Different notions of subdifferentials for nonconvex functions

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Consider a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$, which is proper and closed. The functions we consider here are nonconvex.

Regular subdifferential $\partial^{\mathrm{reg}} f$.

$$\begin{aligned} u \in \partial^{\mathrm{reg}} f(x) &\Leftrightarrow \liminf_{y \to x, y \neq x} \frac{f(y) - f(x) - \langle u \mid y - x \rangle}{\|y - x\|} \geq 0 \\ &\Leftrightarrow (\forall y \in \mathbf{R}^n) \quad f(y) \geq f(x) + \langle u \mid y - x \rangle + o(\|y - x\|) \end{aligned}$$

Another characterization is as follows. We say $u \in \partial^{\mathrm{reg}} f(x)$, if on some neighborhood of x, there is an underestimator function $h \leq f$ with $h(x) = f(x)$, and h is differentiable at x with $\nabla h(x) = u$.

Limiting subdifferential $\partial^{\mathrm{lim}} f$.

$$u \in \partial^{\mathrm{lim}} f(x) \Leftrightarrow \exists_{u_n \in \partial^{\mathrm{reg}} f(x_n)} \left(\begin{bmatrix} x_n \\ f(x_n) \\ u_n \end{bmatrix} \to \begin{bmatrix} x \\ f(x) \\ u \end{bmatrix} \right).$$

Horizon subdifferential $\partial^{\mathrm{hor}} f$.

$$u \in \partial f^{\mathrm{hor}}(z) \Leftrightarrow \exists_{u_n \in \partial^{\mathrm{reg}} f(x_n)} \exists_{\lambda_n \downarrow 0} \left(\begin{bmatrix} x_n \\ f(x_n) \\ \lambda_n u_n \end{bmatrix} \to \begin{bmatrix} x \\ f(x) \\ u \end{bmatrix} \right).$$

In the third row, the interpretation is as follows. If there is a positive scalar sequence $\lambda_n \to 0$ that satisfies $\lambda_n u_n \to u$, then we say $u_n \to \mathbf{dir}(u)$.

Bouligand subdifferential $\partial^{\mathrm{bou}} f$.

$$u \in \partial f^{\mathrm{bou}}(x) \Leftrightarrow \exists_{(x_n): x_n \to x,\ f \text{ differentiable at all } x_n} \left(\nabla f(x_n) \to u \right).$$

Characterization of minimizer.

Consider the aforementioned function f. Then, if x is a local minimizer of f, then it satisfies $0 \in \partial^{\mathrm{reg}} f(x)$.

Basic subdifferential rules.

- For any $t > 0$, we have $\partial^{\mathrm{lim}}(tf)(x) = t\partial^{\mathrm{lim}} f(x)$, and $\partial^{\mathrm{reg}}(tf)(x) = t\partial^{\mathrm{reg}} f(x)$.
- The function f is strictly continuous at x, if and only if $x \in \mathbf{dom}\, f$ and $\partial^{\mathrm{hor}} f(x) = \{0\}$.
- If the function f is strictly continuous at x, then $\partial^{\mathrm{lim}}(f + g)(x) \subseteq \partial^{\mathrm{lim}} f(x) + \partial^{\mathrm{lim}} g(x)$.
- If f is strictly continuous at x and $\partial^{\mathrm{lim}} f(x)$ has at most one element, then f is strictly differentiable at x.

- If f is differentiable at x, then $\partial^{\text{reg}} f(x) = \{\nabla f(x)\}$.
- If f is continuously differentiable around x, and g is a proper function, then we have

$$\begin{aligned}\partial^{\text{lim}} f(x) = \partial^{\text{reg}} f(x) = \{\nabla f(x)\},\\ \partial^{\text{lim}}(f+g)(x) = \nabla f(x) + \partial^{\text{lim}} g(x),\\ \partial^{\text{reg}}(f+g)(x) = \nabla f(x) + \partial^{\text{reg}} g(x).\end{aligned}$$

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