Semicontinuity of set-valued maps

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Consider a set-valued map $F : X \rightrightarrows Y$.

Notation. For the map F and set V in X, we have

$$F(V) = \bigcup_{x \in V} F(x),$$

and $F(V)_\bullet \subseteq W$ means for all $x \in V$, the set $F(x)$ lies in W. The notation $\mathcal{V}(F(x))$ means collection of all the open sets that contain the set $F(x)$. One arbitrary element of the set $F(x)$ is denoted by $\widetilde{F(x)}$. The set of all open neighborhoods of a point x is denoted by $\mathcal{V}(x)$.

Recap on continuity of a function f. A function f is continuous at x, if

$$\forall_{W \in \mathcal{V}(f(x))} \exists_{V \in \mathcal{V}(x)} \quad F(V) \subseteq W.$$

Another definition is through sequence convergence: f is continuous at x, if

$$\forall_{x_n \to x} \quad f(x_n) \to f(x).$$

When we are talking about a set-valued map, these two definitions lead to two different notions of continuity.

Now let's go back to F.

Upper semicontinuous map. The map F is upper semicontinuous at $x \in \mathbf{dom}\, F$ if

$$\forall_{W \in \mathcal{V}(F(x))} \exists_{V \in \mathcal{V}(x)} \quad F(V)_\bullet \subseteq W.$$

Roughly speaking, upper semicontinuity at a point x means that a small perturbation in x does not make the set $F(x)$ to suddenly expand violently (it is allowed to shrink abruptly however).

Lower semicontinuous map. The map F is lower semicontinuous at $x \in \mathbf{dom}\, F$ if

$$\forall_{\widetilde{F(x)}} \forall_{x_n \to x} \exists_{\widetilde{F(x_n)}} \quad \widetilde{F(x_n)} \to \widetilde{F(x)}.$$

Roughly speaking, lower semicontinuity at a point x *eliminates* the following situation: there is some $x_{\text{almost}} \approx x$ such that $F(x_{\text{almost}})$ is far away from some of the points in $F(x)$.

Figure showing the differences. Figure 1 shows different notions of continuity for set-valued maps. The picture is taken from "Real Analysis with Economic Applications" by Efe A. Ok, page 288. In the figure, *hemicontinuity* is an equivalent term for semicontinuity, and the black circles correspond to values taken by the maps, where the white circles correspond to the values *not* taken by the maps.

Some results.

- A set-valued map is continuous at x, if it is both upper semicontinuous and lower semicontinuous at x.
- Set-valued maps with closed graphs with their values in a compact set are upper semicontinuous.

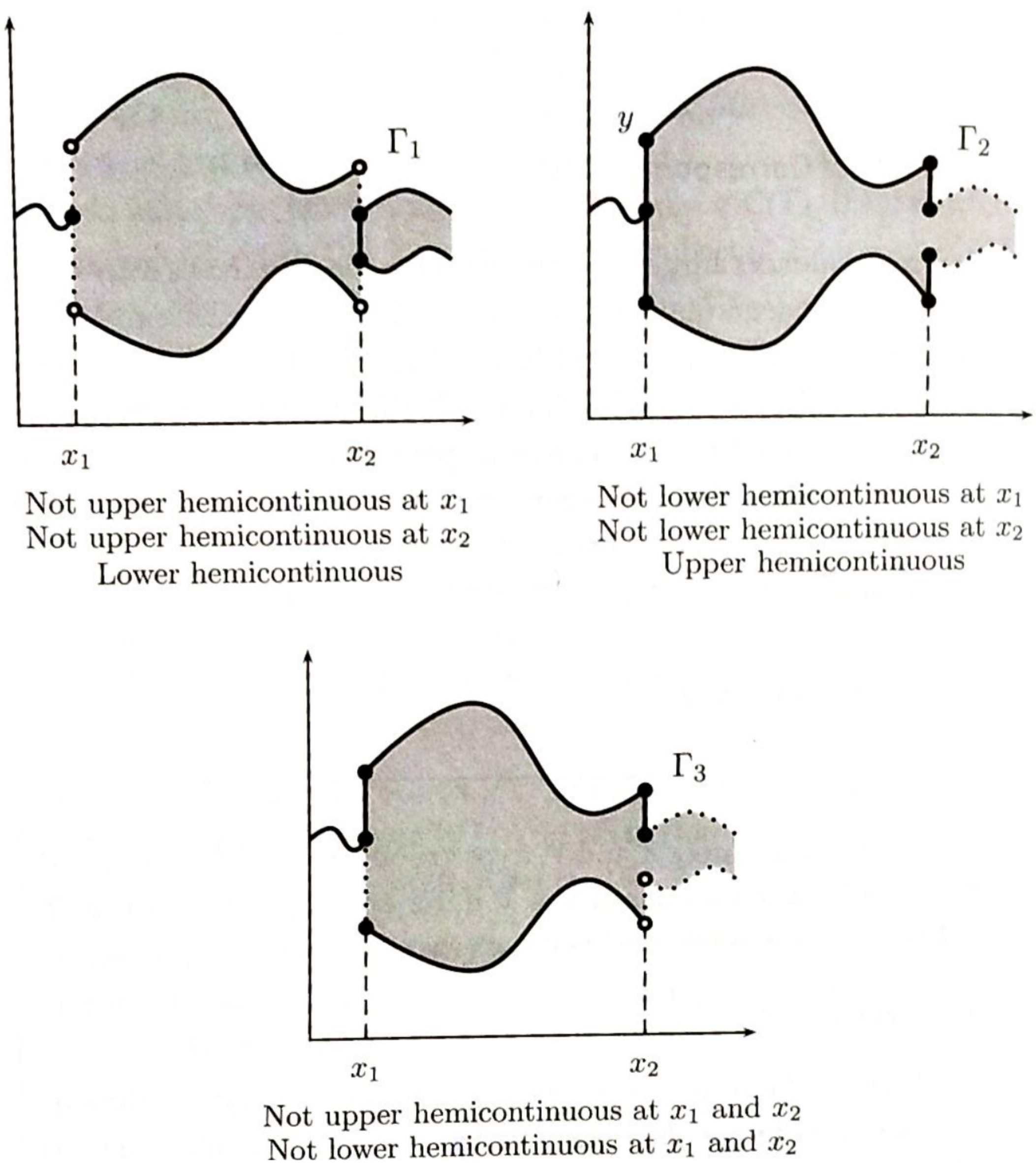

Figure 1: The figure shows different notions of continuity for set-valued maps.