A Proof of The Changepoint Detection Threshold Conjecture in Preferential Attachment Models

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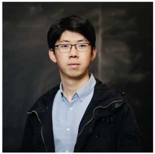
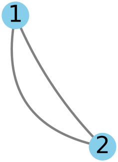



Hang Du (MIT) and Jiaming Xu (Duke)

38th Annual Conference on Learning Theory
Lyon, France

Preferential attachment models

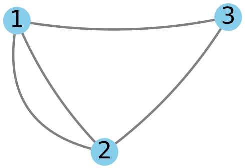
Initial graph G_2 consists of two vertices connected by m parallel edges



Preferential attachment models

At each time t, a new vertex t arrives and forms m edges, one at a time, to existing nodes $v \in [t-1]$:

$$\mathbb{P}\{t \to v\} \propto \mathsf{deg}(v) + \delta_t$$



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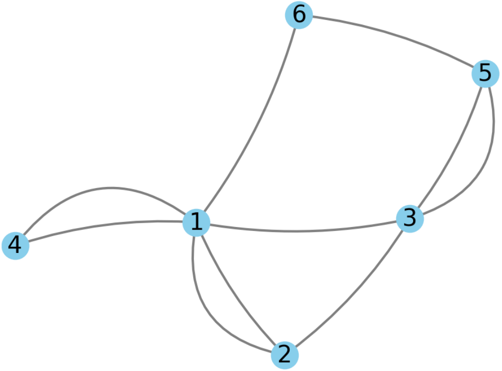

Changepoint detection problem

Definition

$$\mathcal{H}_0 : \delta_t = \delta$$
$$\mathcal{H}_1 : \delta_t = \delta \mathbf{1}_{t \leq \tau_n} + \delta' \mathbf{1}_{\tau_n < t \leq n}$$

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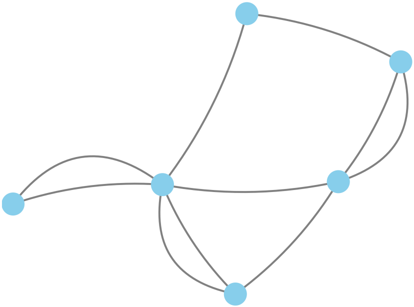

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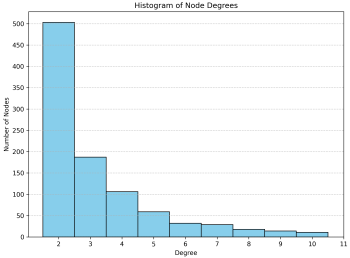
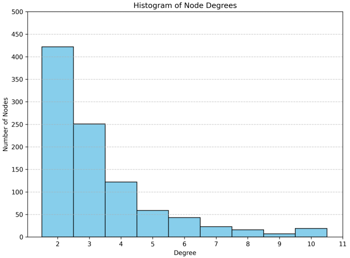
- Changepoint localization: estimate τ_n under $\mathcal{H}_1$ [Bhamidi-Jin-Nobel '18]
- Applications: detect structural changes in various settings, such as communication networks, social networks, financial networks, and biological networks [Cirkovic-Wang-Zhang '24].

A simple test based on minimum-degree





$n = 1000,\ m = 2,\ \delta(t) \equiv 0$ $\qquad$ $n = 1000,\ m = 2,\ \delta(t) = 10 \cdot \mathbf{1}\left(t > n - n^{0.8}\right)$

- [Bet-Bogerd-Castro-van der Hofstad '23] achieves strong detection by thresholding the number of degree m vertices.
- Intuition: Mean difference $= \Theta(n - \tau_n)$, while standard deviation $= O(\sqrt{n})$.

Changepoint detection conjecture

Conjecture (Bet-Bogerd-Castro-van der Hofstad '23)

Suppose $\tau_n = n - cn^\gamma$ *for a constant* c *and* $\gamma < 1/2$.

1. *All tests based on vertex degrees are powerless.*
2. *All tests are powerless.*

- Part 2 of the conjecture is particularly striking, because, if true, neither degree information nor any higher-level graph structure is useful for detection when $\gamma < 1/2$

Significant progress

Theorem (Kaddouri-Naulet-Gassiat '24)

Suppose $\tau_n = n - \Delta$. *If* $\Delta = o(n^{1/3})$ *for* $\delta > 0$ *or* $\Delta = o(n^{1/3}/\log n)$ *for* $\delta = 0$, *then*

$$\mathbb{P}_0(A_n) \to 0 \Longrightarrow \mathbb{P}_1(A_n) \to 0, \quad \textit{for all sequences of events } A_n$$

- As a consequence, $\mathrm{TV}(\mathbb{P}_0, \mathbb{P}_1) \le 1 - \Omega(1) \Rightarrow$ strong detection is impossible
- Does not cover the entire regime $\Delta = o(\sqrt{n})$ and the regime $\delta < 0$
- Does not rule out the possibility of weak detection

Our resolution

Theorem (Du-G.-Xu '25)

Suppose $\tau_n = n - \Delta$. *If* $\Delta = o(n^{1/2})$, *then*

$$\mathrm{TV}(\mathbb{P}_0, \mathbb{P}_1) = o(1)$$

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- As a consequence, all tests are powerless $\Rightarrow$ resolves the changepoint detection conjecture [Bet-Bogerd-Castro-van der Hofstad '23] in positive
- We prove a stronger statement: all tests remain powerless even if, in addition to G_n, the entire network history were observed up to time $n - N$ for $\Delta^2 \ll N \ll n$
- As a corollary, we prove no estimator can locate τ_n within $o(\sqrt{n})$ with $\Omega(1)$ probability $\Rightarrow$ the estimator in [Bhamidi-Jin-Nobel'18], which achieves $|\hat{\tau}_n - \tau_n| = O_P(\sqrt{n})$, is order-optimal

Proof ideas

Challenge of directly bounding second-moment

Define the Likelihood ratio

$$L(G) \triangleq \frac{\mathbb{P}_1(G)}{\mathbb{P}_0(G)}$$

Then

$$\mathbb{E}_{G_n \sim \mathbb{P}_0}\left[L^2(G_n)\right] = 1 + o(1) \implies \mathrm{TV}(\mathbb{P}_1, \mathbb{P}_0) = o(1)$$

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- Widely used to prove impossibility of detection in high-dimensional statistics and network analysis (e.g. community detection)
- However, since only final network snapshot is observed, $L(G_n)$ involves an average over compatible network histories, making it hard to bound its second-moment directly

Our proof strategy

- Interpolation
  - $\mathbb{P}_{n,n-k}$: distribution of G_n with changepoint at time $n-k$
  - reduce to analyzing changepoint $\tau_n = n-1$:

$$\mathrm{TV}(\mathbb{P}_{n,n-1}, \mathbb{P}_{n,n}) = O\left(\frac{1}{\sqrt{N}}\right).$$

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- "Easier" model: reveal the network history up to time $M = n - N$
  - The node arrival times are known.
  - Largely simplifies the form of likelihood ratio.

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- Control the second moment
  - The components are approximately uncorrelated.
  - Tools: Combinatorial coupling, Efron-Stein inequality.

Concluding remarks

- We show changepoint detection threshold is $\tau_n = n - o(\sqrt{n})$, confirming a conjecture of [Bet-Bogerd-Castro-van der Hofstad '23]
- As by-product, we show changepoint localization threshold is also $\sqrt{n}$, matching upper bound in [Bhamidi-Jin-Nobel '18]
- Key proof ideas: reduces to bounding TV when changepoint occurs at $n-1$, reveal network history up to $n - o(n)$, and bound the second-moment of likelihood ratio.

Future directions

- General attachment rule: $\mathbb{P}(t \to v) \propto f(\text{deg}(v))$ [Banerjee-Bhamidi-Carmichael '22]
- Changepoint detection in general dynamic graph models
- Other related reconstruction and estimation problems in PA graphs

References

- Hang Du, Shuyang Gong, & Jiaming Xu. *A Proof of The Changepoint Detection Threshold Conjecture in Preferential Attachment Models*, arXiv:2502.00514, v3.