Lecture04-Wave_Dynamics

September 15, 2025

1 Lecture04 - Wave dynamics

Learning Objectives: linear wave theory (Airy waves), deep vs shallow water waves, dispersive or non-dispersive for deep vs shallow water waves, wave group speed and phase speed for deep vs shallow water waves, wave refraction, wave shoaling, wave breaking depth, wave breaker types

Before class: - watch TedEd "The physics of surfing - Nick Pizzo"

After class:

- Assignment03.ipynb (due 09/25/2025 23:59PM)
- read about Tsunami wave propagation, note that the wave length of a Tsunami wave is much larger than the water depth! Estimate the propagation speed of a Tsunami wave in the open ocean (~4000 m) using the shallow water Airy wave theory your self!

Reference:

- Textbook chapter 4, skip 4.3.2
- optional: Textbook 4.4.2; BS 5.2

1.1 1. Wave theory

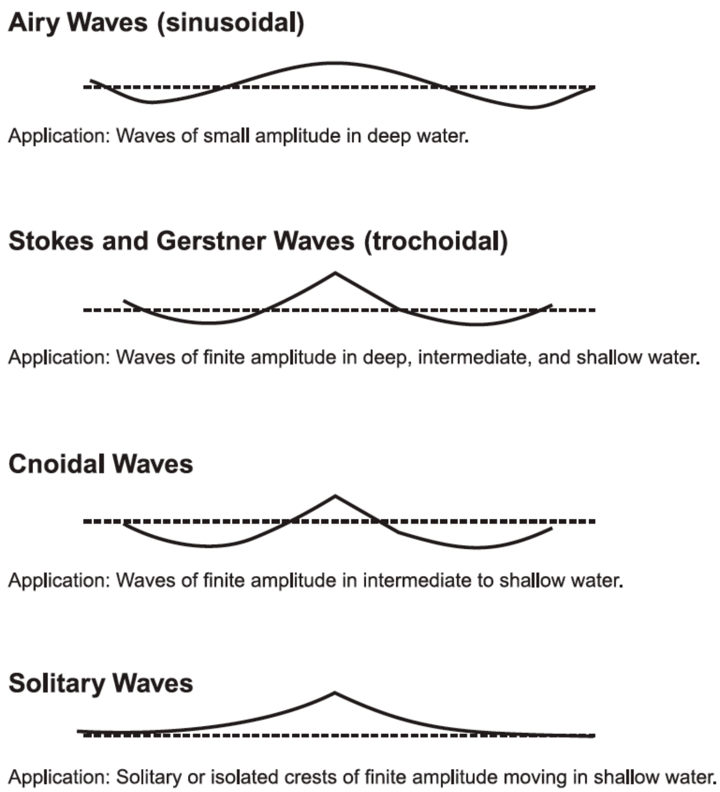


Figure 5.1 Wave form and relation to the still water level for Airy, Stokes, cnoidal and solitary waves.

Airy wave theory: Airy (1845) developed a theory for irrotational waves travelling over a horizontal bottom in any depth of water. The wave profile is assumed to be perfectly sinusoidal and only linear equations are necessary to describe the water surface and orbital motion hence it is often termed **linear wave theory** in comparison to the other higher-order theories. It further assumes that the **water surface slope is small**, and that the **water depth is much greater than the wave amplitude** (hence it is also described as small amplitude wave theory).

The assumptions for Airy wave theory generally hold in describing regular waves in deep water but, as water depth decreases, the assumptions about slope and the relationship between wave height and water depth are increasingly violated. Higher-order theories such as **Stokes' second-order theory** becomes more adequate.

1.1.1 1) Dispersion Relation

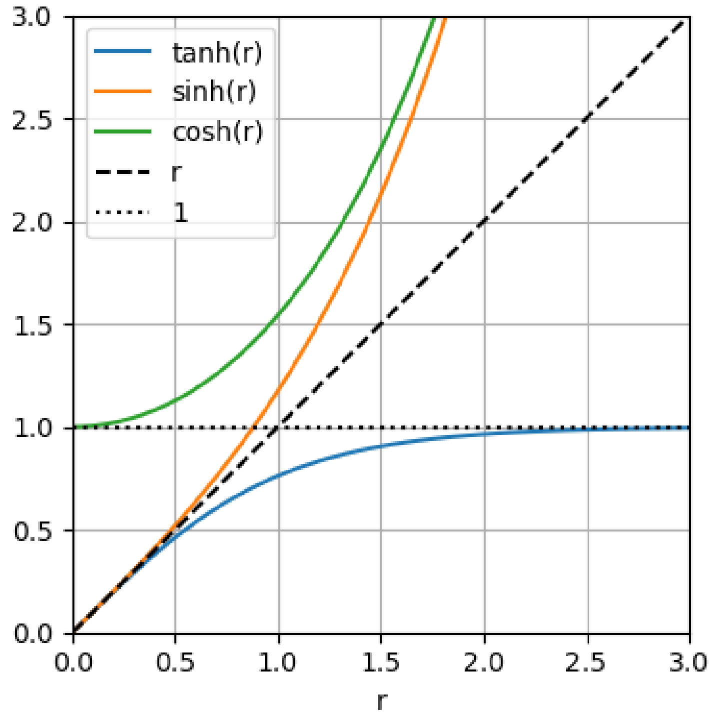
Recall that **Wave dispersion** refers to the phenomenon where the phase velocity of a wave changes with its frequency or wavelength. **Dispersion Relation** connects mathematically the wave frequency σ and wavelength k. Since by definition, phase speed $c = \sigma/k$, the dispersion relation also connects phase speed with frequency/wavelength.

Dispersion Relation for Airy waves is: > $\sigma^2 = gk\tanh(kh)$,
where h is the water depth

or substituting the identities $\sigma = 2\pi/T$ and $k = 2\pi/L$, > $L = \frac{g}{2\pi}T^2\tanh(\frac{2\pi h}{L})$

[1]:
```
# %%capture

# # Install necessary packages and restart the kernel.
# !pip install numpy
# !pip install matplotlib
```

[2]:
```
# The above equation is hard to salve since L is on both side of the equation
# However, tanh(r) can be approximated when r is very small or very large
# which allows us to simplify the above equation
# let's look at a tanh function since together

# import python libraries
import numpy as np
from matplotlib import cm
import matplotlib.pyplot as plt
%matplotlib inline

# define r from 0 to 3
r = np.linspace(0,3,100)

# plot tanh(r)
fig, ax = plt.subplots(nrows=1, ncols=1, figsize=(4,4))
ax.plot(r, np.tanh(r), label = 'tanh(r)')
ax.plot(r, np.sinh(r), label = 'sinh(r)')
ax.plot(r, np.cosh(r), label = 'cosh(r)')
ax.plot(r, r, 'k--', label = 'r')
ax.plot(r, np.ones(r.shape), 'k:', label = '1')
ax.set_xlabel('r')
ax.set_ylabel('')
ax.set_aspect(1)
ax.set_xlim([0,np.max(r)])
ax.set_ylim([0,np.max(r)])
ax.legend()
ax.grid(True)
plt.tight_layout()
```



The above graph suggest that as $r = kh = 2\pi h/L$ becomes large, $\tanh(2\pi h/L) \approx 1$, so that > $L_\infty = \frac{g}{2\pi}T^2$
> $\sigma_\infty = \sqrt{gk}$
> $c_\infty = \frac{gT}{2\pi}$

The subscript ∞ is used here to denote this as the **deep-water approximation**, so-called because of the assumption that h is large compared with the wave length.

Class Discussion: Let's now derive **shallow-water wave approximation** together!

When h is small compared with the wave length, $\tanh(2\pi h/L) \approx 2\pi h/L$, so that > $L_s = T\sqrt{gh}$
> $\sigma_s = k\sqrt{gh}$
> $c_s = \sqrt{gh}$

In contrast to the **deep-water** condition where the wave length and phase velocity depend only on the **wave period**, in **shallow water** the primary dependence is on the **water depth**!

1.1.2 2) Shallow water vs deep water waves

a. Wave theory The regions of application of the approximations: | | | | :——- | :——: | |Deep water | $h/L_\infty > 1/2$| |Intermediate water | $1/4 > h/L_\infty > 1/20$| |Shallow water | $h/L_\infty < 1/20$|

The **Airy wave theory** also provides a description of the water-partical movement at depth beneath the surface and the trajectories of the partical paths. The genral form of orbital horizontal velocity of water partical and horizontal orbital diameter are > $u = \frac{\pi H}{T}\frac{\cosh(k(z+h))}{\sinh(kh)}\cos(kx - \sigma t)$
> $s = H\frac{\cosh(k(z+h))}{\sinh(kh)}$

where $H = 2a$ is the wave height.
At the bed: > $u(z=-h) = \frac{\pi H}{T \sinh(kh)} \cos(kx - \sigma t)$
> $s(z=-h) = \frac{H}{\sinh(kh)}$

For deep water wave ($kh >> 1$): > $u_\infty(z=-h) \approx 0$
> $s_\infty(z=-h) \approx 0$

For shallow water wave ($kh << 1$): > $u_s(z=-h) \approx \frac{\pi H}{Tkh} \cos(kx - \sigma t)$
> $s_s(z=-h) \approx \frac{H}{kh}$

We can use the relations $c = L/T$ and $c_s = \sqrt{gh}$ to reduce to number of unknowns in $u_s(z=-h)$ and $s_s(z=-h)$. After some algebra: > $u_s(z=-h) \approx \frac{H}{2}\sqrt{\frac{g}{h}} \cos(kx - \sigma t)$
> $s_s(z=-h) \approx \frac{HT}{2\pi}\sqrt{\frac{g}{h}}$

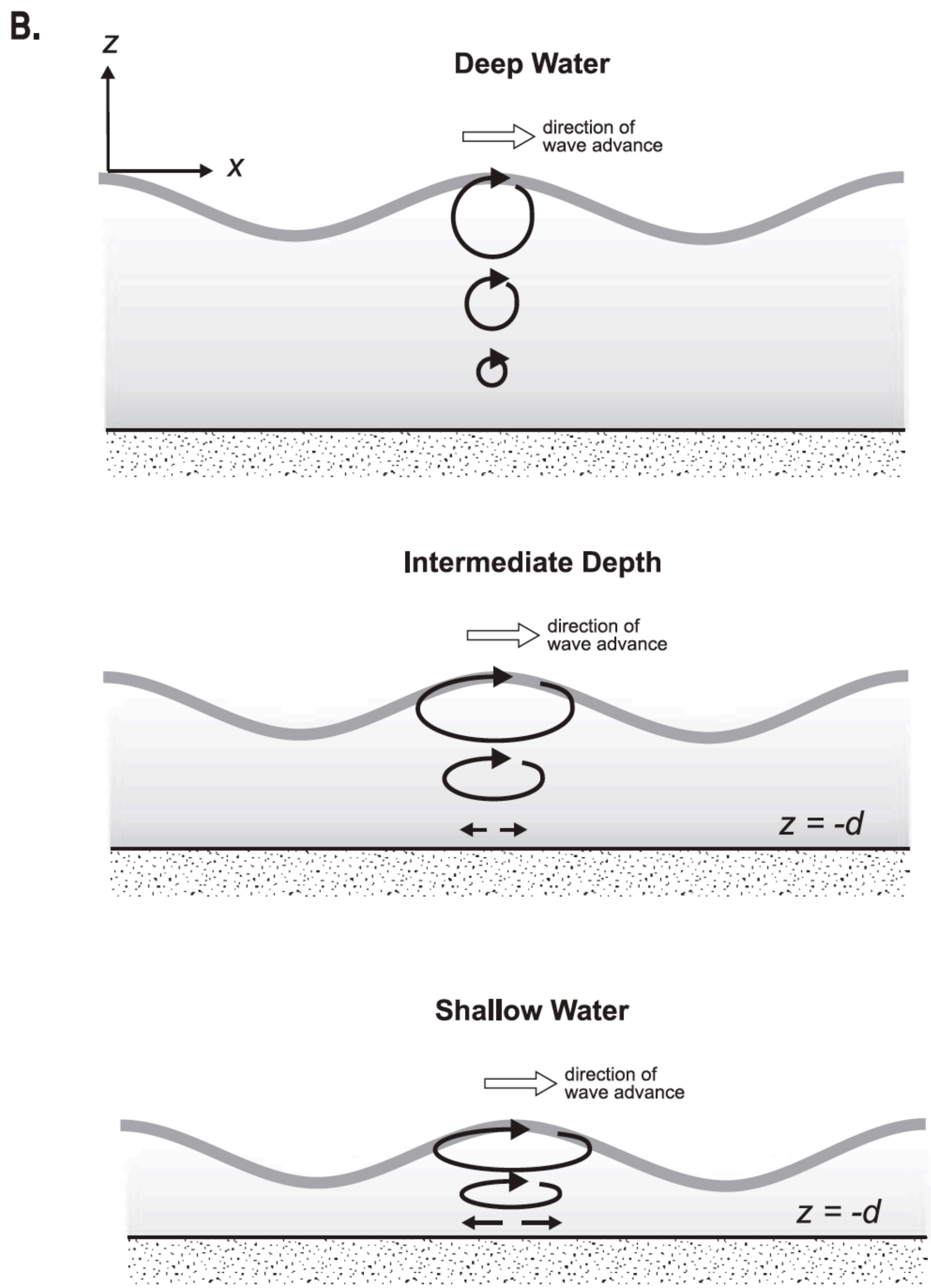


Figure 4.1 Wave characteristics: (A) definition sketch of the characteristics of a simple progressive wind wave in deep water; (B) wave orbital motion in deep, intermediate and shallow water.

b. Revisit sea and swell Recall phase speed of deep water waves $c_\infty = \frac{gT}{2\pi}$. This suggests that the wave celerity increases with the wave period, the longer-period wave travel the fastest as they leave the area of generation. Sea waves generated within a storm area are a superposition of many waves that have a range of periods and heights. As these waves propogate away from their generation site, they sort themselves out by period, this process is termed **wave dispersion**, as this is the primary factor in converting irregular **sea** into a regular **swell** where ther waves have a specific period or a very narrow range of periods.

Let's read a surf forecast together and

Class Discussion: Discuss how the periods of the second, third, etc., swells compare to that of the first swell.

1.1.3 3) Wave energy

The energy in water waves is made up of two components:

(1) **Kinetic energy** associated with the orbital motion of the water particles;

(2) **Potential energy** resulting from displacement of the water surface away from the mean sea level.

According to Airy wave theory, if potential energy is determined relative to the mean sea level and all waves are propagated in the same direction, the two energies are equal, and the total energy per unit crest width (unit: J/m) is given by

$$E = E_k + E_p = \frac{\rho g H^2 L}{16} + \frac{\rho g H^2 L}{16} = \frac{\rho g H^2 L}{8}$$

The energy per unit area, or energy density, $\overline{E}$ (unit: J/m^2) is given by

$$\overline{E} = \frac{E}{L} = \frac{\rho g H^2}{8}$$

The rate at which energy is transmitted in the direction of wave propagation is the energy flux P (sometimes called wave power, unit: J/(m s)), and is given by

$$P = \overline{E} c_g,$$

where c_g is the group veolcity.

1.2 2. Revisit Dispersion Relation for Airy Wave

Recall

Dispersion Relation for Airy waves: $\sigma^2 = gk\tanh(kh)$, where h is the water depth

,

$c = \sigma/k$ (phase speed) $c_g = \frac{\partial\sigma}{\partial k}$ (group speed)

and > non-dispersive wave: $c = c_g$ > dispersive wave: $c \neq c_g$

Class Discussion: Discuss whether Airy waves are dispersive. What about waves in the shallow and deep water limits?

From the dispersion relation, we find that in deep water, energy is propagated at one half of the speed of an individual wave ($c_{g\infty} = c_\infty/2$). In shallow water, the rate of energy propagation is equal to the wave celerity ($c_{gs} = c_s/2$).

1.3 3. Wave Refraction and Diffraction

1.3.1 1) Wave Refraction

When a wave approaches underwater contours at an angle, it is evident that the sections of the crest in the deeper parts travel faster than those in the shallower sectors. This causes the **wave crest to turn towards the depth contour**. This bending effect is called refraction, and is analogous to similar phenomena in physics (light, sound).

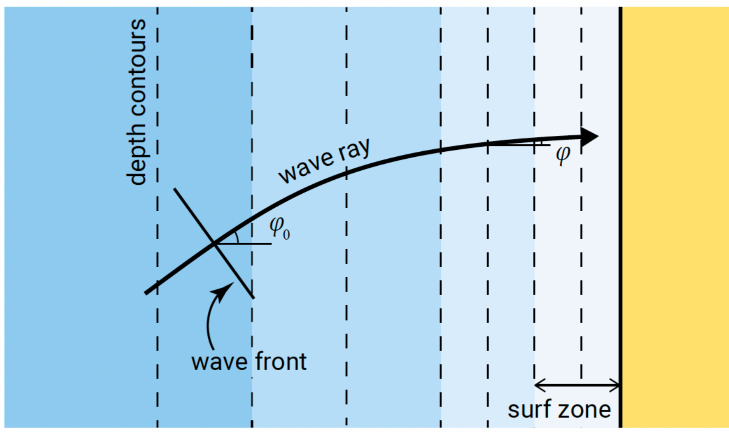


Figure 5.4: Obliquely incident waves propagating on alongshore uniform depth contours.

Snell's law: > $\frac{sin\varphi_2}{c_2} = \frac{sin\varphi_1}{c_1}$

If the wave rays **converge**, there is an **accumulation of energy** and relatively **high wave heights** can be expected. In contrast, if wave rays diverge, the energy is spread over a larger part of the wave crest, so the wave height is reduced.

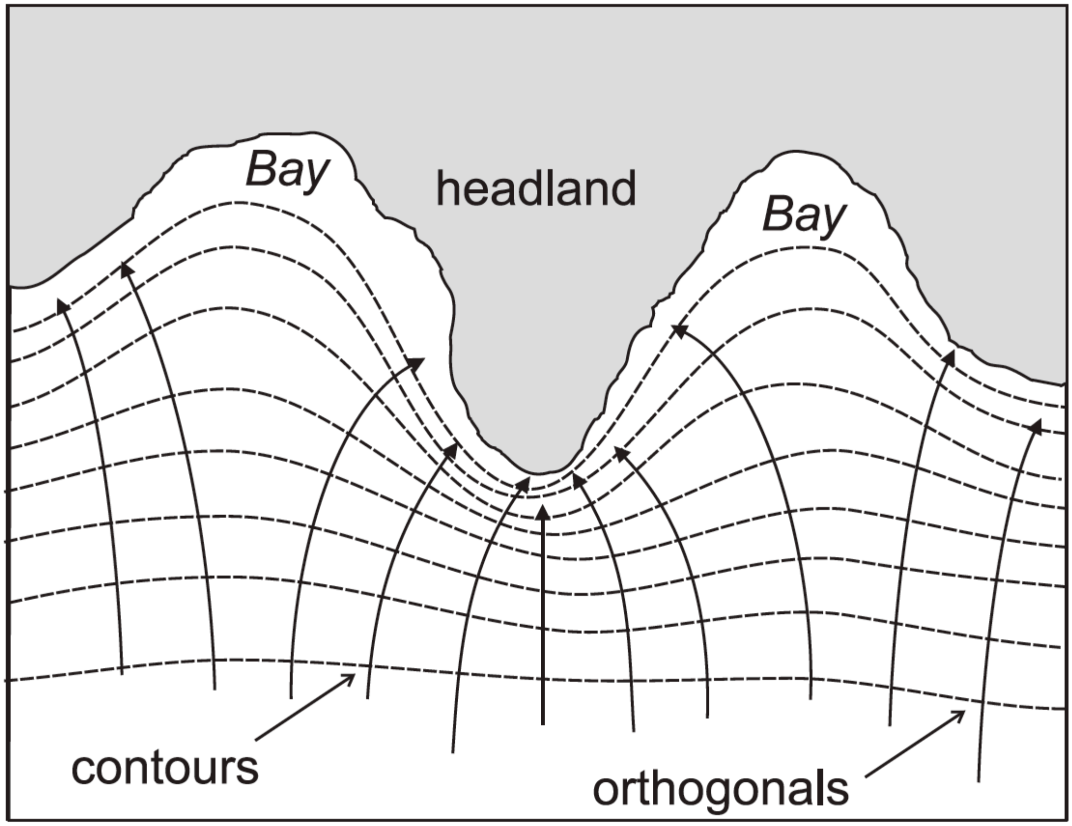


Figure 5.8 Refraction of waves shown by wave orthogonals over mildly complex topography associated with a series of headlands and bays.

1.3.2 2) Wave Diffraction

If obstructions to the wave propagation (an offshore island, a breakwater, a headland) or abrupt changes in the bottom contours are present, there is a large (initial) variation of wave energy along a wave crest, which leads to **transfer of energy along the wave crests**. This phenomenon is called diffraction.

Figure 5.7 shows the diffraction of an incident wave train in case there are no depth changes. A part of the wave front is blocked by the breakwater and is reflected seaward. The remainder of the wave front will bend around the obstacle and thus penetrate into the zone in the lee of the obstacle (shadow zone). The diffracted wave crests will form concentric circular arcs with the wave height decreasing along the crest of each wave.

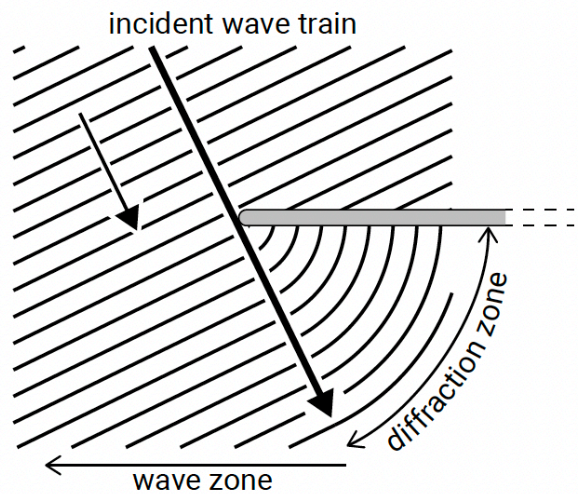


Figure 5.7: Diffraction of an incident wave train.

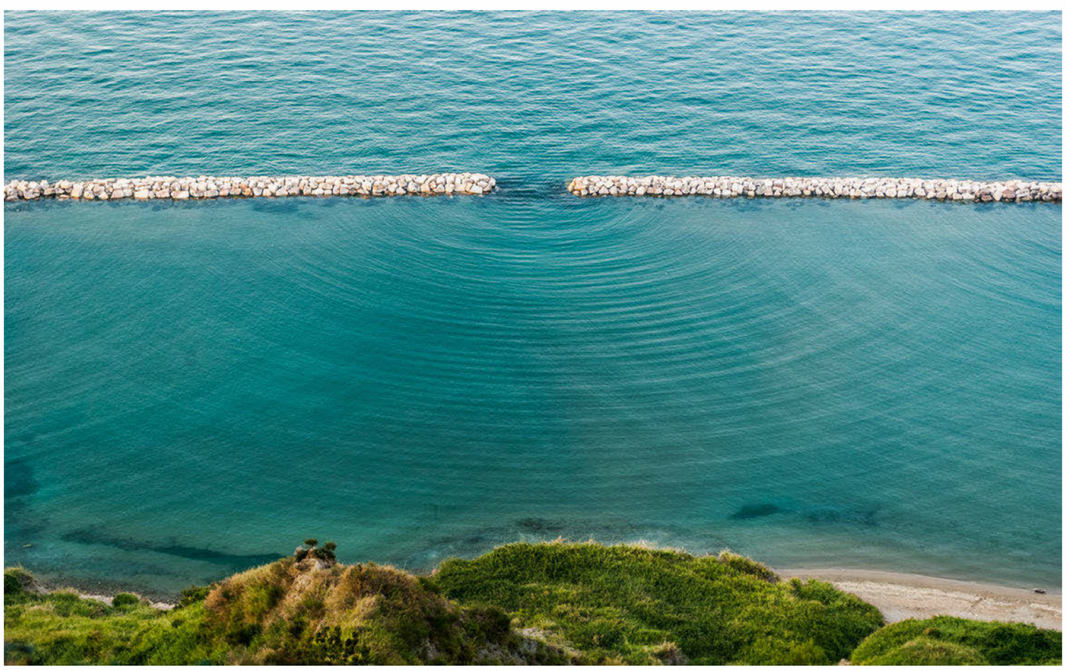


Figure 5.8: Typical diffraction pattern through a gap between two detached breakwaters in Pesaro, Italy. Photo by Roberto Lo Savio ('Credits' on page 575).

1.4 4. Wave Shoaling and Breaking

1.4.1 1) wave shoaling

Recall that the energy flux P is the rate at which energy is transmitted in the direction of wave propagation across a vertical plane perpendicular to the direction of wave propagation and extending over the entire depth. This value can relate the wave heights at two arbitrary locations:

$$P = P_0 \rightarrow \overline{E}c_g = \overline{E_0}c_{g0}$$

since $\overline{E} = \frac{\rho g H^2}{8}$,

$$P = P_0 \rightarrow H^2 c_g = H_0^2 c_{g0}$$

or

$$\frac{H}{H_0} = \sqrt{\frac{c_{g0}}{c_g}}$$

The shoaling factor $K_{sh} = \frac{H}{H_0}$ is 1.0 in deep water, then decreases slightly with water depth to 0.91 and subsequently rises to infinity. In reality the wave height increase in the shoaling zone is limited by dissipation due to wave-breaking.

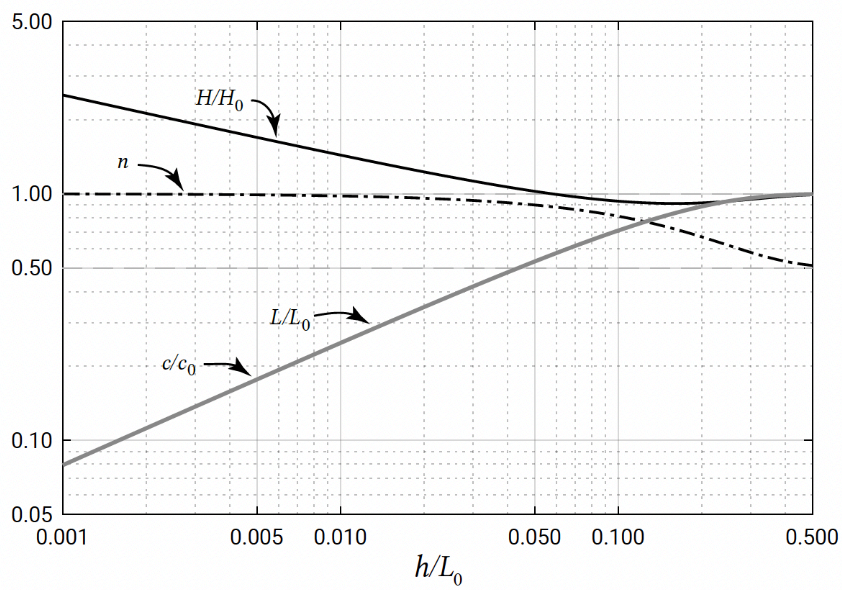


Figure 5.3: The shoaling factor $K_{sh} = H/H_0$ (Eq. 5.7), n (Eq. 3.25) and $c/c_0 = L/L_0 = \tanh kh$ (see Eq. 3.22) as a function of h/L_0.

1.4.2 2) wave breaking

a. breaking depth Section 5.2.2 demonstrated how shoaling would increase the wave height until infinity, at least in the absence of a physical limit to the steepness of waves. A wave crest becomes unstable and starts breaking when the particle velocity exceeds the velocity of the wave crest (the wave celerity). This breaking condition corresponds to a crest angle of about 120° (see Fig. 5.9).

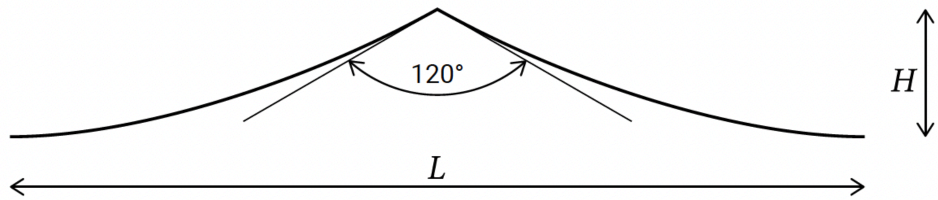


Figure 5.9: Maximum crest angle.

According to Miche (1944), in **deep water**, when $H_{max}/L > 1/7$, the crest angle exceeds its maximum and the steepness-induced wave-breaking (called white-capping) occurs.

In **shallow water**, when $H_{max}/h > 0.88$, wave breaks. $\gamma = H/h$ is the breaker index. The breaker index shows that in the shallow nearshore zone wave-breaking of individual waves starts when the wave height becomes greater than a certain fraction of the water depth. This is called **depth-induced breaking**, since the limiting wave height is governed by a water depth limitation. The maximum wave height H_{max} in a wave record is equal to $2H_S$. The maximum value of H_S/h for which the largest waves are breaking is therefore about 0.4~0.5.

Class Discussion: Consider a deep-water wave height $H_S = 1.5$ m and a wave period of $T = 5$ s. We find $L_\infty = \frac{g}{2\pi}T^2 = 39$ m, and $H_S/L_\infty = 0.04$. Under these conditions, even for the higher waves in the record, little white-capping is expected.

b. breaker type Depending on the wave properties and the **angle of the bed slope**, the process of breaking takes place in various different ways.

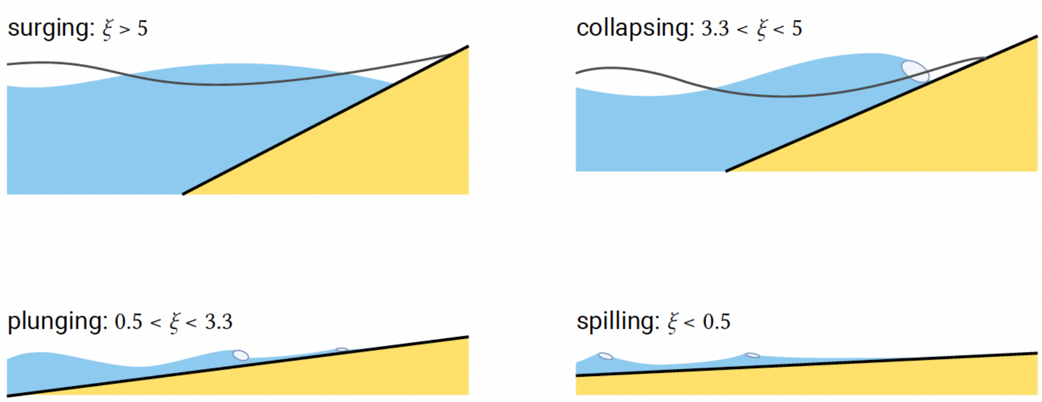


Figure 5.11: Breaker types.

(source: https://www.surfertoday.com/surfing/the-four-types-of-breaking-waves)

Spilling breakers are usually found along flat beaches. These waves begin breaking at a relatively great distance from shore and break gradually (over a distance of 6 to 7 wavelengths) as they approach progressively shallower water. During breaking, a foam line develops at the crest and leaves a thin layer of foam over a considerable distance. There is very little reflection of wave energy back towards the sea. Practically all wave energy is dissipated in the breaking process.

A plunging breaker is a type that is found on moderately-sloped beaches. The curling top is typical of such a wave. When the curling top breaks over the lower part of the wave, a lot of energy is dissipated into turbulence. Some energy is reflected back to the sea, and some is transmitted towards the coast, while forming a 'new' wave.

Surging breakers occur along rather steep shores for relatively long swell waves. The waves surge up and down the slope with minor air entrainment. The breaker zone is very narrow and more than half of the energy is reflected back into deeper water. The breakers form like plunging breakers, but the toe of each wave surges upon the beach before the crest can curl over and fall. It is debated whether surging breakers are actually breakers or rather standing waves (caused by interference of the incoming and reflected wave).

A collapsing breaker is between a plunging and a surging breaker and thus in between breaking and non-breaking.

Miche, R. (1944). Mouvements ondulatoires des mers en profundeur constante on decroisante. Annales des Ponts et Chaussees.

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