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A Stackelberg Game Model for Plug-in Electric Vehicles in a Smart Grid

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Abstract—**In this paper, we consider and analyze a Stackelberg game model for Plug-in Electric Vehicles (followers) charging from a Smart Grid (leader). Our model attempts to account for the time-of-use pricing of the Smart Energy Meter using an indirect penalty approach. We show that a unique Stackelberg Equilibrium exists for the game under realistic conditions. To better understand the evolution of the game, we solve a monopolistic version of the game and then we solve the game for the general case. The solution we obtain is in closed form and tractable, yet reveals several important aspects of the game arising from the interplay between the vehicles and the grid. The results can be applied to both individual vehicles and vehicle groups. We show that, if the battery capacity of a particular vehicle model falls below the threshold battery capacity of the game, that model will be out of the market in the long run. We discuss the relation between the monopolistic game and the general game.**

I. Introduction

Background

A Plug-in Electric Vehicle (**PEV**), which uses electricity as its source of energy, is the most promising alternative to petroleum vehicles [1]. Many developed countries are moving towards large scale penetration of PEVs in the market. For example, by 2020, one in every 20 cars in Ontario, Canada will be PEV and 20% of total vehicle purchases will be PEVs [2]. By then, the power grid is projected to acquire the capability to operate and interact autonomously to efficiently control the generation and distribution of electric power. Such a cyber-physical grid is called Smart Grid (**SG**) [3]. For example, by 2030, all of the power generation and distribution in Ontario, Canada will be controlled by SG [4]. In such a scenario, PEVs connected to the SG compete among themselves to consume as much electrical energy as possible subject to their battery capacities, whereas the SG sells electricity at a particular price to PEVs with the objective to maximize its revenue without overloading the grid. So, a relevant question in this regard is how to determine a suitable pricing strategy for PEVs connected to the SG.

Related Works

Recently, there has been much interest in the design of optimal charging strategies for PEVs connected to the SG. In [5], two algorithms are introduced to determine the economically optimal solution for the PEVs in a deregulated electricity market. The work in [6] augments the optimal charging problem of PEVs into the non convex optimal power flow problem and solves the corresponding convex dual problem. In [7], the non-cooperative interaction between PEVs in a Cournot market is analyzed using a mean field game. A pricing strategy to obtain optimal frequency regulation service from PEVs to the power grid is proposed in [8].

The work in [9] is one of the first few papers that capture the interaction between the PEVs and the SG and the corresponding decision making process in a grid-to-vehicle scenario. The authors have developed a Stackelberg game model for a number of PEV groups connected to the SG using a Lagrangian pricing approach. They have proposed an algorithm based on S-S hyperplane projection method [10] to solve a socially stable refinement of the proposed game. However, the mechanism of a realistic Smart Energy Meter (SEM), that implements *time-of-use pricing* [11], is not considered in [9]. Time-of-use pricing is the rate structure used by SEM that enables the SG to increase or decrease price of electricity depending on its demand and availability. Also, in their model the charging is assumed to be provided by charging stations and the cost function is proposed for PEV groups rather than individual PEVs, which is not always realistic. For example, it is projected that, in Ontario, Canada the PEV owners will charge their PEVs mainly at home [12], [2].

Our Contributions

Our contributions in this paper can be summarized as follows:

- Inspired by [13], we propose a game theoretic model in the form of a Stackelberg game for PEVs (followers) charging under SG (leader) considering realistic conditions. Our model attempts to account for the time-of-use pricing of the SEM by using an indirect penalty approach and is applicable to both individual PEVs and PEV groups. Based on realistic assumptions, we show that, the PEVs' game admits a unique and inner NE and the game in general admits a unique Stackelberg game.
- We solve the game for a monopolistic market condition, because it is related to the general game. We show that, the price of electricity and revenue of the SG in the general case compared to the monopolistic case depend on the ratio of the weighted average of battery capacity in the general case to the single battery capacity in the monopolistic case.

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- We solve the general game in explicit and tractable closed form using an *implicit programming approach* [14]. Such a closed form reveals the interrelation between different parameters of the game in a precise manner. An important result in this section is the condition for survival of a PEV model in a competitive market based on its battery capacity.

Organization

The rest of the paper is organized as followed. In Section II, a Stackelberg game model for PEVs operating under an SG is formulated. In Section III, we show that, under realistic conditions the PEVs' game admits unique and inner Nash Equilibrium. Then, in Section IV, a monopolistic version of the game is solved. In section V, we determine the analytical solution of the game for the general case in closed form. In the same section, we make important remarks regarding the survival of a particular PEV model, and the relation between the game for the general case and the game for the monopolistic case. In section VI, we present and explain our results graphically and compare them to that of [9]. Section VII presents the conclusion.

II. Game theoretic formulation

Consider a scenario where the SG is charging $\mathcal{N}$ PEVs at a particular time. The SG sets a price $p > 0$ for consumption of 1 unit of electrical energy. The PEVs are classified into N different PEV models in the descending order of their battery capacities, where $\mathfrak{N} = \{1, 2, ..., N\}$ is the set of the PEV models. PEV model number $i \in \mathfrak{N}$ has n_i PEVs belonging to it, and $\mathfrak{N}_\mathrm{i} = \{1, 2, ..., n_i\}$ is the set of PEVs who are of the same PEV model number i. So, $\sum_{i=1}^{N} n_i = \mathcal{N}$. Any PEV belonging to the same model number $i \in \mathfrak{N}$ has the same battery capacity b_i, which is in a compact set $\mathfrak{B}_\mathrm{i} = [b_{min}, b_{max}] \subset \mathbb{R}_{++}$, where $\mathbb{R}_{++}$ is the set of positive real numbers. $c \in \mathbb{R}_{++}$ is the total amount of electrical energy provided by the SG for the charging of the PEVs. The consumed energy by the jth PEV of model number i is $u_{ij} \in \Omega_{ij}$, where $\Omega_{ij} = [0, u_{max}] \subset \mathbb{R}_+$ is the set of consumed energy of the mentioned PEV, and $\mathbb{R}_+$ is the set of nonnegative real numbers. Let $\Omega = \prod_{i=1}^{N}\prod_{j=1}^{n_i} \Omega_{ij}$ be the action space in the absence of any constraint. Total electrical energy consumed by all the PEVs with same model i is $\bar{u}_i = \sum_{j=1}^{n_i} u_{ij}$. Total electrical energy consumed by all the PEVs under the SG is then, $\bar{u} = \sum_{i}^{N} \bar{u}_i = \sum_{i=1}^{N}\sum_{j=1}^{n_i} u_{ij}$, and it cannot exceed c. We term the inequality $\bar{u} \leq c$ as the *energy capacity constraint* of the game. Let arithmetic mean of consumed energy of all the PEVS be $\tilde{u} = \frac{\sum_{i=1}^{N}\bar{u}_i}{\sum_{i=1}^{N} n_i} = \frac{\bar{u}}{\mathcal{N}}$. Let the total electrical energy consumed by all PEVs under the SG except those belonging to the model number i be $u_{-i} = \bar{u} - \bar{u}_i$, and the total electrical energy consumed by all PEVs with same model number i except the jth one be $u_{-ij} = \bar{u}_i - u_{ij}$. Thus we can write the energy capacity constraint as $\bar{u} = u_{-i} + u_{-ij} + u_{ij} \leq c$.

The overall action space of the PEVs is, $\bar{\Omega} = \{\{\{u_{ij}\}_{j=1}^{n_i}\}_{i=1}^{N} \in \Omega : \sum_{i=1}^{N}\sum_{j=1}^{n_i} u_{ij} - c \leq 0\}$ and it is compact and convex. The feasible action set of the jth PEV of model number i is, $\hat{\Omega}_{ij}(u_{-i}, u_{-ij}) = \{\zeta \in \Omega_{ij} : \zeta + u_{-i} + u_{-ij} \leq c\}$. $\bar{\Omega}$ is coupled, because charging of jth PEV of model number i from the SG will effect the feasible action sets $\{\{\hat{\Omega}_{rk}\}_{k=1}^{n_r}\}_{r=1}^{N} \backslash \hat{\Omega}_{ij}$ of other PEVs. All the PEVs try to minimize their cost functions due to the consumption of electrical energy in a non cooperative manner at a fixed price set by the SG, and the SG attempts to set a realistic price to maximize its revenue. Such a scenario can be modeled as a **Stackelberg game** with a single leader and multiple Nash followers. The solution to the PEVs' minimization problem is attained by finding the corresponding Nash Equilibrium (**NE**) at a fixed price set by the game and the SG's maximization problem is solved by finding the price that maximizes its revenue. The solution pair that simultaneously fulfills both the objectives of the PEVs and the SG is called the Stackelberg Equilibrium (**SE**).

Inspired by [15], we make the following assumption:

Assumption 1. *Compared to the aggregate consumed energy of all the PEVs charging from the SG ($\bar{u}$), the consumed energy of any single PEV $u_{ij}(\forall j \in \mathfrak{N}_\mathrm{i})(\forall i \in \mathfrak{N})$ is so negligible that it will have no effect on the SE of the game and $u_{-i} + u_{-ij}$ can be considered equal to $\bar{u}$, i.e.,*

$$u_{-i}+u_{-ij} \approx u_{-i}+u_{-ij}+u_{ij} = \bar{u} \quad (\forall j \in \mathfrak{N}_\mathrm{i})(\forall i \in \mathfrak{N}) \tag{1}$$

We justify the assumption as followed. According to Table 2.1 of [16], in 2007 number of light vehicles in Ontario was = 6,957,086. 53% of them were cars (page 21, [16]), so in 2007 number of cars in Ontario was = 3,687,255. Even if the number of cars stays the same, according to [4], in 2020 the number of PEVs in Ontario operating under an SG will be $\frac{3{,}687{,}255}{20} = 184{,}362$. Even if only 10% of them are connected to the SG (as in off peak hour), the number of PEVs participating in the Stackelberg game will be $\mathcal{N} = 18{,}436$. If the jth PEV of ith model consumes maximum possible 65 kWh i.e. $u_{ij} = 0.065$ MWh, and the rest of the PEVs consume 10 kWh/PEV for running 45 miles (less than half of needed average energy 22 kWh for running 100 miles [9]), then $u_{-i}+u_{-ij} = 184.35$ MWh, and $\bar{u} = 184.415$ MWh with a change of 0.0353%. So, even in off peak hour, with the rest of the PEVs consuming less than half of the needed energy, the assumption is quite accurate.

Next, we formulate the cost function for the jth PEV of model number i, $J_{ij} : \hat{\Omega}_{ij} \to \mathbb{R}$. It can be written as the difference between *a pricing function* $\mathcal{P} : \hat{\Omega}_{ij} \to \mathbb{R}$ and *a utility function* $\mathcal{U}_{ij} : \hat{\Omega}_{ij} \to \mathbb{R}$:

$$J_{ij} = \mathcal{P}_{ij} - \mathcal{U}_{ij} \tag{2}$$

The SG implements an approach called **time-of-use pricing** via the SEMs, where the price is increased when the electric energy demand is high and is decreased in the opposite situation[11]. To reflect such a policy, using an

indirect penalty approach [17], we compactly write the price per unit energy p as a ratio of a positive pricing parameter α set by the SG to a penalty term:

$$p = \frac{\alpha}{c - \bar{u}} \tag{3}$$

The denominator in the price penalizes the violation of the energy capacity constraint by approaching zero when $\bar{u}$ approaches c, consequently the price increases without any bound. So, the structure of p forces the PEVs to decrease their energy consumption when the load condition demands such, thus indirectly satisfying the energy capacity constraint and accounting for the time-of-use pricing. Hence, we write the pricing function representing the payment due to the consumption of electrical energy as :

$$\mathcal{P}_{ij}\left(u_{ij}, u_{-ij}, u_{-i}; \alpha\right) = p u_{ij} = \frac{\alpha}{c - \bar{u}} u_{ij} \tag{4}$$

According to [9] the utility function $\mathcal{U}_{ij}\left(u_{ij}, u_{-ij}, u_{-i}; \alpha\right)$ should satisfy the following properties *i)* $\frac{\partial \mathcal{U}_{ij}}{\partial u_{ij}} \geq 0$, *ii)* $\frac{\partial^2 \mathcal{U}_{ij}}{\partial u_{ij}^2} \leq 0$, *iii)* $\frac{\partial \mathcal{U}_{ij}}{\partial b_i} > 0$ and *iv)* $\frac{\partial \mathcal{U}_{ij}}{\partial s_{ij}} < 0$. So, considering all these properties, in contrast to the quadratic utility function of [9], we model the utility function for the jth PEV of model number i as a logarithmic function because of its relation with the concept of proportional fairness [17] and nice conformity to the law of diminishing marginal utility[18].

$$\mathcal{U}_{ij}\left(u_{ij}, u_{-ij}, u_{-i}; \alpha\right) = b_i \log\left(u_{ij} + 1\right) - s_{ij} \tag{5}$$

The dissatisfaction parameter $s_{ij} \in \mathbb{R}_{++}$, is individual user dependent, and is subtracted from the utility because of the probable dissatisfaction of consumer psyche that arises from *consumer confusion* [19]. s_{ij} has no effect on the minimization of the cost functions of the PEVs. So, the cost function for the jth PEV of model number i is:

$$J_{ij}\left(u_{ij}, u_{-ij}, u_{-i}; \alpha\right) = \frac{\alpha\, u_{ij}}{c - \bar{u}} - b_i \log\left(u_{ij} + 1\right) + s_{ij} \tag{6}$$

Revenue of the SG is the price ($p > 0$) per unit electrical energy times the total energy ($\bar{u}$) consumed by all of the $\mathcal{N}$ PEVs charging from the grid:

$$L(p, \bar{u}) = p\, \bar{u} = \frac{\alpha}{c - \bar{u}}\, \bar{u} \tag{7}$$

We call the game in its totality a **Stackelberg game** and express it as $\mathcal{G}(\{\mathcal{N} \cup \mathrm{SG}\}, \{\{J_{ij}\}_{j=1}^{n_i}\}_{i=1}^{N}, L)$. The Nash followers' (PEVs') game is expressed as $\mathcal{G}(\mathcal{N}, \hat{\Omega}_{ij}, J_{ij}; \alpha)$, where we impose two assumptions on the NE solution $\{\{u_{ij}^*\}_{j=1}^{n_i}\}_{i=1}^{N}$.

Assumption 2. *$\{\{u_{ij}^*\}_{j=1}^{n_i}\}_{i=1}^{N}$ must obey the energy capacity constraint strictly because if the SG supplies all its electric energy (c) to the PEVs, it might lead to overloading and consequently power outage. So,*

$$\bar{u}^* = u_{-i}^* + u_{-ij}^* + u_{ij}^* < c \tag{8}$$

Assumption 3. *At NE of the PEVs' game, every participating PEV is able to consume a positive amount of electrical energy from the SG:*

$$(\forall i \in \mathfrak{N})\, (\forall j \in \mathfrak{N}_\mathrm{i}) \quad u_{ij}^* > 0 \tag{9}$$

(8) and (9) implies an inner NE solution. Now we define the NE of the game $\mathcal{G}(\mathcal{N}, \hat{\Omega}_{ij}, J_{ij}; \alpha)$, revenue maximizing condition for the SG and SE of the Stackelberg game $\mathcal{G}(\{\mathcal{N} \cup \mathrm{SG}\}, \{\{J_{ij}\}_{j=1}^{n_i}\}_{i=1}^{N}, L)$ respectively:

Definition 1 (Condition for achieving NE for the PEVs)**.** *Consider the Nash followers' (PEVs') game $\mathcal{G}(\mathcal{N}, \hat{\Omega}_{ij}, J_{ij}; \alpha)$, where J_{ij} is given by* (6)*. For $\forall \alpha > 0$, $\{\{u_{ij}^*\}_{i=1}^{n_i}\}_{i=1}^{N}$ is called the NE of the mentioned game if besides obeying* (8) *and* (9)*, u_{ij}^* satisfies the following condition:*

$$\begin{aligned}&(\forall i \in \mathfrak{N})\, (\forall j \in \mathfrak{N}_\mathrm{i})(\forall u_{ij} \in \hat{\Omega}_{ij})\\ &\qquad J_{ij}\left(u_{ij}^*, u_{-ij}^*, u_{-i}^*; \alpha\right) \leq J_{ij}\left(u_{ij}, u_{-ij}^*, u_{-i}^*; \alpha\right)\end{aligned} \tag{10}$$

Definition 2 (Revenue maximizing condition for the SG)**.** *If the Nash followers' (PEVs') game $\mathcal{G}(\mathcal{N}, \hat{\Omega}_{ij}, J_{ij}; \alpha)$ achieves a unique NE as characterized by Definition 1, the leader's (SG's) objective is to find a pricing parameter $\alpha^* > 0$ such that it maximizes its revenue function L given by* (7)*, i.e.:*

$$(\forall \alpha > 0) \qquad L\left(\alpha^*, \bar{u}^*\right) \geq L\left(\alpha, \bar{u}^*\right) \tag{11}$$

Definition 3 (SE of the game)**.** *The pair $(\{\{u_{ij}^*\}_{j=1}^{n_i}\}_{i=1}^{N},\ p^*)$ is called the SE of the game $\mathcal{G}(\{\mathcal{N} \cup SG\}, \{\{J_{ij}\}_{j=1}^{n_i}\}_{i=1}^{N}, L)$, if it satisfies* (10) *and* (11) *simultaneously.*

III. Existence and uniqueness of NE for the general case

To show the existence and uniqueness of NE, we follow the approach in [20]. We use the following theorem (Theorem 4.4 of [21]) to show that the PEVs' game admits an NE solution.

Theorem 1. *Let $\bar{\Omega}$ be compact and convex and let the cost function $J : \bar{\Omega} \to \mathbb{R}$ be continuous on $\bar{\Omega}$ and convex in $\forall u_{ij} \in \hat{\Omega}_{ij}$, for $\forall j \in \mathfrak{N}_\mathrm{i}, \forall i \in \mathfrak{N}$. Then the associated Nash game $\mathcal{G}\left(\mathcal{N}, \bar{\Omega}, J; \alpha\right)$ admits an NE solution at a fixed positive pricing parameter α.*

The following lemma is used to show the existence and uniqueness of NE and also to calculate NE:

Lemma 1. *Under Assumption 1, $J_{ij}\left(u_{ij}, u_{-ij}, u_{-i}; \alpha\right)$ in* (6) *can be considered equal to (approximated by) the following equivalent augmented cost function that is identical for all PEVs:*

$$\begin{aligned}&J\left(\bar{u}_1, \bar{u}_2, ..., \bar{u}_N; \alpha\right)\\ &= \frac{\alpha \bar{u}}{c - \bar{u}} + \sum_{r=1}^{N} \sum_{k=1}^{n_r} s_{rk} - \sum_{r=1}^{N} b_r \sum_{k=1}^{n_r} \log\left(u_{rk} + 1\right)\end{aligned} \tag{12}$$

and the game $\mathcal{G}(\mathcal{N}, \hat{\Omega}_{ij}, J_{ij}; \alpha)$ is equivalent to $\mathcal{G}(\mathcal{N}, \bar{\Omega}, J; \alpha)$.

Proof: The term below :

$$\begin{aligned}\Psi =& \frac{\alpha}{c - u_{-ij} - u_{-i}} \left(\sum_{k=1;k\neq j}^{n_i} u_{ik} + \sum_{r=1;r\neq i}^{N} \sum_{k=1}^{n_r} u_{rk} \right) \\ &- b_i \sum_{k=1;k\neq j}^{n_i} \log\left(u_{ik}+1\right) - \sum_{r=1;r\neq i}^{N} b_r \sum_{k=1}^{n_r} \log\left(u_{rk}+1\right) \\ &+ \sum_{r=1;r\neq i}^{N} \sum_{k=1}^{n_r} s_{rk} + \sum_{k=1;k\neq j}^{n_i} s_{ik} \end{aligned} \tag{13}$$

is not a function of u_{ij}, so can be added to both sides of (6), yet the optimization of resultant augmented cost function with respect to u_{ij} will yield the same NE of the followers' game. Let $\sum_{r=1}^{N}\sum_{k=1}^{n_r} s_{rk} = \bar{s}$. We add (13) to both sides of (6) and then use (1) under Assumption 1 to obtain:

$$\begin{aligned}&J_{ij} + \Psi \\ &\approx \frac{\alpha}{c-\bar{u}} \left(\sum_{k=1;k\neq j}^{n_i} u_{ik} + \sum_{r=1;r\neq i}^{N} \sum_{k=1}^{n_r} u_{rk} \right) + \frac{\alpha u_{ij}}{c-\bar{u}} \\ &+ \bar{s} - b_i \sum_{k=1}^{n_i} \log\left(u_{ik}+1\right) - \sum_{r=1;r\neq i}^{N} b_r \sum_{k=1}^{n_r} \log\left(u_{rk}+1\right) \\ &= \frac{\alpha\bar{u}}{c-\bar{u}} + \bar{s} - \sum_{r=1}^{N} b_r \sum_{k=1}^{n_r} \log\left(u_{rk}+1\right)\end{aligned}$$

We express the resultant augmented cost function $J\left(\bar{u}_1, \bar{u}_2, ..., \bar{u}_N; \alpha\right)$ as:

$$J\left(\bar{u}_1, ..., \bar{u}_N; \alpha\right) = \frac{\alpha\bar{u}}{c-\bar{u}} + \bar{s} - \sum_{r=1}^{N} b_r \sum_{k=1}^{n_r} \log\left(u_{rk}+1\right)$$

Also $J\left(\bar{u}_1, ..., \bar{u}_N; \alpha\right)$ takes values from $\bar{\Omega}$ to $\mathbb{R}$, so the PEVs' game $\mathcal{G}(\mathcal{N}, \hat{\Omega}_{ij}, J_{ij}; \alpha)$ is equivalent to $\mathcal{G}\left(\mathcal{N}, \bar{\Omega}, J; \alpha\right)$. ■

We now give the following theorem regarding the uniqueness and existence of NE of the PEVs' game:

Theorem 2. *The PEVs' game $\mathcal{G}(\mathcal{N}, \bar{\Omega}, J; \alpha)$ admits a unique inner NE satisfying Assumptions 1, 2 and 3, if $0 < \alpha < \tilde{b}c$, where $\tilde{b}$=weighted mean of all battery capacities=$\frac{\sum_{i=1}^{N} b_i n_i}{\sum_{i=1}^{N} n_i}$.*

Proof:

Outline : The proof has three parts. At first, we prove the existence of the NE by showing that all conditions mentioned in Theorem 1 are met. Then, we find out the condition for the NE to be inner as imposed by Assumption 2 and Assumption 3. Finally, we show that the NE is unique. For this, we exploit the structure of the Hessian of J evaluated at a convex combination of supposedly two different NE solutions, integrate it with respect to the corresponding convex parameter and show that the resultant matrix is nonsingular. This nonsingularity implies that the NE solutions are in fact equal and hence the NE is unique.

Part 1: We calculate first and second order derivative of the augmented cost function J with respect to u_{ij} respectively:

$$\frac{\partial J}{\partial u_{ij}} = \frac{c\alpha}{\left(c-\bar{u}\right)^2} - \frac{b_i}{u_{ij}+1} \tag{14}$$

$$\frac{\partial^2 J}{\partial {u_{ij}}^2} = \frac{2c\alpha}{\left(c-\bar{u}\right)^3} + \frac{b_i}{\left(u_{ij}+1\right)^2} > 0 \tag{15}$$

For any $\alpha > 0$, (15) is greater than zero according to (8), thus proving the convexity of J in $\forall u_{ij}$. So, $\mathcal{G}\left(\mathcal{N}, \bar{\Omega}, J; \alpha\right)$ admits an NE solution.

Part 2: According to Assumption 2 and Assumption 3 the NE is inner, but only $\alpha > 0$ does not imply that, so we find the condition for the existence of an inner NE by using Lemma 1. Due to the convexity of J, the corresponding NE is found by optimizing J with respect to all u_{ij}-s, i.e., by equating (14) to zero and then solving the resultant equation $g\left(\bar{u}\right)$for each u_{ij} at a fixed α:

$$g\left(\bar{u}\right) = \frac{c\alpha}{\left(c-\bar{u}\right)^2} - \frac{b_i}{u_{ij}+1} = 0 \tag{16}$$

$$\Rightarrow u_{ij} = \frac{b_i\left(c-\bar{u}\right)^2}{c\alpha} - 1 \qquad (\forall j \in \mathfrak{N}_{\mathrm{i}})(\forall i \in \mathfrak{N}) \tag{17}$$

So, (17) implies that all PEVs belonging to the same model will have the same energy requirements at the NE of the followers' game:

$$\begin{aligned}\bar{u}_i &= \sum_{i=1}^{n_i} u_{ij} = n_i u_{ij} \Rightarrow \frac{\bar{u}_i}{n_i} = \frac{b_i\left(c-\bar{u}\right)^2}{c\alpha} - 1 \\ \Rightarrow \sum_{i=1}^{N} \bar{u}_i &= \frac{\left(c-\bar{u}\right)^2 \sum_{i=1}^{N} b_i n_i}{c\alpha} - \sum_{i=1}^{N} n_i \\ \Rightarrow \frac{\mathcal{N}\tilde{b}}{\bar{u}+\mathcal{N}} &= \frac{c\alpha}{\left(c-\bar{u}\right)^2} = \frac{b_i}{u_{ij}+1} \; (\forall j \in \mathfrak{N}_{\mathrm{i}})(\forall i \in \mathfrak{N})\end{aligned} \tag{18}$$

So, the function $g(\bar{u})$ in (16) can be written as:

$$g(\bar{u}) = \frac{c\alpha}{\left(c-\bar{u}\right)^2} - \frac{\mathcal{N}\tilde{b}}{\bar{u}+\mathcal{N}} = 0 \tag{19}$$

Now, according to Assumption 2 and Assumption 3, $\bar{u} \in (0, c)$. Again, $\lim_{\bar{u}\to c} g\left(\bar{u}\right) = \infty$ and $g\left(\bar{u}=0\right) = \frac{\alpha}{c} - \tilde{b}$, so for the NE to satisfy both Assumption 2 and Assumption 3, there must be a zero of $g\left(\bar{u}\right)$ in $(0, c)$, yielding $g\left(\tilde{u}=0\right) < 0$, i.e. $\alpha < \tilde{b}c$. So, the condition for the existence of an inner NE is:

$$\boxed{0 < \alpha < \tilde{b}c} \tag{20}$$

Part 3: Now we prove the uniqueness of NE. Let, $u_{i1} = u_{i2} = ... = u_{ij} = ... = u_{in_i} = v_i$.Then (14) and (15) can be written as:

$$\frac{\partial J}{\partial v_i} = \frac{c\alpha}{(c-\bar{u})^2} - \frac{b_i}{v_i+1} \tag{21}$$

$$\frac{\partial^2 J}{\partial v_i^2} = \frac{2c\alpha}{(c-\bar{u})^3} + \frac{b_i}{(v_i+1)^2} = B_i > 0 \tag{22}$$

and
$$\frac{\partial^2 J}{\partial v_i\,\partial v_\chi} = \frac{2c\alpha}{(c-\bar{u})^3} = A \tag{23}$$

The gradient vector of $J : \bar{\Omega} \to \mathbb{R}$ is:

$$\nabla J(v) = \begin{pmatrix} \frac{c\alpha}{(c-\bar{u})^2} - \frac{b_1}{v_1+1} \\ ... \\ ... \\ ... \\ \frac{c\alpha}{(c-\bar{u})^2} - \frac{b_N}{v_N+1} \end{pmatrix} \tag{24}$$

and the Hessian matrix of J is:

$$\nabla^2 J(v) = \begin{pmatrix} B_1 & A & ... & A \\ A & B_2 & ... & ... \\ ... & ... & ... & ... \\ A & A & ... & B_N \end{pmatrix} \tag{25}$$

Let $v^1 = (\ v_1^1 \ \ ... \ \ v_N^1\ )^T$ and $v^0 = (\ v_1^0 \ \ ... \ \ v_N^0\ )^T$ be two inner NE solutions of the followers' game, then $\nabla J(v^1) = 0,\ \nabla J(v^0) = 0$. Let, $0 < \theta < 1$ and $v(\theta)$ be a convex combination of v^1 and v^0, i.e. $v(\theta) = \theta v^1 + (1-\theta)v^0$.

$$\begin{aligned} &\frac{\partial(\nabla J(v(\theta)))}{\partial\theta} = \nabla^2 J(v(\theta))\,\frac{\partial v(\theta)}{\partial\theta} = \nabla^2 J(v)(v^1 - v^0) \\ &\Rightarrow \int_{\theta=0}^{1} d(\nabla J(v(\theta))) = \int_{\theta=0}^{1} \nabla^2 J(v(\theta))(v^1 - v^0)d\theta \\ &\Rightarrow \nabla J\left(v^1\right) - \nabla J\left(v^0\right) = \left(\int_{\theta=0}^{1} \nabla^2 J(v(\theta))d\theta\right)(v^1 - v^0) \\ &\quad = 0 \end{aligned} \tag{26}$$

Let $\bar{A} = \int_{\theta=0}^{1} A(v(\theta))\,d\theta,\ \bar{B}_i = \int_{\theta=0}^{1} B_i(v(\theta))\,d\theta$.
$\because (\forall_i) B_i > A > 0 \therefore (\forall_i)\ \bar{B}_i > \bar{A}$. Then,

$$\gamma = \int_{\theta=0}^{1} \nabla^2 J(v(\theta))d\theta = \begin{pmatrix} \bar{B}_1 & \bar{A} & ... & \bar{A} \\ \bar{A} & \bar{B}_2 & ... & ... \\ ... & ... & ... & ... \\ \bar{A} & \bar{A} & ... & \bar{B}_N \end{pmatrix}$$

Using the definition of γ (26) becomes :

$$\gamma.\left(v^1 - v^0\right) = 0 \tag{27}$$

Let, $y = v^1 - v^0$. Then, i-th row of (27) is:

$$\sum_{i=1}^{N} y_i + \sum_{i=1}^{N} \frac{\bar{A}}{\bar{B}_i - \bar{A}} \sum_{j=1}^{N} y_j = 0 \tag{28}$$

As $\sum_{i=1}^{N} \frac{\bar{A}}{\bar{B}_i - \bar{A}} \neq 0$, (28) is true if and only if γ is a full rank matrix. So, the only solution to $\gamma.y = 0$ is the trivial one, i.e., $y = 0 \Rightarrow v^1 = v^0$. So, the NE is unique. ■

IV. Solution for a monopolistic version of the game

In this section we solve the game for a monopolistic case, where each PEV has the same battery capacity parameter $b_i = b$, each model type has same number of PEVs $n_i = n$, thus $\mathcal{N} = Nn$. We express the corresponding Stackelberg game by $\mathcal{G}_m(\{\mathcal{N} \cup \text{SG}\}, \{\{J_{ij}\}_{j=1}^{n_i}\}_{i=1}^{N}, L)$. The following theorem captures the full solution to the monopolistic version of the game:

Theorem 3. *The monopolistic version of the Stackelberg game $\mathcal{G}_m(\{\mathcal{N} \cup SG\}, \{\{J_{ij}\}_{j=1}^{n_i}\}_{i=1}^{N}, L)$ admits a unique SE given by*

$$\begin{aligned} &(\{\{u_{ij}^*\}_{j=1}^{n_i}\}_{i=1}^{N}, p^*) = \\ &(\{\{\frac{\sqrt{c\mathcal{N} + \mathcal{N}^2} - \mathcal{N}}{\mathcal{N}}\}_{j=1}^{n_i}\}_{i=1}^{N}, \frac{b(\sqrt{c\mathcal{N} + \mathcal{N}^2} - \mathcal{N})}{c}) \end{aligned}$$

Proof: For the monopolistic case, we put $b_i = b$ in (17), that gives: $u_{ij} = \frac{b(c-\bar{u})^2}{c\alpha} - 1 (\forall j \in \mathfrak{N}_\mathrm{i})(\forall i \in \mathfrak{N})$. The right hand side of this equation is independent of i and j, so at the NE all PEVs will be consuming equal amount of electrical energy i.e., $u_{ij}^* = \frac{\bar{u}^*}{\mathcal{N}}\ (\forall j \in \mathfrak{N}_\mathrm{i})(\forall i \in \mathfrak{N})$. The simplified cost function for the monopolistic case is obtained by modifying (6): $J_{ij} = \frac{\alpha u_{ij}}{c - \mathcal{N}u_{ij}} - b\log(u_{ij}+1) + s_{ij} (\forall j \in \mathfrak{N}_\mathrm{i})(\forall i \in \mathfrak{N})$. There is one-to-one correspondence between J_{ij} and u_{ij}, and the terms u_{-ij} and u_{-i} are absent in J_{ij}. Then for the monopolistic case, the condition for achieving NE for the PEVs (10) and the revenue maximizing condition for the SG (11) are simplified as:

$$\begin{aligned} &(\forall i \in \mathfrak{N})\,(\forall j \in \mathfrak{N}_\mathrm{i})(\forall u_{ij} \in \hat{\Omega}_{ij}) \\ &\qquad\qquad J_{ij}\left(u_{ij}^*; \alpha\right) \le J_{ij}\left(u_{ij}; \alpha\right) \end{aligned} \tag{29}$$

and

$$(\forall \alpha > 0) \qquad L\left(\alpha^*, \mathcal{N}u_{ij}^*\right) \ge L\left(\alpha, \mathcal{N}u_{ij}\right) \tag{30}$$

respectively. Also, the condition for the existence of an inner NE for the monopolistic case can be found by substituting $\tilde{b} = b$ in (20):

$$0 < \alpha < bc \tag{31}$$

So, determination of the SE for the monopolistic case is equivalent to the optimization of J_{ij} with respect to u_{ij} at a fixed α, and L with respect to α. We take the derivative of J_{ij} with respect to u_{ij}, set it equal to zero and then find the pricing parameter α at NE:

$$\alpha = \frac{bc^2 - 2bc\mathcal{N}u_{ij} + b\mathcal{N}^2 u_{ij}^2}{c\,(u_{ij}+1)} \tag{32}$$

We substitute this value of α from (32) in (7) to get:

$$L = \frac{\mathcal{N}u_{ij}\left(bc^2 - 2bc\mathcal{N}u_{ij} + b\mathcal{N}^2 u_{ij}^2\right)}{c\,(u_{ij}+1)\,(c - \mathcal{N}u_{ij})} \tag{33}$$

Strict concavity of L ensures the existence of a global maximum and unique SE. To find the SE, we calculate the first order derivative of L with respect to u_{ij}, equate it to zero and then solve for u^*_{ij}:

$$u^*_{ij} = \frac{\sqrt{c\mathcal{N}+\mathcal{N}^2}-\mathcal{N}}{\mathcal{N}} \tag{34}$$

From (32) the pricing parameter set by the SG at SE is:

$$\alpha^* = \frac{b\mathcal{N}\left(c+\mathcal{N}-\sqrt{c\mathcal{N}+\mathcal{N}^2}\right)^2}{c\sqrt{c\mathcal{N}+\mathcal{N}^2}} \tag{35}$$

For an inner NE, α^* must obey (31). Clearly $\alpha^* > 0$. Now, let assume $\alpha^* \not< bc$ i.e., $\alpha^* = \frac{\mathcal{N}b\left(c+\mathcal{N}-\sqrt{c\mathcal{N}+\mathcal{N}^2}\right)^2}{c\sqrt{c\mathcal{N}+\mathcal{N}^2}} \geq bc$, resulting in $c^2\mathcal{N}(c+\mathcal{N})\left(c^2+3c\mathcal{N}+3\mathcal{N}^2\right) \leq 0$, which is a contradiction, as left hand side of the inequality is positive. So, $0 < \alpha^* < bc$ is satisfied, establishing an inner NE.

The optimal price p^* of the SG is:

$$p^* = \frac{b\left(\sqrt{c\mathcal{N}+\mathcal{N}^2}-\mathcal{N}\right)}{c} \tag{36}$$

■

The total consumed energy at SE is:

$$\bar{u^*} = \mathcal{N}u^*_{ij} = \sqrt{c\mathcal{N}+\mathcal{N}^2}-\mathcal{N} \tag{37}$$

We substitute the values of α^* and $\bar{u}^*$ in (33) to determine the revenue of the SG at the SE of the monopolistic case:

$$L\left(\alpha^*, \tilde{u}^*\right) = p\bar{u}^* = \frac{b\left(\sqrt{c\mathcal{N}+\mathcal{N}^2}-\mathcal{N}\right)^2}{c} \tag{38}$$

V. Complete Solution of the Game for the General Case

To find the SE of the game for the general case, we use the fact that, any Stackelberg game can be reformulated as a Mathematical Programming with Equilibrium Constraints (**MPEC**) problem [14].Thus, the objective of finding the SE in this case can be reformulated as followed:

$$\begin{aligned}
&\text{maximize } L(\alpha, \bar{u}^*)\\
&\text{subject to}\\
&0 < \alpha < \tilde{b}c,\\
&\bar{u}^* = \sum_{i=1}^{N}\sum_{j=1}^{n_i} u^*_{ij},\\
&\{\{u^*_{ij}\}_{j=1}^{n_i}\}_{i=1}^{N} \in interior(\bar{\Omega}), \quad \text{and}\\
&\qquad (\forall i \in \mathfrak{N})\,(\forall j \in \mathfrak{N}_\mathrm{i})(\forall u_{ij} \in \hat{\Omega}_{ij})\\
&J_{ij}\left(u^*_{ij}, u^*_{-ij}, u^*_{-i}; \alpha\right) \leq J_{ij}\left(u_{ij}, u^*_{-ij}, u^*_{-i}; \alpha\right)
\end{aligned} \tag{39}$$

If there is a one-to-one relation between α and u^*_{ij} $(\forall j \in \mathfrak{N}_\mathrm{i})(\forall i \in \mathfrak{N})$, and α is Lipschitz continuous and directionally differentiable in u^*_{ij} $(\forall j \in \mathfrak{N}_\mathrm{i})(\forall i \in \mathfrak{N})$, the Stackelberg game can be solved using *implicit programming* (**IMP**) approach [14]. In this method, in the SG's revenue L, we substitute the value of α in terms of $\tilde{u}^*$, and then maximize the resultant function with respect to $\tilde{u}^*$. Finally, we substitute the value of the resultant maximizer $\tilde{u}^*$ in (41) and (42) and thus obtain the SE.

At first, we find the relation between α and u^*_{ij} $(\forall j \in \mathfrak{N}_\mathrm{i})(\forall i \in \mathfrak{N})$ to find out if the IMP approach can be applied. We drop the $*$ superscript. Then substituting $\bar{u} = \mathcal{N}\tilde{u}$ into (19):

$$g(\tilde{u}) = \frac{c\alpha}{(c-\mathcal{N}\tilde{u})^2} - \frac{\tilde{b}}{\tilde{u}+1} = 0 \tag{40}$$

$$\Rightarrow \alpha = \frac{\tilde{b}\,(c-\mathcal{N}\tilde{u})^2}{c\,(\tilde{u}+1)} \tag{41}$$

The relation between u_{ij} and $\tilde{u}$ can be found out from (18):

$$u_{ij} = \frac{(\tilde{u}+1)\,b_i}{\tilde{b}} - 1 \qquad (\forall j \in \mathfrak{N}_\mathrm{i})(\forall i \in \mathfrak{N}) \tag{42}$$

So, the relation between α and u_{ij} satisfies all the necessary condition to apply IMP approach. Using (41), the leader's revenue $L\left(\alpha, \bar{u}\right)$ can be written as:

$$L\left(\alpha, \tilde{u}\right) = \frac{\mathcal{N}\tilde{b}\tilde{u}\,(c-\mathcal{N}\tilde{u})}{c\,(\tilde{u}+1)} \tag{43}$$

then

$$\frac{\partial L}{\partial \tilde{u}} = \frac{\mathcal{N}\tilde{b}\left(-\mathcal{N}\tilde{u}^2-2\mathcal{N}\tilde{u}+c\right)}{c\,(\tilde{u}+1)^2} \tag{44}$$

$$\frac{\partial^2 L}{\partial \tilde{u}^2} = -\frac{2\mathcal{N}\tilde{b}(c+\mathcal{N})}{c\,(\tilde{u}+1)^3} < 0 \tag{45}$$

Strict concavity of L in $\tilde{u}$ implies the existence of a global maxima in $\left(0, \frac{c}{\mathcal{N}}\right)$. We equate (44) to zero and solve the resultant quadratic equation for the positive $\tilde{u}^*$:

$$\tilde{u}^* = \frac{\sqrt{c\mathcal{N}+\mathcal{N}^2}-\mathcal{N}}{\mathcal{N}} \tag{46}$$

To obtain the pricing parameter set by the SG at the SE, we substitute the value of $\tilde{u}^*$ in (41):

$$\alpha^* = \frac{\tilde{b}\,(c-\mathcal{N}\tilde{u}^*)^2}{c\,(\tilde{u}^*+1)} = \frac{\mathcal{N}\tilde{b}\left(c+\mathcal{N}-\sqrt{c\mathcal{N}+\mathcal{N}^2}\right)^2}{c\sqrt{c\mathcal{N}+\mathcal{N}^2}} \tag{47}$$

For an inner NE, α^* must obey (20). Clearly $\alpha^* > 0$. Now, let assume $\alpha^* \not< \tilde{b}c$ i.e., $\alpha^* = \frac{\mathcal{N}\tilde{b}\left(c+\mathcal{N}-\sqrt{c\mathcal{N}+\mathcal{N}^2}\right)^2}{c\sqrt{c\mathcal{N}+\mathcal{N}^2}} \geq \tilde{b}c$ implying $c^2\mathcal{N}(c+\mathcal{N})\left(c^2+3c\mathcal{N}+3\mathcal{N}^2\right) \leq 0$, which leads to a contradiction as left hand side is positive. So, $0 < \alpha^* < bc$ is satisfied, resulting in an inner NE.

The optimal price p^* that maximizes the revenue of the SG is:

$$p^* = \frac{\alpha^*}{c-\mathcal{N}\tilde{u}^*} = \frac{\tilde{b}\left(\sqrt{c\mathcal{N}+\mathcal{N}^2}-\mathcal{N}\right)}{c} \tag{48}$$

We substitute the values of α^* and $\tilde{u}^*$ in (43) to get the revenue of the SG at the SE:

$$L\left(\alpha^*, \tilde{u}^*\right) = p\bar{u}^* = \frac{\tilde{b}\left(\mathcal{N}-\sqrt{c\mathcal{N}+\mathcal{N}^2}\right)^2}{c} \tag{49}$$

From (42) the NE solution at the SE is:

$$
\begin{aligned}
u_{ij}^* &= \frac{(\tilde{u}^* + 1)\, b_i}{\tilde{b}} - 1 \\
&= \frac{b_i \sqrt{c\mathcal{N} + \mathcal{N}^2}}{\mathcal{N}\tilde{b}} - 1 (\forall j \in \mathfrak{N}_{\mathrm{i}})(\forall i \in \mathfrak{N}) \qquad (50)
\end{aligned}
$$

The corresponding cost function at SE is:

$$
\begin{aligned}
&J_{ij}\left(u_{ij}^*, u_{-ij}^*, u_{-i}^*; \alpha^*\right) = \frac{\alpha u_{ij}^*}{c - \bar{u}^*} - b_i \log\left(u_{ij}^* + 1\right) + s_{ij} \\
&= \frac{b_i c - (\sqrt{c\mathcal{N} + \mathcal{N}^2} - \mathcal{N})(\tilde{b} + b_i)}{c} \\
&- b_i \log(\frac{b_i \sqrt{c\mathcal{N} + \mathcal{N}^2}}{\mathcal{N}\tilde{b}}) + s_{ij} \quad (\forall j \in \mathfrak{N}_{\mathrm{i}})(\forall i \in \mathfrak{N}) \quad (51)
\end{aligned}
$$

The following theorem summarizes the complete solution to the general case of the game:

Theorem 4. *Under Assumption 1, Assumption 2 and Assumption 3, the general case of the Stackelberg game $\mathcal{G}(\{\mathcal{N} \cup SG\}, \{\{J_{ij}\}_{j=1}^{n_i}\}_{i=1}^{N}, L)$ admits a unique SE given by:*

$$
\begin{aligned}
&(\{\{u_{ij}^*\}_{j=1}^{n_i}\}_{i=1}^{N}, p^*) = \\
&(\{\{\frac{b_i \sqrt{c\mathcal{N} + \mathcal{N}^2}}{\mathcal{N}\tilde{b}} - 1\}_{j=1}^{n_i}\}_{i=1}^{N}, \frac{\tilde{b}(\sqrt{c\mathcal{N} + \mathcal{N}^2} - \mathcal{N})}{c})
\end{aligned}
$$

Proof: Proof follows readily from the discussion above. ■

Remark 1: From the positivity condition (9):

$$
\begin{aligned}
u_{ij}^* &= \frac{b_i \sqrt{c\mathcal{N} + \mathcal{N}^2}}{\mathcal{N}\tilde{b}} > 0 \quad (\forall j \in \mathfrak{N}_{\mathrm{i}})(\forall i \in \mathfrak{N}) \\
\Rightarrow b_i &> \frac{\tilde{b}}{\sqrt{\frac{c}{\mathcal{N}} + 1}} \quad (\forall i \in \mathfrak{N})
\end{aligned}
$$

We interpret $\frac{c}{\mathcal{N}} = \tilde{c}$ as the average electrical energy per PEV supplied by the SG, and $\frac{\tilde{b}}{\sqrt{\tilde{c}+1}} = b_{th}$ as the threshold battery capacity. Then the inequality above becomes:

$$
\boxed{\therefore b_i > b_{th} = \frac{\tilde{b}}{\sqrt{\tilde{c} + 1}}} \qquad (52)
$$

(52) signifies that, if the battery capacity of a particular model falls below the threshold battery capacity, that model can never achieve SE and in the long run will be out of the market.

Remark 2: Comparison between (34) and (46) reveals that, the arithmetic average of all the PEVs' consumed energy at SE is equal to the energy consumed by a PEV at the SE in a monopolistic market, i.e.:

$$
(u_{ij}^*)_{\text{monopolistic}} = \tilde{u}^* \quad (\forall j \in \mathfrak{N}_{\mathrm{i}})(\forall i \in \mathfrak{N}) \qquad (53)
$$

From (48), (36), (38) and (49) we see that:

$$
\frac{p^*}{(p^*)_{\text{monopolistic}}} = \frac{L(\alpha^*, \bar{u}^*)}{(L(\alpha^*, \bar{u}^*))_{\text{monopolistic}}} = \frac{\tilde{b}}{b} \qquad (54)
$$

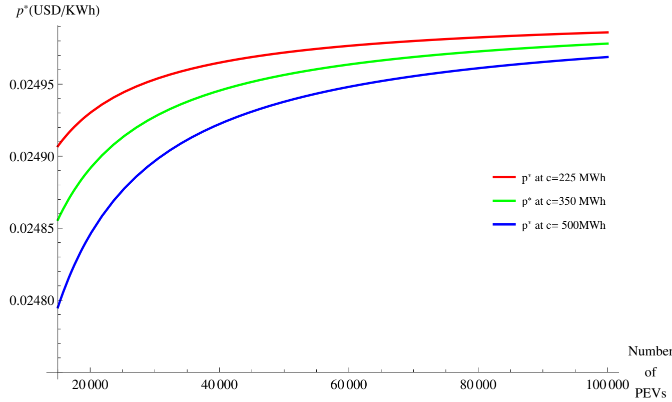

Fig. 1. Change of p^* with respect to $\mathcal{N}$ for different c-s

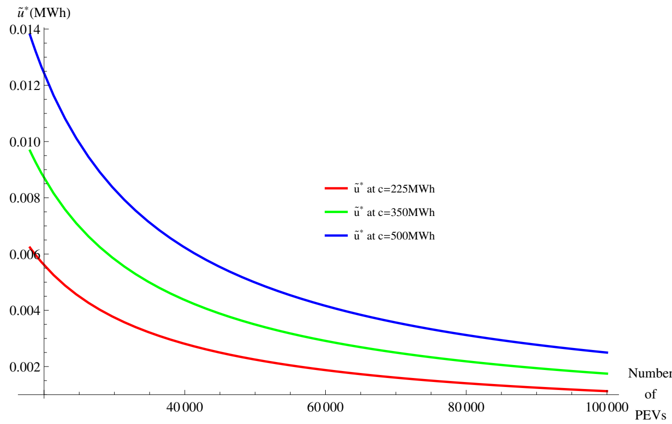

Fig. 2. Change of $\tilde{u}^*$ with respect to $\mathcal{N}$ for different c-s

So, compared to a monopolistic market with same number of PEVs, in a competitive market both the price set and the revenue earned by the SG at SE increase as the ratio of the weighted average of the battery capacity of the PEVs in the competitive market to the battery capacity of the only PEV model for the monopolistic case increases.

VI. Graphical Representation of Results

In this section we represent our results graphically, explain their implications and causes and compare them to [9] when possible. The weighted average of battery capacities is taken as, $\tilde{b}$=50 kWh. We scale the parameters accordingly.

Figure 1 shows the change in the optimal price per unit electrical energy (USD/KWh) set by the SG at SE (p^*) as number of PEVs $\mathcal{N}$ increases for different total electrical energy supplied by the SG $c = 225$, 350 and 500 MWh. The figure shows that, p^* increases with increasing $\mathcal{N}$ at a fixed c and decreases with increasing c at a fixed $\mathcal{N}$. Intuitively, increased number of PEVs result in increased energy demand and it forces the SG to increase the price to discourage connections of more PEVs to the SG. On the other hand, as the total electrical energy c supplied by the SG increases, the SG has surplus energy to sell and thus reduces the price per unit electrical energy to encourage the connected PEVs

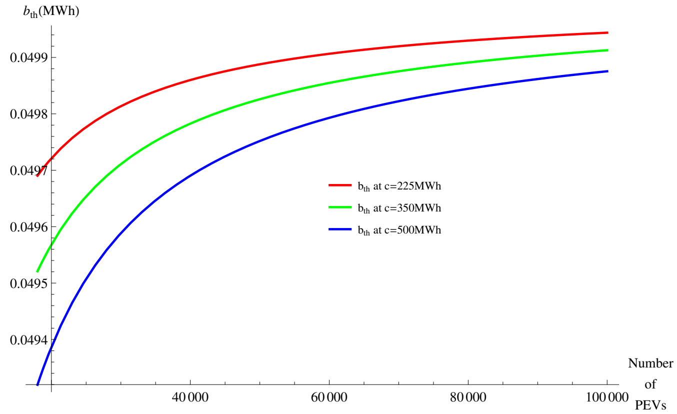

Fig. 3. Change of b_{th} with respect to $\mathcal{N}$ for different c-s

to consume more energy. Both of these observations agree with [9].

Figure 2 shows the change in the average consumed energy at SE (MWh) ($\tilde{u}^*$) as $\mathcal{N}$ increases for different capacities, $c =$ 225, 350 and 500 MWh. The figure shows that, $\tilde{u}^*$ increases with increasing c at a fixed $\mathcal{N}$ and decreases with increasing $\mathcal{N}$ at a fixed c. The first behavior of the PEVs is due to the fact that, as c increases with $\mathcal{N}$ fixed, the price per unit energy drops and on the average the PEVs take the opportunity to consume more energy by paying less. [9] does not discuss such a scenario. The latter behavior can be explained by the fact that, as $\mathcal{N}$ increases with c fixed, the price per unit energy increases, and on aggregate the PEVs reduce their energy consumptions to avoid extra payment. This behavior agrees with [9].

Figure 3 shows, the change in threshold battery capacity as number of PEVs increases for different capacities, $c =$ 225, 350 and 500 MWh. Threshold battery capacity increases with increasing $\mathcal{N}$ at fixed c and decreases with increasing c at fixed $\mathcal{N}$. As $\mathcal{N}$ increases at a fixed c, all PEV models are forced to decrease their energy consumptions and those with small battery capacities will obtain even smaller utilities eventually leading to their disconnections from the SG. Similarly, the increase in consumed energy with increasing c at fixed $\mathcal{N}$ results in more PEV models achieving NE, in effect decreasing b_{th}.

VII. Conclusion

In this paper, a simple but realistic game theoretic model of PEVs charging from an SG is proposed and analyzed. At first, the game is solved for a monopolistic case, where all the PEVs are essentially of the same model. Then, the game is analytically solved for any number of PEVs. We have derived an important inequality regarding the battery state of a PEV to reach SE state hence to survive in the market and an equation corresponding to the change in the SG's revenue as it evolves from a monopolistic to competitive market.

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References

[1] D. Sandalow, *Plug-in Electric Vehicles: What Role for Washington?* Brookings Institution Press, 2009.

[2] "Towards an ontario action plan for plug-in-electric vehicles (pevs)," University of Waterloo, Tech. Rep., 2010. [Online]. Available: http://www.ieso.ca/imoweb/pubs/smart_grid/materials/20100907/WATERLOO%20PHEV_Report%20Final%20May_17.pdf

[3] W. Saad, Z. Han, H. Poor, and T. Başar, "Game-theoretic methods for the smart grid: an overview of microgrid systems, demand-side management, and smart grid communications," *IEEE Signal Processing Magazine*, vol. 29, no. 5, pp. 86–105, 2012.

[4] "Modernizing ontario's electricity system: Next steps," Tech. Rep., 2011, second Report. [Online]. Available: http://www.ieso.ca/imoweb/pubs/smart_grid/Smart_Grid_Forum-Report-May_2011.pdf

[5] N. Rotering and M. Ilic, "Optimal charge control of plug-in hybrid electric vehicles in deregulated electricity markets," *IEEE Transactions on Power Systems*, vol. 26, no. 3, pp. 1021–1029, 2011.

[6] S. Sojoudi and S. H. Low, "Optimal charging of plug-in hybrid electric vehicles in smart grids," *IEEE Power and Energy Society General Meeting*, pp. 1–6, 2011.

[7] R. Couillet, S. M. Perlaza, H. Tembine, and M. Debbah, "Electrical vehicles in the smart grid: A mean field game analysis," *IEEE Journal on Selected Areas in Communications*, vol. 30, no. 6, pp. 1086–1096, 2012.

[8] C. Wu, H. Mohsenian-Rad, and J. Huang, "Vehicle-to-aggregator interaction game," *IEEE Transactions on Smart Grid*, vol. 3, no. 1, pp. 434–442, 2012.

[9] W. Tushar, W. Saad, H. V. Poor, and D. B. Smith, "Economics of electric vehicle charging: A game theoretic approach," *IEEE Transactions on Smart Grid*, vol. 3, no. 4, pp. 1767–1778, 2012.

[10] M. Solodov and B. Svaiter, "A new projection method for variational inequality problems," *SIAM Journal on Control and Optimization*, vol. 37, no. 3, pp. 765–776, 1999.

[11] "Smart meters and time-of-use prices," Website of Ontario Ministry of Energy. [Online]. Available: http://www.energy.gov.on.ca/en/smart-meters-and-tou-prices/

[12] "Website of plug'ndrive." [Online]. Available: www.plugndriveontario.ca

[13] T. Başar and R. Srikant, "Revenue-maximizing pricing and capacity expansion in a many-users regime," *Twenty-First Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM)*, vol. 1, pp. 294–301, 2002.

[14] J. Outrata, M. Kocvara, and J. Zowe, *Nonsmooth approach to optimization problems with equilibrium constraints: theory, applications and numerical results*. Springer, 1998.

[15] J. N. Tsitsiklis and Y. Xu, "Pricing of fluctuations in electricity markets," *IEEE 51st Annual Conference on Decision and Control (CDC)*, pp. 457–464, 2012.

[16] "Summary report, canadian vehicle survey," Tech. Rep., 2007. [Online]. Available: http://oee.nrcan.gc.ca/Publications/statistics/cvs07/pdf/cvs07.pdf

[17] L. Pavel, *Game theory for control of optical networks*. Springer, 2012.

[18] A. Asano, *An Introduction to Mathematics for Economics*. Cambridge University Press, 2012.

[19] G. Walsh and M. Yamin, "Towards a conceptual model of consumer confusion," *Advances in consumer Research*, vol. 32, pp. 143–150, 2005.

[20] T. Alpcan and T. Başar, "Distributed algorithms for nash equilibria of flow control games," *Advances in Dynamic Games*, pp. 473–498, 2005.

[21] T. Başar and G. J. Olsder, *Dynamic noncooperative game theory*. SIAM, 1995.