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Algorithmic Game Theory and Mechanism Design (Graphical Games)

Joint work with mentor Rian Neogi

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Introduction

- **What is Algorithmic Game Theory and Mechanism Design?**
  - Mathematical study of games
  - Found applications in analyzing systems involving strategic agents.
  - eg. Prisoner's dilemma, eBay auctions
- **Purpose**
  Develop an efficient algorithm to compute a Nash Equilibrium for cyclic graphical games.

Multiplayer Games

- n players
- Each player has
  - A finite set of actions
  - Specifications of payoffs
- Define **joint action** $\vec{a}$ as the action vector denoting actions by all n players
- eg. For rock paper scissors, a joint action between two players could be (rock, rock)

Payoff Matrix

- Each player i has a payoff matrix that is indexed by the joint actions of all the players.
- $M_i(\vec{a})$ is the payoff to player i resulting from $\vec{a}$
- We'll assume that $M_i(\vec{a})$ lies in the interval [0, 1]

Player One

Player Two

Actions	Rock	Paper	Scissor
Rock	$1/2, 1/2$	1, 0	0, 1
Paper	0, 1	$1/2, 1/2$	1, 0
Scissor	1, 0	0, 1	$1/2, 1/2$

Table: Payoff Matrix

Player One

Player Two

Actions	Rock	Paper	Scissor
Rock	$1/2, 1/2$	1, 0	0, 1
Paper	0, 1	$1/2, 1/2$	1, 0
Scissor	1, 0	0, 1	$1/2, 1/2$

Table: Payoff Matrix

Pure vs Mixed Strategies

- Actions like Rock, Paper, Scissor are called **pure strategies**
- **Mixed strategies** are given by probabilities
  - ie. There is a 40% probability that I will play rock, a 50% probability that I will play paper, a 10% probability that I will play scissor
- For pure strategies, the best payoff I can get is 0
  - When I play Rock, my opponent can change their strategy to be Paper.

Player One

Player Two

Actions	Rock	Paper	Scissor
Rock	$1/2, 1/2$	1, 0	0, 1
Paper	0, 1	$1/2, 1/2$	1, 0
Scissor	1, 0	0, 1	$1/2, 1/2$

Table: Payoff Matrix

Pure vs Mixed Strategies

However, I can get a better payoff if I use a mixed strategy

- Instead of playing just one of rock, paper, or scissor, I play all of them with a probability of $1/3$
- Then if my opponent plays rock, my payoff becomes $1/3 * 1/2 + 1/3 * 1 + 1/3 * 0 = 1/2$
- We get the same calculation if my opponent plays paper or scissor.

Pure vs Mixed Strategies (Formally)

- To keep things simple, assume that there are binary actions $\{0, 1\}$
- For a player i, a mixed strategy is the probability $p_i \in [0, 1]$ that player i will play 0
- A joint mixed strategy is given by a probability vector $\vec{p}$
- The **expected payoff** to player i is denoted as $M_i(\vec{p}) = E_{\vec{a} \sim \vec{p}} M_i(\vec{a})$, where $\vec{a} \sim \vec{p}$ indicates that each a_j is 0 with probability p_j and 1 with probability $1 - p_j$ independently

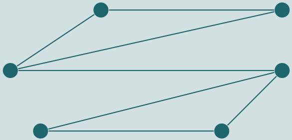
Nash Equilibrium

- A **Nash equilibrium** for the game is a mixed strategy $\vec{p}$ such that for any player i, p_i is a **best response** to the rest of $\vec{p}$.
- ie. player i cannot improve their expected payoff by deviating from the Nash equilibrium $\vec{p}$ by playing an alternate strategy p_i'

Nash Theorem

For any game using mixed strategies, there exists a Nash Equilibrium.

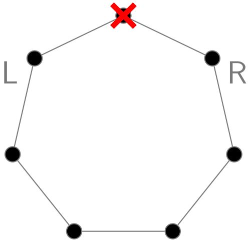
Graphical Games



- Every player is represented by a node on the graph
- Player i's payoff is only dependent on their neighbours' actions

TreeNash (Foundation)

- Algorithm TreeNash for computing NE of tree graphical games [Kearns, 2007]
- using dynamic programming



Nash Equilibrium for Cycle

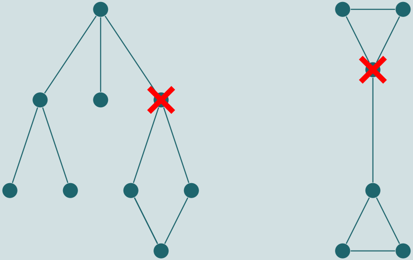
Input: Cycle graphical game G
Output: Nash equilibrium (a^*, ϕ_q)

1. **Initialize:**
   Define $A = \left\{\frac{1}{\epsilon}, \frac{2}{\epsilon}, \ldots, 1\right\}$.
2. **Iterate:**
   For each $a \in A$:
   - Modify payoffs: $R \to M_{R,a}, \quad L \to M_{L,a}$.
   - Compute Nash equilibrium for G_a.
3. **Check Best Response:**
   Find a^* such that it is the best response for player N against ϕ_q.
4. **Output:** Return (a^*, ϕ_q).

Cactus Graph

- A connected graph in which every edge belongs to at most one simple cycle.

Nash Equilibrium for Some Cactus Graph Structure



Conclusion

- **Key Achievement**
  Developed an algorithm to compute Nash Equilibrium (NE) for cyclic graphs in polynomial time.
- **Limitations & Future Work**
  Currently, the algorithm is limited to certain cyclic structures.
  In the future, we aim to extend this approach to compute NE for cactus graphs in polynomial time.

References

Michael Kearns. Graphical games. *Algorithmic Game Theory*, pages 159 – 180, 2007.