On seeking *efficient* Pareto optimal points in *multi-player* minimum cost flow problems

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Outline

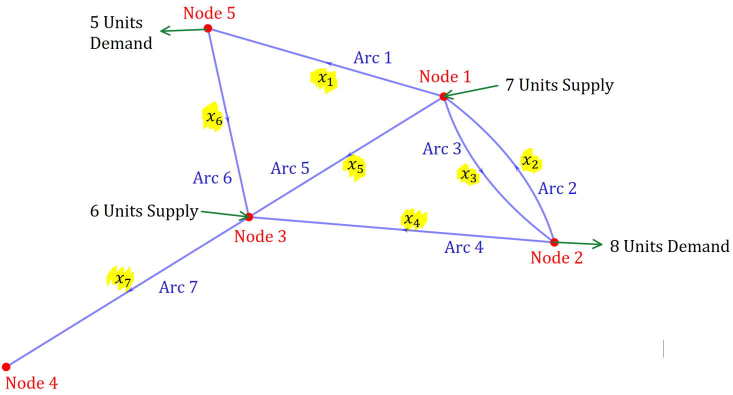
- what happens if the **minimum cost flow problem** is extended to a multi-player setup
- what is a good solution concept in such a multi-player setup
  $\Rightarrow$ ***efficient* Pareto optimal point**
- **under what conditions** these good solutions exist
  - one version always exists in any network
  - existence of a stricter version can be checked
- how to **compute** such solutions

Application

- E-commerce product transportation systems are dominated by **Amazon, Walmart, Alibaba**...
- Amazon uses **FedEx, UPS, AAR** and **other competing organizations** for transportation services

$\Rightarrow$ ***multi-player* minimum cost flow problem**

Minimum cost flow problem



- **Directed connected graph** with nodes and arcs
- **Integer-valued flow** of some material on each arc
- Each arc incurs a **cost**
- **Minimize the total cost** of all flows subject to the network constraints

Minimum cost flow problem

$$\begin{array}{ll} \text{minimize}_x & \sum_{i \in \{1,\ldots,n\}} f_i(x_i) \qquad \texttt{\textbackslash* default:} f_i(x_i) = c_i x_i \ \texttt{*\textbackslash} \\ \text{subject to} & Ax = b \qquad \texttt{\textbackslash* flow conservation constraint *\textbackslash} \\ & 0 \preceq x \preceq u \qquad \texttt{\textbackslash* flow bound constraint *\textbackslash} \\ & x \in \mathbf{Z}^n \qquad \texttt{\textbackslash* flow is integer *\textbackslash}. \end{array}$$

- The network has n arcs, $m+1$ nodes
- A : reduced node-arc incidence matrix, dimension $m \times n$
- b : represents supplies/demands
- u : upper bound on flow

Meaning of the constraints

- **The flow conservation constraint:** for any node, the outflow minus inflow must equal the supply/demand of the node
  - must be maintained
- **The flow bound constraint:** imposes direction and capacity limit on the flow
  - can often be relaxed or omitted in practice
  - in the relaxed case, flow direction is flexible and overflow is allowed

A multi-player extension

- With **each arc** of the network graph we associate **one player**
- Each player tries to minimize its **nonconvex cost function**, subject to the network flow constraints
- Our goal is to seek a **good solution concept** in this multi-player problem

Goal of a player

- The goal of the ith player for $i = 1, \ldots, n$, given other players' strategies $x_{-i} \in \mathbf{Z}^{n-1}$, is to solve:

$$\begin{array}{lll} \text{minimize}_{x_i} & f_i(x_i) & \texttt{\textbackslash* nonconvex *\textbackslash} \\ \text{subject to} & A(x_i, x_{-i}) = b & \texttt{\textbackslash* constraints} \\ & 0 \preceq (x_i, x_{-i}) \preceq u & \texttt{couple the players *\textbackslash} \\ & x \in \mathbf{Z}^n. & \end{array}$$

Back to the application

- **Arcs:** transportation links
- **Nodes:**
  - **Supply nodes:** warehouses
  - **Demand nodes:** retail centers
- **Players:** transportation organizations (FedEx, UPS, AAR etc)
- **Flow:** products transported
- **Each player's goal:**
  - maximize its profit (nonlinear)
  - only controls its own flow

Solution concepts

- A **solution to the optimization problem** would always favor the dominant player ignoring the rest
- A **vector optimal solution** is the best social solution
  - It minimizes all the objectives simultaneously
  - **problem:** violates flow conservation
- The celebrated **Nash equilibrium** is also not very efficient in our setup
- A better solution solution concept is the **Pareto optimal point**

Pareto optimal point

Pareto optimal point: none of the cost functions can be reduced without increasing some other cost function.

A "feasible" point x^{Pareto} is Pareto optimal if it satisfies the following: there *does not* exist another "feasible" point x such that for all $i = 1, \ldots, n$

$$f_i(x_i) \leq f_i(x_i^{\text{Pareto}}),$$

with at least one index j satisfying $f_j(x_j) < f_j(x_j^{\text{Pareto}})$.

Problem: There can be numerous such generic Pareto optimal points, some poor in quality or efficiency.

Efficient Pareto optimal point

- Finds a balance between **vector optimality** and the **generic Pareto optimality**
- It is a Pareto optimal point where
  - the maximum possible number of players minimize their cost functions simultaneously
  - flow is conserved
  - flow bound is maintained for the maximum possible number of arcs

The main result

For any multi-player minimum cost flow problem, there exists one efficient Pareto optimal point such that

- it is **Pareto optimal** and $n-m$ **vector optimal**
  - out of n players, $n-m$ will minimize their cost functions simultaneously
  - the set of $n-m$ vector optimality is maximal (it cannot be made any larger)
- the flow conservation constraints are maintained
- at least $n-m$ of the flow bound constraints are maintained (possibly all)

An existence result

- Out of n flow bound constraints, m of them may not be maintained
- Can we check in advance if all of the flow bound constraints are maintained?
- We provide an existence theorem using algebraic geometry

Why need algebraic geometry?

The flow bound constraints over integers can be formulated as polynomials.

Some necessary concepts

- The **ideal** generated by polynomials $f_1, f_2, \dots, f_m$ is the set

$$\textbf{ideal}\{f_1, \dots, f_m\} = \{\sum_{i=1}^{m} h_i f_i \mid h_1, \dots, h_m \text{ are polynomials}\}.$$

  - analogous to **span of vectors**
- Given an ideal I, **affine variety** is the set

$$\textbf{variety}(I) = \{x \mid f(x) = 0, \text{ for all } f \text{ in } I\}$$

  - analogous to **null-space of a matrix**
- **A Groebner basis** G is particular kind of generating set of an ideal I
  - analogous to **basis** of a span
- **Reduced Groebner basis** G_{reduced} is the most compact Groebner basis for an ideal I
  - analogous to **orthonormal basis** of a span

Statement of the existence theorem

From the structure of the network (after some pre-calculation), we can generate polynomials

$$q_1, \ldots, q_m, r_1, \ldots, r_{n-m}$$

Exactly one of the following holds:
(a) There exists an efficient Pareto optimal point, where all the flow bound constraints are maintained.
(b) We have

$$G_{\text{reduced}} = \{1\},$$

where G_{reduced} is the reduced Groebner basis of $\mathbf{ideal}\{q_1, \ldots, q_m, r_1, \ldots, r_{n-m}\}$.

There are many computer algebra packages (Maple, Mathematica, FGb) that can compute G_{reduced}

Computing efficient Pareto optimal point

We propose an algorithm to compute efficient Pareto optimal points in two stages:

- **Stage 1:** We compute a larger set $\mathcal{F}$, that contains all the efficient Pareto optimal points
- **Stage 2:** From $\mathcal{F}$, we compute efficient Pareto optimal points using algebraic elimination theory
  - Need to solve only single-variable optimization problems

Stage 1

- For $i = 1, 2, \ldots, n-m$ calculate
  - Compute

    $$G_{n-m-i} = G_{\text{reduced}} \cap \mathbf{C}\left[z_{n-m-i+1}, z_{n-m-i+2}, \ldots, z_{n-m}\right]$$

  - Compute $\textbf{variety}(G_{n-m-i})$
- Each G_{n-m-i} results in a single-variable polynomial system
- Finding $\textbf{variety}(G_{n-m-i})$ is just finding the roots of a single variable polynomial
- $\mathcal{F} = \textbf{variety}(G_0)$

Stage 2

for $i = 1, \ldots, m$
$X_i := d_i - h_i^T \mathcal{F}$ `\* The inverse operator is denoted` X_i^{-1} `*\`
end for
`\* We can compute` d_i `and` h_i `from` A `and` u`*\`

Sort the elements of the $\{X_i\}_{i=1}^m$s with respect to cardinality of the elements in a descending order.

Denote the index set of the sorted set by $\{s_1, \ldots, s_m\}$ such that $|X|_{s_1} \geq \cdots \geq |X|_{s_m}$.

for $i = 1, \ldots, m$
$X_{s_i}^* :=$ argmin$_{x_{s_i} \in X_{s_i}} f_{s_i}(x_{s_i})$ `\* Easy to solve single-variable optimization problem *\`
$\mathcal{F}_{s_i}^* := \bigcup_{x_{s_i} \in X_{s_i}^*} (X_{s_i}^*)^{-1}(x_{s_i})$ `\* Lemma. The set` $\mathcal{F}_{s_i}^*$ `is nonempty *\`
if $i \leq m$
$X_{s_{i+1}} := \left\{ d_{s_{i+1}} - h_{s_{i+1}}^T z \mid z \in \mathcal{F}_{s_i}^* \right\}.$
end if
end for
return $\mathcal{F}_{s_m}^*$ `\* Theorem. Any member of` $\mathcal{F}_{s_m}^*$ `is an efficient Pareto optimal point *\`

Concluding Remarks

- ***Multi-player* minimum cost flow problem**: a natural extension of the minimum cost flow problem
- ***Efficient* Pareto optimal point** is a desirable solution concept
  - A soft version always exists
  - A strict version can exist: existence can be checked
- **Algorithms** to compute efficient Pareto optimal points

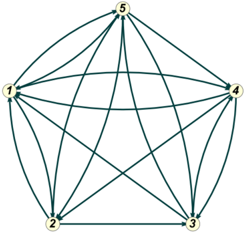
Paper available at

- https://shuvomoy.github.io/site/
- https://link.springer.com/article/10.1007/s10898-019-00750-9

Thank You!

Questions?

Numerical example



<u>A multi-player transportation problem</u>

- 5 nodes, 16 arcs
- Nodes 2, 4 are retail centers
- Nodes 1, 3 are warehouses
- Node 5 transshipment node
- Products need to be shipped from warehouses to retail centers
- Arcs represent different modes of shipment from one node to another and carried out by different organization
- Cost of a certain type of shipment depends on the number of products shipped

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \end{pmatrix}$$

$$b = (9, -13, 15, -11), \; u = (5, 6, 6, 10, 10, 7, 11, 13, 16, 12, 4, 5, 6, 14, 13, 15)$$

Player	Cost function
1	$-\frac{x_1^4}{30}-\frac{13x_1^3}{15}+\frac{259x_1^2}{30}-\frac{263x_1}{15}+1$
2	$\frac{77x_2^5}{120}-\frac{247x_2^4}{24}+\frac{471x_2^3}{8}-\frac{3365x_2^2}{24}+\frac{6779x_2}{60}+1$
3	$\frac{47x_3^4}{24}-\frac{133x_3^3}{4}+\frac{4897x_3^2}{24}-\frac{2123x_3}{4}+485$
4	$\frac{323x_4^5}{3360}-\frac{2179x_4^4}{1120}+\frac{47393x_4^3}{3360}-\frac{48709x_4^2}{1120}+\frac{7885x_4}{168}+5$
5	$(x_5-1)^2$
6	$-\frac{x_6^4}{8}+\frac{25x_6^3}{12}-\frac{71x_6^2}{8}+\frac{95x_6}{12}+10$
7	$\lvert x_7-5\rvert$
8	$\frac{11x_8^7}{1260}-\frac{7x_8^6}{36}+\frac{119x_8^5}{72}-\frac{479x_8^4}{72}+\frac{4609x_8^3}{360}-\frac{803x_8^2}{72}+\frac{155x_8}{28}+1$
9	$-\frac{15}{16}x_9^3+\frac{365x_9^2}{16}-\frac{2865x_9}{16}+\frac{7315}{16}$
10	$(x_{10}-10)^2$
11	$\frac{5x_{11}^4}{6}-\frac{35x_{11}^3}{3}+\frac{355x_{11}^2}{6}-\frac{370x_{11}}{3}+90$
12	$\frac{5x_{12}^4}{6}-\frac{25x_{12}^3}{3}+\frac{175x_{12}^2}{6}-\frac{110x_{12}}{3}+15$
13	$\frac{5x_{13}^4}{6}-15x_{13}^3+\frac{595x_{13}^2}{6}-280x_{13}+285$
14	$\frac{5x_{14}^4}{6}-\frac{85x_{14}^3}{3}+\frac{2155x_{14}^2}{6}-\frac{6020x_{14}}{3}+4165$
15	$\lvert x_{15}-7\rvert$
16	$\begin{cases} x_{16}+1, & \text{if } 0\le x_{16}\le 3 \\ 0, & \text{if } 4\le x_{16}\le 6 \\ (x_{16}+1)^3, & \text{if } 7\le x_{16}\le 9 \\ -\frac{x_{16}^3}{6}+\frac{13x_{16}^2}{2}-\frac{244x_{16}}{3}+330, & \text{else} \end{cases}$

Pareto Optimal Solutions

Our algorithm provides two efficient Pareto optimal points:

$$(1, 3, 5, 4, 11, 10, 2, 1, 3, 7, 7, 5)$$

and

$$(1, 3, 5, 6, 11, 10, 2, 1, 3, 7, 7, 6).$$