Optimality Conditions in Integer Optimization using Integral Basis

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In integer programming we minimize a linear cost function subject to the constraint set being the integer points in a polyhedron. In this tutorial, we study fundamental geometric objects of integer optimization, namely integral generating set and integral basis. They *i)* lead to optimality conditions in integer programming, *ii)* characterize integral polyhedra, and *iii)* play very important role in integer programming algorithms such as cutting plane and integral basis algorithms.

Necessary Background

Integral generating set and integral basis. Consider a set $\mathcal{F}$, which is a set of integer points. For example, $\mathcal{F}$ could be the set

$$\{x \in \mathbf{Z}^n \mid Ax \preceq b\},$$

which is the constraint set for an integer programming problem. We look for a subset of $\mathcal{F}$, preferably much smaller than $\mathcal{F}$, which can generate all the points in $\mathcal{F}$ with respect to nonnegative combinations of its elements. Such sets are called integral generating sets. The integral generating set of minimum size is called the integral basis.

Notation

- Set of integers and real numbers are denoted by $\mathbf{Z}$ and $\mathbf{R}$ respectively.
- If $S \subseteq \mathbf{R}^n$ is a set, then the set of all integer points in S is denoted by $S^{\mathbf{Z}}$, *i.e.*, $S^{\mathbf{Z}} = S \cap \mathbf{Z}^n$.
- $P = \{x \in \mathbf{R}^n \mid Ax \preceq b\}$.
- $P^{\mathbf{Z}} = \{x \in \mathbf{Z}^n \mid Ax \preceq b\}$.
- Convex hull of a set S is denoted by $\mathbf{conv}(S)$.
- $x_i^+ = \max\{0, x_i\}$, $x_i^- = \max\{0, -x_i\}$, $x^+ = (x_1^+, \ldots, x_n^+)$, $x^- = \{x_1^-, \ldots, x_n^-\}$.

Definition 1. *(Integral generating set and integral basis) Consider the set $\mathcal{F}$, which is a set of integer points in $\mathbf{Z}^n$. A subset of $\mathcal{F}$, denoted by $\mathbf{igs}(\mathcal{F})$, is called an integral generating set of $\mathcal{F}$, if*

$$(\forall x \in \mathcal{F})\,(\exists\{h_1, \ldots, h_k\} \subseteq \mathbf{igs}(\mathcal{F}))\,(\exists\{\lambda_1, \ldots, \lambda_k\} \subseteq \mathbf{Z}_+) \quad x = \sum_{i=1}^{k} \lambda_i h_i.$$

An integral generating set $\mathbf{igs}(\mathcal{F})$ of $\mathcal{F}$ is called an integral basis, and denoted by $\mathbf{ib}(\mathcal{F})$ if it is the smallest integral generating set. More specifically,

$$(\forall G : \text{integral generating set of } \mathcal{F}) \quad \mathbf{ib}(\mathcal{F}) \subseteq G.$$

Note that, $\mathcal{F}$ is its own integral generating set, however it is not an integral basis in general.

Cone in polyhedral theory. The concept of cone is very important in integer optimization.

Definition 2. *(Cone) A set $C \subseteq \mathbf{R}^n$ is called a cone if*

$$(\forall x \in C)\,(\forall \lambda \geq 0) \quad \lambda x \in C.$$

Now we define some other types of cone: polyhedral cone, rational cone, and pointed cone.

Definition 3. *(Polyhedral cone) A cone $C \subseteq \mathbf{R}^n$ is called a polyhedral cone if* [1]

$$\left(\exists V \in \mathbf{R}^{n\times k}\right) \quad C = cone(V) = \left\{V\lambda \mid \lambda \in \mathbf{R}^k_+\right\}.$$

The matrix V is called the *generator* of the cone. If the ith column of V is denoted by v_i, then we can write $V\lambda = \sum_{i=1}^k \lambda_i v_i$. So, each element of C is generated by a nonnegative combination of the columns of V. We say that the polyhedral cone C is generated by V, and write it as $C = \text{cone}(V)$. If $V \in \mathbf{Q}^{n\times k}$, *i.e.*, all the elements of V are rational, then we say that C is a *rational polyhedral cone.* Another equivalent representation of a rational polyhedral cone is the set

$$\{x \in \mathbf{R}^n \mid Ax \succeq 0\},$$

where $A \in \mathbf{Q}^{m\times n}$.

[1] A polyhedral cone is always convex.

Definition 4. *(Pointed cone) A cone is pointed if there exists a halfspace $h = \{x|a^T x \leq 0\}$ such that* [2]

$$h \cap C = \{0\}.$$

[2] A pointed cone does not contain a line.

Similar to integral generating set and integral basis, we can define analogous concepts for a set of real numbers.

Definition 5. *(Real generating set and real basis) Consider a set $\mathcal{R}$, a set of real points in $\mathbf{R}^n$. A subset of $\mathcal{R}$, denoted by $\mathbf{rgs}(\mathcal{R})$, is called a real generating set of $\mathcal{R}$, if*

$$(\forall x \in \mathcal{R})\,(\exists\{h_1,\ldots,h_k\} \subseteq \mathbf{rgs}(\mathcal{F}))\,(\exists\{\lambda_1,\ldots,\lambda_k\} \subseteq \mathbf{R}_+) \quad x = \sum_{i=1}^k \lambda_i h_i.$$

An real generating set $\mathbf{rgs}(\mathcal{R})$ of $\mathcal{R}$ is called an real basis, and denoted by $\mathbf{rb}(\mathcal{R})$ if it is the smallest real generating set. More specifically,

$$(\forall G : \textit{real generating set of } \mathcal{R}) \quad \mathbf{rb}(\mathcal{R}) \subseteq G.$$

If $\text{cone}(V)$ is a pointed rational polyhedral cone, then the any element in $\mathbf{rb}\left(\text{cone}(V)\right)$ is an extreme ray.

Integral generating sets and integral bases in cones

Set of integer points in a rational polyhedral cone has a finite integral generating set. In this section we study the following problem. We are given a rational polyhedral cone $C = \text{cone}(V)$, with $V \in \mathbf{Q}^{n\times k}$. The number of integer points in C, *i.e.*, in $C^{\mathbf{Z}} = C \cap \mathbf{Z}^n$, is infinite. What about its integral generating set? It turns out that there is a finite integral generating set. So a finite number of integer points would generate all the elements in a countably infinite set!

Theorem 6. *If $C = \text{cone}(V)$ is rational, then there exists an* $\textbf{igs}\left(C^{\mathbf{Z}}\right)$, *which has a finite number of elements.*

Set of integer points in a rational polyhedral cones has a finite unique integral basis. The theorem above is an encouraging one; it motivates us to ask to look for the smallest of all the integral generating sets - an integral basis, which will contain the smallest and finite number of generating points. However, the integral basis sets of $\text{cone}(V)^{\mathbf{Z}} = \text{cone}(V) \cap \mathbf{Z}^n$ may note be unique, though the cardinality of each of those integral basis sets will be the same. But, if we add the attribute - pointedness to $\text{cone}(V)$, then we have a unique integral basis.

Theorem 7. *Suppose C is a rational polyhedral cone. Then,* $\textbf{ib}\left(C^{\mathbf{Z}}\right)$ *is unique and is given by*

$$\textbf{ib}\left(C^{\mathbf{Z}}\right) = \left\{h \in (C^{\mathbf{Z}} \setminus \{0\}) \mid \left(\forall v, w \in (C^{\mathbf{Z}} \setminus \{0\})\right) \quad h \neq v + w\right\}.$$

In words, the integer points in a rational polyhedral cone has a unique integral basis.

Change in the cone generator. Suppose we are given a rational polyhedral cone $C = \text{cone}(V)$, where $V = \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix} \in \mathbf{Z}^{n \times k}$,[3] and we have calculated an integral generating set of $C^{\mathbf{Z}} = C \cap \mathbf{Z}^n$, denoted by $\textbf{igs}\left(C^{\mathbf{Z}}\right)$. Now we pick a point c from $C^{\mathbf{Z}}$, negate it and add it to V, so we have a new matrix

$$V' = \begin{bmatrix} v_1 & v_2 & \cdots & v_k & -c \end{bmatrix}.$$

Consider the rational polyhedral cone generated by V', *i.e.*, $C' = C(V')$. What would be an integral generating set of $C'^{\mathbf{Z}}$? Turns out that, all we have to do is to take $\textbf{igs}\left(C^{\mathbf{Z}}\right)$ and union the vector $-c$ to it, *i.e.*, $\textbf{igs}\left(C'^{\mathbf{Z}}\right) = \textbf{igs}\left(C^{\mathbf{Z}}\right) \cup \{-c\}$.

Theorem 8. *Suppose, $V = \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix} \in \mathbf{Z}^{n \times k}$, and $C = \text{cone}(V)$ is a rational (integral) polyhedral cone. For any $c \in C^{\mathbf{Z}}$, suppose $V' = \begin{bmatrix} V & -c \end{bmatrix}$ and $C' = C(V')$. Then,*

$$\textbf{igs}\left(C'^{\mathbf{Z}}\right) = \textbf{igs}(C^{\mathbf{Z}}) \cup \{-c\}.$$

Extension of Caratheodory's theorem to integer points in a rational polyhedral cone. Caratheodory's theorem is a key result in linear programming theory. Consider a rational polyhedral cone generated by $V \in \mathbf{Q}^{n \times k}$, denoted by $\text{cone}(V) \subseteq \mathbf{R}^n$. Suppose its real basis is denoted by $\textbf{rb}(\text{cone}(V))$[4]. Now all points in $\text{cone}(V)$ can

[3] In fact C is an integral polyhedral cone.

[4] For $\text{cone}(V)$,

$$\textbf{rb}\left(\text{cone}\left(V\right)\right) = \{v_1, \ldots v_i, \ldots, v_k\},$$

where v_i is the ith column of V.

be represented as a nonnegative combinations of the elements in $\mathbf{rb}\,(\text{cone}(V))$. A natural question is, if we are given an arbitrary element of $\text{cone}(V)$, how many elements from $\mathbf{rb}(\text{cone}(V))$ is required to express it? Caratheodory's theorem says that the magic number is n, same as the dimension of the space where $\text{cone}(V)$ resides.

Theorem 9. *(Caratheodory's theorem) Suppose* $\text{cone}(V) \subseteq \mathbf{R}^n$ *is a rational polyhedral cone. Any element of* $\text{cone}(V) \subseteq \mathbf{R}^n$ *can be represented by a nonnegative combination of at most n elements from* $\mathbf{rb}\,(\text{cone}(V))$.

A natural question is if Caratheodory's theorem holds for the integer points in $\text{cone}(V)$. The answer is no in generally[5]. We need almost twice as many points.

[5] Holds for $n = 2$ and 3, but does not for $n \geq 6$.

Theorem 10. *(Extension of Caratheodory's theorem to integer points in a pointed rational polyhedral cone) Suppose* $\text{cone}(V) \subseteq \mathbf{R}^n$ *is a* <u>*pointed*</u> *rational polyhedral cone. Any element of* $\text{cone}(V)^{\mathbf{Z}} \subseteq \mathbf{Z}^n$ *can be represented by a nonnegative* <u>*integral*</u> *combination at most* $2n - 2$ *elements from* $\mathbf{ib}\,\left(\text{cone}(V)^{\mathbf{Z}}\right)$.

Optimality conditions in integer programming

Review of optimality conditions in linear programming. Consider the standard for linear programming problem given below.

$$\begin{array}{ll} \text{maximize}_x & c^T x \\ \text{subject to} & x \in P = \{x \in \mathbf{R}^n \mid Ax = b, 0 \preceq x \preceq u\} \end{array} \tag{1}$$

Here, $A \in \mathbf{Z}^{m\times n}, b \in \mathbf{Z}^m$.

An orthant in a finite n-dimensional vector space is a set which contains the zero vector and the sign component of all vectors in it is the same. Any finite $n-$dimensional vector space can be divided into 2^n orthants[6]. We denote the jth orthant by O_j for $j = 1, 2, \ldots, 2^n$. Note that, if we multiply any point of an orthant by a nonnegative number, the sign components does not change and the resultant component stays in the same orthant. So, an orthant is also a cone. Now for $j = 1, 2, \ldots, 2^n$, we denote the nullspace of A in orthant O_j by C_j, *i.e.*,

[6] *E.g.*, for $n = 2$, we have 4 orthants.

$$\begin{aligned} C_j &= \{x \in \mathbf{R}^n \mid Ax = 0\} \cap O_j \\ &= \{x \in O_j \mid Ax = 0\}. \end{aligned}$$

The set C_j is a pointed rational polyhedral cone too[7]. Now the set of all extreme rays[8] in C_j, *i.e.*, the real basis of C_j, is denoted by $\mathbf{rb}\,(C_j)$. Using the $\mathbf{rb}(C_j)$s we can come up with the optimality condition for the linear programming problem (1).

[7] C_j is a cone, which we can show as follows. First, by definition,

$$x \in C_j \Leftrightarrow x \in O_j, Ax = 0.$$

Consider any $\lambda \geq 0$. By definition of an orthant, $\lambda x \in O_j$, and $A(\lambda x) = \lambda Ax = 0$. So, $\lambda x \in C_j$, *i.e.*, C_j is a cone. As, C_j is also a pointed rational polyhedron, it is a pointed rational polyhedral cone.

[8] •Extreme rays of a cone are the directions associated with the edges of a cone which extend to infinity.

•The set of extreme rays of a rational polyhedral cone and its real basis are the same set.

Theorem 11. *(Optimality conditions in linear programming) Consider any feasible solution $x \in P$ for the linear programming problem* (1)*. Then x is optimal if and only if*

$$\left(\forall h \in \bigcup_{j=1}^{2^n} \mathbf{rb}\left(C_j\right)\right) \begin{cases} c^T h \le 0, \text{ or} \\ c^T h > 0, \text{ and } x + \lambda h \notin P \quad (\forall \lambda > 0)\,. \end{cases}$$

The first disjunctive statement means that if we move away from x to any other feasible point, then the objective function cannot increase anymore. The second disjunctive statement means that if we can get to a point from x, where we have a larger objective value, then that first point will be infeasible.

Optimality conditions in integer programming. Now let's see how the optimality conditions change for integer programming problem. Consider the integer programming problem[9]:

[9] Compare with linear programming problem (1).

$$\begin{aligned} &\text{maximize}_x && c^T x \\ &\text{subject to} && x \in P^{\mathbf{Z}} = \{x \in \mathbf{Z}^n \mid Ax = b, 0 \preceq x \preceq u\}\,. \end{aligned} \tag{2}$$

The optimality conditions for the problem above has structural similarity to that of the linear programming problem. The optimality conditions are stated in the theorem below.

Theorem 12. *(Optimality conditions in integer programming) Consider any feasible solution $x \in P^{\mathbf{Z}}$ for the integer programming problem* (2)*. Then x is optimal if and only if*

$$\left(\forall h \in \bigcup_{j=1}^{2^n} \mathbf{ib}\left(C_j^{\mathbf{Z}}\right)\right) \begin{cases} c^T h \le 0, \text{ or} \\ c^T h > 0, \text{ and } x + h \notin P. \end{cases}$$

Note the surprising structural similarity between Theorems (11) and (12). For a detailed proof of this theorem see [10].

[10] Dimitris Bertsimas and Robert Weismantel. *Optimization over integers*, volume 13. Dynamic Ideas Belmont, 2005

From cones to polyhedra

Integer points in a polyhedron may not have a finite integral generating set. In this section we discuss the existence and uniqueness of integral bases of the constraint set arising in integer optimization problem:

$$\mathcal{F} = \{x \in \mathbf{Z}^n \mid Ax \preceq b\}\,,$$

where $A \in \mathbf{Z}^{m \times n}$, $b \in \mathbf{Z}^m$. Recall that the integer points in a rational polyhedral cone has a finite integral generating set. However, the set of integer points in a polyhedron may not have a finite integral generating set. This is a disappointing result. So, we relax one of the specifications for the integral generating set - the generators *i.e.*, members of the integral generating set coming from the integer set $\mathcal{F}$ itself, and ask if we can find a finite number of points, not necessarily in $\mathcal{F}$, which generate $\mathcal{F}$. This has a positive answer, which we present next.

Theorem 13. *Consider the polyhedron*

$$P = \{x \in \mathbf{R}^n \mid Ax \preceq b, x \succeq 0\},$$

where $A \in \mathbf{Z}^{m\times n}$ and $b \in \mathbf{Z}^m$. The polyhedral cone associated with P is

$$C = \{x \in \mathbf{R}^n \mid Ax \preceq 0, x \succeq 0\},$$

and the real basis of C (the set of all extreme rays of C) is

$$\mathbf{rb}(C) = \{v_1, \ldots, v_t\} \subseteq \mathbf{Z}^n.$$

Then,

- *There exists a finite set $W \subseteq P^{\mathbf{Z}}$ such that*

$$\left(\forall x \in P^{\mathbf{Z}}\right) (\exists w \in W) (\exists \lambda_1, \ldots, \lambda_t \in \mathbf{Z}_+) \quad x = w + \sum_{i=1}^{t} \lambda_i v_i.$$

- *The convex hull of $P^{\mathbf{Z}}$, denoted by $\mathbf{conv}(P^{\mathbf{Z}})$ satisfies*

$$\mathbf{conv}(P^{\mathbf{Z}}) = \mathbf{conv}(W) + C.$$

The set of all integer points in P is denoted by

$$P^{\mathbf{Z}} = \{x \in \mathbf{Z}^n \mid Ax \preceq b, x \succeq 0\}.$$

The set of all integer points in C is denoted by

$$C^{\mathbf{Z}} = \{x \in \mathbf{Z}^n \mid Ax \preceq 0, x \succeq 0\}.$$

Nonnegative resource vector b leads to a finite integral basis. Is there any condition under which we have a finite integral generating set for the integer points in a polyhedron? The answer is yes, if we have a nonnegative resource vector, then we have a finite integral generating set, more specifically a finite integral basis.

Theorem 14. *Consider the polyhedron*

$$P = \{x \in \mathbf{R}^n \mid Ax \preceq b, x \succeq 0\},$$

where $A \in \mathbf{Z}^{m\times n}$ and $b \in \mathbf{Z}^m$. If $b \succeq 0$, then $P^{\mathbf{Z}}$ has finite integral basis $\mathbf{ib}(P^{\mathbf{Z}})$ given by

$$\begin{aligned}
\mathbf{ib}(P^{\mathbf{Z}}) &= \left\{ v \in P^{\mathbf{Z}} \mid \neg \left((\exists \lambda_1, \ldots, \lambda_k \in \mathbf{Z}_+) \left(\exists v_1, \ldots, v_k \in P^{\mathbf{Z}} \setminus \{v, 0\}\right) \quad v = \sum_{i=1}^{k} \lambda_i v_i \right) \right\} \\
&= \left\{ v \in P^{\mathbf{Z}} \mid (\forall \lambda_1, \ldots, \lambda_k \in \mathbf{Z}_+) \left(\forall v_1, \ldots, v_k \in P^{\mathbf{Z}} \setminus \{v, 0\}\right) \quad v \neq \sum_{i=1}^{k} \lambda_i v_i \right\}.
\end{aligned}$$

A more general condition for the existence of a finite integral basis. Is there a more general theorem than this, which would guarantee the existence of a finite integral basis? The answer is yes and it is related to how many integer points of the polyhedral cone is missing from $P^{\mathbf{Z}}$; if it is finite, then we have a finite integral basis.

Theorem 15. *Consider the polyhedron*

$$P = \{x \in \mathbf{R}^n \mid Ax \preceq b, x \succeq 0\},$$

where $A \in \mathbf{Z}^{m \times n}$ and $b \in \mathbf{Z}^m$. The polyhedral cone associated with P is

$$C = \{x \in \mathbf{R}^n \mid Ax \preceq 0, x \succeq 0\}.$$

There exists a finite integral generating set of $P^{\mathbf{Z}}$ if and only if $P^{\mathbf{Z}}$ contains all but a finite number of points in $C^{\mathbf{Z}}$. Moreover, if a finite integral generating set of $P^{\mathbf{Z}}$ exists, then there is a unique integral basis of $P^{\mathbf{Z}}$.

Algorithms to compute integral generating sets and integral bases

Suppose we are given a set of integer points $\mathcal{F}$, and we want to find out its integral generating sets and integral bases.

Integral generating set when $\mathcal{F}$ is finite and given explicitly. If $\mathcal{F}$ is finite and given explicitly, then $\mathcal{F}$ itself is its integral generating set.

Integral generating set when $\mathcal{F}$ consists of the integer points of a rational polyhedral cone. Suppose we are given a rational polyhedral cone $C = \mathrm{cone}(V)$, where $V = \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix} \in \mathbf{Z}^{n \times k}$, then

$$\mathbf{igs}\left(C^{\mathbf{Z}}\right) = \{v_1, \ldots, v_k\} \bigcup \left\{ \sum_{i=1}^{k} \lambda_i v_i \mid (\lambda_i)_{i=1}^{k} \subseteq [0,1) \right\}.$$

Integral generating set when $\mathcal{F}$ consists of the integer points in a polyhedron. The set in consideration is

$$\mathcal{F} = \{x \in \mathbf{Z}^n \mid Ax \preceq b, x \succeq 0\},$$

where $A \in \mathbf{Z}^{m \times n}$ and $b \in \mathbf{Z}^m$ and $b \succeq 0$. Recall that in this case, we have a finite integral generating set and finite integral basis by Theorem (14). We want to find $\mathbf{igs}(\mathcal{F})$. The algorithm to find it is given by Algorithm (1), which terminates in finite time.

Integral basis when $\mathcal{F}$ consists of the integer points in a polyhedron. After we have found $\mathbf{igs}(\mathcal{F})$ via Algorithm (1). Now, we want to find out the integral basis from the integral generating set. The algorithm is given by (2).

Algorithm 1 Calculating integral generating set of $\{x \in \mathbf{Z}^n \mid Ax \preceq b, x \succeq 0\}$.

input: $A \in \mathbf{Z}^{m \times n}, b \in \mathbf{Z}^m, b \succeq 0$
output: $\mathbf{igs}(\mathcal{F})$, where

$$\mathcal{F} = \{x \in \mathbf{Z}^n \mid Ax \preceq b, x \succeq 0\}.$$

algorithm:
1. initialization:
$T_0 := \emptyset$
$T := \{e_i \mid i = 1, \ldots, n\} \triangleright e_i$ is the ith unit vector.
2. main iteration:
while $T_0 \neq T$
repeat
  $T_0 := T$
  $(\forall v, w \in T_0) \quad z := v + w$
  if $(\exists y \in T) \begin{cases} y \preceq z \\ (Ay)^+ \preceq (Az)^+ \\ (Ay)^- \preceq (Az)^- \end{cases}$
    then $z := z - y$
  end if
  if $z \neq 0$
    then $T := T \cup \{z\}$
  end if
end repeat
3. return $\mathbf{igs}(\mathcal{F}) := T \cap \mathcal{F}$.

Integral basis when $\mathcal{F}$ consists of the integer points in a pointed rational polyhedral cone. When the set in consideration is a pointed rational polyhedral cone, then Algorithm (2) can be simplified using following theorem.

Theorem 16. *Suppose C is a pointed rational polyhedral cone, and $h \in \mathbf{igs}\left(C^{\mathbf{Z}}\right) \setminus \{0\}$. Then $h \notin \mathbf{ib}\left(C^{\mathbf{Z}}\right)$ if and only if there exists a $h_j \in \mathbf{igs}\left(C^{\mathbf{Z}}\right)$ such that $h - h_j \in C$.*

Due to the theorem above, we can simplify the algorithm for calculating the integral basis of a pointed rational polyhedral cone. Checking whether $h - h_j \in C$ is a linear feasibility problem. The algorithm is given by Algorithm (3). If we are given $\mathbf{igs}\left(C^{\mathbf{Z}}\right)$ explicitly, then we can find $\mathbf{ib}\left(C^{\mathbf{Z}}\right)$ in polynomial time by solving $\frac{|\mathbf{igs}(C^{\mathbf{Z}})|(|\mathbf{igs}(C^{\mathbf{Z}})|-1)}{2}$ linear feasibility problems.

Total dual integrality

The concept of total dual integrality is handy to guarantee the integrality of a polyhedron.

Algorithm 2 Calculating integral basis of $\{x \in \mathbf{Z}^n \mid Ax \preceq b, x \succeq 0\}$.

input: $A \in \mathbf{Z}^{m\times n}, b \in \mathbf{Z}^m, b \succeq 0$, $\mathbf{igs}\,(\mathcal{F}) = \{h_1, \ldots, h_k\}$, where

$$\mathcal{F} = \{x \in \mathbf{Z}^n \mid Ax \preceq b, x \succeq 0\}.$$

output: $\mathbf{ib}(\mathcal{F})$.

algorithm:
1. initialization
$T := \mathbf{igs}\,(\mathcal{F})$
2. main iteration:
for $i = 1, \ldots, k$ **do**
Solve the integer feasibility problem

$$\begin{array}{ll} \text{find} & y \\ \text{subject to} & \sum_{j=1, j\neq i}^{k} y_j h_j = h_i \\ & y \in \mathbf{Z}_+^{k-1}. \end{array} \tag{3}$$

if problem (3) is feasible
then $T := T \setminus \{h_i\}$
end if
end for
3. return $\mathbf{ib}(\mathcal{F}) := T$.

Algorithm 3 Calculating integral basis of $\{x \in \mathbf{Z}^n \mid Ax \preceq 0, x \succeq 0\}$.

input: $A \in \mathbf{Z}^{m\times n}, b \in \mathbf{Z}^m$,
$C = \{x \in \mathbf{R}^n \mid Ax \preceq 0, x \succeq 0\}$,
$\mathbf{igs}\left(C^{\mathbf{Z}}\right) = \{h_1, \ldots, h_k\}$.
output: $\mathbf{ib}(C^{\mathbf{Z}})$

algorithm:
1. initialization
$T := \mathbf{igs}\left(C^{\mathbf{Z}}\right)$
2. main iteration:
for $i = 1, \ldots, k$ **do**
Solve the linear feasibility problem

$$\begin{array}{lll} \text{minimize} & 0 & \\ \text{subject to} & (\forall j \in \{1, \ldots, k\} \setminus \{i\}) & h_i - h_j \in C \end{array} \tag{4}$$

if problem (4) is feasible
then $T := T \setminus \{h_i\}$
end if
end for
3. return $\mathbf{ib}(C^{\mathbf{Z}}) := T$.

If the polyhedron $P = \{x \in \mathbf{R}^n \mid Ax \preceq b\}$ with A$\in \mathbf{Z}^{m\times n}, b \in \mathbf{Z}^m$ is integral, then to solve the integer optimization problem

$$\begin{array}{ll} \text{minimize}_x & c^T x \\ \text{subject to} & Ax \preceq b \\ & x \in \mathbf{Z}^n, \end{array}$$

we can just solve the relaxed linear program

$$\begin{array}{ll} \text{minimize}_x & c^T x \\ \text{subject to} & Ax \preceq b \\ & x \in \mathbf{R}^n, \end{array}$$

which will give an integer optimal solution.

Total dual integrality of a polyhedron is associated with the a system of linear inequalities which represents a polyhedron.

Definition 17. *(Totally dual integral system)* Consider the polyhedron

$$P = \{x \in \mathbf{R}^n \mid Ax \preceq b\},$$

where $A \in \mathbf{Z}^{m\times n}, b \in \mathbf{Z}^m$. The system of inequalities $Ax \preceq b$ is called totally dual integral if for every face [11] $F = \{x \in P \mid A_I x = b_I\}$ we have

$$\left\{a_i^T \mid i \in I\right\} = \mathbf{igs}\left(\operatorname{cone}\left(\{a_i^T \mid i \in I\}\right)^{\mathbf{Z}}\right).$$

There may exist different systems of linear inequalities each representing P. Only one among them is totally dual integral, and to get to that system we may need to add many redundant inequalities. There is a nice connection between dual integrality of a system of linear inequalities and duality theory.

Theorem 18. *Suppose $A \in \mathbf{Z}^{m\times n}, b \in \mathbf{Z}^m$. Consider the optimization problem*

$$p^* = \begin{pmatrix} \text{maximize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x \in \mathbf{R}^n \end{pmatrix},$$

and its dual

$$d^* = \begin{pmatrix} \text{minimize} & b^T y \\ \text{subject to} & A^T y = c \\ & y \succeq 0 \\ & y \in \mathbf{R}^m \end{pmatrix}.$$

Then the following are equivalent.

[11] **Face:** Suppose we have a polyhedron

$$P = \{x \in \mathbf{R}^n \mid Ax \preceq b\},$$

where $A \in \mathbf{Z}^{m\times n}, b \in \mathbf{Z}^m$. If $\tilde{a}^T x \geq \tilde{b}$ is a valid inequality for a P, then the set $\{x \in P \mid \tilde{a}^T x = \tilde{b}\}$ is called a face of P. For any face F of P, we can find an index set $I \subseteq \{1, 2, \ldots, m\}$ such that F has the equivalent representation

$$F = \{x \in P \mid A_I x = b_I\},$$

where A_I is the square submatrix consisting of rows a_i^T for $i \in I$, and b_I is the associated subvector of b. For example, if $I = \{1, 10\} \subseteq \{1, 2, \ldots, 15\}$, then $A_I = \begin{bmatrix} a_1^T \\ a_{10}^T \end{bmatrix}$, and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

(i) The system of inequalities associated with the primal problem $Ax \preceq b$ *is totally dual integral.*

(ii) For any $c \in \mathbf{Z}^n$ *such that* d^* *is finite, there is an integral optimal solution* $y^* \in \mathbf{Z}^m$.

The following theorem shows the connection between totally dual integral system and the integrality of the associated polyhedron.

Theorem 19. *Consider the polyhedron*

$$P = \{x \in \mathbf{R}^n \mid Ax \preceq b\},$$

where $A \in \mathbf{Z}^{m \times n}, b \in \mathbf{Z}^m$. *If the system of inequalities* $Ax \preceq b$ *is totally dual integral, then* P *is integral.*

Theorem 20. *Every rational polyhedron* P *can be described by a totally dual integral system of the form* $Ax \preceq b$ *with* A *integral. Additionally, if* b *is integral then* P *is integral, and vice versa.*

References

[1] Dimitris Bertsimas and Robert Weismantel. *Optimization over integers*, volume 13. Dynamic Ideas Belmont, 2005.