Notes on Jointly Convex Nash Equilibrium Problems

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Game setup. Consider a game with N players. Each player i controls variables $x_i \in \mathbf{R}^{n_i}$. The vector formed by concatenating all decision variables is denoted by $x = (x_{1,} \ldots, x_N)$, where $n = \sum_{i=1}^{N} n_i$. The vector formed by the decision variables of all players except player i is denoted by $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. So we have $x = \text{rearrange}(x_i, x_{-i})$, for convenience we will drop the rearrange term and write just $x = (x_i, x_{-i})$. *The feasible set of the game* is denoted by X. The feasible set for player i, given other players strategies x_{-i}, is denoted by $X_i(x_{-i})$; we call such a feasible set *individual feasible set.* The optimization problem for player i is given by

$$\begin{array}{ll} \text{minimize}_{x_i} & f_i(x_i, x_{-i}) \\ \text{subject to} & x_i \in X_i(x_{-i}) \\ & x_i \in \mathbf{R}^{n_i}. \end{array} \tag{1}$$

The set of all solutions of this optimization problem is denoted by $S_i(x_{-i})$.

Generalized Nash Equilibrium Problem (GNEP). The GNEP is the following search problem.

$$\begin{array}{ll} \text{find} & x^* \in \mathbf{R}^n \\ \text{such that} & \forall i \in \{1, \ldots, N\} \quad \forall x_i \in X_i(x^*_{-i}) \quad f_i(x^*_i, x^*_{-i}) \leq f_i(x_i, x^*_i) \\ & x \in \mathbf{R}^n. \end{array}$$

Such a solution of the problem above, denoted by x^* is called the *Generalized Nash Equilibrium (GNE)* of the game. We can also define GNEP as the following search problem.

$$\begin{array}{ll} \text{find} & x^* \in \mathbf{R}^n \\ \text{such that} & x^*_i \in S_i(x^*_{-i}), \quad i = 1, \ldots, N^{\cdot} \end{array}$$

Now we we define the most well studied class of GNEP problems: jointly convex GNEP.

Jointly Convex GNEP. A GNEP is called *jointly convex* if the feasible set of the game is closed and convex, and its relationship with the individual feasible sets is given by $\forall i \in \{1,\ldots,N\}$ $X_i(x_{-i}) = \{x_i \in \mathbf{R}^{n_i} \mid (x_i, x_{-i}) \in X\}$. Intuitively, it means that the individual feasible set for any player is associated with the same feasible set (feasible set of the game). In this case the optimization problem for the ith player can be written as:

$$\begin{array}{ll} \text{minimize}_{x_i} & f_i(x_i, x_{-i}) \\ \text{subject to} & (x_i, x_{-i}) \in X \\ & x_i \in \mathbf{R}^{n_i}. \end{array} \tag{2}$$

Recall, X is a closed convex set. So, without loss of generality, we write X as a conjunction of inequalities:

$$X = \{x \in \mathbf{R}^n \mid \forall j \in \{1,\ldots,m\}\ g_j(x) \leq 0\}, \tag{3}$$

where $g_j(x)$ is a convex function in x for all $j = 1,\ldots,m$. Let us rewrite the optimization problem for ith player:

$$\begin{array}{ll} \text{minimize}_{x_i} & f_i(x_i, x_{-i}) \\ \text{subject to} & g_j(x) \leq 0, \quad j = 1,\ldots,m \\ & x_i \in \mathbf{R}^{n_i}. \end{array} \tag{4}$$

The pseudo-gradient of the game is defined as

$$F(x) = \begin{bmatrix} \nabla_{x_1} f_1(x_1, x_{-1}) \\ \nabla_{x_2} f_2(x_2, x_{-2}) \\ \vdots \\ \nabla_{x_N} f_N(x_N, x_{-1}) \end{bmatrix}. \tag{5}$$

Importance of jointly convex GNEP. Why is jointly convex GNEP an important problem? Because we can solve them! More precisely, if a GNEP problem is jointly convex, then we can solve it by solving a variational inequality (VI) problem.

Variational Inequality. Consider the relation $F : \mathbf{R}^n \to \mathbf{R}^n$, and some constraint set K. The variational inequality problem, denoted by $\mathrm{VI}(K, F)$ is as follows.

$$\begin{array}{ll} \text{find} & x \\ \text{such that} & \forall y \in K \quad F(x)^T(y - x) \geq 0. \end{array}$$

Variational inequality problems arises in many branches of science and engineering. If F is monotone and K is convex, then a variational inequality problem can be solved using monotone operator methods.

Theorem. ***Jointly convex GNEP solving via VI.*** *Consider the jointly convex GNEP described above. If the following assumptions hold for this GNEP:*

- $\forall i \in \{1,\ldots,N\}$ $f_i(x)$ *: continuously differentiable on* X `\* the individual cost functions are continuously differentiable *\`

• $\forall i \in \{1,\dots,N\} \quad \forall x_i : (\exists \bar{x}_{-i}(x_i,\bar{x}_{-i}) \in X) \quad (f_i(x_i,x_{-i})$ *: convex in* $x_i)$, `\* the cost function of a player is convex in his feasible actions *\`

• $\forall j \in \{1,\dots,m\}$ $g_j(x)$ *: continuously differentiable in* x `\* the constraints are continuously differentiable *\`

• $\forall j \in \{1,\dots,m\} \quad \forall i \in \{1,\dots,N\} \quad \forall x_i : (\exists \bar{x}_{-i}(x_i,\bar{x}_{-i}) \in X)$
$(g_j(x_i,x_{-i})$ *: convex in* $x_i)$ `\*For any player all constraints are convex in his feasible actions*\`

Then any solution of the variational inequality problem VI(X,F) *will be a GNE, where* X *is given by Equation 3 and* F *is given by Equation 5.*

It should be noted that, the other direction is not true, i.e., a GNE is not necessarily a solution of VI(X,F). So by solving VI(X,F), we can hope to recover only some of the GNEs. This motivates us to define *variational GNE*, i.e., those GNEs which we can find by solving the VI(X,F). Before proceeding further, let us recall the KKT conditions for a convex optimization problem.

KKT Conditions. Consider the convex optimization problem:

$$\begin{array}{ll} \text{minimize}_x & f_o(x) \\ \text{subject to} & g_j(x) \leq 0, \quad j = 1,\dots,m \\ & x \in \mathbf{R}^n, \end{array}$$

where $f_0, g_1, \dots, g_m$ are differentiable convex functions. The primal dual optimal pair (x^*,λ^*) will satisfy the following KKT conditions:

• $\nabla f_o(x) + \sum_{j=1}^{m} \lambda_j^* \nabla g_j(x^*) = 0$ `\* Vanishing gradient of the lagrangian*\`
• $\forall j \in \{1,\dots,m\} \quad g_j(x^*) \leq 0$ `\* Primal feasibility *\`
• $\forall j \in \{1,\dots,m\} \quad \lambda_j^* \geq 0$ `\*   Dual feasibility *\`
• $\forall j \in \{1,\dots,m\} \quad \lambda_j^* g_j(x^*) = 0$ `\* Complementary slackness *\`

Characterization of variational GNE. Suppose in ith player's optimization problem, the KKT conditions hold and the dual multipliers are unique, which can happen if the dual function is a strongly concave function. If x^* is a solution of the jointly convex GNEP, then for all $i = 1,\dots,N$, strategy x_i^* must be an optimal solution to ith player's optimization problem. Again, we have assumed that the associated dual multiplier $\lambda_i^* = (\lambda_{ij}^*)_{j=1}^m \in \mathbf{R}^m$ is unique. So for ith player's optimization problem we have the KKT conditions:

$$\begin{aligned} \nabla_{x_i} f_i(x_i^*, x_{-i}^*) + \sum_{j=1}^{m} \lambda_{ij}^* \nabla_{x_i} g_j(x_i^*, x_{-i}^*) &= 0 \\ \forall j \in \{1,\dots,m\} \quad g_j(x^*) &\leq 0 \\ \forall j \in \{1,\dots,m\} \quad \lambda_{ij}^* &\geq 0 \\ \forall j \in \{1,\dots,m\} \quad \lambda_{ij}^* g_j(x^*) &= 0 \end{aligned}$$

The dimension of the dual multipliers is same for all the players' optimization problem, however in generally they are not necessarily the same. For example, the Lagrangian for ith player is $L_i(x_i,\lambda_i) = f_i(x_i,x_{-i}) + \sum_{j=1}^{m} \lambda_{ij} g_j(x_i,x_{-i})$.

The dual function for the ith player is $d_i(\lambda_i) = \inf_x L_i(x_i, \lambda_i)$, and his dual problem is maximize$_{\lambda_i \succeq 0} d_i(\lambda_i)$. Clearly, the dual problem for each player is different, so different optimization problem will have different dual multipliers. However, it turns out that when the dual multipliers are the same for all the players, only then the vector $x^* = (x_1^*, \ldots, x_N^*)$ formed by concatenating the solution of the individual optimization problems $x_1^*, \ldots, x_N^*$, will be a variational GNE.