Function

Function: If x and y be two variables, so related that corresponding to every value within a define domain, we get a define value of y then y is said to be the function of x defined in its domain.

If the two real variables x and y are related in such a way that only one real value of y is found for each real value of x, then y is called the function of x. (x এবং y দুটি বাস্তব চল রাশি যদি এমনভাবে সম্পর্কিত হয় যে, x -এর প্রতিটি বাস্তব মানের জন্য y -এর কেবল মাত্র একটি বাস্তবমান পাওয়া যায় তবে y -কে বলা হয় x -এর ফাংশন।)

This process is called a function when the subordinate variable between two or more variables is dependent on the independent variable. (দুই বা ততধিক চলকের মধ্যে অধীন চলক যখন স্বাধীন চলকের উপর নির্ভরশীল হয় তখন এই প্রক্রিয়াকে ফাংশন বলে।)

Mathematically,

$$y= f(x), \text{ where } (x, y) \in \mathbf{R}$$

If R is a relation from set A to set B, then the first set of elements of all the sequences belonging to R is called the domain of R, which is expressed by R_D. R is a subset of domain A.

Similarly, the set of second elements of a sequence is called the range of R, which is expressed by R_R. (যদি A সেট হতে B সেটে R একটি অন্বয় হয়, তবে R -এর অন্তর্গত সকল ক্রমজোড়গুলির প্রথম উপাদানসমুহের সেটকে R -এর ডোমেন বলা হয়, যা RD দ্বারা প্রকাশ করা হয়। R -এর ডোমেন A -এর একটি উপসেট।

একইভাবে, ক্রমজোড়গুলির দ্বিতীয় উপাদানসমূহের সেটকে R -এর রেঞ্জ বলা হয়, যা RR দ্বারা প্রকাশ করা হয়।)

CLASSIFICATION OF FUNCTIONS:

(I) Even Function: If f(x) is a real valued function then f(x) is an even function if the equations hold for all values of x such that x and -x are the domain of the function,

$f(x)=f(-x)$

or, $f(x)-f(-x) =0$.

Example:

(I) $f(x)=x^2$, (II) $f(x) = \cos x$, (III) $f(x)=x^2+1$.

(II)<u>Odd Function</u>: If f(x) is a real valued function then f(x) is an odd function if the equations hold for all values of x such that x and -x are the domain of the function,

$f(-x) = -f(x)$

or, $f(-x) + f(x) =0$.

Example:

(I) $f(x)=x^3$, (II) $f(x) = \sin x$, (III) $f(x)=2x+\sin x$.

(III)<u>Implicit Function</u>: Let (x, y) be two variables where the relation between x and y is expressed by an equation, say $\phi(x, y) = 0$, then it is called as an implicit function.

Example:

(I) $f(x, y) =x^2 + y^2$, (II) $f(x, y) = x^3 + xy+ y^3$.

(IV) <u>Explicit Function:</u> If a function can be expressed in form as, $y=f(x)$ and $x \in D$ where $D \subseteq R$ be domain of the function then the function is called as an explicit function.

Example:

(I) $y =x^3 +x+10$, (II) $y = \sqrt{x^2 + 10}$

(V) <u>Periodic Function</u>**:** If a function f(x) is defined in a domain D then it is called as periodic function of μ when μ be the lest positive real number such, $f(x+\mu) =f(x)$ for all $x \in D$. $[x+ \mu \in D]$

Example:

f(x)=sinx, x ∈d periodic function of 2π since 2π is a least positive number such that f(x+2π) =sin(x+2π) = sinx =f(x).

(VI) Algebraic Function**:** If a function only involves algebraic equations then it is called algebraic function.

Example:

(I)f(x)=x, (II)f(x)=x^2 +x+1, (III)f(x) = $\frac{1}{x+1}$.

(VII) Exponential Function: An exponential function is a function of the form where base is a real number not equal to 1 and the argument x occurs as an exponent.

Example:

(I)f(x)=b^x, (II)f(x)= e^x .

PROBLEMS LIST:

Find the domain and ranges of the following functions:

OK 1. f(x)=$\frac{x^2-4}{x-2}$ **OK** 2. f(x)=$\frac{x-2}{x^2-3x+2}$

OK 3. f(x)=$\frac{x^2-3x+2}{x^2+x-6}$ **OK** 4. f(x)=$\frac{x^2+1}{x^2-5x+6}$

OK 5. f(x)=$\frac{2x-1}{(x-1)^2}$

SOLUTION:

OK 1**.** Given that, f(x)**=** $\frac{x^2-4}{x-2}$ **.**

As the denominator must be ≠0

therefore, x≠2

Here f(x) is defined (সংজ্ঞায়িত) for all values of x except x=2.

Domain of f(x) = **R –** {2}.

let, y= $\frac{x^2-4}{x-2}$

⇒ $xy-2y=x^2 - 4$

⇒ $x^2- xy+2y- 4=0$

⇒ $x^2 - xy +(2y - 4) = 0$

Since x is real, the determinant D= b^2- 4ac will be greater or equal to 0.

∴ $(-y)^2 -4(-2y-4).1\geq 0$

⇒ $y^2 - 8y +16\geq 0$

⇒ $(y - 4)^2 \geq 0$

Since, $(y- 4)^2$ is not defined at $y \geq 4$.

Range of f(x)= $y \in [4, +\infty)$

(Ans)

OK 2**.** Given that,

$$f(x)= \frac{x-2}{x^2-3x+2} = \frac{x-2}{(x-2)(x-1)}$$

As the denominator must be ≠0

therefore, x≠1 and x≠2.

Here f(x) is defined for all values of x except x=2 and x=1

Domain of f(x) = **R** – {1, 2}.

Let, $y = \frac{x-2}{x^2-3x+2}$

⇒ $yx^2 -3yx+2y=x-2$

⇒ $yx^2 -(3y+1) x+2y+2=0$

$\therefore \quad x = \frac{-(-3y-1)\pm\sqrt{(-3y-1)^2-4y(2y+2)}}{2y}$

Since x is real, the determinant D= b^2- 4ac will be greater or equal to 0.

$\therefore$ $(-3y-1)^2$ -4y(2y+2) ≥0

⇒ $9y^2$ +6y+1-$8y^2$-8y≥0

⇒ y^2 -2y+1 ≥0

⇒ $(y-1)^2$ ≥0

Since $(y-1)^2$ ≥0 is not defined at y≥1

let, ⇒ x= g(y)

And denominator of function x=g(y) must be ≠ 0.

therefore,2y≠0 ⇒ y≠0.

Range of f(x) **=** y ∈ (0,1] ∪ [1, +∞)

(Ans)

OK 3. Given that,

$$f(x)= \frac{x^2-3x+2}{x^2+x-6} = \frac{x^2-3x+2}{(x-2)(x+3)}$$

As the denominator must be ≠0

therefore, x≠2 and x≠ -3.

Here f(x) is defined for all values of x except x=2 and x= -3

Domain of f(x) = **R** – {2, -3}.

Let, $y = \frac{x^2-3x+2}{x^2+x-6}$

⇒ yx^2 +yx-6y=x^2 – 3x +2

⇒ (y-1) x^2 +(y+3) x-(6y+2) =0

$\therefore x = \frac{-(y+3)\pm\sqrt{(y+3)^2-4(y-1)(-6y-2)}}{2(y-1)}$

Since x is real, the determinant D= b^2- 4ac will be greater or equal to 0.

∴ $(y+3)^2 -4(y-1)(-6y-1) \geq 0$

⇒ $y^2 +6y+9 + 24y^2 -16y-8 \geq 0$

⇒ $25y^2 -10y+1 \geq 0$

⇒ $(5y-1)^2 \geq 0$

Since $(5y-1)^2 \geq 0$ is not defined at $y \geq \frac{1}{5}$

let, x= g(y)

And denominator of function x=g(y) must be ≠ 0.

therefore, $2(y-1) \neq 0 \Rightarrow y \neq 1$.

Range of f(x) **=** $y \in [\frac{1}{5}, 1) \cup (1, +\infty)$

(Ans)

OK 4. Given that,

$$f(x) = \frac{x^2+1}{x^2-5x+6} = \frac{x^2+1}{(x-2)(x-3)}$$

As the denominator must be ≠0

therefore, x≠2 and x≠ 3.

Here f(x) is defined for all values of x except x=2 and x= 3

Domain of f(x) = **R** – {2, 3}.

Let, $y = \frac{x^2+1}{x^2-5x+6}$

⇒ $yx^2 -5yx+6y=x^2 +1$

⇒ $(y-1)x^2 - 5yx +(6y-1) =0$

∴ $x = \frac{-(-5y) \pm \sqrt{(-5y)^2-4(y-1)(6y-1)}}{2(y-1)}$

Since x is real, the determinant D= b^2- 4ac will be greater or equal to 0.

∴ $(-5y)^2-4(y-1)(6y-1) \geq 0$

⇒ $25y^2 - 24y^2 + 28y - 4 \geq 0$

⇒ $y^2 + 28y - 4 \geq 0$

this inequality will be true if

$y \geq (-14 + 10\sqrt{2})$ Or $y \leq (-14 - 10\sqrt{2})$.

let, x = g(y)

As denominator of function x=g(y) must be ≠ 0.

therefore, $2(y-1) \neq 0$ ⇒ $y \neq 1$.

Range of f(x)= $y \in (-\infty, -14 - 10\sqrt{2}] \cup [-14 + 10\sqrt{2}, 1) \cup (1, +\infty)$

(Ans.)

OK 5**.** Given that,

$$f(x) = \frac{2x-1}{\langle x-1 \rangle^2}$$

As the denominator must be ≠0

therefore, x≠1

Here f(x) is defined for all values of x except x=1

Domain of f(x) = **R** – {1}.

Let, y= $\frac{2x-1}{\langle x-1 \rangle^2}$

⇒ $y(x^2 - 2x + 1) = 2x - 1$

⇒ $yx^2 - 2yx + y - 2x + 1 = 0$

⇒ $yx^2 - (2y+2)x + (y+1) = 0$

∴ x= $\frac{-(-2y-2) \pm \sqrt{(-2y-2)^2 - 4y(y+1)}}{2y}$

Since x is real, the determinant D= $b^2 - 4ac$ will be greater or equal to 0.

$\therefore$ $(-2y-2)^2 - 4y(y+1) \geq 0$

$\Rightarrow 4y^2 + 8y + 4 - 4y^2 - 4y \geq 0$

$\Rightarrow 4y+4 \geq 0$

$\Rightarrow 4y \geq -4$

$\therefore$ $y \geq -1$

let, x= g(y)

As denominator of function x=g(y) must be $\neq 0$.

therefore, $2y \neq 0 \Rightarrow y \neq 0$.

Range of f(x) **=** $y \in [-1,0) \cup (0, +\infty)$

LIMIT AND CONTINUITY

Limit: A function f(x) is to tend to a limit as x tends to a if the difference between f(x) and l is less than any given positive number, however small by making x approach to given constant a.

Limit: Limit, mathematical concept based on the idea of closeness, used primarily to assign values to certain functions at points where no values are defined, in such a way as to be consistent with nearby values.

Mathematically, $\lim\limits_{x \to a} f(x) = l$

which means that | f(x) - l | is less than any given number.

Right Hand Limit: A function is said to be tend to a limit l if x

approaches the value a form right side.

Mathematically, $\lim\limits_{x \to a^+} f(x) = I_1$

Sometimes $\lim\limits_{x \to a^+} f(x)$ is represented by the symbol f(a+ 0) or,

f(a+ h).

Left Hand Limit: A function is said to be tend to a limit l if x approaches the value a form left side.

Mathematically, $\lim_{x \to a^-} f(x) = l_2$

Sometimes $\lim_{x \to a-} f(x)$ is represented by the symbol f (a- 0) or, f(a- h).

PROBLEMS LIST:

1. Prove $\lim_{x \to a} \frac{\mathbf{x^2 - a^2}}{\mathbf{x - a}} = 2a$ by (ε- δ) the definition of limit.

2. Prove $\lim_{x \to 2} \frac{\mathbf{2x^2 - 8}}{\mathbf{x - 2}} = 8$ by (ε- δ) the definition of limit and find δ if **ε** =1.

SOLUTIONS:

1. Let, an arbitrary positive number ε>0, however very small.

by (ε- δ) the definition of limit, For all values of x

we get,

$$|f(x) - l\,| < \varepsilon$$

$\Rightarrow \left| \frac{x^2 - a^2}{x - a} - 2a \right| < \varepsilon$

$\Rightarrow |\, x + a - 2a\,| < \varepsilon$

$\Rightarrow |\, x - a\,| < \varepsilon$ .... (I)

We can determine another positive number δ depending on ε such that

$\Rightarrow |\, x - a\,| < \delta$ ...... (II) [for all values of x]

from (I) and (II),

$$\varepsilon = \delta$$

Where ε=δ, the value of the function f(x) = $\frac{x^2-a^2}{x-a}$ will differ from 2a by a number ε.

Hence, $\lim_{x\to a}\frac{x^2-a^2}{x-a} = 2a$

(Proved)

2. Let, an arbitrary positive number ε>0, however very small.

by (ε- δ) the definition of limit, For all values of x

we get,

$$|f(x) - l\,| < \varepsilon$$

$\Rightarrow \left| \frac{2x^2-8}{x-2} - 8 \right| < \varepsilon$

$\Rightarrow |\ 2x + 4 - 8| < \varepsilon$

$\Rightarrow 2|\ x - 2| < \varepsilon$

$\Rightarrow |\ x - 2| < \frac{\varepsilon}{2}$ .... (I)

We can determine another positive number δ depending on ε such that

$\Rightarrow |\ x - 2| < \delta$ ...... (II) [for all values of x]

from (I) and (II) ,

$\delta = \frac{\varepsilon}{2}$

Where $\delta=\frac{\varepsilon}{2}$, the value of the function f(x) = $\frac{2x^2-8}{x-2}$ will differ from 8 by a number ε.

Hence, $\lim_{x\to 2}\frac{2x^2-8}{x-2} = 8$

(Proved)

Again, if ε =1, $\delta=\frac{1}{2}$

(Ans)

<u>Continuity</u>: A function f(x) is said to be continuous for x=a, provided $\lim_{x\to a} f(x)$ exists, finite and is equal to f(a).

Mathematically, f(x) is continuous at x=a, if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$.

<u>PROBLEMS LIST</u>:

OK 1. A function ø(x) is defined as follows:

ø(x)= x^2 when x<1

=2.5 when x=1

=x^2 + 2 when x>1

Is ø(x) continues at x=1?

OK 2. A function f(x) is defined as follows:

f(x)= -x when x≤0

=x when 0<x<1

=2-x when x≥1

show that it is continuous at x=0 and x=1.

OK 3. A function f(x) is defined as follows:

f(x)= 3+2x for $-\frac{3}{2} \leq x < 0$

=3 - 2x for $0 \leq x < \frac{3}{2}$

= -3-2x for $x \geq \frac{3}{2}$

show that it is continuous at x=0 and discontinuous x=$\frac{3}{2}$.

4. f(x)= 5x -4 for $0 < x \leq 1$

=$4x^2$ -3x for $1 < x < 2$

=3x+4 for $x \geq 2$

Discuss the continuity of f(x) for x=1 and 2, and the existence of $f'(x)$ for these values.

OK 5. $f(x)= x$ for $0<x<1$

$=2-x$ for $1\le x\le 2$

$= x - \frac{x^2}{2}$ for $x>2$

Is f (x) continuous at x=1 and x=2 ? Does $f'(x)$ exist for these values?

<u>SOLUTIONS:</u>

OK 1. Given, $\varnothing(x) = x^2$ when $x<1$

$=2.5$ when $x=1$

$=x^2 + 2$ when $x>1$

Let consider the point x=1,

L. H. L = $\lim\limits_{x\to 1^-} \varnothing(x)$

$= \lim\limits_{h\to 0} \varnothing\,(1-h)$

$=\lim\limits_{h\to 0}\{(1-h)^2\}$

$= (1-0)^2$

$= 1$

R.H.L = $\lim\limits_{x\to 1^+} \varnothing(x)$

$= \lim\limits_{h\to 0} \varnothing\,(1+h)$

$=\lim\limits_{h\to 0}\{(1+0)^2+2\}$

$=(1+0)^2+2$

$=3$

$f(1)=2.5$

Since, $L.H.L \neq R.H.L \neq f(1)$.

Hence the function $\varnothing(x)$ is not continuous at x=1.

OK 2. Given, f(x)= -x when x≤0

=x when 0<x<1

=2-x when x≥1

Let consider the point x=0,

L.H.L = $\lim_{x \to 0^-} f(x)$

= $\lim_{h \to 0} f(0-h)$

=$\lim_{h \to 0}\{-(0-h)\}$

= 0

R.H.L = $\lim_{x \to 0^+} f(x)$

= $\lim_{h \to 0} f(0+h)$

=$\lim_{h \to 0}\{(0+h)\}$

= 0

f (0) = -(0) =0

Since, L.H.L = R.H.L = f (0).

Hence the function f(x) is continuous at x=0.

Again,

Let consider the point x=1,

L.H.L = $\lim_{x \to 1^-} f(x)$

= $\lim_{h \to 0} f(1-h)$

=$\lim_{h \to 0}\{(1-h)\}$

= 1-0

=1

R.H.L = $\lim_{x \to 1^+} f(x)$

= $\lim_{h \to 0} f(1+h)$

=$\lim_{h \to 0}\{2-(1+h)\}$

= 2-1+0

=1

f (1) = 2-1

=1

Since, L.H.L = R.H.L = f (1).

Hence the function f(x) is continuous at x=1.

(Showed)

OK 3. Given, f(x)= 3+2x for $-\frac{3}{2} \leq x < 0$

=3 - 2x for $0 \leq x < \frac{3}{2}$

= -3-2x for $x \geq \frac{3}{2}$

Let consider the point x=0,

L.H.L = $\lim_{x \to 0^-} f(x)$

= $\lim_{h \to 0} f(0-h)$

=$\lim_{h \to 0}\{3+2(0-h)\}$

= 3+2(0-0)

=3

R.H.L = $\lim_{x \to 0^+} f(x)$

= $\lim_{h \to 0} f(0+h)$

$=\lim_{h\to 0}\{3-2(0+h)\}$

= 3-2(0+0)

=3

f (0) = 3-2(0)

=3

Since, L.H.L = R.H.L = f (0).
Hence the function f(x) is continuous at x=0.

Again,

Let consider the point x=$\frac{3}{2}$

L.H.L = $\lim_{x\to\frac{3}{2}^-} f(x)$

= $\lim_{h\to 0} f(\frac{3}{2}-h)$

= $\lim_{h\to 0}\{3-2\left(\frac{3}{2}-h\right)\}$

= $\lim_{h\to 0}(3-3+2h\}$

= 3-3+0

=0

R.H.L = $\lim_{x\to\frac{3}{2}^+} f(x)$

= $\lim_{h\to 0} f(\frac{3}{2}+h)$

= $\lim_{h\to 0}\{-3-2\left(\frac{3}{2}+h\right)\}$

= $\lim_{h\to 0}(-3-3-2h\}$

= -3-3-0

= -6

$f\left(\frac{3}{2}\right) = \{-3-2\left(\frac{3}{2}\right)\}$

$= -3-3 = -6$

Since, L.H.L ≠ R.H.L = $f\left(\frac{3}{2}\right)$.

Hence the function f(x) is discontinuous at $x=\frac{3}{2}$.

(Showed)

4. Given, $f(x)= 5x -4$ for $0< x \leq 1$

$=4x^2 -3x$ for $1< x <2$

$=3x+4$ for $x \geq 2$

Let consider the value x=1,

$$\text{L.H.L} = \lim_{x\to 1^-} f(x)$$
$$= \lim_{h\to 0} f(1-h)$$
$$= \lim_{h\to 0}\{5(1-h)-4\}$$
$$= 5(1-0) -4$$
$$=1$$

$$\text{R.H.L} = \lim_{x\to 1} f(x)$$
$$= \lim_{h\to 0} f(1+h)$$
$$= \lim_{h\to 0}\{4(1+h)^2-3(1+h)\}$$
$$= \{4(1+0)^2-3(1+0)\}$$
$$=1$$

$f(1) = 5(1) -4$

$= 1$

Since,L.H.L=R.H.L=f(1).

Hence the function f(x) is continuous at x=1.

Again,

Let consider the value x=2,

L.H.L $= \lim_{x\to 2^-} f(x)$

$= \lim_{h\to 0} f(2-h)$

$= \lim_{h\to 0} \{4(2-h)^2 - 3(2-h)\}$

$= 4(2-0)^2 - 3(2-0)$

= 16-6

= 10

R.H.L = $\lim_{x\to 2} f(x)$

$= \lim_{h\to 0} f(2+h)$

$= \lim_{h\to 0}\{3(2+h)+4\}$

$= \{3(2+0)+4\}$

= 6+4

= 10

f (2) = 3(2) +4

= 10

Since,L.H.L=R.H.L=f(2).
Hence the function f(x) is continuous at x=2.

Now,

Let consider the value x=1,

$$R.f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{\{4(1+h)^2-3(1+h)\}-\{5(1)-4\}}{h}$$

$$= \lim_{h\to 0} \frac{\{4(1+2h+h^2)-3(1+h)\}-1\}}{h}$$

$$= \lim_{h\to 0} \frac{\{4+8h+4h^2-3-3h\}-1}{h}$$

$= \lim_{h\to 0} \frac{\{4+8h+4h^2-3-3h-1\}}{h}$

$= \lim_{h\to 0} \frac{\{5h+4h^2\}}{h}$

$= \lim_{h\to 0}$ (5+4h)

= 5

$L. f'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$

$= \lim_{h\to 0} \frac{\{5(1-h)-4\}-\{5(1)-4\}}{-h}$

$= \lim_{h\to 0} \frac{\{5-5h-4\}-\{5(1)-4\}}{-h}$

$= \lim_{h\to 0} \frac{\{5-5h-4\}-1}{-h}$

$= \lim_{h\to 0} \frac{-5h}{-h}$

$= \lim_{h\to 0}$ (5)

= 5

Since $R. f'(1)$ = $L. f'(1)$.

Hence the function $f'(x)$ exists at x = 1.

Again,

Let consider the point x=2,

$R. f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$

$= \lim_{h\to 0} \frac{\{3(2+h)+4\}-\{3(2)+4\}}{h}$

$= \lim_{h\to 0} \frac{(6+3h+4)-(6+4)}{h}$

$= \lim_{h\to 0} \frac{3h}{h}$

$= \lim_{h\to 0}$ (3)

= 3

$$L.f'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0} \frac{\{4(2-h)^2-3(2-h)\}-\{3(2)+4\}}{-h}$$

$$= \lim_{h\to 0} \frac{\{4(4-4h+h^2)-3(2-h)\}-10}{-h}$$

$$= \lim_{h\to 0} \frac{(16-16h+4h^2-6+3h-10)}{-h}$$

$$= \lim_{h\to 0} \frac{(-13h+4h^2)}{-h}$$

$$= \lim_{h\to 0} (13-4h)$$

= 13

Since $R.f'(2) \neq L.f'(2)$.

Hence the function f(x) does exist not at x = 2.

(Shwoed)

OK 5. Given, f(x)= x for 0<x<1

=2-x for 1≤x≤2

$= x - \frac{x^2}{2}$ for x>2

Let, consider the value x=1,

L.H.L = $\lim_{x\to 1^-} f(x)$

$= \lim_{h\to 0} f(1-h)$

$= \lim_{h\to 0}(1-h)$ = 1-0 = 1

R.H.L = $\lim_{x\to 1^+} f(x)$

= $\lim_{h \to 0} f(1 + h)$

= $\lim_{h \to 0} \{2 - (1 + h)\}$

= 2-1-0

= 1

f (1) = 2-1

=1

Since, L.H.L = R.H.L = f (1).

Hence the function f(x) is continuous at x=1.

Again,

Let, consider the value x=2,

L.H.L = $\lim_{x \to 2^-} f(x)$

= $\lim_{h \to 0} f(2 - h)$

= $\lim_{h \to 0} \{2 - (2 - h)\}$ = 2 – 2 + 0 = 0

R.H.L = $\lim_{x \to 2^+} f(x)$

= $\lim_{h \to 0} f(2 + h)$

= $\lim_{h \to 0} \{(2 + h) - \frac{(2+h)^2}{2}\}$

= $\{(2 + 0) - \frac{(2+0)^2}{2}\}$

= 2-2

= 0

f (1) = 2-2 =0

Since, L.H.L = R.H.L = f (2).

Hence the function f(x) is continuous at x=2

Now,

Let consider the value x=1,

$$R.f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{\{2-(1+h)\}-(2-1)\}}{h}$$

$$= \lim_{h\to 0} \frac{(2-1-h-1)}{h}$$

$$= \lim_{h\to 0} (\frac{-h}{h})$$

$$= \lim_{h\to 0} (-1)$$

$$= -1$$

$$L.f'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{\{(1-h)\}-(2-1)\}}{-h}$$

$$= \lim_{h\to 0} \frac{(1-h-1)}{-h}$$

$$= \lim_{h\to 0} (\frac{-h}{-h})$$

$$= \lim_{h\to 0} (1)$$

$$= 1$$

Since $R.f'(1) \neq L.f'(1)$.

Hence the function $f'(x)$ does not exist at x = 1.

Again, Let consider the value x=2,

$$R.f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{\left\{(2+h)-\frac{(2+h)^2}{2}\right\}-(2-2)}{h}$$

$$= \lim_{h\to 0} \frac{\{(4+2h)-(4+4h+h^2\}}{2h}$$

$$= \lim_{h\to 0} \frac{(-2h-h^2)}{2h}$$

$$= \lim_{h\to 0} (-1-\frac{h}{2})$$

= (-1-0)

= -1

$$L.f'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0} \frac{\{2-(2-h)\}-(2-2)}{-h}$$

$$= \lim_{h\to 0} \frac{\{2-2+h\}}{-h}$$

$$= \lim_{h\to 0} (\frac{h}{-h})$$

$$= \lim_{h\to 0} (-1)$$

= -1

∴ $R.f'(2)$ = $L.f'(2)$.

Therefore, the function $f'(x)$ exists at x = 2.

Differential Calculus:

Find the differential coefficient of:

1. $\sec(\tan^{-1} x)$...ok
2. $\tan(\sin^{-1} x)$...ok
3. $\cot^{-1}(cosec\, x + \cot x)$
4. $\tan^{-1}(\sec x + \tan x)$
5. $\cot^{-1}(\sqrt{1+x^2} - x)$
6. $\cot^{-1}\frac{1+x}{1-x}$...ok
7. $\cos^{-1}\frac{1-x^2}{1+x^2}$..ok
8. $\tan^{-1}\frac{1}{\sqrt{x^2-1}}$..ok
9. $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$..ok

10. $2\tan^{-1}\sqrt{\frac{x-a}{b-x}}$..ok

11. $x^{\cos^{-1}x}$..ok

12. $(\sin x)^{\tan x}$...ok

13. x^{x^x}...ok

14. $(\sin x)^{\cos x} + (\cos x)^{\sin x}$.........................ok

15. $(\tan x)^{\cot x} + (\cot x)^{\tan x}$........................ok

16. $\cos^{-1}\frac{1-x^2}{1+x^2}$ w. r. t. $\tan^{-1}\frac{2x}{1-x^2}$

17. $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ w. r. t. $\tan^{-1}x$

18. $x^{\sin^{-1}x}$ w. r. t. $\sin^{-1}x$..............................ok

19. If $f(x) = \left(\frac{a+x}{b+x}\right)^{a+b+2x}$, show that $f'(0) = \left(2log\frac{a}{b} + \frac{b^2-a^2}{ab}\right).(\frac{a}{b})^{a+b}$...ok

Solution:

1. Let, y= sec $(\tan^{-1}x)$

$$y = \sqrt{1+\{\tan(\tan^{-1}x)\}^2}$$

$$y = \sqrt{1+x^2}$$

Differentiating both sides with respect to x,

$$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\left(\sqrt{1+x^2}\right)$$

$$= \frac{x}{\sqrt{1+x^2}}$$

(Ans)

2. let, y = $\tan(\sin^{-1} x)$

   Differentiating both sides with respect to x,

   $$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\{\tan(\sin^{-1} x)\}$$
   $$= \{\sec(\sin^{-1} x)\}\}^2 \cdot \frac{1}{\sqrt{1-x^2}}$$
   $$= \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{\{\cos(\sin^{-1} x)\}^2}$$
   $$= \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{1-\{\sin(\sin^{-1} x)\}^2}$$
   $$= \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{1-x^2}$$
   $$= \frac{1}{(1-x^2)^{\frac{3}{2}}}$$

   (Ans.)

3. let, y=$\cot^{-1}(cosecx + cotx)$

   $$= \cot^{-1}\left(\frac{1+cosx}{sinx}\right)$$
   $$= \cot^{-1}\left(\frac{2\cos^2 {x}/{2}}{2sin\,{x}/{2}cos\,{x}/{2}}\right)$$
   $$= \cot^{-1}\left(cot\,\frac{x}{2}\right)$$
   $$= \frac{x}{2}$$

   Differentiating both sides with respect to x,

   $$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{x}{2}\right)$$

$= \frac{1}{2}$

(Ans)

4. let, $y = \tan^{-1}(secx + tanx)$

$= \tan^{-1}\left(\frac{1+sinx}{cosx}\right)$

$= \tan^{-1} \frac{(cos\, {x}/{2} + sin\, {x}/{2})^2}{\cos^2 {x}/{2} - \sin^2 {x}/{2}}$

$= \tan^{-1} \frac{cos\, {x}/{2} + sin\, {x}/{2}}{cos\, {x}/{2} - sin\, {x}/{2}}$

$= \tan^{-1} \frac{1 + tan\, {x}/{2}}{1 - tan\, {x}/{2}}$

$= \tan^{-1} 1 + \tan^{-1}(\tan\frac{x}{2})$

$= \frac{\pi}{4} + \frac{x}{2}$

Differentiating both sides with respect to x,

$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\left[\frac{\pi}{4} + \frac{x}{2}\right]$

$= \frac{1}{2}$

(Ans)

5. let, $y = \cot^{-1}(\sqrt{1 + x^2} - x)$

$= \cot^{-1}(\sec\theta - \tan\theta)$

$= \cot^{-1}\left(\frac{1 - sin\theta}{cos\theta}\right)$

$= \cot^{-1}\left[\frac{(\cos\frac{\theta}{2} - \sin\frac{\theta}{2})^2}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}\right]$

$= \cot^{-1} \frac{(\cos\frac{\theta}{2} - \sin\frac{\theta}{2})}{(\cos\frac{\theta}{2} + \sin\frac{\theta}{2})}$

consider,

x = $\tan\theta$

$\therefore \theta = \tan^{-1} x$

$= \cot^{-1} \frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}$

$= \tan^{-1} \frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}$

$= \tan^{-1} 1 + \tan(\tan^{-1}\frac{\theta}{2})$

$= \frac{\pi}{4} + \frac{\theta}{2}$

$= \frac{\pi}{4} + \frac{1}{2}\tan^{-1} x$

Differentiating both sides with respect to x,

$\therefore \frac{d}{dx}(y) = \frac{d}{dx}[\frac{\pi}{4} + \frac{1}{2}\tan^{-1} x]$

$= \frac{1}{2(1+x^2)}$

(Ans)

6. $\text{let}, y = \cot^{-1} \frac{1+x}{1-x}$

$= \cot^{-1} \frac{1+tan\theta}{1-tan\theta}$

$= \tan^{-1} \frac{1-tan\theta}{1+tan\theta}$

consider,

x + tanθ

$\therefore\ \theta = \tan^{-1} x$

$= \tan^{-1} 1 - \tan^{-1}(tan\theta)$

$= \frac{\pi}{4} - \theta$

$= \frac{\pi}{4} - \tan^{-1} x$

Differentiating both sides with respect to x,

$\therefore \frac{d}{dx}(y) = \frac{d}{dx}[\frac{\pi}{4} - \tan^{-1} x]$

$= -\frac{1}{1+x^2}$

(Ans)

7. $\text{let}, \cos^{-1}\frac{1-x^2}{1+x^2}$ | consider,

$= \cos^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta}$ | x = tanθ

$= \cos^{-1}\frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$

$= \cos^{-1}\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$

$= \cos^{-1}\cos 2\theta$

$= 2\theta$

$= 2\tan^{-1}x$

Differentiating both sides with respect to x,

$\therefore \frac{d}{dx}(y) = 2\,tan^{-1}x$

$= \frac{2}{1+x^2}$

(Ans)

8. $\text{let}, y = \tan^{-1}\frac{1}{\sqrt{(x^2-1)}}$ | consider,

$= \tan^{-1}\frac{1}{\sqrt{\text{cosec}^2\,\theta-1}}$ | x = cosecθ

$= \tan^{-1}\tan\theta$ | $\theta = \text{cosec}^{-1}x$

$= \theta$

$= \text{cosec}^{-1}x$

Differentiating both sides with respect to x,

$$\frac{d}{dx}(y) = \frac{d}{dx}(\operatorname{cosec}^{-1} x)$$

$$= \frac{-1}{x\sqrt{x^2-1}}$$

(Ans)

9.let, $y = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

$= \tan^{-1}\frac{\sin\theta}{1-\sin^2\theta}$

$= \tan^{-1}\tan\theta$

$= \theta$

$= \sin^{-1} x$

consider,

x = sinθ

$\therefore\ \theta = \sin^{-1} x$

Differentiating both sides with respect to x,

$$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\sin^{-1} x$$

$$= \frac{1}{\sqrt{1-x^2}}$$

(Ans)

10.let, $y = 2\tan^{-1}\sqrt{\frac{x-a}{b-x}}$

Differentiating both sides with respect to x,

$$\therefore \frac{d}{dx}(y) = \frac{d}{dx} 2\tan^{-1}\sqrt{\frac{x-a}{b-x}}$$

$$= \frac{2}{1+\frac{x-a}{b-x}} \cdot \frac{1}{2\sqrt{\frac{x-a}{b-x}}} \cdot \frac{(b-x)-(x-a)(-1)}{(b-x)^2}$$

$$= \frac{b-x}{b-a} \cdot \frac{\sqrt{b-x}}{\sqrt{x-a}} \cdot \frac{(b-a)}{(b-x)^2}$$

$$= -\frac{1}{\sqrt{(x-a)(x-b)}}$$

(Ans)

11.let, y= $x^{\cos^{-1}x}$

Differentiating both sides with respect to x,

$\frac{d}{dx}$(y) = $\frac{d}{dx}$ $(x^{\cos^{-1}x})$

$= (x^{\cos^{-1}x})[\frac{d}{dx}(\cos^{-1}x)(\ln x)]$

$= (x^{\cos^{-1}x})[\frac{\cos^{-1}x}{x} - \frac{\ln x}{\sqrt{1-x^2}}]$

(Ans)

12. $y = (\sin x)^{\tan x}$

Differentiating both sides with respect to x ,

$\frac{d}{dx}(y) = \frac{d}{dx}\ (\sin x)^{\tan x}$

= $\{\ (\sin x)^{\tan x}\}[\frac{d}{dx}\{(\tan x)\ \ln(\sin x)\}]$

= $\{\ (\sin x)^{\tan x}\}[\frac{\tan x}{\sin x}.\cos x + (\sec^2 x)\ \ln(\sin x)]$

= $\{(\sin x)^{\tan x}\}[1 + (\sec^2 x)\ \ln(\sin x)]$

(Ans)

13. let, $y = (x^{x^x})$

Differentiating both sides with respect to x ,

$\frac{d}{dx}(y) = \frac{d}{dx}(x^{x^x})$

= $x^{x^x}[\frac{d}{dx}x^x\ln x]$

= $x^{x^x}[\frac{x^x}{x} + x^x\ln x\{\frac{d}{dx}x\ln x\}]$

$= x^{x^x}.x^x[\frac{1}{x} + \ln x(1 + \ln x)]$

(Ans)

14.let,y = $\{(\sin x)^{\cos x} + (\cos x)^{\sin x}\}$

Differentiating both sides with respect to x ,

$\frac{d}{dx}$(y)=$\frac{d}{dx}$ $\{(\sin x)^{\cos x} + (\cos x)^{\sin x}\}$

$= \frac{d}{dx}\{(\sin x)^{\cos x}\} + \frac{d}{dx}\{(\cos x)^{\sin x}\}$

= $\{(\sin x)^{\cos x}\}[\frac{d}{dx}(\cos x)\ln(\sin x)] + \{(\cos x)^{\sin x}\}[\frac{d}{dx}(\sin x)\ln(\cos x)]$

=$(\sin x)^{\cos x}[(\cot x)(\cos x) - (\sin x)\ln(\sin x)] +$
$(\cos x)^{\sin x}[(\tan x)(-\sin x) + (\cos x)\ln(\cos x)]$

=$(\sin x)^{\cos x}[(\cot x)(\cos x) - (\sin x)\ln(\sin x)] +$
$(\cos x)^{\sin x}[(\cos x)\ln(\cos x) - (\tan x)(\sin x)$

(Ans)

15. let, y = $\{(\tan x)^{\cot x} + (\cot x)^{\tan x}\}$

Differentiating both sides with respect to x ,

$\frac{d}{dx}$(y)= $\frac{d}{dx}\{(\tan x)^{\cot x} + (\cot x)^{\tan x}\}$

= $\frac{d}{dx}(\tan x)^{\cot x} + \frac{d}{dx}(\cot x)^{\tan x}$

=$\{(\tan x)^{\cot x}\}$ [$\frac{d}{dx}\cot x \ln\tan x$] + $\{(\cot x)^{\tan x}\}$[$\frac{d}{dx}\tan x \ln\cot x$]

=$(\tan x)^{\cot x}[\cot^2 x \sec^2 x - \csc^2 x \tan x]$+
$(\cot x)^{\tan x}[\tan^2 x(-\csc^2 x) + \sec^2 x \ln\cot x]$

=$(\tan x)^{\cot x}$.$\csc^2 x[1 - \ln(\tan x)]$ + $(\cot x)^{\tan x}.\sec^2 x[\ln(\cot x) - 1]$

(Ans)

16. let, y = $\cos^{-1}\frac{1-x^2}{1+x^2} = 2\tan^{-1}x$

z = $\tan^{-1}\frac{2x}{1-x^2}$ = $2\tan^{-1}x$

Differentiating both sides with respect to z ,

$$\therefore \frac{dy}{dz} = \frac{\frac{d}{dx}(y)}{\frac{d}{dx}(z)}$$

$$= \frac{\frac{d}{dx}(2\tan^{-1}x)}{\frac{d}{dx}(2\tan^{-1}x)}$$

= 1 (Ans)

17. let, $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

$$= \tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{tan\theta}$$

let, x = tanθ

$\theta = \tan^{-1}x$

$$= \tan^{-1}\frac{\sec\theta-1}{\tan\theta}$$

$$= \tan^{-1}\frac{1-\cos\theta}{\sin\theta}$$

$$= \tan^{-1}\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}cos\frac{\theta}{2}}$$

$$= \tan^{-1}\tan\frac{\theta}{2}$$

$$= \frac{1}{2}\theta$$

$$z = \frac{1}{2}\tan^{-1}x$$

Differentiating both sides with respect to z,

$$\therefore \frac{dy}{dz} = \frac{\frac{d}{dx}(y)}{\frac{d}{dx}(z)}$$

$$= \frac{\frac{d}{dx}(\frac{1}{2}\tan^{-1} x)}{\frac{d}{dx}(\tan^{-1} x)}$$

$$= \frac{1}{2}$$

(Ans)

18. let,

$y = x^{\sin^{-1} x}$

$z = \sin^{-1} x$

Differentiating both sides with respect to z,

$$\therefore \frac{dy}{dz} = \frac{\frac{d}{dx}(x^{\sin^{-1} x})}{\frac{d}{dx}(\sin^{-1} x)}$$

$$= \frac{x^{\sin^{-1} x}[\frac{d}{dx}(\sin^{-1} x) lnx]}{\frac{1}{\sqrt{1-x^2}}}$$

$$= \frac{x^{\sin^{-1} x}[\frac{\sin^{-1} x}{x} + \frac{lnx}{\sqrt{1-x^2}}]}{\frac{1}{\sqrt{1-x^2}}}$$

$$\frac{dy}{dz} = x^{\sin^{-1} x}(\sqrt{1-x^2})[\frac{\sin^{-1} x}{x} + \frac{lnx}{\sqrt{1-x^2}}]$$

(Ans)

19.Given,

$$f(x) = (\frac{a+x}{b+x})^{a+b+2x}$$

$$f'(x) = (\frac{a+x}{b+x})^{a+b+2x}[\frac{d}{dx}(a+b+2x)\log\frac{a+x}{b+x}]$$

$$= (\frac{a+x}{b+x})^{a+b+2x}[2\log(\frac{a+x}{b+x}) + (a+b+2x)\frac{d}{dx}\{\log(a+x) - \log(b+x)\}]$$

$$f'(x) = \left(\frac{a+x}{b+x}\right)^{a+b+2x}\left[2\log\left(\frac{a+x}{b+x}\right) + (a+b+2x)\left\{\frac{1}{a+x} - \frac{1}{b+x}\right\}\right]$$

$$\therefore f'(0) = \left(\frac{a}{b}\right)^{a+b}.\left[2\log\frac{a}{b} + (a+b)\left(\frac{1}{a} - \frac{1}{b}\right)\right]$$

$$\therefore f'(0) = \left(2\log\frac{a}{b} + \frac{b^2 - a^2}{ab}\right).\left(\frac{a}{b}\right)^{a+b}$$

(proved)

Taylor's Theorem: If (a + h) be a function of the variable h such that it can be expanded in ascending powers of h and this expansion be differentiable with respect to h in any number of times then,

$$\boldsymbol{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ \ldots\ldots}$$

Proof:

Consider a function –

$$f(a+h) = A_0 + hA_1 + h^2A_2 + h^3A_3 + h^4A_4 + \ldots\ldots\ldots \qquad \text{eqn. (1)}$$

Differentiate with respect to h

$$f'(a+h) = A_1 + 2A_2h + 3A_3h^2 + 4A_4h^3 + \ldots\ldots\ldots \qquad \text{eqn. (2)}$$

$$f''(a+h) = 2A_2 + 6A_3h + 12A_4h^2 + \ldots\ldots\ldots \qquad \text{eqn. (3)}$$

$$f'''(a+h) = 6A_3 + 24A_4h + \ldots\ldots\ldots \qquad \text{eqn. (4)}$$

Put h = 0 in all eqn.

$$f(a) = A_0$$

$$f'(a) = A_1$$

$$f''(a) = 2A_2 \qquad \Rightarrow A_2 = \frac{f''(a)}{2!}$$

$$f'''(a) = 6A_3 \qquad \Rightarrow A_3 = \frac{f'''(a)}{3!}$$

Now put the value of A_0, A_1, A_2, A_3 in eqn. (1)

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \dots \dots \dots$$

Let, $a + h = x \Rightarrow h = x - a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots \dots \dots$$

(proved)

<u>Expand in Taylor's series:</u>

1. $f(x) = \log x, \quad a = 3$

<u>Solution</u>: Given f(x)=log x

$f(x) = \log x$	$f(3) = \log 3$
$f'(x) = \frac{1}{x}$	$f'(3) = \frac{1}{3}$
$f''(x) = -\frac{1}{x^2}$	$f''(3) = -\frac{1}{3^2}$
$f'''(x) = \frac{1}{x^3}$	$f'''(3) = \frac{1}{3^3}$

Using Taylors th^m ,we get

$$f(3 + x - 3) = f(3) + (x-3)f'(3) + \frac{(x-3)^2}{2!}f''(3) + \frac{(x-3)^3}{3!}f'''(3) + \dots$$

$$\therefore \ f(x) = f(3) + (x-3)f'(3) + \frac{(x-3)^2}{2!}f''(3) + \frac{(x-3)^3}{3!}f'''(3) + \dots \dots \dots$$

$$\therefore f(x) = \log 3 + (x-3)\left(\frac{1}{3}\right) - \frac{(x-3)^2}{2!}\left(\frac{1}{3^2}\right)$$

$$+\frac{(x-3)^3}{3!}\left(\frac{1}{3^3}\right) + \dots\dots\dots$$

(Ans.)

OK 2. $f(x) = \cos x, \quad a = \frac{\pi}{4}$

<u>Solution:</u> Given,

$$f(x) = \cos x, \quad a = \frac{\pi}{4}$$

$f(x) = \cos x$	$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
$f'(x) = -\sin x$	$f'\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$f''(x) = -\cos x$	$f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$f'''(x) = \sin x$	$f'''\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Using Taylors th^m, we get,

$$f\left(\frac{\pi}{4} + x - \frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right)$$

$$+\frac{(x-\frac{\pi}{4})^3}{3!} f'''\left(\frac{\pi}{4}\right) + \dots\dots\dots$$

$$f(x) = f\left(\frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) +$$

$$\frac{(x-\frac{\pi}{4})^3}{3!}f'''\left(\frac{\pi}{4}\right) + \dots\dots\dots$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}}\frac{(x-\frac{\pi}{4})^2}{2!} + \frac{1}{\sqrt{2}}\frac{(x-\frac{\pi}{4})^3}{3!} + \dots\dots\dots$$

$$\therefore f(x) = \frac{1}{\sqrt{2}}[1 - \left(x - \frac{\pi}{4}\right) - \frac{(x-\frac{\pi}{4})^2}{2!} + \frac{(x-\frac{\pi}{4})^3}{3!} + \dots\dots\dots$$

(Ans.)

<u>Maclaurin's Theorem:</u> If f(x) be a function of the variable x such that it can be expanded in ascending power of x and this expansion be differentiable with respect to x in any number of times then,

$$\boldsymbol{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots\dots + \frac{x^n}{n!}f^n(0) + \dots\dots\dots}$$

<u>Proof:</u>

Consider a function –

$f(x) = A_0 + xA_1 + x^2A_2 + x^3A_3 + x^4A_4 + \dots\dots\dots$ eqn. (1)

Differentiate with respect to x

$f'(x) = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \dots\dots\dots$ eqn. (2)

$f''(x) = 2A_2 + 6A_3x + 12A_4x^2 + \dots\dots\dots$ eqn. (3)

$f'''(x) = 6A_3 + 24A_4x + \dots\dots\dots$ eqn. (4)

Put x = 0 in all eqn.

$f(0) = A_0$

$f'(0) = A_1$

$f''(0) = 2A_2 \qquad \Rightarrow A_2 = \frac{f''(0)}{2!}$

$f'''(0) = 6A_3 \qquad \Rightarrow A_3 = \frac{f'''(0)}{3!}$

The same can be written, $A_n = \frac{f^n(0)}{n!}$

Now put the value of A_0, A_1, A_2, A_3 in eqn. (1)

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots + \frac{x^n}{n!}f^n(0) + \ldots\ldots.$$

(proved)

Leibnitz's Theorem: If u and v are two functions of x, each possessing derivatives up to n_{th} order, then the n_{th} derivative of their product,

$$(uv)_n = u_n v + {}^nc_1 u_{n-1}v_1 + {}^nc_2 u_{n-2}v_2 + \ldots\ldots + {}^nc_r u_{n-r}v_r + \ldots\ldots + uv_n$$

Where the suffixes of u and v denote the order of differentiations of u and v with respect to x.

Proof:

Let y = uv

By actual differentiation, we have

$y_1 = u_1 v + uv_1$

$y_2 = u_2 v + 2u_1v_1 + uv_2 = u_2v + {}^2c_1u_1v_1 + uv_2$

$y_3 = u_3v + 3u_2v_1 + 3u_1v_2 + uv_3$

$\quad = u_3v + {}^3c_1u_2v_1 + {}^3c_2u_1v_2 + uv_3$

The theorem is thus seen to be true when n = 2 and 3.

Let us assume, therefore, that

$(uv)_n = u_nv + {}^nc_1u_{n-1}v_1 + {}^nc_2u_{n-2}v_2 + \ldots\ldots + {}^nc_ru_{n-r}v_r + \ldots\ldots + uv_n$

∴ differentiating,

$y_{n+1} = u_{n+1}v + ({}^nc_1 + 1)\, u_nv_1 + ({}^nc_2 + {}^nc_1)\, u_{n-1}v_2 + ... + ({}^nc_r + {}^nc_{r-1})\, u_{n-r+1}v_r + uv_{n+1}$

Since, ${}^nc_r + {}^nc_{r-1} = {}^{n+1}c_r$ and ${}^nc_1 + 1 = {}^{n+1}c_1$

$\therefore y_{n+1} = u_{n+1}v + {}^{n+1}c_1\, u_nv_1 + {}^{n+1}c_2u_{n-1}v_2 + ... + {}^{n+1}c_ru_{n-r+1}v_r + ...\, uv_{n+1}$

Thus, if the theorem holds for n differentiations, it also holds for n+1. But it is probed to hold for 2 and 3 differentiations; hence it holds for four, and so on, and thus the theorem is true for every positive integral value of n.

Successive Differentiation:

OK 1. If $y = \ tan^{-1}x$ prove that, $(1 + x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$

Solution:

Given,

$y = tan^{-1}x$......(1)

Differentiating eq^n (1) with respect to x,

$y_1 = \frac{1}{1+x^2}$

Or,$(1 + x^2)y_1 = 1$ . . . . (2)

Differentiating eq^n (2) n times with respect to x,

$(1 + x^2)y_{n+1} + n_{c_1}y_n 2x + n_{c_2}y_{n-1}.2 = 0$

Or,$(1 + x^2)y_{n+1} + 2nxy_n + \frac{n(n-1)}{2}.2y_{n-1} = 0$

$\therefore (1 + x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$

[Proved]

OK 2. If $y = sin^{-1}x$ show that, $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0$

Solution:

Given, $y = sin^{-1}x$.......(1)

Differentiating equation (1) with respect to x, (2 times)

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

Or, $\sqrt{1-x^2}y_1 = 1$

Or, $y_1^2(1-x^2) = 1$

Or, $y_1^2(-2x) + (1-x^2)2y_1y_2 = 0$

Or,$2y_1\{(1-x^2)y_2 - xy_1\} = 0$

Or,$(1-x^2)y_2 - xy_1 = 0$ $\quad$ . . . . (2)

Differentiating equation (2) n times with respect to x with the help of Leibnitz theorem,

$y_{n+2}(1-x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2) - y_{n+1}x - n_{C_1}y_n.1 = 0$

Or,$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + \frac{(-2)n(n-1)}{2}y_n - ny_n = 0$

Or,$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - y_n(n^2 - n + n) = 0$

$\therefore$ $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

[Showed]

OK 3. $y = e^{tan^{-1}x}$, prove that, $(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$

<u>Solution</u>: Given,

$$y = e^{tan^{-1}x} \text{ . . . . (1)}$$

Differentiating equation (1) with respect to x, (2 times)

$$y_1 = e^{tan^{-1}x}.\frac{1}{1+x^2}$$

Or,$(1+\mathrm{x}^2)\mathrm{y}_1 = \mathrm{e}^{\tan^{-1}\mathrm{x}}$

Or,$(1+\mathrm{x}^2)\mathrm{y}_1 = \mathrm{y}$ $\quad$ [From (1)]

Or, $(1+\mathrm{x}^2)\mathrm{y}_2 + 2\mathrm{x}\mathrm{y}_1 = \mathrm{y}_1$

Or, $(1+\mathrm{x}^2)y_2 + y_1(2x-1) = 0 \ldots\ldots\ldots\ldots(2)$

Differentiating equation (2) with respect to x n times by Leibnitz theorem, we get,

$y_{n+2}(1+x^2) + n_{C_1}y_{n+1}2x + n_{C_2}y_n 2 + y_{n+1}(2x-1) + n_{C_1}y_n.2 = 0$

Or,$(1+x^2)y_{n+2} + 2nxy_{n+1} + (2x-1)y_{n+1} + \frac{n(n-1)}{2}2y_n + 2ny_n = 0$

Or,$(1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + y_n(n^2-n+2n) = 0$

$\therefore (1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + (n^2+n)y_n = 0$

[Proved]

4. If $y = e^{asin^{-1}x}$ then show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

Solution: Given

$$y = e^{asin^{-1}x} \ldots\ldots\ldots\ldots(1)$$

Differentiating equation (1) with respect to x, (2 times)

$\therefore y_1 = e^{sin^{-1}x}.\frac{a}{\sqrt{1-x^2}}$

Or, $(\sqrt{1-x^2}\, y_1 = e^{sin^{-1}x}.a$

Or, $(\sqrt{1-x^2}\, y_1 = y.a\, , Using\ (1)$

Or, $(1-x^2)y_1^2 = a^2y^2$

Or, $(1-x^2)2y_1y_2 + y_1^2(-2x) = a^2.2yy_1$

Or, $(1-x^2)2y_1y_2 - 2xy_1^2 = a^2 2yy_1$

Or, $(1-x^2)y_2 - xy_1 = a^2y \quad [\therefore 2y_1 \neq 0]$

Or, $(1-x^2)y_2 - xy_1 - a^2y = 0 \ldots\ldots\ldots(2)$

Differentiating equation (2) n times with respect to x by the help of Leibnitz's theorem,

$$(1-x^2)y_{n+2} + n_{C_1} y_{n+1}(-2x) + n_{C_2} y_n(-2) - (xy_{n+1} + n_{C_1} y_n(1)) - a^2 y_n = 0$$

Or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n - a^2 y_n = 0$

Or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - n^2 y_n + ny_n - xy_{n+1} - ny_n - a^2 y_n = 0$

Or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

[Showed]

OK 5. If $y = \sin(m\sin^{-1}x)$ then show that, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Solution: Given,

$y = \sin(m\sin^{-1}x)$ . . . . (1)

Differentiating equation (1) with respect to x, (2 times)

$\therefore y_1 = \cos(m\sin^{-1}x).m.\frac{1}{\sqrt{1-x^2}}$, using (1)

Or, $\sqrt{1-x^2}\, y_1 = m\cos(m\sin^{-1}x)$

Or, $(1-x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x)$

Or, $(1-x^2)y_1^2 = m^2[1-\sin^2(m\sin^{-1}x)]$

Or, $(1-x^2)y_1^2 = m^2(1-y^2)$

Or, $(1-x^2)2yy_1 + y_1^2(-2x) = m^2(-2yy_1)$

Or, $(1-x^2)2yy_1 - 2xy_1^2 = -m^2.2yy_1$

Or, $(1-x^2)y_2 - xy_1 + m^2 y = 0$ $[\therefore 2y_1 \neq 0]$

Or, $(1-x^2)y_2 - xy_1 + m^2 y = 0$ . . . . (2)

Differentiating equation (2) n times with respect to x by the help of Leibnitz theorem,

$$(1-x^2)y_{n+2} + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2) - (xy_{n+1} + n_{C_1}y_n(1)) + m^2y_n = 0$$

Or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$

Or,$(1-x^2)y_{n+2} - 2nxy_{n+1} - n^2y_n + ny_n - xy_{n+1} - ny_n + m^2y_n = 0$

Or,$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

[Showed]

OK 6. $y = e^x \cos x$ show that $y_4 + 4y = 0$

<u>Solution:</u>

Given, $y = e^x \cos x$ ....(i)

Differentiating equation (i) with respect to x (4 times),

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \cos x)$$

Or, $y_1 = (-e^x \sin x + e^x \cos x)$

Or, $\frac{d}{dx}(y_1) = \frac{d}{dx}(-e^x \sin x + e^x \cos x)$

Or, $y_2 = (-e^x \sin x + e^x \cos x - e^x \sin x - e^x \cos x)$

Or, $(y_2) = (-2e^x \sin x)$

Or, $\frac{d}{dx}(y_2) = \frac{d}{dx}(-2e^x \sin x)$

Or, $y_3 = -2(e^x \sin x + e^x \cos x)$

Or, $\frac{d}{dx}(y_3) = -2\frac{d}{dx}(e^x \sin x + e^x \cos x)$

Or, $y_4 = -2(-e^x \sin x + e^x \cos x + e^x \cos x + e^x \sin x)$

Or, $y_4 = -2(2e^x \cos x)$

Or, $y_4 = -4e^x \cos x$

Or, $y_4 = -4y$ [From equation (i)]

$\therefore y_4 + 4y = 0$

(showed).

7. $y = e^{ax} \sin bx$ show, $y_2 - 2ay_1 + a^2y + b^2y = 0$

Solution:

Given, $y = e^{ax} \sin bx$ ......(i)

Differentiating equation (i) with respect to x, (2 times),

$y_1 = be^{ax} \cos bx + ae^{ax} \sin bx$

Or, $y_1 - ay = be^{ax} \cos bx$

Or, $y_1 - ay = be^{ax} \cos bx$

Or, $y_2 - ay_1 = b\{be^{ax}(-\sin bx) + ae^{ax} \cos bx\}$

Or, $y_2 - ay_1 = abe^{ax} \cos bx - b^2e^{ax} \sin bx$

Or, $y_2 - ay_1 = -b^2 + a(y_1 - ay)$

Or, $y_2 - ay_1 - a(y_1 - ay) + b^2y = 0$

Or, $y_2 - 2ay_1 + a^2y + b^2y = 0$

$\therefore y_2 - 2ay_1 + a^2y + b^2y = 0$

(showed)

OK 8. If $y = e^x sinx$, show that $y_4 + 4y = 0$

Solution:

Given, $y = e^x sinx$ ....(i)

Differentiating equation (i) with respect to x (4 times),

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \ \sin x)$$

Or, $y_1 = (e^x \cos x + e^x \sin x)$

Or, $\frac{d}{dx}(y_1) = \frac{d}{dx}(e^x \cos x + e^x \sin x)$

Or, $y_2 = (-e^x \sin x + e^x \cos x + e^x \, cos\, x + e^x \, sin\, x)$

Or, $(y_2) = (2e^x \cos x)$

Or, $\frac{d}{dx}(y_2) = \frac{d}{dx}(2e^x \cos x)$

Or, $y_3 = 2(-e^x \sin x + e^x \cos x)$

Or, $\frac{d}{dx}(y_3) = 2\frac{d}{dx}(-e^x \sin x + e^x \cos x)$

Or, $y_4 = 2(-e^x \cos x - e^x \sin x - e^x \sin x + e^x \cos x)$

Or, $y_4 = 2(-2e^x \sin x)$

Or, $y_4 = -4e^x \sin x)$

Or, $y_4 = -4y$ [From equation (i)] $\therefore y_4 + 4y = 0$

[showed]

Rolle's Theorem: Let a function f(x) be a real valued function in interval [a, b] such that,

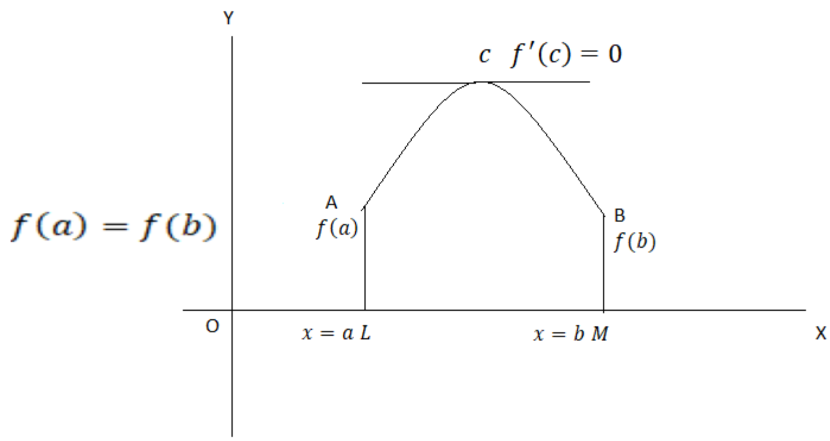
(i) f(x) is continuous in closed interval [a, b]

(ii) f(x) is differentiable in open interval (a, b)

(iii) f(a) = f(b)

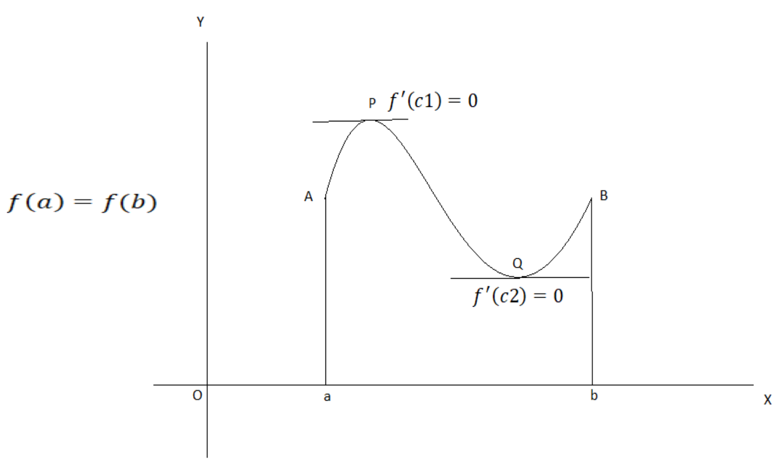
Then there exist at least one-point c ϵ (a, b) such that $f'(c) = 0$.

Geometrical Interpretation:



Let l, M be the points on the number axis $\overrightarrow{OX}$ representing the real numbers a, b respectively. We draw the graph of the function y = f(x) and let A, B be the points in it corresponding to L, M respectively, that is, LA = f(a) and MB = f(b).

From the condition (i) of Rolle's theorem, we say that the graph is a continuous curve between the points A and B; the condition (ii) says that the curve has tangents at every point between A and B and the third condition implies that LA = MB.



Now, f(c) is the gradient of the tangent of the curve at x = c. By Rolle's theorem $f'(x)$ vanishes at least once between x = a and x = b. Geometrically we say that we get at least one-point C on the graph between A and B such that the tangent at C is parallel to $\overrightarrow{OX}$.

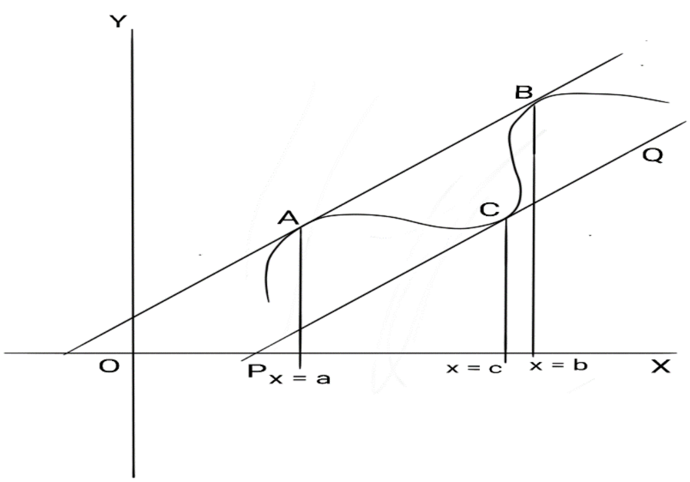
Lagrange's Mean Value Theorem: Let, f(x) be defined in [a, b] such that,

(i) f(x) is continuous in [a, b]

(ii) f(x) is differentiable in (a, b)

Then, there exist at least one-point c ϵ (a, b) such that, $f'(c) = \frac{f(b)-f(a)}{b-a}$

Geometrical Interpretation:



Let A and B are two point on the graph of f(x) corresponding to x = a and x = b respectively. Then coordinates of A and B are A (a, f(a)) and B (b, f(b)).

Slope of line AB, $m_1 = \frac{f(b)-f(a)}{b-a}$

Now there is a point c ϵ (a, b) where the slope is parallel to AB.

Since f(x) is continuous and differentiable in (a, b), we will get a tangent at point c.

Let, the slope in point c = PQ = $f'(c)$

PQ is parallel to AB.

Therefore,

$f'(c) = m_1$

$\therefore f'(c) = \frac{f(b)-f(a)}{b-a}$ (proved)

Expansion of Functions:

1. Find the value of c in the mean value theorem. $f(b) - f(a) = (b - a)f'(c)$

If, $f(x) = x(x-1)(x-2)$ $a = 0$ and $b = \frac{1}{2}$

Solution:

Given that,

$f(x) = x(x-1)(x-2)$

$= (x^2 - x)(x-2)$

$\therefore f(x) = x^3 - 3x^2 + 2x$

$\therefore f'(x) = 3x^2 - 6x + 2$

$\therefore f'(c) = 3c^2 - 6c + 2$

$f(a) = 0,\quad f(b) = \frac{1}{8} - \frac{3}{4} + \frac{2}{2}$

$= \frac{1}{8} - \frac{3}{4} + 1$

$= \frac{3}{8}$

Now,

$3c^2 - 6c + 2 = \frac{\frac{3}{8}-0}{\frac{1}{2}-0} = \frac{3}{8} \times 2 = \frac{3}{4}$

$\Rightarrow 12c^2 - 24c + 5 = 0$

$\therefore c = \frac{-(-24)\pm\sqrt{(-24)^2-4.5.12}}{2\times 12}$

$= \frac{24\pm\sqrt{336}}{24}$

$$= 1 \pm \sqrt{\frac{7}{12}}$$

Since, $0 < c < 1/2$, the +ve sign is to be rejected

$\therefore$ $c=1-\sqrt{\frac{7}{12}}$

(Ans.)

OK 2. In the mean value theorem,

$$f(a+h) = f(a) + hf'(a+\theta h)$$

If $a = 1$ and $h = 3$ and $f(x) = \sqrt{x}$, find $\theta = ?$

Solution:

Given that,

$f(x) = \sqrt{x}$ Here, a=1, h =3

$f'(x) = \frac{1}{2\sqrt{x}}$

$f(a+h) = \sqrt{1+3} = 2,\quad f(a) = \sqrt{1} = 1$

$f(a+\theta h) = \sqrt{a+\theta h}$

$f(a+h) = f(a) + hf'(a+\theta h)$

$\therefore 2 = 1 + 3.\frac{1}{2\sqrt{a+\theta h}}$

$\Rightarrow 2\sqrt{a+\theta h} = 3 \quad \Rightarrow 1 + 3\theta = \frac{9}{4}$

$$\Rightarrow 3\theta = \frac{9}{4} - 1 = \frac{5}{4}$$

$\Rightarrow \theta = \frac{5}{12}$

(Ans.)

OK 3. In the mean value theorem,

if $f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h), 0 < \theta < 1$, find θ, when $h = 7$ and $f(x) = \frac{1}{1+x}$

Solution:

Given that,

$f(x) = \frac{1}{1+x}$ $\qquad$ $f(0) = 1$

$f'(x) = -\frac{1}{(1+x)^2}$ $\qquad$ $f'(0) = -1$

$f''(x) = \frac{2}{(1+x)^3}$ $\qquad$ $f''(\theta h) = \frac{2}{1+\theta h}$

Given equation is,

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h)$$

$$\Rightarrow \frac{1}{1+h} = 1 - h + \frac{h^2}{2!} \ \frac{2}{(1+\theta h)^3}$$

When h=7,

$$\frac{1}{1+7} = 1 - 7 + \frac{7^2}{2!} \ \frac{2}{(1+\theta 7)^3}$$

$$\Rightarrow \frac{1}{8} = -6 + \frac{49}{2} \ \frac{2}{(1+\theta 7)^3}$$

$$\Rightarrow \frac{1}{8} + 6 = \frac{49}{(1+\theta 7)^3}$$

$$\Rightarrow \frac{49}{8} = \frac{49}{(1+\theta 7)^3}$$

$$\Rightarrow (1 + \theta 7)^3 = 8$$

$$\Rightarrow (1 + \theta 7)^3 = 2^3$$

$\Rightarrow 1 + \theta 7 = 2$

$\therefore \theta = \frac{1}{7}$ (Ans.)

OK 4. In the mean value theorem, $f(a+h) - f(a) = hf'(a + \theta h),\ 0 < \theta < 1$

$f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$ and a = 0, h = 3. Show that θ has got two values and find them.

Solution:

Given,

$f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$; a = 0, h = 3

$f'(x) = x^2 - 3x + 2$

$f(a) = 0$

$f(a+h) = \frac{3^3}{3} - \frac{3}{2}.3^2 + 2.3$

$= 9 - \frac{27}{2} + 6 = \frac{3}{2}$

Given equation is,

$f(a+h) - f(a) = hf'(a + \theta h)$

$\Rightarrow \frac{3}{2} - 0 = 3[(3\theta)^2 - 3.(3\theta) + 2$

$\Rightarrow \frac{3}{2} - 0 = 3(9\theta^2 - 9\theta + 2)$

$\Rightarrow \frac{1}{2} = 9\theta^2 - 9\theta + 2$

$\Rightarrow 9\theta^2 - 9\theta + \frac{3}{2} = 0$

$\Rightarrow \theta^2 - \theta + \frac{1}{6} = 0$

$\Rightarrow \theta = \frac{1}{6}(3 \pm \sqrt{3})$

Thus, θ has got two values.

(showed)

Maxima and Minima:

OK 1. Find for what value of x, the following expression is maximum and minimum respectively: $2x^3 - 21x^2 + 36x - 20$. Find also the maximum and minimum values of the expression.

Solution:

Let,

$f(x) = 2x^3 - 21x^2 + 36x - 20 \ldots\ldots\ldots\ldots\ldots$(i)

$\therefore f'(x) = 6x^2 - 42x + 36 \ldots\ldots\ldots\ldots\ldots$(ii) [Differentiating with respect to x]

Now, when (x) is a maximum or a minimum,

$f'(x) = 0$

Or, $6x^2 - 42x + 36 = 0$

Or, $x^2 - 7x + 6 = 0$

Or, $x^2 - 6x - x + 6 = 0$

Or, $x(x-6) - 1(x-6) = 0$

Or, $(x-1)(x-6) = 0$

$\therefore x = 1$ or 6

From (ii),

Again,

$f''(x) = 12x - 42 \ldots\ldots\ldots\ldots\ldots$(iii) [Differentiating with respect to x]

Now,

when, $x = 1$, $f''(x) = -30$, which is negative.

when, $x = 6$, $f''(x) = 30$, which is positive.

Hence, the given expression is maximum for x=1 and minimum for x=6.

The maximum and minimum values of the given expression are respectively,

For, $x = 1, f(1) = 2(1)^3 - 21(1)^2 + 36 \times 1 - 20 = -3$

For, $x = 6, f(6) = 2(6)^3 - 21(6)^2 + 36 \times 6 - 20 = -128$

(Ans.)

2. Investigate for what values of x, $f(x) = 5x^6 - 18x^5 + 15x^4 - 10$

Is a maximum or minimum.

<u>Solution:</u>

Given that,

$f(x) = 5x^6 - 18x^5 + 15x^4 - 10$ . . . . (i)

$\therefore f'(x) = 30x^5 - 90x^4 + 60x^3$ . . . . (ii) [Differentiating with respect to x]

When *f(x)* is a maximum or a minimum,

$f'(x) = 0$

Or, $30x^5 - 90x^4 + 60x^3 = 0$

Or, $30x^3(x^2 - 3x + 2) = 0$

Or, $x^3(x^2 - 2x - x + 2) = 0$

Or, $x^3\{x(x - 2) - 1(x - 2)\} = 0$

Or, $x^3(x - 1)(x - 2) = 0$

$\therefore\ x = 0,1\ or\ 2$

From (ii) again, differentiating with respect to x,

$f''(x) = 30(5x^4 - 12x^3 + 6x^2)$ . . . . (iii)

When, $x = 1, f''(x) = -30$ which is negative and hence $f(x)$ is a maximum value.

When, $x = 2, f''(x) = 240$ which is positive and hence $f(x)$ is a minimum value.

When, $x = 0$, $f''(x) = 0$, so the test fails and we have to examine higher order derivatives.

From(iii) again differentiating with respect to x,

$f'''(x) = 120(5x^3 - 9x^2 + 3x)$ . . . . (iv)

Now,

When, $x = 0$, $f'''(x) = 0$, again the test fails and we have to examine higher order derivatives.

From(iv), again differentiating with respect to x,

$f^{iv}(x) = 360(5x^2 - 6x + 1)$ . . . .(v)

Now,

When, $x = 0$, $f^{iv}(x) = 360$, which is positive and hence $f(x)$ is a minimum value.

Now,

For, $x = 0$, $f(x)$ is a minimum value.

For, $x = 1$, $f(x)$ is a maximum value.

For, $x = 2$, $f(x)$ is a minimum value.

(Ans.)

OK 3. Examine $f(x) = x^3 - 9x^2 + 24x - 12$ for maximum or minimum values.

Solution:

Given that,

$f(x) = x^3 - 9x^2 + 24x - 12$ . . . . (i)

$\therefore f'(x) = 3x^2 - 18x + 24$ . . . . (ii) [Differentiating with respect to x]

When *f(x)* is a maximum or a minimum,

$f'(x) = 0$

Or, $3x^2 - 18x + 24 = 0$

Or, $x^2 - 6x + 8 = 0$

Or, $x^2 - 4x - 2x + 8 = 0$

Or, $x(x-4) - 2(x-4) = 0$

Or, $(x-2)(x-4) = 0$

$\therefore x = 2$ or 4

From (ii), again differentiating with respect to x,

$f''(x) = 6x - 18$ . . . .(iii)

Now,

when, $x = 2$, $f''(x) = -6$, which is negative.

when, $x = 4$, $f''(x) = 6$, which is positive.

Hence, the given expression is maximum for $x = 2$ and minimum for $x = 4$.

The maximum and minimum values of the given expression are respectively,

For, $x = 2$, $f(2) = (2)^3 - 9(2)^2 + 24 \times 2 - 12 = 8$

For, $x = 4$, $f(4) = (4)^3 - 9(4)^2 + 24 \times 4 - 12 = 4$

(Ans.)

OK 4. Find the maxima and minima of $1 + 2\sin x + 3cos^2x$ $(0 \le x \le \frac{1}{2}\pi)$

Solution:

Let, $f(x) = 1 + 2sinx + 3cos^2x$ . . . .(i)

$\therefore f'(x) = 2\cos x - 6\sin x \cos x$ . . . .(ii) [Differentiating with respect to x]

When *f(x)* is a maximum or a minimum,

$f'(x) = 0$

Or, $2\cos x - 6\sin x \cos x = 0$

Or, $\cos x(1 - 3\sin x) = 0$

$\therefore \cos x = 0$ and $\sin x = \frac{1}{3}$

From(ii), again differentiating with respect to x,

$f''(x) = -2\sin x + 6(sin^2x - cos^2x)$ . . . .(iii)

When, $\cos x = 0$, then $x = \frac{\pi}{2}$

$f''(x) = 4$, which is positive.

When, $\sin x = \frac{1}{3}$

$f''(x) = -2\sin x + 6(2sin^2x - 1) = -\frac{2}{3} + 6\left(\frac{2}{9} - 1\right) = -\frac{2}{3} - \frac{14}{3} = -\frac{16}{3}$, which is negative.

Hence, the given expression is maximum for $\sin x = \frac{1}{3}$ and minimum for $\cos x = 0$

The maximum and minimum values of the given expression are respectively,

Now,

For, $\sin x = \frac{1}{3}, f(x) = 1 + 2\sin x + 3(1 - sin^2x) = 1 + \frac{2}{3} + 3(1 - \frac{1}{9}) = \frac{13}{3}$

For, $\cos x = 0$ which means, $= \frac{\pi}{2}, f(x) = 1 + 2 + 0 = 3$ (Ans.)

OK 5. Examine whether $x^{\frac{1}{x}}$ possesses a maximum or a minimum and determine the same.

Solution:

Let,

$y = x^{\frac{1}{x}}$

$\therefore \ln y = \frac{1}{x}\ln x$ . . . . (i)

Differentiating equation (i) with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2} - \frac{1}{x^2}\ln x$$

Or, $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2}(1 - \ln x)$. . . . . (ii)

$\therefore \frac{dy}{dx} = \frac{x^{\frac{1}{x}}}{x^2}(1 - \ln x)$

For maxima and minima $\frac{dy}{dx} = 0$, we have,

$\frac{x^{\frac{1}{x}}}{x^2}(1 - \ln x) = 0$

Or, $1 - \ln x = 0$

Or, $\ln x = 1$

Or, $\ln x = \ln e$

$\therefore x = e$

Again, differentiating equation (ii) with respect to x,

$-\frac{1}{y^2}(\frac{dy}{dx})^2 + \frac{1}{y}\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(-\frac{1}{x}\right) - \frac{2}{x^3}(1 - \ln x)$

Or, $-\frac{1}{y^2}(\frac{dy}{dx})^2 + \frac{1}{y}\frac{d^2y}{dx^2} = \frac{-3+2\ln x}{x^3}$

$\therefore \frac{d^2y}{dx^2} = x^{\frac{1}{x}}\frac{-3+2\ln x}{x^3}$ (for, $\frac{dy}{dx} = 0$)

When, $x = e$, $\frac{d^2y}{dx^2} = e^{\frac{1}{e}}\frac{-3+2}{e^3} = -\frac{e^{\frac{1}{e}}}{e^3}$, which is negative.

For, $x = e$, the function is maximum.

Now, the maximum value is $e^{\frac{1}{e}}$.

(Ans.)

OK 6. Find the maximum and minimum values of u where,

$u = \frac{4}{x} + \frac{36}{y}$ and $x + y = 2$

<u>Solution:</u>

Given that,

$$u = \frac{4}{x} + \frac{36}{y}$$

$x + y = 2$

Eliminating y between the two given relations,

$u = \frac{4}{x} + \frac{36}{2-x}$ . . . .(i)

Differentiating equation (i) with respect to x,

$\frac{du}{dx} = -\frac{4}{x^2} + \frac{36}{(2-x)^2}$ . . . . (ii)

Or, $\frac{du}{dx} = \frac{-4(2-x)^2+36x^2}{x^2(2-x)^2}$

$$\therefore \frac{du}{dx} = \frac{16(2x^2 + x - 1)}{x^2(2 - x)^2}$$

For maxima and minima $\frac{du}{dx} = 0$,

$\frac{16(2x^2+x-1)}{x^2(2-x)^2} = 0$

$2x^2 + x - 1$=0

Or, $2x^2 + 2x - x + 1$=0

Or, $2x(x + 1) - 1(x + 1)$=0

Or, $(x + 1)(2x - 1) = 0$ $\therefore x = -1$ or $\frac{1}{2}$

Again, differentiating equation (ii) with respect to x,

$\frac{d^2u}{dx^2} = \frac{8}{x^3} + \frac{72}{(2-x)^3}$

Now,

When,$x = -1$,

$\frac{d^2u}{dx^2} = \frac{8}{(-1)^3} + \frac{72}{(2+1)^3} = -8 + \frac{72}{27}$, which is negative.

When, $x = -\frac{1}{2}$,

$\frac{d^2u}{dx^2} = \frac{8}{(\frac{1}{2})^3} + \frac{72}{(2-\frac{1}{2})^3} = 64 + \frac{576}{27}$, which is positive.

Hence, the given expression is maximum for $x = -1$ and minimum for $x = \frac{1}{2}$.

The maximum and minimum values of the given expression are respectively,

For, $x = -1$,

Maximum value of $u = -4 + \frac{36}{2+1} = -4 + 12 = 8$

For, $x = \frac{1}{2}$,

Minimum value of $u = \frac{4}{\frac{1}{2}} + \frac{36}{2-\frac{1}{2}} = 8 + 24 = 32$ (Ans.)

OK 7. Show that the maximum value of $x + \frac{1}{x}$ is less than its minimum value.

<u>Solution:</u>

Let,

$y = x + \frac{1}{x}$..................(i)

Differentiating equation (i) with respect to x(2 times)

Or, $\frac{dy}{dx} = 1 - \frac{1}{x^2}$

Or, $\frac{d^2y}{dx^2} = \frac{2}{x^3}$

For maxima and minima $\frac{dy}{dx} = 0$,

$\therefore 1 - \frac{1}{x^2} = 0$

Or, x=1 or -1

when, x = 1, $\frac{d^2y}{dx^2} = \frac{2}{1} = 2$ which is positive

for x = 1, y is minimum.

$\therefore$minimum value of y = $1 + \frac{1}{2} = \frac{3}{2}$

when, $x = -1$, $\frac{d^2y}{dx^2} = -2$ which is negative.

for $x = -1$, y is a maximum

$\therefore$ maximum value of $y = -1 - \frac{1}{1} = -2$

$\therefore$ The maximum value of $x + \frac{1}{x}$ is less than its minimum value.

(showed).

OK 8. Show that the following function possess neither a maximum nor a minimum.

(i) $x^3 - 3x^2 + 6x + 3$ (ii) $x^3 - 3x^2 + 9x - 1$

(iii) $\sin(x + a) / \sin(x + b)$ (iv) $(ax + b)/(cx + d)$

Solution:

(i)Let, $x^3 - 3x^2 + 6x + 3 = f(x)$

Differentiating with respect to x(2 times),

$\therefore f'(x) = 3x^2 - 6x + 6$

$\therefore f''(x) = 6x - 6$

For maximum and minimum value,

f '(x) = 0

$\therefore 3x^2 - 6x + 6 = 0 \;\; or, \;\; x^2 - 2x + 2 = 0$

x = $\frac{-(-2)\pm\sqrt{(-2)^2-4.1.2}}{2.1}$

=$\frac{2\pm\sqrt{-4}}{2}$

we can see considering f '(x) = 0 x doesn't have any real value,

so, $x^3-3x+6x+3$ doesn't have maximum and minimum value.

(ii) Let, $x^3-3x^2+9x-1=f(x)$

Differentiating with respect to x,

∴ f '(x) = $3x^2-6x+9$

for maximum and minimum values,

f '(x) = 0

∴ $3x^2-6x+9=0$ or, $x^2-2x+3=0$

∴ $x=\frac{-(-2)\pm\sqrt{(-2)^2-4.1.3}}{2.1}=\frac{2\pm\sqrt{-8}}{2}$

we can see that, considering f '(x) = 0 x doesn't have any real value.

so, x^3-3x^2+9x-1 neither have a maximum nor a minimum value.

(iii) Let, $f(x)=\sin(x+a)/\sin(x+b)$

Differentiating with respect to x,

$$f'(x)=\frac{\sin(x+b)\cos(x+a)-\cos(x+b)\sin(x+a)}{\sin^2(x+b)}$$

$$=\frac{\sin(x+b-x-a)}{\sin^2(x+b)}$$

$$=\frac{\sin(b-a)}{\sin^2(x+b)}$$

for, maximum and minimum value,

f ‘(x) = 0

Or, $\frac{\sin(b-a)}{\sin^2(x+b)} = 0$

$\therefore \sin(b-a) = 0$

$\sin(x+a)/\sin(x+b)$ neither have a maximum nor a minimum value.

(iv) Let, $(ax+b)/(cx+d) = f(x)$

Differentiating with respect to x,

$$f'(x) = \frac{a(cx+d)-c(ax+b)}{(cx+d)^2}$$

$$= \frac{acx+ad-acx-bc}{(cx+d)^2}$$

$= \frac{ad-bc}{(cx+d)^2}$, that will not be zero for any real value of x.

so, $(ax+b)/(cx+d)$ neither have a maximum nor a minimum value.

OK 9. Show that $x^5 - 5x^4 + 5x^3 - 1$ is a maximum when x = 1, a minimum when x = 3; neither when x = 0.

Solution:

Let,

$$f(x) = x^5 - 5x^4 + 5x^3 - 1$$

Differentiating with respect to x,

$$f'(x) = 5x^4 - 20x^3 + 15x^2$$

for maximum and minimum value,

f ‘(x)= 0

$\therefore 5x^4 - 20x^3 + 15x^2 = 0$

Or, $5x^2(x^2 - 4x + 3) = 0$

$5x^2 = 0$ Or, $x^2 - 4x + 3 = 0$

$\therefore\ x = 0$ Or, $x^2 - 3x - x + 3 = 0$

Or, $(x - 3)(x - 1) = 0$

$\therefore$ x = 1, 3

Again, differentiating with respect to x,

$f''(x) = 20x^3 - 60x^2 + 30x$

when, x = 1,

$f''(1) = 20 \times 1^3 - 60 \times 1^2 + 30 \times 1$

$= 50 - 60$

$= -10 < 0$

So, we will get maximum value of f(x) at x = 1.

at x = 3,

$f''(3) = 20 \times 3^3 - 60 \times 3^2 + 30 \times 3$

$= 540 - 540 + 90$

$= 90 > 0$

We will get minimum value of f(x) at x = 3.

at x = 0,

$f''(0) = 20 \times 0^2 - 60 \times 0^2 + 30 \times 0 = 0$

So, test fails.

We have to examine high order derivatives,

$f'''(x) = 60x^2 - 120x + 30$

at x=0, $f'''(0) = 30 \neq 0$

Therefore, f(x) is neither a maximum or a minimum value when x = 0.

(showed)

Partial Differentiation:

OK 1. If $v = x^2 + y^2 + z^2$, then show that, $xv_x + yv_y + zv_z = 2v$.

Solution:

Given that,

$v = x^2 + y^2 + z^2$

L.H.S $= xv_x + yv_y + zv_z$

$= x\left\{\frac{\partial}{\partial x}(v)\right\} + y\left\{\frac{\partial}{\partial y}(v)\right\} + z\left\{\frac{\partial}{\partial z}(v)\right\}$

$= x\left\{\frac{\partial}{\partial x}(x^2 + y^2 + z^2)\right\} + y\left\{\frac{\partial}{\partial y}(x^2 + y^2 + z^2)\right\} + z\left\{\frac{\partial}{\partial z}(x^2 + y^2 + z^2)\right\}$

$= x\{2x + 0 + 0\} + y\{0 + 2y + 0\} + z\{0 + 0 + 2z\}$

$= 2x^2 + 2y^2 + 2z^2$

$= 2(x^2 + y^2 + z^2)$

$= 2v \qquad [v = x^2 + y^2 + z^2]$

= R.H.S

$\therefore$ L. H. S = R. H. S

(showed)

OK 2. If $u = sin^{-1}\frac{x}{y} + tan^{-1}\frac{y}{x}$ show that, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Solution:

Given that,

$u = sin^{-1}\frac{x}{y} + tan^{-1}\frac{y}{x}$

L.H.S $= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$= \mathrm{x}\frac{\partial}{\partial \mathrm{x}}\left(\sin^{-1}\frac{\mathrm{x}}{\mathrm{y}} + \tan^{-1}\frac{\mathrm{y}}{\mathrm{x}}\right) + \mathrm{y}\frac{\partial}{\partial \mathrm{y}}\left(\sin^{-1}\frac{\mathrm{x}}{\mathrm{y}} + \tan^{-1}\frac{\mathrm{y}}{\mathrm{x}}\right)$$

$$= \mathrm{x}\left\{\frac{1}{\sqrt{\left(1-\frac{\mathrm{x}^2}{\mathrm{y}^2}\right)}}.\frac{1}{\mathrm{y}} + \frac{1}{\left(1+\frac{\mathrm{y}^2}{\mathrm{x}^2}\right)}.\left(\frac{-\mathrm{y}}{\mathrm{x}^2}\right)\right\} + \mathrm{y}\left\{\frac{1}{\sqrt{\left(1-\frac{\mathrm{x}^2}{\mathrm{y}^2}\right)}}.\left(\frac{-\mathrm{x}}{\mathrm{y}^2}\right) + \frac{1}{\left(1+\frac{\mathrm{y}^2}{\mathrm{x}^2}\right)}.\frac{1}{\mathrm{x}}\right\}$$

$$= \mathrm{x}\left\{\frac{1}{\mathrm{y}}.\frac{\mathrm{y}}{\sqrt{(\mathrm{y}^2-\mathrm{x}^2)}}.\left(\frac{-\mathrm{y}}{\mathrm{x}^2+\mathrm{y}^2}\right)\right\} + \mathrm{y}\left\{\frac{1}{\mathrm{x}}.\frac{\mathrm{x}^2}{\mathrm{x}^2+\mathrm{y}^2} - \frac{\mathrm{xy}}{\mathrm{y}^2\sqrt{(\mathrm{y}^2-\mathrm{x}^2)}}\right\}$$

$$= \frac{\mathrm{x}}{\sqrt{(\mathrm{y}^2-\mathrm{x}^2)}} - \frac{\mathrm{xy}}{\mathrm{x}^2+\mathrm{y}^2} + \frac{\mathrm{xy}}{\mathrm{x}^2+\mathrm{y}^2} - \frac{\mathrm{x}}{\sqrt{(\mathrm{y}^2-\mathrm{x}^2)}}$$

$= 0$

= R.H.S

$\therefore$ L. H. S = R. H. S

(showed)

OK 3. Show that $\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{u}}{\partial \mathrm{y}^2} = 0$, if $\mathrm{u} = \log(\mathrm{x}^2 + \mathrm{y}^2)$

Solution:

Given that,

$\mathrm{u} = \log(\mathrm{x}^2 + \mathrm{y}^2)$

Partially differentiating u with respect to y (2 times),

$$\frac{\partial \mathrm{u}}{\partial \mathrm{x}} = \frac{\partial}{\partial \mathrm{x}}\{\log(\mathrm{x}^2 + \mathrm{y}^2)\} = \frac{2\mathrm{x}}{\mathrm{x}^2+\mathrm{y}^2}$$

$$\Rightarrow \frac{\partial}{\partial \mathrm{x}}\left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\right) = \frac{\partial}{\partial \mathrm{x}}\left(\frac{2\mathrm{x}}{\mathrm{x}^2+\mathrm{y}^2}\right)$$

$$= \frac{\mathrm{x}^2+\mathrm{y}^2.\frac{\partial}{\partial \mathrm{x}}(2\mathrm{x}) - 2\mathrm{x}.\frac{\partial}{\partial \mathrm{x}}(\mathrm{x}^2+\mathrm{y}^2)}{(\mathrm{x}^2+\mathrm{y}^2)^2}$$

$$= \frac{2(x^2+y^2)-2x.2x}{(x^2+y^2)^2}$$

$$= \frac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2}$$

Similarly, partially differentiating u with respect to y (2 times),

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\{\log(x^2+y^2)\} = \frac{2y}{x^2+y^2}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right) = \frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2}$$

L. H. S. $= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$$= \frac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2} + \frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2}$$

$$= \frac{4x^2-4x^2+4y^2-4y^2}{(x^2+y^2)^2}$$

$$= 0$$

= R.H.S

$\therefore \mathrm{L.H.S} = \mathrm{R.H.S}$

(showed)

OK 4. Show that, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, if $u = \tan^{-1}\left(\frac{y}{x}\right)$

Solution:

Given that,

$u = \tan^{-1}\left(\frac{y}{x}\right)$

Partially differentiating u with respect to y (2 times),

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot (-yx^{-2})$$

$$= -\frac{y}{x^2+y^2}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right)$$

$$= -\left\{\frac{x^2+y^{2.0} - y.\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)}\right\}$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

Similarly, partially differentiating u with respect to y (2 times),

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

$$= \frac{x}{x^2+y^2}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{-2xy}{(x^2+y^2\ )^2}$$

L. H. S. $= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$$= \frac{2xy}{(x^2+y^2)^2} + \left(\frac{-2xy}{(x^2+y^2)^2}\right)$$

$$= 0$$

$$= \mathrm{R.H.S}$$

$\therefore \mathrm{L.H.S} = \mathrm{R.H.S}$

(showed)

5. If, $u = \log(x^3 + y^3 + z^3 - 3xyz)$ then showed that,

OK (i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

(ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{3}{(x+y+z)^2}$

(iii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

Solution:

(i) Given that,

$u = \log(x^3 + y^3 + z^3 - 3xyz)$

Partially differentiating u with respect to x, y, z respectively,

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$$

$$\text{L. H. S.} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$= \frac{3}{x+y+z}$$

$$= \text{R. H. S}$$

$\therefore$ L. H. S = R. H. S

(showed)

(ii)Let,

$$x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega^4 z)$$

Where ω is the imaginary root.

$$u=\log(x^3+y^3+z^3-3xyz)$$

$$\therefore u=\log(x+y+z)+\log(x+\omega y+\omega^2 z)+\log(x+\omega^2 y+\omega^4 z)$$

Partially differentiating u with respect to x, y and z respectively,

$$\frac{\partial u}{\partial x}=\frac{1}{x+y+z}+\frac{1}{x+\omega y+\omega^2 z}+\frac{1}{x+\omega^2 y+\omega^4 z} \;\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{\partial u}{\partial y}=\frac{1}{x+y+z}+\frac{\omega}{x+\omega y+\omega^2 z}+\frac{\omega^2}{x+\omega^2 y+\omega^4 z} \ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial u}{\partial z}=\frac{1}{x+y+z}+\frac{\omega^2}{x+\omega y+\omega^2 z}+\frac{\omega^4}{x+\omega^2 y+\omega^4 z} \ldots\ldots\ldots\ldots\ldots (3)$$

Partially differentiating equation 1, 2 and 3 with respect to x, y and z respectively,

$$\frac{\partial^2 u}{\partial x^2}=\frac{-1}{x+y+z}+\frac{-1}{x+\omega y+\omega^2 z}+\frac{-1}{x+\omega^2 y+\omega^4 z}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{-1}{x+y+z}+\frac{-\omega}{x+\omega y+\omega^2 z}+\frac{-\omega^2}{x+\omega^2 y+\omega^4 z}$$

$$\frac{\partial^2 u}{\partial z^2}=\frac{-1}{x+y+z}+\frac{-\omega^2}{x+\omega y+\omega^2 z}+\frac{-\omega^4}{x+\omega^2 y+\omega^4 z}$$

$$\text{L. H. S.}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

$$=\frac{3}{(x+y+z)^2}+\frac{1+\omega^2+\omega^4}{(x+\omega y+\omega^2 z)}-\frac{1+\omega^2+\omega^4}{(x+\omega y+\omega^2 z)}$$

$$=\frac{3}{(x+y+z)^2}$$

$$=\mathrm{R.H.S}$$

$$\therefore \mathrm{L.H.S}=\mathrm{R.H.S}$$

(showed)

(iii) Given that,

$$u=\log(x^3+y^3+z^3-3xyz)$$

Differentiating u with respect to x, y and z respectively,

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$= \frac{3}{x+y+z}$$

$$\text{L. H. S.} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right).\frac{3}{x+y+z}$$

$$= 3.\frac{\partial}{\partial x}(x+y+z)^{-1} + 3\frac{\partial}{\partial y}(x+y+z)^{-1} + 3\frac{\partial}{\partial z}(x+y+z)^{-1}$$

$$= -3(x+y+z)^{-2} - 3(x+y+z)^{-2} - 3(x+y+z)^{-2}$$

$$= \frac{-9}{(x+y+z)^2}$$

$$= \text{R. H. S}$$

$$\therefore \text{L. H. S} = \text{R. H. S}$$

(showed)

OK 6. If, $v = \sqrt{(x^2 + y^2 + z^2)}$, then show that, $v_{xx} + v_{yy} + v_{zz} = \frac{2}{v}$

Solution:

Given that,

$$v = \sqrt{(x^2 + y^2 + z^2)} \dots \dots \dots \dots \dots (1)$$

Partially differentiating equation (1) with respect to x (2 times)

$$\frac{\partial v}{\partial x} = \frac{2x}{2\sqrt{(x^2+y^2+z^2)}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{x}{\sqrt{(x^2+y^2+z^2)}}$$

$$= \frac{\sqrt{(x^2+y^2+z^2)} - \frac{2.x.x}{\sqrt{(x^2+y^2+z^2)}}}{\left(\sqrt{(x^2+y^2+z^2)}\right)^2}$$

$$= \frac{x^2+y^2+z^2-x^2}{\sqrt{(x^2+y^2+z^2)}.(x^2+y^2+z^2)}$$

$$\therefore v_{xx} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

Similarly, by partially differentiating 1 with respect to y and z respectively,

$$v_{yy} = \frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$v_{zz} = \frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\text{L. H. S.} = v_{xx} + v_{yy} + v_{zz}$$

$$= \frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$= \frac{2}{\sqrt{(x^2+y^2+z^2)}}$$

$$= \frac{2}{v} \qquad [\because v = \sqrt{(x^2 + y^2 + z^2)}]$$

$$= R.H.S$$

$\therefore L.H.S = R.H.S$

(showed)

OK 7. If, $v = \frac{1}{\sqrt{(x^2+y^2+z^2)}}$, then show that, $v_{xx} + v_{yy} + v_{zz} = 0$

<u>Solution:</u>

Given that,

$$v = \frac{1}{\sqrt{(x^2+y^2+z^2)}}$$

$$\Rightarrow v = (x^2 + y^2 + z^2)^{\frac{-1}{2}}$$

Partially differentiating v with respect to x (2 times)

$$\frac{\partial v}{\partial x} = \frac{-1}{2} \ (x^2 + y^2 + z^2)^{\frac{3}{2}}.2x = -x(x^2 + y^2 + z^2)^{\frac{3}{2}}$$

$$\frac{\partial^2 v}{\partial x^2} = -\left[x\left\{\frac{-3}{2}(x^2 + y^2 + z^2)^{\frac{-5}{2}}.2x\right\} + (x^2 + y^2 + z^2)^{\frac{-3}{2}}.(-1)\right]$$

$$= 3x^2(x^2 + y^2 + z^2)^{\frac{-5}{2}} - (x^2 + y^2 + z^2)^{\frac{-3}{2}}$$

$$v_{xx} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Similarly, differentiating v with respect to y and z (2 times) respectively,

$$v_{yy} = \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$v_{zz} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$\text{L. H. S.} = v_{xx} + v_{yy} + v_{zz}$

$= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$

$= \frac{2(x^2+y^2+z^2)-2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$

$= \frac{0}{(x^2+y^2+z^2)^{\frac{5}{2}}}$

$= 0$

$= \text{R. H. S}$

$\therefore \text{L. H. S} = \text{R. H. S}$

(showed)

8. If, $u = e^{xyz}$, then prove that, $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$

Solution:

Given that,

$u = e^{xyz}$

Partially differentiating u with respect to z,

$\frac{\partial u}{\partial z} = xy.\, e^{xyz}$ … … … … (1)

Partially differentiating equation 1 with respect to y,

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \{xy.\,(xz.\,e^{xyz}) + e^{xyz}.\,x\}$

$\therefore \frac{\partial^2 u}{\partial y\,\partial z} = e^{xyz}(x^2yz + x)$ … … … … (2)

Partially differentiating equation 2 with respect to x,

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\,\partial z}\right) = e^{xyz}.(2xyz + 1) + (x^2yz + x)(yz.e^{xyz})$$

$$\therefore \frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = 2xyz.e^{xyz} + e^{xyz} + x^2y^2z^2.e^{xyz} + xyz.e^{xyz}$$

$$\text{L. H. S.} = \frac{\partial^3 u}{\partial x\,\partial y\,\partial z}$$

$$= 2xyz.e^{xyz} + e^{xyz} + x^2y^2z^2.e^{xyz} + xyz.e^{xyz}$$

$$= (1 + 3xyz + x^2y^2z^2)e^{xyz}$$

$$= \text{R.H.S}$$

$$\therefore \text{L.H.S} = \text{R.H.S}$$

(showed)

OK 9. If, $u = \log r$ and $r^2 = x^2 + y^2 + z^2$, prove that,

$$r^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2\,u}{\partial y^2} + \partial^2\,\frac{u}{\partial z^2}\right) = 1$$

<u>Solution:</u>

Given that,

$$r^2 = x^2 + y^2 + z^2$$

$$\Rightarrow r = \sqrt{x^2 + \ y^2 + \ z^2}$$

$$\Rightarrow r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

Again, given that,

$$u = \log r$$

$$= \log(x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}\log(x^2 + y^2 + z^2)$$

Now,

$$\text{L.H.S} = r^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$= r^2\left\{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)\right\}$$

$$= r^2\left[\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\left[\frac{1}{2}\log(x^2 + y^2 + z^2)\right]\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}\left[\frac{1}{2}\log(x^2 + y^2 + z^2)\right]\right\} + \frac{\partial}{\partial z}\left\{\frac{\partial}{\partial z}\left[\frac{1}{2}\log(x^2 + y^2 + z^2)\right]\right\}\right]$$

$$= r^2\left\{\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2+z^2}.2x\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2+z^2}.2y\right) + \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{1}{x^2+y^2+z^2}.2z\right)\right\}$$

$$= r^2\left\{\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right)\right\}$$

$$= r^2\left[\frac{(x^2+y^2+z^2)\frac{\partial}{\partial x}(x) - x\frac{\partial}{\partial x}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2+z^2)\frac{\partial}{\partial y}(y) - y\frac{\partial}{\partial y}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2+z^2)\frac{\partial}{\partial z}(z) - z\frac{\partial}{\partial z}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}\right]$$

$$= r^2\left\{\frac{x^2+y^2+z^2-(x.2x)}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2-(y.2y)}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2-(z.2z)}{(x^2+y^2+z^2)^2}\right\}$$

$$= r^2\left(\frac{x^2+y^2+z^2-2x^2+x^2+y^2+z^2-2y^2+x^2+y^2+z^2-2z^2}{(x^2+y^2+z^2)^2}\right)$$

$$= r^2\left(\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2}\right)$$

$$= r^2 \times \frac{1}{x^2+y^2+z^2}$$

$$= x^2 + y^2 + z^2 \times \frac{1}{x^2+y^2+z^2}$$

$$= 1$$

$$= \text{R.H.S}$$

$\therefore$ L. H. S = R. H. S

(proved)

OK 10. If, $u = \log r$ and $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, then prove that,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2}$.

<u>Solution:</u>

Given that,

$u = \log r$

$$= \log(\sum(x-a)^2)^{\frac{1}{2}} = \frac{1}{2}\log(\sum(x-a)^2)$$

Partially differentiating u with respect to x,

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{2(x-a)}{\sum(x-a)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\sum(x-a)^2 - (x-a).2.(x-a)}{(\sum(x-a)^2)^2}$$

$$= \frac{\sum(x-a)^2 - 2(x-a)^2}{(\sum(x-a)^2)^2}$$

Similarly, partially differentiating u with respect to y, z respectively (2 times)

$$\frac{\partial^2 u}{\partial y^2} = \frac{\sum(x-a)^2 - 2(y-b)^2}{(\sum(x-a)^2)^2}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\sum(x-a)^2 - 2(z-c)^2}{(\sum(x-a)^2)^2}$$

L. H. S. $= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

$$= \frac{\sum(x-a)^2 - 2(x-a)^2}{(\sum(x-a)^2)^2} + \frac{\sum(x-a)^2 - 2(y-b)^2}{(\sum(x-a)^2)^2} + \frac{\sum(x-a)^2 - 2(z-c)^2}{(\sum(x-a)^2)^2}$$

$$= \frac{\sum(x-a)^2 - 2(x-a)^2 + \sum(x-a)^2 - 2(y-b)^2 + \sum(x-a)^2 - 2(z-c)^2}{(\sum(x-a)^2)^2}$$

$$= \frac{(x-a)^2 + (y-b)^2 + (z-c)^2 - 2(x-a)^2 + (x-a)^2 + (y-b)^2 + (z-c)^2 - 2(y-b)^2 + (x-a)^2 + (y-b)^2 + (z-c)^2 - 2(z-c)^2}{((x-a)^2 + (y-b)^2 + (z-c)^2)^2}$$

$$= \frac{(x-a)^2+(y-b)^2+(z-c)^2}{((x-a)^2+(y-b)^2+(z-c)^2)^2}$$

$$= \frac{1}{r^2} = \mathrm{R.H.S}$$ (proved)