三重积分. 03/04/2025 孔舒雅

(I) Framework and definition of triple integral (n-multiple integral)

① Similar to double integral both in terms of def & tech. but calculation is more compliacted.

② In many cases. we can use some properties such as symmetry to simplify the calculation

1.1. Definition.

Step 1: Discretize $\Omega$ by $\Omega_i$. $i=1,\cdots,n$. with $|\Omega_i| \leq \eta$

Step 2: Pick any point $(x_i, y_i, z_i) \in \Omega_i$

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)|\Omega_i| \quad (|\Omega_i| \text{ is the volume of } \Omega_i)$$

Step 3: Taking limit. Let $\eta \to 0$.

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \lim_{\eta \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)|\Omega_i| .$$

(Recall the definition of Riemann integral on 1-d)

1.2. Calculating integral in 3-D rectangular coordinates

**Method**: From 3-d integral to iterated integral

① Fix $(x,y)$, integrate $z$.

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iint_D \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\right) dx\,dy.$$

② Fix $z$, integrate $(x,y)$

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \int_z \left(\iint_{D_z} f(x,y,z)\,dx\,dy\right) dz.$$

Practical steps:

Step 1: draw the area.

Step 2: choose method ① or ②

Step 3: integrate according to ① or ②.

Some principles to chouse ① or ②:

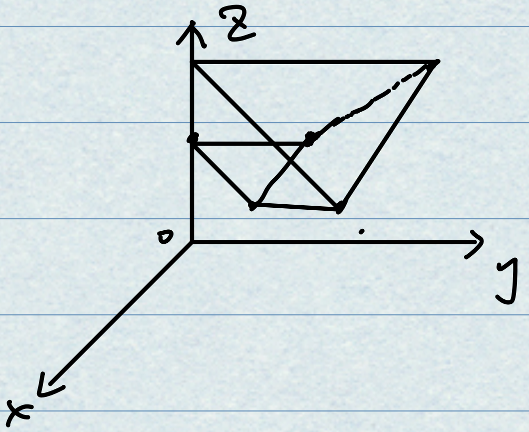
1. The form of integral area (try to avoid too many discussion)

**Ex 1**: $I = \iiint_{\Omega} (y^2+z^2)^{-1}\,dx\,dy\,dz$

$\Omega$ is formed by $\begin{cases} (0,0,1),\ (0,1,1),\ (1,1,1) \\ (0,0,2),\ (0,2,2),\ (2,2,2) \end{cases}$

Proof:

① Draw the area

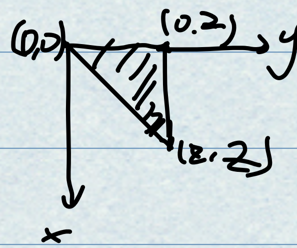



parallel to xoy plane.

② fix z. $D^z(x,y)$ is the triangle by $(0,0,z)$ $(0,z,z)$ $(z,z,z)$

③ $$\int_1^2 \left( \iint_{D^z} (y^2+z^2)^{-1}\, dx\, dy \right) dz$$

$D^z$




$$\Rightarrow \iint_{D^z} (y^2+z^2)^{-1}\, dx\, dy$$

$$= \int_0^z \left( \int_0^y (y^2+z^2)^{-1}\, dx \right) dy = \int_0^z y(y^2+z^2)^{-1}\, dy$$

$$= \int_0^z (y^2+z^2)^{-1}\, d\left(\frac{y^2+z^2}{2}\right) = \frac{1}{2}\log(y^2+z^2)\Big|_0^z$$

$$= \frac{1}{2}\log(2z^2) - \log(z^2) = \frac{\log 2}{2}$$

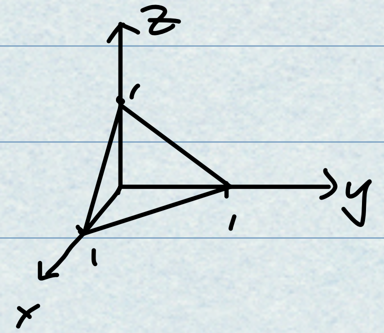
$$\Rightarrow I = \int_1^2 \frac{\log 2}{2}\, dz = \frac{\log 2}{2}$$

∎

2. The order of integral should take the closed form of the integral into accomt.

Ex 2: $I = \iiint_{\Omega} (1-y) e^{-(1-y-z)^2} dx\,dy\,dz$, where $\Omega$ is formed by $\begin{cases} x+y+z=1 \\ x=0 \\ y=0 \\ z=0 \end{cases}$

① Draw.




② Observation: we can't integrate $y$ or $z$ first. due to the Gaussian integral.

$\Rightarrow$ integrate $x$ first.

$$I = \iint_{D(y,z)} (1-y-z)(1-y) e^{-(1-y-z)^2} dy\,dz$$

$$= \int_0^1 \int_0^{1-y} (1-y-z)(1-y) e^{-(1-y-z)^2} dz\,dy$$

$$= \int_0^1 (1-y) \int_0^{1-y} -\frac{1}{2} e^{-(1-y-z)^2} d\left[(1-y-z)^2\right] dy$$

$$= \int_0^1 (1-y) \left( \frac{1}{2} e^{-(1-y-z)^2} \Big|_0^{1-y} \right) dy$$

$$= \int_0^1 (1-y)\frac{1}{2}\left(1 - e^{-(1-y)^2}\right) dy$$

$$= -\frac{1}{4}\left((1-y)^2 + e^{-(1-y)^2}\right)\Big|_0^1 = -\frac{1}{4}\left(1 - 1 - e^{-1}\right)$$

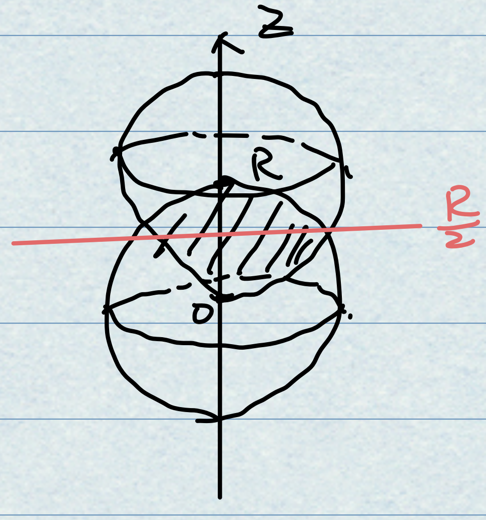
$$= \frac{1}{4e}$$

3. Some special functions. only depend on one of the x, y z variables. If the function only depends on z, then integrate x, y first.

Ex3 $\iiint_\Omega z^2 \, dx\,dy\,dz$, $\Omega$ is formed by.

$$\Omega := \begin{cases} x^2+y^2+z^2 \le R^2 \\ x^2+y^2+(z-R)^2 \le R^2 \end{cases}$$

① Draw.




details refer to Yantong Xie's notes.

Calculating by symmetry. (first consideration)

Ex 4 :

$$I = \iiint_\Omega (x+1)(y+1) \, dx\,dy\,dz$$

$$\Omega: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

proof:

$$(x+1)(y+1) = xy + y + x + 1$$

$I = I_1 + I_2 + I_3 + I_4$ , where

$$I_1 = \iiint_\Omega xy \, dx\, dy\, dz$$

$$I_2 = \iiint_\Omega y \, dx\, dy\, dz$$

$$I_3 = \iiint_\Omega x \, dx\, dy\, dz$$

$$I_4 = \iiint_\Omega dx\, dy\, dz$$

By symmetry, $I_1 = I_2 = I_3 = 0$

$$I_4 = V = \frac{4}{3}\pi abc$$

1.3. change of variables.

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$

three conditions:

① bijection : $\Omega' \to \Omega$

② $x(\,)$, $y(\,)$, $z(\,)$ continuous partial derivatives

③ Jacobian: $J \triangleq \frac{D(x,y,z)}{D(u,v,w)} \neq 0$

Then,

$$\iiint_{\Omega} f(x,y,z)\,dV = \iiint_{\Omega'} f(x(u,v,w), y(u,v,w), z(u,v,w))\,|J|\,du\,dv\,dw.$$

Two important coordinates:

① Cylindrical coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$\Rightarrow J = r$

$$\iiint_{\Omega} f(x,y,z)\,dV = \iiint_{\Omega} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

<u>Ex 5</u> $I = \iiint_{\Omega} (x^2+y^2)^{\frac{1}{2}}\,dx\,dy\,dz$.

$$\Omega = \begin{cases} x^2+y^2 = 9 \\ x^2+y^2 = 16 \\ z = 0 \\ z = \sqrt{x^2+y^2} \end{cases}$$

<u>Pf</u>: $I = \iiint_{\Omega} r^2\,dr\,d\theta\,dz$

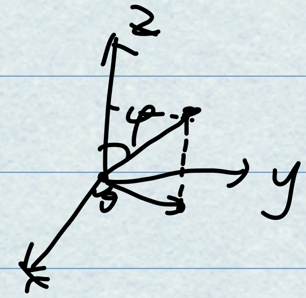
$$\Omega = \begin{cases} 3 \le r \le 4 \\ 0 \le z \le r \end{cases}$$

$$I = \int_0^{2\pi}\int_3^4 \left(\int_0^r r^2\,dz\right) dr\,d\theta = \int_0^{2\pi}\int_3^4 r^3\,dr\,d\theta$$

$$= 2\pi \cdot \frac{r^4}{4}\Big|_3^4 = \frac{175}{2}\pi$$

② Spherical coordinates

$$\begin{cases} z = \rho\cos\varphi \\ x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \end{cases}$$




$$J = \rho^2\sin(\varphi)$$

$$\Rightarrow \iiint_\Omega f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_{\Omega'} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

**Ex 6.** $\iiint_D y^2\,dx\,dy\,dz$ $\quad D: x^2+y^2+z^2 \le 2z$

**pf:**

**Obs:** $D \leftrightarrow D'$ where $D': x^2+y^2+z^2 \le 1$

the integral is invariant

$$\Rightarrow \iiint_D \rho^2\sin^2\varphi\sin^2\theta\;\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \iiint_D \rho^4\sin^3\varphi\sin^2\theta\,d\rho\,d\varphi\,d\theta$$

$$= \left(\int_0^1 \rho^4 \, d\rho\right)\left(\int_0^{\pi} \sin^3\varphi \, d\varphi\right)\cdot\left(\int_0^{2\pi} \sin^2\theta \, d\theta\right)$$

$$= \frac{1}{5}\cdot\frac{4}{3}\cdot\pi = \frac{4}{15}\pi$$

(Recall the integral $I = \int_0^{\frac{\pi}{2}} \sin^n(\theta)\, d\theta$ )

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n(\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-1}(\theta)\, d(-\cos\theta)$$

$$= \int_0^{\frac{\pi}{2}} \cos^2\theta\,(n-1)\sin^{n-2}(\theta)\, d\theta$$

$$= (n-1)\, I_{n-2} - (n-1)\, I_n$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2}$$

$$\Rightarrow I_n = \begin{cases} \dfrac{(n-1)!!}{n!!} & n \text{ odd} \\ \dfrac{(n-1)!!}{n!!}\cdot\dfrac{\pi}{2} & n \text{ even} \end{cases}$$