❑ Damped harmonic motion/oscillation:

In real systems, masses on springs do not continue to oscillate forever at the same amplitude, eventually the oscillations die away and object stops. This type of motion/oscillation is known as damped harmonic motion/oscillation. This is due to the fact that springs are not very efficient for storing and releasing energy. Much of the energy is lost as heat due to friction within the spring. In order to better model these kinds of systems, we can consider the damped harmonic oscillator.

❑ Differential equation of damped harmonic oscillator:

The equation of motion of the damped harmonic oscillation is

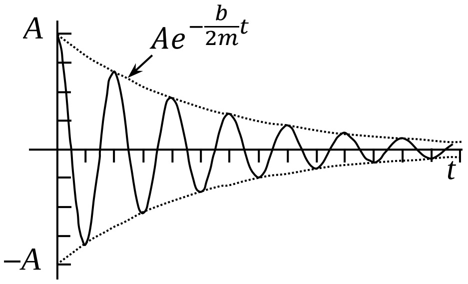
$$F = m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

or, $$m\frac{d^2x}{dt^2} + kx + b\frac{dx}{dt} = 0$$

or, $$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Where we have taken the differential equation for the simple harmonic oscillator and added a damping term, $-b\frac{dx}{dt}$, where b is called damping constant or drag coefficient.

Since this damping term acts in the opposite direction of motion and is proportional to velocity, it causes objects with high velocity to slow down quickly.



By solving Eq. (1), we have

$$x = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$$

where, $\omega = \sqrt{\omega_o^2 - \frac{b^2}{4m^2}}$

and $\omega_o = \sqrt{\frac{k}{m}}$ is the frequency of simple harmonic oscillator.

If $\quad b = 0,\ \omega = \omega_o$

We can see that if we add damping to a simple harmonic oscillator, the frequency will change and the amplitude of the oscillation will exponentially decay with time.

Damped physical system can be three types-

(i) Under damped:

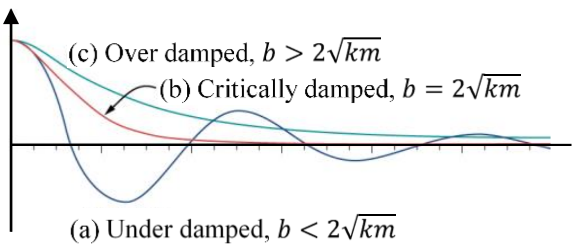
$$b < 2\sqrt{km}$$

$\therefore \frac{b^2}{4m^2} < \frac{k}{m}$, for which ω is positive. $\qquad$ [since $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$]

(ii) Critically damped:

$$b = 2\sqrt{km}$$

$\therefore \frac{b^2}{4m^2} \approx \frac{k}{m} \approx \omega_o^2$ for which $\omega \approx 0$.



(ii) Over damped:

$$b > 2\sqrt{km}$$

or, $\frac{b^2}{4m^2} > \frac{k}{m}$

or, $\frac{b^2}{4m^2} > \omega_o^2$ for which ω is imaginary.

❑ Forced vibration:

For a vibrating system, when the frequency of the driving and driven systems is not the same, the natural frequency of the oscillator dies out soon and it begins to oscillate with the frequency of the impressed periodic force. The vibrations so kept up are called forced vibration.

Differential equation of forced vibration or forced harmonic oscillator:

For forced harmonic oscillations, there are three forces acting on the vibrating particle-

(i) Restoring force, $F_r = -kx$
(ii) Damping force, $F_d = -b\frac{dx}{dt}$
(iii) Driving force, $F_D = F_o \sin \omega t$

where, F_o is the amplitude of driving force and ω is the angular frequency of driving force. The equation of motion becomes

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_o \sin \omega t$$

or, $\frac{d^2x}{dt^2} = -\frac{k}{m}x - \frac{b}{m}\frac{dx}{dt} + \frac{F_o}{m}\sin \omega t$

or, $\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_o^2 x = f_o \sin \omega t$………… (1)

where, $\frac{b}{m} = 2r,\quad \frac{k}{m} = \omega_o^2,\quad \frac{F_o}{m} = f_o$

The complete solution of Eq. (1) is

$$A_o e^{-rt}\sin(\omega' t + \phi) + B\sin(\omega t + \phi)$$

The first part of the solution ceases after some time. It is called the transient solution. In the transient state, the oscillator oscillates neither with its natural frequency nor with the frequency of the impressed force.

The second part continues indefinitely as there is driving force. It is called the steady state term. In the steady state, the oscillator oscillates with the frequency of the external force.