03/11/2025 孔舒阳

Exercise 1:

$$I = \iiint_{\Omega} (z + z^2)\, dV \qquad \Omega: x^2+y^2+z^2 \le 1$$

Pf: (Use symmetry)

$$I = \iiint_{\Omega} z^2\, dV$$

symmetry.

$$\boxed{=} \frac{1}{3} \iiint_{\Omega} (x^2+y^2+z^2)\, dV$$

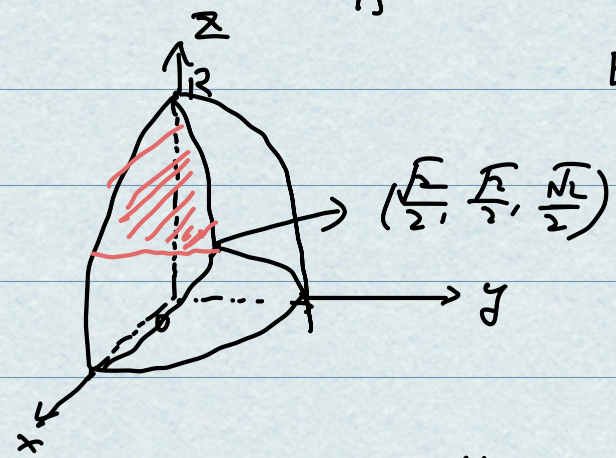
$$= \frac{1}{3} \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^2 \cdot \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= \frac{1}{3} \cdot 2\pi \cdot \frac{1}{5} \cdot \int_0^{\pi} \sin\varphi \, d\varphi = \frac{4\pi}{15}.$$

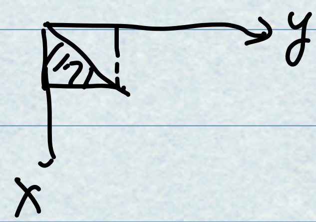
Exercise 2. (☆)

Calculate the surface measure

$$\begin{cases} x^2+y^2=R^2 \\ x^2+y^2=R^2 \\ y^2+z^2=R^2 \end{cases}$$



$S = 6\, S'$



$$z = \sqrt{R^2 - x^2} =: f_2(x,y)$$

$$\frac{\partial z}{\partial x} = \frac{(-x)}{\sqrt{R^2-x^2}}$$

$$S' = \iint_{D_{xy}} \sqrt{1 + \frac{x^2}{R^2-x^2}}\, dx\,dy = \iint_{D_{xy}} R \cdot \frac{1}{\sqrt{R^2-x^2}}\, dx\,dy$$

$$= R\int_0^{\frac{\sqrt{2}}{2}R} \frac{x}{\sqrt{R^2-x^2}}\,dx = -\frac{R}{2}\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{R^2-x^2}}\,d(R^2-x^2)$$

$$= -\frac{R}{2}\cdot 2\,(R^2-x^2)^{\frac{1}{2}}\Big|_0^{\frac{\sqrt{2}}{2}R} = -\left(R\left(R^2-\frac{R^2}{2}\right)^{\frac{1}{2}} - R^2\right)$$

$$= R^2\cdot\left(1-\frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow S = 6S' = 6R^2\left(1-\frac{\sqrt{2}}{2}\right)$$

<u>Ex3</u>: $V = 6V'$

$$V' = \iiint_\Omega 1\,dx\,dy\,dz \qquad \Omega: \begin{cases} x^2+y^2 \le R^2 \\ x \ge y \\ z \le y \end{cases}$$

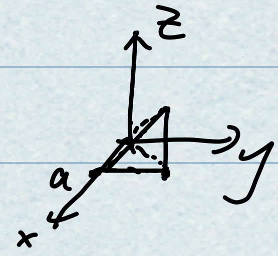
$$V' = \int_0^{\frac{\pi}{4}}\int_0^R\int_0^{r\sin\theta} r\,dz\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}}\int_0^R r^2\sin\theta\,dr\,d\theta$$

$$= \frac{R^3}{3}\cdot(-\cos\theta)\Big|_0^{\frac{\pi}{4}} = \frac{R^3}{3}\left(1-\frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow V = 2R^3\left(1-\frac{\sqrt{2}}{2}\right)$$

Exercise 4.



$$I = \int_0^a dx \int_0^x dy \int_0^y f(z)\,dz$$

$$= \iiint_\Omega f(z)\,dx\,dy\,dz$$

$$= \int_0^a f(z)\underbrace{\frac{1}{2}(a-z)^2}_{\text{area of triangle.}}dz$$

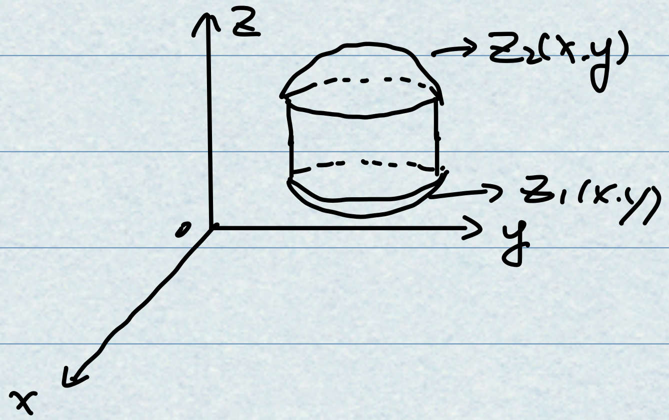
知识点回顾.

① 区域 Ω 的体积 V.

$$V = \iiint_\Omega 1\,dx\,dy\,dz$$

常用方法.

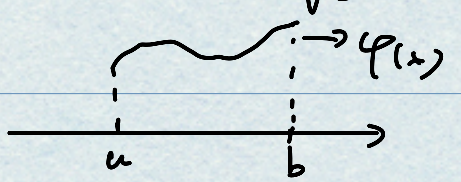
$$V = \iint_D \left(z_2(x,y) - z_1(x,y)\right)dx\,dy$$



② 旋转体的体积

设 V 由 $y=\varphi(x)$ $a\le x\le b$ 绕 x 轴旋转的体积

$$V = \pi\int_a^b \varphi^2(x)\,dx$$



③ 曲面面积

$$z=f(x,y),\quad (x,y)\in D$$

$$S=\iint_D \sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,dxdy$$

④ 质量

$$M=\iint_D \rho(x,y)\,dxdy \quad \text{平面}$$

$$M=\iiint_D \rho(x,y,z)\,dxdydz \quad \text{三维空间.}$$

⑤ 质心

$$x_0=\frac{\iiint_\Omega x\rho(x,y,z)\,dV}{M}$$

y_0, z_0 类似.

二重积分, 三重积分 头等大事
是计算: 算对!

<u>Exercise 5</u>:

求第一卦限中的部分椭球体

$$\Omega:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1 \quad (x\ge 0,\ y\ge 0,\ z\ge 0)$$

的重心坐标. (设密度 $\rho=1$)

<u>解</u>:

<u>Step1</u>: 先算体积,

$$M=\iiint_\Omega dxdydz=\int_0^c\left(\iint_{D(z)}dxdy\right)dz$$

其中 $D(z): \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1-\frac{z^2}{c^2}$

$$\Rightarrow S(D(z)) = \frac{1}{4}\pi ab\left(1-\frac{z^2}{c^2}\right)$$

$$\Rightarrow M = \int_0^c \frac{1}{4}\pi ab\left(1-\frac{z^2}{c^2}\right)dz$$

$$= \frac{\pi ab}{4}\left(z-\frac{z^3}{3c^2}\right)\Big|_0^c = \frac{\pi ab}{4}\frac{2c}{3} = \frac{\pi abc}{6}$$

Step 2:

$$\iiint_V z\,dx\,dy\,dz$$

$$= \int_0^c z\,dz \cdot \left(\iint_{D(z)} dx\,dy\right)$$

$$= \int_0^c z\,\frac{1}{4}\pi ab\left(1-\frac{z^2}{c^2}\right)dz$$

$$= \frac{\pi ab}{4}\int_0^c\left(z-\frac{z^3}{c^2}\right)dz$$

$$= \frac{\pi ab}{4}\left(\frac{z^2}{2}-\frac{z^4}{4c^2}\right)\Big|_0^c = \frac{\pi ab}{4}\cdot\frac{c^2}{4} = \frac{\pi abc^2}{16}.$$

$$\Rightarrow z_0 = \frac{\frac{\pi abc^2}{16}}{\frac{\pi abc}{6}} = \frac{3}{8}c$$

同理: $x_0 = \frac{3}{8}a \quad y_0 = \frac{3}{8}b$

證

<u>Exercise 6</u> 设 f 在 $[0,1]$ 上为正连续函数，证明

记 $I = \int_0^1 f(x)\,dx \int_0^1 \frac{1}{f(x)}\,dx$

(i) $1 \le I = \dfrac{m^2+M^2}{2mM}$

(ii) $I \le \dfrac{(m+M)^2}{4mM}$

<u>pf</u>:

(i) $I = \int_0^1 \frac{1}{f(x)}\,dx \int_0^1 f(x)\,dx$

$$= \int_{[0,1]\times[0,1]} \frac{f(y)}{f(x)}\,dx\,dy$$

$$= \frac{1}{2}\int_{[0,1]\times[0,1]} \left(\frac{f(y)}{f(x)} + \frac{f(x)}{f(y)}\right) dx\,dy$$

令 $t = \dfrac{f(y)}{f(x)}$, $\dfrac{m}{M} \le t \le \dfrac{M}{m}$.

考虑 $f(t) = t + \dfrac{1}{t}$ $\quad t \in \left[\dfrac{m}{M}, \dfrac{M}{m}\right]$

$$\Longrightarrow \quad 1 \le I \le \frac{1}{2}\left(\frac{M}{m} + \frac{m}{M}\right) = \frac{m^2+M^2}{2mM}$$

$\square$

(ii)

$$I = \int_0^1 \frac{\sqrt{mM}}{f(x)}\,dx \int_0^1 \frac{f(x)}{\sqrt{mM}}\,dx$$

$$\le \frac{1}{4}\left[\int_0^1 \left(\frac{\sqrt{mM}}{f(x)} + \frac{f(x)}{\sqrt{mM}}\right) dx\right]^2$$

令 $t = \dfrac{\sqrt{mM}}{f(x)}$ $\quad \sqrt{\dfrac{m}{M}} \le t \le \sqrt{\dfrac{M}{m}}$

$$\Longrightarrow I \le \frac{1}{4}\left(\sqrt{\frac{m}{M}} + \sqrt{\frac{M}{m}}\right)^2 = \frac{(m+M)^2}{4mM}$$

$\square$

(Hölder 不等式)

<u>Exercise 7</u>, $f=f(x,y)$ on Ω.

令 $\|f\|_p = \left(\iint_\Omega |f(x,y)|^p \, dx\, dy\right)^{\frac{1}{p}}$ 　设 $\|f\|_p < +\infty$

↓

L_p norm of f

设 $p>1$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, 则

$$\|uv\|_1 \le \|u\|_p \|v\|_q$$

Hint: $ab \le \frac{a^p}{p}+\frac{b^q}{q}$ (Young 不等式)

<u>Pf</u>:

设 $a=\frac{|u|}{\|u\|_p}$ 　$b=\frac{|v|}{\|v\|_q}$

$$\Rightarrow \frac{|u|\cdot|v|}{\|u\|_p\|v\|_q} \le \frac{1}{p}\frac{|u|^p}{\|u\|_p^p}+\frac{1}{q}\frac{|v|^q}{\|v\|_q^q}$$

$$\iint_\Omega \frac{|u||v|}{\|u\|_p\|v\|_q}\, dx\, dy$$

$$\le \frac{1}{p}\iint_\Omega \frac{|u|^p}{\|u\|_p^p}\, dx\, dy + \frac{1}{q}\iint_\Omega \frac{|v|^q}{\|v\|_q^q}\, dx\, dy$$

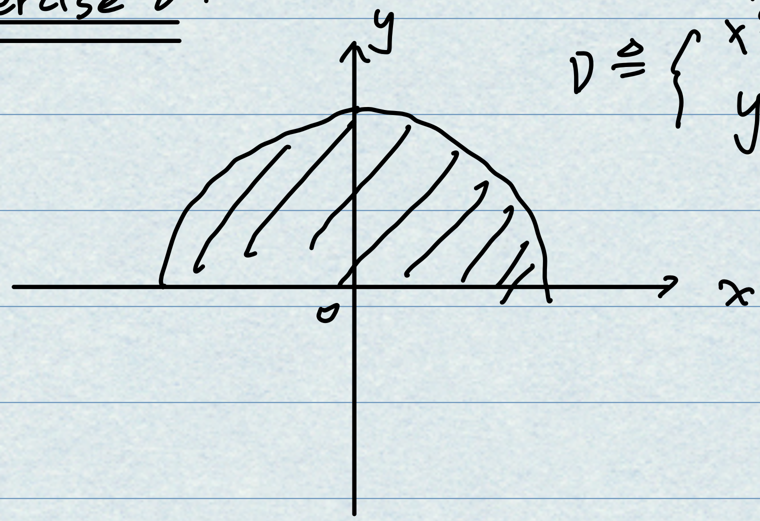
$$= \frac{1}{p}+\frac{1}{q}=1$$

$\square$

更一般地, 有 $\sum_{i=1}^m \frac{1}{p_i}=1$

$$\iint_\Omega u_1 u_2 \cdots u_m \, dx\, dy \le \|u_1\|_{p_1} \cdots \|u_m\|_{p_m}$$

Exercise 8:



$$D \triangleq \begin{cases} x^2+y^2 \leq R^2 \\ y \geq 0 \end{cases}$$

求 D 的质心. (x_0, y_0)

Pf: 由对称性 $x_0 = 0$

① $S = \frac{1}{2}\pi R^2$

② $\iint_D y \, dxdy = \int_0^{\pi} \int_0^R r\sin\theta \, r\,dr\,d\theta$

$$= \frac{2R^3}{3}$$

$$y_0 = \frac{\iint_D y\,dxdy}{S} = \frac{2}{3}R^3 \cdot \frac{2}{\pi R^2} = \frac{4R}{3\pi}$$

$\boxed{\text{证}}$

三重积分. 03/04/2025 孔舒阳

(I) Framework and definition of triple integral (n-multiple integral)

① Similar to double integral both in terms of def & tech. but caculation is more compliacted.

② In many cases. we can use some properties such as symmetry to simplify the calculation

1.1. Definition.

Step 1: Discretize Ω by Ω_i. $i=1,\cdots,n$. with $|\Omega_i| \leq \eta$

Step 2: Pick any point $(x_i, y_i, z_i) \in \Omega_i$

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\,|\Omega_i| \quad (|\Omega_i| \text{ is the volume of } \Omega_i)$$

Step 3: Taking limit. Let $\eta \to 0$.

$$\iiint_\Omega f(x, y, z)\,dx\,dy\,dz = \lim_{\eta\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,|\Omega_i|.$$

(Recall the definition of Riemann integral on 1-d)

1.2. Calculating integral in 3-D rectangular coordinates

Method: From 3-d integral to iterated integral

① Fix (x,y), integrate z.

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iint_{D}\left(\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\right) dx\,dy.$$

② Fix z, integrate (x,y)

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \int_{z}\left(\iint_{D_z} f(x,y,z)\,dx\,dy\right) dz.$$

Practical steps:

Step 1: draw the area.

Step 2: choose method ① or ②

Step 3: integrate according to ① or ②.

Some principles to choose ① or ②:

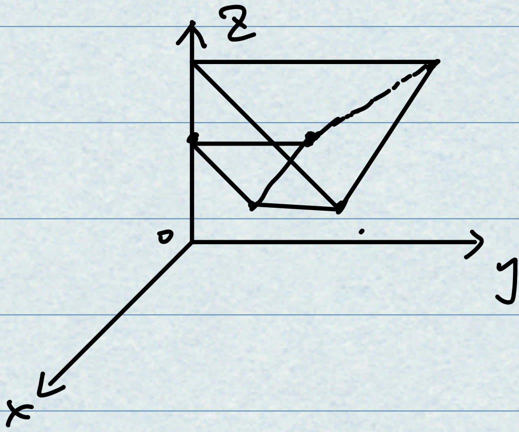
1. The form of integral area (try to avoid too many discussion)

Ex 1: $I = \iiint_{\Omega} (y^2+z^2)^{-1}\,dx\,dy\,dz$

Ω is formed by $\begin{cases} (0,0,1),\ (0,1,1),\ (1,1,1) \\ (0,0,2),\ (0,2,2),\ (2,2,2) \end{cases}$

Proof:

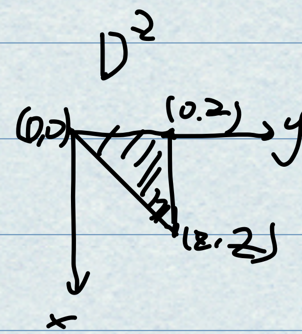
① Draw the area



parallel to xoy plane.

② fix z. $D^z(x,y)$ is the triangle by $(0,0,z)$ $(0,z,z)$ (z,z,z)

③ $$\int_1^2 \left(\iint_{D^z} (y^2+z^2)^{-1} \, dx\, dy \right) dz$$



$$\Rightarrow \iint_{D^z} (y^2+z^2)^{-1} \, dx\, dy$$

$$= \int_0^z \left(\int_0^y (y^2+z^2)^{-1} \, dx \right) dy = \int_0^z y (y^2+z^2)^{-1} \, dy$$

$$= \int_0^z (y^2+z^2)^{-1} \, d\left(\frac{y^2+z^2}{2}\right) = \frac{1}{2} \log(y^2+z^2) \Big|_0^z$$

$$= \frac{1}{2} \log(2z^2) - \log(z^2) = \frac{\log 2}{2}$$

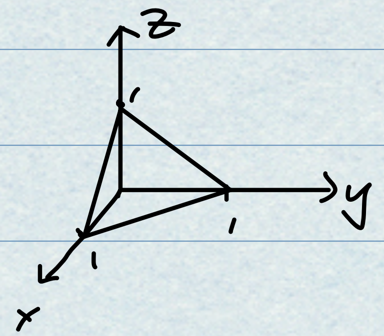
$$\Rightarrow I = \int_1^2 \frac{\log 2}{2} \, dz = \frac{\log 2}{2}$$

$\square$

2. The order of integral should take the closed form of the integral into account.

Ex 2: $I = \iiint_{\Omega} (1-y) e^{-(1-y-z)^2} \, dx\,dy\,dz$, where Ω is formed by $\begin{cases} x+y+z=1 \\ x=0 \\ y=0 \\ z=0 \end{cases}$

① Draw.



② Observation: we can't integrate y or z first. due to the Gaussian integral.

$\Rightarrow$ integrate x first.

$$I = \iint_{D(y,z)} (1-y-z)(1-y) e^{-(1-y-z)^2} \, dy\,dz$$

$$= \int_0^1 \int_0^{1-y} (1-y-z)(1-y) e^{-(1-y-z)^2} \, dz \, dy$$

$$= \int_0^1 (1-y) \int_0^{1-y} \frac{1}{-2} e^{-(1-y-z)^2} \, d\left[(1-y-z)^2\right] \, dy$$

$$= \int_0^1 (1-y) \left(\frac{1}{2} e^{-(1-y-z)^2} \Big|_0^{1-y} \right) dy$$

$$= \int_0^1 (1-y)\frac{1}{2}\left(1 - e^{-(1-y)^2}\right) dy$$

$$= -\frac{1}{4}\left((1-y)^2 + e^{-(1-y)^2}\right)\Big|_0^1 = -\frac{1}{4}\left(1 - 1 - e^{-1}\right)$$

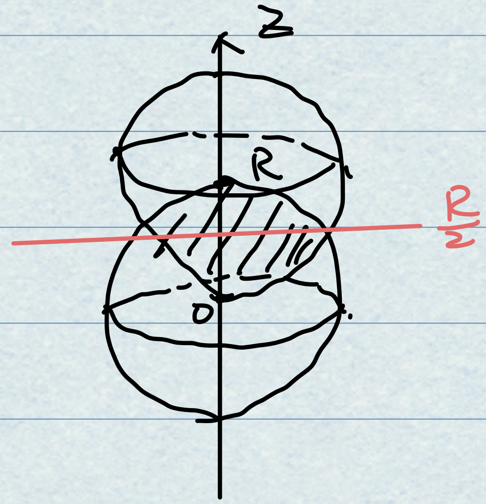
$$= \frac{1}{4e}$$

3. Some special functions. only depend on one of the $x, y\ z$ variables. If the function only depends on z, then integrate x, y first.

Ex3 $\iiint_\Omega z^2\, dx\,dy\,dz$, Ω is formed by.

$$\Omega := \begin{cases} x^2+y^2+z^2 \le R^2 \\ x^2+y^2+(z-R)^2 \le R^2 \end{cases}$$

① Draw.



details refer to Yantong Xie's notes.

Calculating by symmetry. (first consideration)

Ex 4 :

$$I = \iiint_\Omega (x+1)(y+1)\, dx\,dy\,dz$$

$$\Omega: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

proof:

$$(x+1)(y+1) = xy + y + x + 1$$

$I = I_1 + I_2 + I_3 + I_4$, where

$$I_1 = \iiint_\Omega xy \, dx\, dy\, dz$$

$$I_2 = \iiint_\Omega y \, dx\, dy\, dz$$

$$I_3 = \iiint_\Omega x \, dx\, dy\, dz$$

$$I_4 = \iiint_\Omega dx\, dy\, dz$$

By symmetry, $I_1 = I_2 = I_3 = 0$

$$I_4 = V = \frac{4}{3}\pi abc$$

1.3. change of variables.

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$

three conditions:

① bijection : $\Omega' \to \Omega$

② $x(\,)$. $y(\,)$, $z(\,)$ continuous partial derivatives

③ Jacobian: $J \triangleq \dfrac{D(x,y,z)}{D(u,v,w)} \neq 0$

Then,

$$\iiint_{\Omega} f(x,y,z)\,dV = \iiint_{\Omega'} f(x(u,v,w), y(u,v,w), z(u,v,w))\,|J|\,du\,dv\,dw.$$

Two important coordinates:

① Cylindrical coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\Rightarrow J = r$$

$$\iiint_{\Omega} f(x,y,z)\,dV = \iiint_{\Omega} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

<u>Ex 5</u> $I = \iiint_{\Omega} (x^2+y^2)^{\frac{1}{2}}\,dx\,dy\,dz$.

$$\Omega = \begin{cases} x^2+y^2 = 9 \\ x^2+y^2 = 16 \\ z = 0 \\ z = \sqrt{x^2+y^2} \end{cases}$$

<u>Pf:</u> $I = \iiint_{\Omega} r^2\,dr\,d\theta\,dz$

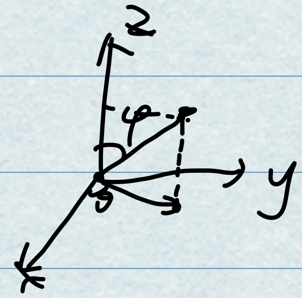
$$\Omega = \begin{cases} 3 \le r \le 4 \\ 0 \le z \le r \end{cases}$$

$$I = \int_0^{2\pi}\int_3^4 \left(\int_0^r r^2\,dz\right) dr\,d\theta = \int_0^{2\pi}\int_3^4 r^3\,dr\,d\theta$$

$$= 2\pi \cdot \frac{r^4}{4}\Big|_3^4 = \frac{175}{2}\pi$$

② Spherical coordinates

$$\begin{cases} z = \rho\cos\varphi \\ x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \end{cases}$$



$$J = \rho^2\sin(\varphi)$$

$$\Rightarrow \iiint_\Omega f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_{\Omega'} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

Ex 6. $\iiint_D y^2\,dx\,dy\,dz \qquad D: x^2+y^2+z^2 \le 2z$

pf:

Obs: $D \mapsto D'$ where $D': x^2+y^2+z^2 \le 1$

the integral is invariant

$$\Rightarrow \iiint_D \rho^2\sin^2\varphi\sin^2\theta\;\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \iiint_D \rho^4\sin^3\varphi\sin^2\theta\,d\rho\,d\varphi\,d\theta$$

$$= \left(\int_0^1 \rho^4 \, d\rho\right) \left(\int_0^{\pi} \sin^3\varphi \, d\varphi\right) \cdot \left(\int_0^{2\pi} \sin^2\theta \, d\theta\right)$$

$$= \frac{1}{5} \cdot \frac{4}{3} \cdot \pi = \frac{4}{15}\pi$$

(Recall the integral $I = \int_0^{\frac{\pi}{2}} \sin^n(\theta)\, d\theta$)

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n(\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-1}(\theta)\, d(-\cos\theta)$$

$$= \int_0^{\frac{\pi}{2}} \cos^2\theta\, (n-1) \sin^{n-2}(\theta)\, d\theta$$

$$= (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2}$$

$$\Rightarrow I_n = \begin{cases} \dfrac{(n-1)!!}{n!!} & n \text{ odd} \\[2ex] \dfrac{(n-1)!!}{n!!} \cdot \dfrac{\pi}{2} & n \text{ even} \end{cases}$$