❑ Simple harmonic motion:

The motion of a particle moving along a straight line with an acceleration whose direction is always towards a foxed point on the line and whose magnitude is proportional to the distance from the fixed point is called simple harmonic motion. A vibrating system that execute simple harmonic motion is sometimes called simple harmonic oscillator.

❑ Characteristics of simple harmonic motion:

1. The motion is periodic motion.
2. The acceleration of the body is proportional to its displacement from its equilibrium position.
3. The acceleration of the body is always directed towards a certain fixed point.
4. The motion is isochronous because the expression for the time period ($T = 2\pi\sqrt{\frac{m}{k}}$) is independent of the amplitude of the motion.
5. The equation does not conclude the motion of the particle along the circle.

❑ Differential equation of a simple harmonic oscillator and its solution:

Let us consider a motion of the vibration of a system of mass m, attached to a spring of negligible mass, as the mass slides on a frictionless surface, as shown in Fig. 1.

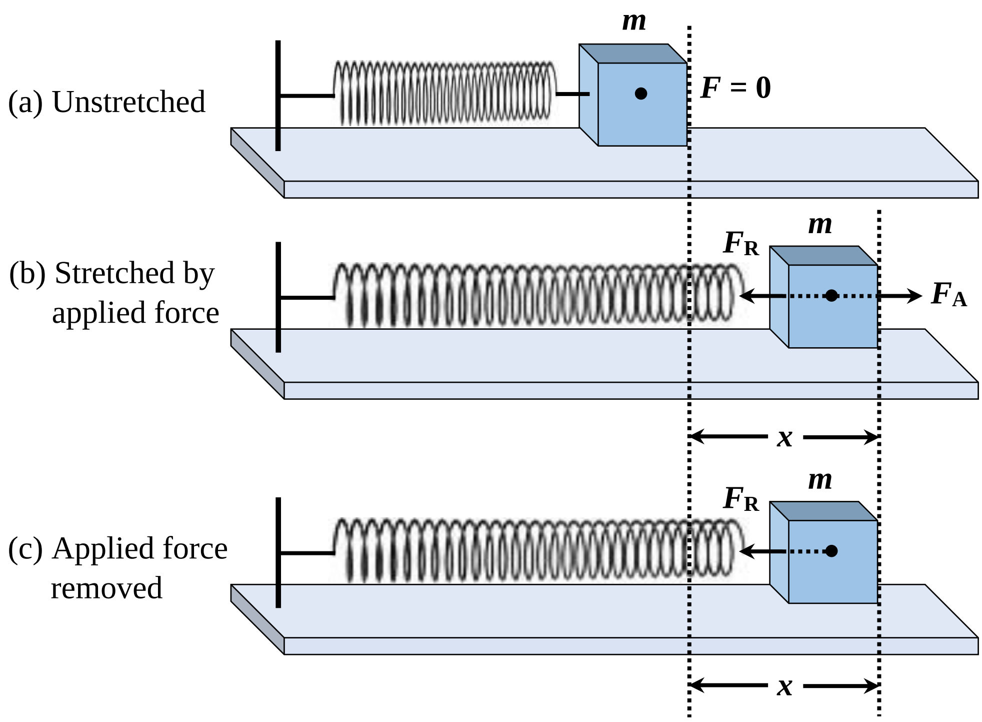

Fig.1. Vibrating spring.

Let us consider that the mass m, in unstretched position [Fig. 1(a)], is in its equilibrium position. If an applied force F_A acts on the mass, it will be displaced to the right of its equilibrium position be a distance x as shown in Fig. 1(b).

From Hook's law

$$F_A = kx \quad \text{where, k is the force constant}$$

By Newton's third law, there is an equal but opposite elastic restoring force exerted by the spring on the mass which pulling the mass towards the left.

$F_R = -F_A$

or, $F_R = -kx \ldots\ldots\ldots\ldots\ldots\ldots.(1)$

When F_A is removed, the elastic restoring force F_R is then the only force acting on the mass *m* [Fig. 1(c)], and it tries to restore *m* to its equilibrium position. We can then find the acceleration of the mass from Newton's second law as

$ma = F_R$

or, $ma = -kx \ \ldots\ldots\ldots\ldots\ldots.(2)$

Since the acceleration $a = \frac{d^2x}{dt^2}$, the Eq. (2) can be written as

$m\frac{d^2x}{dt^2} = -kx$

or, $\frac{d^2x}{dt^2} = -\frac{k}{m}x$

$\therefore \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \ \ldots\ldots\ldots\ldots\ldots(3)$

The Eq. (3) is the required differential equation of a simple harmonic oscillator.

Solving the Eq. (3) produce a solution that is a sinusoidal function

$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$

This equation can be written in the form

$x(t) = A\cos(\omega t + \phi)$

where, $\omega = \sqrt{\frac{k}{m}}$, $A = \sqrt{c_1^2 + c_2^2}$, $\tan\phi = \frac{c_2}{c_1}$

where, *A* is the amplitude (maximum displacement from equilibrium), $\omega = 2\pi f$ is the angular frequency, f is the frequency, and ϕ is a constant known as phase.

Velocity $v = \frac{dx}{dt} = -A\sin(\omega t + \phi).\omega = -A\omega\sin(\omega t + \phi)$

Acceleration $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -A\omega\cos(\omega t + \phi).\omega = -A\omega^2\cos(\omega t + \phi)$

or, $a = -\omega^2 x$, where, $\omega^2 = \frac{k}{m}$

Since $\omega = 2\pi f$

$\therefore$ **frequency** $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

and since **time period** $T = \frac{1}{f}$

$\therefore \quad T = 2\pi\sqrt{\frac{m}{k}}$

❑ **Energy of simple harmonic oscillator:**

Kinetic energy $K = \frac{1}{2}mv^2 = \frac{1}{2}m\,[-A\omega\sin(\omega t + \phi)]^2$ [since $v = -A\omega\sin(\omega t + \phi)$]

$$= \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi) = \frac{1}{2} m \; \frac{k}{m} \; A^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2} kA^2 \sin^2(\omega t + \phi)$$

and potential energy $U = \frac{1}{2} kx^2$

$$= \frac{1}{2} k \, A^2 \cos^2(\omega t + \phi) \qquad [\text{since } x = A\cos(\omega t + \phi)]$$

The total energy $E = K + U$

$$= \frac{1}{2} kA^2 \sin^2(\omega t + \phi) + \frac{1}{2} k \, A^2 \cos^2(\omega t + \phi)$$

$$= \frac{1}{2} kA^2 \, [\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

$= \frac{1}{2} kA^2$ where, *k* and *A* are constant.

Therefore, in the absence of friction and other energy loss, the total mechanical energy of a simple harmonic oscillator is constant.