Mathematics Assignment (Geometry, Group-A)

(A) Change of axes:

Ok 1. Determine the equation of the curve $2x^2+3y^2-8x+6y-7=0$. When origin is transferred to the point $(2,-1)$.

Solution:

Hold to,

$$x = x' + 2$$
$$y = y' - 1$$

The given equation,

$$2x^2+3y^2-8x+6y-7=0 \ldots\ldots\ldots\ldots\ldots (1)$$

Putting the value of $x = x' + 2$ and $y = y' - 1$ in equation number (1)→

$$2(x'+2)^2+3(y'-1)^2-8(x'+2)+6(y'-1)-7=0$$

$$\Rightarrow 2\left(x'^2+2.x'.2+2^2\right)+3\left(y'^2-2.y'.1+1^2\right)-(8x'+16)+(6y'-6)-7=0$$

$$\Rightarrow 2\left(x'^2+4x'+4\right)+3\left(y'^2-2y'+1\right)-(8x'+16)+(6y'-6)-7=0$$

$$\Rightarrow 2x'^2+8x'+8+3y'^2-6y'+3-8x'-16+6y'-6-7=0$$

$$\Rightarrow 2x'^2+3y'^2-18=0$$

Hence,

Remove suffixes the equation referred to new axes is→

$$2x^2+3y^2-18=0.$$

Ok 2. Determine the equation of the parabola $x^2 - 2xy + y^2 + 2x - 4y + 3 = 0$ after rotating of axes through 45°.

Solution:

Given that, $x^2-2xy+y^2+2x-4y+3=0$ …. …(1)

If we rotated the axes through $\theta = 45°$

$$x = x_1\cos 45° - y_1\sin 45° = \frac{x_1-y_1}{\sqrt{2}}$$

$$\therefore y = x_1\sin 45° + y_1\cos 45° = \frac{x_1+y_1}{\sqrt{2}}$$

Putting these values in (1) we get,

$$\left(\frac{x_1-y_1}{\sqrt{2}}\right)^2-2\left(\frac{x_1-y_1}{\sqrt{2}}\right)\left(\frac{x_1+y_1}{\sqrt{2}}\right)+\left(\frac{x_1+y_1}{\sqrt{2}}\right)^2+2\left(\frac{x_1-y_1}{\sqrt{2}}\right)-4\left(\frac{x_1+y_1}{\sqrt{2}}\right)+3=0$$

$$\Rightarrow \frac{1}{2}(x_1{}^2-2x_1y_1+y_1{}^2)-(x_1{}^2-y_1{}^2)+\frac{1}{2}(x_1{}^2+2x_1y_1+y_1{}^2)+\sqrt{2}(x_1-y_1) \\ -2\sqrt{2}(x_1+y_1)+3=0$$

$$\Rightarrow \frac{1}{2}\left(x_1{}^2-2x_1y_1+y_1{}^2+x_1{}^2+2x_1y_1+y_1{}^2\right)-\left(x_1{}^2-y_1{}^2\right)+\sqrt{2}x_1-\sqrt{2}y_1-2\sqrt{2}x_1 \\ -2\sqrt{2}y_1+3=0$$

$$\Rightarrow \frac{1}{2}\left(2x_1{}^2+2y_1{}^2\right)-\left(x_1{}^2-y_1{}^2\right)-\sqrt{2}x_1-3\sqrt{2}y_1+3=0$$

$$\Rightarrow x_1{}^2+y_1{}^2-x_1{}^2+y_1{}^2-\sqrt{2}x_1-3\sqrt{2}y_1+3=0$$

$$\Rightarrow 2y_1{}^2-\sqrt{2}x_1-3\sqrt{2}y_1+3=0$$

Removing suffixes, we get,

$$2y^2-\sqrt{2}x-3\sqrt{2}y+3=0 \qquad [Proved]$$

Ok 3. If the direction of the axes is turned through an angle 30° and the origin remain unchanged. Transform the axes inclined at 30° to the original axes the equation, $x^2+2\sqrt{3}xy-y^2=2a^2$.

Solution:

Given that, $x^2+2\sqrt{3}xy-y^2=2a^2$-------------(i)

When the axes have been rotated through an angle 30° and origin be unchanged,

So,

$x = x'\cos30° - y'\sin30° = \dfrac{(\sqrt{3}x'-y')}{2}$

And, $y = x'\sin30° + y'\cos30° = \left(\dfrac{x'+\sqrt{3}.y'}{2}\right)$

Putting this value in equation (i), we get,

$$\frac{(\sqrt{3}.x'-y')^2}{4}+2\sqrt{3}\left(\frac{(\sqrt{3}x'-y')}{2}.\frac{(x'+\sqrt{3}y')}{2}\right)-\left(\frac{(x'+\sqrt{3}y')}{2}\right)^2=2a^2$$

$$\Rightarrow \frac{\left(3x'^2 - 2\sqrt{3}x'y' + y'^2\right)}{4} + 2\sqrt{3}\left(\frac{\sqrt{3}.x'^2 + 3x'y' - x'y' - \sqrt{3}.y'^2}{4}\right) - \left(\frac{x'^2 + 2.\sqrt{3}.x'.y' + 3y'^2}{4}\right) = 2a^2$$

$$\Rightarrow 3x'^2 - 2\sqrt{3}x'y' + y'^2 + 6x'^2 + 4\sqrt{3}x'y' - 6y'^2 - x'^2 - 2\sqrt{3}x'y' - 3y'^2 = 8a^2$$

$$\Rightarrow 8x'^2 - 8y'^2 - 8a^2 = 0$$

$$\Rightarrow x'^2 - y'^2 = a^2$$

Removing suffixes, we get

$x^2 - y^2 = a^2$ which is the equation of rectangular hyperbola.

Ok **4. Transform to the parallel axes through the new origin of the equation**

(a) $2x^2+y^2-4x+4y=0$. Origin (1, -2).

Solution:

$2x^2+y^2-4x+4y=0$ -------(i)

Transforming to the parallel axes through the origin (1, -2) we get from the equation (i),

$2(x+1)^2+(y-2)^2-4(x+1)+4(y-2)=0$

$\Rightarrow 2x^2+4x+2+y^2-4y+4-4x-4+4y-8=0$

$\Rightarrow 2x^2+y^2=6$

The Transformed to parallel axes through the origin is (1, -2) is $2x^2+y^2=6$. (Answer)

b) Origin (3, 1), $x^2 + 2y^2 - 6x + 7 = 0$.

Solution:

Transform to the parallel axes through the new origin of the equation $x^2+2y^2-6x+7=0$. Origin(3, -1).

Sol^n: $x^2+2y^2-6x+7=0$-------(i)

Transforming to the parallel axes through the origin (3,1) we get from the equation (i),

$(x+3)^2+2(y+1)^2-6(x+3)+7=0$

$\Rightarrow x^2+6x+9+2y^2+4y+2-6x-18+7=0$

$\Rightarrow x^2+2y^2+4y=0$

The Transformed equation to parallel axes through the origin is (3,1) is $x^2+2y^2+4y=0$.(Answer)

5. Transform the equation $11x^2 + 24xy + 4y^2 - 20x - 40y - 5 = 0$ to the rectangular axes through the point (2, -1) and inclined at an angle $\tan^{-1}(-\frac{4}{3})$.

<u>Solution:</u>

Given that, $11x^2 + 24xy + 4y^2 - 20x - 40y - 5 = 0$ .... (1)

Transformation the equation to rectangular axes through the point $(2,-1)$ $x = x_1 + 2$, $y = y_1 - 1$

Putting these values in (1) we get,

$$11(x_1 + 2)^2 + 24(x_1 + 2)(y_1 - 1) + 4(y_1 - 1)^2 - 20(x_1 + 2) - 40(y_1 - 1) - 5 = 0$$

$$\Rightarrow 11({x_1}^2 + 4x_1 + 4) + 24(x_1y_1 - x_1 + 2y_1 - 2) + 4({y_1}^2 - 2y_1 + 1) - 20(x_1 + 2) - 40(y_1 - 1) - 5 = 0$$

$$\Rightarrow 11{x_1}^2 + 44x_1 + 44 + 24x_1y_1 - 24x_1 + 48y_1 - 48 + 4{y_1}^2 - 8y_1 + 4 - 20x_1 + 40 - 40y_1 - 40 - 5 = 0$$

$$\Rightarrow 11{x_1}^2 + 44x_1 - 44x_1 + 24x_1y_1 + 48y_1 - 48y_1 + 4{y_1}^2 + 88 - 93 = 0$$

$$\Rightarrow 11{x_1}^2 + 24x_1y_1 + 4{y_1}^2 - 5 = 0$$

$$\Rightarrow 11{x_1}^2 + 4{y_1}^2 + 24x_1y_1 - 5 = 0$$

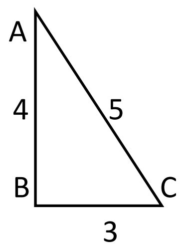
Removing suffixes, we get,

$$11x^2 + 4y^2 + 24xy - 5 = 0 \text{------------------- (2)}$$

If we rotated the axes through an angle $\theta = \tan^{-1}(-\frac{4}{3})$

$\therefore \tan\theta = -\frac{4}{3}$.

Here,



$AC = \sqrt{4^2 + 3^2}$ =5

$\therefore \sin\theta = \frac{-4}{5}$

$\therefore \cos\theta = \frac{3}{5}$

$\therefore x = x_1\cos\theta - y_1\sin\theta = \frac{3x_1}{5} + \frac{4y_1}{5} = \frac{3x_1 + 4y_1}{5}$

$\therefore y = x_1\sin\theta + y_1\cos\theta = \frac{-4x_1}{5} + \frac{3y_1}{5} = \frac{-4x_1 + 3y_1}{5}$

Putting these values in (2) we get,

$$11\left(\frac{3x_1 + 4y_1}{5}\right)^2 + 4\left(\frac{-4x_1 + 3y_1}{5}\right)^2 + 24\left(\frac{3x_1 + 4y_1}{5}\right)\left(\frac{-4x_1 + 3y_1}{5}\right) - 5 = 0$$

$$\Rightarrow 99{x_1}^2 + 264x_1y_1 + 176{y_1}^2 + 36{y_1}^2 - 96x_1y_1 + 64{x_1}^2 - 288{x_1}^2 + 288{y_1}^2 - 168x_1y_1 - 125 = 0$$

$$\Rightarrow -125{x_1}^2 + 500{y_1}^2 - 125 = 0$$

$$\Rightarrow -{x_1}^2 + 4{y_1}^2 - 1 = 0$$

$$\therefore \quad {x_1}^2 - 4{y_1}^2 + 1 = 0$$

Removing suffixes, the required equation is,

$$x^2 - 4y^2 + 1 = 0$$

Ok 06. Transform the axes inclined at 45° to the original axes the equation:

i)$\mathbf{x^2 - y^2 = a^2}$

Solution:

Given that, $x^2 - y^2 = a^2 \ldots\ldots\ldots\ldots(i)$

When the axes have been rotated through on angle $45°$ then,

$x = x'cos45° - y'sin45° = \frac{1}{\sqrt{2}}(x' - y')$

$y = x'sin45° + y'cos45° = \frac{1}{\sqrt{2}}(x' + y')$

Putting them on equation (i), then we get,

$\frac{1}{2}(x' - y')^2 - \frac{1}{2}(x' + y')^2 = a^2$

$\Rightarrow x'^2 - 2x'y' + y'^2 - x'^2 - 2x'y' - y'^2 = 2a^2$

$\Rightarrow -4x'y' = 2a^2$

$\therefore 2x'y' + a^2 = 0$

Removing suffixes, the equation is,
$2xy + a^2 = 0$

$\boldsymbol{ii)x^2 - y^2 - 2\sqrt{2}x - 10\sqrt{2}y + 2 = 0}$
Solution:

Given that, x^2-y^2-2√2x- 10√2y+2=0-----------(i)
When the axes have been rotated through on angle 45° then,

$$x = x'cos45° - y'sin45° = \frac{1}{\sqrt{2}}(x' - y')$$
$$y = x'sin45° + y'cos45° = \frac{1}{\sqrt{2}}(x' + y')$$

Putting them on equation (i), then we get,

$$\frac{(x' - y')^2}{2} - \frac{(x' + y')^2}{2} - 2(x' - y') - 10(x' + y') + 2 = 0$$

⇨ $½\left(x'^2 - 2x'y' + y'^2 - x'^2 - 2x'y' - y'^2\right) - 2x' + 2y' - 10x' - 10y' + 2 = 0$

⇨ $-2x'y' - 12x' - 8y' + 2 = 0$

⇨ $x'y' + 6x' + 4y' = 1\ (ans)$

Removing suffixes, the equation is,
$xy + 6x + 4y = 1.$

Ok 07. Transform the axes inclined at 45° to the original axes the equation,
$\mathbf{x^2 - y^2 - 2\sqrt{2}x - 10\sqrt{2}y + 2 = 0}$
Solution:

Given that, $x^2 - y^2 - 2\sqrt{2}x - 10\sqrt{2}y + 2 = 0$-----------(i)

When the axes have been rotated through on angle 45° then,

$$x = x'cos45° - y'sin45° = \frac{1}{\sqrt{2}}(x' - y')$$
$$y = x'sin45° + y'cos45° = \frac{1}{\sqrt{2}}(x' + y')$$

Putting them on equation (i), then we get,

$$\frac{(x'-y')^2}{2}-\frac{(x'+y')^2}{2}-2(x'-y')-10(x'+y')+2=0$$

$\Rightarrow$ $\frac{1}{2}\left(x'^2-2x'y'+y'^2-x'^2-2x'y'-y'^2\right)-2x'+2y'-10x'-10y'+2=0$

$\Rightarrow$ $-2x'y'-12x'-8y'+2=0$

$\Rightarrow$ $x'y'+6x'+4y'=1$

Removing the suffixes, we get
$xy+6x+4y=1$ (Ans) ...

Ok 08. Transform the axes inclined at 30° to the original axes the equation, $x^2+2\sqrt{3}xy-y^2=2a^2$.

<u>Solution:</u>

Given that, $x^2+2\sqrt{3}xy-y^2=2a^2$-------------(i)
When the axes have been rotated through an angle 30° and origin be unchanged,

$$x=x'\cos 30°-y'\sin 30°=\frac{(\sqrt{3}x'-y')}{2}$$

And,

$$y=x'\sin 30°+y'\cos 30°=\left(\frac{x'+\sqrt{3}.y'}{2}\right)$$

Putting this value in equation (i), we get,

$$\frac{(\sqrt{3}.x'-y')^2}{4}+2\sqrt{3}\left(\frac{(\sqrt{3}x'-y')}{2}.\frac{(x'+\sqrt{3}y')}{2}\right)-\left(\frac{(x'+\sqrt{3}y')}{2}\right)^2=2a^2$$

$\Rightarrow$ $\frac{(3x'^2-2\sqrt{3}.x'y'+y'^2)}{4}2\sqrt{3}\left(\frac{\sqrt{3}.x'^2+3x'y'-x'y'-\sqrt{3}.y'^2}{4}\right)-\left(\frac{x'^2+2.\sqrt{3}.x'.y'+3y'^2}{4}\right)=2a^2$

$\Rightarrow$ $3x'^2-2\sqrt{3}x'y'+y'^2+6x'^2+4\sqrt{3}x'y'-6y'^2-x'^2-2\sqrt{3}x'y'-3y'^2=8a^2$

$\Rightarrow$ $8x'^2-8y'^2-8a^2=0$

$\Rightarrow$ $x'^2-y'^2=a^2$

Removing suffixes, we get $x^2-y^2=a^2$

which is the equation of rectangular hyperbola.

(B) Pair of straight line:

1. Prove that pair of lines joining the origin to the point of intersecting of the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by the line $lx+my+n=0$ are coincident $a^2l^2+b^2m^2=n^2$.

Solution:

We know,

The main pair of straight line is $ax^2+2hxy+by^2=0\ldots\ldots\ldots\ldots(A)$

Given that,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ldots\ldots\ldots\ldots(1)$$

$$lx+my+n=0$$

$$\Rightarrow lx+my=-n$$

$$\Rightarrow \frac{lx+my}{-n}=1$$

$$\Rightarrow \left(\frac{lx+my}{-n}\right)^2=1\ldots\ldots\ldots\ldots(2)$$

Hence, equation… (1) and equation… (2) are equal,

$$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{lx+my}{-n}\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{l^2x^2+2lmxy+m^2y^2}{n^2}$$

$$\Rightarrow \frac{1}{a^2}x^2+\frac{1}{b^2}y^2=\frac{l^2}{n^2}x^2+\frac{2ml}{n^2}xy+\frac{m^2}{n^2}y^2$$

$$\Rightarrow \frac{1}{a^2}x^2+\frac{1}{b^2}y^2-\frac{l^2}{n^2}x^2-\frac{2ml}{n^2}xy-\frac{m^2}{n^2}y^2=0$$

$$\Rightarrow \left(\frac{1}{a^2}-\frac{l^2}{n^2}\right)x^2-\frac{2ml}{n^2}xy+\left(\frac{1}{b^2}-\frac{m^2}{n^2}\right)y^2=0$$

$\Rightarrow ax^2+2hxy+by^2=0\ldots\ldots\ldots\ldots(3)$[Note: This equation can compare with equation (A)]

So, equation (3) represented a pair of straight line. Which is passing through the origin. This line will be coincident.

If,

$$ab=h^2$$

$$\Rightarrow \left(\frac{1}{a^2}-\frac{l^2}{n^2}\right)\left(\frac{1}{b^2}-\frac{m^2}{n^2}\right)=\left(-\frac{ml}{n^2}\right)^2$$

$$\Rightarrow \frac{1}{a^2b^2} - \frac{l^2}{b^2n^2} - \frac{m^2}{a^2n^2} + \frac{m^2l^2}{n^4} = \frac{m^2l^2}{n^4}$$

$$\Rightarrow \frac{1}{a^2b^2} - \frac{l^2}{b^2n^2} - \frac{m^2}{a^2n^2} = 0$$

$$\Rightarrow \frac{n^2 - a^2l^2 - b^2m^2}{a^2b^2n^2} = 0$$

$$\Rightarrow n^2 - a^2l^2 - b^2m^2 = 0$$

$$\Rightarrow -a^2l^2 - b^2m^2 = -n^2$$

$$\Rightarrow -(a^2l^2 + b^2m^2) = -n^2$$

$$\therefore (a^2l^2 + b^2m^2) = n^2$$

[Proved]

2. Show that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel lines if a : h = h : b = g : f . Also show that the distance between them is $2\sqrt{\left\{\frac{(g^2-ac)}{a(a+b)}\right\}}$

Solution:

We know the condition that the general equation represents two straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and it will represent two parallel straight lines if in addition.

$ab - h^2 = 0$

Now,

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\Rightarrow h^2c + 2fgh - af^2 - bg^2 - ch^2 = 0, Since\ ab = h^2$$

$$\Rightarrow bg^2 - 2fgh + af^2 = 0$$

$$\Rightarrow bg^2 - 2fg\sqrt{(ab)} + af^2 = 0 \quad (\therefore h^2 = ab)$$

$$\Rightarrow \left(\sqrt{b}g - \sqrt{a}f\right)^2 = 0$$

$$\Rightarrow \sqrt{b}g = \sqrt{a}f$$

$$\Rightarrow \frac{g}{f} = \sqrt{\frac{a}{b}} = \sqrt{\frac{ab}{b^2}} = \frac{h}{b}$$

$\Rightarrow \frac{g}{f} = \frac{h}{b}$

$\therefore h^2 = ab.$

$\Rightarrow \frac{a}{h} = \frac{h}{b}$

$\therefore \frac{a}{h} = \frac{h}{b} = \frac{g}{f}$ [Which are the required conditions]

Let AB and CD be the lines $lx + my + n = 0, lx + my + n_1 = 0$ respectively. Take any point $p(x_1 y_1)$ on AB. Draw PM perpendicular to CD.

$l^2 = a, m^2 = b, nn_1 = c, l(n + n_1) = 2g.$

$\therefore lx_1 + my_1 + n_1 = 0$.................................(1)

$$\therefore PM = \frac{lx_1 + my_1 + n_1}{\sqrt{(l^2 + m^2)}} = \frac{n_1 - n}{\sqrt{(l^2 + m^2)}} \quad from(1)$$

$$= \frac{\sqrt{\{(n+n_1)^2 - 4nn_1\}}}{\sqrt{(l^2+m^2)}}$$

$$= \frac{\sqrt{\left(\frac{4g^2}{l^2} - 4c\right)}}{\sqrt{(a+b)}}$$

$$= \frac{\sqrt{\left(\frac{4g^2}{a} - 4c\right)}}{\sqrt{(a+b)}}$$

$$= \frac{2\sqrt{\{(g^2 - ac)\}}}{\sqrt{\{a(a+b)\}}}$$

$$= 2\sqrt{\left(\frac{g^2 - ac}{a(a+b)}\right)} \text{...[}\mathit{Showed}\text{]}$$

3. Prove that the straight lines represents by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be equidistance from the origin if $f^4 - g^4 = c(bf^2 - ag^2)$.

Solve: Let the lines represented by the given equation be lx+my+n=0 and l'x+m'y+n'=0 since they are equidistant from the origin

$$\frac{n}{\sqrt{(l^2+m^2)}} = \frac{n'}{\sqrt{(l'^2+m'^2)}}$$

or, $n^2\left(l'^2 + m'^2\right) = n'^2\left(l^2 + m^2\right)$

or, $n^2 l'^2 - n'^2 l^2 = ,n'^2 m^2 - n^2 m'^2$

or, (nl′ + n′l) (nl′ -n′l) = (nm′+ n′m) (n′m - nm′)

or, $(\mathrm{nl}' + \mathrm{n}'\mathrm{l})^2\{(\mathrm{nl}' + \mathrm{n}'\mathrm{l})^2 - 4\mathrm{nn}'\mathrm{ll}'\} = (\mathrm{nm}' + \mathrm{n}'\mathrm{m})^2\{(\mathrm{nm}' + \mathrm{n}'\mathrm{m})^2 - 4nn'mm'\}$----------(1)

let, ax^2 + 2hxy + by^2 + 2gx + 2fy+ c =(lx+ my+ n) (l'x+ m'y+ n') compairing the co-efficient we have

$ll' = a,$
$mm' = b,$
$nn' = c,$
$m'n + mn' = 2f,$
$n'l + nl' = 2g,$
$l'm + lm' = 2h$

putting the values in (1) we have

$g^2(g^2$- ac) = f^2 (f^2- bc)
or, f^4-g^4=c(bf^2 - ag^2) [Proved]

(C) Circle:

Radical Axis:

In geometry, the radical axis of two non-concentric circles is a line defined from the two circles, perpendicular to the line connecting the centers of the circles. If the circles cross, their radical axis is the line through their two crossing points, and if they are tangent, it is their line of tangency.

Coaxial circles:

A system of circles is coaxial if every pair of circles from the system have the same radical axis. Therefore, a coaxial system of circles is defined by the radical axis and any one of the circles.

Chord of contact:

The chord joining the points of contact of the two tangents to a conic drawn from a given point, outside it, is called the chord of contact of tangents. ... Then equation of PQ is known as equation of chord of contact.

Conjugate Diameter:

In geometry, two diameters of a conic section are said to be conjugate if each chord parallel to one diameter is bisected by the other diameter. For example, two diameters of a circle are conjugate if and only if they are perpendicular.

Orthogonal circles:

Orthogonal circles are orthogonal curves, i.e., they cut one another at right angles. By the Pythagorean theorem, two circles of radii and whose centers are a distance apart are orthogonal if. (1) Two circles with Cartesian equations.

OK 1. Find the equation to the circle which cuts the circles, $x^2 + y^2 + 2x + 4y + 6 = 0$ and $x^2 + y^2 + 4x + 6y + 2 = 0$ orthogonally and passes through the point (1,1).

Solution:

Let the circle equation,

$X_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c_1 = 0$--------(i)

Given that,

$x^2 + y^2 + 2x + 4y + 6 = 0$-------(ii)

$x^2 + y^2 + 4x + 6y + 2 = 0$------(iii)

circle (ii) and (iii) are orthogonally cut the circle (i)

$\therefore (-1, -2) and\ (-g, -f)$ cut orthogonally;

$2.(-1).(-g) + 2.(-2).(-f) = c_1 + 6$

$=2g + 4f - c_1 - 6 = 0$-----------(iv)

Again, (-2, -3) and (-g, -f) are orthogonally cut the circle (i)

$2.(-2).(-g) + 2.(-3).(-f) = c_1 + 2$

$= 4g + 6f - c_1 - 2 = 0$-----------(v)

Circle (i) passes through (1,1) point

$1^2 + 1^2 + 2g.1 + 2f.1 + c_1 = 0$

$=2g+2f+c_1+2= 0$---------(vi)

(iv) + (vi) – (v), we get;

$-4 + c_1 + 2 = 0$

$c_1 = 2$

putting this value in circle (vi) & (iv), and subtract this circle we get,

$2f-2c_1-8=0$

$= f- c_1=4$

$f=6$

In circle (vi), putting the value of f, we get

$2g + 12 + 2+2 = 0$

$g = -8$

So, the required circle equation is

$X^2 + y^2 - 16x + 12y + 2 = 0$ (Answer)

OK **2. Find the value of k for which the circles $\mathbf{x^2 + y^2 + 5x + 3y + 7 = 0}$ and $\mathbf{x^2 + y^2 - 8x + 6y + k = 0}$ are orthogonal.**

Solution:

Given that,

$x^2 + y^2 + 5x + 3y + 7 = 0$----------(1)

$x^2 + y^2 - 8x + 6y + k = 0$-------------(2)

Now Center of (1) is $\left(-\frac{5}{2}, -\frac{3}{2}\right)$ $and\ c_1 = 7$

Center of (2) is $(4, -3)$ $and\ c_2 = k.$

Since Circle (1) & (2) are orthogonal then we get,

$2\left(-\frac{5}{2}\right)(4) + 2\left(-\frac{3}{2}\right)(-3) - 7 - k = 0.$

⇨ $-20 + 9 - 7 - k = 0.$

⇨ k=-18. [Ans]

OK **3. Find the equation of the circle through the intersection of the circle $x^2 + y^2 + 2x + 3y + 4 = 0$ and the line $2x + 3y + 4 = 0$ and which passes through the point (2,3).**

Solution:

Equation of given circle : $x^2 + y^2 + 2x + 3y + 4 = 0$…………………(i)

Equation of given line : $2x + 3y + 4 = 0$……………………………..…(ii)

The equation of the circle through the intersection of the circle $x^2 + y^2 + 2x + 3y + 4 = 0$ and the line $2x + 3y + 4 = 0$ is

$x^2 + y^2 + 2x + 3y + 4 + k(2x + 3y + 4) = 0$

Since, it passes through (2,3) ,

$2^2 + 3^2 + 2*2 + 3*3 + 4 + k(2*2 + 3*3 + 4) = 0$

⇨ $30 + 17k = 0$

⇨ $K = -\frac{30}{17}$

Therefore, the equation is

$x^2 + y^2 + 2x + 3y + 4 - \frac{30}{17}(2x + 3y + 4) = 0$

⇨ $17x^2 + 17y^2 + 34x + 51y + 68 - 60x - 90y - 120 = 0$

⇨ $17x^2 + 17y^2 - 26x - 39y - 52 = 0$

It is the required equation.

4. Find the equation of the circle passing through the points of intersection of the line 2x + 3y + 4 = 0 and the circle $x^2 + y^2 + 2x + 3y + 4 = 0$ and cutting the circle $x^2 + y^2 + 5x + 6y + 7 = 0$ orthogonally.

Solution:

$S_1 \equiv x^2 + y^2 + 2x + 3y + 4 = 0.$

$S_2 \equiv x^2 + y^2 + 5x + 6y + 7 = 0.$

Let,

$x^2+y^2+2gx+2fy+c=0$ ....(i)

Now, intersect a circle and the line is,

$S_1 + K (2x + 3y + 4) = 0.$

⇨ $x^2 + y^2 + 2x + 3y + 4 + K (2x + 3y + 4) = 0.$

⇨ $x^2 + y^2 + 2x + 3y + 4 + 2Kx + 3Ky + 4K = 0.$

⇨ $x^2 + y^2 + 2(1+K)x + (3+3K)y + 4 + 4K = 0$ ....(ii)

Here, circle (ii) cut an another circle orthogonally. So,

$2 g_1 g_2 + 2 f_1 f_2 = c_1 + c_2$.

⇨ $2 (1+K)\frac{5}{2} + 2\frac{(3+3K)}{2}3 = 4 + 4K + 7.$

⇨ $5 + 5K + 9 + 9K = 11 + 4K.$

⇨ $10K = - 3.$

⇨ $K = -\frac{3}{10}.$

Putting the value of K in circle (ii),

$x^2 + y^2 + 2(1 - \frac{3}{10})x + (3 - \frac{9}{10})y + 4 + 4 * (-\frac{3}{10}) = 0.$

⇨ $x^2 + y^2 + \frac{14}{10}x + \frac{21}{10}y + 4 - \frac{12}{10} = 0$

⇨ $10x^2 + 10y^2 + 14x + 21y + 28 = 0.$ (Answer)...

OK **5. Find the length of the common chord of the circles.**

i. $x^2 + y^2 – 12x + 16y – 69 = 0$

ii. $x^2 + y^2 – 9x + 12y – 59 = 0.$

Solution:

Let,

$x^2 + y^2 – 12x + 16y – 69 = 0$--------(i)

$x^2 + y^2 – 9x + 12y – 59 = 0$----------(ii)

The common chord equation of this two circle is (i) – (ii);

$-3x + 4y – 10 = 0$

=> $3x - 4y + 10 = 0$-------(iii)
Equation (iii) is the common cord equation.

Now, circle (i) center (6, -8)
Circle (i) center to common chord distance,

$$\frac{|(3*6)-4(-8)+10|}{\sqrt{9+16}}$$

$$= \frac{18+32+10}{5}$$

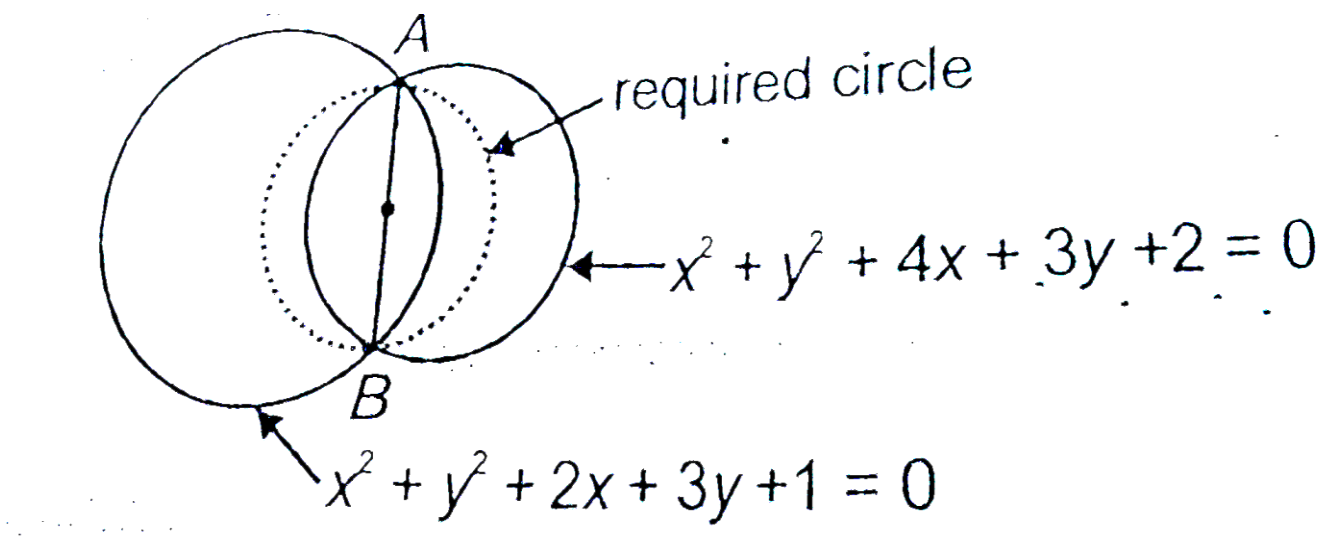
$$= \frac{60}{5} = 12$$

And circle (i) radius length,
$=\sqrt{36+64+69}$
$=\sqrt{169}$
$= 13$

∴The length of the common chord is $= 2\sqrt{13^2 - 12^2}$
$= 2\sqrt{25}$
$= 10.$

The length of the common chord is 10.

OK 6. Find the equation of the circle whose diameter is a common cord of the circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$.
Solution:



$S_1 \equiv x^2+y^2+2x+3y+1=0$ ....(1)
$S_2 \equiv x^2+y^2+4x+3y+2=0$ ....(2)

Passes through their points of intersection and has its centre on the radical axis of (1) and (2). The radical axis of (1) and (2) is,

$S_1 - S_2 = 0.$
Or, $x^2+y^2+2x+3y+1-(x^2+y^2+4x+3y+2)=0.$
Or, $2x + 1 = 0.$ ....(3)

Any equation of the circle which cut the common chord (3) of (1) and (2) circles and passing through the circle (1) is

$x^2+y^2+2x+3y+1+K(2x+1)=0.$
Or, $x^2+y^2+2x+3y+1+2Kx+K=0.$
Or, $x^2+y^2+2x(1+K)+3y+K+1=0.$ ....(4)

So, centre = $\{-(K+1), -\frac{3}{2}\}$.

If the equation (3) be the diameter of the circle (4) ,
the centre $\{-(K+1), -\frac{3}{2}\}$ lie on the equation (3).

$2. \{-(K+1)\} + 1 = 0.$
Or, $-2K - 2 + 1 = 0.$
Or, $-2K – 1 = 0.$
Or, $K = -\frac{1}{2}$.
Putting the value of K in equation (4),
$x^2+y^2+2x(1-\frac{1}{2})+3y - \frac{1}{2} +1=0.$
Or, $x^2 + y^2 + 2x – x + 3y - \frac{1}{2} + 1 =0.$
Or,$2x^2 + 2y^2 + 2x + 6y + 1 = 0.$

$\therefore 2x^2 + 2y^2 + 2x + 6y + 1 = 0.$
This is the required equation of circle.

7. Prove that the two circles will pass through the two points (0, a) and (0, -a) and touch the striate line y = mx + c will cut orthogonally if $c^2 = (2 + m^2)x^2$.

Solution:
Let the equation of the circle be
$\boldsymbol{x^2}+\boldsymbol{y^2}+2gx+2fy+\lambda=0.$

Since it passes through (0, a) and (0, −a).

$\therefore a^2$+2fa+λ=0
a^2−2fa+λ=0.

Subtracting, we get 4fa=0, ∴f=0; and λ=−a^2.
Hence the circle becomes

x^2+y^2+2gx−a^2=0.
Centre is (−g,0)
and radius =√(g^2−λ)=√(g^2+a^2)

It is given that the circle touches the line
y=mx+c or mx−y+c=0.

Hence perpendicular from centre should be equal to radius

$\frac{-mg+c}{\sqrt{(m^2+1)}}$=√($g^2$+$a^2$)

or, $(c-mg)^2$=(g^2+a^2)(m^2+1)

or, g^2+2gmc+{a^2 (1+m^2)− c^2}=0.

Let g and g′ be two the roots of this quadratic equation. Hence
g + g′ = -2cm
gg′ = a^2 (a + m^2) - c^2
Now, the circles x^2+y^2+2gx- a^2= 0 and x^2+y^2+2g′x-a^2= 0 cut orthogonally implies that
2gg =c1+c2
2{a^2 (1+m^2)− c^2}+0=-2a^2
or a^2 (1+m^2)− c^2=−a^2
or a^2 (2+m^2)= c^2
is the required condition.

8. If two circles cut a third circle orthogonally the radical axis of the two circles passes through the center of the third circles.

Solution:
Let the equation of two circle are,

$X^2 + y^2 +2g_1x + 2f_1y + c_1 = 0$ ----------(i)

$X^2 + y^2 +2g_2x + 2f_2y + c_2 = 0$ ----------(ii)

And the equation of the third circle $X^2 + y^2 +2g_3x + 2f_3y + c_3 = 0$ ----------(iii)

The equation of the radical axis of first two circle,

$2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$ --------- (iv)

Since, it passes through the center of the third circle ($-g_3$, $-f_3$), then,

$$-2(g_1 - g_2)g_3 - 2(f_1 - f_2)f_3 + (c_1 - c_2) = 0$$

$$\Longrightarrow 2(g_1 - g_2)g_3 + 2(f_1 - f_2)f_3 - (c_1 - c_2) = 0 \text{----------(v)}$$

Circle (i), (iii) and (ii), (iii) cut orthogonally

$$2g_1g_3 + 2f_1f_3 = c_1 + c_3 \text{ ------------ (vi)}$$

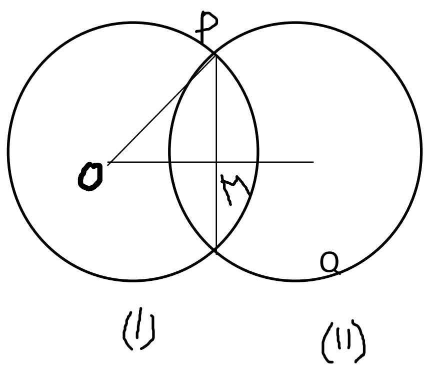
$$2g_2g_3 + 2f_2f_3 = c_2 + c_3 \text{ ------------ (vii)}$$

Subtracting equation (vi) & (vii),

$$2(g_1 - g_2)g_3 + 2(f_1 - f_2)f_3 - (c_1 - c_2) = 0 \text{----------(viii)}$$

Equation (v) and (viii) are same, so we can say that the radical axis of the two circle passes through the center of the third circle if they are cur orthogonally.

OK 9. **Prove that the length of the common cord of the two circles, $x^2 + y^2 + 2px + c$** = 0 and $x^2 + y^2 + 2qy - c = 0$ is, $2\sqrt{\frac{(p2-c)(q2+c)}{p2+q2}}$



Solution:

$$x^2 + y^2 + 2px + c = 0 \text{--------(i)}$$

$$x^2 + y^2 + 2qy - c = 0 \text{---------(ii)}$$

center(-p,0), (0,-q)

radius $\sqrt{p^2 - c}$, $\sqrt{q^2 + c}$

The equation of the common chord of the circle (i) and (ii) is,

$$x^2 + y^2 + 2px + c - (x^2 + y^2 + 2qy - c) = 0$$

$$\Rightarrow 2px - 2qy + 2c = 0$$

$$\Rightarrow px - qy + c = 0 \text{ ------------(iii)}$$

Now we get the perprndicular distance from (-p, 0) is,

$$OM = \frac{-p^2+c}{\sqrt{p^2+q^2}} = \frac{c-p^2}{\sqrt{p^2+q^2}}.$$

And,

$$OP^2 = OM^2 + PM^2$$

$$PM^2 = p^2 - c - \left(\frac{c-p^2}{\sqrt{p^2+q^2}}\right)^2$$

$$= (p^2 - c) - \frac{(c-p^2)^2}{p^2+q^2}$$

$$= \frac{(p^2 - c)(p^2 + q^2) - (p^2 - c)^2}{p^2 + q^2}$$
$$= \frac{(p^2 - c)(q^2 + c)}{p^2 + q^2}$$

$$PM = \sqrt{\frac{(p^2 - c)(q^2 + c)}{p^2 + q^2}}$$

From the figure
PQ = 2PM

$$= 2. \sqrt{\frac{(p^2 - c)(q^2 + c)}{p^2 + q^2}}$$

So, the length of the common cord of the two circles, $x^2 + y^2 + 2px + c = 0$ and $x^2 + y^2 + 2qy - c = 0$ is $2\sqrt{\frac{(p2-c)(q2+c)}{p2+q2}}$
. (Proved)

Ok 10. Which required condition that the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^{2+} 2g'x + 2f'y + c' = 0$ should touch if $(2gg' + 2ff' -c -c')^2 = 4(g^2 + f^2 - c)(g'^2 + f'^2 - c')$.

Solution:
Given equation,
$x^2 + y^2 + 2gx + 2fy + c = 0$--------(1)
$x^2 + y^{2+} 2g'x + 2f'y + c' = 0$-------(ii)
center (-g, -f), (-g', -f')
Radius $\sqrt{g^2 + f^2 - c}$, $\sqrt{g'^2 + f'^2 - c'}$

For the given condition if two circles touches the distance between the center of two circle is equal to the sum of the radius,

$$\sqrt{(-g + g')^2 + (-f + f')^2} = \sqrt{g^2 + f^2 - c} + \sqrt{g'^2 + f'^2 - c'}$$

$$\Rightarrow (-g + g)^2 + (-f + f)^2 = g^2 + f^2 - c + g^2 + f^2 - c' + 2\sqrt{(g^2 + f^2 - c)(g'^2 + f'^2 - c')}$$

$$\Rightarrow -2gg' - 2ff' = -c - c' + + 2\sqrt{(g^2 + f^2 - c)(g'^2 + f'^2 - c')}$$

$$\Rightarrow (-2gg' + 2ff' = -c - c')^2 = 4((g^2 + f^2 - c))\,({g'}^2 + {f'}^2 - c')$$

(Proved)

(D) CONIC:

1. $\boldsymbol{16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0}$**. Identify this equation (Conic or Circle).**

Solution:

$$16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0$$
$$=> 16x^2 + 2(-12)xy + 9y^2 + 2(-52)x + 2(-86)y + 44 = 0$$

Comparing the equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
we get,

$$a = 16, \quad h = -12, \quad b = 9, \quad g = -52, \quad f = -86, \quad c = 44$$

We know,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= (16 * 9 * 44) + 2(-86)(-52)(-12) - 16(-86)^2 - 9(-52)^2 - 44(-12)^2$$
$$= 6336 - 107328 - 118336 - 24336 - 6336$$
$$= -250000$$
$$\therefore \Delta \neq 0$$

Now, $ab - h^2 = 16 * 9 - (-12)^2$
$$= 144 - 144 = 0$$

Since $\Delta \neq 0$ and $ab - h^2 = 0$, so the given equation represents a parabola,

$$16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0 \ldots\ldots\ldots\ldots (1)$$

$$=> (4x)^2 - 2 * 4x * 3y + (3y)^2 = 104x + 172y - 44$$
$$=> (4x - 3y)^2 = 104x + 172y - 44$$
$$=> (4x - 3y + a)^2 = a^2 + 2(4x - 3y)a + 104x + 172y - 44$$
$$=> (4x - 3y + a)^2 = a^2 + 8ax - 6ay + 104x + 172y - 44$$
$$=> (4x - 3y + a)^2 = (8a + 104)x + (172 - 6a)y + (a^2 - 44) \ldots\ldots\ldots (2)$$

Where a is a scalar.
Choose a in such a way that the lines:
$4x - 3y + a = 0$ and $(8a + 104)x + (172 - 6a)y + (a^2 - 44) = 0$ are perpendicular so,

$$4(8a + 104) + (-3)(172 - 6a) = 0$$
$$=> 32a + 416 - 516 + 18a = 0$$
$$=> 50a - 100 = 0$$
$$\therefore a = 2$$

Putting the value of a in equation (2),

$$(4x - 3y + 2)^2 = (16 + 104)x + (172 - 12)y + (4 - 44)$$

$$=> (4x - 3y + 2)^2 = 120x + 160y - 40$$

$=> (4x - 3y + 2)^2 = 4(30x + 40y - 10) \ldots\ldots\ldots (3)$

Take,

$Y = \frac{4x-3y+2}{\sqrt{4^2+3^2}} = \frac{4x-3y+2}{5}$

$=> 5Y = 4x - 3y + 2$

And $X = \frac{30x+40y-10}{50}$

$=> 50X = 30x + 40y - 10$

Putting the values in equation(3),

$(5Y)^2 = 4.50X$

$Y^2 = 4.2X$

$Y^2 = 4AX \;\; [A = 2]$

Which is the standard form of parabola.

Properties:

(i) Vertex (0,0)

$X = 0$, $Y = 0$

$30x + 40y - 10 = 0$, $=> 4x - 3y + 2 = 0$

$=> 3x + 4y - 1 = 0$

Solving the equations: $x = \frac{-1}{5}, \;\; y = \frac{2}{5}$

(ii) Focus $(A, 0)$

$X = A$, $Y = 0$

$30x + 40y - 10 = 2$, $4x - 3y + 2 = 0$

$=> 30x + 40y = 12$

Solving the equations: $x = \frac{-22}{125}, \quad y = \frac{54}{125}$

(iii)Directrix $X + A = 0$

$=> 30x + 40y - 10 + 2 = 0$

$=> 30x + 40y - 8 = 0$

$=> 15x + 20y - 4 = 0$

(iv)Length of the latus rectum $= 4a = 4 * 2 = 8$

(v)Equation of the latus rectum $=> X - A = 0$

$=> 30x + 40y - 10 - 2 = 0$

$\therefore 15x + 20y - 6 = 0$

2. $\boldsymbol{25x^2 + 2xy + 25y^2 - 130x - 130y + 169 = 0}$ **Identify the equation and show the properties.**

<u>Solution:</u>

$25x^2 + 2xy + 25y^2 - 130x - 130y + 169 = 0 \ldots\ldots\ldots (1)$

$$\Rightarrow 25x^2 + 2(1)xy + 25y^2 + 2(-65)x + 2(-65)y + 169 = 0$$

Comparing the equation with the general equation of second degree,
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$,
We get, $a = 25$, $h = 1$, $b = 25$, $g = -65$, $f = -65$, $c = 169$

We know,
$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$
$= (25 * 25 * 169) + 2(-65)(-65)(1) - 25(-65)^2 - 25(-65)^2 - 169(1)^2$
$= 105625 + 8450 - 105625 - 105625 - 169$
$= -97344$
$\therefore \Delta \neq 0$
Now, $ab - h^2 = (25 * 25) - (1)^2$
$= 624$
Since $\Delta \neq 0$ and $ab - h^2 > 0$, so the given equation represents an ellipse.

$$25x^2 + 2xy + 25y^2 + \frac{\Delta}{ab-h^2} = 0$$
$$\Rightarrow 25x^2 + 2xy + 25y^2 + \frac{-97344}{624} = 0$$
$$\Rightarrow 25x^2 + 2xy + 25y^2 - 156 = 0$$
$$\Rightarrow 25x^2 + 2xy + 25y^2 = 156 \ldots\ldots\ldots (2)$$

Rotating the axis such that xy term removed then,

$$a'x^2 + b'y^2 = 156 \ldots\ldots\ldots (3)$$

Where $a' + b' = 50$

And $a'b' - h'^2 = ab - h^2 = 624$

$\Rightarrow a'b' = 624 + h'^2$ $[h' = 0]$

$\therefore a'b' = 624$

$$a' - b' = \sqrt{(a' + b')^2 - 4a'b'}$$
$$= \sqrt{(50)^2 - 4 * 624}$$
$$= \sqrt{2500 - 2496}$$
$$= \sqrt{4}$$
$$= 2$$

$\therefore a' - b' = 2$
$a' + b' = 50$

Solving these two equations , $a' = 26$, $b' = 24$
Putting the values in equation (3) we get ,

$$26x^2 + 24y^2 = 156$$
$$\Rightarrow 13x^2 + 12y^2 = 78$$
$$\Rightarrow \frac{13}{78}x^2 + \frac{12}{78}y^2 = 1$$
$$\Rightarrow \frac{x^2}{6} + \frac{y^2}{\frac{13}{2}} = 1$$

Which is the standard form of ellipse.
Properties:

$a^2 = 6 \quad and \quad b^2 = \dfrac{13}{2}$

$\therefore a = \sqrt{6} \qquad \therefore b = \sqrt{\frac{13}{2}}$

Properties:

(1) Ecentricity

$e^2 = 1 - \frac{a^2}{b^2} \qquad \Rightarrow e^2 = 1 - \frac{6}{\frac{13}{2}}$

$\Rightarrow e^2 = 1 - \frac{12}{13}$

$\Rightarrow e^2 = \frac{1}{13}$

$\therefore e = \frac{1}{\sqrt{13}}$

(2) Center (0,0)

$x = 0 \quad , \quad y = 0$

(3) Focus $(\pm ae, 0)$

$x = \pm\sqrt{6} * \frac{1}{\sqrt{13}} \quad , \quad y = 0$

$\therefore \sqrt{13}x = \pm\sqrt{6}$

(4) Vertexes $(\pm a, 0)$

$x = \pm\sqrt{6} \quad , \quad y = 0$

(5) Length of major axis $= 2b$

$= 2 * \sqrt{\frac{13}{2}}$

$= \sqrt{26}$

(6) Length of minor axis $= 2a$

$= 2\sqrt{6}$

(7) Length of latus rectum $= \frac{2b^2}{a}$

$= \frac{2\frac{13}{2}}{\sqrt{6}}$

$= \frac{13}{\sqrt{6}}$

(8) Equation of directrix :

$x = \pm\frac{a}{e}$

$\Rightarrow x = \pm\frac{\sqrt{6}}{\frac{1}{\sqrt{13}}}$

$\Rightarrow x = \pm\sqrt{6 * 13}$

$$\therefore x \pm \sqrt{78} = 0$$

3. $32x^2+52xy-7y^2-64x-52y-145=0$**. Identify the equation and show the properties.**

Solution:

We know that the general equation of the second degree is,
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ .... (1)

Hence,

$32x^2+52xy-7y^2-64x-52y-145 = 0$ .... (2)

⇨ $32x^2+2(26)xy-7y^2+2(-32)x+2(-26)y-145=0$.
Now it Compare the equation (1),

$a = 32,\quad h = 26,\quad b = -7,\quad g = -32,\quad f = -26,\quad c = -145$
We know that the condition of second degree is,

$\Delta = abc+2fgh-af^2-bg^2-ch^2$.

$$= (32)(-7)(-145) + 2(-26)(-32)(26) - (32.-26)^2 - (-7.-32)^2 - (-145)(26)2.$$
$$= 159300.$$

Here, $\Delta \neq 0$
Now, $ab-h^2 = (32)(-7) - (26)^2 = -900$.

Since, $ab-h^2 < 0$ and $\Delta \neq 0$ so that given equation represent a Hyperbola.

$$32x^2 + 52xy - 7y^2 + \frac{\Delta}{ab-h^2} = 0$$

⇨ $32x^2 + 52xy - 7y^2 + \frac{159300}{-900} = 0$

⇨ $32x^2+52xy-7y^2 = 177$.

Rotate the axis such that, xy, term has been removed than,

$a'x^2+b'y^2 = 177$........................... (3).

Where, $a'+ b' = 32+ (-7) =25$. $a'b'-h'^2 = ab-h^2 = -900$.

⇨ $a'b' = -900+h'^2$ $[h' = 0]$

⇨ $a'b' = -900$.

$$a' - b' = \sqrt{((a' + b')^2 - 4a'b')}$$

$$\Rightarrow a' - b' = \sqrt{(25^2) - 4.(-900)}$$

$$= 65$$

Hence,

$a' - b' = 65.$

$a' + b' = 25.$ So,

$a' = 45,$

$b' = -20.$

Putting the value of a' and b' in the equation (3).

$45x^2$-$20y^2$ =177.

$$\frac{45}{177}x^2 - \frac{20}{177}x^2 = 1$$

$$\Rightarrow \frac{x^2}{\frac{177}{45}} + \frac{y^2}{\frac{177}{20}} = 1$$

$a^2 = \frac{177}{45}$ and $b^2 = \frac{177}{20}$

1. Vertex (0, 0)

X = 0 Y = 0

Vertex (0, 0).

Length of the *major* axis = 2a = $2\sqrt{\frac{177}{45}}$.

Length of the *minor* axis = 2b = $2\sqrt{\frac{177}{20}}$.

4. Find the condition that, $y = mx+c$ will touch the parabola $y^2=4ax$ find the point of contact.

Solution:

Given that,

$y = mx + c$................................ (1)

$y^2 = 4ax$..............................(2)

The line (1) touch parabola (2) then,

$(mx + c)^2 = 4ax$

$\Rightarrow m^2x^2 + 2.mx.c + c^2 - 4ax = 0$

$\Rightarrow m^2x^2 + 2x(mc - 2a) + c^2 = 0$-------------------(3)

The line (1) will be a tangent of (2) if the equation are equal.

$$(2(mc - 2a))^2 - 4m^2c^2 = 0$$

$$c = \frac{a}{m}$$

Putting the value of $c = \frac{a}{m}$ in equation (1) and (3).

$y = mx + \frac{a}{m}$ .... (4)

$$\Rightarrow m^2x^2 + 2x\left(m\frac{a}{m} - 2a\right) + \left(\frac{a}{m}\right)^2 = 0$$

$$\Rightarrow x = \frac{a}{m^2}$$

Form (4) we get,

$$y = \frac{2a}{m}$$

The point is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

5. Show that $4x - 3y = 9$ touches the hyperbola $4x^2 - 9y^2 = 27$.

Solution:

We know that,

$y = mx + c$ should touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $[if\ c^2 = m^2a^2 - b^2]$

Given the line is,

$4x - 3y = 9$

⇨ $3y = 4x - 9$

⇨ $y = \frac{(4x-9)}{3}$

⇨ $y = \frac{4}{3}.x + (-3)$, So, $m = \frac{4}{3}$ and $c = -3$

Now,

⇨ $4x^2 - 9y^2 = 27$

⇨ $\frac{x^2}{\frac{27}{4}} - \frac{y^2}{3} = 1$

here, $a^2 = \frac{27}{4}$ and $b^2 = 3$

So we can write,

$$(-3)^2 = \left(\frac{4}{3}\right)^2.\frac{27}{4} - 3$$

$$⇨ 9 = \frac{16}{9}.\frac{27}{4} - 3$$

$$⇨ 9 = 4.3 - 3$$

$$⇨ 9 = 12 - 3$$

$$⇨ 9 = 9_{(Proved)}$$

6. If the poles of (x_1, y_1), (x_2, y_2) with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are right angles. Then show that → $\frac{x_1x_2}{y_1y_2}$ + $\frac{a^4}{b^4}$ = 0.

Solution:

Given, hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Polar equation of the hyperbola at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$------------------------(1)

and

Polar equation of the hyperbola at (x_2, y_2) is $\frac{xx_2}{a^2} - \frac{yy_2}{b^2} = 1$------------------------(2)

From equation (1).

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

$$\Rightarrow \frac{yy_1}{b^2} = \frac{xx_1}{a^2} - 1$$

$$\Rightarrow yy_1 = \frac{b^2}{a^2}.xx_1 - b^2$$

$$\Rightarrow \mathrm{y} = \left(\frac{b^2}{a^2}\frac{x_1}{y_1}\right).x - \frac{b^2}{y_1}$$

The slop of equation (1) is, $m_1 = \frac{b^2x_1}{a^2y_1}$

Similarly, the slop of equation (2) is, $m_2 = \frac{b^2x_2}{a^2y_2}$

Now, we can write, $\quad m_1m_2 = -1$

$$\Rightarrow \frac{b^2x_1}{a^2y_1}.\frac{b^2x_2}{a^2y_2} = -1$$

$$\Rightarrow b^4x_1x_2 = -a^4y_1y_2$$

$$\Rightarrow \frac{x_1x_2}{y_1y_2} = -\frac{a^4}{b^4}$$

$$\Rightarrow \frac{x_1x_2}{y_1y_2} + \frac{a^4}{b^4} = 0 \qquad [Proved.]$$

7. Find the locus of the middle points if the chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which subtend a right angle at the center is→

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \left(\frac{a^2+b^2}{a^2b^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2$$

Solution:

Let ($x_{1,}$ y_1) be the middle point of the chord which middle point = k. [Say]

$$(x_1, y_1) \ is \ \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}$$

Now,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=k \dots\dots\dots.(1)$$

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=k \dots\dots\dots\dots\dots(2)$$

$$\Rightarrow \frac{xx_1}{ka^2}+\frac{yy_1}{kb^2}=1$$

Given,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\text{-------}(3)$$

Marking homogeneous equation (3) with the help of equation (2) we get,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{xx_1}{ka^2}+\frac{yy_1}{kb^2}\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x^2{x_1}^2}{a^4k^2}+\frac{y^2{y_1}^2}{b^4k^2}+2.\left(\frac{xx_1}{a^2b^2}.\frac{yy_1}{k^2}\right)$$

According to the condition, *the coefficient of x^2+ Coefficient of y^2+.*

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{{x_1}^2}{a^4k^2}+\frac{{y_1}^2}{b^4k^2}$$

$$\Rightarrow \left(\frac{b^2+a^2}{a^2b^2}\right)=\frac{1}{k^2}\left(\frac{{x_1}^2}{a^4}+\frac{{y_1}^2}{b^4}\right)$$

Putting the value of K from equation (1).

$$\Rightarrow \left(\frac{{x_1}^2}{a^2}+\frac{{y_1}^2}{b^2}\right)^2\left(\frac{b^2+a^2}{a^2b^2}\right)=\left(\frac{{x_1}^2}{a^4}+\frac{{y_1}^2}{b^4}\right)$$

Hence, the locus of (x, y) is,

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\left(\frac{a^2+b^2}{a^2b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$$

$$[Proved]$$

8. The locus of polar of normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the curve

$$\boldsymbol{\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2}$$

<u>Solution:</u>

At (x_1, y_1) point, the polar equation of the ellipse,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}= \quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

(1) Equation of normal at θ is,

$$\frac{ax}{cos\theta}-\frac{by}{sin\theta}=a^2-b^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

Since, equation (1) and (2) are identical.

$$\frac{\frac{x_1}{a^2}}{\frac{a}{cos\theta}}=\frac{\frac{y_1}{b^2}}{\frac{-b}{sin\theta}}=\frac{1}{a^2-b^2}$$

$$\Rightarrow \frac{x_1}{a^2}.\frac{cos\theta}{a}=\frac{y_1 sin\theta}{b^2(-b)}=\frac{1}{a^2-b^2}$$

$$So, \qquad Cos\theta=\frac{a^3}{y_1(a^2-b^2)} \quad And \quad Sin\theta=\frac{-b^3}{y_1(a^2-b^2)}$$

We know that,

$$sin^2\theta+cos^2\theta=1$$

$$So, \left\{\frac{-b^3}{y_1(a^2-b^2)}\right\}^2 + \left\{\frac{a^3}{y_1(a^2-b^2)}\right\}^2 = 1$$

$$\Rightarrow \frac{b^6}{{y_1}^2} + \frac{a^6}{{x_1}^2} = (a^2-b^2)^2$$

$Removing\ the\ suffixes, we\ can\ write,$

$$\Rightarrow \frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2-b^2)^2 \qquad [Proved]$$

*****The End*****