Davis-Yin three operator splitting

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Consider solving the following problem:

$(A+B+C)(x) \ni 0 : (A, B, C : \text{maximal monotone}, \; C : \text{single valued})$

\# Define: $T = R_B \,//\, -\alpha C \boxed{z} \,//\, (C_B + \boxed{z}) \,//\, C_A \boxed{z} \,//\, \boxed{z} + \boxed{\cdot} = (C_A((C_B - \alpha C_{R_B}) - \alpha C R_B)$

We have the following theorem:

For α ≠ 0, we have $(A+B+C)(x) \ni 0 \leftrightarrow Tz = z, \; z = R_B(z)$

Justification of $(A+B+C)(x) \ni 0$

$\leftrightarrow \alpha(A+B+C)(x) \ni 0$

$\leftrightarrow x - x + \alpha(A+B+C)x = (1+\alpha A)(x) - \underbrace{(1-\alpha B)}_{C_B(1+\alpha B) = (1-\alpha B)}(x) + \alpha C x \ni 0 \leftrightarrow (1+\alpha A)(x) - C_B(1+\alpha B)x + \alpha C x \ni 0 \leftrightarrow \exists_{z \in (1+\alpha B)x} \; (1+\alpha A)(x) - C_B z + \alpha C x \ni 0$

\# Recall that, $(A : \text{maximal monotone}, \alpha > 0) \rightarrow C_A(1+\alpha A) = (1-\alpha A)$

Proof: 저기 증명. resolvent of a maximal monotone operator is a function

$\therefore (1+\alpha A)^{-1}$: function

$\forall_{x \in \mathrm{dom} A} \; C_A(1+\alpha A)(x)$ # $C_A = 2R_A - 1 = 2(1+\alpha A)^{-1} - 1$

$= (2(1-\alpha A)^{-1} - 1)(1+\alpha A)(x)$

$= 2(1-\alpha A)^{-1}\underbrace{(1+\alpha A)(x)}_{\text{this may be a set}} - 1(1+\alpha A)(x)$

but as $(1-\alpha A)^{-1}$ is a function, when it acts on $(1+\alpha A)(x)$ we will get back the singleton x.

$= 2x - x - \alpha A(x) = x - \alpha A(x) = (1-\alpha A)(x)$

$\therefore \forall_{x \in \mathrm{dom} A} \; C_A(1+\alpha A)(x) = (1-\alpha A)(x)$

$\leftrightarrow z \in (1+\alpha B)x, \; (1+\alpha A)x - C_B z + \alpha C x \ni 0$

\# $(x, z) \in (1+\alpha B)$

$\updownarrow$

$(z, x) \in (1+\alpha B)^{-1} = R_B$

$\updownarrow$

$R_B(z)(1+\alpha B)^{-1} z \ni x$

$\updownarrow$

$x = R_B(z)$ # R_B is a function as B : maximal monotone

$\leftrightarrow x = R_B(z) \wedge (1+\alpha A)x - C_B z + \alpha C x \ni 0$

$\updownarrow$

$(1+\alpha A)R_B z \underbrace{- C_B z + \alpha C R_B z}_{-(C_B - \alpha C R_B)z} \ni 0$

$\leftrightarrow x = R_B(z) \wedge (1+\alpha A)R_B z - (C_B - \alpha C R_B)z \ni 0$

\# now C : single-valued, B : maximal-monotone $\rightarrow R_B, C_B$: function $\Rightarrow (C_B - \alpha C R_B)$: function

$\therefore (C_B - \alpha C R_B)z$: singleton

$\leftrightarrow \exists_{\eta \in (1+\alpha A)R_B z} \; (\eta - \underbrace{(C_B - \alpha C R_B)z}_{\text{single vector}} = 0 \leftrightarrow \eta = (C_B - \alpha C R_B)z)$

$\leftrightarrow (C_B - \alpha C R_B)z \in (1+\alpha A)R_B z$

$\leftrightarrow x = R_B(z), \; \underbrace{(C_B - \alpha C R_B)}_{\text{single vector}} z \in (1+\alpha A)(R_B z) \leftrightarrow x = R_B(z), \; R_A(C_B - \alpha C R_B)z = R_B(z)$

\# $(R_B z) \boxed{(1+\alpha A)} ((C_B - \alpha C R_B)(z)$

$\leftrightarrow (R_B z, (C_B - \alpha C R_B)z) \in (1+\alpha A)$

$\leftrightarrow ((\underbrace{(C_B - \alpha C R_B)z}_{\text{single vector}}, \underbrace{R_B z}_{\text{single vector}}) \in \underbrace{(1+\alpha A)^{-1}}_{} = R_A$ $\quad R_A$: function as A : maximal monotone

$\leftrightarrow \underbrace{R_A}_{\text{function}} \underbrace{(C_B - \alpha C R_B)z}_{\text{single vector}} \ni R_B(z)$, $\leftrightarrow R_A(C_B - \alpha C R_B)z = R_B(z)$ → will become equal

$x = R_B z, \; R_A(C_B - \alpha C R_B)z = R_B(z)$

\# $2R_A(C_B - \alpha C R_B)z = 2R_B z$

$\leftrightarrow 2R_A(C_B - \alpha C R_B)z - 1z = 2R_B z - 1z = \underbrace{(2R_B - 1)}_{C_B} z = C_B z$

$\leftrightarrow \underbrace{2R_A(C_B - \alpha C R_B)z - (C_B - \alpha C R_B)z}_{} + (C_B - \alpha C R_B)z - z = C_B z$

$\leftrightarrow \underbrace{(2R_A - 1)}_{C_A}(C_B - \alpha C R_B)z + \cancel{C_B z} - \alpha C R_B z - z = \cancel{C_B z}$

$\leftrightarrow C_A(C_B - \alpha C R_B)z - \alpha C R_B z = z$

$\leftrightarrow (C_A(C_B - \alpha C R_B) - \alpha C R_B)z = z$

$\leftrightarrow x = R_B z, \; \underbrace{(C_A(C_B - \alpha C R_B) - \alpha C R_B)}_{T} z = z$ # question: how do I know that T is nonexpansive?

$\leftrightarrow x = R_B(z), \; Tz = z$

* Simplify $T = (C_A(C_B - \alpha C R_B) - \alpha C R_B)$

$$=(C_A C_B - \alpha C_A C R_B - \alpha C R_B) = (C_A C_B - \alpha (C_A + I) C R_B)$$

the fixed point iteration can be defined as

for now assume that, T: nonexpansive.

so, damped iteration scheme # or averaged iteration scheme অনুযায়ী:

অনুযায়ী the iteration will be

$$z^{k+1} = \frac{1}{2} z^k + \frac{1}{2} T z^k$$ # convex factor $= \frac{1}{2}$ ধরে

$$= \frac{1}{2} z^k + \frac{1}{2}(C_A C_B - \alpha (C_A + I) C R_B) z^k$$

$$= \frac{1}{2} z^k + \frac{1}{2}(C_A C_B)(z^k) - \frac{1}{2}\alpha ((C_A + I) C \underbrace{(R_B z^k)}_{x^{k+1/2}}$$

$$\boxed{x^{k+1/2} = R_B(z^k)}$$ # intermediate iteration same as D-R splitting

$$= \frac{1}{2} z^k + \frac{1}{2}((C_A C_B)(z^k) - \frac{\alpha}{2}(C_A + I) C x^{k+\frac{1}{2}}$$

$$= \frac{1}{2} z^k + \frac{1}{2} C_A (2R_B - I) z^k - \frac{\alpha}{2}((C_A + I) C x^{k+\frac{1}{2}}$$

$$= \frac{1}{2} z^k + \frac{1}{2} C_A (2\underbrace{R_B z^k}_{x^{k+\frac{1}{2}}} - z^k) - \frac{\alpha}{2}((C_A + I) C x^{k+\frac{1}{2}}$$

$$= \frac{1}{2} z^k + \frac{1}{2} C_A (\underbrace{2x^{k+\frac{1}{2}} - z^k}_{\tilde{z}^{k+1/2}}) - \frac{\alpha}{2} C_A C x^{k+\frac{1}{2}} - \frac{\alpha}{2} C x^{k+\frac{1}{2}}$$

this is another intermedite operation.

$$\boxed{\tilde{z}^{k+\frac{1}{2}} = 2x^{k+\frac{1}{2}} - z^k}$$

$$= \frac{1}{2} z^k + \underbrace{\frac{1}{2} C_A \tilde{z}^{k+\frac{1}{2}} - \frac{\alpha}{2} C_A C x^{k+\frac{1}{2}}}_{\frac{1}{2} C_A (\tilde{z}^{k+\frac{1}{2}} - \alpha C x^{k+1/2})} - \frac{\alpha}{2} C x^{k+\frac{1}{2}}$$

$$= \frac{1}{2} z^k + \frac{1}{2} \underbrace{C_A}_{(2R_A - I)} (\tilde{z}^{k+\frac{1}{2}} - \alpha C x^{k+\frac{1}{2}}) - \frac{\alpha}{2} C x^{k+\frac{1}{2}}$$

$$= \frac{1}{2} z^k + \frac{1}{2}(2R_A - I)(\tilde{z}^{k+\frac{1}{2}} - \alpha C x^{k+\frac{1}{2}}) - \frac{\alpha}{2} C x^{k+\frac{1}{2}}$$

$$= \frac{1}{2} z^k + R_A(\tilde{z}^{k+\frac{1}{2}} - \alpha C x^{k+\frac{1}{2}}) - \frac{1}{2}\tilde{z}^{k+\frac{1}{2}} + \cancel{\frac{\alpha}{2} C x^{k+\frac{1}{2}}} - \cancel{\frac{\alpha}{2} C x^{k+\frac{1}{2}}}$$

$$= \frac{1}{2} z^k + \underbrace{R_A(\tilde{z}^{k+\frac{1}{2}} - \alpha C x^{k+\frac{1}{2}})}_{x^{k+1}} - \frac{1}{2}\tilde{z}^{k+\frac{1}{2}}$$

another intermediate iteration $$\boxed{x^{k+\frac{1}{2}} = R_A(\tilde{z}^{k+\frac{1}{2}} - \alpha C x^{k+\frac{1}{2}})}$$

$$= \frac{1}{2} z^k + x^{k+1} - \frac{1}{2}\underbrace{\tilde{z}^{k+\frac{1}{2}}}_{(2x^{k+1/2} - z^k)}$$ # $\because \tilde{z}^{k+\frac{1}{2}} = 2x^{k+\frac{1}{2}} - z^k$

$$= \frac{1}{2} z^k + x^{k+1} - \frac{1}{2}(2x^{k+\frac{1}{2}} - z^k)$$

$$= \frac{1}{2} z^k + x^{k+1} - x^{k+1/2} + \frac{1}{2} z^k$$

$$= z^k + x^{k+1} - x^{k+1/2}$$

$$\therefore \boxed{z^{k+\frac{1}{2}} = z^k + x^{k+1} - x^{k+\frac{1}{2}}}$$

So in iteration, Davis-Yin scheme becomes:

$$\boxed{\begin{aligned} x^{k+\frac{1}{2}} &= R_B(z^k) \\ \tilde{z}^{k+\frac{1}{2}} &= 2x^{k+\frac{1}{2}} - z^k \\ x^{k+1} &= R_A(\tilde{z}^{k+\frac{1}{2}} - \alpha C x^{k+\frac{1}{2}}) \\ z^{k+1} &= z^k + x^{k+1} - x^{k+1/2} \end{aligned}}$$