Part 1: Charecterization

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Proposition 25.1. $[\![A:\mathcal{H}\to 2^{\mathcal{H}},\ B:\mathcal{H}\to 2^{\mathcal{H}},\ \gamma\in\mathbb{R}_{++}]\!] \Rightarrow$

(i) $\operatorname{zer}(A+B)=\operatorname{dom}(A\cap(-B))$

(ii) A,B: monotone $\Rightarrow \operatorname{zer}(A+B)=J_{\gamma B}(\operatorname{Fix} R_{\gamma A} R_{\gamma B})$

(iii) (C: closed affine subspace of $\mathcal{H}$; $V:=C-C$; $A=N_C$) $\Rightarrow$

$\operatorname{zer}(A+B)=\{x\in C \mid V^{\perp}\cap Bx\neq\emptyset\}$

(iv) (A: monotone , B: almost single-valued) $\Rightarrow \operatorname{zer}(A+B)=\operatorname{Fix} J_{\gamma A}\circ(\mathrm{Id}-\gamma B)$

Proof:

(i) $\forall x\in\mathcal{H}$

$x\in\operatorname{zer}(A+B) \Leftrightarrow 0\in(A+B)x=Ax+Bx \Leftrightarrow \exists_{\underset{\exists_u u\in Ax}{u\in Ax}}\ u+Bx\ni 0 \Leftrightarrow \exists_{u\in Ax}\ \underset{Bx\ni -u}{\overset{(-B)x\ni u}{\updownarrow}} \Leftrightarrow \exists_u\ u\in Ax \wedge u\in -Bx$

$\Leftrightarrow \exists_u\ u\in(Ax)\cap((-B)x)=(A\cap(-B))(x)$

$\therefore \operatorname{zer}(A+B)=\operatorname{dom}(A\cap(-B))$ $\qquad \Leftrightarrow x\in\operatorname{dom}(A\cap(-B))$

(ii) $\forall x\in\mathcal{H}$

$x\in\operatorname{zer}(A+B) \Leftrightarrow 0\in(A+B)x=Ax+Bx \Leftrightarrow \gamma Ax+\gamma Bx\ni 0 \Leftrightarrow \exists_{u\in\mathcal{H}}\ u\in\gamma Ax,\ -u\in\gamma Bx$ /* let, $x-u=\overset{\exists y}{y} \Leftrightarrow u=x-y$ */

$\Leftrightarrow \exists_{y\in\mathcal{H}}\ x-y\in\gamma Ax,\ \boxed{y-x\in\gamma B x}$

$\updownarrow$

$x\in J_{\gamma B}y$

$\updownarrow$

$x=J_{\gamma B}y$

Proposition 23.2 · Basic relationships between A, ${}^{\gamma}A$, $J_{\gamma A}$

(ii) $p\in J_{\gamma A}x \Leftrightarrow x\in p+\gamma Ap \Leftrightarrow x-p\in\gamma Ap$

/* $\because B$: monotone $\Rightarrow J_{\gamma B}$: firmly nonexpansive $\to$ single-valued */

$\Leftrightarrow \exists_{y\in\mathcal{H}}\ \boxed{\underbrace{x+x-y}_{2x-y}\in\underbrace{x+\gamma Ax}_{(\mathrm{Id}+\gamma A)x}},\ x=J_{\gamma B}y$

/* $2x=2J_{\gamma B}y \Leftrightarrow 2x-y=2J_{\gamma B}y-y=\underbrace{(2J_{\gamma B}-\mathrm{Id})}_{R_{\gamma B}\ //\text{By definition}}y=R_{\gamma B}y$ */

$2x-y\in(\mathrm{Id}+\gamma A)x$

$\Leftrightarrow \underbrace{(\mathrm{Id}+\gamma A)^{-1}}_{J_{\gamma A}\ //\text{By definition}}(2x-y)\ni x \Leftrightarrow J_{\gamma A}(\underbrace{2x-y}_{R_{\gamma B}y})=x \Leftrightarrow J_{\gamma A}(R_{\gamma B}y)=x$

$\Leftrightarrow \exists_{y\in\mathcal{H}}\ x=J_{\gamma A}(R_{\gamma B}y)\ \wedge\ x=J_{\gamma B}y\ \wedge\ 2x-y=R_{\gamma B}y$

$2J_{\gamma A}(R_{\gamma B}y)-y=R_{\gamma B}y$

$\Leftrightarrow\ y=2J_{\gamma A}(R_{\gamma B}y)-R_{\gamma B}y=\underbrace{(2J_{\gamma A}-\mathrm{Id})}_{R_{\gamma A}}(R_{\gamma B}y)=R_{\gamma A}R_{\gamma B}y$ /* By definition */

$\Leftrightarrow \exists_{y\in\mathcal{H}}\ y=R_{\gamma A}R_{\gamma B}y\ \wedge\ x=J_{\gamma B}y$

$\Leftrightarrow \exists_{y\in\mathcal{H}}\ y\in\operatorname{Fix} R_{\gamma A}R_{\gamma B},\ x=J_{\gamma B}y$

$\Leftrightarrow \exists_{y\in\operatorname{Fix} R_{\gamma A}R_{\gamma B}}\ x=J_{\gamma B}y \Leftrightarrow x\in J_{\gamma B}(\operatorname{Fix} R_{\gamma A}R_{\gamma B})$

$\therefore \operatorname{zer}(A+B)=J_{\gamma B}(\operatorname{Fix} R_{\gamma A}R_{\gamma B})$

(iii) $\forall x\in\mathcal{H}$

$x\in\operatorname{zer}(\underset{N_C}{A}+B)=\operatorname{zer}(N_C+B) \Leftrightarrow (N_C+B)x=\underset{\exists_u\ \ni -u}{N_Cx}+\underset{\ni u}{Bx}\ni 0 \Leftrightarrow \exists_{u\in Bx}\ (-u\in N_Cx=V^{\perp}\ \wedge\ x\in C)$ /*:

Example 6.42 · [C: affine subspace of $\mathcal{H}$; $V=C-C$; $x\in\mathcal{H}$] $\Rightarrow$

$$T_Cx=\begin{cases}\overline{V}, & \text{if } x\in C\\ \emptyset, & \text{otherwise}\end{cases}$$

$$N_Cx=\begin{cases}V^{\perp}, & \text{if } x\in C\\ \emptyset, & \text{otherwise}\end{cases}$$ */

this is not possible as $N_Cx\ni -u$

$\Leftrightarrow (x\in C\ \wedge\ \exists_{u\in Bx}\ u\in V^{\perp})$ // as $V^{\perp}$ linear $u,-u\in V^{\perp}$

$\therefore \operatorname{zer}(A+B)=\{x\in C \mid V^{\perp}\cap Bx\neq\emptyset\}$

(iv) $\forall x\in\mathcal{H}$

$x\in\operatorname{zer}(A+B)$

$\Leftrightarrow (A+B)x=Ax+Bx\ni 0$

$\Leftrightarrow Ax\ni -Bx$ /* as B: single valued $Bx\ni Ax$ not possible for general A */

$\Leftrightarrow \gamma Ax\ni\gamma Bx \Leftrightarrow \underbrace{x-\gamma Bx}_{(\mathrm{Id}-\gamma B)x}\in\underbrace{x+\gamma Ax}_{(\mathrm{Id}+\gamma A)x} \Leftrightarrow \underbrace{\underbrace{(\mathrm{Id}+\gamma A)^{-1}}_{J_{\gamma A}}\underbrace{(\mathrm{Id}-\gamma B)}_{\text{single-valued as } B \text{ single-valued}}}_{\text{single-valued}}x\ni x \Leftrightarrow J_{\gamma A}(\mathrm{Id}-\gamma B)x=x \Leftrightarrow x\in\operatorname{Fix}(J_{\gamma A}(\mathrm{Id}-\gamma B))$

$\therefore \operatorname{zer}(A+B)=(J_{\gamma A}\circ(\mathrm{Id}-\gamma B))$

■

Proposition 25.2. [$A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone ; $B:\mathcal{H}\to\mathcal{H}$; $\gamma\in\mathbb{R}_{++}$] $\operatorname{Fix} J_A\circ(\mathrm{Id}+\gamma(B-\mathrm{Id}))=\operatorname{Fix} J_{\gamma^{-1}A}\circ B$

Proof:

$\forall x\in\mathcal{H}$

$x\in\operatorname{Fix} J_{\gamma^{-1}A}\circ B$

$\Leftrightarrow (J_{\gamma^{-1}A}\circ B)x=\underbrace{J_{\gamma^{-1}A}}_{(\mathrm{Id}+\gamma A)^{-1}\ //\text{By definition}}(Bx)\ni x \Leftrightarrow \overbrace{J_{\gamma^{-1}A}\overbrace{(Bx)}^{\text{single-valued}}}^{\text{single-valued as } A\text{: maximally monotone}}=x \Leftrightarrow \underbrace{Bx-x}_{(B-\mathrm{Id})x}\in\gamma^{-1}Ax$ // using $p\in J_{\gamma A}x \Leftrightarrow x-p\in\gamma Ap$

$\Leftrightarrow \gamma(B-\mathrm{Id})x\in Ax$

$\Leftrightarrow \underbrace{x+\gamma(B-\mathrm{Id})x}_{(\mathrm{Id}+\gamma(B-\mathrm{Id}))x=\underbrace{((1-\gamma)\mathrm{Id}+B)x}_{\text{single-valued}}}\in x+Ax=(\mathrm{Id}+A)x$

$\Leftrightarrow \underbrace{\underbrace{(\mathrm{Id}+A)^{-1}}_{=J_A\ //\text{By definition : single-valued}}\underbrace{((1-\gamma)\mathrm{Id}+B)}_{\text{: single-valued}}}_{\text{single-valued}}x=x \Leftrightarrow (J_A\circ((1-\gamma)\mathrm{Id}+B))x=x$

$\Leftrightarrow x\in\operatorname{Fix}(J_A\circ((1-\gamma)\mathrm{Id}+B))$

$\therefore \operatorname{Fix} J_{\gamma A+B} = \operatorname{Fix} J_A\circ(\mathrm{Id}+\gamma(B-\mathrm{Id}))$

∎

Proposition 25.3.

$[\![V$: closed linear subspace of $\mathcal{H}$

$B:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone

$(x_n,u_n)_{n\in\mathbb{N}} \subseteq \operatorname{gra} B$

$(x,u)\in\mathcal{H}\times\mathcal{H}: x_n \rightharpoonup x, u_n \rightharpoonup u, P_{V^\perp}x_n\to 0, P_V u_n \to 0]\!] \Rightarrow$

(i) $x\in\operatorname{zer}(N_V+B)$

(ii) $(x,u)\in(V\times V^\perp)\cap\operatorname{gra} B$

(iii) $\langle x_n|u_n\rangle\to\langle x|u\rangle=0$

Proof: we will (ii) and (iii) first

\# Proposition 20.60. $\quad V-V=V$ // ∵ for a linear subspace $V=V-V$

$[\![\underset{=V}{\overset{\therefore=V^\perp}{C}}, D$: closed ~~affine~~ subspaces of $\mathcal{H}$: linear ⇒ hence affine ; $\underset{V}{D}-\underset{V}{D} = (\underbrace{\underset{V^\perp}{C}-\underset{V^\perp}{C}}_{V^\perp-V^\perp=V^\perp})^\perp$ ∴ $V^\perp=V^\perp$: vacuously true ✓

$\underset{=B}{A}$: maximally monotone

$(x_n,y_n)_{n\in\mathbb{N}}\subseteq \operatorname{gra} \cancel{A}B$

$(x,u)\in\mathcal{H}\times\mathcal{H}]\!]$

$(x_n\rightharpoonup x ✓;\ u_n\rightharpoonup u ✓;\ x_n-P_{\underset{V^\perp}{C}}x_n\to 0 ✓;\ u_n-P_{\underset{V}{D}}u_n\to 0 ✓)\Rightarrow \begin{cases}(x,u)\in(\overset{V^\perp}{C}\times\overset{V}{D})\cap\operatorname{gra}\overset{B}{A} \Leftrightarrow (x,u)\in(V^\perp\times V)\cap\operatorname{gra} B & \therefore \text{(ii) proved ✓}\\ \langle x_n|y_n\rangle\to\langle x|u\rangle \Leftrightarrow \langle x_n|y_n\rangle\to\langle x|u\rangle & \therefore \text{(iii) proved ✓}\end{cases}$

(i)

/* Proposition 25.1.(iii). $[\![\overset{V}{C}$: closed $\overset{\text{linear}}{\text{affine}}$ subspace of $\mathcal{H}$; $\underbrace{V=C-C}_{V=V-V \text{ // as } V:\text{linear}}$; $A=N_{\underset{V}{C}}$. $]\!]$ $\operatorname{zer}(A+\overset{:\mathcal{H}\to 2^{\mathcal{H}},\text{ general}}{B})=\{\tilde{x}\in C \mid V^\perp\cap B\tilde{x}\neq\emptyset\}$ */

$\Updownarrow$

$\operatorname{zer}(N_V+B)=\{\tilde{x}\in V\mid V^\perp\cap B\tilde{x}\neq\emptyset\}$

from (ii)

$(x,u)\in V^\perp\times V \ \wedge\ (x,u)\in\operatorname{gra} B$

$\Downarrow \qquad\qquad \Downarrow$

$x\in V^\perp \qquad Bx\ni u\Rightarrow Ax\neq\emptyset$

$Bx\cap V^\perp\neq\emptyset \Rightarrow x\in\operatorname{zer}(N_V+B)$

∎

Proposition 25.4. // Notation here $\tilde{x}=(\overbrace{\tilde{x}_i}^{\in\mathcal{H}})_{i\in I}\in\tilde{\mathcal{H}}=\bigoplus_{i\in I}\mathcal{H}$

$[\![m\in\mathbb{N}: m\geq 2;\ I=\{1,\dots,m\};\ (A_i)_{i\in I}$: maximally monotone, $\mathcal{H}\to 2^{\mathcal{H}}$;

$\tilde{\mathcal{H}}=\bigoplus_{i\in I}\mathcal{H}_i$; $\tilde{D}=\{(x,\dots,x)\in\tilde{\mathcal{H}}: x\in\mathcal{H}\}$; $\tilde{j}: X\to\tilde{D}: x\mapsto(x,x,\dots,x)$; $\tilde{B}=\times_{i\in I} A_i]\!]$

$\forall \tilde{x}=(\tilde{x}_i)_{i\in I}\in\mathcal{H}$

(i) $\tilde{D}^\perp=\{\tilde{u}\in\mathcal{H}\mid\sum_{i\in I}\tilde{u}_i=0\}$

(ii) $N_{\tilde{D}}\tilde{x}=\begin{cases}\{\tilde{u}\in\tilde{\mathcal{H}}:\sum_{i\in I}u_i=0\}, & \text{if } \tilde{x}\in\tilde{D}\\ \emptyset, & \text{otherwise}\end{cases}$

(iii) $P_{\tilde{D}}\tilde{x}=\tilde{j}(\frac{1}{m}\sum_{i\in I}\tilde{x}_i)$

(iv) $P_{\tilde{D}^\perp}\tilde{x}=(\tilde{x}_i-\frac{1}{m}\sum_{j\in I}\tilde{x}_j)_{i\in I}$

(v) $J_{\gamma\tilde{B}}\tilde{x}=(J_{\gamma A_i}\tilde{x}_i)_{i\in I}$

(vi) $\tilde{j}(\operatorname{zer}\sum_{i\in I}A_i)=\operatorname{zer}(N_{\tilde{D}}+\tilde{B})$

Proof.

(i) $\tilde{D}=\{(x,x,\dots,x)\in\tilde{\mathcal{H}}: x\in\mathcal{H}\}$ now by definition $\tilde{D}^\perp=\{\tilde{u}\in\tilde{\mathcal{H}}\mid\forall_{\tilde{x}\in D}\ \langle\tilde{x}|\tilde{u}\rangle=\sum\langle \underbrace{x_i}_{=x \text{ for } \tilde{x}\in D}|\tilde{u}_i\rangle=\sum\langle x|\tilde{u}_i\rangle=\langle x|\sum_i\tilde{u}_i\rangle=0\Leftrightarrow\sum\tilde{u}_i=0\}$ ∴ $\tilde{D}^\perp=\{\tilde{u}\in\mathcal{H}\mid\sum_{i\in I}\tilde{u}_i=0\}$;)

$(\tilde{x}_i)_{i\in I}=(x,\dots,x)=(x)_{i\in I}$

Example 2.1. (Hilbert direct sum notation)

$\bigoplus_{i\in I}\mathcal{H}_i=\{\tilde{x}=(x_i)_{i\in I}\in\times_{i\in I}\mathcal{H}_i \mid \sum_{i\in I}\|x_i\|^2<+\infty\}$: Hilbert direct sum // when I finite ⇒ $\bigoplus_{i\in I}\mathcal{H}_i=\times_{i\in I}\mathcal{H}_i$

• addition on Hilbert direct sum: $(\tilde{x},\tilde{y})\mapsto(x_i+y_i)_{i\in I}$ // $\tilde{x}+\tilde{y}=(x_i+y_i)_{i\in I}$

• scalar multiplication on Hilbert direct sum: $(\alpha,\tilde{x})\mapsto(\alpha x_i)_{i\in I}$ // $\alpha\tilde{x}=(\alpha x_i)_{i\in I}$

• scalar product on Hilbert direct sum: $(\tilde{x},\tilde{y})\mapsto\sum_{i\in I}\langle x_i|y_i\rangle$; // $\langle\tilde{x}|\tilde{y}\rangle=\sum_{i\in I}\langle x_i|y_i\rangle$

• $[\![\forall_{i\in I}\ f_i:\mathcal{H}_i\to\,]-\infty,+\infty]$; I: infinite ⇒ $\inf_{i\in I} f_i\geq 0]\!]$ ⇒ $\bigoplus_{i\in I} f_i:\bigoplus_{i\in I}\mathcal{H}_i\to\,]-\infty,+\infty]:(x_i)_{i\in I}\mapsto\sum_{i\in I}f_i(x_i)$

(ii) first note that $\tilde{D}$: linear subspace of $\tilde{\mathcal{H}}$ as $\forall_{\underbrace{\tilde{x}}_{(x,\dots,x)},\underbrace{\tilde{y}}_{(y,\dots,y)}\in\tilde{D}}\ \forall_{\alpha,\beta\in\mathbb{R}}\ z=\alpha\tilde{x}+\beta\tilde{y}=(\alpha x+\beta y,\dots,\alpha x+\beta y)\in\tilde{D}$, By structure $\tilde{D}$: closed too

now let us use Example 6.42.

Example 6.42. $[\![\overset{\tilde{D}:\text{linear}}{C}$: affine subspace of $\mathcal{H}$; $\underset{\tilde{D}}{V}=\underset{\tilde{D}}{C}-C$ // as $\tilde{D}$ linear; $\tilde{x}\in\tilde{\mathcal{H}}]\!]\Rightarrow$

$\cancel{T_C x=\begin{cases}V, & \text{if } x\in C\\ \emptyset, & \text{otherwise}\end{cases}}$

$N_{\tilde{D}}\tilde{x}=\begin{cases}\overset{\tilde{D}^\perp}{\cancel{V^\perp}}, & \text{if } \tilde{x}\in\tilde{D}\\ \emptyset, & \text{otherwise}\end{cases}$

$\therefore \forall_{\tilde{x}\in\tilde{\mathcal{H}}}\ N_{\tilde{D}}\tilde{x}=\begin{cases}\tilde{D}^\perp, & \text{if } \tilde{x}\in\tilde{D}\\ \emptyset, & \text{else}\end{cases}$;)

(iii) take any $\tilde{x}=(\tilde{x}_i)_{i\in I}\in\tilde{\mathcal{H}}$, we want to show that the operator $\tilde{x}=(\tilde{x}_i)_{i\in I}\mapsto\tilde{j}(\frac{1}{m}\sum_{i\in I}\tilde{x}_i)=P_{\tilde{D}}\tilde{x}$

$p=\frac{1}{m}\sum_{i\in I}\tilde{x}_i$

$\therefore \tilde{p}=\tilde{j}(p)=(p,p,\dots,p)\in\tilde{D}$

denote, $\tilde{y}=\tilde{j}(y): y\in\mathcal{H}$

$=(y,\dots,y)\in\tilde{D}$ ∴ so, $\tilde{y}$: arbitrary point in $\tilde{D}$

\#(Corollary 3.22) set $\tilde{V}:=\tilde{D}$

* Projection onto a closed linear subspaces.

$[\![V$: closed linear subspace of $\mathcal{H}$. $x\in\mathcal{H}]\!]$

(i) $P_V x$: $P_V x\in V$, $x-P_V x\perp V$

want to see if $\langle\tilde{x}-\tilde{p}|\tilde{y}\rangle=0$

lets test this one:

now: $\langle\tilde{x}-\tilde{p}|\tilde{y}\rangle=\sum_{i\in I}\langle\tilde{x}_i-\underset{=p=\frac{1}{m}\sum_{i\in I}x_i}{\tilde{p}_i}|\underset{=y}{\tilde{y}_i}\rangle=\sum_{i\in I}\langle\tilde{x}_i-p|y\rangle=\overbrace{\sum_{i\in I}\langle\tilde{x}_i|y\rangle}^{}-\underbrace{\sum_{i\in I}\langle p|y\rangle}_{\langle p|y\rangle\sum_{i\in I}1=m\langle p|y\rangle=\langle mp|y\rangle}=\langle\sum_{i\in I}\tilde{x}_i|y\rangle-\langle mp|y\rangle=\langle\underbrace{(\sum_{i\in I}\tilde{x}_i)-mp}_{0}|y\rangle=0$ $\therefore P_{\tilde{D}}\tilde{x}=\tilde{j}(\frac{1}{m}\sum_{i\in I}\tilde{x}_i)=(\frac{1}{m}\sum_{i\in I}\tilde{x}_i,\dots,\frac{1}{m}\sum_{i\in I}\tilde{x}_i)$

$\tilde{x}_1^T y+\tilde{x}_2^T y+\dots+\tilde{x}_m^T y=(\tilde{x}_1+\tilde{x}_2+\dots+\tilde{x}_m)^T y=(\sum_{i\in I}\tilde{x}_i)^T y=\langle\sum_{i\in I}\tilde{x}_i|y\rangle$

(iv)

$\underset{i\in I}{} \quad p = \frac{1}{m}\sum_{i\in I} x_i \quad y \qquad \langle p|y\rangle \underbrace{\sum_{i\in I} 1}_{m} = m\langle p|y\rangle = \langle mp|y\rangle \qquad 0$

(iv)

*(Corollary 3.22)

* Projection onto a closed linear subspaces.
[V: closed linear subspace of $\mathcal{H}$, $x\in\mathcal{H}$]

(i) $P_V x$: $P_V x\in V$, $x - P_V x \perp V$

(ii) $\|P_V x\|^2 = \langle P_V x | x\rangle$

(iii) $(P_V \in \mathcal{B}(\mathcal{H}),\ \|P_V\| = 1) \Leftarrow V \neq \{0\}$

$\|P_V\| = 0 \Leftarrow V = \{0\}$

(iv) $V^{\perp\perp} = V$

(v) $P_{V^\perp} = \mathrm{Id} - P_V$ // i.e., $\forall_{x\in\mathcal{H}}\ x = P_V x + P_{V^\perp} x$ (Linear subspace decomposition rule)

(vi) $P_{V^*} = P_V$

(vii) $\|x\|^2 = \|P_V x\|^2 + \|P_{V^\perp} x\|^2 = d_V^2(x) + d_{V^\perp}^2(x)$

using the closed linear subspace decomposition rule: $\forall_{\tilde{x}\in\tilde{\mathcal{H}}}\ \tilde{x} = P_{\tilde{D}}\tilde{x} + P_{\tilde{D}^\perp}\tilde{x}$

$\leftrightarrow P_{\tilde{D}^\perp}\tilde{x} = \tilde{x} - P_{\tilde{D}}\tilde{x} = \left(\tilde{x}_i - \frac{1}{m}\sum_{i\in I} x_i\right)_{i\in I}$:)

(v) Recall

Proposition 23.16 (Resolvent on Hilbert sum space)
[$(\mathcal{H}_i)_{i\in I}$: totally ordered finite family of real Hilbert spaces
$\mathcal{H} = \bigoplus_{i\in I}\mathcal{H}_i$
$\forall_{i\in I}\ A_i: \mathcal{H}_i \to 2^{\mathcal{H}_i}$: maximally monotone
$A = \times_{i\in I} A_i$] $\Rightarrow$
| A : maximally monotone
| $J_A = \times_{i\in I} J_{A_i}$

using this $\underset{\underset{(\times_{i\in I} A_i)}{\parallel}}{J_{\gamma\tilde{B}}}\tilde{x} = \left(\times_{i\in I} J_{\gamma A_i}\right)(\tilde{x}_1,\ldots,\tilde{x}_m) = \Big(J_{\gamma A_i}\underbrace{\tilde{x}_i}_{\in\mathcal{H}}\Big)_{i\in I}$:)

(vi) $\tilde{j}\left(\operatorname{zer}\sum_{i\in I} A_i\right) = \tilde{j}\left(\{x\in\mathcal{H} \mid x\in\operatorname{zer}\sum_{i\in I} A_i\}\right) = \{\tilde{j}(x) = (x,\ldots,x)\in\tilde{\mathcal{H}} \mid x\in\operatorname{zer}\sum_{i\in I} A_i\}$

$\therefore\ \tilde{j}(x) \in \tilde{j}\left(\operatorname{zer}\sum_{i\in I} A_i\right)$

$\leftrightarrow\ x \in \operatorname{zer}\left(\sum_{i\in I} A_i\right) \leftrightarrow \left(\sum_{i\in I} A_i\right)x = \left(\sum_{i\in I} A_i x\right) \ni 0 \leftrightarrow \exists_{u_1\in A_1 x,\ldots,u_m\in A_m x}\ u_1+\ldots+u_m = 0$

$\leftrightarrow \exists_{\underset{u}{\underbrace{(u_i)_{i\in I}}}} \in \left(\times_{i\in I}(A_i x)\right)\ \sum_{i\in I} u_i = 0$ /* now from (ii) $\tilde{D}^\perp = \{\tilde{u}\in\tilde{\mathcal{H}} \mid \sum_{i\in I}\tilde{u}_i = 0\}$ $\therefore u\in\tilde{D}^\perp \leftrightarrow -u\in\tilde{D}^\perp$ */
(as $\tilde{D}^\perp$: linear subspace)

$\Big(\underbrace{\times_{i\in I} A_i}_{\tilde{B}}\Big)\underbrace{(x, x, \ldots, x)}_{\tilde{j}(x)} = \tilde{B}\tilde{j}(x)$ (1, 2, …, mth)

$\tilde{j}(x) = (x,\ldots,x)\in\tilde{\mathcal{H}}$ by def ($x\in\mathcal{H}$)

$\leftrightarrow \exists_{u\in\tilde{\mathcal{H}}}: u\in\tilde{B}\tilde{j}(x)$, $-u\in\tilde{D}^\perp$

/* now by definition, $\tilde{j}(x)\in\tilde{D} \Rightarrow N_{\tilde{D}}\tilde{j}(x) = \{u\in\tilde{\mathcal{H}} \mid \sum_{i\in I}\tilde{u}_i = 0\} = \tilde{D}^\perp$ by (ii) */

$\leftrightarrow \exists_{u\in\tilde{\mathcal{H}}}\ u\in\tilde{B}\tilde{j}(x)$, $-u\in\tilde{D}^\perp = N_{\tilde{D}}\tilde{j}(x)$

$\leftrightarrow N_{\tilde{D}}\tilde{j}(x) + \tilde{B}\tilde{j}(x) \ni 0$

$\leftrightarrow (N_{\tilde{D}} + \tilde{B})(\tilde{j}(x)) \ni 0 \leftrightarrow \tilde{j}(x) \in \operatorname{zer}(N_{\tilde{D}} + \tilde{B})$

$\therefore\ \tilde{j}\left(\operatorname{zer}\sum_{i\in I} A_i\right) = \operatorname{zer}(N_{\tilde{D}} + \tilde{B})$:)

∎

Corollary 25.5. [$m\in\mathbb{N}: m\geq 2$; $I = \{1,\ldots,m\}$; $(A_i)_{i\in I}$: maximally monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$;
$\forall_{i\in I}\ (x_{i,n}, u_{i,n})_{n\in\mathbb{N}} \subseteq \operatorname{gra} A_i$; $(x_i, u_i)\in\mathcal{H}\times\mathcal{H}$;
$\sum_{i\in I} u_{i,n} \to 0$; $\forall_{i\in I}$ | $x_{i,n} \rightharpoonup x_i$ | $u_{i,n} \rightharpoonup u_i$ | $m x_{i,n} - \sum_{j\in I} x_{j,n} \to 0$] $\Rightarrow \exists_{\check{x}\in\operatorname{zer}\sum_{i\in I} A_i}$
| (i) $\check{x} = x_1 = \ldots = x_m$
| (ii) $\sum_{i\in I} u_i = 0$
| (iii) $\forall_{i\in I}\ (\check{x}, u_i)\in\operatorname{gra} A_i$
| (iv) $\sum_{i\in I}\langle x_{i,n} | u_{i,n}\rangle \to \langle\check{x} | \sum_{i\in I} u_i\rangle = 0$

Proof:

define: $\tilde{\mathcal{H}} = \bigoplus_{i\in I}\mathcal{H}$

$\tilde{D} = \{(x,\ldots,x)\in\tilde{\mathcal{H}} : x\in\mathcal{H}\}$: consensus linear space

$\tilde{j}: \mathcal{H}\to\tilde{D}: x\mapsto(x, x, \ldots, x)$; $\tilde{B} = \times_{i\in I} A_i$ // $\tilde{B}\underbrace{\tilde{x}}_{(\tilde{x}_1,\ldots,\tilde{x}_m)} = (A_1\tilde{x}_1, A_2\tilde{x}_2, \ldots, A_m\tilde{x}_m)$ ($\tilde{x}_1\in\mathcal{H}$, $\tilde{x}_m\in\mathcal{H}$)

now $\tilde{B} = \underbrace{\times_{i\in I} A_i}_{\text{maximally monotone}}$: maximally monotone using:

Proposition 23.16 (Resolvent on Hilbert sum space) **
[$(\mathcal{H}_i)_{i\in I}$: totally ordered finite family of real Hilbert spaces
$\mathcal{H} = \bigoplus_{i\in I}\mathcal{H}_i$
$\forall_{i\in I}\ A_i: \mathcal{H}_i\to 2^{\mathcal{H}_i}$: maximally monotone

Define. $(x_i)_{i\in I} = x$, $(u_i)_{i\in I} = u$

now $\tilde{B} = \underbrace{\times_{i\in I} A_i}_{\text{maximally monotone}}$: maximally monotone using:

Proposition 23.16 (Resolvent on Hilbert sum space) ★★
[$(\mathcal{H}_i)_{i\in I}$: totally ordered finite family of real Hilbert spaces
$\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$
$\forall_{i\in I}\ A_i : \mathcal{H}_i \to 2^{\mathcal{H}_i}$: maximally monotone
$A = \times_{i\in I} A_i$] ⇒
A : maximally monotone   product of maximally monotone operators is maximally monotone
$J_A = \times_{i\in I} J_{A_i}$ i.e. $J_{(\times_{i\in I} A_i)} x = (J_{A_i} x_i)_{i\in I}$, $x = (x_i)_{i\in I} \in \mathcal{H}$, $x_i \in \mathcal{H}_i$

Define. $(x_i)_{i\in I} = x$, $(u_i)_{i\in I} = u$

$$\forall_{n\in\mathbb{N}} \left.\begin{array}{l} x_n = (x_{i,n})_{i\in I}, \\ u_n = (u_{i,n})_{i\in I} \end{array}\right\} \Rightarrow (x_n, u_n) \subseteq \operatorname{gra} B$$

now given

$$\left(\sum_{i\in I} u_{i,n} \to 0 \ ; \ \forall_{i\in I} \left| \begin{array}{l} x_{i,n} \rightharpoonup x_i \\ u_{i,n} \rightharpoonup u_i \\ m x_{i,n} - \sum_{j\in I} x_{j,n} \to 0 \end{array} \right. \right) \xleftrightarrow{\text{applying notation}} \boxed{x_n \rightharpoonup x} \ \boxed{u_n \rightharpoonup u}$$

$\Rightarrow \sum_{i\in I} u_{i,n} \rightharpoonup \sum_{i\in I} u_i$, but $\sum_{i\in I} u_{i,n} \to 0$ by given $\Rightarrow \boxed{\sum_{i\in I} u_i = 0}$ // as the other possibility is not possible → (ii) proved. :)

$$\left[\forall_{i\in I} \left(x_{i,n} - \frac{1}{m} \sum_{j\in I} x_{j,n} \right) \to 0 \right] \leftrightarrow \underbrace{\left(x_{i,n} - \frac{1}{m} \sum_{j\in I} x_{j,n} \right)_{i\in I}}_{P_{\tilde{D}^\perp} x_n} \to 0 \quad \therefore \boxed{P_{\tilde{D}^\perp} x_n \to 0}$$

$$\leftrightarrow \underbrace{\left(\frac{1}{m} \sum_{i\in I} u_{i,n}, \ldots, \frac{1}{m} \sum_{i\in I} u_{i,n} \right)}_{\tilde{j}_{\tilde{D}}\left(\frac{1}{m}\sum_{i\in I} u_{i,n}\right)} \to (0, \ldots, 0)$$

$$\leftrightarrow \underbrace{\tilde{j}_{\tilde{D}} \left(\frac{1}{m} \sum_{i\in I} u_{i,n} \right)}_{P_{\tilde{D}} u_n} \to 0 \in \tilde{\mathcal{H}} \quad \therefore \boxed{P_{\tilde{D}} u_n \to 0}$$

Proposition 25.4. ★★
[$m \in \mathbb{N}$: $m \geq 2$; $I = \{1, \ldots, m\}$; $(A_i)_{i\in I}$: maximally monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$;
$\tilde{\mathcal{H}} = \bigoplus_{i\in I} \mathcal{H}$; $\tilde{D} = \{(x, \ldots, x) \in \tilde{\mathcal{H}} : x \in \mathcal{H}\}$; $\tilde{j} : \mathcal{H} \to \tilde{D} : x \mapsto (x, x, \ldots, x)$; $B = \times_{i\in I} A_i$]
we can name $\tilde{j}$ as the duplication operator over consensus space
we can name $\tilde{D}$ as the consensus space, as every component vectors are the same
$\forall \tilde{x} = (\tilde{x}_i)_{i\in I} \in \mathcal{H}$
(i) $\tilde{D}^\perp = \{\tilde{u} \in \tilde{\mathcal{H}} \mid \sum_{i\in I} \tilde{u}_i = 0\}$; (ii) $N_{\tilde{B}} \tilde{x} = \begin{cases} \{\tilde{u} \in \tilde{\mathcal{H}} \mid \sum_{i\in I} \tilde{u}_i = 0\}, & \text{if } \tilde{x} \in \tilde{D} \\ \emptyset & \text{, otherwise} \end{cases}$; (iii) $P_{\tilde{B}} \tilde{x} = \tilde{j}\left(\frac{1}{m} \sum_{i\in I} \tilde{x}_i\right)$;
(iv) $P_{\tilde{D}^\perp} \tilde{x} = \left(\tilde{x}_i - \frac{1}{m} \sum_{j\in I} \tilde{x}_j\right)_{i\in I}$; (v) $J_{\gamma \tilde{B}} \tilde{x} = (J_{\gamma A_i} \tilde{x}_i)_{i\in I}$; (vi) $\tilde{j}(\operatorname{zer} \sum_{i\in I} A_i) = \operatorname{zer}(N_{\tilde{D}} + \tilde{B})$

Now we use Proposition 25.3.

Proposition 25.3. [V : closed linear subspace of $\mathcal{H}$; $B : \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone ;
$(x_n, u_n) \subseteq \operatorname{gra} B$; $(x, u) \in \mathcal{H} \times \mathcal{H}$: $x_n \rightharpoonup x$, $u_n \rightharpoonup u$, $P_{V^\perp} x_n \to 0$, $P_V u_n \to 0$]
(i) $x \in \operatorname{zer}(N_V + B)$
(ii) $(x, u) \in (V \times V^\perp) \cap \operatorname{gra} B$
(iii) $\langle x_n | u_n \rangle \to \langle x | u \rangle = 0$

then in our case $\underbrace{x}_{(x_1, \ldots, x_m)} \in \operatorname{zer}(N_{\tilde{D}} + \tilde{B}) = \tilde{j}(\operatorname{zer} \sum_{i\in I} A_i) = \tilde{j}\{\check{x} \in \mathcal{H} \mid \operatorname{zer}(\sum_{i\in I} A_i)(\check{x}) \ni 0\}$
$= \{(\check{x}, \ldots, \check{x}) \mid \operatorname{zer}(\sum_{i\in I} A_i \check{x}) \ni 0\}$

$$\leftrightarrow \underbrace{\exists_{\check{x}}\ x_1 = \ldots = x_m = \check{x}}_{\text{(i) proved :)}} \wedge \underbrace{\check{x} \in \operatorname{zer}(\sum_{i\in I} A_i)}_{\text{initial goal proven}}$$

now given

$\forall_{i\in I}\ (x_{i,n}, u_{i,n})_{n\in\mathbb{N}} \subseteq \operatorname{gra} A_i$; $(x_i, u_i) \in \mathcal{H} \times \mathcal{H}$; $x_{i,n} \rightharpoonup x_i = \check{x}$, $u_{i,n} \rightharpoonup u_i$

$(x, u) \in (\tilde{D}^\perp \times \tilde{D}) \cap \operatorname{gra} \tilde{B}$

$\Rightarrow (x, u) \in \operatorname{gra} \tilde{B}$, $x = (\check{x}, \ldots, \check{x})$, $u = (u_1, \ldots, u_m)$, $\tilde{B} = \times_{i\in I} A_i$

$\leftrightarrow (\times_{i\in I} A_i)(\check{x}, \ldots, \check{x}) \ni (u_1, \ldots, u_m)$

$\leftrightarrow (A_1 \check{x}, A_2 \check{x}, \ldots, A_m \check{x}) \ni (u_1, \ldots, u_m)$

$\leftrightarrow \forall_{i\in I}\ A_i \check{x} \ni u_i \leftrightarrow \forall_{i\in I} (\check{x}, u_i) \in \operatorname{gra} A_i$ . . . (iii) proved :)

$\langle x_n | u_n \rangle \to \langle x | u \rangle = 0$

$\langle x_n | u_n \rangle = \langle (x_{i,n})_{i\in I} | (u_{i,n})_{i\in I} \rangle = \langle (x_{1,n}, \ldots, x_{m,n}) | (u_{1,n}, \ldots, u_{m,n}) \rangle = (x_{1,n}, \ldots, x_{m,n})^T (u_{1,n}, \ldots, u_{m,n})$
$= x_{1,n}^T u_{1,n} + \ldots + x_{m,n}^T u_{m,n}$
$= \sum_{i\in I} \langle x_{i,n} | u_{i,n} \rangle$

$\langle x | u \rangle = (\underbrace{(x_i)_{i\in I}}_{\check{x}} | (u_i)_{i\in I}) \overset{\text{similarly}}{=} \check{x}^T u_1 + \ldots + \check{x}^T u_m = \check{x}^T (u_1 + \ldots + u_m)$
$= \langle \check{x} | \sum_{i\in I} u_i \rangle$

$\sum_{i\in I} \langle x_{i,n} | u_{i,n} \rangle \to \langle \check{x} | \sum_{i\in I} u_i \rangle = 0$:) (iv) proved

∎

Part 2: Douglas Rachford Algorithm

4:32 PM

Theorem 25.6. (Douglas-Rachford algorithm)

[A, B: maximally monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$; $\operatorname{zer}(A+B) \neq \emptyset$;
$(\lambda_n)_{n\in\mathbb{N}}$: sequence in $[0,2]$, $\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$; $\gamma \in \mathbb{R}_{++}$; $x_0 \in \mathcal{H}$

$$\forall_{n\in\mathbb{N}} \left| \begin{array}{l} y_n = J_{\gamma B} x_n \\ z_n = J_{\gamma A}(2y_n - x_n) \\ x_{n+1} = x_n + \lambda_n(z_n - y_n) \end{array} \right\} \text{(eq:25.9)} \;] \Rightarrow \exists_x \; x \in \operatorname{Fix} R_{\gamma A} R_{\gamma B}$$

(i) $J_{\gamma B} x \in \operatorname{zer}(A+B)$

(ii) $(y_n - z_n)_{n\in\mathbb{N}}$: $(y_n - z_n) \to 0$

(iii) $(x_n)_{n\in\mathbb{N}}$: $x_n \rightharpoonup x$

(iv) $(y_n)_{n\in\mathbb{N}}$: $y_n \rightharpoonup J_{\gamma B} x$

(v) $(z_n)_{n\in\mathbb{N}}$: $z_n \rightharpoonup J_{\gamma B} x$

(vi) (C: closed affine subspace of $\mathcal{H}$; $A = N_C$) $\Rightarrow$ $(P_C x_n)_{n\in\mathbb{N}}$: $P_C x_n \rightharpoonup J_{\gamma B} x$

(vii) (one of the following holds:

(a) A : uniformly monotone on every nonempty bounded subset of $\operatorname{dom} A$

(b) B : uniformly monotone on every nonempty bounded subset of $\operatorname{dom} B$) $\Rightarrow$

$(y_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$: converge strongly to the unique point in $\operatorname{zer}(A+B)$

proof:

A, B : maximally monotone // * now

$\Rightarrow R_{\gamma A}, R_{\gamma B}$: nonexpansive

$\therefore T = R_{\gamma A} R_{\gamma B}$: nonexpansive

// composition of two nonexpansive
// operators are nonexpansive *|

now recall Proposition 25.1 (ii)

Proposition 25.1. [$A: \mathcal{H} \to 2^{\mathcal{H}}$, $B: \mathcal{H} \to 2^{\mathcal{H}}$, $\gamma \in \mathbb{R}_{++}$] $\Rightarrow$
(i) $\operatorname{zer}(A+B) = \operatorname{dom}(A \cap (-B))$
(ii) A, B : monotone $\Rightarrow$ $\operatorname{zer}(A+B) = J_{\gamma B}(\operatorname{Fix} R_{\gamma A} R_{\gamma B})$ // Note: very general, can it be used in nonconvex setup?

$\therefore \operatorname{zer}(A+B) = J_{\gamma B}(\operatorname{Fix} R_{\gamma A} R_{\gamma B}) \neq \emptyset$ . . . (eq:C)
$\;\;\;\;\not\parallel$
$\;\;\;\;\emptyset$ /* By given */

$\therefore \operatorname{Fix} T = \operatorname{Fix} R_{\gamma A} R_{\gamma B} \neq \emptyset$ // as otherwise $(\cdot)$ cannot hold

$\leftrightarrow \exists_x \; x \in \operatorname{Fix} T = \operatorname{Fix} R_{\gamma A} R_{\gamma B}$ (eq:D) $\therefore$ this proves the initial consequent

now using

Proposition 4.21: /* a special composition of two firmly nonexpansive operators being firmly nonexpansive */
[$T_1: \mathcal{H} \to \mathcal{H}$, firmly nonexpansive
$T_2: \mathcal{H} \to \mathcal{H}$, firmly nonexpansive
$T = T_1(2T_2 - \mathrm{Id}) + \mathrm{Id} - T_2$] $\Rightarrow$
(i) $2T - \mathrm{Id} = (2T_1 - \mathrm{Id})(2T_2 - \mathrm{Id})$
(ii) T : firmly nonexpansive
(iii) $\operatorname{Fix} T = \operatorname{Fix}(2T_1 - \mathrm{Id})(2T_2 - \mathrm{Id})$
(iv) T_1 : projector onto a closed affine subspace $\Rightarrow$ $\operatorname{Fix} T = \{x \in \mathcal{H} \mid T_1 x = T_2 x\}$

set $J_{\gamma A}, J_{\gamma B}$ both firmly nonexpansive as A, B : maximal monotone

define. $T_1 = J_{\gamma A}$, $T_2 = J_{\gamma B}$ in Proposition 4.21.

define $\tilde{T} = T_1(2T_2 - \mathrm{Id}) + \mathrm{Id} - T_2 = J_{\gamma A}(\underbrace{2J_{\gamma B} - \mathrm{Id}}_{R_{\gamma B}}) + (\mathrm{Id} - J_{\gamma B}) = J_{\gamma A} R_{\gamma B} + \mathrm{Id} - J_{\gamma B}$

then: $2\tilde{T} - \mathrm{Id} = 2(J_{\gamma A} R_{\gamma B} + \mathrm{Id} - J_{\gamma B}) - \mathrm{Id} = 2J_{\gamma A} R_{\gamma B} + 2\mathrm{Id} - 2J_{\gamma B} - \mathrm{Id} = 2J_{\gamma A} R_{\gamma B} - 2J_{\gamma B} + \mathrm{Id} = 2(J_{\gamma A} R_{\gamma B} - J_{\gamma B}) + \mathrm{Id}$

but $2\tilde{T} - \mathrm{Id} = (2T_1 - \mathrm{Id})(2T_2 - \mathrm{Id}) = \underbrace{(2J_{\gamma A} - \mathrm{Id})}_{R_{\gamma A}} \underbrace{(2J_{\gamma B} - \mathrm{Id})}_{R_{\gamma B}} = 2(J_{\gamma A} R_{\gamma B} - J_{\gamma B}) + \mathrm{Id}$

this sort of operators can be manipulated via $\pm$ like normal numbers

$\leftrightarrow R_{\gamma A} R_{\gamma B} = 2(J_{\gamma A} R_{\gamma B} - J_{\gamma B}) + \mathrm{Id} \leftrightarrow J_{\gamma A} R_{\gamma B} - J_{\gamma B} = \frac{1}{2}(R_{\gamma A} R_{\gamma B} - \mathrm{Id})$

/* check : $R_{\gamma A} R_{\gamma B} x = (2(J_{\gamma A} R_{\gamma B} - J_{\gamma B}) + \mathrm{Id})(x) = 2(J_{\gamma A} R_{\gamma B} - J_{\gamma B})x + x$

$\leftrightarrow (J_{\gamma A} R_{\gamma B} - J_{\gamma B})x = \frac{1}{2}(R_{\gamma A} R_{\gamma B} x - x) = \frac{1}{2}(R_{\gamma A} R_{\gamma B} - \mathrm{Id})x$ */

from (eq.25.9):

$$\left.\begin{array}{l} y_n = J_{\gamma B} x_n \\ z_n = J_{\gamma A}(2y_n - x_n) = J_{\gamma A}(2J_{\gamma B} x_n - x_n) = J_{\gamma A}\overbrace{(2J_{\gamma B} - \mathrm{Id})}^{R_{\gamma B} \text{ /* By definition */}} x_n = J_{\gamma A} R_{\gamma B} x_n \end{array}\right\}$$

$x_{n+1} = x_n + \lambda_n(z_n - y_n)$

$= x_n + \lambda_n(J_{\gamma A} R_{\gamma B} x_n - J_{\gamma B} x_n)$

$= x_n + \lambda_n (J_{\gamma A} R_{\gamma B} - J_{\gamma B})(x_n)$

$= x_n + \underbrace{\lambda_n \tfrac{1}{2}}_{// = \mu_n \text{ (let)}} (R_{\gamma A} R_{\gamma B} - \mathrm{Id})(x_n)$

$= x_n + \mu_n (\underbrace{R_{\gamma A} R_{\gamma B}}_{= T \text{ // By our notation}} - \mathrm{Id}) x_n = x_n + \mu_n(Tx_n - x_n)$ $\therefore \forall_{n\in\mathbb{N}} \; x_{n+1} = x_n + \mu_n(Tx_n - x_n)$ $\{\mu_n = \frac{\lambda_n}{2}\}$ . . (25.10) (eq: 25.10)

$= (1 - \mu_n)x_n + \mu_n T x_n$

For convenience set:

$$\forall_{n\in\mathbb{N}} \left| \begin{array}{l} v_n = x_n - y_n \\ w_n = 2y_n - x_n - z_n \end{array} \right. \quad . . . (25.11)$$

(Eq:25.11)

$v_n + w_n = y_n - z_n$ . . . (eq:A)

(eq: A)

Proposition 23.2: Basic relationships between A, R_A, $J_{\gamma A}$
[$A: \mathcal{H} \to 2^{\mathcal{H}}$, $\gamma \in \mathbb{R}_{++}$, ...
(i) $\operatorname{dom} J_{\gamma A} = \ldots$; $\operatorname{ran} J_{\gamma A} = \operatorname{dom} A$
(ii) ...
(iii) ...

Again from (eq:25.9):

$$\forall_{n\in\mathbb{N}} \left| \begin{array}{l} y_n = J_{\gamma B} x_n \leftrightarrow x_n - y_n \in \gamma B y_n \leftrightarrow (y_n, \overbrace{x_n - y_n}^{v_n}) = (y_n, v_n) \in \operatorname{gra} \gamma B \\ z_n = J_{\gamma A}(2y_n - x_n) \leftrightarrow \underbrace{2y_n - x_n - z_n}_{w_n} \in \gamma A z_n \leftrightarrow (z_n, 2y_n - x_n - z_n) = (z_n, w_n) \in \operatorname{gra} \gamma A \end{array} \right\} . . \text{(eq:B)}$$

(Eq: B)

(i) from (eq:C) and (eq:D) : $\left| \begin{array}{l} \operatorname{zer}(A+B) = J_{\gamma B}(\operatorname{Fix} \underbrace{R_{\gamma A} R_{\gamma B}}_{= T}) \neq \emptyset \\ \exists_{x\in\mathbb{N}} \; x \in \operatorname{Fix} T \end{array} \right.$

$\operatorname{zer}(A+B) = J_{\gamma B}(\{\tilde{x} : \tilde{x} \in \operatorname{Fix} T\}) = \{J_{\gamma B} \tilde{x} \mid \tilde{x} \in \operatorname{Fix} T\} \ni J_{\gamma B} x$

$\therefore J_{\gamma B} x \in \operatorname{zer}(A+B)$ . . . (i) proved.

(ii) From (eq:25.10):

$\forall_{n\in\mathbb{N}} \; x_{n+1} = x_n + \mu_n(Tx_n - x_n)$ $\{\mu_n = \frac{\lambda_n}{2}\}$
$\mu_n \in [0,1]$ as $\lambda_n \in [0,2]$; T: nonexpansive

So, via KM iteration convergence

$\therefore (\underbrace{Tx_n}_{R_{\gamma A} R_{\gamma B}} - x_n)_{n\in\mathbb{N}}$: strongly converges to 0.

Theorem 5.15 (KM algorithm):
[D: nonempty closed convex subset of $\mathcal{H}$,
$T: D \to D$, nonexpansive, $\operatorname{Fix} T \neq \emptyset$
$(\lambda_n)_{n\in\mathbb{N}}$: sequence in $[0,1]$, $\sum_{n\in\mathbb{N}} \lambda_n(1-\lambda_n) = +\infty$
$x_0 \in D$
$\forall_{n\in\mathbb{N}} \; x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n$ /* KM iteration */] $\Rightarrow$
$= x_n + \lambda_n(Tx_n - x_n)$
(i) $(x_n)_{n\in\mathbb{N}}$: Fejér monotone w.r.t. $\operatorname{Fix} T$
(ii) $(Tx_n - x_n)_{n\in\mathbb{N}}$: converges strongly to 0
(iii) $(x_n)_{n\in\mathbb{N}}$: converges weakly to a point in $\operatorname{Fix} T$

now: $(R_{\gamma A} R_{\gamma B} x_n - x_n)$

$= (R_{\gamma A} R_{\gamma B} - \mathrm{Id}) x_n$ // says: $(R_{\gamma A} R_{\gamma B} - \mathrm{Id}) = 2(J_{\gamma A} R_{\gamma B} - J_{\gamma B})$

$= 2(J_{\gamma A} R_{\gamma B} - J_{\gamma B}) x_n$

$= 2(\underbrace{J_{\gamma A} R_{\gamma B} x_n}_{z_n} - \underbrace{J_{\gamma B} x_n}_{y_n})$ // using, says

$= 2(z_n - y_n)$

$\therefore 2(z_n - y_n)$: strongly converges to zero

$\leftrightarrow (z_n - y_n)$: strongly converges to zero (ii) proved.

(iii)

(eq: 25.10) and (*) says : $(x_n)_{n\in\mathbb{N}}$: converges weakly to a point in $\operatorname{Fix} T$.

$x_{n+1} = x_n + \underbrace{\mu_n}_{\in[0,1]}(Tx_n - x_n)$

(iv) want to show: $y_n \rightharpoonup J_{\gamma B} x$

now B : maximally monotone, also recall $y_n = J_{\gamma B} x_n$

$\Rightarrow J_{\gamma B}$: firmly nonexpansive, maximally monotone, $\mathcal{H} \to \mathcal{H}$

$x_{n+1} = x_n + \underbrace{\lambda_n}_{\in[0,1]} (T x_n - x_n)$

(iv) want to show: $y_n \rightharpoonup J_{\gamma B} x$

now B: maximally monotone, also recall $y_n = J_{\gamma B} x_n$

$\Rightarrow J_{\gamma B}$: firmly nonexpansive, maximally monotone, $\mathcal{H} \to \mathcal{H}$

$\Downarrow$

$J_{\gamma B}$: nonexpansive on $\mathcal{H}$ $\therefore \forall_{n \in \mathbb{N}} \ \|y_n - y_0\| = \|J_{\gamma B} x_n - J_{\gamma B} x_0\| \leq \|x_n - x_0\|$ /* slightly modified triangle inequality: $\|x-y\| \geq \|x\| - \|y\|$ /* $\|x-y\| + \|y\| \geq \|x-y+y\| = \|x\|$ $\|x-y\| \geq \|y\| - \|x\|$ /* similarly */ */

$\Rightarrow \|y_n\| - \|y_0\| \leq \|y_n - y_0\| \leq \|x_n - x_0\|$

$\therefore \forall_{n \in \mathbb{N}} \ \|y_n\| \leq \|y_0\| + \underbrace{\|x_n - x_0\|}_{\text{bounded}}$ // now, $x_n \rightharpoonup x \in \operatorname{Fix} T \Rightarrow (x_n)_{n\in\mathbb{N}}$: bounded

$\Rightarrow (y_n)_{n \in \mathbb{N}}$: bounded

$\Rightarrow (y_n)_{n \in \mathbb{N}}$: will have a sequential cluster point // using Lemma 2.45: [$(x_n)_{n\in\mathbb{N}}$: bounded sequence in $\mathcal{H}$] $\Rightarrow \exists_{(x_{k_n})_{n\in\mathbb{N}}: \text{subsequence of } (x_n)_{n\in\mathbb{N}}} (x_{k_n})_{n\in\mathbb{N}}$: weakly convergent /* so a bounded sequence possesses a weak sequential cluster point */

$\leftrightarrow \exists_{(y_{k_n})_{n \in \mathbb{N}}} \boxed{y_{k_n} \rightharpoonup y}$

Now: (Eq:25.11): $v_n = x_n - y_n$

$w_n = 2y_n - x_n - z_n$

from (ii), (iii): $y_n - z_n \to 0 \Rightarrow y_{k_n} - z_{k_n} \to 0$

$x_n \rightharpoonup x \Rightarrow x_{k_n} \rightharpoonup x$

$\boxed{v_{k_n} = x_{k_n} - y_{k_n} \rightharpoonup x - y}$

$\boxed{z_{k_n} \rightharpoonup y}$ // as $(a_n - b_n \to 0,\ b_n \rightharpoonup b) \Rightarrow a_n \rightharpoonup b$

$w_{k_n} = 2y_{k_n} - x_{k_n} - z_{k_n} \rightharpoonup 2y - x - y = y - x$ $\therefore \boxed{w_{k_n} \rightharpoonup y - x}$

So collecting the results above:

$y_{k_n} \rightharpoonup y,\ z_{k_n} \rightharpoonup y,\ v_{k_n} \rightharpoonup x - y,\ w_{k_n} \rightharpoonup y - x$ ... (eq: 25.13)

Now we are going to use Corollary 25.5 — read it first

$v_n + w_n = y_n - z_n$ // $\because$ (eq: A)

$(z_n, w_n) \in \operatorname{gra} \gamma A$, $(y_n, v_n) \in \operatorname{gra} \gamma B$ // $\because$ (eq: B) (Eq: 25.12)

$y_n - z_n \to 0$ // from (ii)

$v_n + w_n \to 0 \Rightarrow v_{k_n} + w_{k_n} \to 0$ ✓, $y_{k_n} \rightharpoonup y$ ✓, $z_{k_n} \rightharpoonup y$ ✓, $2y_{k_n} - (y_{k_n} + z_{k_n}) = y_{k_n} - z_{k_n} \to 0$ // from (ii)

and $2z_{k_n} - (y_{k_n} + z_{k_n}) = z_{k_n} - y_{k_n} \to 0$ ✓

So, all of the antecedents of Corollary 25.5 holds:

$\therefore \exists_{\check{y} \in \operatorname{zer}(\gamma A + \gamma B) = \operatorname{zer}(A+B)}$

$\check{y} = y = y$

$(\check{y}, y - x) = (y, y - x) \in \operatorname{gra} \gamma A$

$(\check{y}, x - y) = (y, x - y) \in \operatorname{gra} \gamma B$

/* Proposition 23.2: basic relationships between A, $^\gamma A$, $J_{\gamma A}$ ... (iii) ... */

$\leftrightarrow x - y \in \gamma B y \leftrightarrow y \in J_{\gamma B} x$ // But $J_{\gamma B}$: firmly nonexpansive i.e., single-valued

$\leftrightarrow y = J_{\gamma B} x \ \wedge\ y \in \operatorname{dom} \gamma A \leftrightarrow y \in \operatorname{dom} A$ ... (eq: 25.14) (eq: 25.14)

$\therefore y_{k_n} \rightharpoonup y = J_{\gamma B} x$

But $(y_{k_n})_{n\in\mathbb{N}}$: arbitrary convergent subsequence converging to one point $J_{\gamma B} x$

$\leftrightarrow (y_n)_{n \in \mathbb{N}}$: possesses one weak sequential cluster point

$(y_n)_{n\in\mathbb{N}}$: bounded, $(y_n)_{n\in\mathbb{N}}$: possesses one weak sequential cluster point /* Lemma 2.46: [$(x_n)_{n\in\mathbb{N}}$: sequence in $\mathcal{H}$] $(x_n)_{n\in\mathbb{N}}$: converges weakly $\Leftrightarrow$ $(x_n)_{n\in\mathbb{N}}$: bounded, possesses at most one weak sequential cluster */

$\leftrightarrow y_n \rightharpoonup y = J_{\gamma B} x$ ☺

(v) from (ii): $y_n - z_n \to 0$, $y_n \rightharpoonup J_{\gamma B} x$ $\Rightarrow z_n \rightharpoonup J_{\gamma B} x$ // $(a_n - b_n \to 0,\ b_n \rightharpoonup b) \Rightarrow a_n \rightharpoonup b$ ☺

(vi) from (iii): $x_n \rightharpoonup x$

/* recall Proposition 4.11 (i): Proposition 4.11: [C: closed affine subspace of $\mathcal{H}$] $\Rightarrow$ (i) P_C: weakly continuous (ii) $\forall_{x \in \mathcal{H}} \forall_{y \in \mathcal{H}} \ \|P_C x - P_C y\|^2 = \langle x - y \mid P_C x - P_C y \rangle$ */

as C: closed affine subspace of $\mathcal{H}$

$\Rightarrow P_C$: weakly continuous /* $x_n \rightharpoonup x \Rightarrow P_C x_n \rightharpoonup P_C x$ */ $\therefore P_C x_n \rightharpoonup P_C x$

C: closed affine subspace

/* now Example 23.4: [C: nonempty closed convex subset of $\mathcal{H}$] ...

$P_C x = J_{N_C} x$ // now given: $N_C = A$, also: $N_C = \gamma N_C$

as $N_C x = \begin{cases} \{u \in \mathcal{H} \mid \sup \langle C - x \mid u \rangle \leq 0\}, & x \in C \\ \emptyset; & \text{else} \end{cases}$

$\gamma N_C x = \begin{cases} \{\gamma u \in \mathcal{H} \mid \sup \langle C - x \mid u \rangle \leq 0 \leftrightarrow \sup \langle C - x \mid \underbrace{\gamma u}_{\in \mathbb{R}_{++}} \rangle \leq 0\} = \{\tilde{u} \in \mathcal{H} \mid \sup \langle C - x \mid \tilde{u} \rangle \leq 0\} = N_C x \\ \emptyset \quad \text{else} \end{cases}$ $\therefore N_C = \gamma N_C$

$= J_{\gamma N_C} x = J_{\gamma A} x$ $\therefore \boxed{P_C = J_{\gamma A}}$

$\boxed{P_C x_n \rightharpoonup P_C x = J_{\gamma A} x}$

now let us use: $T_1 := J_{\gamma A},\ T_2 = J_{\gamma B}$

/* Proposition 4.31: /* A specific composition of two firmly nonexpansive operators being firmly nonexpansive */ [$T_1: \mathcal{H} \to \mathcal{H}$, firmly nonexpansive; $T_2: \mathcal{H} \to \mathcal{H}$, firmly nonexpansive; $T = T_1(2T_2 - \mathrm{Id}) + \mathrm{Id} - T_2$] $\Rightarrow$ $T = J_{\gamma A}(2J_{\gamma B} - \mathrm{Id}) + \mathrm{Id} - J_{\gamma B}$

(i) $2T - \mathrm{Id} = (2T_1 - \mathrm{Id})(2T_2 - \mathrm{Id})$

(ii) T: firmly nonexpansive

(iii) $\operatorname{Fix} T = \operatorname{Fix}(2T_1 - \mathrm{Id})(2T_2 - \mathrm{Id})$ $\Leftrightarrow \operatorname{Fix}(\underbrace{J_{\gamma A}(2J_{\gamma B} - \mathrm{Id}) + \mathrm{Id} - J_{\gamma B}}_{\check{T}}) = \operatorname{Fix}(\underbrace{(2J_{\gamma A} - \mathrm{Id})}_{R_{\gamma A}} \underbrace{(2J_{\gamma B} - \mathrm{Id})}_{R_{\gamma B}}) = \operatorname{Fix} R_{\gamma A} R_{\gamma B} = \operatorname{Fix} T$

(iv) T_1: projector onto a closed affine subspace $\Rightarrow \operatorname{Fix} T = \{x \in \mathcal{H} \mid T_1 x = T_2 x\}$ */

$\therefore x \in \operatorname{Fix} T \Leftrightarrow x \in \operatorname{Fix} \check{T} \Leftrightarrow x \in \{\check{x} \in \mathcal{H} \mid \underbrace{J_{\gamma A}}_{= P_C} \check{x} = J_{\gamma B} \check{x}\} = \{\check{x} \in \mathcal{H} \mid P_C \check{x} = J_{\gamma B} \check{x}\} \Leftrightarrow \boxed{P_C x = J_{\gamma B} x}$

$\boxed{P_C x_n \rightharpoonup J_{\gamma B} x}$ ☺

(vii) let $y = J_{\gamma B} x$

from (i): $J_{\gamma B} x \in \operatorname{zer}(A+B)$

from (iii): $x_n \rightharpoonup x$

now as uniform monotonicity $\Rightarrow$ strong monotonicity

as $A \vee B$: uniformly monotone (by (a) or (b)) we have $A \vee B$: strictly monotone

$\therefore A + B$: strictly monotone // $\check{A}$: monotone, $\check{B}$: strictly monotone $\Rightarrow \check{A} + \check{B}$: strictly monotone

$\Rightarrow \operatorname{zer}(A+B) = $ singleton /* $\because$ Proposition 23.35: [$A: \mathcal{H} \to 2^{\mathcal{H}}$] $\operatorname{zer} A$: singleton */

$\operatorname{zer}(A+B) = \{y\} = \{J_{\gamma B} x\}$

Now from (iii): $x_n \rightharpoonup x$

" (iv): $y_n \rightharpoonup J_{\gamma B} x = y$

" (v): $z_n \rightharpoonup J_{\gamma B} x = y$

(eq: 25.15)

(Eq:25.11) says: $v_n = x_n - y_n \rightharpoonup x - y$

$w_n = 2y_n - x_n - z_n \rightharpoonup 2y - x - y = y - x$ } (25.15)

from (Eq: B):

$(z_n, w_n) \in \operatorname{gra} \gamma A$: $(z_n, w_n) \rightharpoonup (y, y - x)$

$(y_n, v_n) \in \operatorname{gra} \gamma B$: $(y_n, v_n) \rightharpoonup (y, x - y)$ } using Corollary 25.5 $\to$ $(y, y - x) \in \operatorname{gra} \gamma A$, $(y, x - y) \in \operatorname{gra} \gamma B$ } (eq: 25.16)

Corollary 25.5: [$m \in \mathbb{N}: m \geq 2$; $I = \{1, \dots, m\}$; $(A_i)_{i \in I}$: maximally monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$; $\forall_{i \in I} \ ((x_{i,n}, u_{i,n})_{n \in \mathbb{N}} \in \operatorname{gra} A_i$; $(x_i, u_i) \in \mathcal{H} \times \mathcal{H}$;
$\sum_{i \in I} u_{i,n} \to 0$; $\forall_{i \in I} \ x_{i,n} \rightharpoonup x_i$, $u_{i,n} \rightharpoonup u_i$, $m x_{i,n} - \sum_{j \in I} x_{j,n} \to 0$] $\Rightarrow \exists_{x \in \operatorname{zer} \sum_{i \in I} A_i}$
(i) $x = x_1 = \dots = x_m$
(ii) $\sum_{i \in I} u_i = 0$
(iii) $\forall_{i \in I} \ (x, u_i) \in \operatorname{gra} A_i$
(iv) $\sum_{i \in I} \langle x_{i,n} \mid u_{i,n} \rangle \to \langle x \mid \sum_{i \in I} u_i \rangle = 0$

Now let us consider possibility (vii)(a): A uniformly monotone on every nonempty bounded subset of $\operatorname{dom} A$

set. $C = \{y = J_{\gamma B} x\} \cup \{z_n\}_{n \in \mathbb{N}}$

from (v): $z_n \rightharpoonup J_{\gamma B} x \Rightarrow z_n$: bounded sequence

from (eq: A): $(z_n, w_n) \in \operatorname{gra} \gamma A \leftrightarrow \gamma A z_n \ni w_n \leftrightarrow A z_n \ni \frac{1}{\gamma} w_n \leftrightarrow z_n \in \operatorname{dom} A$

$\Rightarrow (z_n)_{n \in \mathbb{N}}$: bounded sequence on $\operatorname{dom} A$

From (eq: 25.14): $y \in \operatorname{dom} A$ $\rightsquigarrow$ $C = \{y\} \cup \{z_n\}_{n \in \mathbb{N}}$: bounded subset on $\operatorname{dom} A$

*/

set. $C = \{y = J_{\gamma B} x\} \cup \{z_n\}_{n\in\mathbb{N}}$

from (v): $z_n \to J_{\gamma B} x \Rightarrow z_n$: bounded sequence

from (eq: A): $(z_n, w_n) \in \operatorname{gra} \gamma A \leftrightarrow \gamma A z_n \ni w_n \leftrightarrow A z_n \ni \frac{1}{\gamma} w_n \leftrightarrow z_n \in \operatorname{dom} A$

$\Rightarrow (z_n)_{n\in\mathbb{N}}$: bounded sequence on dom A

From (eq: 25.14): $y \in \operatorname{dom} A \rightsquigarrow C = \{y\} \cup \{z_n\}_{n\in\mathbb{N}}$: bounded subset on dom A

Now: recall

$\gamma A z_n \ni w_n \leftrightarrow A z_n \ni \frac{1}{\gamma} w_n \Rightarrow z_n \in \operatorname{dom} A,\ \frac{1}{\gamma} w_n \in A z_n$

$\gamma A y \ni y - x \leftrightarrow A y \ni \frac{1}{\gamma}(y-x) \Rightarrow y \in \operatorname{dom} A,\ \frac{1}{\gamma}(y-x) \in A y$

set $C := \operatorname{dom} A$,

$\phi_A : \mathbb{R}_+ \to [0, +\infty]$, vanishes only at 0

$\langle z_n - y \mid \frac{1}{\gamma}(w_n - y + x) \rangle \geq \phi_A(\|z_n - y\|) \quad \forall_{n\in\mathbb{N}}$

$\leftrightarrow \langle z_n - y \mid w_n - y + x \rangle \geq \gamma \phi_A(\|z_n - y\|) \quad \forall_{n\in\mathbb{N}} \quad \dots$ (eq: 25.17)

As γB : monotone

$\Rightarrow \forall_{n\in\mathbb{N}} \quad \langle y_n - y \mid v_n - x + y \rangle \geq 0 \quad \dots$ (eq: 25.18)

+

$\forall_{n\in\mathbb{N}} \quad \gamma \phi_A(\|z_n - y\|) \leq \langle \underbrace{z_n - y}_{(z_n - y_n) + (y_n - y)} \mid w_n - y + x \rangle + \langle y_n - y \mid v_n - x + y \rangle$

//again the one change at a time rule from real analysis

$= \langle z_n - y_n \mid w_n - y + x \rangle + \underbrace{\langle y_n - y \mid w_n - y + x \rangle + \langle y_n - y \mid v_n - x + y \rangle}_{\langle y_n - y \mid w_n + v_n - y + x - x + y \rangle = \langle y_n - y \mid w_n + v_n \rangle = \langle y_n - y \mid y_n - z_n \rangle} = \langle z_n - y_n \mid w_n - y + x \rangle - \langle z_n - y_n \mid y_n - y \rangle = \langle z_n - y_n \mid w_n - \cancel{y} + x - y_n + \cancel{y} \rangle = \langle z_n - y_n \mid w_n - y_n + x \rangle$

$\therefore \forall_{n\in\mathbb{N}} \quad 0 \leq \gamma \phi_A(\|z_n - y\|) \leq \langle z_n - y_n \mid w_n - y_n + x \rangle$

/* now: from (ii): $z_n - y_n \to 0$

from (25.15): $y_n \to y$

$w_n \to y - x,\ y_n \to y$

$\Rightarrow w_n - y_n \to -y + y - x = -x$

$\leftrightarrow w_n - y_n + x \to 0$

Lemma 2.41.

$\langle z_n - y_n \mid w_n - y_n + x \rangle \to 0$ */

$\Rightarrow$ as $n \to \infty$,

$0 \leq \gamma \phi_A(\|z_n - y\|) \leq 0$

$\therefore$ as $n \to \infty$, $\phi_A(\|z_n - y\|) = 0$

$\leftrightarrow$ as $n \to \infty$ $\|z_n - y\| \to 0$

$\leftrightarrow z_n \to y = J_{\gamma B} x$

$\therefore z_n \to J_{\gamma B} x,\ y_n \to J_{\gamma B} x$ /* as $z_n - y_n \to 0$ */ ¡

When possibility (vii)(b): the logic is same except roles of A, B are interchanged ¡

■

Part 3: Parallel splitting alg, Forward-backward alg

6:37 PM

Proposition 25.7. (Parallel splitting algorithm)

[$m \in \mathbb{N}: m \geq 2$; $I = \{1, \dots, m\}$; $(A_i)_{i\in I}$: maximally monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$; $\operatorname{zer}\sum_{i\in I} A_i \neq \emptyset$;

$(\lambda_n)_{n\in\mathbb{N}}$: $\subseteq [0,2]$, $\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$; $\gamma \in \mathbb{R}_{++}$; $(x_{i,0})_{i\in I} \in \mathcal{H}^m$;

$$\forall_{n\in\mathbb{N}} \left| \begin{array}{l} p_n = \frac{1}{m}\sum_{i\in I} x_{i,n} \\ \forall_{i\in I}\ y_{i,n} = J_{\gamma A_i} x_{i,n} \\ q_n = \frac{1}{m}\sum_{i\in I} y_{i,n} \\ \forall_{i\in I}\ x_{i,n+1} = x_{i,n} + \lambda_n(2q_n - p_n - y_{i,n}) \end{array} \right. \quad \text{(eq: 25.21)} \quad] \Rightarrow$$

$(p_n)_{n\in\mathbb{N}}$ converges weakly to a point in $\operatorname{zer}\sum_{i\in I} A_i$

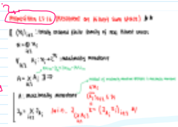
Proof:

define: $\tilde{\mathcal{H}} = \bigoplus_{i\in I} \mathcal{H}$

$\tilde{D} = \{(x, \dots, x) \in \tilde{\mathcal{H}} : x \in \mathcal{H}\}$: consensus closed linear space $\Rightarrow$ closed, convex set

$\tilde{j} : \mathcal{H} \to \tilde{D} : x \mapsto (x, x, \dots, x)$; $\tilde{B} = \times_{i\in I} A_i$ // $\tilde{B}\underbrace{\tilde{x}}_{(\tilde{x}_1, \dots, \tilde{x}_m)} = (A_1\tilde{x}_1, A_2\tilde{x}_2, \dots, A_m\tilde{x}_m)$, $\tilde{x}_1, \dots, \tilde{x}_m \in \mathcal{H}$

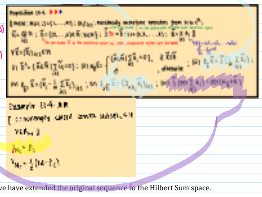
now $\tilde{B} = \times_{i\in I} \underbrace{A_i}_{\text{maximally monotone}}$: maximally monotone using:



define $\tilde{A} = N_{\tilde{D}}$: maximally monotone

// $\because$ the normal cone operator of a
// nonempty closed convex set is maximally
// monotone : Example 20.41.

Using Proposition 25.4:



$\forall \tilde{x} = (\tilde{x}_i)_{i\in I} \in \tilde{\mathcal{H}}$

$\boxed{J_{\gamma\tilde{A}}\tilde{x} = P_{\tilde{D}}\tilde{x}} = \tilde{j}\left(\frac{1}{m}\sum_{i\in I}\tilde{x}_i\right)$ (eq: 25.22 (i))

$J_{\gamma\tilde{B}}\tilde{x} = (J_{\gamma A_i}\tilde{x}_i)_{i\in I}$ (eq: 25.22 (ii)) // using Proposition 23.4

$\tilde{j}(\operatorname{zer}\sum_{i\in I} A_i) = \operatorname{zer}(\tilde{A} + \tilde{B})$

set: $\forall_{n\in\mathbb{N}} \left| \begin{array}{l} \tilde{x}_n = (x_{i,n})_{i\in I} \in \tilde{\mathcal{H}} \\ \tilde{p}_n = \tilde{j}(p_n) \in \tilde{\mathcal{H}} \\ \tilde{y}_n = (y_{i,n})_{i\in I} \in \tilde{\mathcal{H}} \\ \tilde{q}_n = \tilde{j}(q_n) \in \tilde{\mathcal{H}} \end{array} \right.$ // So we have extended the original sequence to the Hilbert Sum space.

now using (eq: 25.22) and (eq: 25.21):

setting $\tilde{x} := \tilde{x}_n$ in (eq: 25.22 (i))

$J_{\gamma\tilde{A}}\tilde{x}_n = \tilde{j}\big(\underbrace{\tfrac{1}{m}\sum x_{i,n}}_{p_n \text{ // by (25.21)}}\big) = \tilde{j}(p_n) = \tilde{p}_n$

setting $\tilde{x} := \tilde{x}_n \in \tilde{\mathcal{H}}$ in eq: 25.22 (ii) we have: $J_{\gamma\tilde{B}}\tilde{x}_n = (\underbrace{J_{\gamma A_i} x_{i,n}}_{y_{i,n} \text{ // by eq: 25.21}})_{i\in I} = (y_{i,n})_{i\in I} = \tilde{y}_n$

" $\tilde{x} := \tilde{y}_n \in \tilde{\mathcal{H}}$ " eq: 25.22 (i) we have:

$J_{\gamma\tilde{A}}\tilde{y}_n = \tilde{j}\big(\underbrace{\tfrac{1}{m}\sum_{i\in I} y_{i,n}}_{q_n \text{ // by eq: 25.21}}\big) = \tilde{j}(q_n) = \tilde{q}_n$

Finally concatenating $(x_{i,n+1} = x_{i,n} + \lambda_n(2q_n - p_n - y_{i,n}))_{i\in I}$ // the individual components are the last given iteration

we have: $\tilde{x}_{n+1} = \tilde{x}_n + \lambda_n(2\tilde{j}(q_n) - \tilde{j}(p_n) - \tilde{y}_n) = \tilde{x}_n + \lambda_n(2\tilde{q}_n - \tilde{p}_n - \tilde{y}_n)$

so, we have the concatenated sequence:

$$\forall_{n\in\mathbb{N}} \left| \begin{array}{l} \tilde{p}_n = J_{\gamma\tilde{A}}\tilde{x}_n \\ \tilde{y}_n = J_{\gamma\tilde{B}}\tilde{x}_n \\ \tilde{q}_n = J_{\gamma\tilde{A}}\tilde{y}_n \\ \tilde{x}_{n+1} = \tilde{x}_n + \lambda_n(2\tilde{q}_n - \tilde{p}_n - \tilde{y}_n) = \tilde{x}_n + \lambda_n(\underbrace{J_{\gamma\tilde{A}}(2\tilde{y}_n - \tilde{x}_n)}_{\tilde{z}_n} - \tilde{y}_n) \end{array} \right.$$

these two are used in $\tilde{x}_{n+1}$, so not needed again

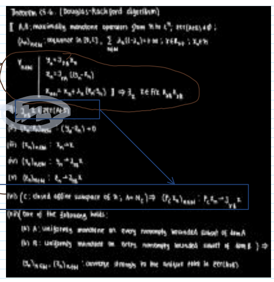
$2J_{\gamma\tilde{A}}\tilde{y}_n - J_{\gamma\tilde{A}}\tilde{x}_n - \tilde{y}_n$
$= J_{\gamma\tilde{A}}(2\tilde{y}_n - \tilde{x}_n) - \tilde{y}_n$

/* now $J_{\gamma\tilde{A}} = P_{\tilde{A}}$ wher $\tilde{A}$: closed linear subspace

$\Rightarrow J_{\gamma\tilde{A}} = P_{\tilde{A}}$: linear // Corollary 3.22 (iii) : projector on a closed linear subspace is a linear operator */

$\therefore 2J_{\gamma\tilde{A}}\tilde{y}_n - J_{\gamma\tilde{A}}\tilde{x}_n = J_{\gamma\tilde{A}}(2\tilde{y}_n - \tilde{x}_n)$

$$\Leftrightarrow \forall_{n\in\mathbb{N}} \left\{ \begin{array}{l} \tilde{y}_n = J_{\gamma\tilde{B}}\tilde{x}_n \\ \tilde{z}_n = J_{\gamma\tilde{A}}(2\tilde{y}_n - \tilde{x}_n) \\ \tilde{x}_{n+1} = \tilde{x}_n + \lambda_n(\tilde{z}_n - \tilde{y}_n) \end{array} \right.$$



So, all the results of D-R splitting will hold:

now in (vii) we have:

$\tilde{D}$: closed linear subspace ($\Downarrow$ affine), $\tilde{A} = N_{\tilde{D}}$, $\tilde{B}$ is the other operator

$\therefore \underbrace{P_{\tilde{D}}\tilde{x}_n}_{J_{\gamma\tilde{A}}\tilde{x}_n = \tilde{p}_n} \to \underbrace{J_{\gamma\tilde{B}}\tilde{x}^*}_{} \in \operatorname{zer}(\tilde{A} + \tilde{B}) = \tilde{j}(\operatorname{zer}\sum_{i\in I} A_i)$

// where $\tilde{x}^* \in \operatorname{Fix} R_{\gamma\tilde{A}}R_{\gamma\tilde{B}}$

now $J_{\gamma\tilde{B}}\tilde{x}^* \in \tilde{j}(\operatorname{zer}\sum_{i\in I} A_i) = (\operatorname{zer}\sum_{i\in I} A_i, \dots, \operatorname{zer}\sum_{i\in I} A_i)$

$\Leftrightarrow \exists_{x \in \operatorname{zer}\sum_{i\in I} A_i}\ J_{\gamma\tilde{B}}\tilde{x}^* = \tilde{j}(x)$

$\Rightarrow \underbrace{(\tilde{p}_n) = \tilde{j}(p_n)}_{\text{// by def}} \to \tilde{j}(x) : x \in \operatorname{zer}\sum_{i\in I} A_i$

$\Leftrightarrow p_n \to x : x \in \operatorname{zer}\sum_{i\in I} A_i$ ∎

Theorem 25.8. (Forward-backward algorithm)

[$A : \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone ; $\beta \in \mathbb{R}_{++}$; $B : \mathcal{H} \to \mathcal{H}$, β-cocercive ; $\gamma \in]0, 2\beta[$; $\delta = \min\{1, \frac{\beta}{\gamma}\} + \frac{1}{2}$;

$(\lambda_n)_{n\in\mathbb{N}}$: sequence in $[0, \delta]$, $\sum_{n\in\mathbb{N}} \lambda_n(\delta - \lambda_n) = +\infty$; $x_0 \in \mathcal{H}$; $\operatorname{zer}(A+B) \neq \emptyset$;

$$\forall_{n\in\mathbb{N}} \left[\begin{array}{l} y_n = x_n - \gamma B x_n \\ x_{n+1} = x_n + \lambda_n(J_{\gamma A} y_n - x_n) \end{array} \right\} \text{(eq: 25.26)} \quad] \Rightarrow$$

(i) $(x_n)_{n\in\mathbb{N}}$: converges weakly to a point in $\operatorname{zer}(A+B)$

(ii) $(\inf_{n\in\mathbb{N}} \lambda_n > 0$, $x \in \operatorname{zer}(A+B)) \Rightarrow (Bx_n)_{n\in\mathbb{N}} : Bx_n \to Bx$

(iii) ($\inf_{n\in\mathbb{N}} \lambda_n > 0$, one of the following holds:

(a) A: uniformly monotone on every nonempty bounded subset of dom A

(b) B: uniformly monotone on every nonempty bounded subset of $\mathcal{H}$) $\Rightarrow$

$(x_n)_{n\in\mathbb{N}}$: converges strongly to the unique point in $\operatorname{zer}(A+B)$

Proof:

set $T = J_{\gamma A} \circ (Id - \gamma B)$

$\alpha = \frac{1}{\delta}$

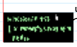
Recall:

/* Remark 4.24. (iii) : $\tilde{T}$: firmly nonexpansive $\Leftrightarrow \tilde{T}$: $\frac{1}{2}$ averaged $\overset{\text{def}}{\Leftrightarrow} \exists_{R : \text{nonexpansive}}\ \tilde{T} = \frac{1}{2}Id + \frac{1}{2}R$ */

/* Corollary 23.8. ($\tilde{T} : \mathcal{H} \to \mathcal{H}$, firmly nonexpansive) $\Leftrightarrow \exists_{A : \mathcal{H} \to 2^{\mathcal{H}}, \text{ maximally monotone}}\ \tilde{T} = J_{\gamma A}$, $\gamma \in \mathbb{R}_{++}$ */

Now A: maximally monotone $\Rightarrow J_{\gamma A}$: firmly nonexpansive /* Proposition 23.7 ii), (i) */

$\Leftrightarrow J_{\gamma A}$: $\frac{1}{2}$ averaged

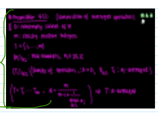


using this

$(Id - \gamma B)$: $\frac{\gamma}{2\beta}$ averaged

$\frac{1}{2}$ averaged

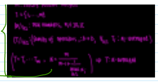
using Proposition 4.32



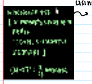
$\therefore T = J_{\gamma A} \circ (Id - \gamma B)$: ...

monotone $\quad \gamma \in \mathbb{R}_{++}$

Now A: maximally monotone $\Rightarrow J_{\gamma A}$: firmly nonexpansive /* Proposition 23.7 ii),(i) */



$\Leftrightarrow J_{\gamma A} : \frac{1}{2}$ averaged



using this

$(Id - \gamma B) : \frac{\gamma}{2\beta}$ averaged

$\therefore T = J_{\gamma A} \circ (Id - \gamma B) :$ ($\frac{1}{2}$ averaged, $\frac{\gamma}{2\beta}$-averaged) using Proposition 4.32

$$\frac{2}{2-1+\frac{1}{\max\{\frac{1}{2}, \frac{\gamma}{2\beta}\}}} = \frac{2}{1+2\min\{1, \beta/\gamma\}} = \frac{\cancel{2}\,1}{\cancel{2}\underbrace{(\frac{1}{2}+\min\{1,\frac{\beta}{\gamma}\})}_{\delta}} = \frac{1}{\delta} = \kappa : \text{averaged}$$

$\frac{1}{2}\max\{1, \frac{\gamma}{\beta}\} = \frac{1}{2} \cdot \frac{1}{\min\{1, \frac{\beta}{\gamma}\}}$

// as $\max\{a,b\} = \frac{1}{\min\{\frac{1}{a}, \frac{1}{b}\}}$

$\Leftrightarrow \boxed{T : \kappa\text{-averaged}}$

/* Proposition 25.1 (iv) :

$(A$: monotone, B: almost single-valued$) \Rightarrow \operatorname{zer}(A+B) = \operatorname{Fix} J_{\gamma A} \circ (Id - \gamma B)$ */ $\therefore$ $\boxed{\operatorname{Fix} \underbrace{J_{\gamma A} \circ (Id-\gamma B)}_{T} = \operatorname{Fix} T = \operatorname{zer}(A+B)}$

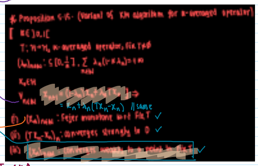
from (eq:25.26) :

$$\forall_{n\in\mathbb{N}} \left[\begin{array}{l} y_n = x_n - \gamma B x_n = (Id - \gamma B) x_n \\ x_{n+1} = x_n + \lambda_n (J_{\gamma A} y_n - x_n) \end{array} \right.$$

$$= x_n + \lambda_n \big(\underbrace{(J_{\gamma A} \circ (Id - \gamma B))}_{T : \kappa\text{-averaged}} x_n - x_n \big)$$

$$= x_n + \lambda_n (T x_n - x_n) \quad \ldots \text{(eq:A)}$$

now we use Proposition 5.15 (all the notations are the same)



$= x_n + \lambda_n (T x_n - x_n)$ // same ✓ ✓ ✓

thus (i), (ii), (iii) will hold.

(i) From Proposition 5.15 (iii) :

$(x_n)_{n\in\mathbb{N}}$: converges weakly to a point in $\operatorname{zer}(A+B)$ "Fix T" and $(x_n)_{n\in\mathbb{N}}$ is fejer monotone w.r.t Fix T. and

$T x_n - x_n \to 0 \quad \ldots$ (eq:B) (eq:B)

$\therefore x_n \rightharpoonup x ; x \in \operatorname{zer}(A+B)$

(ii) given, $\varepsilon = \inf_{n\in\mathbb{N}} \lambda_n > 0$, $x \in \operatorname{zer}(A+B)$

$$\forall_{n\in\mathbb{N}} \; \|\underbrace{T}_{J_{\gamma A} \circ (Id - \gamma B)} x_n - x\|^2 = \| J_{\gamma A} (x_n - \gamma B x_n - x) - x \|^2$$

/* $x \in \operatorname{Fix} T = \operatorname{Fix} J_{\gamma A} \circ (Id - \gamma B)$

$\Leftrightarrow \underbrace{\underbrace{J_{\gamma A}}_{\text{single-valued}} \circ \underbrace{(Id - \gamma B)}_{\text{single-valued}}}_{\text{single-valued}} x = x$ */

$$= \| J_{\gamma A} (x_n - \gamma B x_n) - J_{\gamma A} (x - \gamma B x) \|^2 \quad \text{/* } J_{\gamma A} \text{ : nonexpansive. */}$$

$$\leq \| (x_n - \gamma B x_n) - (x - \gamma B x) \|^2 \quad \text{//applying the nonexpansiveness to } (x_n - \gamma B x_n), (x - \gamma B x)$$

$$= \| (Id - \gamma B) x_n - (Id - \gamma B) x \|^2 \quad \text{// now } (Id - \gamma B) : \tilde{\kappa} = \tfrac{\gamma}{2\beta} \text{ averaged}$$

/* Proposition 4.25 (iii). [D: nonempty subset of $\mathcal{H}$; $\tilde{T} : D \to \mathcal{H}$, nonexpansive ; $\kappa \in]0,1[$]

$\forall_{\tilde{x}\in D} \forall_{\tilde{y}\in D} \; \|\tilde{T}\tilde{x} - \tilde{T}\tilde{y}\|^2 \leq \|\tilde{x} - \tilde{y}\|^2 - \frac{1-\kappa}{\kappa} \|(Id - \tilde{T})\tilde{x} - (Id - \tilde{T})\tilde{y}\|^2$ */

set $\tilde{T} := (Id - \gamma B)$, $\tilde{x} := x_n$, $\tilde{y} := x$

$$\leq \|x_n - x\|^2 - \frac{1-\tilde{\kappa}}{\tilde{\kappa}} \| \underbrace{(Id - (Id - \gamma B))}_{\gamma B} x_n - \underbrace{(Id - (Id - \gamma B))}_{= \gamma B} x \|^2$$

$\frac{1-\tilde{\kappa}}{\tilde{\kappa}} = (1 - \frac{\gamma}{2\beta}) / \frac{\gamma}{2\beta} = \frac{2\beta - \gamma}{2\beta} \cdot \frac{2\beta}{\gamma} = \frac{2\beta - \gamma}{\gamma}$

$$= \|x_n - x\|^2 - \frac{2\beta - \gamma}{\gamma} \underbrace{\| \gamma B x_n - \gamma B x \|^2}_{\gamma^2 \|B x_n - B x\|^2} \quad \text{// } \gamma \text{: linear operator LOL}$$

$$= \|x_n - x\|^2 - \frac{2\beta - \gamma}{\cancel{\gamma}} \cancel{\gamma^2}\,\gamma \|B x_n - B x\|^2$$

$$= \|x_n - x\|^2 - \gamma(2\beta - \gamma) \|B x_n - B x\|^2$$

$$\therefore \forall_{n\in\mathbb{N}} \; \|T x_n - x\|^2 \leq \|x_n - x\|^2 - \gamma(2\beta - \gamma) \|B x_n - B x\|^2 \quad \ldots \text{(eq:25.27)}$$

$\delta = \min\{1, \frac{\beta}{\gamma}\} + \frac{1}{2}$ ($\gamma \in]0, 2\beta[$) now note that $\delta > 1$ as $\delta = \underbrace{1 + \frac{1}{2}}_{>1}$ or $\frac{\beta}{\gamma} + \frac{1}{2} = \frac{\beta}{\kappa 2\beta} + \frac{1}{2} = \overbrace{\left(\frac{1}{\kappa}\right)}^{>1} \frac{1}{2} + \frac{1}{2} > 1$ ($\gamma \in]0, 2\beta[\Rightarrow \exists_{\kappa \in]0,1[} \; \gamma = \kappa 2\beta$)

//another way of seeing this is $\delta = \frac{1}{\alpha}$ now $\alpha \in]0,1[$ as T: κ averaged $\Rightarrow \delta > 1$ */

So, we have: $\forall_{n\in\mathbb{N}}$

$$\|x_{n+1} - x\|^2$$

$$= \| x_n + \lambda_n (T x_n - x_n) - x \|^2$$

$$= \| \underbrace{x_n + \lambda_n T x_n - \lambda_n x_n - (1 - \lambda_n + \lambda_n) x}_{(1-\lambda_n) x_n - (1-\lambda_n) x + \lambda_n T x_n - \lambda_n x} \|^2$$

$$= \| (1-\lambda_n)(x_n - x) + \lambda_n (T x_n - x) \|^2$$

/* Using Corollary 2.14. $\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}} \forall_{\alpha\in\mathbb{R}} \|\alpha x + (1-\alpha) y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2$ */

$$= (1-\lambda_n) \|x_n - x\|^2 + \lambda_n \|T x_n - x\|^2 - \lambda_n (1-\lambda_n) \|T x_n - x - x_n + x\|^2$$

$$\leq \underbrace{(1-\lambda_n) \|x_n - x\|^2 + \lambda_n \|x_n - x\|^2}_{\|x_n - x\|^2} - \lambda_n \gamma (2\beta - \gamma) \|B x_n - B x\|^2 - \underbrace{\lambda_n}_{[0,\delta]} (1-\lambda_n) \|T x_n - x_n\|^2$$

($\delta > 1$)

$$= \|x_n - x\|^2 - \gamma(2\beta - \gamma) \lambda_n \|B x_n - B x\|^2 + \lambda_n (\lambda_n - 1) \|T x_n - x_n\|^2$$

// as $(\lambda_n)_{n\in\mathbb{N}}$: is bounded below $\Rightarrow$ inf will exist

// $\varepsilon = \inf_{n\in\mathbb{N}} \lambda_n > 0 \Leftrightarrow \forall_{n\in\mathbb{N}} \; \varepsilon \leq \lambda_n \Leftrightarrow \forall_{n\in\mathbb{N}} \; -\varepsilon \geq -\lambda_n \Leftrightarrow \boxed{\forall_{n\in\mathbb{N}} \; -\gamma(2\beta - \gamma)\varepsilon \geq -\gamma(2\beta - \gamma)\lambda_n}$ ($2\beta - \gamma > 0$ as $\gamma \in]0, 2\beta[$)

// $\lambda_n \in [0, \delta] \Leftrightarrow \lambda_n \leq \delta \Leftrightarrow \lambda_n - 1 \leq \delta - 1$ // as $\delta > 1$, $\lambda_n \leq \delta$, $\lambda_n - 1$, $\delta - 1$ are positive

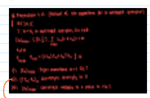
$\boxed{\lambda_n (\lambda_n - 1) \leq \delta(\delta - 1)}$ //recall, $a, b, c, d \in \mathbb{R}_+$; $(a \geq b, c \geq d) \Rightarrow (ac \geq bd)$

$$\leq \|x_n - x\|^2 - \gamma(2\beta - \gamma)\varepsilon \|B x_n - B x\|^2 + \delta(\delta - 1) \|T x_n - x_n\|^2$$

$$\therefore \forall_{n\in\mathbb{N}} \; \|x_{n+1} - x\|^2 \leq \|x_n - x\|^2 - \gamma(2\beta - \gamma)\varepsilon \|B x_n - B x\|^2 + \delta(\delta - 1) \|T x_n - x_n\|^2 \quad \text{(eq:25.28)}$$

$$\Leftrightarrow \gamma(2\beta - \gamma)\varepsilon \|B x_n - B x\|^2 \leq \underbrace{\|x_n - x\|^2 - \|x_{n+1} - x\|^2}_{\to 0} + \delta(\delta - 1) \underbrace{\|T x_n - x_n\|^2}_{\to 0}$$

// now in (i) we have shown that $(x_n)_{n\in\mathbb{N}}$: Fejer monotone w.r.t Fix T



$\therefore \forall_{x\in \operatorname{Fix} T} \; \|x_n - x\|$: converges

$\Leftrightarrow \boxed{\|x_n - x\|^2 - \|x_{n+1} - x\|^2 \to 0}$

$\boxed{T x_n - x_n \to 0}$

$\therefore$ as $n \to \infty$

$$\gamma(2\beta - \gamma)\varepsilon \|B x_n - B x\|^2 \to 0$$

$\Leftrightarrow \boxed{B x_n \to B x}$:)

(iii) in (i) we have proven that $\exists_{x\in \operatorname{zer}(A+B)} \; x_n \rightharpoonup x$

$\Downarrow$

$A x + B x \ni 0 \Leftrightarrow \boxed{\gamma A x \ni -\gamma B x \text{ // as } Bx \text{ : singleton}}$

$\boxed{x \in \operatorname{dom} A, \; -Bx \in Ax}$

suppose (iii)(a) holds:

set, $\forall_{n\in\mathbb{N}} \; z_n = J_{\gamma A} y_n \Leftrightarrow \gamma A z_n \ni y_n - z_n$ /* as $p \in J_{\gamma A} x \Leftrightarrow x - p \in \gamma A p$ using Proposition 23.2.(ii) */

$\Rightarrow \boxed{\forall_{n\in\mathbb{N}} \; \underbrace{x_n - \gamma B x_n}_{y_n \text{ // via iteration definition}} - z_n = y_n - z_n \in \gamma A z_n}$

$\boxed{z_n \in \operatorname{dom} A}$

$\frac{1}{\gamma}(x_n - \gamma B x_n - z_n) \in A z_n$

now :

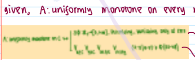
$$z_n - x_n = J_{\gamma A} \underbrace{y_n}_{(Id - \gamma B) x_n \text{ // by iteration scheme}} - x_n = \overbrace{(J_{\gamma A} \circ (Id - \gamma B))}^{T} x_n - x_n = T x_n - x_n \to 0 \quad \text{// from (eq:B)}$$

$\Leftrightarrow z_n \to x_n$, but $x_n \rightharpoonup x$

$\Rightarrow \boxed{z_n \rightharpoonup x} \Rightarrow$ clearly $\boxed{(z_n)_{n\in\mathbb{N}} : \text{bounded}}$ $\quad (z_n)_{n\in\mathbb{N}}$: bounded on dom A

now: $C = \{x\} \cup \{z_n\}_{n\in\mathbb{N}}$: bounded subset of dom A

given, A: uniformly monotone on every nonempty bounded subset of dom A



set $C := \operatorname{dom} A$,

$\exists \phi_A : \mathbb{R}_+ \to [0, +\infty]$, increasing, vanishing only at zero

$(z_n \in C, \; x \in C, \; \frac{1}{\gamma}(x_n - \gamma B x_n - z_n) \in A z_n, \; -Bx \in Ax) \Rightarrow \langle z_n - x \mid \frac{1}{\gamma}(x_n - \gamma B x_n - z_n) + Bx \rangle \geq \phi_A(\|z_n - x\|)$

$\Leftrightarrow \langle z_n - x \mid \frac{1}{\gamma}(x_n - \gamma B x_n - z_n + \gamma B x) \rangle \geq \phi_A(\|z_n - x\|)$

$\therefore \forall_{n\in\mathbb{N}} \; \langle z_n - x \mid x_n - z_n + \gamma(Bx - B x_n) \rangle \geq \gamma \phi_A(\|z_n - x\|) \geq 0$

$\curvearrowright (z_n \in C \; , x \in C \; , \frac{1}{\gamma}(x_n - \gamma B x_n - z_n) \in A z_n , -Bx \in Ax) \Rightarrow \langle z_n - x \mid \frac{1}{\gamma}(x_n - \gamma B x_n - z_n) + Bx \rangle \geqslant \phi_A(\|z_n - x\|)$

$\Leftrightarrow \langle z_n - x \mid \frac{1}{\gamma}(x_n - \gamma B x_n - z_n + \gamma B x) \rangle \geqslant \phi_A(\|z_n - x\|)$

$\therefore \forall_{n\in\mathbb{N}} \quad \langle z_n - x \mid x_n - z_n + \gamma(Bx - Bx_n) \rangle \geqslant \gamma \phi_A(\|z_n - x\|) \geqslant 0$

now: $\boxed{z_n \rightharpoonup x, \quad x_n \rightharpoonup z_n, \quad \underbrace{Bx_n \to Bx}_{\text{from (i)}} \quad \Downarrow \quad x_n - z_n + \gamma(Bx - Bx_n) \to 0}$ $\Rightarrow \langle z_n - x \mid x_n - z_n + \gamma(Bx - Bx_n) \rangle \to 0$ // using

As $n \to \infty$, $\phi_A(\|z_n - x\|) \to 0$

$\Leftrightarrow \|z_n - x\| \to 0$ /* as ϕ_A: vanishes only at zero */

$\Leftrightarrow z_n \to x : x \in \operatorname{zer}(A+B)$ // A: uniformly monotone $\Rightarrow$ A: strictly monotone, B: monotone $\} \oplus\Rightarrow$ A+B: strictly monotone $\Rightarrow \operatorname{zer}(A+B)$: almost singleton

// But $x_n \rightharpoonup z_n$ — given: $\operatorname{zer}(A+B) \neq \emptyset$

$\therefore \boxed{x_n \to x : \operatorname{zer}(A+B) = \{x\}}$ — $\therefore \operatorname{zer}(A+B) = \{x\}$ */

(iii)(b) holds:
Now B: uniformly monotone on every nonempty bounded subset of $\mathcal{H}$
in (i) we have proven that $\boxed{x_n \rightharpoonup x : x \in \operatorname{zer}(A+B)}$

Clearly $(x_n)_{n\in\mathbb{N}}$: bounded w.r.t. $\operatorname{zer}(A+B)$

$\Rightarrow (x_n)_{n\in\mathbb{N}}$ " " $\mathcal{H}$

$\Rightarrow C = \{x\} \cup \{x_n\}_{n\in\mathbb{N}}$: bounded on $\mathcal{H}$

$\curvearrowright \exists_{\phi_B} : \mathbb{R}_+ \to [0, +\infty]$, increasing, vanishes only at 0

$(x \in C, \; x_n \in C, \; Bx \in Bx, \; Bx_n \in Bx_n) \Rightarrow \langle x_n - x \mid Bx_n - Bx \rangle \geqslant \phi_B(\|x_n - x\|) \geqslant 0$
trivial right!

now $\begin{cases} x_n \rightharpoonup x & \text{// in (i)} \\ Bx_n \to Bx & \text{// in (ii)} \end{cases}$ using

$\Rightarrow \langle x_n - x \mid Bx_n - Bx \rangle \to 0$

$\therefore$ as $n \to \infty$, $\phi_B(\|x_n - x\|) \to 0$

$\Leftrightarrow \|x_n - x\| \to 0$ //as ϕ_B vanishes only at zero

$\Leftrightarrow x_n \to x : \operatorname{zer}(A+B) = \{x\}$ // B: uniformly monotone $\Rightarrow$ B: strictly monotone, A: monotone $\} \oplus\Rightarrow$ A+B: strictly monotone $\Rightarrow \operatorname{zer}(A+B)$: almost singleton

// A strictly monotone operator has a zero set that is at most singleton

given: $\operatorname{zer}(A+B) \neq \emptyset$
$\therefore \operatorname{zer}(A+B) = \{x\}$ */

Proposition 25.9. (Linear convergence of forward-backward algorithm)

[D: nonempty closed subset of $\mathcal{H}$;
A: maximally monotone, $\operatorname{dom} A \subseteq D$;
$B: D \to \mathcal{H}$; $\alpha \in \mathbb{R}_{++}$; $\beta \in \mathbb{R}_{++}$;
One of the following holds:
(i) A: α-strongly monotone; B: β-cocoercive; $\gamma \in]0, 2\beta[$
(ii) $\alpha \leqslant \beta$; B: α-strongly monotone, β-Lipschitz continuous; $\gamma \in]0, \frac{2\alpha}{\beta^2}[$;
$x_0 \in D$

$\forall_{n\in\mathbb{N}} \left| \begin{array}{l} y_n = x_n - \gamma B x_n \\ x_{n+1} = J_{\gamma A} y_n \end{array} \right\}$ (eq. 25.34)] $\Rightarrow$

$(x_n)_{n\in\mathbb{N}}$: converges linearly to the unique point in $\operatorname{zer}(A+B)$.

Proof: set, $T = J_{\gamma A} \circ (Id - \gamma B)$ /* now */

$\Rightarrow \operatorname{Fix} T = \operatorname{zer}(A+B)$

so, $\operatorname{dom} J_{\gamma A} = \mathcal{H}$, $\operatorname{ran} J_{\gamma A} \subseteq \mathcal{H}$ $\}$ as $J_{\gamma A}$: single valued

single valued operator: $T: \mathcal{X} \to \mathcal{Y} \overset{\text{def}}{\Leftrightarrow} \forall_{x\in\mathcal{X}} \exists_{y\in\mathcal{Y}} \; y = Tx$; so $\operatorname{dom} T = \mathcal{X}$, $\operatorname{ran} T \subseteq \mathcal{Y}$

also by def (1.5): $T = J_{\gamma A} \circ (Id - \gamma B)$.

composition of operators: $B \circ A : x \mapsto B(Ax) = \bigcup_{y \in Ax} By$; $\operatorname{dom}(B \circ A) = \ldots$; $\operatorname{ran}(B \circ A) = (B \circ A)(\mathcal{X}) = \bigcup_{x\in\mathcal{X}} (B\circ A)x = \bigcup_{x\in\mathcal{X}} B(Ax) = \bigcup_{x\in\mathcal{X}} \bigcup_{y\in Ax} By$

$\operatorname{dom} T = \underbrace{\operatorname{dom} J_{\gamma A}}_{= \mathcal{H}} \cap \underbrace{\operatorname{dom}(Id - \gamma B)}_{= \operatorname{dom} B = D}$

$= \mathcal{H} \cap D = D$

$\therefore \operatorname{dom} T = D$

$\operatorname{ran} T = \bigcup_{x\in D} (J_{\gamma A} \circ (Id - \gamma B))x = \bigcup_{x\in D} J_{\gamma A}((Id - \gamma B)x) \subseteq \bigcup_{x\in D} J_{\gamma A} x$ /* because in $(Id - \gamma B)x$ only those $x \in \operatorname{dom}(Id - \gamma B) \subseteq D$ will be unioned */

$\subseteq \operatorname{ran} J_{\gamma A}$ /* as by definition $\operatorname{ran} J_{\gamma A} = J_{\gamma A}(\mathcal{H}) = \bigcup_{x\in\mathcal{H}} J_{\gamma A}(x) \supseteq \bigcup_{x\in D} J_{\gamma A}(x)$ */

$= \operatorname{dom} A$ /* as $\operatorname{ran} J_{\gamma A} = \operatorname{dom} A$ by Proposition 23.2 */

$\subseteq D$ /* given $\operatorname{dom} A \subseteq D$ */

$\therefore \operatorname{ran} T \subseteq D$

$\therefore$ T: single valued operator, $D \to D$ /* i.e., $\operatorname{dom} T = D$, $\operatorname{ran} T \subseteq D$ */

Now D: closed subset of $\mathcal{H} \Rightarrow$ D: complete metric space equipped with Euclidean norm
i.e., $(D, \|\cdot\|)$: complete metric space

Now from (eq: 25.34):

$\forall_{n\in\mathbb{N}} \left| \begin{array}{l} y_n = x_n - \gamma B x_n = (Id - \gamma B) x_n \\ x_{n+1} = J_{\gamma A} y_n = \underbrace{J_{\gamma A} (Id - \gamma B)}_{T} x_n = T x_n \end{array} \right\} \Leftrightarrow \forall_{n\in\mathbb{N}} \; x_{n+1} = T x_n$: note that this iteration looks like Banach-Picard iteration, so we are going to use the associated theorem

(Banach-Picard iteration for contraction operator): set $\mathcal{X} := D$, $d = \|\cdot\|$, our goal is proving that T: Lipschitz continuous with constant $[0,1[$, then the results of Banach-Picard iteration will hold and we will be done.

Let us assume that (i) holds i.e., A: α-strongly monotone; B: β-cocoercive; $\gamma \in]0, 2\beta[$

set $\tau := \frac{1}{\alpha\gamma + 1}$, $\alpha\gamma \in \mathbb{R}_{++}$, $\gamma \in]0, 2\beta[$

using this and slightly modifying the theorem

$J_{\gamma A}$: Lipschitz continuous with constant $\tau = \frac{1}{\alpha\gamma+1}$ and $J_{\gamma A}: D \to D \overset{\text{def}}{\Leftrightarrow} \forall_{x,y\in D} \; \|J_{\gamma A} x - J_{\gamma A} y\| \leqslant \tau \|x - y\|$
clearly $\tau \in]0,1[$ as $\alpha\gamma > 0$

now B: β-cocoercive, $\gamma \in]0, 2\beta[$

$(Id - \gamma B)$: $\frac{\gamma}{2\beta}$-averaged $\Rightarrow$ $(Id - \gamma B)$: nonexpansive, // An averaged operator is nonexpansive.

$(Id - \gamma B): D \to \mathcal{H}$

$\overset{\text{def}}{\Leftrightarrow} \forall_{x,y\in D} \; \|(Id - \gamma B)x - (Id - \gamma B)y\| \leqslant \|x - y\|$

We have shown that $T = J_{\gamma A} \circ (Id - \gamma B) : D \to D$

$\therefore \forall_{x\in D} \forall_{y\in D} \; \|Tx - Ty\| = \|J_{\gamma A} \underbrace{(Id - \gamma B)x}_{\in D} - J_{\gamma A} \underbrace{(Id - \gamma B)y}_{\in D}\| \leqslant \tau \underbrace{\|(Id - \gamma B)x - (Id - \gamma B)y\|}_{\leqslant \|x-y\|} \leqslant \tau \|x - y\| \Leftrightarrow$ T: Lipschitz continuous with constant $\tau \in]0,1[$

// $\operatorname{dom} B = \operatorname{dom}(Id - \gamma B)$

· Now let us assume that: (ii) holds, i.e., $\alpha \leqslant \beta$, B: α-strongly monotone, β-Lipschitz continuous, $\gamma \in]0, 2\alpha/\beta^2[$

A: maximally monotone $\Rightarrow J_{\gamma A}: \mathcal{H} \to \mathcal{H}$, firmly nonexpansive $\Leftrightarrow \forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}} \; \|J_{\gamma A} x - J_{\gamma A} y\|^2 \leqslant \|x - y\|^2$

now $\forall_{x\in D} \forall_{y\in D} \; \|(Id - \gamma B)x - (Id - \gamma B)y\|^2 = \|(x - y) - \gamma(Bx - By)\|^2$

$= \|x - y\|^2 + \gamma^2 \underbrace{\|Bx - By\|^2}_{\leqslant \beta^2 \|x-y\|^2} \underbrace{- 2\gamma\langle x - y \mid Bx - By\rangle}_{\leqslant -2\gamma\alpha \|x-y\|^2}$ // B: α-strongly monotone $\Rightarrow \langle x - y \mid Bx - By \rangle \geqslant \alpha \|x - y\|^2$ */

$\leqslant \|x - y\|^2 + \beta^2\gamma^2 \|x - y\|^2 - 2\gamma\alpha \|x - y\|^2$

$= \|x - y\|^2 (1 + \beta^2\gamma^2 - 2\gamma\alpha)$ /* now: given, $\gamma \in]0, \frac{2\alpha}{\beta^2}[\Rightarrow \gamma < \frac{2\alpha}{\beta^2} \Leftrightarrow \beta^2\gamma < 2\alpha \Leftrightarrow \beta^2\gamma^2 < 2\alpha\gamma \Leftrightarrow 1 + \beta^2\gamma^2 - 2\alpha\gamma < 1$ $\Rightarrow 0 \leqslant \underbrace{1 + \beta^2\gamma^2 - 2\alpha\gamma}_{} < 1$

also, $\alpha \leqslant \beta \Leftrightarrow -\alpha \geqslant -\beta \Leftrightarrow -2\beta\gamma \leqslant -2\alpha\gamma$ — say $\tau^2 = 1 + \beta^2\gamma^2 - 2\alpha\gamma \in [0,1[$ */

$\therefore 1 + \beta^2\gamma^2 - 2\alpha\gamma \geqslant 1 + \beta^2\gamma^2 - 2\beta\gamma = (1 - \beta\gamma)^2 \geqslant 0$

$= \tau^2 \|x - y\|^2 : \tau \in [0,1)$

$\therefore \forall_{x\in D} \forall_{y\in D} \; \|(Id - \gamma B)x - (Id - \gamma B)y\|^2 \leqslant \tau^2 \|x - y\|^2$

We have shown that $T = J_{\gamma A} \circ (Id - \gamma B) : D \to D$

$\|Tx - Ty\| = \|J_{\gamma A} (Id - \gamma B)x - J_{\gamma A} (Id - \gamma B)y\| \leqslant \|(Id - \gamma B)x - (Id - \gamma B)y\| \leqslant \tau \|x - y\| \Leftrightarrow$ T: Lipschitz continuous with constant $\tau \in]0,1[$

$\therefore \forall_{x \in D} \forall_{y \in D}$

We have shown that $T = J_{\gamma A} \circ (\mathrm{Id} - \gamma B) : D \to D$

$$\therefore \forall_{x \in D} \forall_{y \in D} \quad \|Tx - Ty\| = \|J_{\gamma A} \underbrace{(\mathrm{Id}-\gamma B)x}_{\in D} - J_{\gamma A} \underbrace{(\mathrm{Id}-\gamma B)y}_{\in D}\| \leq \underbrace{\|(\mathrm{Id}-\gamma B)x - (\mathrm{Id}-\gamma B)y\|}_{\leq \tau \|x-y\|,\ \tau \in [0,1[} \leq \tau \|x-y\| \Leftrightarrow T : \text{Lipschitz continuous with constant } \tau \in]0,1[$$

$\| \operatorname{dom} B = \operatorname{dom}(\mathrm{Id} - \gamma B)$

∎

Part 4: Tseng's algorithm

8:14 AM

Theorem 25.10 (Tseng's algorithm)

[D: nonempty subset of $\mathcal{H}$; $A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone, $\operatorname{dom} A \subseteq D$; $B: \mathcal{H} \to 2^{\mathcal{H}}$, monotone, single-valued on D;
$A+B$: maximally monotone; C: closed convex, $\subseteq D$, $C \cap \operatorname{zer}(A+B) \neq \emptyset$;
B: $\frac{1}{\beta}$ Lipschitz continuous relative to $C \cup \operatorname{dom} A$, $\beta \in \mathbb{R}_{++}$;
$\gamma \in]0, \beta[$;

$$\forall_{n \in \mathbb{N}} \left| \begin{array}{l} y_n = x_n - \gamma B x_n \\ z_n = J_{\gamma A} y_n \\ r_n = z_n - \gamma B z_n \\ x_{n+1} = P_C(x_n - y_n + r_n) \end{array} \right. \quad \text{(eq: 25.36)} \quad] \Rightarrow$$

(i) $(x_n - z_n)_{n \in \mathbb{N}}$: converges strongly to 0

(ii) $(x_n)_{n \in \mathbb{N}}$ and $(z_n)_{n \in \mathbb{N}}$ converges weakly to a point in $C \cap \operatorname{zer}(A+B)$

(iii) (A or B: uniformly monotone on every nonempty bounded subset of $\operatorname{dom} A$) $\Rightarrow$
$(x_n)_{n \in \mathbb{N}}$, $(z_n)_{n \in \mathbb{N}}$: converge strongly to the unique point in $C \cap \operatorname{zer}(A+B)$

<u>Proof:</u>

(Setup)

First, we will show that in (eq: 25.36) the iterations are well defined and "=" makes sense //as in general they will be "$\in$"

// x_(n) is projection on C

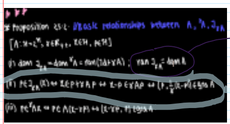
$\forall_{n \in \mathbb{N}}$ $x_n \in C \subseteq D$ // $\because C \subseteq D$

$\Rightarrow \boxed{x_n \in D}$

now. $y_n = \underbrace{\underbrace{x_n}_{\in D} - \gamma \underbrace{B x_n}_{\text{one vector as } B \text{: single valued on } D}}_{\text{one vector}}$

$\to$ so the equality is one defined //as otherwise $\in$ will be more general case.

now $z_n = J_{\gamma A} y_n \Rightarrow \boxed{z_n \in \operatorname{ran} J_{\gamma A} = \operatorname{dom} A \subseteq D} \Rightarrow B z_n$: single vector //as B: single valued on D



$\Rightarrow \underbrace{z_n - \gamma B z_n}_{\text{single vector}} = r_n$: well-defined // = makes sense

finally, $x_{n+1} = P_C(r_n + x_n - y_n) \in C$ // projection on a closed convex set is unique

$\therefore$ By induction, $(x_n)_{n \in \mathbb{N}}$, $(y_n)_{n \in \mathbb{N}}$, $(z_n)_{n \in \mathbb{N}}$: well defined

Define: $\forall_{n \in \mathbb{N}}$ $u_n = \gamma^{-1}(x_n - z_n) + B z_n - B x_n$ [eq: 25.37] /* from (eq: 25.36) $z_n = J_{\gamma A} y_n \Leftrightarrow \gamma A z_n \ni y_n - z_n \Leftrightarrow \boxed{(z_n, y_n - z_n) \in \operatorname{gra} \gamma A}$
$\Leftrightarrow A z_n \ni \frac{1}{\gamma}(y_n - z_n)$ */

$\therefore \forall_{n \in \mathbb{N}}$ $u_n = \frac{1}{\gamma}(x_n - z_n) + B z_n - B x_n = \frac{1}{\gamma}(y_n + \gamma B x_n - z_n) + B z_n - B x_n = \underbrace{\gamma^{-1}(y_n - z_n)}_{\in A z_n} + B z_n \in A z_n + B z_n = (A+B) z_n \Leftrightarrow (z_n, u_n) \in \operatorname{gra}(A+B)$ (eq:25.38)

$x_n = y_n + \gamma B x_n$ //from eq: 25.36

given $C \cap \operatorname{zer}(A+B) \neq \emptyset \Rightarrow \exists_z$ (say) $z \in C \cap \operatorname{zer}(A+B) \Rightarrow$ $z \in C \Leftrightarrow z = P_C z$; $C \subseteq D$ $\therefore \boxed{z \in D}$

$\Rightarrow z \in \operatorname{zer}(A+B) \Leftrightarrow Az + Bz \ni 0$

$\Leftrightarrow \gamma A z + \gamma B z \ni 0 \Leftrightarrow \gamma A z \ni -\gamma B z \Leftrightarrow \boxed{(z, -\gamma B z) \in \operatorname{gra} \gamma A}$

// single vector //as B: single-valued on D; $C \subseteq D$

A: maximally monotone

$\Rightarrow \gamma A$: " " $\Rightarrow \forall_{(x,u),(y,v) \in \operatorname{dom} A}$ $\langle x - y \mid u - v \rangle \geq 0$

$\langle z_n - z \mid y_n - z_n + \gamma B z \rangle \geq 0$

$\Leftrightarrow \boxed{\langle z_n - z \mid z_n - y_n - \gamma B z \rangle \leq 0}$

Again. B: monotone, single-valued on $D \Rightarrow \gamma B$: monotone, single valued on D

also $\boxed{z_n \in D}$, $\boxed{z \in D}$

$\to (z, \gamma B z) \in \operatorname{gra} \gamma B$, $(z_n, \gamma B z_n) \in \operatorname{gra} \gamma B$ $\xRightarrow{B\text{: monotone}}$ $\langle z_n - z \mid \gamma B z_n - \gamma B z \rangle \geq 0$

$\Leftrightarrow \boxed{\langle z_n - z \mid \gamma B z - \gamma B z_n \rangle \leq 0}$

$+$

$\boxed{\langle z_n - z \mid z_n - y_n - \gamma B z_n \rangle \leq 0}$

Now:

$2\gamma \langle z_n - z \mid B x_n - B z_n \rangle$

$= 2 \langle z_n - z \mid \gamma B x_n - \gamma B z_n \rangle$

$= 2 \langle z_n - z \mid \underbrace{\gamma B x_n + y_n - z_n}_{} + \underbrace{z_n - y_n - \gamma B z_n}_{} \rangle$

$= 2 \langle z_n - z \mid \gamma B x_n + y_n - z_n \rangle + \underbrace{2 \langle z_n - z \mid z_n - y_n - \gamma B z_n \rangle}_{}$ //negative number, so removing it will give a larger number

$\leq 2 \langle z_n - z \mid \underbrace{\gamma B x_n + y_n - z_n}_{\gamma B x_n + x_n - \gamma B x_n - z_n = x_n - z_n} \rangle$ /* from (eq: 25.36) : $y_n = x_n - \gamma B x_n$ */

$= 2 \langle z_n - z \mid x_n - z_n \rangle$ // now $\|(z_n - z) + (x_n - z_n)\|^2 = \|z_n - z\|^2 + \|x_n - z_n\|^2 + \underbrace{2 \langle z_n - z \mid x_n - z_n \rangle}_{}$ */

$= \|x_n - z\|^2 - \|z_n - z\|^2 - \|x_n - z_n\|^2$

$\therefore 2\gamma \langle z_n - z \mid B x_n - B z_n \rangle \leq \|x_n - z\|^2 - \|z_n - z\|^2 - \|x_n - z_n\|^2$ ... (eq: 25.41)

Now Proposition 4.8 says: [C: nonempty closed convex, $\subseteq \mathcal{H}$] P_C: firmly nonexpansive
$\Downarrow$
nonexpansive

So. $\|\underbrace{x_{n+1}}_{P_C(r_n + x_n - y_n)} - \underbrace{z}_{P_C z = z}\|^2 = \|P_C(r_n + x_n - y_n) - P_C z\|^2 \leq \|\underbrace{r_n}_{z_n - \gamma B z_n} + \underbrace{x_n - y_n}_{\gamma B x_n} - z\|^2 = \|z_n - \gamma B z_n + \gamma B x_n - z\|^2 = \|(z_n - z) + \gamma(B x_n - B z_n)\|^2$

//by iteration scheme (eq: 25.36)

$= \|z_n - z\|^2 + \gamma^2 \|B x_n - B z_n\|^2 + 2\gamma \langle z_n - z \mid B x_n - B z_n \rangle$

$\leq \|z_n - z\|^2 + \gamma^2 \|B x_n - B z_n\|^2 + \|x_n - z\|^2 - \|z_n - z\|^2 - \|x_n - z_n\|^2$

$= \|x_n - z\|^2 - \|x_n - z_n\|^2 + \gamma^2 \underbrace{\|B x_n - B z_n\|^2}_{\leq \frac{1}{\beta^2}\|x_n - z_n\|^2}$ /* $\because B$: $\frac{1}{\beta}$ Lipschitz continuous w.r.t. $\underbrace{C}_{\subseteq D} \cup \underbrace{\operatorname{dom} A}_{\subseteq D}$ (given). now $x_n \in C$, $z_n \in \operatorname{dom} A$
$\therefore \|B x_n - B z_n\| \leq \frac{1}{\beta} \|x_n - z_n\|$ */

$\leq \|x_n - z\|^2 - \underbrace{\left(1 - \frac{\gamma^2}{\beta^2}\right) \|x_n - z_n\|^2}_{}$ /* now $\gamma \in]0, \beta[$, $\left(1 - \frac{\gamma^2}{\beta^2}\right) \geq 0$ */ ... (eq: 25.42)

/*negative number, so removing it will give only larger number*/

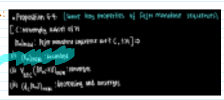
$\leq \|x_n - z\|^2$

$\therefore \forall_{z \in C \cap \operatorname{zer}(A+B)}$ $\|x_{n+1} - z\|^2 \leq \|x_n - z\|^2 \overset{\text{def}}{\Leftrightarrow} (x_n)_{n \in \mathbb{N}}$: Fejér monotone w.r.t. $C \cap \operatorname{zer}(A+B)$

$\boxed{\text{(i)}}$ from (eq: 25.42) $\forall_{z \in C \cap \operatorname{zer}(A+B)} \forall_{n \in \mathbb{N}}$ $\|x_{n+1} - z\|^2 \leq \|x_n - z\|^2 - \left(1 - \frac{\gamma^2}{\beta^2}\right) \|x_n - z_n\|^2$

$\Rightarrow \forall_{n \in \mathbb{N}}$ $0 \leq \left(1 - \frac{\gamma^2}{\beta^2}\right) \|x_n - z_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2$

as $n \to +\infty$, $\|x_n - z\|^2 - \|x_{n+1} - z\|^2 \to 0$ /* as:



*/

$\therefore$ as $n \to +\infty$, $\left(1 - \frac{\gamma^2}{\beta^2}\right) \|x_n - z_n\|^2 \to 0$

$\Leftrightarrow$ as $n \to +\infty$, $\|x_n - z_n\|^2 \to 0$

$\therefore x_n - z_n \to 0$ ■

$\boxed{\text{(ii)}}$ $\forall_{n \in \mathbb{N}}$ $0 \leq \|B z_n - B x_n\| \leq \frac{1}{\beta} \|x_n - z_n\|$ $\to 0$ //from (i)

$\therefore$ as $n \to +\infty$, $(B z_n - B x_n) \to 0$

now, from [eq: 25.37] $u_n = \frac{1}{\gamma}\underbrace{(x_n - z_n)}_{\to 0} + \underbrace{(B z_n - B x_n)}_{\to 0}$

$\therefore u_n \to 0$ (eq: 25.44)

now, $(x_n)_{n \in \mathbb{N}}$: bounded //as $(x_n)_{n \in \mathbb{N}}$: Fejér monotone w.r.t. $C \cap \operatorname{zer}(A+B)$ and Fejér monotone sequences are bounded

// $x_n \in C$ in (eq: 25.36)

x: a weak sequential cluster point of $(x_n)_{n \in \mathbb{N}}$ // Any bounded sequence has a weakly convergent subsequence: Lemma 2.45

$\Leftrightarrow \exists_{(x_{k_n})_{n \in \mathbb{N}}}$ $x_{k_n} \rightharpoonup x$,

$\therefore\ u_n \to 0$ (eq: 25.44)

now, $(x_n)_{n\in\mathbb{N}}$: bounded // as $(x_n)_{n\in\mathbb{N}}$: fejer monotone w.r.t $C\cap \mathrm{zer}(A+B)$, and fejer monotone sequences are bounded

via // $x_n \in C$ in (eq: 25.36)

x : a weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ // Any bounded sequence has a weakly convergent subsequence : Lemma 2.37

$\leftrightarrow \exists\, (x_{k_n})_{n\in\mathbb{N}}$ $x_{k_n} \rightharpoonup x$,

now we will show $x \in C\cap \mathrm{zer}(A+B)$

first C: closed, convex, $\subseteq D$ /*

Theorem 3.32

*/

$\Rightarrow$ C: sequentially closed

$\Rightarrow$ $\boxed{x\in C}$

now we will show that:

$x\in \mathrm{zer}(A+B) \leftrightarrow (A+B)x = Ax+Bx \ni 0$

$\leftrightarrow (x,0)\in \mathrm{gra}(A+B)$

in (i) we have shown: $x_n - z_n \to 0$

$\Rightarrow$ $\boxed{x_{k_n} - z_{k_n} \to 0}$ // now, $(a_n - b_n \to 0,\ a_n \rightharpoonup a) \Rightarrow b_n \rightharpoonup a$

$\therefore$ $\boxed{z_{k_n} \rightharpoonup x}$

$\boxed{u_{k_n} \to 0}$

(eq:25.38) $\therefore$ $(z_n, u_n)\in \mathrm{gra}(A+B) \Rightarrow (z_{k_n}, u_{k_n}) \in \mathrm{gra}(A+B)$ /*

Proposition 20.33

*/

$(x,0)\in \mathrm{gra}(A+B) \leftrightarrow \boxed{x\in \mathrm{zer}(A+B)}$

$x\in C\cap \mathrm{zer}(A+B)$

$\therefore \forall \underbrace{(x_{k_n})_{n\in\mathbb{N}}}$ $x_{k_n} \rightharpoonup x \in C\cap\mathrm{zer}(A+B)$

any subsequence of

$(x_n)_{n\in\mathbb{N}}$: Fejer monotone w.r.t $C\cap \mathrm{zer}(A+B)$

$\therefore$ $x_n \rightharpoonup x \in C\cap \mathrm{zer}(A+B)$

as $x_n - z_n \to 0$ // from (i)

$\boxed{z_n \rightharpoonup x \in C\cap \mathrm{zer}(A+B)}$ // $\because (a_n - b_n \to 0,\ a_n \rightharpoonup a) \Rightarrow b_n \rightharpoonup b$:)

(iii) $A+B$: strictly monotone // $\because$ $\tilde{A}$: uniformly monotone $\Rightarrow$ strictly monotone

$\Rightarrow$ $\mathrm{zer}(A+B)$: atmost a singleton // $\tilde{B}$: monotone

// $\tilde{A}+\tilde{B}$: strictly monotone

// $\because$ A strictly monotone operator's zero set is atmost singleton

given $\mathrm{zer}(A+B) :\neq \emptyset$

$\mathrm{zer}(A+B)$: singleton $\Rightarrow$ $C\cap \mathrm{zer}(A+B)$: singleton

// $\because$ given, $C\cap\mathrm{zer}(A+B) \neq \emptyset$

in (ii) we have shown that: $\exists\, x\in C\cap \mathrm{zer}(A+B) \subseteq \mathrm{dom}A$ $z_n \rightharpoonup x$, and $u_n \to 0$ (eq: 25.44)

given, $A \vee B$: uniformly monotone on every nonempty bounded subset of $\mathrm{dom}A$

$\Rightarrow$ $A+B$: uniformly " " $\{x\}\cup\{z_n\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}A$ // as $(z_n)_{n\in\mathbb{N}} \subseteq \mathrm{dom}A$, $x\in \mathrm{dom}A$

now, $x\in \mathrm{zer}(A+B)$

$\leftrightarrow (x,0)\in \mathrm{gra}(A+B)$

(eq:25.38) $(z_n, u_n)\in \mathrm{gra}(A+B)$

set $\tilde{A} := (A+B)$, $\tilde{C} := \mathrm{dom}A$

(eq.22.5) :

$\exists\, \phi : \mathbb{R}_+ \to [0,+\infty]$, vanishes only at zero, increasing

$(z_n \in \mathrm{dom}A,\ x\in\mathrm{dom}A,\ u_n \in (A+B)z_n,\ 0\in (A+B)x)$ $\langle z_n - x \mid u_n - 0\rangle \geq \gamma\phi(\|z_n - x\|) \geq 0 \Rightarrow \|z_n - x\| \to 0 \leftrightarrow z_n \to x : x\in C\cap\mathrm{zer}(A+B) = \{x\}$

$\underbrace{\langle z_n - x \mid u_n - 0\rangle}_{0 \quad 0} \to 0$ $\because$ */ :)

Part 5: Variational inequality problem

9:05 AM

Definition 25.12. (Variational inequality problem)

[$f \in \Gamma_0(\mathcal{H})$; $B : \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone]

Associated variational inequality problem:

$$\begin{array}{ll} \text{find} & x \in \mathcal{H} \\ \text{such that} & \exists_{u \in Bx} \ \forall_{y \in \mathcal{H}} \ \langle x - y \mid u \rangle + f(x) \leq f(y) \qquad \text{(eq: 25.49)} \end{array}$$

Example 25.13. [$z \in \mathcal{H}$; $B : x \mapsto x - z$] Associated variational inequality problem:

$$\begin{array}{ll} \text{find} & x \in \mathcal{H} \\ \text{such that} & \forall_{y \in \mathcal{H}} \ \langle x - y \mid x - z \rangle + f(x) \leq f(y) \end{array}$$

• solution: $x = \operatorname{Prox}_f z$

Example 25.14. [set: $f = \iota_C$ in Definition 25.12., C: nonempty, closed, convex, $\subseteq \mathcal{H}$; $B : \mathcal{H} \to \mathcal{H}$, maximally monotone] Classical variational inequality problem:

$$\begin{array}{ll} \text{find} & x \in C \\ \text{such that} & \forall_{y \in C} \ \langle x - y \mid Bx \rangle \leq 0 \end{array} \quad \ldots \text{(eq: 25.51)}$$

/* Dual cone:

[$C \subseteq \mathcal{H}$] $C^{\oplus}$: dual cone of $C = \{ u \in \mathcal{H} \mid \inf \langle C \mid u \rangle \geq 0 \}$ $\Leftrightarrow \forall_{x \in C} \langle x \mid u \rangle \geq 0$ */

Example 25.15. (complementarity problem) [In example 25.14., set $C := K$: K : nonempty closed convex cone in $\mathcal{H}$] $B : \mathcal{H} \to \mathcal{H}$, maximally monotone. complementarity problem:

$$\begin{array}{ll} \text{find} & x \in K \\ \text{such that} & x \perp Bx, \ Bx \in K^{\oplus} \end{array}$$

Proof: We want to show that, under our antecedent, Example 25.14 reduces to our goal.

K: nonempty closed convex cone $\overset{\text{def}}{\Leftrightarrow} K = \mathbb{R}_{++} K$

$\therefore x \in K \Rightarrow \begin{cases} x/2 \in K \\ 2x \in K \end{cases}$

from (eq: 25.51) $\forall_{y \in C} \langle x - y \mid Bx \rangle \leq 0$ // now $C := K$

$\Leftrightarrow \forall_{y \in K} \langle x - y \mid Bx \rangle \leq 0$ (eq: 1)

$y := x/2 \Rightarrow \langle x - \frac{x}{2} \mid Bx \rangle = \langle \frac{x}{2} \mid Bx \rangle = \frac{1}{2} \langle x \mid Bx \rangle \leq 0 \Leftrightarrow \langle x \mid Bx \rangle \leq 0$

$y := 2x \Rightarrow \langle x - 2x \mid Bx \rangle = \langle -x \mid Bx \rangle = (-1) \langle x \mid Bx \rangle \leq 0 \Leftrightarrow \langle x \mid Bx \rangle \geq 0$

$\Rightarrow \langle x \mid Bx \rangle = 0 \Leftrightarrow \boxed{x \perp Bx}$ //now we can use this as a given $\because (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow \left[\begin{array}{l} A \Rightarrow B \\ A \Rightarrow C \end{array} \right. \Leftrightarrow A \Rightarrow (B \wedge C)$

$\forall_{y \in C = K} \langle x - y \mid Bx \rangle = \underbrace{\langle x \mid Bx \rangle}_{= 0} - \langle y \mid Bx \rangle = - \langle y \mid Bx \rangle \leq 0$

$\Leftrightarrow \forall_{y \in C = K} \langle y \mid Bx \rangle \geq 0$ // $K^{\oplus} = \{ u \in \mathcal{H} \mid \forall_{x \in C = K} \langle x \mid u \rangle \geq 0 \}$ $\therefore u \in K^{\oplus} \Leftrightarrow \forall_{\tilde{x} \in C = K} \langle \tilde{x} \mid u \rangle \geq 0$

$\Leftrightarrow \boxed{Bx \in K^{\oplus}}$

So, (eq: 25.51) becomes:

$$\begin{array}{ll} \text{find} & x \in K \\ \text{such that} & x \perp Bx, \ Bx \in K^{\oplus} \end{array}$$

∎

Remark 25.16. Restructuring of the variational inequality problem:

[$f \in \Gamma_0(\mathcal{H})$; $B : \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone]

Associated variational inequality problem: // Restructured version of (eq: 25.49)

find $x \in \mathcal{H}$

such that $x \in \operatorname{zer}(\partial f + B)$

Proof:

$x \in \operatorname{zer}(\partial f + B)$

$\Leftrightarrow (\partial f + B) x \ni 0 \Leftrightarrow \partial f(x) + Bx \ni 0 \Leftrightarrow \exists_{u \in \mathcal{H}} \ u \in Bx, \ -u \in \partial f(x)$

$\Leftrightarrow \exists_{u \in Bx} \ -u \in \partial f(x)$ // By definition $\tilde{u} \in \partial f(x) \overset{\text{def}}{\Leftrightarrow} \forall_{y \in \mathcal{H}} \langle y - x \mid \tilde{u} \rangle + f(x) \leq f(y)$

$\therefore -u \in \partial f(x) \Leftrightarrow \forall_{y \in \mathcal{H}} \underbrace{\langle y - x \mid -u \rangle}_{} + f(x) \leq f(y)$

/* $(-1) \langle y - x \mid u \rangle = \langle x - y \mid u \rangle$ */

$\Leftrightarrow \forall_{y \in \mathcal{H}} \langle x - y \mid u \rangle + f(x) \leq f(y)$

$\Leftrightarrow \exists_{u \in Bx} \langle x - y \mid u \rangle + f(x) \leq f(y)$ . . . (eq: 1)

$\therefore$ variational inequality problem := // from (eq: 25.49)

$$\left(\begin{array}{ll} \text{find} & x \\ \text{such that} & \exists_{u \in Bx} \ \forall_{y \in \mathcal{H}} \ \langle x - y \mid u \rangle + f(x) \leq f(y) \end{array} \right) \text{ // now using (eq: 1)}$$

$$= \left(\begin{array}{ll} \text{find} & x \\ \text{such that} & x \in \operatorname{zer}(\partial f + B) \end{array} \right)$$

∎

Proposition 25.17. (Douglas-Rachford algorithm for variational inequality)

[$f \in \Gamma_0(\mathcal{H})$; $B: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone; the variational inequality problem: find $x \in \mathcal{H}$ such that $\exists_{u \in Bx} \forall_{y \in \mathcal{H}} \langle x-y \mid u \rangle + f(x) \leq f(y)$ has atleast one solution; $(\lambda_n)_{n \in \mathbb{N}}: \subseteq [0,2]$, $\sum_{n \in \mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$; $\gamma \in \mathbb{R}_{++}$; $x_0 \in \mathcal{H}$; set

$$\forall_{n \in \mathbb{N}} \left| \begin{array}{l} y_n = J_{\gamma B} x_n \\ z_n = \mathrm{Prox}_{\gamma f}(2y_n - x_n) \\ x_{n+1} = x_n + \lambda_n(z_n - y_n) \end{array} \right. \quad \text{(eq:25.59)}$$

] $\Rightarrow \exists_{x \in \mathcal{H}}$

(i) $J_{\gamma B} x$: solution to the variational inequality problem

(ii) $(x_n)_{n \in \mathbb{N}}$: $x_n \rightharpoonup x$

(iii) $(y_n)_{n \in \mathbb{N}}$: $y_n \rightharpoonup J_{\gamma B} x$

$(z_n)_{n \in \mathbb{N}}$: $z_n \rightharpoonup J_{\gamma B} x$

(iv) (One of the following holds:

(a) f: uniformly convex on every nonempty bounded subset of dom ∂f

(b) B: uniformly monotone on every nonempty bounded subset of dom B) $\Rightarrow$

$(y_n)_{n \in \mathbb{N}}$: $y_n \to J_{\gamma B} x$: unique solution of the variational inequality

$(z_n)_{n \in \mathbb{N}}$: $z_n \to J_{\gamma B} x$: unique solution of the variational inequality

Proof: Proof is straightforward and follows from the following results:

Definition 25.12. (Variational inequality problem)

[$f \in \Gamma_0(\mathcal{H})$; $B: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone]

Associated variational inequality problem:

find $x \in \mathcal{H}$

such that $\exists_{u \in Bx} \forall_{y \in \mathcal{H}} \quad \langle x-y \mid u \rangle + f(x) \leq f(y)$ (eq:25.49)

Remark 25.16. (Restructuring of Variational Inequality Problem)

[$f \in \Gamma_0(\mathcal{H})$, $B: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone]

Associated variational inequality problem: // restructured version of (eq:25.49)

find $x \in \mathcal{H}$

such that $x \in \mathrm{zer}(\partial f + B)$ // $f \in \Gamma_0(\mathcal{H}) \Rightarrow \partial f$: maximally monotone

Example 23.3.

[$f \in \Gamma_0(\mathcal{H})$, $\gamma \in \mathbb{R}_{++}$] $\Rightarrow \left| \begin{array}{l} \mathrm{Prox}_{\gamma f} = J_{\gamma \partial f} \\ \nabla({}^{\gamma}f) = {}^{\gamma}(\partial f) \end{array} \right.$ // here ${}^{\gamma}f$: Moreau envelope of f ;

Example 22.4. set $C := \mathrm{dom}\, \partial f$

[$f: \mathcal{H} \to]-\infty, +\infty]$, proper, convex, uniformly convex on C

C: nonempty, $\subseteq \mathrm{dom}\, \partial f$]

∂f : uniformly monotone on C

Theorem 25.6. (Douglas-Rachford algorithm) set $A := \partial f$: maximally monotone

[A, B: maximally monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$; $\mathrm{zer}(A+B) \neq \emptyset$;

$(\lambda_n)_{n \in \mathbb{N}}$: sequence in $[0,2]$, $\sum_{n \in \mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$; $\gamma \in \mathbb{R}_{++}$; $x_0 \in \mathcal{H}$

$$\forall_{n \in \mathbb{N}} \left| \begin{array}{l} y_n = J_{\gamma B} x_n \\ z_n = J_{\gamma A}(2y_n - x_n) = \mathrm{Prox}_{\gamma f}(2y_n - x_n) \\ x_{n+1} = x_n + \lambda_n(z_n - y_n) \end{array} \right.] \Rightarrow \exists_x \; x \in \mathrm{Fix}\, R_{\gamma A} R_{\gamma B}$$

(i) $J_{\gamma B} x \in \mathrm{zer}(A+B)$

(ii) $(y_n - z_n)_{n \in \mathbb{N}}$: $(y_n - z_n) \to 0$

(iii) $(x_n)_{n \in \mathbb{N}}$: $x_n \rightharpoonup x$

(iv) $(y_n)_{n \in \mathbb{N}}$: $y_n \rightharpoonup J_{\gamma B} x$

(v) $(z_n)_{n \in \mathbb{N}}$: $z_n \rightharpoonup J_{\gamma B} x$

(vi) C: closed affine subspace of $\mathcal{H}$; $A = N_C$) $\Rightarrow$ $(P_C x_n)_{n \in \mathbb{N}}$: $P_C x_n \rightharpoonup J_{\gamma B} x$

(vii) (One of the following holds:

(vi) C: closed affine subspace of $\mathcal{H}$; $A = N_C$) $\Rightarrow$ $(P_C x_n)_{n\in\mathbb{N}}$: $P_C x_n \rightharpoonup J_{\gamma B} x$

(vii) (One of the following holds:

(a) A : uniformly monotone on every nonempty bounded subset of $\operatorname{dom} A$

(b) B : uniformly monotone on every nonempty bounded subset of $\operatorname{dom} B$) $\Rightarrow$

$(y_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$: converge strongly to the unique point in $\operatorname{zer}(A+B)$

Proposition 25.18. (Forward-backward algorithm for variational inequality)

[$f \in \Gamma_0(\mathcal{H})$; $\beta \in \mathbb{R}_{++}$; $B : \mathcal{H} \to \mathcal{H}$, β-cocoercive ; $\gamma \in]0, 2\beta[$; $\delta = \min\{1, \beta/\gamma\} + \frac{1}{2}$; the variational inequality problem: find $x \in \mathcal{H}$ such that $\forall_{y\in\mathcal{H}}$ $\langle x-y \mid Bx\rangle + f(x) \leq f(y)$ admits atleast one solution ;

$(\lambda_n)_{n\in\mathbb{N}} : \subseteq [0, \delta]$, $\sum_{n\in\mathbb{N}} \lambda_n(\delta - \lambda_n) = +\infty$; $x_0 \in \mathcal{H}$; set

$$\forall_{n\in\mathbb{N}} \left| \begin{array}{l} y_n = x_n - \gamma B x_n \qquad \text{(eq: 25.56)} \\ x_{n+1} = x_n + \lambda_n(\operatorname{Prox}_{\gamma f} y_n - x_n) \end{array} \right.$$

] $\Rightarrow$

(i) $(x_n)_{n\in\mathbb{N}}$: converges weakly to a solution to the variational inequality

(ii) ($\inf_{n\in\mathbb{N}} \lambda_n > 0$; x : solution to the variational inequality) $\Rightarrow$ $(Bx_n)_{n\in\mathbb{N}}$: $Bx_n \to Bx$

(iii) ($\inf_{n\in\mathbb{N}} \lambda_n > 0$; One of the following holds:

(a) f : uniformly convex on every nonempty bounded subset of $\operatorname{dom} \partial f$

(b) B : uniformly convex on every nonempty bounded subset of $\mathcal{H}$) $\Rightarrow$

$(x_n)_{n\in\mathbb{N}}$: x_n converges strongly to the unique solution to the variational inequality.

Proof: Proof is strightforward again using the following results:

Example 23.3.

[$f \in \Gamma_0(\mathcal{H})$, $\gamma \in \mathbb{R}_{++}$] $\Rightarrow$ $\left| \begin{array}{l} \operatorname{Prox}_{\gamma f} = J_{\gamma \partial f} \\ \nabla({}^{\gamma}f) = {}^{\gamma}(\partial f) \end{array} \right.$ // here ${}^{\gamma}f$: Moreau envelope of f ;

Example 22.4. set $C := \operatorname{dom} \partial f$

[$f : \mathcal{H} \to]-\infty, +\infty]$, proper, convex, uniformly convex on C

C : nonempty, $\subseteq \operatorname{dom} \partial f$]

∂f : uniformly monotone on C

Remark 25.16. (Restructuring of Variational Inequality Problem)

[$f \in \Gamma_0(\mathcal{H})$, $B : \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone]

Associated variational inequality problem: // restructured version of (eq 25.49)

find $x \in \mathcal{H}$

such that $x \in \operatorname{zer}(\partial f + B)$ // $f \in \Gamma_0(\mathcal{H}) \Rightarrow \partial f$: maximally monotone

Theorem 25.8. (Forward-backward algorithm) set $A := \partial f$ // $J_{\gamma A}$: firmly nonexpansive

[$A : \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone ; $\beta \in \mathbb{R}_{++}$; $B : \mathcal{H} \to \mathcal{H}$, β-cocoercive ; $\gamma \in]0, 2\beta[$; $\delta = \min\{1, \frac{\beta}{\gamma}\} + \frac{1}{2}$;

$(\lambda_n)_{n\in\mathbb{N}}$: sequence in $[0, \delta]$, $\sum_{n\in\mathbb{N}} \lambda_n(\delta - \lambda_n) = +\infty$; $x_0 \in \mathcal{H}$; $\operatorname{zer}(A+B) \neq \emptyset$;

$$\forall_{n\in\mathbb{N}} \left| \begin{array}{l} y_n = x_n - \gamma B x_n \\ x_{n+1} = x_n + \lambda_n(\overbrace{J_{\gamma A}}^{\operatorname{Prox}_{\gamma f}} y_n - x_n) \end{array} \right.] \Rightarrow$$

(i) $(x_n)_{n\in\mathbb{N}}$: converges weakly to a point in $\operatorname{zer}(A+B)$

(ii) ($\inf_{n\in\mathbb{N}} \lambda_n > 0$, $x \in \operatorname{zer}(A+B)$) $\Rightarrow$ $(Bx_n)_{n\in\mathbb{N}}$: $Bx_n \to Bx$

(iii) ($\inf_{n\in\mathbb{N}} \lambda_n > 0$; one of the following holds:

(a) A : uniformly monotone on every nonempty bounded subset of $\operatorname{dom} A$

(b) B : uniformly monotone on every nonempty bounded subset of $\mathcal{H}$) $\Rightarrow$

$(x_n)_{n\in\mathbb{N}}$: converges strongly to the unique point in $\operatorname{zer}(A+B)$

Example 25.19. [C : nonempty closed convex subset of $\mathcal{H}$; $\beta \in \mathbb{R}_{++}$; $B : \mathcal{H} \to \mathcal{H}$, β-cocoercive ; the variational inequality problem:

find x such that $\forall_{y\in C}$ $\langle x - y \mid Bx\rangle \leq 0$ admits atleast one solution x to the variational inequality ; $x_0 \in \mathcal{H}$; set:

$\forall_{n\in\mathbb{N}}\ x_{n+1}=P_C(x_n-\beta B x_n)\ ] \Rightarrow \left| \begin{array}{l} (x_n)_{n\in\mathbb{N}} : x_n \rightharpoonup x \\ (Bx_n)_{n\in\mathbb{N}} : Bx_n \to Bx \end{array}\right.$

Proof: Proof follows from the following results:

here set $f := \iota_C \in \Gamma_0(\mathcal{H})$; $\gamma := \beta$; $\lambda_n = 1$; // recall example 12.25. C: nonempty closed convex, $\subseteq \mathcal{H} \Rightarrow (\mathrm{Prox}_{\underbrace{\gamma\iota_C}_{\in\mathbb{R}_{++}}} = P_C)$

> Proposition 25.18. (Forward-backward algorithm for variational inequality)
> [$f\in\Gamma_0(\mathcal{H})$; $\beta\in\mathbb{R}_{++}$; $B:\mathcal{H}\to\mathcal{H}$, β-cocoercive; $\gamma\in\,]0,2\beta[$; $\delta=\min\{1,\underbrace{\beta/\gamma}_{=1}\}+\frac{1}{2}$; the variational inequality problem: find $x\in\mathcal{H}$ such that $\forall_{y\in\mathcal{H}}\ \langle x-y\mid Bx\rangle+f(x)\leq f(y)$ admits atleast one solution;
> $\underbrace{\forall_{y\in C}\ \langle x-y\mid Bx\rangle\leq 0}$ // this is the modified // VI problem
> $(\lambda_n)_{n\in\mathbb{N}} : \subseteq [0,\underbrace{\delta}_{=1.5}]$, $\sum_{n\in\mathbb{N}}\underbrace{\lambda_n}_{=1}(\underbrace{\delta-\lambda_n}_{1/2})=+\infty$; $x_0\in\mathcal{H}$; set
>
> $$\forall_{n\in\mathbb{N}} \left| \begin{array}{l} y_n = x_n-\gamma Bx_n = x_n-\beta Bx_n \\ x_{n+1} = x_n+\underbrace{\lambda_n}_{=1}(\underbrace{\mathrm{Prox}_{\gamma f}}_{\mathrm{Prox}_{\gamma\iota_C}=P_C} y_n - x_n) = x_n + P_C y_n - x_n = P_C y_n = P_C(x_n-\beta Bx_n) \end{array}\right.$$
>
> (eq: 25.56) ✓ this is our iteration scheme
>
>] $\Rightarrow$
>
> (i) $(x_n)_{n\in\mathbb{N}}$: converges weakly to a solution to the variational inequality ✓ // this is (i)
>
> (ii) $(\inf_{n\in\mathbb{N}}\lambda_n>0$; x: solution to the variational inequality$)$ $\Rightarrow$ $(Bx_n)_{n\in\mathbb{N}} : Bx_n\to Bx$ ✓ // this is // (ii)
>
> (iii) $(\inf_{n\in\mathbb{N}}\lambda_n>0$; One of the following holds:
>
> (a) f: uniformly convex on every nonempty bounded subset of $\mathrm{dom}\,\partial f$
>
> (b) B: uniformly convex on every nonempty bounded subset of $\mathcal{H}$ $)\Rightarrow$
>
> $(x_n)_{n\in\mathbb{N}}$: x_n converges strongly to the unique solution to the variational inequality

Example 25.20.

[C: nonempty closed convex subset of $\mathcal{H}$; $\beta\in\mathbb{R}_{++}$; $B:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone operator, single-valued and β-Lipschitz continuous relative to C; $\mathrm{cone}(C-\mathrm{dom}B)=\overline{\mathrm{span}}\,(C-\mathrm{dom}B)$;
variational inequality problem: find $x\in C$ such that $\forall_{y\in C}\ \langle x-y\mid Bx\rangle\leq 0$ admits atleast solution;
$x_0\in C$; $\gamma\in\,]0,1/\beta[$; set:

$$\forall_{n\in\mathbb{N}} \left| \begin{array}{l} y_n = x_n-\gamma Bx_n \\ z_n = P_C y_n \\ r_n = z_n-\gamma B z_n \\ x_{n+1} = P_C(z_n+y_n-r_n) \end{array}\right.] \Rightarrow$$

(i) $(x_n-z_n)\to 0$

(ii) $(x_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}$: converge weakly to a solution of the variational inequality

(iii) B: uniformly monotone on every nonempty bounded subset of $C\Rightarrow$

$(x_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}$: converge strongly to the unique solution to variational inequality.

Proof:

First recall the following results:

/* Example 20.41. [C: nonempty closed convex, $\subseteq\mathcal{H}$] $\left| \begin{array}{l} J_{N_C}=P_C \text{ (Example 23.4)} \\ N_C: \text{maximally monotone ; By definition: } \mathrm{dom}\,N_C=C \end{array}\right.$

> Theorem 24.5. [A, B: maximally monotone from $\mathcal{H}$ to $2^{\mathcal{H}}$, set $A := N_C$
> $\underbrace{\mathrm{cone}(\underbrace{\mathrm{dom}A}_{C}-\mathrm{dom}B)=\overline{\mathrm{span}}(\underbrace{\mathrm{dom}A}_{=C}-\mathrm{dom}B)}_{\text{this is given}}$] $\Rightarrow$ $A+B$: maximally monotone $\rightsquigarrow$ $\therefore N_C+B$: maximally monotone */

Now recall Tseng's algorithm:

> Theorem 25.10. (Tseng's algorithm) set $A=N_C$. $D=C$ $\Rightarrow$ $\mathrm{dom}\,N_C=C$
> [$\cancel{D}^{C}$: nonempty subset of $\mathcal{H}$; $\cancel{A}^{N_C}:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone, $\underbrace{\mathrm{dom}A}_{C}\subseteq \underbrace{D}_{C}$ ✓; $B:\mathcal{H}\to 2^{\mathcal{H}}$, monotone, single-valued on D; ✓
> $\cancel{A}^{N_C}+B$: maximally monotone ✓; C: closed convex ✓, $\subseteq \cancel{D}^{C}$ ✓, $C\cap\mathrm{zer}(A+B)\neq\varnothing$;
> B: $\frac{1}{\beta}$ Lipschitz continuous relative to $\underbrace{C\cup\mathrm{dom}\,\underbrace{A}_{C}}_{C}$, $\beta\in\mathbb{R}_{++}$; $\underbrace{\mathrm{dom}\,A}_{\neq\varnothing}$
> $\Uparrow$ follows from the existence of solution of VI
> $\gamma\in\,]0,\beta[$; ✓
>
> $$\forall_{n\in\mathbb{N}} \left| \begin{array}{l} y_n = x_n-\gamma Bx_n ✓ \\ z_n = J_{\gamma \cancel{A}^{N_C}} y_n = P_C y_n \quad \text{(eq: 25.56)} \\ r_n = z_n-\gamma B z_n ✓ \\ x_{n+1} = P_C(x_n-y_n+r_n) ✓ \end{array}\right.] \Rightarrow$$
>
> (i) $(x_n-z_n)_{n\in\mathbb{N}}$: converges strongly to 0 ✓
>
> (ii) $(x_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ converges weakly to a point in $C\cap\underbrace{\mathrm{zer}(A+B)}_{\mathrm{zer}(N_C+B)}$ ✓
>
> (iii) $(A\vee B$: uniformly monotone on every nonempty bounded subset of $\mathrm{dom}\,A)\Rightarrow$
>
> $(x_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}$: converge strongly to the unique point in $C\cap\underbrace{\mathrm{zer}(A+B)}_{\mathrm{zer}(N_C+B)}$ ✓

(iii) $\left(A \vee B : \text{uniformly monotone on every nonempty bounded subset of } \operatorname{dom} A\right) \Rightarrow$

$(x_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}$: converge strongly to the unique point in $\underbrace{C \cap \operatorname{zer}(A+B)}_{\operatorname{zer}(N_C+B)}$ ✓

∎