Math 1201: Integral Calculus and Ordinary Differential Equations and Series Solutions

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Constant:

A constant is a quantity which has a fixed value. Constants are usually written in numbers.

Constant are two types: 1) Absolute/fixed value Constant.
2) Arbitrary Constant.

Arbitrary Constant: A symbol to which various values may be assigned but which remains unaffected by the changes in the values of the variables of the equation.

Variable:

A variable is a quantity which take any value. Typically, we use a single letter to represent a variable. The letters x, y, and z are common generic symbols used for variables.

Variable are two types: 1) Independent Variable($x=5$)
2) Dependent Variable($x+y=5$)

Differential Equation:

A differential equation is an equation that contains one or more functions with its derivatives.

A differential equation is an equation which contains one or more terms and the derivatives of one variable (i.e., dependent variable) with respect to the other variable (i.e., independent variable).

Differential Equation are two types:
1) ODE (Ordinary Differential Equation)
2) PDE (Partially Differential Equation)

ODE: A differential equation that involves only one independent variable.

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 1$

[নিচে যদি একটা Variable থাকে তাহলে সেটা ODE । অর্থাৎ নিচে যদি x থাকে তাহলে শুধু x ই থাকতে পারবে তাহলেই সেটা ODE হবে]

- One independent Variable
- Must have $\frac{d}{dx}$ operator.

PDE: A differential equation that involves more than one independent variables. Exempla….

$\frac{dz}{dx} + \frac{dz}{dy} = 1$
$\frac{dz}{dx} + \frac{dz}{dy} + \frac{dz}{dt} = 1$

[নিচে একাধিক Variable থাকবে]

- More than one independent variable.
- Must have ($\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$,etc) operator.

ODE are two types: 1) Linear ODE.
2) Non-Linear ODE.

Linear ODE: A linear equation, with one or more terms, consisting of the derivatives of the dependent variable with respect to one or more independent variables is known as a linear differential equation.

- No product of the dependent variable [y^2,y^3,y^4,etc]
- No product of the dependent variable and its derivatives [y.$\frac{dy}{dx}$, etc] but it may [x.$\frac{dy}{dx}$, etc] stay.
- No product of the derivative of the dependent variable [$(\frac{dy}{dx})^2 = \frac{dy}{dx} . \frac{dy}{dx}$, etc]
- No transcendental (Series) functions of the dependent variables. [e^y, siny, cosy, etc]

Order: Order of a differential equation is the order of the highest derivative (also known as differential coefficient) present in the equation. Example

i) $\frac{d^2y}{dx^2} + \frac{dy}{dx}$ +y=1

The order of this equation is 2.

ii) $(\frac{d^3y}{dx^3})^4 + \frac{dy}{dx} =1$

The order of this equation is 3.

Degree: The degree of the differential equation is represented by the power of the highest order derivative in the given differential equation. Example

i) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 1$

The order of this equation is 2 and the degree is 1.

ii) $\left(\frac{d^3y}{dx^3}\right)^4 + \frac{dy}{dx} = 1$

The order of this equation is 3 and the degree is 4.

iii) $\left(\frac{d^2x}{dx^2}\right)^{20} + \left(\frac{dy}{dx}\right)^{700} + y^{909} = 1$

The order of this equation is 2 and the degree is 20.

iv) $(x^3 + y^3)\, dx + 2x^2y\, dy = 0$

The order of this equation is 1(for dx or dy) and the degree is 3(for x^3 or y^3 or x^2y(2+1=3).

Formation of the ODE:

Arbitrary Constant: A symbol to which various values may be assigned but which remains unaffected by the changes in the values of the variables of the equation.

[একটি প্রতীক যার জন্য বিভিন্ন মান বরাদ্দ করা যেতে পারে কিন্তু যা সমীকরণের ভেরিয়েবলের মানগুলির পরিবর্তন দ্বারা প্রভাবিত হয় না]

Rule of the ODE formation:

- যে Equation এ যতটা Arbitrary Constant থাকবে ঠিক তত বার Derivative করতে হবে।
- Derivative করার পর সেই Arbitrary Constant এর মান গুলো বের করতে হবে।
- সেই মান গুলো মূল Equation এ বসাতে হবে।
- সমাধানে কোনো Arbitrary Constant থাকতে পারবে না।

$Y=ax^3 + bx + c$

In this equation a, b, c are arbitrary constant.

Order = Number of arbitrary

Math-01: $Y=ax^2 + bx + c$ transfer this equation in ODE formation.

Solve: Follow the ODE formation rule.

[Ans: $\frac{d^3y}{dx^3}=0$]

Math-02: $Y=ax^3 + bx + c$ transfer this equation in ODE formation.

Solve: Follow the ODE formation rule.

Math-03: $Y=a\cos(bx+c)$ transfer this equation in ODE formation.

Solve: Follow the ODE formation rule.

Separable Equations:

A differential equation is said to be **separable** if the variables can be separated. A separable differential equation is any differential equation that we can write in the following form.

$f_1(x)g_1(y)dx= f_2(x)g_2(y)dy$

Separable Equation Solve:

- প্রথমে সমীকরণের যত x অথবা y আছে সব আলাদা করতে হবে।
- Format: $f(x)dx + g(y)dy = 0$
- তারপর পরো সমীকরণটা Integration করলেই উত্তর চলে আসবে।

- Format: $\int f(x)dx + \int g(x)dy = 0$

Math-01: Solve **$(1+x^2)dy = (1+y^2)dx$** Equation using variable Separation method.

Solve:

Given that,

$$(1+x^2)dy = (1+y^2)dx$$

This is a Separable Equation. Separating Variable we get,

$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$................ (1)

Integrating equation (1) we get,

$\int \frac{1}{1+y^2}\,dy = \int \frac{1}{1+x^2}\,dx$

or, $\tan^{-1}(y) = \tan^{-1}(x) + c$ [ANSWER]

or, $\tan^{-1}(y) = \tan^{-1}(x) + \tan^{-1}(c)$

or, $\tan^{-1}y - \tan^{-1}x = \tan^{-1}c$

or, $\tan^{-1}(\frac{y-x}{1+xY}) = \tan^{-1}c$

or, $(\frac{y-x}{1+xY}) = c$

or, $(y-x) = c(1+xy)$ [ANSWER]

Math-02: Solve **$(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$** Equation using variable Separation method.

Solve:

Hint: $\frac{cosx}{sinx}\,dx = -\frac{e^y}{1+e^y}dy$

Ans-1: $\log(\sin x) = -\log(1+e^y)+c$

Ans-2: $\log(\sin x) = -\log(1+e^y)+\log c$

Ans-3: $\sin x(1+e^y)=c$

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Formula: $\int \frac{f^1(x)}{f(x)}$ **dx = log[f(x)]**

Math-03: Solve $\frac{dy}{dx}$ **= e^{x-y} + x^2 e^{-y}** Equation using variable Separation method.

Solve:

Hint: e^y dy = (e^x + x^2) dx

Ans-1: $e^y = e^x + \frac{x^3}{3} + c$

Math-04: Solve $3e^x$ tany dx + (1-e^x)sec^2y dy =0 Equation using variable Separation method.

Solve:

Hint: $\frac{e^x dx}{e^x-1} = \frac{sec^2 y\, dy}{tany}$

Ans-1: log(e^x -1) = log(tany) + c

Ans-2: log(e^x -1) = log(tany) + logc

Ans-3: $(e^x-1)^3$ = c tany

Math-05: Solve y - x$\frac{dy}{dx}$ = a(y^2 + $\frac{dy}{dx}$) Equation using variable Separation method.

Solve:

Hint-1: $\frac{dx}{a+x} = \frac{dy}{y-ay^2}$

Hint-2: $\frac{dx}{a+x} = \frac{dy}{y(1-ay)}$

Hint-3: $\frac{dx}{a+x} = \frac{1}{y}$ dy + $\frac{a}{1-ay}$dy

Ans-1: log(x+a) = log(y) – log(1-ay) + c

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Ans-2: $\log(x+a) = \log(y) - \log(1-ay) + \log c$

Ans-3: $(x+a)(1-ay) = cy$

Math-06: Solve **$x \sin y\, dx + (1+x^2) \cos y\, dy = 0$** Equation using variable Separation method.

Solve:

Hint: $\frac{x}{1+x^2}\, dx + \frac{\cos y}{\sin y}\, dy = 0$

Ans-1: $\frac{1}{2} \log(1+x^2) + \log(\sin y) = c$

Ans-2: $\frac{1}{2} \log(1+x^2) + \log(\sin y) = \log c$

Ans-3: $(1+x^2) \sin^2 y = c$ Or $2c$

Math-07: Solve **$(x-4)y^4\, dx - x^3(y^2 - 3)\, dy = 0$** Equation using variable Separation method.

Homogeneous Function:

Homogeneous function is a function with multiplicative scaling behaving. The function $f(x, y)$, if it can be expressed by writing $x = kx$, and $y = ky$ to form a new function $f(kx, ky) = k^n f(x, y)$ such that the constant k can be taken as the n^{th} power of the exponent, is called a homogeneous function. Here the exponent n is called the degree of homogeneity.

A function $f(x,y)$ is said to be a homogeneous function of degree n if it can be expressed in the form $x^n \varphi(\frac{y}{x})$ or $y^n \varphi(\frac{x}{y})$.

Homogeneous Equation:

A differential equation of the form **m(x,y)dx + n(x,y)dy = 0** is said to be homogeneous differential equation if ***the degree of f(x,y) and g(x, y) is same.*** A function of form F(x,y) which can be written in the form k^n F(x,y) is said to be a homogeneous function of degree n, for k≠0.

দুটি homogeneous function ভাগ করলে degree শূন্য(0) হবেI

$$\frac{m(x,y)}{n(x,.y)} = x^0\ \varphi(\frac{y}{x}) \text{ or } y^0\ \varphi(\frac{x}{y})$$

Math-01: Show that y=vx transform a homogeneous equation into a separable equation.

Or, Transform a homogeneous equation into a separable equation.

Solve:

Here, Homogeneous equation format,

m(x,y)dx + n(x,y)dy = 0 ……………..(i)

or, $\frac{dy}{dx} = -\frac{m(x,y)}{n(x,.y)}$

or, $\frac{dy}{dx} = x^0\ \varphi(\frac{y}{x})$ **[A function f(x,y) is said to be a homogeneous function of degree n if it can be expressed in the form $x^n\ \varphi(\frac{y}{x})$ or $y^n\ \varphi(\frac{x}{y})$.]**

…………..(ii)

Let, y = vx ……………….(iii)

Derivative equation (iii) we get,

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ ……………..(iv)

Putting (iii) and (iv) into (ii) than we get,

$v + x\frac{dv}{dx} = \varphi(\frac{vx}{x})$

or, v + x $\frac{dv}{dx}$ = φ(v)

or, x $\frac{dv}{dx}$ = φ(v) - v

or, $\frac{dv}{\varphi(v) - v}$ = $\frac{dx}{x}$ [This is a separable]

Homogeneous Equation Solve:

প্রদত্ত সমীকরনের Degree অবশ্যই একই হতে হবে।

- সমীকরণ হতে $\frac{dy}{dx}$ অথবা $\frac{dx}{dy}$ বের করা লাগবে।
- Format: $\frac{dy}{dx}$ = f($\frac{y}{x}$) অথবা $\frac{dx}{dy}$ = f($\frac{x}{y}$) [Equ-01]
- যদি $\frac{dy}{dx}$ বের হয় তবে y = vx ধরতে হবে অথবা $\frac{dx}{dy}$ বের হয় তবে x = vy ধরতে হবে। [Equ-02]
- Derivative Equation-02 and get Equation-03.
- Putting Equ-02 and Equ-03 value into Equ-01.
- Follow Separable Equation Solve. (A to Z)
- After answer putting the value of v into this answer and get final answer.

Math-02: Solve **(x^2 – $3y^2$) dx + 2xy dy = 0** homogeneous equation.

Solve:

Given that,

(x^2 – $3y^2$) dx + 2xy dy = 0

or, $\frac{dy}{dx}$ = - $\frac{x^2-3y^2}{2xy}$ ………….(i)

Let, y = vx ……………….(ii)

Derivative Equation-(ii) and we get,

$\frac{dy}{dx}$ = v + x $\frac{dv}{dx}$ ……………..(iii)

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Putting (ii) and (iii) into (i) than we get,

$$v + x\frac{dv}{dx} = -\frac{x^2-3v^2x^2}{2vx^2}$$

or, $\frac{2v}{v^2-1}$ dv = $\frac{dx}{x}$ ……………..(iv)

Integrating equation (iv) we get,

$$\int\frac{2v}{v^2-1}\,dv = \int\frac{dx}{x}$$

or, $\log(v^2 - 1) = \log(x) + c$

or, $\log(v^2 - 1) = \log(x) + \log(c)$

or, $\log(v^2 - 1) - \log(x) = \log(c)$

or, $\log(\frac{v^2-1}{x}) = \log(c)$

or, $\frac{v^2-1}{x} = c$

or, $v^2 - 1 = cx$

Putting the value of v from Eq-(ii) v = y/x

$$\frac{y^2}{x^2} - 1 = cx$$

or, $y^2 - x^2 = cx^3$ [Answer]

Math-02: Solve $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$ homogeneous equation.

Solve:

Answer: $\frac{y}{x} - \log(y) = c$

Math-02: Solve $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$ homogeneous equation.

Solve:

Answer: $y + \sqrt{x^2+y^2} = cx^2$

Non-homogeneous equations:

Non homogeneous differential equations are simply differential equations that do not satisfy the conditions for homogeneous equations.

Format: $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

or, $(a_1x + b_1y + c_1)$**dx = $(a_2x + b_2y + c_2)$dy**

Non-Homogeneous Equation Solve:

Non-homogeneous -> Homogeneous -> Separable

Way-01:

- $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ [Equ-1] তে $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ সমান হলে $a_1x + b_1y = v$ অথবা $a_2x + b_2y = v$ ধরতে হবে। [Equ-2]
- Derivative Equation-02 and get Equation-03.
- Putting Equ-02 and Equ-03 value into Equ-01.
- Follow Separable Equation Solve. (A to Z)
- After answer putting the value of v into this answer and get final answer.

Way-02:

- $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ [Equ-1] তে $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ হলে $x = x_1 + h$ and $y = y_1+k$ ধরতে হবে। [Equ-2]
- Derivative Equation-02 and divided them to get $\frac{dy}{dx}$ [Equ-3]
- Putting Equ-02 and Equ-03 value into Equ-01.
- Follow Math-02.

Math-01: Solve the differential equation $\frac{dy}{dx} = \frac{6x-2y-7}{3x-y+4}$

Solve:

Given that,

$$\frac{dy}{dx} = \frac{6x-2y-7}{3x-y+4} \ldots\ldots\ldots\ldots\ldots(i)$$

This is a non-homogeneous equation.

Here, $a_1 = 6$, $b_1 = -2$

$a_2 = 3$, $b_2 = -1$

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{2}{1}$

Let, $3x-y = v \ldots\ldots\ldots\ldots\ldots(ii)$

or, $3 - \frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dy}{dx} = 3 - \frac{dv}{dx} \ldots\ldots\ldots\ldots\ldots(iii)$

Putting Equ-02 and Equ-03 value into Equ-01 we get,

$3 - \frac{dv}{dx} = \frac{2v-7}{v+4}$

or, $\frac{dv}{dx} = \frac{v+19}{v+4}$

or, $\frac{v+4}{v+19}$ dv = dx

This is a Separable equation. Integrating this equation we get,

$\int \frac{v+4}{v+19} \, dv = \int dx$

or, $\int \frac{v+19-15}{v+19} \, dv = \int dx$

or, $\int dv - 15\int \frac{1}{v+19} \, dv = \int dx$

or, $v - 15 \log(v+19) = x + c$

or, $3x-y - 15 \log(3x-y+19) = x+c$ [From Equ-(ii) $v = 3x-y$]

or, $2x - 15 \log(3x-y+19) = c + y$ [Answer]

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Math-02: Solve the differential equation $\frac{dy}{dx} = \frac{6x-2y-7}{2x+3y-6}$

Solve:

Given that,

$$\frac{dy}{dx} = \frac{6x-2y-7}{2x+3y-6} \text{ ....(i)}$$

This is a non-homogeneous equation.

Here, $a_1 = 6$, $b_1 = -2$

$a_2 = 2$, $b_2 = 3$

$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Let, $x = x_1+h$ and $y = y_1+k$ ....(ii)

or, $dx = dx_1$ or, $dy = dy_1$

$\frac{dy}{dx} = \frac{dy_1}{dx_1}$ ....(iii)

Putting Equ-(ii) and Equ-(iii) value into Equ-(i) we get,

$$\frac{dy_1}{dx_1} = \frac{6(x_1+h)-2(y_1+k)-7}{2(x_1+h)+3(y_1+k)-6}$$

or, $\frac{dy_1}{dx_1} = \frac{6x_1-2y_1+(6h-2k-7)}{2x_1+3y_1+(2h+3k-6)}$ ....(iv)

Equation-(iv) become homogeneous so,

$6h - 2k - 7 = 0$ and $2h + 3k - 6 = 0$

Solving these two equation we get, h = -3/2 and k = -1.

$\frac{dy_1}{dx_1} = \frac{6x_1-2y_1}{2x_1+3y_1}$ ....(v)

Let, $y_1 = vx_1$ ....(vi)

or, $\frac{dy_1}{dx_1} = v + x_1 \frac{dv}{dx_1}$ ....(vii)

Putting Equ-(vi) and Equ-(vii) value into Equ-(v) we get,

$v + x_1 \frac{dv}{dx_1} = \frac{6x_1 - 2vx_1}{2x_1 + 3vx_1}$

or, $\frac{dx_1}{x_1} = \frac{2+3v}{6-4v-3v^2}$ dv

This is a separable equation. Integrating this equation we get,

$\int \frac{dx_1}{x_1} = \int \frac{2+3v}{6-4v-3v^2}$ dv

or, $\log(x_1) = \int \frac{-1/2(-4-6v)}{6-4v-3v^2}$

or, $\log(x_1) = -1/2 \log(6-4v-3v^2) + \log(c)$ $[\int \frac{f^1(x)}{f(x)} dx = \log[f(x)]]$

or, $\log(x_1) + 1/2 \log(6-4v-3v^2) = \log(c)$

or, $x_1 (6-4v-3v^2)^{1/2} = c$

Putting the value of $x_1 = x+3/2$, $y_1 = y+1$ and $v = y_1/x_1$ for getting final answer.

First Order Linear Differential Equations:

A first order linear differential equation is a differential equation of the form $\frac{dy}{dx}$ **+ p(x) y = Q(x).** Here x is independent variable and y is dependent variable.

Format-1:

$\frac{dy}{dx} + p(x)\, y = Q(x)$……………………..(i)

Here p and Q are constant or function of x.

Format-1 Solve:

- Make co-efficient of $\frac{dy}{dx} = 1$
- IF (integrating factor) = $e^{\int p(x)dx}$
- Solution of equation (i) y.IF = $\int Q(x).IF\ dx$

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Format-2:

$\frac{dx}{dy}$ + p(y) x = Q(y)……………………..(i)

Here p and Q are constant or function of y.

Format-2 Solve:

- Make co-efficient of $\frac{dx}{dy}$ = 1
- IF (integrating factor) = $e^{\int p(y)dy}$
- Solution of equation (i) x.IF = $\int Q(y).IF\ dy$

Math-01: Solve the differential equation **(1+x²) $\frac{dy}{dx}$ + 2xy – 4x² = 0.**

Solve:

Given that,

$(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0.$

Or, $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2.$

Or, $\frac{dy}{dx} + (\frac{2x}{1+x^2})y = \frac{4x^2}{1+x^2}$……………………..(i)

Equation-(i) is in the form of linear differential equation.

IF = $e^{\int p(x)dx}$

Or, IF = $e^{\int \frac{2x}{1+x^2}dx}$

Or, IF = $e^{\log(1+x^2)}$

Or, IF = $1+x^2$ …………………………..(ii)

Solution of equation-(i),

y.IF = $\int(\frac{4x^2}{1+x^2})$.IF dx

Or, y.(1+x^2) = $\int(\frac{4x^2}{1+x^2})$.(1+x^2) dx

Or, y.(1+x^2) = $\int 4x^2$ dx

Or, y.(1+x^2) = $\frac{4x^3}{3}$ + c [Answer]

Formula-1: $\int(uv)dx = u\int vdx - \int(\frac{du}{dv}\int vdx)$ dx

Formula-2: $\int xe^x$ dx = e^x(x-1) + c

Math-02: Solve the differential equation **$\cos^2 x \frac{dy}{dx}$ + y = tanx**

Solve:

Ans: **y = tanx – 1 + c e^{-tanx}.**

Math-03: Solve the differential equation **(1+y^2) dx = ($\tan^{-1}$y – x) dy.**

Solve:

Ans: **x = $\tan^{-1}$y – 1 + c $e^{-\tan^{-1}y}$**

Math-04: Solve the differential equation **$\frac{dy}{dx}$ + ($\frac{3}{x}$)y = x**

Solve:

Ans: **y = $\frac{x^3}{5} + \frac{c}{x^3}$**

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Math-05: Solve the differential equation $\frac{dy}{dx}$ **-y = xy⁵.**

Solve:

Hint: Divided by y^5 both side.

Ans: $\mathbf{y^{-4}.e^{4x} = \int(-4x)e^{4x}dx}$

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