Stable Marriage with Ties and Incomplete Lists (SMTI)

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Stable Marriage with Ties and Incomplete Lists

- Classic version introduced by David Gale and Lloyd Shapley in 1962 [2]
- They also came up with a fantastic algorithm, today known as Gale-Shapely (GS) algorithm
- In 1984 Alvin E. Roth applied GS algorithm to National Resident Matching Program US, which matches medical interns to hospitals [5]
- Roth has also applied GS algorithm to:
  - School admission process (e.g., in New York and Boston public school systems) [1]
  - Kidney transplant (since 2004, already 2000 transplants have been done this way) [5]
- Gale passed away in 2008. Roth and Shapley were awarded Nobel Prize in Economics in 2012.

Structure of the presentation

- Problem Description: Evolution of classical Stable Marriage to Stable Marriage with Ties and Incomplete Lists
- The GS algorithm
- Constraint programming model by Pascal Van Hentenryck
    - What we did
- Constraint programming model by Ian Philip Gent
- A integer programming formulation by John Vande Vate

Problem Description: Classic Stable Marriage Problem

- Two finite disjoint set of men and women
- Number of men and women are identical
- Each man has strict preferences over all women
- Each woman has strict preferences over all men
- The goal is to marry men and women $\Leftrightarrow$ find a matching
- Condition for a stable matching: There is no incentive for individuals to divorce and elope $\Leftrightarrow$ there is no rogue pair

Rogue Pair in SM

Man m and woman w are not in the matching, but

- m prefers w to his current partner;
- w prefers m to her current partner;

Problem Description: Stable Marriage with Ties and Incomplete Lists

- Two finite disjoint set of men and women
- Number of men and women are $\underbrace{\text{not necessarily identical}}_{\substack{\text{add universally hated hateful} \\ \text{fictitious men/women}}}$
- Each man has not necessarily strict preferences over a subset women
- Each woman has not necessarily strict preferences over a subset men
- The goal is to marry men and women $\Leftrightarrow$ find a matching
- Condition for a **weakly** stable matching: There is no incentive for individuals to divorce and elope $\Leftrightarrow$ there is no rogue pair

Rogue pair in SMTI

Man m and woman w are not paiblue in the matching, but

- m has included w in his preference list;
- w has included m in her preference list;
- m is single or prefers w to his current partner;
- w is single or prefers m to her current partner;

Gale Shapely Algorithm (Classic Stable Marriage) [4]

Data: Set of men and women and their preference lists
Result: A stable marriage, S

```
Assign each person to be free; /* not a member of a pair in the matching */
while There is a free man m and has not proposed to every woman on his list do
    m proposes to w, the first woman on his list to whom he has not proposed yet;
    if w is free then
        add (m, w) to S; /* w and m get engaged */
    else if w is already engaged to another man μ, but likes m more then
        remove (μ, w) from S; /* w calls off her engagement with μ, setting μ free */
        add (m, w) to S; /* w and m get engaged */
    else
        S remains unchanged; /* w rejects m */
    end
end
```

Some key results about Classic Stable Marriage

- For every instance, there is at least one stable marriage
- GS algorithm runs in $O(n^2)$
- Number of stable matchings can be multiple, but GS algorithm produces the man-optimal one where
  - Each man gets his best possible wife
  - Each woman gets her worst possible husband
- For one instance all stable matchings have the same size

Gale Shapely Algorithm applied SMTI

- No modification needed, GS algorithm will still produce a man-optimal stable marriage!
- Ties are broken arbitrarily

Some key results about SMTI

- For every instance of Stable Marriage, there is at least one stable marriage
- For SMTI, GS algorithm runs in $O(a)$, where a is the total length of all the preference lists
- Number of stable matchings can be multiple, but GS algorithm produces the man-optimal one where
  - Each man gets his best possible wife
  - Each woman gets her worst possible husband
- Different stable matchings can have different sizes

Model by Pascal Van Hentenryck (Classic Stable Marriage) [6]

Data: Set of men $\mathcal{M}$, and women $\mathcal{W}$ and their preference lists $(\forall i \in \mathcal{M})\mathcal{P}_i$ and $(\forall j \in \mathcal{W})\mathcal{P}_j$ /* $\mathcal{P}_{i,2}$ is the rank of the second woman in man i's preference list and so on, and lower ranking denotes higher preference */
Variables: $\forall i \in \mathcal{M}$ Wife of man i: x_i and $\forall j \in$ husband of woman j: y_j
Domain: $D(x_i) = \{j : j \in \mathcal{W}\}$, $D(y_j) = \{i : i \in \mathcal{M}\}$
Result: The completely determined vectors, $x = (x_i)_{i \in \mathcal{M}} = (x_1, \ldots, x_{|\mathcal{M}|})$ and $y = (y_j)_{j \in \mathcal{W}} = (y_1, \ldots, y_{|\mathcal{W}|})$
/* $x_3 = 4$ means man 3 gets married to woman 4, $y_5 = 6$ means woman 5 gets married to man 5 etc. */

```
Constraints begin
    for i ← 1 to |M| do
        y_{x_i} = i
    end
    /* Monogamy Constraint: Husband of wife of i has to be i */
    for j ← 1 to |W| do
        x_{y_j} = j
    end
    /* Monogamy Constraint: Wife of husband of j has to be j */
    for i ← 1 to |M| do
        for w ← 1 to |W| do
            P_{i,w} < P_{i,x_i} ⇒ P_{w,y_w} < P_{w,i} /* Stability Constraint: If man i prefers woman w to his
                current partner x_i, then w must prefer her current partner y_w to man i */
        end
    end
    for j ← 1 to |W| do
        for m ← 1 to |M| do
            P_{j,m} < P_{j,y_j} ⇒ P_{m,x_m} < P_{m,j} /* Stability Constraint: If woman j prefers man m to her
                current partner y_j, then m must prefer his current partner x_m to woman j */
        end
    end
end
```

Data: Set of men $\mathcal{M}$, and women $\mathcal{W}$ and their preference lists $(\forall i \in \mathcal{M})\mathcal{P}_i$ and $(\forall j \in \mathcal{W})\mathcal{P}_j$ /* $\mathcal{P}_{i,2}$ is the rank of the second woman in man i's preference list and so on, and lower ranking denotes higher preference */
Variables: $\forall i \in \mathcal{M}$ Wife of man i: x_i and $\forall j \in$ husband of woman j: y_j
Preprocessing begin

```
for i ← 1 to |M| do
    for w ← 1 to |W| do
        woman w is unacceptable to man i ⇒ (P_{i,w} ← M) /* M is a large enough number */
    end
end
for j ← 1 to |W| do
    for m ← 1 to |M| do
        man m is unacceptable to woman j ⇒ (P_{j,m} ← M) /* M is a large enough number */
    end
end
```

end
Domain: $D(x_i) = \{j : j \in \mathcal{W}\}$, $D(y_j) = \{i : i \in \mathcal{M}\}$
Result: The completely determined vectors, $x = (x_i)_{i\in\mathcal{M}} = (x_1, \ldots, x_{|\mathcal{M}|})$ and $y = (y_j)_{j\in\mathcal{W}} = (y_1, \ldots, y_{|\mathcal{W}|})$
, $\forall i \in \mathcal{M}$ $(\mathcal{A}_i \in \{0, 1\})$ /* $x_3 = 4$ means man 3 gets married to woman 4, $y_5 = 6$ means woman 5 gets married to man 5 etc. */
/* man i gets a woman x_i as his wife where both of them are each others preference lists $\Leftrightarrow \mathcal{A}_i = 1$ (else $\mathcal{A}_i = 0$) */

Modification of the Model by Pascal Van Hentenryck (SMTI) II

Constraints begin

- **for** $i \leftarrow 1$ *to* $|\mathcal{M}|$ **do**
  - $y_{x_i} = i$ /* Monogamy Constraint: Husband of wife of i has to be i */
- **end**
- **for** $j \leftarrow 1$ *to* $|\mathcal{W}|$ **do**
  - $x_{y_j} = j$ /* Monogamy Constraint: Wife of husband of j has to be j */
- **end**
- **for** $i \leftarrow 1$ *to* $|\mathcal{M}|$ **do**
  - $(\mathcal{P}_{i,x_i} \neq M \wedge \mathcal{P}_{x_i,i} \neq M) \Leftrightarrow \mathcal{A}_i = 1$ /* Acceptability of a marriage */
- **end**
- /* Stability constraint for men: */
- **for** $i \leftarrow 1$ *to* $|\mathcal{M}|$ **do**
  - **for** $w \leftarrow 1$ *to* $|\mathcal{W}|$ **do**
    - $(\mathcal{P}_{i,w} \neq M \wedge \mathcal{P}_{w,i} \neq M)$ /* man i and woman w are in each others preference lists */
    - $\Rightarrow ((\mathcal{A}_i = 0$ /* man i is single */
    - $\vee \mathcal{P}_{i,w} < \mathcal{P}_{i,x_i})$ /* or man i prefers w to his current partner */
    - $\Rightarrow (\mathcal{A}_{y_w} = 1$ /* woman w is married */
    - $\wedge \mathcal{P}_{w,y_w} \leq \mathcal{P}_{w,i}))$ /* and woman w likes her current partner y_w at least as much as m */
  - **end**
- **end**
- /* Stability constraint for women: */
- **for** $j \leftarrow 1$ *to* $|\mathcal{W}|$ **do**
  - **for** $m \leftarrow 1$ *to* $|\mathcal{M}|$ **do**
    - $(\mathcal{P}_{i,w} \neq M \wedge \mathcal{P}_{w,i} \neq M) \Rightarrow ((\mathcal{A}_{y_j} = 0 \vee \mathcal{P}_{j,m} < \mathcal{P}_{j,y_j}) \Rightarrow (\mathcal{A}_m = 1 \wedge \mathcal{P}_{m,x_m} \leq \mathcal{P}_{m,j}))$
  - **end**
- **end**

end

Gent's Model

- Formulated as a binary Constraint Satisfaction Problem
- Gent uses a Conflict Matrix based encoding which ensures that binary Arc Consistency (AC) is enough to obtain stable matching
- **Arc Consistency**: Given x_i and y_j and a value $w \in D(x_i)$, a value $m \in D(y_j)$ is a support of w if the partial assignment $\{x_i \leftarrow w, y_j \leftarrow m\}$ is consistent

Gent's Model [3]

Data: Set of men $\mathcal{M}$, and women $\mathcal{W}$ and their preference lists $(\forall i \in \mathcal{M})\mathcal{P}_i$ and $(\forall j \in \mathcal{W})\mathcal{P}_j$

Variables: $\forall i \in \mathcal{M}$ Wife of man i: x_i and $\forall j \in$ husband of woman j: y_j

Domain: $D(x_i) = \{j : j \in \mathcal{P}_i\} \cup \{n+1\}$, $D(y_j) = \{i : i \in \mathcal{P}_j\} \cup \{n+1\}$

The index $n+1$ stands for a fictitious person who is least preferable by everyone in the opposite sex

Result: The completely determined vectors, $x = (x_i)_{i \in \mathcal{M}} = (x_1, ..., x_{|\mathcal{M}|})$ and $y = (y_j)_{j \in \mathcal{W}} = (y_1, ..., y_{|\mathcal{W}|})$

Gent's Model continued

Constraints: $(\forall i \in \mathcal{M})(\forall j \in \mathcal{W})$ *Gent Stable Marriage constraint* is a set of *nogoods* represented by the Conflict Matrix C^{ij} with dimension $|\mathcal{P}_i| \times |\mathcal{P}_j|$

Nogood: In a Constraint Satisfaction Problem, any partial instantiation that does not appear in any solution is called a nogood.

Gent's Model continued: Structure of the Conflict matrix C^{ij} for man i and woman j

k is controlled by man i, l is controlled by woman j

$$C^{ij}_{kl} = \begin{cases} A, & \text{when } k = j \text{ and } l = i \qquad \texttt{\\\\Allowed} \\ I, & \text{when } (k = j \text{ but } l \neq i) \text{ or } (k \neq j \text{ but } l = i) \qquad \texttt{\\\\Illegal} \\ B, & \text{when } \mathcal{P}_{i,j} < \mathcal{P}_{i,k} \text{ and } \mathcal{P}_{j,i} < \mathcal{P}_{j,l} \;\; \texttt{\\\\(i,j) will be a rogue pair} \\ & \texttt{\\\\for couples } (i,k) \texttt{ and } (j,l) \\ S, & else \end{cases}$$

Gent's Model continued: Propagation of Arc Consistency on *Gent Stable Marriage constraint*

- Enforcement of AC on *Gent Stable Marriage constraint* corresponding the Conflict Matrix C^{ij} deletes rows and column consisting of only I and/or B
- After AC propagation, every row or column of the conflict matrix has at least one A or S
- **Finding man-optimal stable matching:** After AC propagation, assign each x_i the most preferable woman in $D(x_i)$
  - No search needed
  - Shown to be equivalent to the output of Gale-Shapely algorithm

A integer programming approach by Vande Vate: Notations [7]

- Set of men $\mathcal{M}$, set of women $\mathcal{W}$
- Decision vector, $\xi \in \{0,1\}^{|\mathcal{M}|\times|\mathcal{W}|}$, where $\overbrace{(x_i = j \wedge y_j = i)}^{\text{defined previously}} \Leftrightarrow \xi(i,j) = 1$
- $\xi(i > \square, j) = \sum_{k\in\{\chi:\mathcal{P}_{j,i}<\mathcal{P}_{j,\chi}\}} \xi(k,j)$ `Significance:`$\xi(i > \square, j) = 1$ `means woman` j `is married to someone who she finds less desirable than man` i
- $\xi(i > \square > i', j) = \sum_{k\in\{\chi:\mathcal{P}_{j,i}<\mathcal{P}_{j,\chi}<\mathcal{P}_{j,i'}\}} \xi(k,j)$ `Significance:`$\xi(i > \square > i', j) = 1$ `means woman` j `is married to someone who she finds less desirable than man` i `and more desirable than` i'
- $\xi(\mathcal{M}, j) = \sum_{k\in\{\chi:\chi\in\mathcal{M}\}} \xi(k,j)$ `Significance:` $\xi(\mathcal{M},j)$`=1, since woman` j `is married to some man`
- $\xi(i, \square \geq j) = \sum_{w\in\{\lambda:\mathcal{P}_{i,\lambda}\leq\mathcal{P}_{i,w}\}} \xi(i,w)$ `Significance:`$\xi(i, \square \geq j) = 1$ `means man` i `is married to someone who he finds at least as desirable as woman` j

minimize 0
subject to
$\forall j \in \mathcal{W} \quad \xi(\mathcal{M}, j) = 1$ `:Monogamy constraint for every woman`
$\forall i \in \mathcal{M} \quad \xi(i, \mathcal{W}) = 1$ `:Monogamy constraint for every man`
$\forall (i,j) \in \mathcal{M} \times \mathcal{W} \quad \xi(i,j) \geq 0$ `:Ensures integrality`
$\forall (i,j) \in \mathcal{M} \times \mathcal{W} \quad \xi(i > \square, j) - \xi(i, \square \geq j) \leq 0$ `:If woman` j `marries some man less desirable than man` i`, then man` i `must marry some woman at least as desirable as` j

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