Progressive Party Problem

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Progressive Party Problem

- Problem Description
- Models:
    - Smith and Hooker's Constraint Programming Model [4, 2]
    - Hooker's Integer Programming Model [2]
        - Will talk about [3] Jeroslow's representability theorem
    - Another Constraint Programming Model [1]
- Comparison of our results

Progressive Party Problem: Description

- Progressive Party takes place for a given number of periods
- Two kinds of boats:
  - Guest boats
  - Host boats
- If any crew is not guest, then that crew as a unit will visit a different host boat in each period.
- If a crew is host, then they will remain on their boat for all periods

Progressive Party Problem: Description

- If some guest boat is visiting another boat, then the second boat must be host boat.
- The capacity of the host boats must be respected. At any time period, the number of people on board has to be less than or equal to the maximum capacity of the host boat.
- In any period if two guest crews do not meet each other, then they must be visiting different host boats.

However...

- Two crews cannot meet each other more than once

Progressive Party Problem: Objective

- Objective is to minimize total number of host boats

Progressive Party Problem: Notations

- Set of all periods: $\mathcal{T} = \{1, ..., T\}$
- Set of all boats: $\mathcal{B} = \{1, ..., n\}$
- Capacity of i th boat: C_i
- Number of crews from boat i: b_i

Progressive Party Problem: Notations

- $x_i = \begin{cases} 1, & \text{iff boat } i \text{ is a host boat} \\ 0, & \text{else} \end{cases}$
- y_{it}=The boat that is visited by crew i in period t
- $v_{ijt} = \begin{cases} 1, & \text{iff crew } i \text{ visit boat } j \text{ in period } t \text{ i.e.} y_{it} = j \\ 0, & \text{else} \end{cases}$
- $m_{ijt} = \begin{cases} 1, & \text{when crews } i \text{ and } j \text{ visit the same host boat in period } t \\ 0, & \text{else} \end{cases}$

Progressive Party Problem: Objective

`Objective is to minimize total number of host boats`

---

minimize $\sum_{i \in \mathcal{B}} x_i$

Progressive Party Problem: Smith and Hooker's Constraint Programming formulation

$$v_{ijt} = \begin{cases} 1, & \texttt{only if crew } i \texttt{ visit boat } j \texttt{ in period } t \texttt{ i.e.} y_{it} = j \\ 0, & \texttt{else} \end{cases}$$

`Definition of` v_{ijt}`:`

$$\boxed{\forall i, j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad v_{ijt} = 1 \Leftrightarrow y_{it} = j} \tag{1}$$

If any ($\forall i \in \mathcal{B}$) crew is not host ($x_i = 0$), then that crew as a unit will visit a different host boat ($\texttt{alldifferent}(y_{i1}, y_{i2}, ..., y_{iT})$) in each period ($\forall t \in \mathcal{T}$)

$$\boxed{\forall t \in \mathcal{T}\ \forall i \in \mathcal{B} \quad (x_i = 0) \Rightarrow \texttt{alldifferent}(y_{i1}, y_{i2}, ..., y_{iT})} \tag{2}$$

Progressive Party Problem: Smith and Hooker's Constraint Programming formulation

```
If a crew is host, then they will remain on their boat for
all periods
```

$$\boxed{\forall i \in \mathcal{B}\, \forall t \in \mathcal{T} \quad (x_i = 1) \Rightarrow (y_{it} = i)} \tag{3}$$

Progressive Party Problem: Smith and Hooker's Constraint Programming formulation

`If in any period` $\forall t \in \mathcal{T}$ `any` $(\forall i \in \mathcal{B})$ `boat is visiting another boat` $(j \in \mathcal{B} \wedge y_{it} = j)$`, then the second boat must be host boat` $x_j = 1$.

$$\boxed{\forall t \in \mathcal{T}\ \forall i, j \in \mathcal{B}\ y_{it} = j \Rightarrow x_j = 1} \tag{4}$$

At any time period ($\forall t \in \mathcal{T}$) on any boat ($j \in \mathcal{B}$) the number of people on board

$$\Big(\underbrace{b_j}_{\text{number of host crews on host boat } j} + \overbrace{\sum_{i \in \mathcal{B} \setminus \{j\}} b_i v_{ijt}}^{\text{guests visiting from the other boats}} \Big)$$

has to be less than or equal to the maximum capacity of the host boat (C_j)

$$\boxed{\forall j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad \Big(b_j + \sum_{i \in \mathcal{B} \setminus \{j\}} b_i v_{ijt}\Big) \leq C_j} \qquad (5)$$

Progressive Party Problem: Smith and Hooker's Constraint Programming formulation

In any period ($\forall t \in \mathcal{T}$), if any two ($\forall i, j \in \mathcal{B} : i < j$) guest crews ($x_i = 0 \wedge x_j = 0$) do not meet each other ($m_{ijt} = 0$), then they must be visiting different host boats ($y_{it} \neq y_{jt}$).

$$
\begin{aligned}
&\forall t \in \mathcal{T} \\
&\forall (i, j) \in \{(k, l) \in \mathcal{B} \times \mathcal{B} : k < l, x_k = 0, x_l = 0\} \\
&m_{ijt} = 0 \Rightarrow y_{it} \neq y_{jt}
\end{aligned}
\tag{6}
$$

`Two crews` ($\forall i, j \in \mathcal{B} : i < j$) `cannot meet each other more than once over all the periods` ($\sum_{t \in \mathcal{T}} m_{ijt} \leq 1$)

$$\forall (i,j) \in \{(k,l) \in \mathcal{B} \times \mathcal{B} : k < l\}$$
$$\sum_{t \in \mathcal{T}} m_{ijt} \leq 1 \tag{7}$$

Progressive Party Problem: Smith and Hooker's Constraint Programming formulation

$$
\begin{aligned}
&\text{minimize} \sum_{i\in\mathcal{B}} x_i \qquad \text{subject to} \\
&\forall i,j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad v_{ijt} = 1 \Leftrightarrow y_{it} = j \\
&\forall t \in \mathcal{T}\ \forall i \in \mathcal{B} \quad (x_i = 0) \Rightarrow \texttt{alldifferent}(y_{i1}, y_{i2}, \ldots, y_{iT}) \\
&\forall t \in \mathcal{T}\ \forall i,j \in \mathcal{B}\ y_{it} = j \Rightarrow x_j = 1 \\
&\forall i \in \mathcal{B}\ \forall t \in \mathcal{T} \quad x_i = 1 \Rightarrow y_{it} = i \\
&\forall j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad (b_j + \sum_{i\in\mathcal{B}\setminus\{j\}} b_i v_{ijt}) \le C_j \\
&\forall t \in \mathcal{T}\ \forall (i,j) \in \{(k,l) \in \mathcal{B}\times\mathcal{B} : k < l, x_k = 0, x_l = 0\} \\
&\qquad\qquad\qquad\qquad m_{ijt} = 0 \Rightarrow y_{it} \neq y_{jt} \\
&\forall (i,j) \in \{(k,l) \in \mathcal{B}\times\mathcal{B} : k < l\} \sum_{t\in\mathcal{T}} m_{ijt} \le 1
\end{aligned} \tag{8}
$$

There is some redundancy in the model!
The constraints

$$\begin{cases} \forall t \in \mathcal{T}\ \forall i,j \in \mathcal{B}\ y_{it} = j \Rightarrow x_j = 1 & \texttt{//:A crew remain on their boat} \\ \forall i \in \mathcal{B}\ \forall t \in \mathcal{T} \quad x_i = 1 \Rightarrow y_{it} = i & \texttt{//iff they are host} \end{cases}$$

$$\forall (i,j) \in \{(k,l) \in \mathcal{B} \times \mathcal{B} : k < l\} \sum_{t \in \mathcal{T}} m_{ijt} \leq 1; \quad \texttt{//No crew can visit}$$

`//the same boat more than once`

imply the `alldifferent` constraint!

minimize $\sum_{i \in \mathcal{B}} x_i$ subject to

$\forall i, j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad v_{ijt} = 1 \Leftrightarrow y_{it} = j$

$\forall t \in \mathcal{T}\ \forall i \in \mathcal{B} \quad (x_i = 0) \Rightarrow \texttt{alldifferent}(y_{i1}, y_{i2}, ..., y_{iT})$ `// redundant`

$\forall t \in \mathcal{T}\ \forall i, j \in \mathcal{B}\ y_{it} = j \Rightarrow x_j = 1$

$\forall i \in \mathcal{B}\ \forall t \in \mathcal{T} \quad x_i = 1 \Rightarrow y_{it} = i$

$$\forall j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad (b_j + \sum_{i \in \mathcal{B} \setminus \{j\}} b_i v_{ijt}) \leq C_j$$

$$\forall t \in \mathcal{T}\ \forall (i, j) \in \{(k, l) \in \mathcal{B} \times \mathcal{B} : k < l, x_k = 0, x_l = 0\}$$
$$m_{ijt} = 0 \Rightarrow y_{it} \neq y_{jt}$$

$$\forall (i, j) \in \{(k, l) \in \mathcal{B} \times \mathcal{B} : k < l\} \sum_{t \in \mathcal{T}} m_{ijt} \leq 1$$

- The red constraints are not mathematical programming constraint, how can we can express them as integer programming constraint?
- More importantly, **when can we express some constraint programming constraint as a mathematical programming constraint and vice versa**?

Jeroslow's Representability Theorem: A Simplified Version [3]

- In 1987 Jeroslow came up with his fundamental representability theorem for integer programming
- **Any optimization problem can be formulated as an integer programming problem if and only if the constraint set is the union of finitely many polyhedrons (also called Jeroslow's disjunctive form)**
- Once we can write constraint programming constraints in *Jeroslow's disjunctive form*, we can transform it into a integer programming constraints

Jeroslow's disjunctive form [3]

$$\bigvee_{k \in \mathcal{K}} \overbrace{y_k}^{\text{a boolean variable}}$$

$$\forall k \in \mathcal{K} \quad y_k \Rightarrow (A_k^T x \succeq b_k) \tag{9}$$

Jeroslow's Representability Theorem: A Simplified Version [3]

$$
\bigvee_{k\in\mathcal{K}} \overbrace{y_k}^{\text{a boolean variable}}
$$

$$
\forall k \in \mathcal{K} \quad y_k \Rightarrow (A_k^T x \succeq b_k) \tag{10}
$$

is equivalent to :

$$
\begin{aligned}
&\sum_{k\in\mathcal{K}} y_k \geq 1\\
&\forall k \in \mathcal{K} \quad A_k x \succeq b_k - M_k(1-y_k)\mathbf{1}\\
&\forall k \in \mathcal{K} \quad y_k \in \{0,1\}
\end{aligned} \tag{11}
$$

$\forall i, j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad v_{ijt} = 1 \Leftrightarrow y_{it} = j$
can be represented as:
$\forall i \in \mathcal{B}\ \forall t \in \mathcal{T} \quad y_{it} = \sum_{j \in \mathcal{B}} j v_{ijt}$

If in any period ($\forall t \in \mathcal{T}$) if any crew ($\forall i \in \mathcal{B}$) is visiting j th boat, then j must be a host boat ($x_j = 1$).

$$\forall i \in \mathcal{B}\, \forall t \in \mathcal{T} \qquad \overbrace{(v_{ijt} = 1)}^{\text{Boolean Variable}} \;\Rightarrow\; \overbrace{(x_j = 1)}^{\text{Boolean Varible}}$$

If a boat is host ($x_i = 1$), then for all period ($\forall t \in \mathcal{T}$) they will stay on their own boat ($v_{iit} = 1$)

$$\forall i \in \mathcal{B}\ \forall t \in \mathcal{T} \qquad \overbrace{(x_i = 1)}^{\text{Boolean Variable}} \quad \Rightarrow \quad \overbrace{(v_{iit} = 1)}^{\text{Boolean Varible}}$$

Nice to know

If both P and Q are boolean variables, then $P \Rightarrow Q$ is equivalent to $P - Q \leq 0$

$\forall i \in \mathcal{B}\ \forall t \in \mathcal{T} \quad (v_{ijt} = 1) \Rightarrow (x_j = 1)$
is equivalent to
$\forall i \in \mathcal{B}\ \forall t \in \mathcal{T} \quad v_{ijt} \leq x_j$

$\forall i \in \mathcal{B} \, \forall t \in \mathcal{T} \quad (x_i = 1) \Rightarrow (v_{iit} = 1)$
is equivalent to
$\forall i \in \mathcal{B} \, \forall t \in \mathcal{T} \quad x_i \leq v_{iit}$

Last but not least...

What about

$$\forall t \in \mathcal{T}\ \forall (i,j) \in \{(k,l) \in \mathcal{B} \times \mathcal{B} : k < l, x_k = 0, x_l = 0\}$$
$$m_{ijt} = 0 \Rightarrow y_{it} \neq y_{jt} \tag{12}$$

?

Some Preliminary Mathematical Logic

$$
\begin{aligned}
&\forall t \in \mathcal{T}\ \forall (i,j) \in \{(k,l) \in \mathcal{B} \times \mathcal{B} : k < l, x_k = 0, x_l = 0\} \\
&\qquad\qquad\qquad\qquad m_{ijt} = 0 \Rightarrow y_{it} \neq y_{jt} \\
\Leftrightarrow &\forall t \in \mathcal{T}\ \forall (i,j) \in \{(k,l) \in \mathcal{B} \times \mathcal{B} : k < l, x_k = 0, x_l = 0\} \\
&\qquad\qquad\qquad\qquad (m_{ijt} = 1 \vee y_{it} \neq y_{jt}) \\
\Leftrightarrow &\forall t \in \mathcal{T}\ \forall i \in \mathcal{B}\ \forall j \in \mathcal{B}/\{i\}\ (x_k = 0 \wedge x_l = 0) \Rightarrow (m_{ijt} = 1 \vee y_{it} \neq y_{jt}) \\
\Leftrightarrow &\forall t \in \mathcal{T}\ \forall i \in \mathcal{B}\ \forall j \in \mathcal{B}/\{i\}\ \neg(x_k = 0 \wedge x_l = 0) \vee (m_{ijt} = 1 \vee y_{it} \neq y_{jt}) \\
\Leftrightarrow &\forall t \in \mathcal{T}\ \forall i \in \mathcal{B}\ \forall j \in \mathcal{B}/\{i\}\ (x_k = 1 \vee x_l = 1) \vee (m_{ijt} = 1 \vee y_{it} \neq y_{jt}) \\
\Leftrightarrow &\forall t \in \mathcal{T}\ \forall i \in \mathcal{B}\ \forall j \in \mathcal{B}/\{i\}\ x_k = 1 \vee x_l = 1 \vee m_{ijt} = 1 \vee y_{it} \neq y_{jt}
\end{aligned}
$$

Some Preliminary Mathematical Logic continued

$$
\begin{aligned}
&\forall t \in \mathcal{T}\ \forall i \in \mathcal{B}\ \forall j \in \mathcal{B}/\{i\}\ (x_k = 1 \vee x_l = 1 \vee m_{ijt} = 0 \vee y_{it} \neq y_{jt}) \\
\Leftrightarrow &\forall t \in \mathcal{T}\ \forall i \in \mathcal{B}\ \forall j \in \mathcal{B}/\{i\} \\
&\qquad (x_k = 1 \vee x_l = 1 \vee m_{ijt} = 1 \vee (y_{it} > y_{jt}) \vee (y_{it} < y_{jt})) \\
\Leftrightarrow &\forall t \in \mathcal{T}\ \forall i \in \mathcal{B}\ \forall j \in \mathcal{B}/\{i\} \\
&\qquad (x_k = 1 \vee x_l = 1 \vee m_{ijt} = 1 \vee \overbrace{(y_{it} \geq y_{jt} + 1)}^{\text{not boolean}} \vee \underbrace{(y_{it} \leq y_{jt} - 1))}_{\text{not boolean}}
\end{aligned}
$$

Some Preliminary Mathematical Logic continued

$$\forall t \in \mathcal{T}\ \forall i \in \mathcal{B}\ \forall j \in \mathcal{B}/\{i\}$$

$$(x_k = 1 \vee x_l = 1 \vee m_{ijt} = 1 \vee \overbrace{(y_{it} \geq y_{jt} + 1)}^{\text{not boolean}} \vee \underbrace{(y_{it} \leq y_{jt} - 1))}_{\text{not boolean}}$$

Let introduce two boolean variables which will imply them

- $(\psi_{ijt} = 1) \Rightarrow (y_{it} \geq y_{jt} + 1)$
- $(\phi_{ijt} = 1) \Rightarrow (y_{it} \leq y_{jt} - 1)$

Some Preliminary Mathematical Logic continued

$$(x_k = 1 \vee x_l = 1 \vee m_{ijt} = 1 \vee (y_{it} \geq y_{jt} + 1) \vee (y_{it} \leq y_{jt} - 1))$$

is equivalent to:

$$(x_k = 1 \vee x_l = 1 \vee m_{ijt} = 1 \vee (\psi_{ijt} = 1) \vee (\phi_{ijt} = 1))$$
$$(\psi_{ijt} = 1) \Rightarrow (y_{it} \geq y_{jt} + 1);$$
$$(\phi_{ijt} = 1) \Rightarrow (y_{it} \leq y_{jt} - 1)$$

Jeroslow's Representability Theorem: A Simplified Version [3]

$$
\begin{aligned}
&\bigvee_{k\in\mathcal{K}} \overbrace{y_k}^{\text{a boolean variable}} \\
&\forall k \in \mathcal{K} \quad y_k \Rightarrow (A_k^T x \succeq b_k)
\end{aligned} \tag{13}
$$

is equivalent to :

$$
\begin{aligned}
&\sum_{k\in\mathcal{K}} y_k \geq 1 \\
&\forall k \in \mathcal{K} \quad A_k x \succeq b_k - M_k(1-y_k)\mathbf{1} \\
&\forall k \in \mathcal{K} \quad y_k \in \{0,1\}
\end{aligned} \tag{14}
$$

Some Preliminary Mathematical Logic continued

$$
\begin{aligned}
&(x_k = 1 \vee x_l = 1 \vee m_{ijt} = 1 \vee (\psi_{ijt} = 1) \vee (\phi_{ijt} = 1)) \\
&(\psi_{ijt} = 1) \Rightarrow (y_{it} \geq y_{jt} + 1) \\
&(\phi_{ijt} = 1) \Rightarrow (y_{it} \leq y_{jt} - 1)
\end{aligned}
$$

is equivalent to:

$$
\begin{aligned}
&x_i + x_j + m_{ijt} + \psi_{ijt} + \phi_{ijt} \geq 1 \\
&y_{it} - y_{jt} \geq 1 - |\mathcal{B}|(1 - \psi_{ijt}) \\
&y_{jt} - y_{it} \geq 1 - |\mathcal{B}|(1 - \phi_{ijt})
\end{aligned}
$$

Progressive Party Problem: John Hooker's Integer Programming formulation

$$\text{minimize} \sum_{i\in\mathcal{B}} x_i \qquad \text{subject to}$$

$$\forall i \in \mathcal{B}\ \forall t \in \mathcal{T} \quad y_{it} = \sum_{j\in\mathcal{B}} j v_{ijt}$$

$$\forall i \in \mathcal{B}\ \forall t \in \mathcal{T} \quad v_{ijt} \leq x_j$$

$$\forall i \in \mathcal{B}\ \forall t \in \mathcal{T} \quad x_i \leq v_{iit}$$

$$\forall j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad (b_j + \sum_{i\in\mathcal{B}\setminus\{j\}} b_i v_{ijt}) \leq C_j$$

$$\forall t \in \mathcal{T}\ \forall i \in \mathcal{B}\ \forall j \in \mathcal{B}/\{i\} \quad x_i + x_j + m_{ijt} + \psi_{ijt} + \phi_{ijt} \geq 1$$

$$\forall t \in \mathcal{T}\ \forall i \in \mathcal{B}\ \forall j \in \mathcal{B}/\{i\} \quad y_{it} - y_{jt} \geq 1 - |\mathcal{B}|(1 - \psi_{ijt})$$

$$\forall t \in \mathcal{T}\ \forall i \in \mathcal{B}\ \forall j \in \mathcal{B}/\{i\} \quad y_{jt} - y_{it} \geq 1 - |\mathcal{B}|(1 - \phi_{ijt})$$

$$\forall (i,j) \in \{(k,l) \in \mathcal{B}\times\mathcal{B} : k < l\} \quad \sum_{t\in\mathcal{T}} m_{ijt} \leq 1$$

Some modifications found on the web of the Constraint Programming Model

- Removing the boolean variables introduced v_{ijt}, m_{ijt}
- Writing the constraints more compactly

Some modifications of the Constraint Programming Model

$$\text{minimize} \sum_{i \in \mathcal{B}} x_i \qquad \text{subject to}$$

$$\forall i, j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad v_{ijt} = 1 \Leftrightarrow y_{it} = j$$

$$\forall t \in \mathcal{T}\ \forall i \in \mathcal{B} \quad (x_i = 0) \Rightarrow \texttt{alldifferent}(y_{i1}, y_{i2}, ..., y_{iT})$$

$$\forall t \in \mathcal{T}\ \forall i, j \in \mathcal{B}\ y_{it} = j \Rightarrow x_j = 1$$

$$\forall i \in \mathcal{B}\ \forall t \in \mathcal{T} \quad x_i = 1 \Rightarrow y_{it} = i$$

$$\forall j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad (b_j + \sum_{i \in \mathcal{B} \setminus \{j\}} b_i v_{ijt}) \leq C_j$$

$$\forall t \in \mathcal{T}\ \forall (i, j) \in \{(k, l) \in \mathcal{B} \times \mathcal{B} : k < l, x_k = 0, x_l = 0\}$$

$$(m_{ijt} = 0 \Rightarrow y_{it} \neq y_{jt}) \Leftrightarrow \underbrace{(y_{it} = y_{jt} \Rightarrow m_{ijt} = 1)}_{\textit{Contrapositive}}$$

$$\forall (i, j) \in \{(k, l) \in \mathcal{B} \times \mathcal{B} : k < l\} \sum_{t \in \mathcal{T}} m_{ijt} \leq 1 \tag{15}$$

Some modifications of the Constraint Programming Model

$$\forall i, j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad v_{ijt} = 1 \Leftrightarrow y_{it} = j$$
$$\forall j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad (b_j + \sum_{i \in \mathcal{B} \setminus \{j\}} b_i v_{ijt}) \leq C_j$$

can be simplified as:

$$\forall j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad (b_j \overbrace{v_{jjt}}^{1} + \sum_{i \in \mathcal{B} \setminus \{j\}} b_i v_{ijt}) \leq C_j$$
$$\Leftrightarrow \forall j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad (\sum_{i \in \mathcal{B}} b_i v_{ijt}) \leq C_j$$
$$\Leftrightarrow \forall j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad (\sum_{i \in \mathcal{B} \wedge v_{ijt} = 1} b_i) \leq C_j$$
$$\Leftrightarrow \boxed{\forall j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad (\sum_{i \in \mathcal{B} \wedge y_{it} = j} b_i) \leq C_j}$$

Known as packing constraint

$$\forall t \in \mathcal{T}\ \forall (i,j) \in \{(k,l) \in \mathcal{B} \times \mathcal{B} : k < l, x_k = 0, x_l = 0\}$$
$$(y_{it} = y_{jt} \Rightarrow m_{ijt} = 1)$$
$$\forall (i,j) \in \{(k,l) \in \mathcal{B} \times \mathcal{B} : k < l\} \sum_{t \in \mathcal{T}} m_{ijt} \leq 1$$

can be simplified as:

$$\boxed{\begin{aligned}&\forall t \in \mathcal{T}\ \forall (i,j) \in \{(k,l) \in \mathcal{B} \times \mathcal{B} : k < l, x_k = 0, x_l = 0\}\\ &\qquad (y_{it} = y_{jt}) \Rightarrow \forall t' \in \mathcal{T} \backslash \{t\} \quad (y_{it'} \neq y_{jt'})\end{aligned}}$$

Some modifications of the Constraint Programming Model

$$\forall t \in \mathcal{T}\ \forall i,j \in \mathcal{B}\ y_{it} = j \Rightarrow x_j = 1$$
$$\forall i \in \mathcal{B}\ \forall t \in \mathcal{T} \quad x_i = 1 \Rightarrow y_{it} = i$$

can be simplified as:

$$\boxed{\forall i \in \mathcal{B}\ \forall t \in \mathcal{T}\ x_{y_{it}} = 1} \tag{16}$$

Modified Constraint Programming Model

$$\text{minimize} \sum_{i \in \mathcal{B}} x_i \qquad \text{subject to}$$

$$\begin{aligned}
&\forall j \in \mathcal{B}\ \forall t \in \mathcal{T} \quad \Big(\sum_{i \in \mathcal{B} \wedge y_{it}=j} b_i\Big) \leq C_j \quad \texttt{:Packing Constraint} \\
&\forall t \in \mathcal{T}\ \forall i \in \mathcal{B} \quad (x_i = 0) \Rightarrow \texttt{alldifferent}(y_{i1}, y_{i2}, \ldots, y_{iT}) \\
&\forall i \in \mathcal{B}\ \forall t \in \mathcal{T}\ x_{y_{it}} = 1 \\
&\forall t \in \mathcal{T}\ \forall (i,j) \in \{(k,l) \in \mathcal{B} \times \mathcal{B} : k < l, x_k = 0, x_l = 0\} \\
&\qquad\qquad (y_{it} = y_{jt}) \Rightarrow \forall t' \in \mathcal{T} \setminus \{t\} \quad (y_{it'} \neq y_{jt'})
\end{aligned} \tag{17}$$

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