I=1,L
      W(I)=I*10
      DO 5 J=1,4
      ZR=-W(I)**2*RJ(J)
      ZI=W(I)*RC(J)
      CRJ(J)=CMPLX(ZR,ZI)
      ZR=RK(J)
      ZI=W(I)*RG(J)
      CRK(J)=CMPLX(ZR,ZI)
    5 CONTINUE
      DE(I,1)=1.
      T(I,1)=CRJ(1)*DE(I,1)
      DO 10 J=2,M
      DE(I,J)=DE(I,J-1)+T(I,J-1)/CRK(J)
      T(I,J)=CRJ(J)*DE(I,J-1)+(1.+CRJ(J)/CRK(J))*T(I,J-1)
   10 CONTINUE
   20 CONTINUE
      DO 25 J=1,M
      PRINT24,J,RJ(J),RK(J),RC(J),RG(J)
   24 FORMAT(20X,I3,5X,F8.3,5X,E12.4,5X,E12.4,5X,E12.4)
   25 CONTINUE
      DO 40 I=1,L
      PRINT30,W(I),DE(I,M),T(I,M)
   30 FORMAT(10X,F8.2,10X,2E12.4,10X,2E12.4)
   40 CONTINUE
      STOP
      END
```

12-16 Cont.

ω	θ_{real}	θ_{imag}	T_{real}	T_{imag}
10.00	0.9963E 00	0.1994E-01	-0.1599E 05	-0.2798E 03
20.00	0.9852E 00	0.3950E-01	-0.6378E 05	-0.2233E 04
30.00	0.9667E 00	0.5833E-01	-0.1429E 06	-0.7505E 04
40.00	0.9409E 00	0.7604E-01	-0.2525E 06	-0.1769E 05
50.00	0.9077E 00	0.9226E-01	-0.3916E 06	-0.3429E 05
60.00	0.8671E 00	0.1066E 00	-0.5585E 06	-0.5873E 05
70.00	0.8193E 00	0.1188E 00	-0.7517E 06	-0.9225E 05
80.00	0.7642E 00	0.1284E 00	-0.9689E 06	-0.1360E 06
90.00	0.7019E 00	0.1350E 00	-0.1208E 07	-0.1908E 06
100.00	0.6324E 00	0.1384E 00	-0.1466E 07	-0.2575E 06
110.00	0.5558E 00	0.1381E 00	-0.1740E 07	-0.3365E 06
120.00	0.4722E 00	0.1339E 00	-0.2026E 07	-0.4280E 06
130.00	0.3816E 00	0.1253E 00	-0.2322E 07	-0.5320E 06
140.00	0.2840E 00	0.1120E 00	-0.2623E 07	-0.6480E 06
150.00	0.1797E 00	0.9379E-01	-0.2925E 07	-0.7754E 06
160.00	0.6853E-01	0.7024E-01	-0.3223E 07	-0.9131E 06
170.00	-0.4925E-01	0.4107E-01	-0.3514E 07	-0.1060E 07
180.00	-0.1736E 00	0.5975E-02	-0.3791E 07	-0.1213E 07
190.00	-0.3044E 00	-0.3535E-01	-0.4051E 07	-0.1372E 07
200.00	-0.4416E 00	-0.8320E-01	-0.4286E 07	-0.1532E 07
210.00	-0.5850E 00	-0.1378E 00	-0.4492E 07	-0.1692E 07
220.00	-0.7346E 00	-0.1996E 00	-0.4663E 07	-0.1847E 07
230.00	-0.8901E 00	-0.2686E 00	-0.4793E 07	-0.1993E 07
240.00	-0.1052E 01	-0.3452E 00	-0.4875E 07	-0.2126E 07
250.00	-0.1219E 01	-0.4297E 00	-0.4902E 07	-0.2241E 07
260.00	-0.1392E 01	-0.5222E 00	-0.4869E 07	-0.2332E 07
270.00	-0.1570E 01	-0.6229E 00	-0.4768E 07	-0.2394E 07
280.00	-0.1753E 01	-0.7322E 00	-0.4593E 07	-0.2419E 07
290.00	-0.1942E 01	-0.8501E 00	-0.4336E 07	-0.2400E 07
300.00	-0.2136E 01	-0.9768E 00	-0.3990E 07	-0.2331E 07
310.00	-0.2334E 01	-0.1113E 01	-0.3547E 07	-0.2201E 07
320.00	-0.2537E 01	-0.1257E 01	-0.3001E 07	-0.2004E 07
330.00	-0.2745E 01	-0.1412E 01	-0.2344E 07	-0.1729E 07
340.00	-0.2956E 01	-0.1575E 01	-0.1568E 07	-0.1367E 07
350.00 $\leftarrow \omega_1$	-0.3172E 01	-0.1748E 01	-0.6651E 06	-0.9071E 06
360.00	-0.3392E 01	-0.1931E 01	0.3715E 06	-0.3387E 06
370.00	-0.3616E 01	-0.2123E 01	0.1550E 07	0.3497E 06
380.00	-0.3842E 01	-0.2325E 01	0.2877E 07	0.1170E 07
390.00	-0.4072E 01	-0.2537E 01	0.4362E 07	0.2135E 07
400.00	-0.4306E 01	-0.2758E 01	0.6011E 07	0.3256E 07
410.00	-0.4542E 01	-0.2990E 01	0.7832E 07	0.4549E 07
420.00	-0.4780E 01	-0.3231E 01	0.9832E 07	0.6025E 07
430.00	-0.5021E 01	-0.3481E 01	0.1202E 08	0.7700E 07
440.00	-0.5264E 01	-0.3742E 01	0.1440E 08	0.9588E 07
450.00	-0.5508E 01	-0.4012E 01	0.1698E 08	0.1170E 08
460.00	-0.5755E 01	-0.4291E 01	0.1977E 08	0.1406E 08
470.00	-0.6002E 01	-0.4580E 01	0.2277E 08	0.1668E 08
480.00	-0.6251E 01	-0.4877E 01	0.2599E 08	0.1958E 08
490.00	-0.6500E 01	-0.5184E 01	0.2944E 08	0.2276E 08
500.00	-0.6750E 01	-0.5500E 01	0.3312E 08	0.2625E 08
510.00	-0.7000E 01	-0.5824E 01	0.3704E 08	0.3006E 08
520.00	-0.7250E 01	-0.6157E 01	0.4121E 08	0.3422E 08
530.00	-0.7500E 01	-0.6498E 01	0.4562E 08	0.3873E 08
540.00	-0.7748E 01	-0.6846E 01	0.5028E 08	0.4361E 08

12 - 18

After reduction to equivalent single shaft system, the freq. eq. of the degenerate 3 - DOF system may be used.



$$\omega^4 - \left[\frac{K_1}{J_1} + \frac{K_2}{J_2}\left(1 + \frac{K_1}{K_2} + \frac{J_2}{J_3}\right)\right]\omega^2 + \frac{K_1}{J_1}\frac{K_2}{J_2}\left(\frac{J_1 + J_2 + J_3}{J_3}\right) = 0$$

$$\omega^4 - \left[\left(\frac{62.8}{24}\right) + \frac{59.7}{14}\left(1 + \frac{.628}{.597} + \frac{14}{40}\right)\right]10^4\omega^2 + \left[\frac{62.8}{24}\cdot\frac{59.7}{14}\left(\frac{24+14+40}{40}\right)\right]10^{8} = 0$$

$$\omega^4 - 12.82\times 10^4\,\omega^2 + 21.8\times 10^6 = 0$$

$$\omega^2 = \begin{cases} 10.86\times 10^4 \\ 2.01\times 10^4 \end{cases} \qquad \omega = \begin{cases} 329.2 \text{ r/s} \\ 141.8 \text{ "} \end{cases} = \begin{cases} 52.30 \text{ Hz} \\ 22.55 \text{ "} \end{cases}$$

---

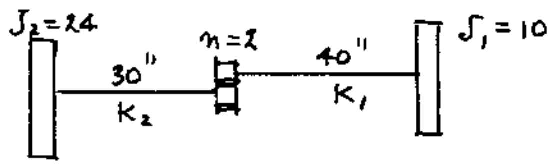
12 - 17

$$\omega^2 = \frac{(J_2 + m^2 J_1) K_2 (m^2 K_1)}{(m^2 J_1)\, J_2\, (K_2 + m^2 K_1)}$$

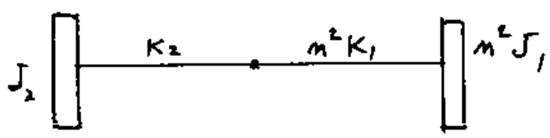
$$K_1 = G\frac{\pi D^4}{32\ell} = \frac{(12\times 10^6)\pi (1.5)^4}{32\times 40} = 0.1492\times 10^6$$

$$K_2 = \frac{(12\times 10^6)\pi (2)^4}{32\times 30} = 0.628\times 10^6$$

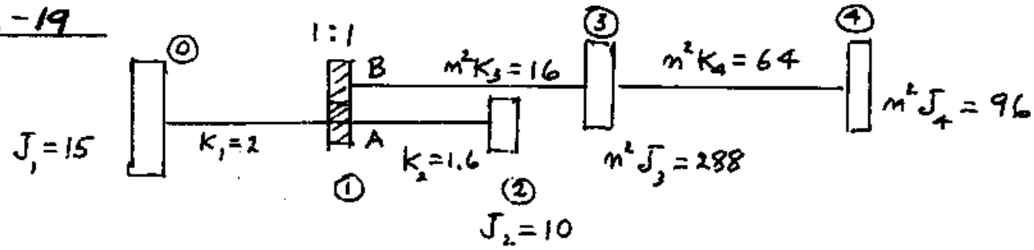
$$\omega^2 = \frac{(24 + 40)(.628)(.1492)\times 10^{12}}{(240)(.628 + .5975)\times 10^6} = 2.04\times 10^4$$



$\omega = 22.7$ Hz



12-19



J & K along shaft B are multiplied by $n^2=16$ so that the gear ratio is reduced to 1:1

Then $\theta_{A_1}^R = -\theta_{B_1}^R = \theta_1^L$ and $T_{A_1}^R + T_{B_1}^R = T_1^L$

$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_0^R = \begin{Bmatrix}1\\-15\omega^2\end{Bmatrix} = \text{starting eq.}$$

$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_1^L = \begin{bmatrix}1 & .5\times10^{-6}\\0 & 1\end{bmatrix}\begin{Bmatrix}1\\-15\omega^2\end{Bmatrix} = \begin{Bmatrix}1-7.5\times10^{-6}\omega^2\\-15\omega^2\end{Bmatrix}$$

$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_3^R = \begin{bmatrix}1 & .0625\times10^{-6}\\-288\omega^2 & (1-18\times10^{-6}\omega^2)\end{bmatrix}\begin{Bmatrix}\theta\\T\end{Bmatrix}_{B1}^R$$

$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_4^R = \begin{bmatrix}1 & .01562\times10^{-6}\\-96\omega^2 & (1-1.5\times10^{-6}\omega^2)\end{bmatrix}\begin{Bmatrix}\theta\\T\end{Bmatrix}_3^R$$

$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_2^R = \begin{bmatrix}1 & .625\times10^{-6}\\-10\omega^2 & (1-6.25\times10^{-6}\omega^2)\end{bmatrix}\begin{Bmatrix}\theta\\T\end{Bmatrix}_{A1}^R$$

Solution

$\omega_1 = 377.2$ $\quad \begin{Bmatrix}\theta_1\\\theta_2\end{Bmatrix}^{(1)} = \begin{Bmatrix}1.0\\-4.35\end{Bmatrix}$

$\omega_2 = 427.0$ $\quad \begin{Bmatrix}\theta_1\\\theta_2\end{Bmatrix}^{(2)} = \begin{Bmatrix}1.0\\2.605\end{Bmatrix}$

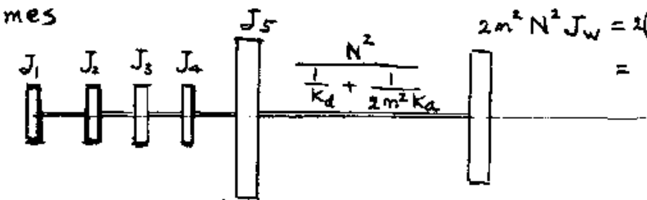
$\omega_3 = 940.0$ $\quad \begin{Bmatrix}\theta_1\\\theta_2\end{Bmatrix}^{(3)} = \begin{Bmatrix}1.0\\1.725\end{Bmatrix}$

12-20

With $n = \frac{1}{3.5}$ the rear wheels and axle are replaced by $2n^2 J_w$ and $2n^2 K_a$. In series with the drive shaft the stiffness is

$$\frac{1}{\frac{1}{K_d} + \frac{1}{2n^2 K_a}}$$

. Let $N = \frac{1}{3}$ = transmission gear ratio, and refer all stiffnesses and inertias to engine speed by multiplying by N^2. System then becomes



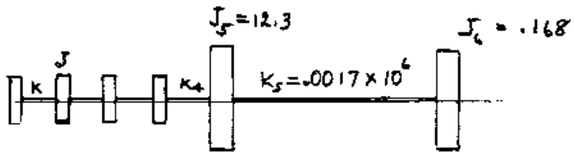
$$2n^2 N^2 J_w = 2\left(\frac{1}{3.5}\right)^2\left(\frac{1}{3}\right)^2 9.2 = .167$$

$$K_a = \frac{(12\times 10^6)\pi(1.25)^4}{32 \times 25} = .115 \times 10^6$$

$$K_d = \frac{(12\times 10^6)\pi(1.5)^4}{32 \times 74} = .0806 \times 10^6$$

$$K_5 = \frac{\left(\frac{1}{3}\right)^2 \times 10^6}{\frac{1}{.0806} + \frac{3.5^2}{.115}} = .00170 \times 10^6$$

12-21



$J = J_1 = J_2 = J_3 = J_4 = 0.20$ lb in sec²

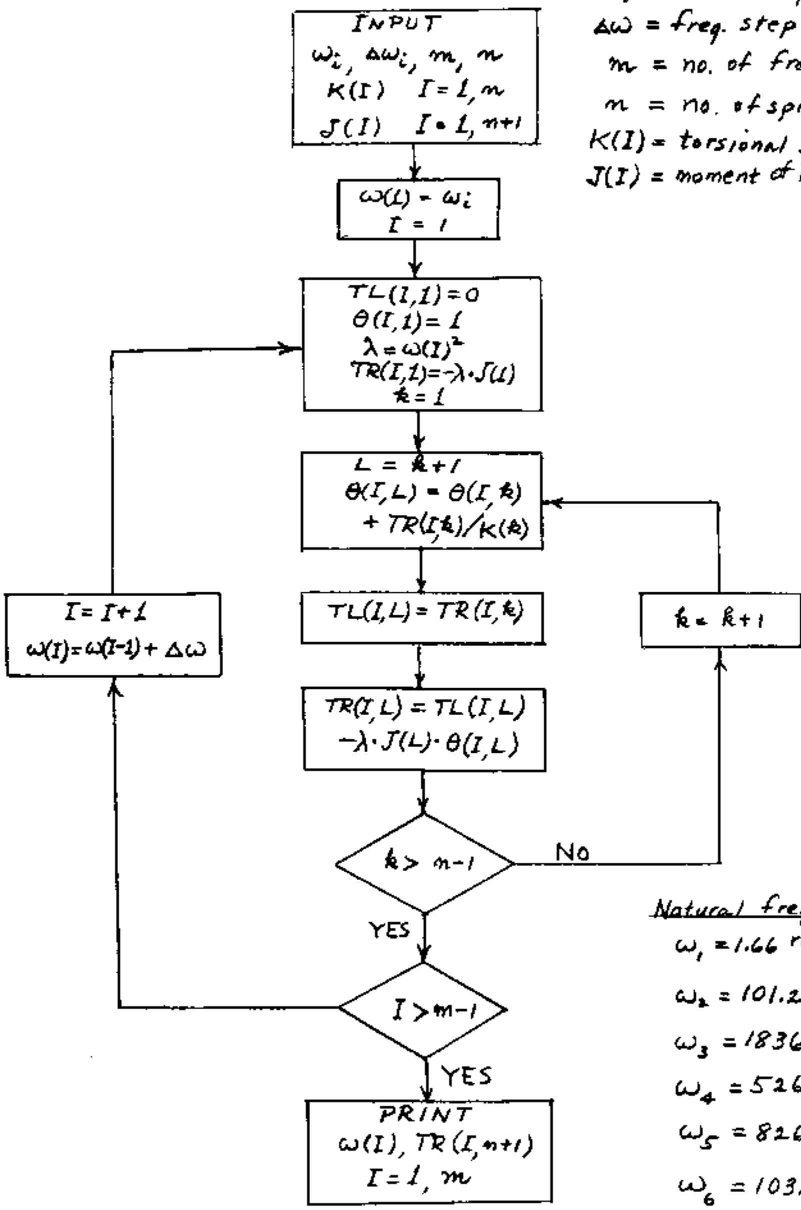
$K = 6.1 \times 10^6$ lb in/rad $\qquad K_4 = 4.5 \times 10^6$ lb in/rad

Approximation to first nat. freq. as two mass system

$$\omega = \sqrt{\frac{K(J_1 + J_2)}{J_1 J_2}} = 10^3 \sqrt{\frac{.0017(13.1 + .168)}{13.1 \times .168}}$$

$$= 10^3 \sqrt{.0102} \cong 100 \text{ rad/sec.}$$

computer solution follows



ω_i = initial freq.
$\Delta\omega$ = freq. step
m = no. of freq.
n = no. of springs
$K(I)$ = torsional springs
$J(I)$ = moment of inertia

Natural frequencies

$\omega_1 = 1.66$ rad/sec $\cong 0$

$\omega_2 = 101.2$

$\omega_3 = 1836$

$\omega_4 = 5265$

$\omega_5 = 8266$

$\omega_6 = 10320$

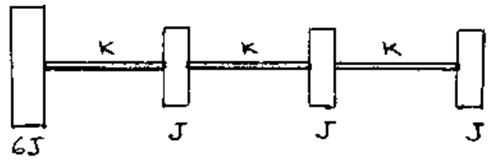
Flow diagram for digital computation of Prob 12-21

12-21 Cont.

FORTRAN IV G LEVEL 21

```
      DIMENSION SJ(6),SK(5),THETA(2000,6),TR(2000,6),TL(2000,6),
     S OMEG(2000)
      M=2000
      OM=50.
      STEP=20.
      DO 10 I=1,3
      SJ(I)=.2
   10 SK(I)=0.61E 07
      SJ(4)=.2
      SJ(5)=12.3
      SJ(6)=.168
      SK(4)=0.45E 07
      SK(5)=0.17E 04
      DO 40 I=1,M
      TL(I,1)=0.
      THETA(I,1)=1.
      OMSQ=OM**2
      TR(I,1)=-OMSQ*SJ(1)
      DO 30 K=1,5
      L=K+1
      THETA(I,L)=THETA(I,K)+TR(I,K)/SK(K)
      TL(I,L)=TR(I,K)
   30 TR(I,L)=TL(I,L)-OMSQ*SJ(L)*THETA(I,L)
      OMEG(I)=OM
   40 OM=OM+STEP
      N=M/2
      PRINT 50
      PRINT 60,((OMEG(I),TR(I,6),OMEG(I+N),TR(I+N,6)),I=1,N)
   50 FORMAT('1',' OMEGA',10X,'TR(OM,6)',33X,'OMEGA',10X,'TR(OM,6)'
   60 FORMAT(' ',F7.1,5X,E15.4,30X,F7.1, 5X,E15.4)
      STOP
      END
```

12-22



The system is first reduced to the above model

The equations in matrix form are

$$K\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = -\omega^2 J \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix}$$

Rearrange to

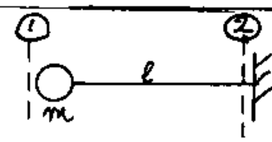
$$\frac{K}{\omega^2 J}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -.1667 & .1667 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix}$$

Results of iteration with sweeping matrix

$\omega_0 = 0, \quad \begin{Bmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \end{Bmatrix}$	$\omega_2 = 1.272\sqrt{\frac{K}{J}}, \quad \begin{Bmatrix} 1.00 \\ -.614 \\ -1.237 \\ .1416 \end{Bmatrix}$
$\omega_1 = .5375\sqrt{\frac{K}{J}}, \quad \begin{Bmatrix} 1.00 \\ .714 \\ .239 \\ -.326 \end{Bmatrix}$	$\omega_3 = 1.805\sqrt{\frac{K}{J}}, \quad \begin{Bmatrix} 1.00 \\ -2.27 \\ 1.87 \\ -.1005 \end{Bmatrix}$

12-23

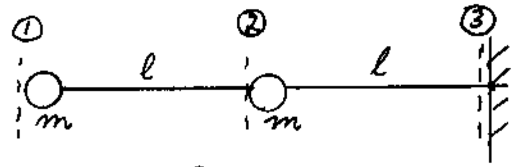
From Eq. 12.9-5



$$\begin{Bmatrix} -V \\ M \\ 0 \\ 0 \end{Bmatrix}_2 = \left[\begin{array}{cc|cc} & & & \\ & & & \\ & & 1 & \frac{m\omega^2 \ell^2}{2EI} \\ & & \ell & \left(1+\frac{m\omega^2\ell^3}{6EI}\right) \end{array}\right]\begin{Bmatrix} 0 \\ 0 \\ \theta \\ 1 \end{Bmatrix}_1$$

$$\therefore \begin{vmatrix} 1 & \frac{m\omega^2\ell^2}{2EI} \\ \ell & \left(1+\frac{m\omega^2\ell^3}{6EI}\right) \end{vmatrix} = 0 \qquad \omega = \sqrt{\frac{3EI}{m\ell^3}}$$

12-24



From Eq. 12.9-5

Let $\alpha = \ell/6EI$

$$\begin{Bmatrix} -V \\ M \\ 0 \\ 0 \end{Bmatrix}_3 = \begin{bmatrix} 1 & 0 & 0 & m\omega^2 \\ \ell & 1 & 0 & m\omega^2\ell \\ 3\alpha\ell & 6\alpha & 1 & 3m\omega^2\alpha\ell \\ \alpha\ell^2 & 3\alpha\ell & \ell & (1+m\omega^2\alpha\ell^2) \end{bmatrix}^2 \begin{Bmatrix} 0 \\ 0 \\ \theta \\ 1 \end{Bmatrix}_1$$

We need only to calculate the last two columns of the last two rows, which are:

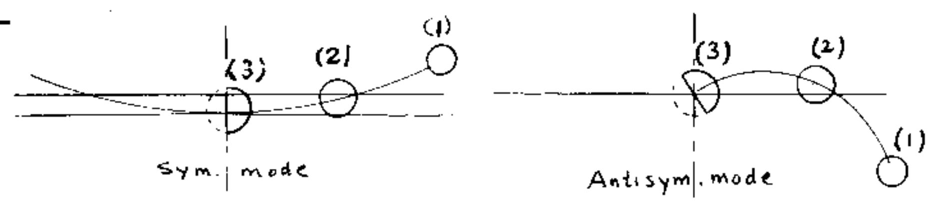
$$\begin{bmatrix} (1+3m\omega^2\alpha\ell^2) & [15m\omega^2\alpha\ell + 3(m\omega^2\alpha)^2\ell^3] \\ (2\ell + m\omega^2\alpha\ell^3) & [1+9m\omega^2\alpha\ell^2 + (m\omega^2\alpha\ell^2)^2] \end{bmatrix} \begin{Bmatrix} \theta \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Det. set to zero gives

$$(m\omega^2\alpha\ell^2)^2 - 3(m\omega^2\alpha\ell^2) + \frac{1}{6} = 0$$

solving $\omega = \begin{cases} 0.583\sqrt{\dfrac{EI}{m\ell^3}} \\ 4.20\sqrt{\dfrac{EI}{m\ell^3}} \end{cases}$

12-25



$$\begin{Bmatrix} 0 \\ M \\ 0 \\ y \end{Bmatrix}_3 = \begin{bmatrix} & u_{ij} & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta \\ y \end{Bmatrix}_1 \qquad \begin{Bmatrix} -V \\ 0 \\ \theta \\ 0 \end{Bmatrix}_3 = \begin{bmatrix} & u_{ij} & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta \\ y \end{Bmatrix}_1$$

Freq. eq.

$$\begin{vmatrix} u_{13} & u_{14} \\ u_{33} & u_{34} \end{vmatrix} = 0$$

Freq. eq.

$$\begin{vmatrix} u_{23} & u_{24} \\ u_{43} & u_{44} \end{vmatrix} = 0$$

12-26

$$-J\omega^2\theta_m = K(\theta_{m-1}-\theta_m) - K(\theta_m - \theta_{m-1})$$

$$\theta_{m-1} - 2\left(1-\frac{\omega^2 J}{2K}\right)\theta_m + \theta_{m+1} = 0 \qquad \text{Let } \theta_n = e^{\lambda n}$$

$$e^{-\lambda} - 2\left(1-\frac{\omega^2 J}{2K}\right) + e^{\lambda} = 0$$

$$\therefore \frac{\omega^2 J}{K} = 2(1-\cosh\lambda) \qquad \lambda = i\beta$$

$$= 2(1-\cos\beta) \qquad \theta_m = e^{i\beta m} = \cos\beta m + i\sin\beta m$$

Assume sol. $\theta_m = A\sin\beta m + B\cos\beta m$

Boundary $\quad \theta_0 = 0 \quad \therefore B = 0 \quad$ and $\quad \theta_m = A\sin\beta m$

Boundary N $\qquad -\omega^2 J\theta_N = -K(\theta_N - \theta_{N-1})$

$$\left(1-\frac{J\omega^2}{K}\right)\theta_N = \theta_{N-1}$$

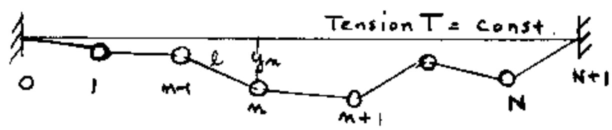
$$[1-2(1-\cos\beta)]\sin\beta N = \sin\beta(N-1)$$

Reduces to $\quad 2\sin\beta N\cos\beta = \sin\beta(N-1) + \sin\beta N$

$$2\cos\beta\left(N+\tfrac{1}{2}\right)\sin\frac{\beta}{2} = 0 \qquad \therefore \beta = 0, 2\pi, \cdots$$

$$\omega_k = 2\sqrt{\frac{K}{J}}\,\sin\frac{(2k-1)\pi}{2(2N+1)} \qquad k = 1, 2, 3 \cdots N$$

12-27



$$m_m\ddot{y}_m = -\frac{T}{l}(y_m - y_{m-1}) + \frac{T}{l}(y_{m+1}-y_m) \qquad \text{rearrange to}$$

$$y_{m+1} - 2\left(1-\frac{\omega^2 m l}{2T}\right)y_m + y_{m-1} = 0$$

Let $y_m = e^{i\beta m}$ & subst. into above eq.

$$\cancel{e^{i\beta m}}e^{i\beta} - 2\left(1-\frac{\omega^2 m l}{2T}\right)\cancel{e^{i\beta m}} + \cancel{e^{i\beta m}}e^{-i\beta} = 0$$

12-27 Cont:

$$1 - \frac{\omega^2 m\ell}{2T} = \frac{e^{i\beta}+e^{-i\beta}}{2} = \cos\beta$$

$$\frac{\omega^2 m\ell}{2T} = 1-\cos\beta = 2\sin^2\frac{\beta}{2}$$

freq. eq. $\omega = 2\sqrt{\frac{T}{m\ell}}\sin\frac{\beta}{2}$ Evaluate β at boundaries starting with gen. sol.

$$y_m = A\cos\beta m + B\sin\beta m$$

$y_0 = 0 \quad \therefore A = 0\,, \qquad y_{N+1} = 0 \quad \therefore \sin\beta(N+1) = 0$

$$\beta(N+1) = 0,\ \pi,\ 2\pi,\ 3\pi\ \cdots = k\pi$$

$$\therefore \omega_k = 2\sqrt{\frac{T}{m\ell}}\sin\frac{k\pi}{2(N+1)} \qquad k = 1, 2, 3, \cdots$$

For $N = 2$
$$\omega_1 = 2\sqrt{\frac{T}{m\ell}}\sin\frac{\pi}{6} = \sqrt{\frac{T}{m\ell}}$$
$$\omega_2 = 2\sqrt{\frac{T}{m\ell}}\sin\frac{2\pi}{6} = \sqrt{\frac{3T}{m\ell}}$$

12-28

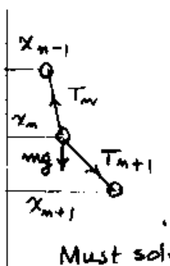
$$m\ddot{x}_m = k(x_{m+1}-x_m) - k_m(x_m - x_{m-1})$$

$$x_{m+1} - 2\left(1-\frac{\omega^2 m}{k}\right)x_m + x_{m-1} = 0$$

This problem is identical to Prob. **12-27** with **changes in** boundary conditions $\therefore \omega_k = 2\sqrt{\frac{k}{m}}\sin\frac{k\pi}{2(N+1)} \qquad k = 1, 2 \cdots$

12-29

$$m\ddot{x}_m = \frac{T_{m+1}}{\ell}(x_{m+1}-x_m) - \frac{T_m}{\ell}(x_m - x_{m-1})$$



$$T_m \cong T'_{m+1} + mg$$

$$x_{m+1} - 2\left(1-\frac{\omega^2 m\ell}{T_m}\right)x_m + x_{m-1} = \frac{mg\ell}{T'_m}(x_{m+1} - x_m)$$

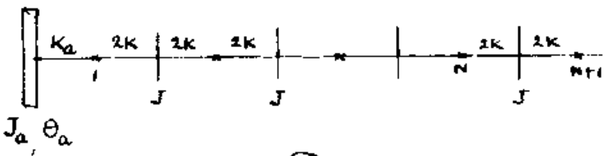
T_m changes ie $T_N = mg$, $T_{N-1} = 2mg$ etc.

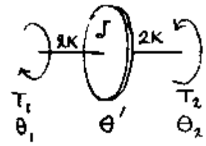
$\therefore$ Usual diff eq with const. coef. cannot be used.

Must solve as a system of N eqs.

$$-\omega^2 m x_1 = -\frac{Nmg}{\ell}x_1 + \frac{(N-1)mg}{\ell}(x_2 - x_1)$$

$$-\omega^2 m x_N = -\frac{mg}{\ell}(x_N - x_{N-1})$$





$$T_1 = 2K(\theta' - \theta_1) \longrightarrow \begin{Bmatrix} T_1 \\ \theta_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-1}{2K} & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ \theta' \end{Bmatrix}$$

$$T_2 - T_1 = -\omega^2 J \theta' \longrightarrow \begin{Bmatrix} T_1 \\ \theta' \end{Bmatrix} = \begin{bmatrix} 1 & \omega^2 J \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} T_2 \\ \theta' \end{Bmatrix}$$

$$T_2 = 2K(\theta_2 - \theta') \longrightarrow \begin{Bmatrix} T_2 \\ \theta' \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-1}{2K} & 1 \end{bmatrix} \begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} T_1 \\ \theta_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-1}{2K} & 1 \end{bmatrix} \begin{bmatrix} 1 & \omega^2 J \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{-1}{2K} & 1 \end{bmatrix} \begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} \left(1 - \frac{\omega^2 J}{2K}\right) & \omega^2 J \\ -\left(\frac{1}{K} - \frac{\omega^2 J}{4K}\right) & \left(1 - \frac{\omega^2 J}{2K}\right) \end{bmatrix} \begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} A & B \\ C & A \end{bmatrix} \begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix}$$

Note that $A^2 - BC = 1$ $\qquad B = \frac{A^2 - 1}{C}$

Let $A = \left(1 - \frac{\omega^2 J}{2K}\right) = \cosh\lambda$

$C = -\left(\frac{1}{K} - \frac{\omega^2 J}{4K}\right) = \frac{1}{Z}\sinh\lambda$

$B = \omega^2 J = \frac{\cosh^2\lambda - 1}{\frac{1}{Z}\sinh\lambda} = Z\sinh\lambda$

$$\begin{Bmatrix} T_1 \\ \theta_1 \end{Bmatrix} = \begin{bmatrix} \cosh\lambda & Z\sinh\lambda \\ \frac{1}{Z}\sinh\lambda & \cosh\lambda \end{bmatrix} \begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix}$$

If replaced by exponentials it can be proved that

$$\begin{bmatrix} \cosh\lambda & Z\sinh\lambda \\ \frac{1}{Z}\sinh\lambda & \cosh\lambda \end{bmatrix}^n = \begin{bmatrix} \cosh n\lambda & Z\sinh n\lambda \\ \frac{1}{Z}\sinh n\lambda & \cosh n\lambda \end{bmatrix}$$

$$\therefore \begin{Bmatrix} T_1 \\ \theta_1 \end{Bmatrix} = \begin{bmatrix} \cosh n\lambda & Z\sinh n\lambda \\ \frac{1}{Z}\sinh n\lambda & \cosh n\lambda \end{bmatrix} \begin{Bmatrix} T_{n+1} \\ \theta_{n+1} \end{Bmatrix}$$

$$\left.\begin{aligned} &\text{From } A = \left(1 - \frac{\omega^2 J}{2K}\right) = \cosh\lambda \\ &\omega^2 J = B = 2K(1 - \cosh\lambda) = -4K\sinh^2\frac{\lambda}{2} \end{aligned}\right\} \therefore \text{freq. eq} = \omega = 2\sqrt{\frac{K}{J}}\sinh\frac{\lambda}{2}$$

where λ must be determined from boundary

Boundary conditions

$$\theta_1 - \theta_a = \frac{T_1}{K_a} = -\frac{\omega^2 J_a \theta_a}{K_a}, \qquad T_{N+1} = 0$$

$$\therefore T_1 = z \sinh N\lambda \cdot \theta_{N+1} = -\omega^2 J_a \theta_a$$

$$\theta_1 = \cosh N\lambda \cdot \theta_{N+1} = \left(1 - \frac{\omega^2 J_a}{K_a}\right)\theta_a$$

Divide $$\frac{z \sinh N\lambda}{\cosh N\lambda} = \frac{-\omega^2 J_a}{\left(1 - \frac{\omega^2 J_a}{K_a}\right)}$$

$$\frac{2k(1-\cosh\lambda)}{\sinh\lambda} \cdot \frac{\sinh N\lambda}{\cosh N\lambda} = \frac{-\omega^2 J_a}{\left(1 - \frac{\omega^2 J_a}{K_a}\right)}$$

$$-2k\left[\sinh N\lambda \cosh\lambda - \sinh N\lambda\right] = \frac{4k \frac{J_a}{J} \sinh^2\frac{\lambda}{2}}{1 + \frac{4k}{K_a}\frac{J_a}{J}\sinh^2\frac{\lambda}{2}} \cdot \sinh\lambda \cosh N\lambda$$

$$\left(-2\sinh N\lambda \cdot \cosh\lambda + 2\sinh N\lambda\right)\left(1 - 4\frac{k}{K_a}\frac{J_a}{J}\sinh^2\frac{\lambda}{2}\right) = 2\frac{J_a}{J}\sinh^2\frac{\lambda}{2}\sinh\lambda \cosh N\lambda$$

$$\left(-\sin N\beta \cdot \cos\beta + \sin N\beta\right)\left(1 + 4\frac{k}{K_a}\frac{J_a}{J}\sin^2\frac{\beta}{2}\right) = -2\frac{J_a}{J}\sin^2\frac{\beta}{2}\sin\beta \cos N\beta$$

solve for β & subst. into freq. eq.

$$\omega = 2\sqrt{\frac{k}{J}}\sin\frac{\beta}{2}$$

This problem can also be solved by using stations at the disks instead of at points midway between disks. The equations

$$\theta_m = A \sin\beta m + B\cos\beta m, \qquad m = 0, 1, 2 \cdots N$$

can be used provided $k_0 = \dfrac{1}{1/K_a + 1/2K}$ is used.

12-31

With $m = 0$ fixed, solution is

$$X_m = B \sin\beta m$$

At top of the bldg, $m\ddot{x}_N = -k(x_N - x_{N-1}) - K_N x_N$

$$\text{or } X_{N-1} = \left(1 + \frac{K_N}{k} - \frac{m\omega^2}{k}\right)X_N$$

$$\therefore \sin\beta(N-1) = \left[1 + \frac{K_N}{k} - 2(1-\cos\beta)\right]\sin\beta N$$

$$\sin\beta(N-1) - \sin\beta N + 2(1-\cos\beta)\sin\beta N = \frac{K_N}{k}\sin\beta N$$

12-31 Cont.

$$\sin\beta N\cos\beta - \cos\beta N\sin\beta - \sin\beta N + 2\sin\beta N - 2\sin\beta N\cos\beta = \frac{K_N}{k}\sin\beta N$$

$$-\sin\beta N\cos\beta - \cos\beta N\sin\beta + \sin\beta N = \frac{K_N}{k}\sin\beta N$$

$$\therefore\ -\sin\beta(N+1) + \sin\beta N = \frac{K_N}{k}\sin\beta N$$

$$2\cos\beta(N+\tfrac{1}{2})\cdot\sin\frac{\beta}{2} = 2\left[\cos\beta N\cos\tfrac{\beta}{2}\sin\tfrac{\beta}{2} - \sin\beta N\sin^2\tfrac{\beta}{2}\right] = \left[\cos\beta N\sin\beta - \sin\beta N(1-\cos\beta)\right]$$

$$= \cos\beta N\sin\beta - \sin\beta N + \sin\beta N\cos\beta$$

$$\therefore\ \underline{-2\cos\beta(N+\tfrac{1}{2})\cdot\sin\frac{\beta}{2} = \frac{K_N}{k}\sin\beta N}$$

12-32

System is same as those of Figs. P12-27 & P12-28

$$\therefore\quad \omega_m = 2\sqrt{\frac{k}{m}}\sin\frac{m\pi}{2(N+1)}$$

12-33

Let h = height between stories

$$m\ddot{y}_n = -k\left[(y_n - nh\theta) - (y_{n-1} - (n-1)h\theta)\right] + k\left[(y_{n+1} - (n+1)h\theta) - (y_n - nh\theta)\right]$$

$$= -k\left[y_n - y_{n-1} - h\theta\right] + k\left[y_{n+1} - y_n - h\theta\right]$$

$$= k\left[y_{n+1} - 2y_n + y_{n-1}\right]$$

$\therefore$ for harmonic motion

$$Y_{n+1} - 2\left(1 - \frac{m\omega^2}{2k}\right)Y_n + Y_{n-1} = 0$$

gen. sol. $\quad Y_n = Y_0\cos\beta n + B\sin\beta n$

boundary $\quad Y_N = Y_0\cos\beta N + B\sin\beta N$

$$B = \frac{Y_N - Y_0\cos\beta N}{\sin\beta N}$$

Gen sol. becomes

$$Y_n = Y_0\frac{\sin\beta(N-n)}{\sin\beta N} + Y_n\frac{\sin\beta n}{\sin\beta N}$$

boundary eq. for N^{th} mass

$$m\ddot{y}_N = -k\left[y_N - y_{N-1} + h\theta\right]$$

$$-\omega^2 m Y_N = -k\left[Y_N - Y_{N-1} + h\theta\right]$$

12-33 Cont:

$$\left(1 - \frac{\omega^2 m}{k}\right) Y_N = Y_{N-1} - h\theta$$

Torque Eq.

$$\sum_{n=1}^{N} nh\,(m\ddot{y}_N) - K_\theta \theta = (N+1)\, m\rho^2 \ddot{\theta}$$

$$-\omega^2 m h \sum_{n=1}^{N} n Y_n - K_\theta \theta = -(N+1)\, m\rho^2 \omega^2 \theta$$

or

$$\omega^2 m h \sum_{n=1}^{N} n Y_n - \left(K_\theta - (N+1)\, m\rho^2 \omega^2\right)\theta = 0$$

12-34

Flow diagram of FIG. 12.2-3 may be used From Prob. 12-3 range of ω may be from 0 to 5000 ie $\omega_1 \cong 264$, $\omega_2 \cong 550$ $\omega_3 < 5000$. We may choose $\Delta\omega = 20$ which will require 30 steps to get past ω_2. For $\omega > 550$ it may be advisable to choose Larger steps, ie $\Delta\omega = 100$ to locate ω_3 after which a finer interval could be used.

Computer program may be written somewhat after that of Prob 12-3

12-35

Eq. 10-8 gives $\beta_k = \frac{(2k-1)\pi}{(2N+1)}$

Eq. 10-9 gives $\omega_k = 2\sqrt{\frac{k}{m}}\,\sin\frac{(2k-1)\pi}{2(2N+1)}$

$\left.\right\}$ N = DOF, k = mode number

Then $X_n = B \sin \beta_k n$ where n = floor number

13 – 1

(1) noise in radio reception – wide freq. range & probably stationary.

(2) ground motion of earthquake – non stationary

(3) ocean wave heights – non stationary and dependent on wind or sea state

---

13 – 2

see p

---

13 – 3

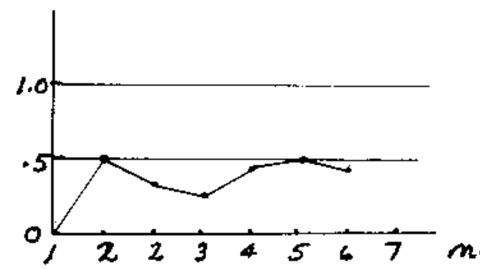
see p for definition $= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i$

$E(h)$ for	800	coins	thrown	100 times	= 400
" "	8	"	"	" "	= 4
" "	8	"	"	1000 "	= 4

13-4

Make table as suggested here

Throw	H	T	sum	sum ÷ n
n = 1		0	0	0
2	1		1	1/2
3		0	1	1/3
4		0	1	1/4
5	1		2	2/5
6	1		3	3/6
7		0	3	3/7
etc				



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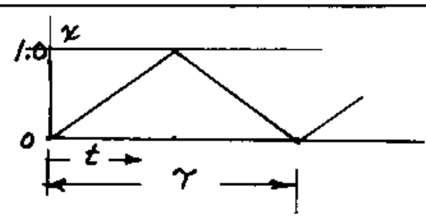
13 – 5



$x = \frac{2t}{\tau} \qquad t \le \tau/2$

$x = 2 - \frac{2t}{\tau} \qquad t \ge \tau/2$

Mean value = 0.50

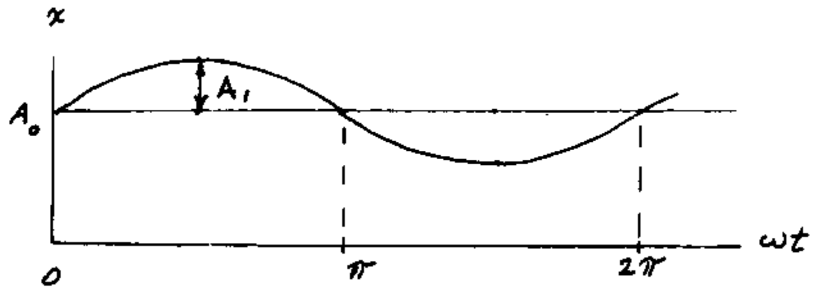
Mean square value

$$\overline{x^2} = \frac{1}{\tau}\left\{ \int_0^{\tau/2} \left(\frac{2t}{\tau}\right)^2 dt + \int_{\tau/2}^{\tau} \left[2 - \frac{2t}{\tau}\right]^2 dt \right\}$$

$$= \frac{1}{\tau}\, \frac{4}{\tau^2}\, \frac{t^3}{3}\Big|_0^{\tau/2} + \left(4t - \frac{8}{\tau}\frac{t^2}{2} + \frac{4}{\tau^2}\frac{t^3}{3}\right)\Big|_{\tau/2}^{\tau}$$

$$= 1/3$$

13-6



$x = A_0 + A_1 \sin \omega t$ — Mean val. $= A_0$ by inspection

Let $\omega t = \theta$

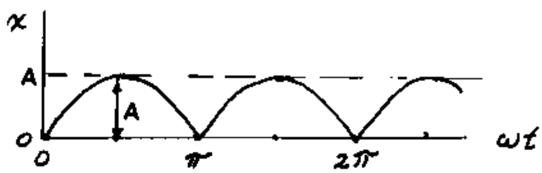
$$\overline{x^2} = \frac{1}{2\pi}\int_0^{2\pi}(A_0^2 + 2A_0A_1\sin\theta + A_1^2\sin^2\theta)\,d\theta$$

$$= \frac{1}{2\pi}\left\{A_0^2\theta - 2A_0A_1\cos\theta + \frac{A_1^2}{2}\theta - \frac{A_1^2}{2}\frac{\sin 2\theta}{2}\right\}_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{A_0^2 2\pi - 2A_0A_1(1-1) + \frac{A_1^2}{2}2\pi - 0\right\} = A_0^2 + \frac{1}{2}A_1^2$$

13-7

$x = |A \sin \omega t|$



$$\bar{x} = \frac{A}{\pi}\int_0^{\pi}\sin\theta\,d\theta = \frac{A}{\pi}(-\cos\theta)\Big|_0^{\pi} = \frac{2A}{\pi}$$

$$\overline{x^2} = \frac{A^2}{\pi}\int_0^{\pi}\sin^2\theta\,d\theta = \frac{A^2}{\pi}\frac{\theta}{2}\Big|_0^{\pi} = \frac{A^2}{2}$$

13-8 Peak values are positive quantities ∴ cannot have a probability in the negative region. Also probability of zero or infinite peaks is zero.

13-9

Area under normal probability curve = 1.0

$$\int_{-\infty}^{\infty}\frac{e^{-\frac{x^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\,dx = \frac{\sigma\sqrt{2\pi}}{\sigma\sqrt{2\pi}} = 1.0 \qquad (a)$$

13-9 Cont.

For higher moments let $\alpha = \frac{1}{2\sigma^2}$ and

$$I = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \quad - - - - - - - \quad (b)$$

From (a) $I = \sigma\sqrt{2\pi} = \sqrt{\pi}\,\alpha^{-\frac{1}{2}}$

Start with $I = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi}\,\alpha^{-\frac{1}{2}}$ (c)

Differentiate w.r.t. α

$$\frac{\partial I}{\partial \alpha} = -\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = -\frac{1}{2}\sqrt{\pi}\,\alpha^{-3/2} \quad (d)$$

$$= -\frac{1}{2}\sqrt{\pi}\,(2\sigma^2)^{3/2} = -\sqrt{2\pi}\,\sigma^3$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 e^{-\alpha x^2}}{\sigma\sqrt{2\pi}} dx = \underline{\underline{\sigma^2 = E(x^2)}}$$

Differentiate w.r.t. α again starting with (d)

$$\int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = -\frac{1}{2}\sqrt{\pi}\left\{-\frac{3}{2}\alpha^{-5/2}\right\} = \frac{3}{4}\sqrt{\pi}\,(2\sigma^2)^{5/2}$$

$$\therefore \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = \underline{\underline{E(x^4) = 3\sigma^4}}$$

Repeat

$$\frac{\partial}{\partial\alpha}\left\{\frac{3}{4}\sqrt{\pi}\,\alpha^{-5/2}\right\} = -\frac{3}{4}\cdot\frac{5}{2}\sqrt{\pi}\,\alpha^{-7/2} = -\frac{3}{4}\cdot\frac{5}{2}\sqrt{\pi}\,(2\sigma^2)^{7/2}$$

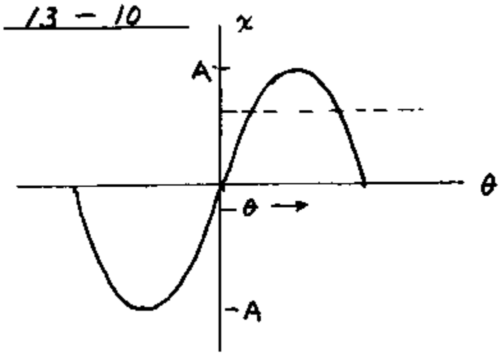
$$= -3\cdot 5\sqrt{2\pi}\,\sigma^7$$

$\div$ by $\sigma\sqrt{2\pi}$ $\quad \underline{\underline{E(x^6) = 3.5\,\sigma^6}}$

General Eq. is $\quad \underline{\underline{E(x^n) = 1\cdot 3\cdot 5\cdot \cdots (n-1)\,\sigma^n}}$ (for n even)

by inspection $E(x^n) = 0$ for n odd.

13 - 10

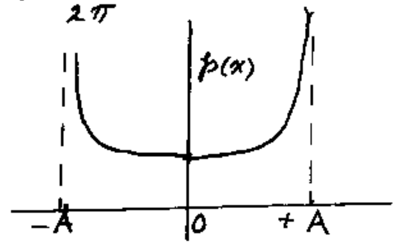


$x = A\sin\theta$

When $x=0$ $\quad P(x) = \frac{1}{2}$

i.e. half the time x is less than $x = 0$.

As we increase x from $x=0$ we add $\frac{2\theta}{2\pi}$ to the probability



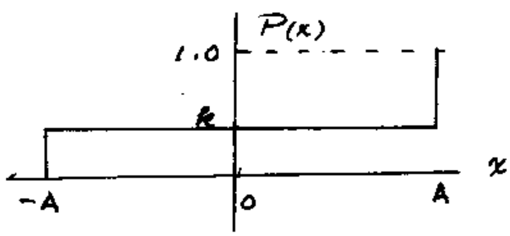
$$\theta = \sin^{-1}\frac{x}{A}$$

$$P(x) = \frac{1}{2} \pm \sin^{-1}\frac{x}{A}$$

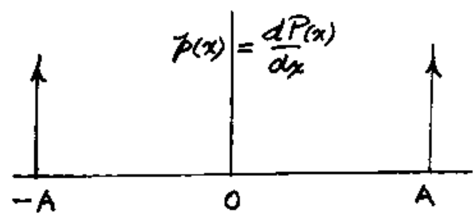
$$p(x) = \frac{dP(x)}{dx} = \frac{1}{\pi}\frac{d}{dx}\left(\sin^{-1}\frac{x}{A}\right) = \frac{1}{\pi\sqrt{A^2-x^2}}$$

---

13 - 11 Measure the total length of line at $-A$ and divide by total length at A and $-A$. Let this fraction be k; then the cumulative prob. curve will appear as in (a). Density curve is shown in (b)



(a)



(b)

---

13 - 12 $\quad x(t) = A\cos t \qquad x(t+\tau) = A\cos(t+\tau)$

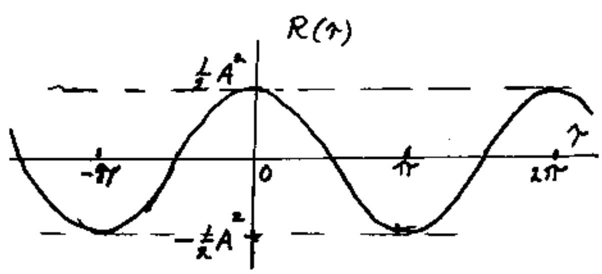
$$= A[\cos t\cos\tau - \sin t\sin\tau]$$

$$x(t)\,x(t+\tau) = A^2[\cos^2 t\cos\tau - \cos t\sin t\sin\tau]$$

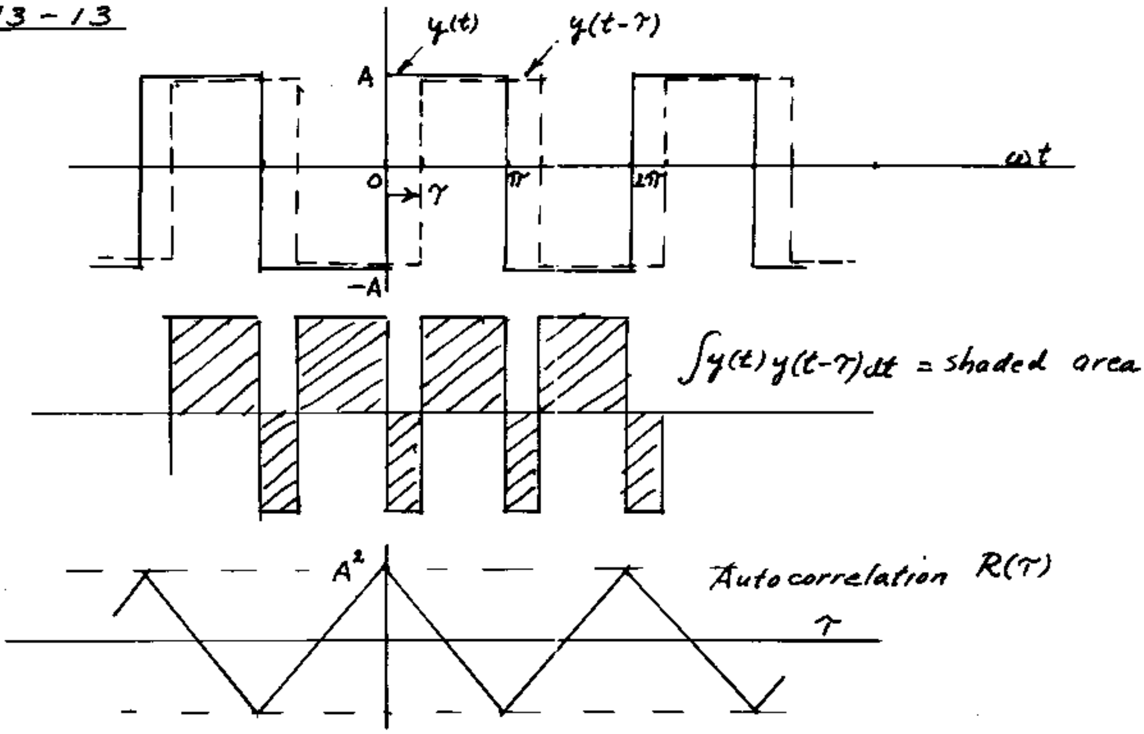
$$R(\tau) = \lim_{T\to\infty}\frac{A^2}{T}\int_{-T/2}^{T/2}\left[\cos\tau\cdot\tfrac{1}{2}(1+\cos 2t) - \sin\tau\cdot\sin t\cos t\right]dt$$

$$= \lim_{T\to\infty}\frac{A^2}{T}\left[\cos\tau\cdot\left(\tfrac{T}{2}\right) - \sin\tau\cdot(0)\right]$$

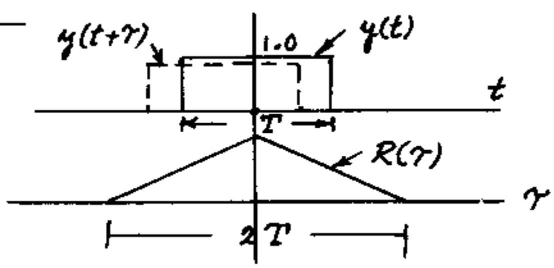
$$= \frac{A^2}{2}\cos\tau$$



13-13

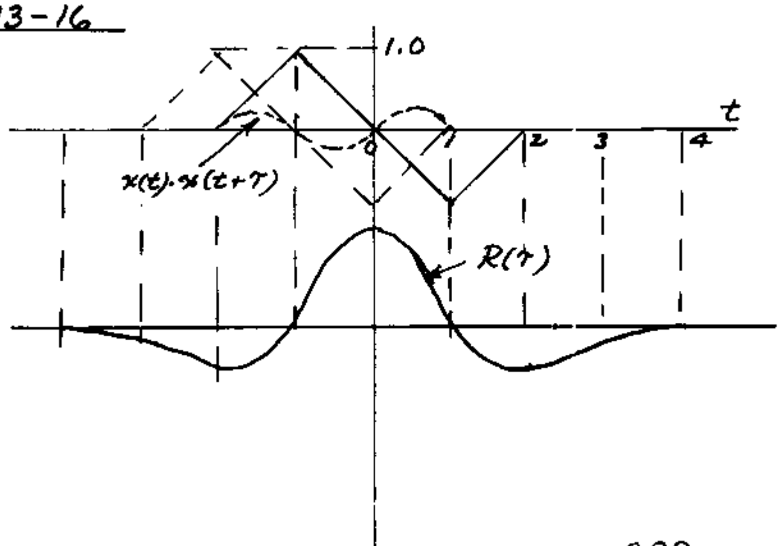


13-14



13-15 This problem is similar to Prob. 13-14. Integrate the area under $y(t)\cdot y(t+\tau)$ curve. Start curve with $R(\tau) = 5$ at $\tau = 0$ and linearly decrease to $R(\tau) = 1.0$ at $\tau = 1$, etc. shift traced curve as suggested.

13-16



13-17

$$\overline{x^2} = .20\ \frac{g^2}{Hz} \times 500\ Hz = 100\ g^2$$

$$= 100 \times 9.81^2 = 9623.6$$

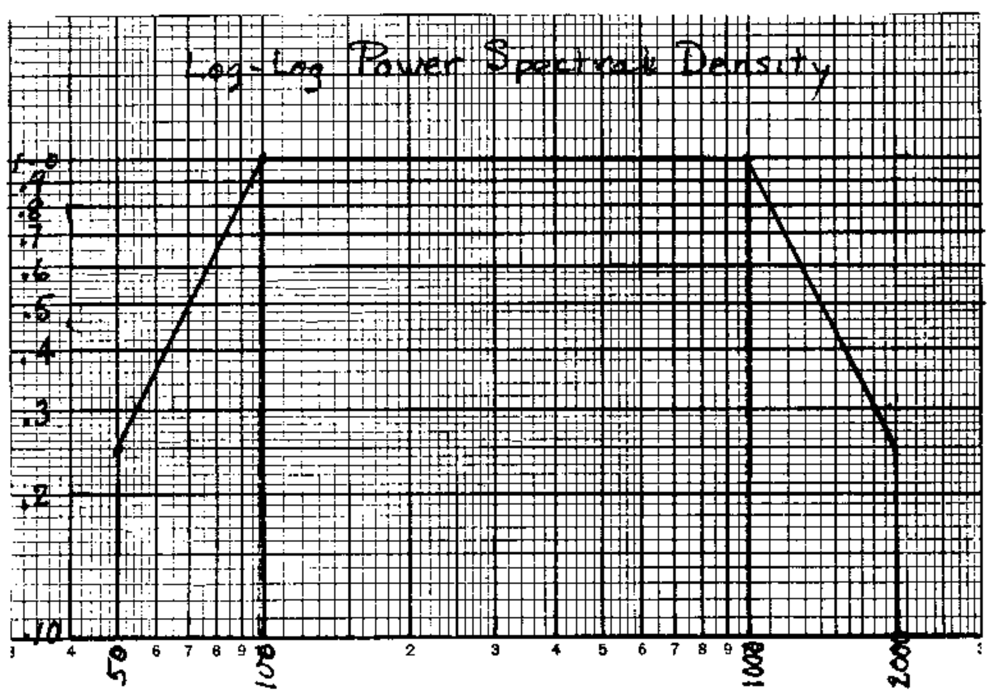
$$RMS = \tilde{x} = \sqrt{9623.6} = 98.1\ m/s^2$$

13-18

$$Area = 1 \times 900 + 2 \times 1000 = 2900\ g^2$$

$$RMS = 9.81\sqrt{2900} = 528.3\ m/s^2$$

13-19

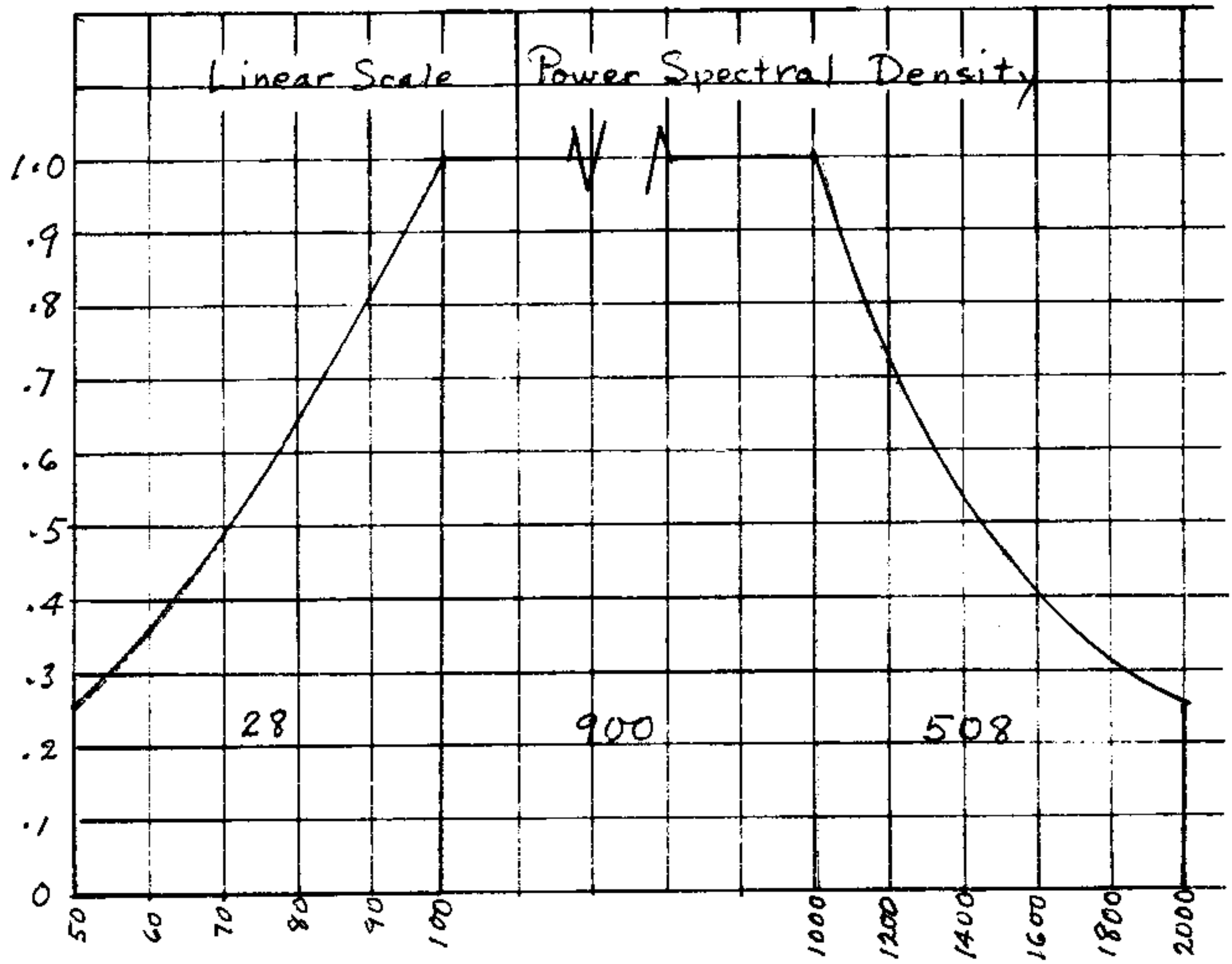


$$DB = 10 \log_{10} \frac{1.0}{.25} = 6.02\ \text{db/octave}$$

The log-log plot is replotted on linear scale

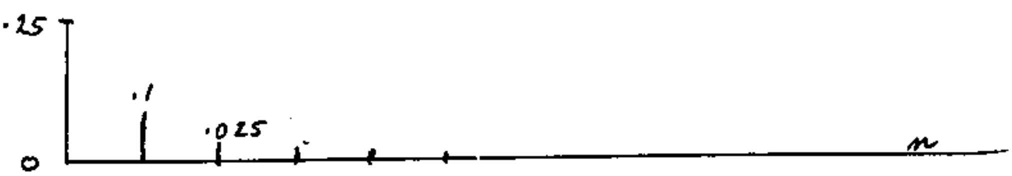
$\overline{x^2}$ = total area = $\underline{\underline{1436}}$ RMS = 37.9 m/s^2

13-19 Cont:

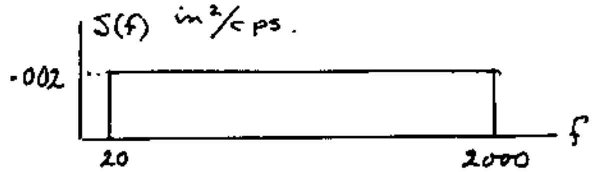


13-20 Procedure here is to determine the Fourier series and plot the quantity $\frac{1}{2} C_m C_m^*$ where $C_n = a_m - i b_m$. Note that $C_m C_m^* = a_m^2 + b_m^2$ i.e. for FIG. 13-20 (a), $x(t) = \frac{1}{2} - \frac{1}{\pi}\left[\sin\omega_1 t + \frac{1}{2}\sin 2\omega_1 t + \cdots\right]$ (see prob. 1-12.) $\therefore$ $a_m = 0$, $b_m = \frac{1}{m\pi}$

$$\sum \tfrac{1}{2} C_n C_n^* = \frac{1}{4} + \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} \cdots$$



13-21

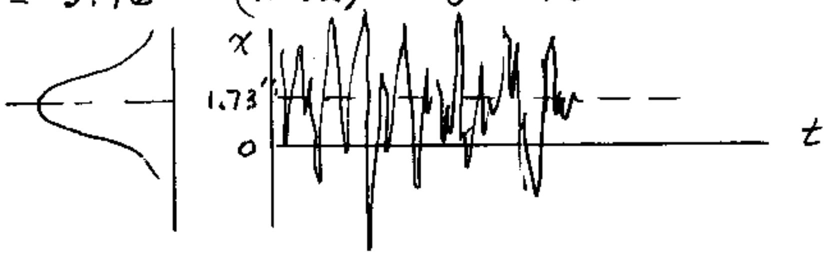


$\overline{x^2} = .002 \times 1980 = 3.96 \ \text{in}^2$

$\text{RMS} = \sqrt{\overline{x^2}} = 1.99 \ \text{in}$ $\qquad \bar{x} = \sqrt{3.0} = 1.732''$

$\sigma^2 = \overline{x^2} - (\bar{x})^2$

$= 3.96 - (1.732)^2 = 0.9602 \qquad \sigma = 0.9799$

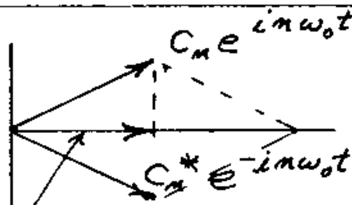


13-22

C_n found by multiplying $f(t)$ by $e^{-in\omega_0 t}$ and integrating over one period

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-in\omega_0 t}\, dt$$

13-23



$C_n = \frac{1}{2}(a_n - i\, b_n)$

$C_0 = \frac{1}{2} a_0$

$$Re\left(C_n e^{in\omega_0 t}\right) = \frac{1}{2}\left(C_n e^{in\omega_0 t} + C_n^* e^{-in\omega_0 t}\right)$$

$$\therefore \ f(t) = \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left(C_n e^{in\omega_0 t} + C_n^* e^{-in\omega_0 t}\right)$$

but $C_n^* = C_{-n} \quad \therefore$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2} C_n e^{in\omega_0 t} = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$$

$C_n = 2 c_n$ (see Eq. 13.2-9)

13-24 Same procedure as Prob 13-20

ie Fig P13-24 (See Prob 1-11)

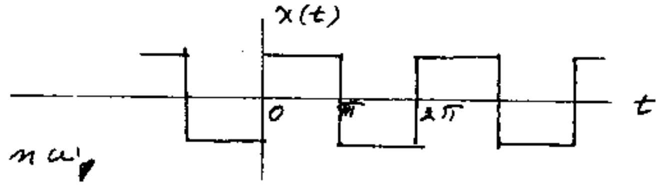
$$x(t) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos\omega_1 t + \frac{1}{3^2}\cos 3\omega_1 t + \cdots\right)$$

$$S(f) = \sum \frac{C_n C_n^*}{2} = \sum \bar{C}_n^2$$

13-25 See Ch. 1 Sec. 1.2

13-26

$$x = \sum_{n=-\infty}^{\infty} C_n e^{i\omega_n t} \qquad \omega_n = n\omega_1$$



$$C_n = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\, e^{-i\omega_n t}\, dt = \frac{1}{2}(a_n - i b_n), \qquad C_0 = \frac{a_0}{2}$$

Let $\omega_n t = n\omega_1 t = n\theta$ where $\begin{cases} \theta = \omega_1 t \\ d\theta = \omega_1 dt \end{cases}$

Limits $n\omega_1 \frac{\tau}{2} = n\theta$

$$n\frac{2\pi}{\tau}\frac{\tau}{2} = n\pi = n\theta \qquad \therefore\ \theta \text{ goes betw. } \pm\pi$$

$$C_n = \frac{1}{\tau}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\frac{d\theta}{\omega_1} \qquad \omega_1\tau = \frac{2\pi}{\tau}\cdot\tau = 2\pi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta = \frac{2A}{i n\pi} \quad \text{for } n \text{ odd}$$

$$= 0 \quad \text{for } n \text{ even}$$

$$\therefore\ x(t) = \frac{2A}{\pi}\sum_{n=-\infty}^{\infty}\frac{e^{in\omega_1 t}}{in} = \frac{4A}{\pi}\sum_{n=0}^{\infty}\frac{1}{n}\left(\frac{e^{in\omega_1 t} - e^{-in\omega_1 t}}{2i}\right)$$

$$= \frac{4A}{\pi}\left[\sin\omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \frac{1}{5}\sin 5\omega_1 t + \cdots\right]$$

plot $\frac{1}{2}$ (square of ampl.)

13-27

$$\left(\frac{xk}{F}\right) = \frac{1}{\sqrt{(1-\eta^2)^2+(2\zeta\eta)^2}} \quad \text{where } \eta = \frac{f}{f_m} = \frac{\omega}{\omega_m}$$

At $\eta = 1$ $\quad \frac{xk}{F} = \frac{1}{2\zeta}$ $\qquad$ At half power pt. $\frac{xk}{F} = \frac{1}{\sqrt{2}}\cdot\frac{1}{2\zeta}$

$$\therefore \quad \frac{1}{2}\left(\frac{1}{2\zeta}\right)^2 = \frac{1}{(1-\eta^2)^2+(2\zeta\eta)^2}$$

$$\eta^4 - 2\eta^2 + 1 + 4\zeta^2\eta^2 = 8\zeta^2$$

$$\eta^4 - 2(1-2\zeta^2)\eta^2 + (1-8\zeta^2) = 0$$

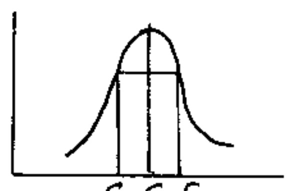
$$\therefore \quad \eta^2 = (1-2\zeta^2 \pm 2\zeta\sqrt{1+\zeta^2}$$

$$\therefore \eta \cong (1 \pm 2\zeta)^{1/2} = 1 \pm \zeta + \text{negl. term}$$

$\left\{ \begin{array}{l} \text{for } \zeta << 1 \\ \eta^2 \cong 1 \pm 2\zeta \\ \text{by neglecting } \zeta^2 \end{array} \right.$

$$\therefore f_1 = f_m(1-\zeta) = f_m\left(1 - \frac{1}{2Q}\right)$$

$$f_2 = f_m(1+\zeta) = f_m\left(1 + \frac{1}{2Q}\right)$$



13-28

$$\int_0^\infty \frac{d\eta}{(1-\eta^2)^2 + (2\zeta\eta)^2} \quad \text{where } \eta = \frac{f}{f_m}$$

1st find poles from zeros of denominator

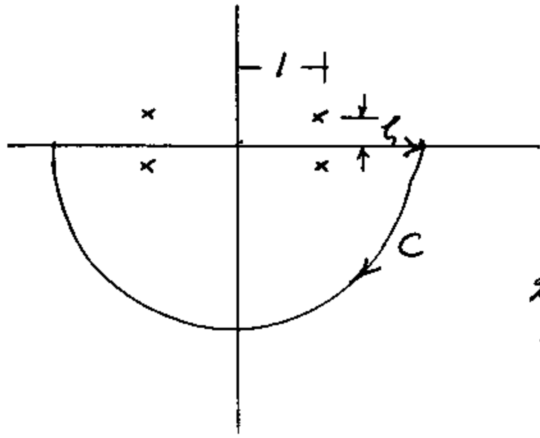
$$F(\eta) = \eta^4 - 2(1-2\zeta^2)\eta^2 + 1 = 0$$

$$\eta^2 = (1-2\zeta^2) \pm i\,2\zeta\sqrt{1-\zeta^2} \cong 1 \pm i2\zeta$$

for $\zeta << 1$

Sea Theory of Residues p 135-137 Laplace Transformation by W. T. Thomson - Prentice-Hall, Inc.

13-28 Cont:



Form contour with infinite circle. Poles are $\eta \cong 1 \pm i\zeta$

Integrate around contour C

$$2\int_0^\infty \cdots \int_C = 2\pi i \sum \text{Residues within contour}$$

Residues at the two poles are

$$\frac{1}{F'(\eta)} = \frac{1}{2(1-\eta^2)(-2\eta)+8\zeta^2\eta}$$ evaluated at $\eta_1 = 1 - i\zeta$
$= -(1+i\zeta)$

$$\cong i\,\frac{1}{4\zeta}$$

and $\eta_1^2 = 1 - i2\zeta$

$\eta_2^2 = 1 + i2\zeta$

Since $\int_C = 0$ we have

$$-2\int_0^\infty = 2\pi i\left(\frac{i}{4\zeta}\right) = -\frac{\pi}{2\zeta}$$

$$\therefore \int_0^\infty \frac{d\eta}{(1-\eta^2)^2+(2\zeta\eta)^2} = \frac{\pi}{4\zeta}$$

Can also be checked by numerical integration (must use very small $\Delta\eta$)

13-29 from definition

$$\bar{X}(s) = \frac{F(s)}{ms^2 + k(1+i\gamma)} = \bar{H}(s)\,\bar{F}(s)$$

13-30

$$H(\omega) = \frac{1}{k} \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta \frac{\omega}{\omega_n}\right]^2}}$$

Each component can be treated separately. For 1^{st} component $F \cos(.5\omega_n t - \theta_1)$, the mean square response is

$$\frac{1}{\left[1-(.5)^2\right]^2 + \left[.2 \times .5\right]^2} \times \frac{1}{2}\left(\frac{F}{k}\right)^2 = 1.746 \times \frac{1}{2}\left(\frac{F}{k}\right)^2$$

similarly other components are

2^{nd} comp $\quad \dfrac{1}{[.2]^2} \dfrac{1}{2}\left(\dfrac{F}{k}\right) = 25 \times \dfrac{1}{2}\left(\dfrac{F}{k}\right)^2$

3^{rd} comp $\quad \dfrac{1}{\left[1-2^2\right]^2 + \left[.2 \times 2\right]^2} \dfrac{1}{2}\left(\dfrac{F}{k}\right)^2 = .109 \times \dfrac{1}{2}\left(\dfrac{F}{k}\right)^2$

$$\therefore \overline{x}^2 = \left[1.746 + 25 + .109\right] \frac{1}{2}\left(\frac{F}{k}\right)^2 = 26.85 \times \frac{1}{2}\left(\frac{F}{k}\right)^2$$

$$= 13.43 \left(\frac{F}{k}\right)^2$$

13-31 From Sec 13.4 & Ex. 13.8-1, $\sigma = 26.6g$

$2\sigma = 53.2g$

$$P\left[-2\sigma \le x(t) \le 2\sigma\right] = 95.4\%$$

$$\therefore P\left[x(t) \ge 2\sigma\right] = 100 - 95.4 = \underline{\underline{4.6\%}}$$

$$P\left[X \ge 2\sigma\right] = \underline{\underline{13.5\%}}$$

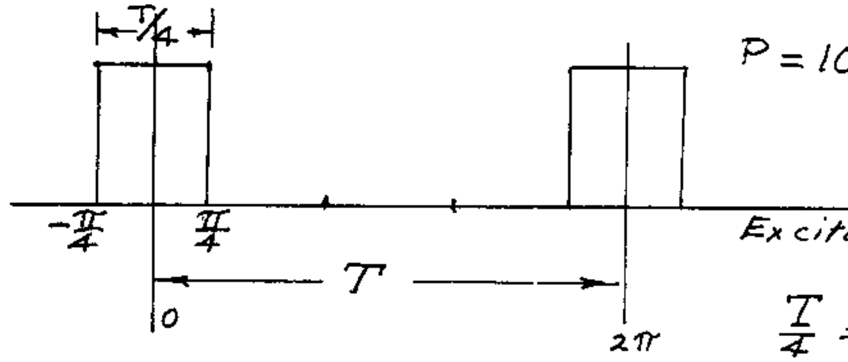
13-32

$m = 40 \ \text{Kg.}$ $\qquad \omega_n = 2\pi \times 2.20 = 13.82$

Assume $\zeta = .15$ $\qquad k = m\omega_n^2 = 40 \times 13.82^2 = 7643.$

$f_n = 2.20 \ Hz$ $\qquad \omega_1 = \frac{2\pi}{T}$

$$X = \frac{F_0}{k} \frac{\sin(\omega t - \phi)}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$



$P = 10^6 \ N$

Excitation Pulses

$$\frac{T}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$$

F.S. of the excitation is a cosine series

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos n\omega_1 t \, dt = \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} P \cos n\theta \frac{d\theta}{\omega_1}$$

$$= \frac{P}{\pi\omega_1} \left. \frac{\sin n\theta}{n} \right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \left(\frac{P}{n\pi\omega_1}\right) 2 \sin\frac{n\pi}{4}$$

$$= \left(\frac{2P}{\pi\omega_1}\right)\left(\frac{\sin n\frac{\pi}{4}}{n}\right)$$

$$a_0 = \frac{2}{T}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} P \, dt = \frac{P}{2}$$

n	$\frac{\sin n\frac{\pi}{4}}{n}$
1	$\sqrt{2}/2$
2	$1/2$
3	$\sqrt{2}/6$
4	0
5	$-\sqrt{2}/10$
6	$-1/6$

13-32 Cont:

Excitation Spectrum $S_p(\omega)$

$$S_p(\omega) = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \frac{a_n^2}{2} = \frac{P^2}{8} + \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{2P}{\pi\omega_1}\right)^2 \frac{\sin^2 n\frac{\pi}{4}}{n^2}$$

$$= 10^{12}\left[\frac{1}{8} + \left(\frac{2}{\pi^2\omega_1^2}\right)\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{18} + 0 + \frac{1}{50} + \cdots\right)\right]$$

Response Spectrum $\overline{y^2} = \int_0^{\infty} HH^* S_p \, df = \overline{x^2}$

$$= \sum HH^* S_p$$

$$HH^* = \frac{1}{k^2} \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta \frac{\omega}{\omega_n}\right]^2}$$

$\omega = n\omega_1 = n\frac{2\pi}{T}$ $\qquad \omega_n = 2\pi f_n = 2\pi\, 2.2$

$\frac{\omega}{\omega_n} = n\left(\frac{1}{2.2T}\right)$ subst. into $\overline{x^2}$

13-33 For base motion, the relative motion is given by Eq. 3.5-4, which can be written as

$$Z = \frac{\left(\frac{\omega}{\omega_n}\right)^2 Y}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta \frac{\omega}{\omega_n}\right]^2}}$$

Thus with $S(f)$ = Spectral density of excitation

$$\overline{Z^2} = \int_0^{\infty} S_Y(f_r) \frac{\left(\frac{f}{f_n}\right)^4 df}{\left[1 - \left(\frac{f}{f_n}\right)^2\right]^2 + \left[2\zeta \frac{f}{f_n}\right]^2} = S_Y(f_n) \cdot f_n \cdot \int_0^{\infty} \frac{\xi^4 d\xi}{[1-\xi^2]^2 + [2\zeta\xi]^2}$$

$\therefore$ must evaluate $\int_0^{\infty} \frac{\xi^4 d\xi}{[1-\xi^2]^2 + [2\zeta\xi]^2}$

13 - 34

$$\ddot{x} = \frac{(k + i\omega c)}{k - m\omega^2 + i\omega c}\,\ddot{y} = \frac{1 + i\left(25\frac{f}{f_m}\right)}{1 - \left(\frac{f}{f_m}\right)^2 + i\left(25\frac{f}{f_m}\right)}\,\ddot{y}$$

$$\overline{\ddot{x}^2} = \int_0^\infty f_m\,\overline{\ddot{y}^2}\;\frac{1 + i\left(25\frac{f}{f_m}\right)}{1 - \left(\frac{f}{f_m}\right)^2 + i\left(25\frac{f}{f_m}\right)} \cdot \frac{1 - i\left(25\frac{f}{f_m}\right)}{1 - \left(\frac{f}{f_m}\right)^2 - i\left(25\frac{f}{f_m}\right)}\;\frac{df}{f_m}$$

$$= \int_0^\infty S_{\ddot{y}}(f) \cdot f_m \cdot \frac{1 + \left(25\frac{f}{f_m}\right)^2}{\left[1 - \left(\frac{f}{f_m}\right)^2\right]^2 + \left[25\frac{f}{f_m}\right]^2} \cdot d\left(\frac{f}{f_m}\right)$$

---

13 - 35

$$\omega^2 = \frac{k}{m} = \frac{k}{60} = (2\pi \cdot 4)^2$$

$$k = 60 \times (8\pi)^2 = 37{,}899. \ \text{N/m}$$

$$k^2 = 1436. \times 10^6 \ \text{N}^2/\text{m}^2$$

$$H^2 = \frac{1}{1436 \times 10^6 \left\{\left[1 - \left(\frac{\omega}{8\pi}\right)^2\right]^2 + \left[2(.05)\frac{\omega}{8\pi}\right]^2\right\}}$$

$$\overline{y^2} = \int_0^\infty H^2 S(\omega)\,d\omega \cong \frac{S(\omega_n)}{k^2}\, f_n\, \frac{\pi}{4\zeta}$$

$$= \frac{100 \times 10^3}{1436 \times 10^6} \cdot \cancel{4} \cdot \frac{\pi}{\cancel{4} \times .05} = .00438$$

$$\sigma^2 = \overline{y^2} = .00438 \ \text{m}^2, \qquad \sigma = .0662 \ \text{m}$$

$$y = .132 = 1.99\,\sigma$$

$$P[\,y > 1.99\sigma\,] = \underline{\underline{4.6\,\%}}$$

13 - 36

$m = 272$ kg $\qquad f_n = 26$ Hz

$\omega_n = 2\pi f_n = 163.36$

$k = m\omega^2 = 272 \times (163.36)^2 \qquad \zeta = 0.10$

$= 7258 \times 10^3$ N/m

$k^2 = 52.689 \times 10^{12}$

$$\overline{x^2} = \sigma^2 = \frac{S(f_n)}{k^2} \cdot f_n \frac{\pi}{4\zeta}$$

$$\sigma^2 = \frac{4 \times 10^6}{52.689 \times 10^{12}} \times 26 \times \frac{\pi}{4 \times .10} = 15.50 \times 10^{-6}$$

$\sigma = .003937 \qquad 0.012\ m = 3.05\ \sigma$

$P[|x| > 3.05\sigma] = 0.3\ \%$

13-37

$S(f) = 5 \times 10^6\ N^2/Hz, \qquad \zeta = 0.03$

$\omega_n = 30 \qquad f_n = \frac{\omega_n}{2\pi} = 4.775$ Hz

$m = 1500$ kg, $\qquad k = m\omega^2 = 1500 \times 30^2 = 1.350 \times 10^6$

$k^2 = 1.823 \times 10^{12}$

$$\sigma^2 = \overline{x^2} = \frac{S(f_n)}{k^2} f_n \cdot \frac{\pi}{4\zeta} = \frac{5 \times 10^6}{1.823 \times 10^{12}} \times 4.775 \times \frac{\pi}{4 \times .03}$$

$= 342.9 \times 10^{-6} \qquad \sigma = 0.01852$

$.037 = 2\sigma$

$P[A > .037] = P[A > 2\sigma] = 13.5\ \%$

13-38

$$x(t) = \int_0^\infty f(t-\xi)\, g(\xi)\, d\xi$$

$$X(i\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty}\int_0^\infty f(t-\xi) g(\xi) d\xi\, e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty}\int_0^\infty f(t-\xi) e^{-i\omega(t-\xi)} dt\;\; g(\xi) e^{-i\omega\xi} d\xi$$

$$X(i\omega) = \int_0^\infty \left[\int_{-\infty}^{\infty} f(t-\xi) e^{-i\omega(t-\xi)} dt\right] g(\xi) e^{-i\omega\xi} d\xi$$

Let $(t-\xi) = \tau \qquad dt = d\tau$

$$= \int_0^\infty \left[\int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} d\tau\right] g(\xi) e^{-i\omega\xi} d\xi$$

$$= \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} d\tau \int_0^\infty g(\xi) e^{-i\omega\xi} d\xi = F(i\omega)\, H(i\omega)$$

$$\overline{x^2} = \int_0^\infty \lim_{T\to\infty} \frac{1}{2\pi T} X(i\omega) X^*(i\omega)\, d\omega$$

$$= \int_0^\infty \lim_{T\to\infty} \frac{1}{2\pi T} F(i\omega) F^*(i\omega)\, H(i\omega) H^*(i\omega)\, d\omega$$

$$= \int_0^\infty S_F(\omega)\, |H(i\omega)|^2\, d\omega$$

13-39

$$H(i\omega) = |H(i\omega)|\, e^{i\phi(\omega)}$$

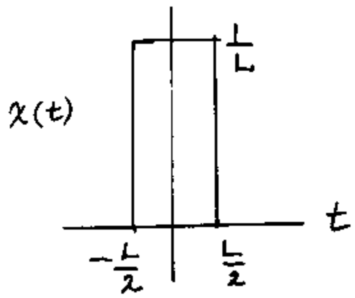
$$H^*(i\omega) = |H(i\omega)|\, e^{-i\phi(\omega)}$$

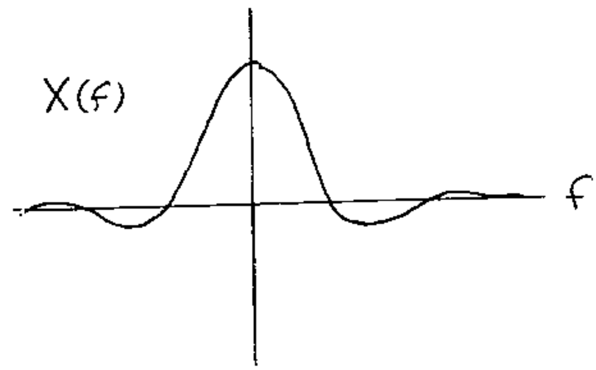
$$\therefore \quad \frac{H(i\omega)}{H^*(i\omega)} = e^{i2\phi(\omega)}$$

13-40 Let $X(f)$ = F.T. of rectangular pulse

$$X(f) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\frac{1}{L}\right) e^{-2\pi f t} d = \left(\frac{1}{L}\right) \frac{e^{-i2\pi ft}}{-i2\pi f} \Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \frac{1}{L} \frac{1}{\pi f}\left[\frac{e^{i2\pi f\frac{L}{2}} - e^{-i2\pi f\frac{L}{2}}}{2i}\right] = \frac{\sin \pi f L}{\pi f L}$$





13-41

$$\int_{-\infty}^{\infty} |f(t)|\, dt = \int_{-\infty}^{\infty} u(t)\, dt = \int_{0}^{\infty} 1\, dt = \infty$$

∴ Unit step function cannot have a F.T.

13-42

$$S_{FX}(\omega) = \lim_{T\to\infty} \frac{1}{2\pi T} F^*(i\omega) X(i\omega)$$

$$= \left[\lim_{T\to\infty} \frac{1}{2\pi T} F^*(i\omega) F(i\omega)\right] H(i\omega) = S_F(i\omega) H(i\omega)$$

$$S_{XF}(\omega) = \lim_{T\to\infty} \frac{1}{2\pi T} X^* F = \lim_{T\to\infty} \frac{1}{2\pi T} F^* H^* F = S_F(i\omega) H^*(i\omega)$$

$$\frac{S_{FX}}{S_{XF}} = \frac{S_F |H| e^{i\phi}}{S_F |H| e^{-i\phi}} = e^{i2\phi(\omega)}$$

13-42 Cont:

$$\frac{S_F}{S_{xF}} = \frac{\cancel{S_x}}{\cancel{S_x} H^*} \qquad \frac{S_{Fx}}{S_F} = \frac{\cancel{S_F} H}{\cancel{S_F}} = H$$

$$\frac{S_F}{S_{xF}} = \frac{1}{a-ib} = \frac{(a+ib)}{(a^2+b^2)} = \frac{H(i\omega)}{|H|^2}$$

13-43

$$\frac{d^2\bar{u}(x,s)}{dx^2} = \left(\frac{s}{c}\right)^2 \bar{u}(x,s)$$

$$\bar{u}(x,s) = C_1 e^{\frac{sx}{c}} + C_2 e^{-\frac{sx}{c}}$$

$$\bar{F}(x,s) = AE\frac{d\bar{u}}{dx} = AE\frac{s}{c}\left[C_1 e^{\frac{sx}{c}} - C_2 e^{-\frac{sx}{c}}\right]$$

$$\bar{F}(0,s) = AE\frac{s}{c}\left[C_1 - C_2\right]$$

$$\bar{F}(\ell,s) = 0 = C_1 e^{\frac{s\ell}{c}} - C_2 e^{-\frac{s\ell}{c}} \qquad \therefore \quad C_1 = C_2 e^{-\frac{2s\ell}{c}}$$

$$C_2 = \frac{-c\,\bar{F}(0,s)}{AEs\left(1-e^{-\frac{2s\ell}{c}}\right)}$$

$$\bar{u}(x,s) = \frac{-c\,\bar{F}(0,s)}{AEs\left(1-e^{-\frac{2s\ell}{c}}\right)}\left[e^{-\frac{2s\ell}{c}} e^{\frac{sx}{c}} + e^{-\frac{sx}{c}}\right]$$

$$= \frac{-c\,\bar{F}(0,s)\,e^{-\frac{s\ell}{c}}}{AEs\left(1-e^{-\frac{2s\ell}{c}}\right)}\left[e^{\frac{s}{c}(x-\ell)} + e^{-\frac{s}{c}(x-\ell)}\right]$$

13-44

$$p(x,t) = F_0 e^{i\omega t}\delta(x)$$

$$\bar{p}(x,s) = \frac{F_0}{s-i\omega}\delta(x), \qquad \bar{F}(0,s) = \int_0^\ell \bar{p}(x,s)\,dx = \frac{F_0}{s-i\omega}$$

$$\bar{u}(x,s) = \frac{-cF_0\,e^{\frac{s\ell}{c}}}{s(s-i\omega)AE\left(1-e^{-\frac{2s\ell}{c}}\right)}\left[e^{\frac{s}{c}(x-\ell)} + e^{-\frac{s}{c}(x-\ell)}\right]$$

$$\therefore \; u(x,t) = \frac{-cF_0\,e^{i\omega t}}{\omega AE\sin\frac{\omega\ell}{c}}\cos\frac{\omega\ell}{c}\left(\frac{x}{\ell}-1\right)$$

$$\sigma = E\frac{du}{dx} = \frac{-cF_0 e^{i\omega t}}{\omega A\sin\frac{\omega\ell}{c}}\cdot\frac{\omega\ell}{c\ell}\sin\frac{\omega\ell}{c}\left(\frac{x}{\ell}-1\right) = \frac{-F_0 e^{i\omega t}}{A\sin\frac{\omega\ell}{c}}\sin\frac{\omega\ell}{c}\left(\frac{x}{\ell}-1\right)$$

$$u(x,t) = \sum_{n=0}^{\infty} \phi_n(x)\, q_n(t)$$

$$\sigma(x,t) = E\,\frac{du}{dx} = E\sum_{n=1}^{\infty} \phi_n'(x)\, q_n(t) \qquad \text{where } \phi' = \frac{d\phi}{dx}$$

$$\overline{\sigma(x,t)\sigma(x',t)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sigma(x,t)\,\sigma(x',t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2\pi T}\,\sigma^*(x,i\omega)\,\sigma(x',i\omega)\,d\omega$$

$$\sigma(x,i\omega) = E\sum Q_m(i\omega)\,\phi_m'(x)$$

$$\overline{\sigma(x,t)\sigma(x',t)} = \frac{E^2}{2}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \phi_m'(x)\phi_k'(x')\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2\pi T}\, Q_m^*(i\omega)\, Q_k(i\omega)\, d\omega$$

$$\ddot{q}_m + \omega_m^2(1+i\gamma)q_m = \frac{1}{m\ell}\int_0^{\ell} p(x,t)\,\delta(x)\,\phi_m(x)\,dx = \frac{1}{m\ell}\,F(0,t)\,\phi_m(0)$$

$$Q_m(i\omega) = \frac{F(0,i\omega)\,\phi_m(0)}{m\ell\left[(\omega_m^2-\omega^2)+i\gamma\omega_m^2\right]} = \frac{F(0,i\omega)\,\phi_m(0)}{m\ell\omega_m^2\left[1-\left(\frac{\omega}{\omega_m}\right)^2+i\gamma\right]}$$

$$\int_{-\infty}^{\infty} \lim_{T\to\infty}\frac{1}{2\pi T}\;\frac{F^*(0,i\omega)\,F(0,i\omega)\,\phi_m(0)\,\phi_k(0)\;d\omega}{m^2\ell^2\omega_m^2\omega_k^2\left[1-\left(\frac{\omega}{\omega_m}\right)^2-i\gamma\right]\left[1-\left(\frac{\omega}{\omega_m}\right)^2+i\gamma\right]}$$

$$= \int_{-\infty}^{\infty} \frac{S(i\omega)\,\phi_m(0)\,\phi_k(0)\;d\omega}{m^2\ell^2\omega_m^2\omega_k^2\left[1-\left(\frac{\omega}{\omega_m}\right)^2-i\gamma\right]\left[1-\left(\frac{\omega}{\omega_m}\right)^2+i\gamma\right]}$$

Greatest contribution occurs when $k=m$ $\therefore$ change double $\sum$ to single summation with $k=m$.

13-45 Cont:

$$\overline{\sigma(x,t)\sigma(x',t)} = \frac{E^2}{2}\sum_n \phi_n'(x)\phi_n'(x')\phi_n^2(0)\frac{1}{m^2\ell^2\omega_n^4}\int_{-\infty}^{\infty}\frac{S(i\omega)\,d\omega}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\gamma^2}$$

$$\overline{\sigma^2(x,t)} = \frac{E^2}{2}\sum_n \phi_n'^2(x)\phi_n^2(0)\frac{1}{m^2\ell^2\omega_n^3}S(\omega_n)\frac{\pi}{\gamma}$$

where;

$$\phi_n = \sqrt{2}\cos n\pi\left(\frac{x}{\ell}-1\right)$$

$$\omega_n = n\pi\frac{c}{\ell}$$

$$c = \sqrt{\frac{AE}{m}}$$

$$E = \frac{c^2 m}{A}$$

$$\therefore\ \overline{\sigma^2(x,t)} \cong \frac{2\pi}{\gamma}\sum \frac{c}{A^2 n\pi\ell}S(\omega_n)\sin^2\frac{n\pi x}{\ell}$$

13-46

Prove $FT[x(t-t_0)] = e^{-i2\pi f t_0} X(f)$

where $X(f) = FT[x(t)]$

From Eq.(13.6-1) $\quad x(t-t_0) = \int_{-\infty}^{\infty} X(f)\, e^{i2\pi f(t-t_0)} df$

$$= \int_{-\infty}^{\infty} \left[e^{-i2\pi f t_0} X(f) \right] e^{i2\pi f t}\, df$$

Comparison with Eq.(13.6-2) shows $e^{-i2\pi f t_0} X(f) = FT[x(t-t_0)]$

13-47

Prove $FT[x(t) * y(t)] = X(f)\,Y(f)$

$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$ = convolution of $x(t) * y(t)$

From Eq(13.6-2)

$$FT[x(t) * y(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \right] e^{-i2\pi f t}\, dt$$

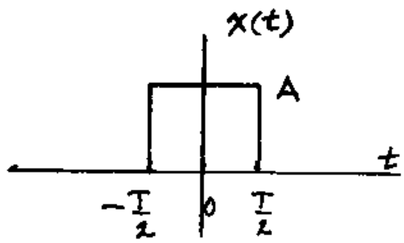
$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y(t-\tau)\, e^{-i2\pi f t}\, dt \right] x(\tau)\, d\tau$$

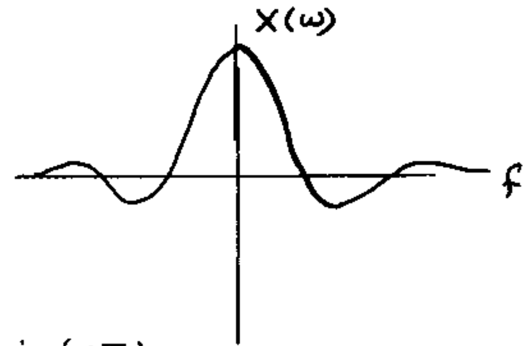
Let $(t-\tau) = \xi$, $\quad t = \xi + \tau \qquad dt = d\xi$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y(\xi)\, e^{-i2\pi f \xi}\, d\xi \right] x(\tau)\, e^{-i2\pi f \tau}\, d\tau$$

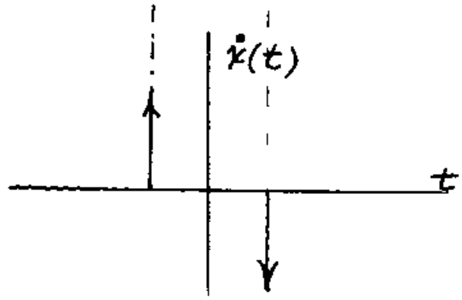
$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-i2\pi f \tau}\, d\tau \cdot \int_{-\infty}^{\infty} y(\xi)\, e^{-i2\pi f \xi}\, d\xi = X(f)\,Y(f)$$

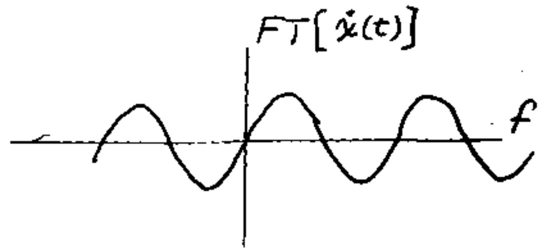
13-48





$$X(\omega) = FT[x(t)] = A\,T\,\frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)}$$





$$FT[\dot{x}(t)] = i\omega\, FT[x(t)] = i\cancel{\omega}\cdot A\cancel{T}\,\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\cancel{\omega}\cancel{T}}{2}}$$

$$= i\,2A\sin\left(\frac{\omega T}{2}\right) = \text{sine wave}$$

14 - 1 If $x_1 = \varphi_1(t)$ and $x_2 = \varphi_2(t)$ are solutions of the equation $\ddot{x} + x^3 = 0$, then they will satisfy

$$\ddot{\varphi}_1 + \varphi_1^3 = 0 \qquad \text{and} \qquad \ddot{\varphi}_2 + \varphi_2^3 = 0$$

Adding, the following is also true

$$(\ddot{\varphi}_1 + \ddot{\varphi}_2) + (\varphi_1^3 + \varphi_2^3) = 0 \qquad (a)$$

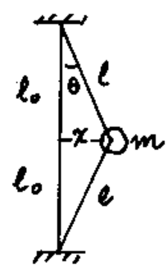
If we assume $x = \varphi_1 + \varphi_2$ and substitute into the D.E. we would obtain

$$(\ddot{\varphi}_1 + \ddot{\varphi}_2) + (\varphi_1^3 + 3\varphi_1^2\varphi_2 + 3\varphi_1\varphi_2^2 + \varphi_2^3) = 0$$

which does not agree with the correct result (a)

Superposition of solutions are in general not solutions of nonlinear equations.

14-2
$$\sum F_x = -2T\sin\theta = m\ddot{x}$$



$$\sin\theta = \frac{x}{\ell} = \frac{x}{\sqrt{\ell_0^2 + x^2}} \cong \frac{x}{\ell_0}\left[1 - \frac{1}{2}\left(\frac{x}{\ell_0}\right)^2\right]$$

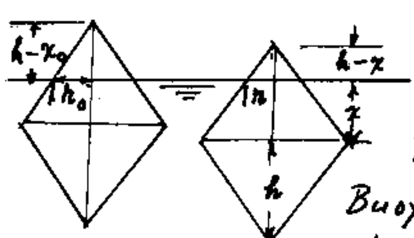
$$T = T_0 + k(\ell - \ell_0) = T_0 + k\left[\ell_0\left(1 + \frac{x^2}{\ell_0^2}\right)^{1/2} - \ell_0\right]$$

$$\cong T_0 + k\,\frac{1}{2}\left(\frac{x}{\ell_0}\right)^2$$

$$\therefore \quad m\ddot{x} + 2\left[T_0 + \frac{k}{2}\left(\frac{x}{\ell_0}\right)^2\right]\frac{x}{\sqrt{\ell_0^2 + x^2}} = 0$$

$$m\ddot{x} + \frac{2}{\ell_0}\left[T_0 + \frac{k}{2}\left(\frac{x}{\ell_0}\right)^2\right]\left[1 - \frac{1}{2}\left(\frac{x}{\ell_0}\right)^2\right]x = 0$$

14-3



Vol. of cone $= \frac{1}{3}\pi r^2(h - x)$

From similar triangles $\frac{r_0}{h - x_0} = \frac{r}{h - x}$

$$\therefore r = r_0\left(\frac{h - x}{h - x_0}\right)$$

Difference in vol. $= \frac{1}{3}\pi\left[r^2(h - x) - r_0^2(h - x_0)\right]$

Buoyant force $= \rho\Delta V = \frac{\pi}{3}\rho r_0^2\left[\left(\frac{h - x}{h - x_0}\right)^2(h - x) - (h - x_0)\right]$

= weight of water displaced.

$$\therefore \; m\ddot{x} = \frac{\pi}{3}\rho r_0^2\left[\frac{(h - x)^3}{(h - x_0)^2} - (h - x_0)\right]$$

$$= \frac{\pi}{3}\rho\frac{r_0^2}{(h - x_0)^2}\left[(h - x)^3 - (h - x_0)^3\right]$$

14-4

For $x > x_0$ Eq. of motion is

$$m\ddot{x} + k(x - x_0) = c \qquad \text{------ (a)}$$

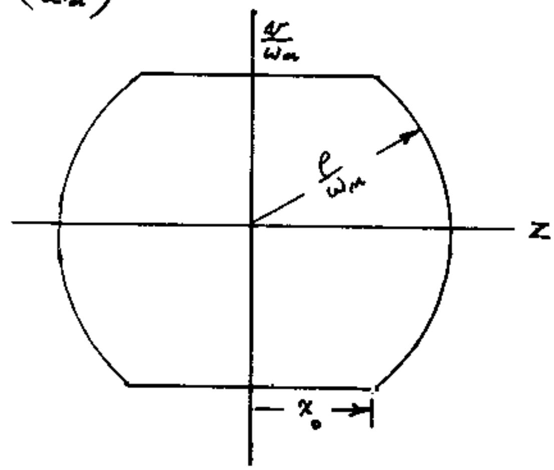
For $x < x_0$ $\quad \ddot{x} = \dot{x}\dfrac{d\dot{x}}{dx} = 0 \quad \therefore \dfrac{dv}{dx} = 0 \quad$ where $v = \dot{x}$

Let $z = (x - x_0)$ in Eq. (a); then

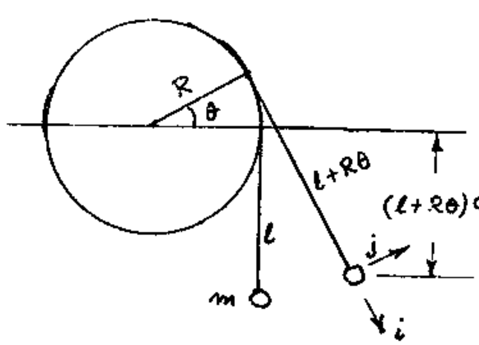
$$\ddot{z} + \omega_n^2 z = 0 \qquad \text{or} \qquad \dot{z}\frac{d\dot{z}}{dz} + \omega_n^2 z = 0$$

Integrating $\quad \dot{z}^2 + \omega_n^2 z^2 = \left(\dfrac{c}{\omega_n}\right)^2 \quad$ a circle

Phase-plane trajectory will look like this



14-5



Since system is conservative

$$\frac{d}{dt}(T+U) = 0 \qquad (a)$$

Vel. of m is

$$\vec{v} = (\ell + R\theta)\dot{\theta}\,\vec{j}$$

$$T = \tfrac{1}{2} m\,\vec{v}\cdot\vec{v} = \tfrac{1}{2} m\,\dot{\theta}^2(\ell + R\theta)^2$$

$$U = mg\left[\ell - (\ell + R\theta)\cos\theta + R\sin\theta\right]$$

Subst. into (a)

$$\dot{\theta}\left\{\ddot{\theta}(\ell+R\theta)^2 + \dot{\theta}(\ell+R\theta)R + g(\ell+R\theta)\sin\theta \cancel{- R\cos\theta} + \cancel{R\cos\theta}\right\} = 0$$

$$\ddot{\theta} + \frac{R\dot{\theta}^2}{\ell + R\theta} + \frac{g}{\ell + R\theta}\sin\theta = 0$$

14-6

$$\ddot{x} + \omega_m^2 x = 0 ,$$

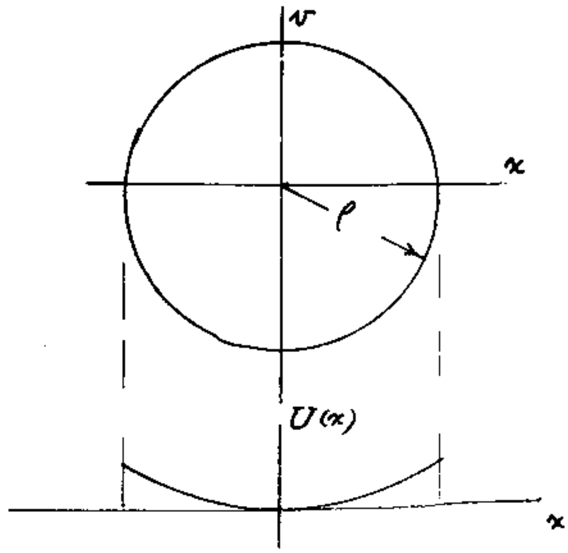
$$\ddot{x} = \dot{x}\frac{d\dot{x}}{dx} , \quad \text{Let } v = \frac{\dot{x}}{\omega_m}$$

then above eq. becomes

$$\frac{dv}{dx} = -\frac{x}{v} \quad \text{or} \quad v^2 + x^2 = \rho^2$$

phase-plane traj. is a circle

$U = \frac{1}{2}kx^2$ a parabola



14-7

$$\tau = 4\int_0^{x_{max}} \frac{dx}{\sqrt{2[E - U(x)]}}$$

$U(x) = \frac{1}{2}\frac{k}{m}x^2$ per unit mass

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\frac{k}{m}x^2$$

$\dot{x} = 0$ when $x = x_{max}$ $\therefore$ $x_{max} = \sqrt{\frac{2Em}{k}}$

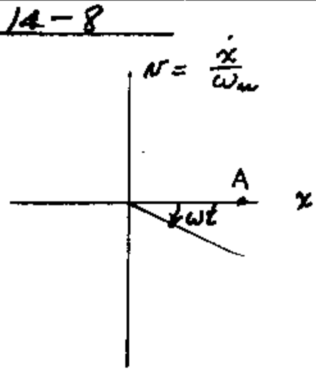
$$\tau = 4\int_0^{\sqrt{\frac{2Em}{k}}} \frac{dx}{\sqrt{2\left[E - \frac{1}{2}\frac{k}{m}x^2\right]}}$$

but $\frac{k}{m}x^2 = \omega_m^2 x^2$

$C^2 = 2E$

$$\tau = \frac{4}{\omega_m}\int_0^{u=\omega_m x_{max} = C} \frac{du}{\sqrt{C^2 - u^2}} = \frac{4}{\omega_m}\sin^{-1}\left(\frac{u}{C}\right)\Big|_0^C = \frac{4}{\omega_m}\frac{\pi}{2} = \frac{2\pi}{\omega_m}$$

14-8



$x(0) = A \qquad \dot{x}(0) = 0$

Let $y = \dot{x} \qquad \dot{y} = -\omega^2 x$

$$V = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{y^2 + \omega_m^4 x^2} = 0$$

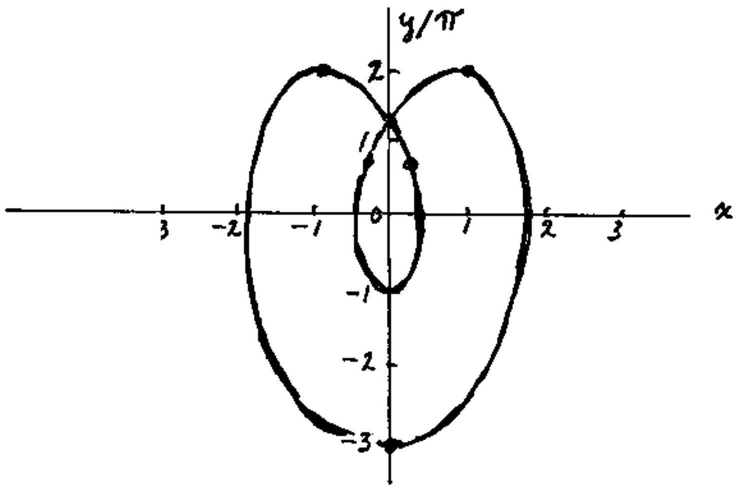
$V = 0$ only if $x = y = 0$

14-9

$$x = \cos \pi t + \sin 2\pi t$$

$$y = \dot{x} = -\pi \sin \pi t + 2\pi \cos 2\pi t$$

t	x	y
0	1	2π
.25	−.3	$.7\pi$
.50	0	$-\pi$
.75	.3	$.7\pi$
1.0	−1	2π
1.25	−1.7	$-.7\pi$
1.5	0	-3π
1.75	1.7	$-.7\pi$
2.0	1	2π



14-10

$$\ddot{x} + 2\zeta\omega_m \dot{x} + \omega_m^2 x = 0$$

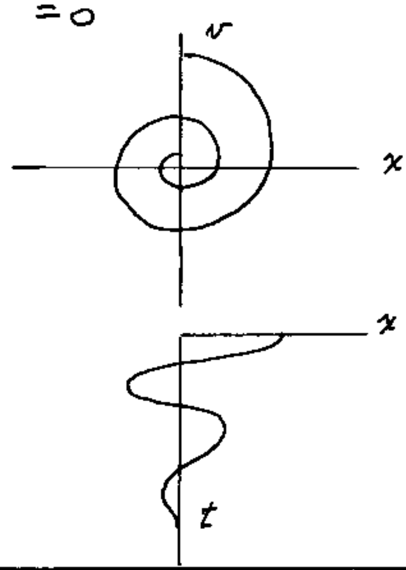
$$\dot{x}\frac{d\dot{x}}{dx} = -2\zeta\omega_m \dot{x} - \omega_m^2 x$$

$$\frac{1}{\omega_m}\frac{d\dot{x}}{dx} = -2\zeta - \omega_m \frac{x}{\dot{x}}$$

Let $v = \frac{\dot{x}}{\omega_m}$

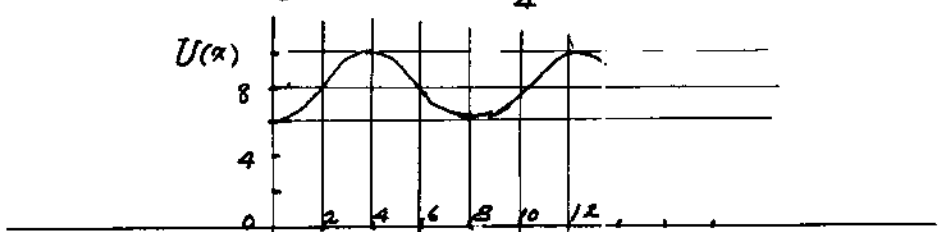
$$\frac{dv}{dx} = -2\zeta - \frac{x}{v}$$

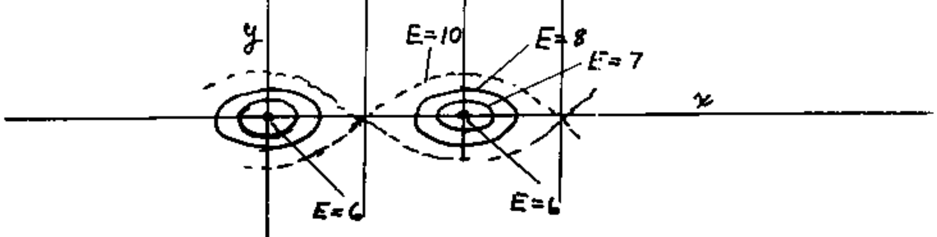
Trajectory is a spiral.



14-12

$$U = 8 - 2\cos\frac{\pi x}{4}$$

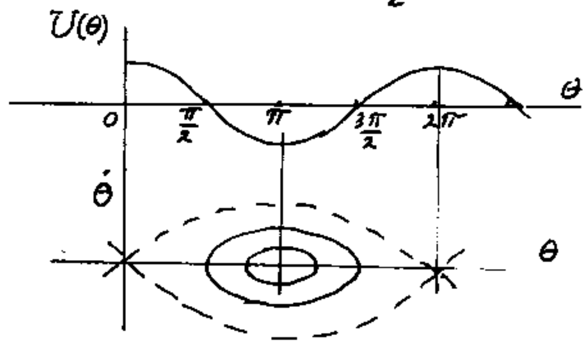




see next page for Prob 14-11

14-11

$U = \frac{g}{l}\cos\theta$



The origin of phase plane is shifted to π, as compared to Fig 14.4-2

$\theta = 0$ and 2π are unstable points

14-13

$$\frac{dy}{dx} = \frac{2x+2y}{5x-y} = \frac{P}{Q}$$

singular points are $\frac{P}{Q} = \frac{0}{0}$

$$2x + 2y = 0$$
$$5x - y = 0$$
$$12x = 0$$

$\therefore\ x = 0,\quad y = 0$

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \quad \text{Eq. 14.3-8}$$

$(a+e) > 0 \quad \therefore$ system is unstable and aperiodic

$\lambda_1 = 3.0 \qquad u = e^{3t}$

$\lambda_2 = 4.0 \qquad v = e^{4t}$

14-14

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \qquad \begin{vmatrix} (5-\lambda) & -1 \\ 2 & (2-\lambda) \end{vmatrix} = 0$$

gives $\lambda = \begin{cases} 3 \\ 4 \end{cases}$

subst λ into eq.

$(5-\lambda)x = y$ gives $x^{(1)} = .5\,y^{(1)}$
$x^{(2)} = 1.0\,y^{(2)}$

$$P = \begin{bmatrix} .5 & 1 \\ 1 & 1 \end{bmatrix}$$

Transformation to decouple is

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} .5 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

$$\begin{bmatrix} .5 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\xi} \\ \dot{\eta} \end{Bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} .5 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{\xi} \\ \dot{\eta} \end{Bmatrix} = \begin{bmatrix} .5 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} .5 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

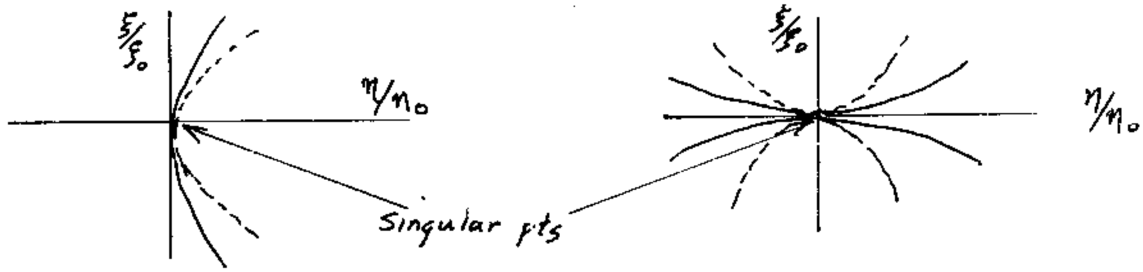
$\therefore\ \dot{\xi} = 3\xi$ and $\dot{\eta} = 4\eta$ decoupled eqs.

1.-15 Uncoupled eqs. $\begin{cases} \dot{\xi} = \lambda_1 \xi \\ \dot{\eta} = \lambda_2 \eta \end{cases}$ can be written as

$$\frac{d\xi}{d\eta} = \left(\frac{\lambda_1}{\lambda_2}\right)\frac{\xi}{\eta} \qquad \text{Integrating} \quad \xi = \xi_0\left(\frac{\eta}{\eta_0}\right)^{\frac{\lambda_1}{\lambda_2}}$$

For $\frac{\lambda_1}{\lambda_2} = .5$, the traj. are tangent to ξ as shown

If $\frac{\lambda_1}{\lambda_2} = 2$, the traj. are Tangent to η



1-16 $\quad \frac{d\xi}{d\eta} = 2\frac{\xi}{\eta} \qquad \therefore \quad \xi = \xi_0\left(\frac{\eta}{\eta_0}\right)^2$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} P \end{bmatrix}\begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = \begin{bmatrix} P \end{bmatrix}\begin{Bmatrix} \xi_0\left(\frac{\eta}{\eta_0}\right)^2 \\ \eta \end{Bmatrix}$$

$\therefore$ Eq for u & v are of the form

$$\left.\begin{aligned} u &= A\eta^2 + B\eta = x - x_s \\ v &= C\eta^2 + D\eta = y - y_s \end{aligned}\right\} \text{ only a linear shift of origin}$$

Need original eq.

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} a & b \\ c & e \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix} \quad \text{corresponding to } \frac{\lambda_1}{\lambda_2} = 2$$

a, b, c, e must be known before plotting.

Use Eq. 12.3-9

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = P \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

From Eq. 14.3-8

$$\begin{vmatrix} (a-\lambda) & b \\ c & (e-\lambda) \end{vmatrix} = 0 \quad \therefore \quad \frac{u_1}{v_1} = \frac{-b}{a-\lambda_1}, \qquad \frac{u_2}{v_2} = \frac{-b}{a-\lambda_2}$$

14-16 Cont. Since only relative values of u, v are essential, Let $v_1 = v_2 = 1.0$ and $u_1 = \frac{-b}{a-\lambda_1}$, $u_2 = \frac{-b}{a-\lambda_2}$

Using Eq. 14.3-8 we have

$$P\begin{Bmatrix}\dot{\xi}\\ \dot{\eta}\end{Bmatrix} = \begin{bmatrix} a & b \\ c & e\end{bmatrix} P \begin{Bmatrix}\xi \\ \eta\end{Bmatrix}$$

$$\begin{Bmatrix}\dot{\xi}\\ \dot{\eta}\end{Bmatrix} = P^{-1}\begin{bmatrix} a & b \\ c & e\end{bmatrix} P \begin{Bmatrix}\xi \\ \eta\end{Bmatrix} = \begin{bmatrix}\Lambda\end{bmatrix}\begin{Bmatrix}\xi \\ \eta\end{Bmatrix}$$

$$\therefore\ P^{-1}\begin{bmatrix} a & b \\ c & e\end{bmatrix} P = \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix} \text{ where } P = \begin{bmatrix}\frac{-b}{a-\lambda_1} & \frac{-b}{a-\lambda_2} \\ 1 & 1\end{bmatrix}$$

Equating the two elements on the two sides of the above eq.

$$\left.\begin{aligned}
&\left[(au_2+b)-(cu_2+e)\right] = 0 \\
&\left[-u_2(au_1+b)+u_1(cu_1+e)\right] = 0 \\
&\frac{1}{(u_1v_2-u_2v_1)}\left[(au_1+b)-(cu_1+e)\right] = \lambda_1 \\
&\frac{1}{(u_1v_2-u_2v_1)}\left[-u_2(au_2+b)+u_1(cu_2+e)\right] = \lambda_2
\end{aligned}\right\}$$

These 4 eqs. can be solved for a, b, c, e, then subst. into Eq. 14.3-9

Other alternative is to solve for u & v from Eq. 14.3-12 for chosen values of t.

14-17 From given eq.

$$\begin{Bmatrix}\dot{v}\\ \dot{u}\end{Bmatrix} = \begin{bmatrix}\alpha & \beta \\ -\beta & \alpha\end{bmatrix}\begin{Bmatrix}v \\ u\end{Bmatrix} \qquad \begin{vmatrix}(\alpha-\lambda) & -\beta \\ -\beta & (\alpha-\lambda)\end{vmatrix} = 0$$

$$\therefore\ \lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2 = 0 \quad \text{and} \quad \lambda = -\alpha \pm i\beta$$

14-18 $u = \rho\cos\theta$ $\qquad v = \rho\sin\theta$

$$du = d\rho\cos\theta - \rho\sin\theta\, d\theta \qquad dv = d\rho\sin\theta + \rho\cos\theta\, d\theta$$

subst. u & v in Eq. for Prob. 14-17

$$\frac{dv}{du} = \frac{\beta\cancel{\rho}\cos\theta + \alpha\cancel{\rho}\sin\theta}{\alpha\cancel{\rho}\cos\theta - \beta\cancel{\rho}\sin\theta} = \frac{d\rho\sin\theta + \rho\cos\theta\, d\theta}{d\rho\cos\theta - \rho\sin\theta\, d\theta}$$

$$\left(\frac{d\rho}{\rho}\sin\theta + \cos\theta\, d\theta\right)(\alpha\cos\theta - \beta\sin\theta) = \left(\frac{d\rho}{\rho}\cos\theta - \sin\theta\, d\theta\right)(\beta\cos\theta + \alpha\sin\theta)$$

$$\frac{d\rho}{\rho}\left[\alpha\cancel{\cos\theta\sin\theta} - \beta(\sin^2\theta + \cos^2\theta) - \alpha\cancel{\sin\theta\cos\theta}\right] = -d\theta\left[\alpha\cos^2\theta - \beta\cancel{\sin\theta\cos\theta} + \cancel{\beta\sin\theta\cos\theta} + \alpha\sin^2\theta\right]$$

$$\therefore\ \frac{d\rho}{\rho} = \frac{\alpha}{\beta}\, d\theta \qquad \rho = e^{\frac{\alpha}{\beta}\theta}$$

14-19 Eq. in x,y plane is $xy = \pm C$

node point at origin is unstable

$x\,dy + y\,dx = 0 \quad \therefore \quad \frac{dy}{dx} = -\frac{y}{x}$

$$\begin{Bmatrix} \dot{y} \\ \dot{x} \end{Bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} y \\ x \end{Bmatrix} \qquad \begin{vmatrix} -(1+\lambda) & 0 \\ 0 & (1-\lambda) \end{vmatrix} = 0$$

$(1-\lambda)(1+\lambda) = 0 \qquad \lambda = \pm 1 \qquad \xi = \xi_0 e^{t}$

$\eta = \eta_0 e^{-t}$

$\frac{d\xi}{d\eta} = \frac{\lambda_1}{\lambda_2}\frac{\xi}{\eta} = -\frac{\xi}{\eta}$ or $\eta\,d\xi + \xi\,d\eta = 0$

$\therefore$ ξ, η plot is $\xi\eta = \pm$ const. $\therefore$ same

---

14-20 The form of the eq. is

$\frac{dv}{du} = \frac{v+u}{u} \qquad \begin{Bmatrix} \dot{v} \\ \dot{u} \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} v \\ u \end{Bmatrix}$

characteristic eq.

$\begin{vmatrix} (1-\lambda) & 1 \\ 0 & (1-\lambda) \end{vmatrix} = 0$ Leads to two equal roots $\lambda = 1$

Transformation eq. 14.3-9 cannot be applied

14-21 Differentiate the eq. $x^2 + 2xy + 3y^2 = C$

$2x\,dx + 2x\,dy + 2y\,dx + 6y\,dy = 0$

$(2x + 6y)\,dy = -(2x + 2y)\,dx \quad \therefore \quad \frac{dy}{dx} = \frac{-x-y}{x+3y}$

14-22 Let $v = \dot{x}$ then the eq. $\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$

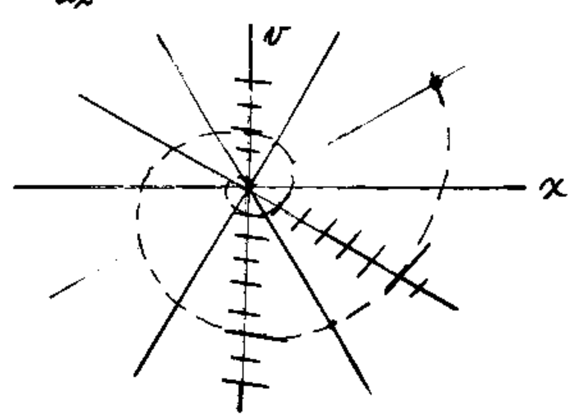
becomes $\dot{v} + 2\zeta\omega_n v + \omega_n^2 x = 0$, $\quad v\frac{dv}{dx} + 2\zeta\omega_n v + \omega_n^2 x = 0$

Let $\frac{dv}{dx}$ = constant for isocline = C_1

$v(C_1 + 2\zeta\omega_n) + \omega_n^2 x = 0$

= straight line through origin

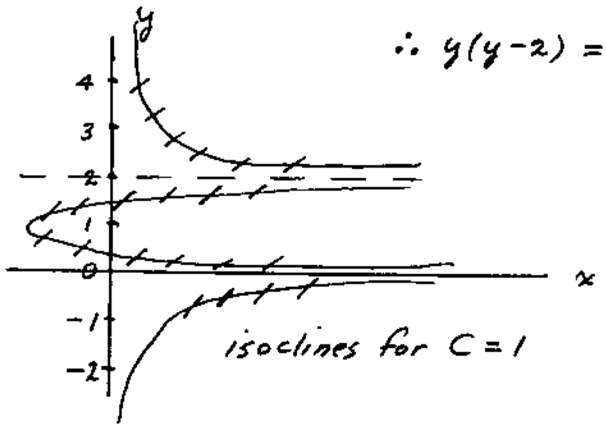
As $t \to \infty$ the points on the isoclines move towards the origin $\therefore$ stable



14-23 $\quad \dfrac{dy}{dx} = xy(y-2) = \text{constant for isocline}$

$\therefore\ y(y-2) = \dfrac{C}{x}$

y	$y(y-2)$	x for $C=1$
-3	15	.066
-2	8	.125
-1	3	.333
0	0	∞
1	-1	-1
2	0	∞
3	3	.333
4	8	.125



14-24 $\quad y\dfrac{dy}{dx} + \omega_m^2 x + \mu x^3 = 0$

integrating $\quad y^2 + \omega_m^2 x^2 + \frac{1}{2}\mu x^4 = 2E$

or $\quad 2T + 2U = 2E$

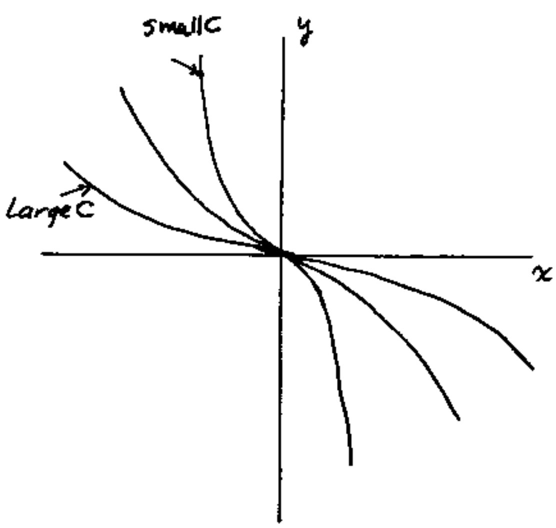
since $y = \dfrac{dx}{dt} \qquad dt = \dfrac{dx}{y} = \dfrac{dx}{\sqrt{2E - 2U}}$

For $\frac{1}{4}$ period $\qquad \dfrac{\tau}{4} = \displaystyle\int_0^A \dfrac{dx}{\sqrt{2(E-U)}}$

14-25 $\quad$ From Prob. 14-24 $\qquad \dfrac{dy}{dx} = \dfrac{-\omega_m^2 x - \mu x^3}{y} = C$

Let $\dfrac{\omega_m^2}{C} = 4 + \dfrac{\mu}{C} = 2$

x	$4+2x^2$	$y=-x(4+2x^2)$
-1	6	6
0	4	0
1	6	-6
2	12	-24
3	22	-66

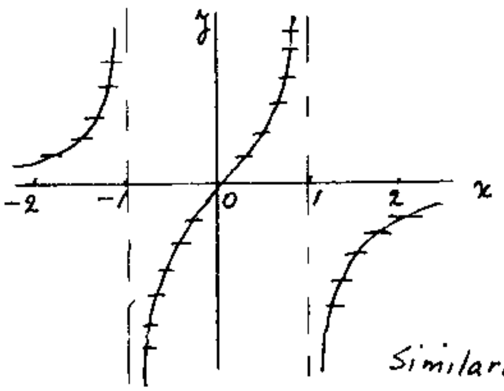


14-26 $\ddot{x} - \mu\dot{x}(1-x^2) + x = 0$

$$y\frac{dy}{dx} - \mu y(1-x^2) + x = 0$$

$$\frac{dy}{dx} = \frac{\mu y(1-x^2) - x}{y} = \mu(1-x^2) - \frac{x}{y} = \text{const}$$

For $\frac{dy}{dx} = C = 0$ and $\mu = 2$



x	y
0	0
$\pm .2$	$\pm .104$
$\pm .4$	$\pm .239$
$\pm .6$	$\pm .45$
$\pm .8$	± 1.11
$\pm .9$	± 2.37
± 1.0	$\pm \infty$
± 2	$\mp .333$
± 4	$\mp .13$

Similarly solve for $C = \pm 1$ and other values

14-27 Let $\omega_n^2 = \frac{k}{m}$ $\quad \tau = \omega_n t$ $\quad y = \frac{dx}{d\tau}$

Then $\dot{x} = \omega_n y$ $\quad \ddot{x} = \omega_n^2 y \frac{dy}{dx}$

the eq $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x + \frac{\mu}{m}x^3 = 0$ becomes

$$y\frac{dy}{dx} + \frac{c}{m}y + x + \frac{\mu}{\omega_n^2 m}x^3 = 0$$

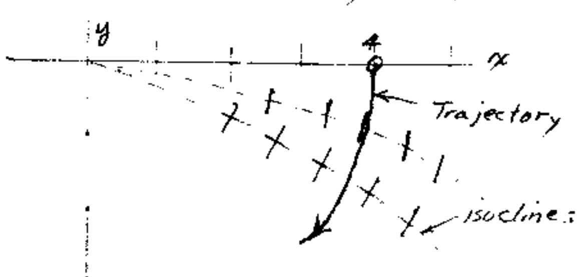
$$\frac{dy}{dx} = \frac{-\left[x + \left(\frac{c}{m}y + \frac{\mu}{\omega_n^2 m}x^3\right)\right]}{y} = \alpha$$

14-28 Above eq. can be written as $\quad$ given: $\omega_n^2 = 25$, $\frac{\mu}{m} = 5$

$\left(\alpha + \frac{c}{m}\right)y = -\left(x + \frac{\mu}{\omega_n^2 m}x^3\right)$ where $\alpha = \frac{dy}{dx}$ $\quad \frac{c}{m} = 2.0$

$\therefore (\alpha + 2)y = -(x + 0.2x^3)$ Assign different values for α and plot the field of isoclines. From prob. 14-27 $\dot{x}(0) = \omega_n y(0) = 0$

$\therefore$ Start with $x(0) = 4$, $\dot{x}(0) = 0 \therefore y(0) = 0$ and fill in trajectory



All isoclines are vertical along x axis. The right side of eq. is straight line + cubic which must be $\div$ by $(\alpha+2)$

14 - 29

$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$ Let $x = \theta$, $\omega_0^2 = g/l$

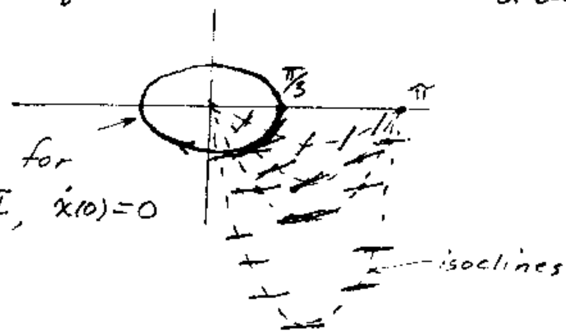
$y = \frac{\dot{\theta}}{\omega_0} = \frac{\dot{x}}{\omega_0} = \frac{dx}{d\tau}$

$\therefore \omega_0^2 y \frac{dy}{dx} + \frac{g}{l}\sin x = 0$ with $\alpha = \frac{dy}{dx}$ = constant

$\alpha y = -\left(\frac{g}{l\omega_0^2}\right)\sin x$ or $y = -\frac{g}{\alpha l\omega_0^2}\sin x$ = eq. for isocline with α = const.

since $\frac{g}{l\omega_0^2} = 1$

$y = -\frac{1}{\alpha}\sin x$ = isocline



14-30

$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$ $\dot{\theta}\frac{d\dot{\theta}}{d\theta} + \frac{g}{l}\sin\theta = 0$

Integrate $\frac{\dot{\theta}^2}{2} - \frac{g}{l}\cos\theta = E$ $\therefore U = -\frac{g}{l}\cos\theta$

At $t=0$ $\theta = 60°$ $\dot{\theta} = 0$ $\therefore$ $0 - \frac{g}{l}\cos 60° = E$

Since $\dot{\theta} = \frac{d\theta}{dt}$ $dt = \frac{d\theta}{\dot{\theta}}$ $E = -\frac{g}{2l} = -\frac{g}{l}\cos\theta_0$

$$dt = \frac{d\theta}{\sqrt{2E + 2\frac{g}{l}\cos\theta}} \qquad t = \int\frac{d\theta}{\sqrt{2(E + \frac{g}{l}\cos\theta)}}$$

$$t = \sqrt{\frac{l}{g}}\int\frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}} \qquad \cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$\cos\theta - \cos\theta_0 = 2\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)$$

Let $\sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\sin\phi = k\sin\phi$ (i)

then $\cos\theta - \cos\theta_0 = 2\sin^2\frac{\theta_0}{2}(1 - \sin^2\phi) = 2\sin^2\frac{\theta_0}{2}\cos^2\phi$ (ii)

Diff (i) $\frac{1}{2}\cos\frac{\theta}{2}\,d\theta = k\cos\phi\,d\phi$ $\therefore$ $d\theta = \frac{2k\cos\phi\,d\phi}{\cos\frac{\theta}{2}}$ (iii)

Subst (ii) + (iii) into t

$$t = \sqrt{\frac{l}{g}}\int\frac{2k\cos\phi\,d\phi}{\cos\frac{\theta}{2}\,2\sin\frac{\theta_0}{2}\cos\phi} = \sqrt{\frac{l}{g}}\int\frac{d\phi}{\cos\frac{\theta}{2}}$$

14-30 Cont:

$$t = \sqrt{\frac{l}{g}} \int \frac{d\phi}{\sqrt{1-\sin^2\frac{\theta}{2}}} = \sqrt{\frac{l}{g}} \int \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

$\therefore$ period $\tau = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$ $\qquad k = \sin\frac{\theta_0}{2}$

since $\sin\phi = \dfrac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$, when $\theta = 0$, $\phi = 0$

when $\theta = \theta_0$, $\phi = \frac{\pi}{2}$ $\quad = \frac{1}{4}$ cycle

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14-31 $\qquad \ddot{x} + \omega_n^2 x + C\,\mathrm{sgn}(\dot{x}) = 0 \qquad \tau = \omega_n t$

$$\dot{x}\frac{d\dot{x}}{dx} + \omega_n^2 x + C\,\mathrm{sgn}(\dot{x}) = 0 \qquad \text{Let } \frac{dx}{d\tau} = y = \frac{\dot{x}}{\omega_n}$$

$$\omega_n^2 y \frac{dy}{dx} + \omega_n^2 x + C\,\mathrm{sgn}(y) = 0$$

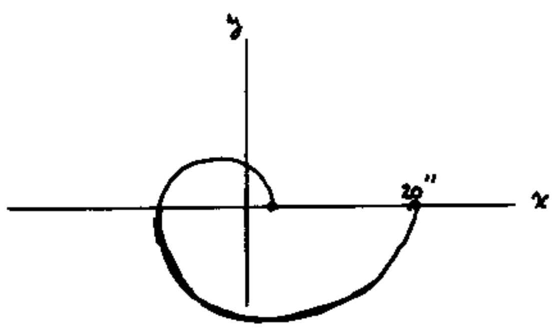
$$\frac{dy}{dx} = -\frac{\frac{1}{\omega_n}C\,\mathrm{sgn}(y) + x}{y} = -\frac{f(y) + x}{y}$$

where $f(y) = \frac{1}{\omega_n}C\,\mathrm{sgn}(y)$

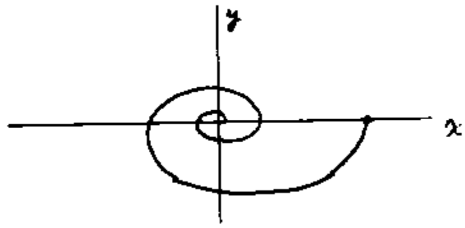
14-32 $\qquad$ Initial values $\quad x(0) = 20° \qquad y(0) = 0$

$$\omega_n = \sqrt{\frac{3.60}{.10}} = 6, \qquad \mu = 0.20$$

$$\frac{\mu g}{\omega_n^2} = \frac{.20 \times 386}{36} = 2.145 \text{ in}$$



14-33 For the undamped pendulum the trajectory is an ellipse, For the damped pendulum the curve is inside of the ellipse, as shown



14-34 $\ddot{\theta} + \omega_m^2 \sin\theta = 0 \qquad \sin\theta \cong \theta - \frac{\theta^3}{6}$

$$\ddot{\theta} + \omega_m^2\left(\theta - \frac{\theta^3}{6}\right) = 0$$

Let $\theta = \theta_0 + \mu\theta_1$ and $\omega^2 = \omega_m^2 + \mu\alpha$

From Eq. 12.6 − 9

$$\omega^2 = \omega_m^2 + \frac{3}{4}\mu A^2 = \omega_m^2\left[1 + \frac{3}{4}\times\frac{1}{6}\theta_0^2\right]$$

$$\therefore \quad \omega = \omega_m\sqrt{1 + \frac{1}{8}\theta_0^2} \cong \omega_m\left(1 + \frac{1}{16}\theta_0^2\right) \qquad \omega_m = \sqrt{\frac{g}{\ell}}$$

14-35 From Prob 14-34

$$\omega = \frac{2\pi}{\tau} \cong \frac{2\pi}{\tau_m}\left(1 + \frac{1}{16}\theta_0^2\right) \quad \therefore \tau \cong \tau_m\left(\frac{1}{1 + \frac{1}{16}\theta_0^2}\right)$$

14-36 $\ddot{x} + 0.15\dot{x} + 10x + x^3 = 5\cos(\omega t + \varphi)$

$$\omega_m^2 = 10, \qquad c = .15, \qquad \mu = 1.0, \qquad F = 5$$

Eq. 14.6 − 11 becomes

$$25 = \left[(10 - \omega^2)A + \frac{3}{4}A^3\right]^2 + \left[.15\,\omega A\right]^2$$

Rearrange to

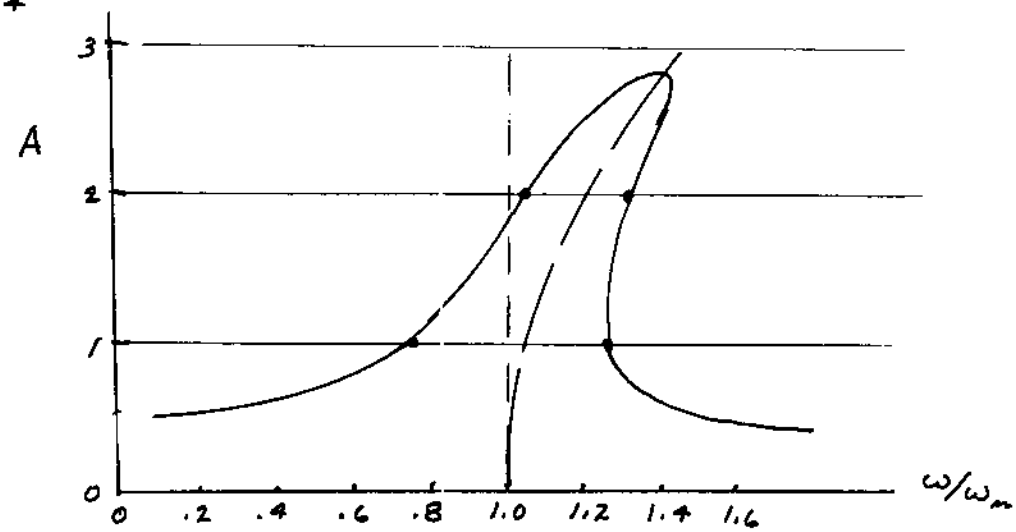
$$\omega^4 - \left(20 + \frac{3}{2}A^2 - .0225\right)\omega^2 + \left(100 + 15A^2 + \frac{9}{16}A^4 - \frac{25}{A^2}\right) = 0$$

$$\omega^4 - b\,\omega^2 + c = 0$$

14-36 Cont.

A	b	c	ω^2	ω/ω_n
0	19.98	$-\infty$	∞	∞
1	21.48	90.5	$\begin{cases} 5.7 \\ 15.78 \end{cases}$	$\begin{cases} .755 \\ 1.255 \end{cases}$
2	25.98	162.7	$\begin{cases} 10.69 \\ 15.29 \end{cases}$	$\begin{cases} 1.036 \\ 1.24 \end{cases}$
3	32.48	272.7	complex ←	this indicates the peak to be below A = 3
$\cong$ .5				0

$\sqrt{\frac{b^2}{4} - c} = 0$ gives $A \cong 2.9$



14-37

ω/ω_n vs φ_1 & φ_2

φ_1 = branch 1

φ_2 = " 2

$$\tan\phi = \frac{B_0}{A_0} = \frac{c\omega}{(\omega_n^2 - \omega^2) + \frac{3}{4}\mu A^2} \quad \text{Eq. 14.6-10}$$

$$= \frac{.15\omega}{(10-\omega^2) + \frac{3}{4}A^2}$$

subst. ω & A from Prob. 14-36

$$(\tan\phi)_{A=1} = \frac{.15\sqrt{5.7}}{(10-5.7)+.75} = \frac{.358}{5.05} = .071, \quad \phi = 4^\circ 4'$$

14-37 Cont:

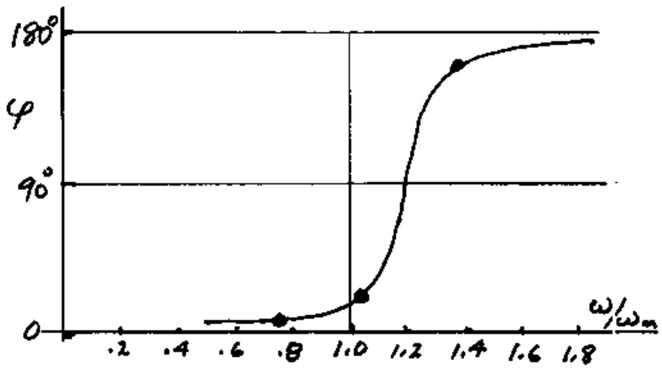
$$(\tan\phi)_{A=1} = \frac{.15\sqrt{15.78}}{(10-15.78)+.75} = \frac{.60}{-5.03} = -.119 \qquad \phi = 173^\circ 12'$$

$$(\tan\phi)_{A=2} = \frac{.15\sqrt{10.69}}{(10-10.69)+3} = \frac{.490}{2.31} = .213 \qquad \phi = 12^\circ$$

$$= \frac{.15\sqrt{15.29}}{(10-15.29)+3} = \frac{.590}{-2.29} = -.257 \qquad \phi = 185^\circ 35'$$

$$(\tan\phi)_{A=0} = \frac{\infty}{-\infty^2} = -\frac{1}{\infty} = -0 \qquad \phi = 180^\circ -$$

$$(\tan\phi)_{A\equiv .5} = \frac{0}{(10-0)+.375} = 0 \qquad \phi = 0^\circ$$



ω/ω_m	φ_1	φ_2
0	0	
.755	4° 4′	
1.036	12°	
1.25		173° 12′
1.24		185° 35′
∞		180°

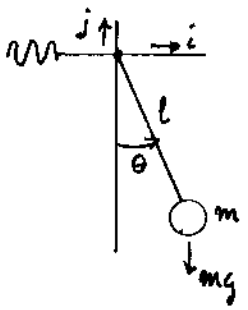
14-38 Moment eq. about an accelerating point A is

$$\vec{M}_A = I_A\,\dot{\vec{\omega}} + \vec{\rho}_{AC} \times m\,\vec{a}_A$$ (see Dynamics by Pestel & Thomson p 213 McGraw Hill.)

where c is the center of mass and $\vec{\rho}_{AC}$ is a vector from A to C. For this problem $y_A = y_0 \cos 2\omega t$

$I_A = m\ell^2 \qquad |\rho_{AC}| = \ell \qquad |a_A| = -4y_0\omega^2\cos 2\omega t$

$$\vec{\rho}_{AC} \times m\vec{a}_A = \ell(\sin\theta\,\vec{i} - \cos\theta\,\vec{j}) \times m(-4y_0\omega^2\cos 2\omega t)\,\vec{j}$$



$$= -4y_0\omega^2\ell\, m\cos 2\omega t \cdot \vec{k}$$

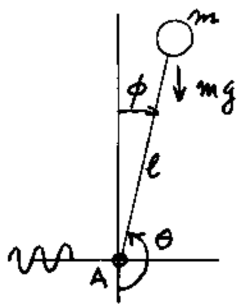
$$\dot{\vec{\omega}} = \ddot{\theta}\,\vec{k} \qquad \vec{M}_A = -mg\ell\sin\theta\;\vec{k}$$

$$\therefore -mg\ell\sin\theta = m\ell^2\ddot{\theta} - 4y_0\omega^2\ell m\cos 2\omega t$$

$$\ddot{\theta} + \left(\frac{g}{\ell} - \frac{4y_0\omega^2\cos 2\omega t}{\ell}\right)\sin\theta = 0$$

14-39 For the inverted pendulum $\theta = \pi - \phi$

$\sin\theta = \sin\phi$



$$\therefore\ -\ddot{\phi} + \left(\frac{g}{\ell} - \frac{4\omega_o^2 y_o}{\ell}\cos 2\omega t\right)\sin\phi = 0$$

For small ϕ

$$\ddot{\phi} + \left(-\frac{g}{\ell} + \frac{4\omega_o^2 y_o}{\ell}\cos 2\omega t\right)\phi = 0$$

Compare with Eq. (g) Sec. 14.5 which is

$$\frac{d^2y}{dz^2} + (a - 2b\cos 2z)\,y = 0$$

which results in

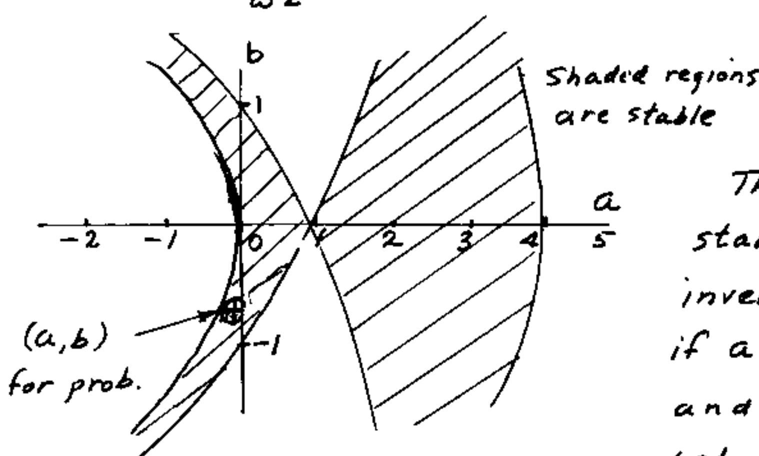
$$z = \omega t, \qquad y = \phi, \qquad dz^2 = \omega^2 dt$$

Rewrite eq.

$$\omega^2\frac{d^2\phi}{dz^2} + \left(-\frac{g}{\ell} + \frac{4\omega^2 y_o}{\ell}\cos 2z\right)\phi = 0$$

$$\frac{d^2\phi}{dz^2} + \left(-\frac{g}{\omega^2\ell} + \frac{4y_o}{\ell}\cos 2z\right)\phi = 0$$

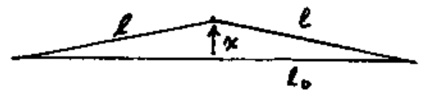
$$\therefore\ a = -\frac{g}{\omega^2\ell} \quad \text{and} \quad b = -\frac{2y_o}{\ell}$$



Mathieu Plot.

The plot indicates that stable oscillations of the inverted pendulum is possible if a is a small number and b is also negative between 0 & −1, as shown by the point ⊕.

14-40 New length after displ is



$$\ell = \ell_0\left(1+\frac{x^2}{\ell_0^2}\right)^{1/2} \cong \ell_0\left(1+\frac{1}{2}\frac{x^2}{\ell_0^2}\right)$$

Let T_0 = initial tension. Since increase in length is $\frac{x^2}{\ell_0}$, the increase in tension is $K\frac{x^2}{\ell_0}$. Total tension $=\left(T_0 + K\frac{x^2}{\ell_0}\right)$

Eq. of motion

$$m\ddot{x} = -\frac{x}{\ell}\,2\left(T_0 + K\frac{x^2}{\ell_0}\right) \qquad \frac{x}{\ell} \cong \frac{x}{\ell_0}$$

$$m\ddot{x} + \left(\frac{2T_0}{\ell_0}\right)x + \left(\frac{2K}{\ell_0^2}\right)x^3 = 0$$

Assume sol. $x = x_1 + \mu x_2 + \cdots$ where μ = arbitrary parameter and $x_2 \ll x_1$. Then $x^3 \cong x_1^3 + 3\mu x_1^2 x_2$. Subst. in D.E.

$$m\ddot{x}_1 + \frac{2T_0}{\ell_0}x_1 + \alpha x_1^3 = 0 \qquad \alpha = \frac{2K}{\ell_0^2}$$

$$\mu\left[m\ddot{x}_2 + \frac{2T_0}{\ell_0}x_2 + 3\alpha x_1^2 x_2\right] = 0$$

If α is small, then $x_1 = A\cos\omega_n t$, $\omega_n = \sqrt{\frac{2T_0}{\ell_0}}$ and 2^{nd} eq. becomes

$$m\ddot{x}_2 + \frac{2T_0}{\ell_0}x_2 + (3\alpha A^2\cos^2\omega_n t)x_2 = 0$$

$$m\ddot{x}_2 + \left[\left(\frac{2T_0}{\ell_0} + \frac{3}{2}\alpha A^2\right) + \frac{3}{2}\alpha A^2\cos 2\omega_n t\right]x_2 = 0$$

which is Mathieu eq.

Runge - Kutta Program

$$\ddot{\theta} + \sin\theta = 0$$

$$\theta_0 = \frac{\pi}{3} = 60^\circ$$

$$\dot{\theta}_0 = 0$$

```
PROBLEM 14.41 THOMSON
      DIMENSION T(100),T1(100),T2(100),T3(100),T4(100),X(100),X1(100),
     1X2(100),X3(100),X4(100),Y(100),Y1(100),Y2(100),Y3(100),Y4(100),
     1F(100),F1(100),F2(100),F3(100),F4(100)
      N=70
      DH=0.1
      X(1)=3.1415/3.
      Y(1)=0.0
      T(1)=0.0
      PRINT5
    5 FORMAT(20X,'J',5X,'TIME',9X,'DISPL',5X,'ACCELERATION',11X,'F(J)')
      DO 10 J=1,N
      F(J)=FXY(T(J),X(J),Y(J))
      PRINT8,J,T(J),X(J),Y(J),F(J)
    8 FORMAT(18X,I3,2X,F7.3,2X,E12.3,5X,E12.3,3X,E12.3)
      T1(J)=T(J)
      X1(J)=X(J)
      Y1(J)=Y(J)
      F1(J)=FXY(T1(J),X1(J),Y1(J))
      T2(J)=T(J)+DH/2.
      X2(J)=X(J)+Y1(J)*DH/2.
      Y2(J)=Y(J)+F1(J)*DH/2.
      F2(J)=FXY(T2(J),X2(J),Y2(J))
      T3(J)=T(J)+DH/2.
      X3(J)=X(J)+Y2(J)*DH/2.
      Y3(J)=Y(J)+F2(J)*DH/2.
      F3(J)=FXY(T3(J),X3(J),Y3(J))
      T4(J)=T(J)+DH
      X4(J)=X(J)+Y3(J)*DH
      Y4(J)=Y(J)+F3(J)*DH
      F4(J)=FXY(T4(J),X4(J),Y4(J))
      X(J+1)=X(J)+DH/6.*(Y1(J)+2.*Y2(J)+2.*Y3(J)+Y4(J))
      Y(J+1)=Y(J)+DH/6.*(F1(J)+2.*F2(J)+2.*F3(J)+F4(J))
      T(J+1)=T(J)+DH
   10 CONTINUE
      STOP
      END


      FUNCTION FXY(T,X,Y)
      FXY=-SIN(X)
      RETURN
      END
```

14-41 Cont.

J	TIME	DISPL	ACCELERATION	F(J)
1	0.000	0.105E 01	0.000E 00	-0.866E 00
2	0.100	0.104E 01	-0.865E-01	-0.864E 00
3	0.200	0.103E 01	-0.173E 00	-0.857E 00
4	0.300	0.101E 01	-0.258E 00	-0.846E 00
5	0.400	0.978E 00	-0.342E 00	-0.830E 00
6	0.500	0.940E 00	-0.424E 00	-0.808E 00
7	0.600	0.894E 00	-0.503E 00	-0.779E 00
8	0.700	0.840E 00	-0.579E 00	-0.744E 00
9	0.800	0.778E 00	-0.652E 00	-0.702E 00
10	0.900	0.709E 00	-0.719E 00	-0.651E 00
11	1.000	0.634E 00	-0.782E 00	-0.593E 00
12	1.100	0.553E 00	-0.838E 00	-0.526E 00
13	1.200	0.467E 00	-0.886E 00	-0.450E 00
14	1.300	0.376E 00	-0.927E 00	-0.367E 00
15	1.400	0.282E 00	-0.960E 00	-0.278E 00
16	1.500	0.185E 00	-0.983E 00	-0.184E 00
17	1.600	0.856E-01	-0.996E 00	-0.855E-01
18	1.700	-0.143E-01	-0.100E 01	0.143E-01
19	1.800	-0.114E 00	-0.993E 00	0.114E 00
20	1.900	-0.213E 00	-0.977E 00	0.211E 00
21	2.000	-0.309E 00	-0.951E 00	0.304E 00
22	2.100	-0.403E 00	-0.917E 00	0.392E 00
23	2.200	-0.492E 00	-0.873E 00	0.473E 00
24	2.300	-0.577E 00	-0.822E 00	0.546E 00
25	2.400	-0.656E 00	-0.764E 00	0.610E 00
26	2.500	-0.730E 00	-0.701E 00	0.667E 00
27	2.600	-0.796E 00	-0.631E 00	0.715E 00
28	2.700	-0.856E 00	-0.558E 00	0.755E 00
29	2.800	-0.909E 00	-0.481E 00	0.788E 00
30	2.900	-0.952E 00	-0.400E 00	0.814E 00
31	3.000	-0.988E 00	-0.318E 00	0.835E 00
32	3.100	-0.102E 01	-0.234E 00	0.850E 00
33	3.200	-0.103E 01	-0.148E 00	0.860E 00
34	3.300	-0.104E 01	-0.619E-01	0.865E 00
35	3.400	-0.105E 01	0.247E-01	0.866E 00
36	3.500	-0.104E 01	0.111E 00	0.862E 00
37	3.600	-0.102E 01	0.197E 00	0.855E 00
38	3.700	-0.100E 01	0.282E 00	0.842E 00
39	3.800	-0.968E 00	0.365E 00	0.824E 00
40	3.900	-0.929E 00	0.446E 00	0.800E 00

14-41 Cont.

41	4.000	-0.879E 00	0.525E 00	0.770E 00
42	4.100	-0.823E 00	0.600E 00	0.733E 00
43	4.200	-0.759E 00	0.671E 00	0.688E 00
44	4.300	-0.689E 00	0.738E 00	0.636E 00
45	4.400	-0.612E 00	0.798E 00	0.574E 00
46	4.500	-0.529E 00	0.852E 00	0.505E 00
47	4.600	-0.442E 00	0.899E 00	0.427E 00
48	4.700	-0.350E 00	0.937E 00	0.343E 00
49	4.800	-0.254E 00	0.967E 00	0.252E 00
50	4.900	-0.157E 00	0.988E 00	0.156E 00
51	5.000	-0.572E-01	0.998E 00	0.572E-01
52	5.100	0.428E-01	0.999E 00	-0.427E-01
53	5.200	0.142E 00	0.990E 00	-0.142E 00
54	5.300	0.240E 00	0.971E 00	-0.238E 00
55	5.400	0.336E 00	0.942E 00	-0.330E 00
56	5.500	0.429E 00	0.905E 00	-0.416E 00
57	5.600	0.517E 00	0.859E 00	-0.494E 00
58	5.700	0.600E 00	0.806E 00	-0.565E 00
59	5.800	0.678E 00	0.747E 00	-0.627E 00
60	5.900	0.749E 00	0.681E 00	-0.681E 00
61	6.000	0.814E 00	0.611E 00	-0.727E 00
62	6.100	0.871E 00	0.536E 00	-0.765E 00
63	6.200	0.921E 00	0.458E 00	-0.796E 00
64	6.300	0.963E 00	0.377E 00	-0.821E 00
65	6.400	0.996E 00	0.294E 00	-0.840E 00
66	6.500	0.102E 01	0.209E 00	-0.853E 00
67	6.600	0.104E 01	0.124E 00	-0.862E 00
68	6.700	0.105E 01	0.372E-01	-0.866E 00
69	6.800	0.105E 01	-0.494E-01	-0.865E 00
70	6.900	0.104E 01	-0.136E 00	-0.861E 00

14-42 Damped Pendulum (large angles)

$$\ddot{\theta} + \sin\theta + 0.30\dot{\theta} = 0, \quad \frac{g}{l} = 1.0$$

$$\theta(0) = \frac{\pi}{3} = 60°, \quad \dot{\theta}(0) = 0$$

Runge-Kutta Program

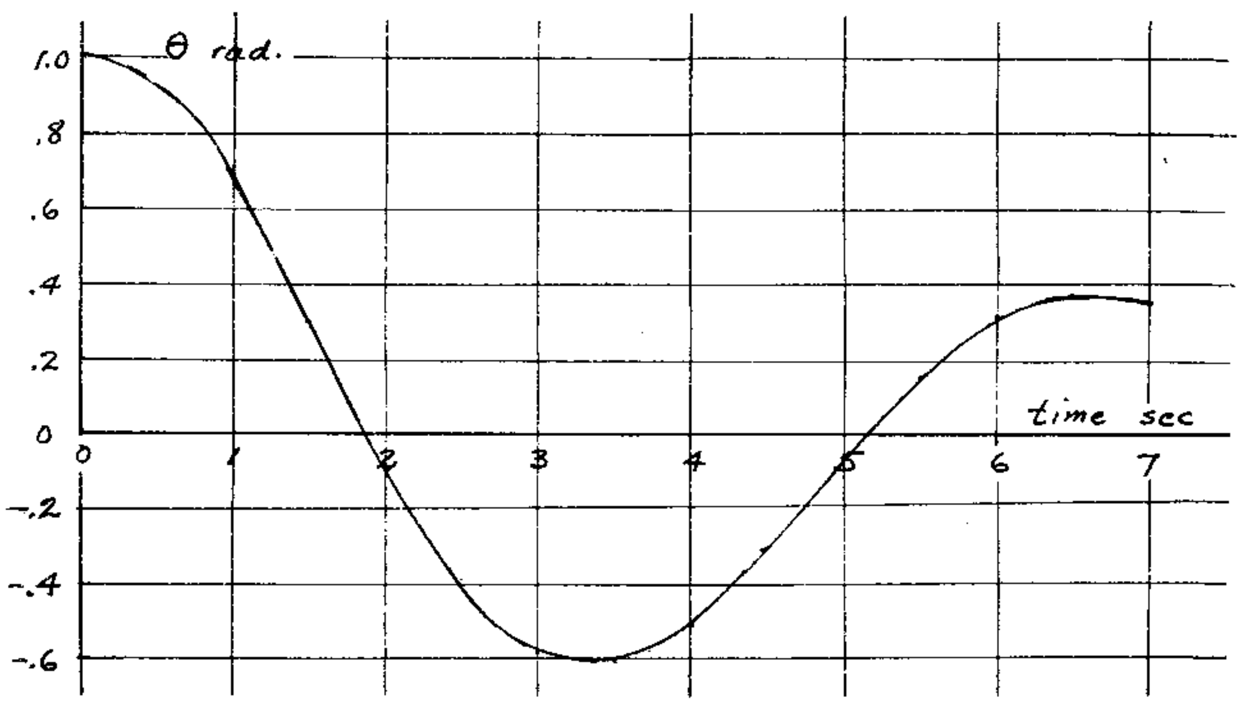
```
PROBLEM 14.42 THOMSON
      DIMENSION T(100),T1(100),T2(100),T3(100),T4(100),X(100),X1(100),
     1X2(100),X3(100),X4(100),Y(100),Y1(100),Y2(100),Y3(100),Y4(100),
     1F(100),F1(100),F2(100),F3(100),F4(100)
      N=70
      DH=0.1
      X(1)=3.1415/3.
      Y(1)=0.0
      T(1)=0.0
      PRINT5
    5 FORMAT(20X,'J',5X,'TIME',9X,'DISPL',5X,'ACCELERATION',11X,'F(J)')
      DO 10 J=1,N
      F(J)=FXY(T(J),X(J),Y(J))
      PRINT8,J,T(J),X(J),Y(J),F(J)
    8 FORMAT(18X,I3,2X,F7.3,2X,E12.3,5X,E12.3,3X,E12.3)
      T1(J)=T(J)
      X1(J)=X(J)
      Y1(J)=Y(J)
      F1(J)=FXY(T1(J),X1(J),Y1(J))
      T2(J)=T(J)+DH/2.
      X2(J)=X(J)+Y1(J)*DH/2.
      Y2(J)=Y(J)+F1(J)*DH/2.
      F2(J)=FXY(T2(J),X2(J),Y2(J))
      T3(J)=T(J)+DH/2.
      X3(J)=X(J)+Y2(J)*DH/2.
      Y3(J)=Y(J)+F2(J)*DH/2.
      F3(J)=FXY(T3(J),X3(J),Y3(J))
      T4(J)=T(J)+DH
      X4(J)=X(J)+Y3(J)*DH
      Y4(J)=Y(J)+F3(J)*DH
      F4(J)=FXY(T4(J),X4(J),Y4(J))
      X(J+1)=X(J)+DH/6.*(Y1(J)+2.*Y2(J)+2.*Y3(J)+Y4(J))
      Y(J+1)=Y(J)+DH/6.*(F1(J)+2.*F2(J)+2.*F3(J)+F4(J))
      T(J+1)=T(J)+DH
   10 CONTINUE
      STOP
      END


      FUNCTION FXY(T,X,Y)
      FXY=-SIN(X)-0.3*Y
      RETURN
      END
```

J	TIME	DISPL	ACCELERATION	F(J)
1	0.000	0.105E 01	0.000E 00	-0.866E 00
2	0.100	0.104E 01	-0.852E-01	-0.838E 00
3	0.200	0.103E 01	-0.168E 00	-0.807E 00
4	0.300	0.101E 01	-0.247E 00	-0.773E 00
5	0.400	0.981E 00	-0.322E 00	-0.734E 00
6	0.500	0.945E 00	-0.393E 00	-0.693E 00
7	0.600	0.903E 00	-0.460E 00	-0.647E 00
8	0.700	0.853E 00	-0.523E 00	-0.597E 00
9	0.800	0.798E 00	-0.580E 00	-0.542E 00
10	0.900	0.738E 00	-0.631E 00	-0.483E 00
11	1.000	0.672E 00	-0.676E 00	-0.420E 00
12	1.100	0.603E 00	-0.715E 00	-0.352E 00
13	1.200	0.530E 00	-0.746E 00	-0.281E 00
14	1.300	0.454E 00	-0.771E 00	-0.207E 00
15	1.400	0.376E 00	-0.788E 00	-0.131E 00
16	1.500	0.296E 00	-0.797E 00	-0.529E-01
17	1.600	0.216E 00	-0.798E 00	0.247E-01
18	1.700	0.137E 00	-0.792E 00	0.101E 00
19	1.800	0.583E-01	-0.778E 00	0.175E 00
20	1.900	-0.185E-01	-0.757E 00	0.246E 00
21	2.000	-0.929E-01	-0.729E 00	0.311E 00
22	2.100	-0.164E 00	-0.695E 00	0.372E 00
23	2.200	-0.232E 00	-0.655E 00	0.426E 00
24	2.300	-0.295E 00	-0.610E 00	0.474E 00
25	2.400	-0.354E 00	-0.561E 00	0.514E 00
26	2.500	-0.407E 00	-0.507E 00	0.548E 00
27	2.600	-0.455E 00	-0.451E 00	0.575E 00
28	2.700	-0.497E 00	-0.393E 00	0.595E 00
29	2.800	-0.533E 00	-0.332E 00	0.608E 00
30	2.900	-0.564E 00	-0.271E 00	0.616E 00
31	3.000	-0.583E 00	-0.210E 00	0.617E 00
32	3.100	-0.605E 00	-0.148E 00	0.613E 00
33	3.200	-0.617E 00	-0.870E-01	0.605E 00
34	3.300	-0.623E 00	-0.271E-01	0.591E 00
35	3.400	-0.623E 00	0.312E-01	0.574E 00
36	3.500	-0.617E 00	0.875E-01	0.552E 00
37	3.600	-0.605E 00	0.141E 00	0.527E 00
38	3.700	-0.588E 00	0.193E 00	0.497E 00
39	3.800	-0.567E 00	0.241E 00	0.465E 00
40	3.900	-0.540E 00	0.286E 00	0.429E 00
41	4.000	-0.510E 00	0.326E 00	0.390E 00
42	4.100	-0.475E 00	0.363E 00	0.349E 00

14-42 Cont:

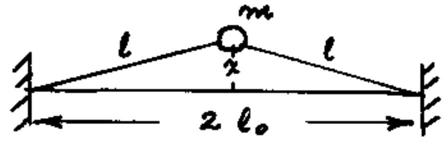
43	4.200	-0.437E 00	0.396E 00	0.305E 00
44	4.300	-0.396E 00	0.424E 00	0.259E 00
45	4.400	-0.353E 00	0.448E 00	0.211E 00
46	4.500	-0.307E 00	0.466E 00	0.162E 00
47	4.600	-0.259E 00	0.480E 00	0.112E 00
48	4.700	-0.211E 00	0.489E 00	0.627E-01
49	4.800	-0.162E 00	0.493E 00	0.133E-01
50	4.900	-0.113E 00	0.492E 00	-0.352E-01
51	5.000	-0.636E-01	0.486E 00	-0.821E-01
52	5.100	-0.156E-01	0.475E 00	-0.127E 00
53	5.200	0.313E-01	0.460E 00	-0.169E 00
54	5.300	0.764E-01	0.441E 00	-0.209E 00
55	5.400	0.119E 00	0.419E 00	-0.245E 00
56	5.500	0.160E 00	0.393E 00	-0.277E 00
57	5.600	0.198E 00	0.363E 00	-0.306E 00
58	5.700	0.233E 00	0.332E 00	-0.330E 00
59	5.800	0.264E 00	0.298E 00	-0.350E 00
60	5.900	0.292E 00	0.262E 00	-0.366E 00
61	6.000	0.316E 00	0.224E 00	-0.379E 00
62	6.100	0.337E 00	0.186E 00	-0.386E 00
63	6.200	0.354E 00	0.147E 00	-0.390E 00
64	6.300	0.366E 00	0.108E 00	-0.391E 00
65	6.400	0.375E 00	0.693E-01	-0.387E 00
66	6.500	0.380E 00	0.309E-01	-0.380E 00
67	6.600	0.381E 00	-0.671E-02	-0.370E 00
68	6.700	0.379E 00	-0.431E-01	-0.357E 00
69	6.800	0.373E 00	-0.780E-01	-0.341E 00
70	6.900	0.363E 00	-0.111E 00	-0.322E 00



14-43 **Runge-Kutta Program**

$$\ddot{\xi} + \xi + 0.2\xi^3 = 0$$

$$\xi = \frac{x}{L_0}, \quad \xi(0) = 0.20, \quad \dot{\xi}(0) = 0$$



Runge-Kutta Program

```
PROBLEM 14.43 THOMSON
      DIMENSION T(100),T1(100),T2(100),T3(100),T4(100),X(100),X1(100),
     1X2(100),X3(100),X4(100),Y(100),Y1(100),Y2(100),Y3(100),Y4(100),
     1F(100),F1(100),F2(100),F3(100),F4(100)
      N=70
      DH=0.1
      X(1)=0.2
      Y(1)=0.0
      T(1)=0.0
      PRINT5
    5 FORMAT(20X,'J',5X,'TIME',9X,'DISPL',5X,'ACCELERATION',11X,'F(J)')
      DO 10 J=1,N
      F(J)=FXY(T(J),X(J),Y(J))
      PRINT8,J,T(J),X(J),Y(J),F(J)
    8 FORMAT(18X,I3,2X,F7.3,2X,E12.3,5X,E12.3,3X,E12.3)
      T1(J)=T(J)
      X1(J)=X(J)
      Y1(J)=Y(J)
      F1(J)=FXY(T1(J),X1(J),Y1(J))
      T2(J)=T(J)+DH/2.
      X2(J)=X(J)+Y1(J)*DH/2.
      Y2(J)=Y(J)+F1(J)*DH/2.
      F2(J)=FXY(T2(J),X2(J),Y2(J))
      T3(J)=T(J)+DH/2.
      X3(J)=X(J)+Y2(J)*DH/2.
      Y3(J)=Y(J)+F2(J)*DH/2.
      F3(J)=FXY(T3(J),X3(J),Y3(J))
      T4(J)=T(J)+DH
      X4(J)=X(J)+Y3(J)*DH(
      Y4(J)=Y(J)+F3(J)*DH
      F4(J)=FXY(T4(J),X4(J),Y4(J))
      X(J+1)=X(J)+DH/6.*(Y1(J)+2.*Y2(J)+2.*Y3(J)+Y4(J))
      Y(J+1)=Y(J)+DH/6.*(F1(J)+2.*F2(J)+2.*F3(J)+F4(J))
      T(J+1)=T(J)+DH
   10 CONTINUE
      STOP
      END


      FUNCTION FXY(T,X,Y)
      FXY=-X-0.2*X**3
      RETURN
      END
```

14-43 Cont.

J	TIME	DISPL	ACCELERATION	F(J)
1	0.000	0.200E 00	0.000E 00	-0.202E 00
2	0.100	0.199E 00	-0.201E-01	-0.201E 00
3	0.200	0.196E 00	-0.400E-01	-0.197E 00
4	0.300	0.191E 00	-0.596E-01	-0.192E 00
5	0.400	0.184E 00	-0.785E-01	-0.185E 00
6	0.500	0.175E 00	-0.966E-01	-0.176E 00
7	0.600	0.165E 00	-0.114E 00	-0.166E 00
8	0.700	0.153E 00	-0.130E 00	-0.153E 00
9	0.800	0.139E 00	-0.144E 00	-0.139E 00
10	0.900	0.124E 00	-0.157E 00	-0.124E 00
11	1.000	0.107E 00	-0.169E 00	-0.108E 00
12	1.100	0.901E-01	-0.179E 00	-0.902E-01
13	1.200	0.717E-01	-0.187E 00	-0.718E-01
14	1.300	0.527E-01	-0.193E 00	-0.527E-01
15	1.400	0.331E-01	-0.198E 00	-0.331E-01
16	1.500	0.132E-01	-0.200E 00	-0.132E-01
17	1.600	-0.679E-02	-0.200E 00	0.679E-02
18	1.700	-0.268E-01	-0.199E 00	0.268E-01
19	1.800	-0.464E-01	-0.195E 00	0.465E-01
20	1.900	-0.657E-01	-0.189E 00	0.657E-01
21	2.000	-0.842E-01	-0.182E 00	0.844E-01
22	2.100	-0.102E 00	-0.172E 00	0.102E 00
23	2.200	-0.119E 00	-0.161E 00	0.119E 00
24	2.300	-0.134E 00	-0.149E 00	0.135E 00
25	2.400	-0.148E 00	-0.135E 00	0.149E 00
26	2.500	-0.161E 00	-0.119E 00	0.162E 00
27	2.600	-0.172E 00	-0.102E 00	0.173E 00
28	2.700	-0.181E 00	-0.844E-01	0.183E 00
29	2.800	-0.189E 00	-0.657E-01	0.190E 00
30	2.900	-0.195E 00	-0.464E-01	0.196E 00
31	3.000	-0.198E 00	-0.266E-01	0.200E 00
32	3.100	-0.200E 00	-0.649E-02	0.201E 00
33	3.200	-0.200E 00	0.137E-01	0.201E 00
34	3.300	-0.197E 00	0.337E-01	0.199E 00
35	3.400	-0.193E 00	0.533E-01	0.194E 00
36	3.500	-0.187E 00	0.724E-01	0.188E 00
37	3.600	-0.178E 00	0.908E-01	0.179E 00
38	3.700	-0.168E 00	0.108E 00	0.169E 00
39	3.800	-0.157E 00	0.125E 00	0.158E 00
40	3.900	-0.144E 00	0.140E 00	0.144E 00
41	4.000	-0.129E 00	0.153E 00	0.129E 00
42	4.100	-0.113E 00	0.166E 00	0.113E 00

14-43 Cont:

	time	displ.	accel.	F(J)
43	4.200	-0.958E-01	0.176E 00	0.960E-01
44	4.300	-0.777E-01	0.185E 00	0.778E-01
45	4.400	-0.589E-01	0.192E 00	0.589E-01
46	4.500	-0.395E-01	0.196E 00	0.395E-01
47	4.600	-0.197E-01	0.199E 00	0.197E-01
48	4.700	0.335E-03	0.200E 00	-0.335E-03
49	4.800	0.203E-01	0.199E 00	-0.203E-01
50	4.900	0.401E-01	0.196E 00	-0.402E-01
51	5.000	0.595E-01	0.191E 00	-0.596E-01
52	5.100	0.783E-01	0.184E 00	-0.784E-01
53	5.200	0.964E-01	0.176E 00	-0.965E-01
54	5.300	0.113E 00	0.165E 00	-0.114E 00
55	5.400	0.129E 00	0.153E 00	-0.130E 00
56	5.500	0.144E 00	0.139E 00	-0.145E 00
57	5.600	0.157E 00	0.124E 00	-0.158E 00
58	5.700	0.169E 00	0.108E 00	-0.170E 00
59	5.800	0.179E 00	0.902E-01	-0.180E 00
60	5.900	0.187E 00	0.718E-01	-0.188E 00
61	6.000	0.193E 00	0.527E-01	-0.194E 00
62	6.100	0.197E 00	0.330E-01	-0.199E 00
63	6.200	0.200E 00	0.130E-01	-0.201E 00
64	6.300	0.200E 00	-0.717E-02	-0.201E 00
65	6.400	0.198E 00	-0.272E-01	-0.200E 00
66	6.500	0.194E 00	-0.470E-01	-0.196E 00
67	6.600	0.189E 00	-0.664E-01	-0.190E 00
68	6.700	0.181E 00	-0.850E-01	-0.182E 00
69	6.800	0.172E 00	-0.103E 00	-0.173E 00
70	6.900	0.161E 00	-0.119E 00	-0.161E 00