1 Q1

(a) In the game shown in Figure 1, there are exactly two subgames. One starts from player 2's node, and the other one is the whole game graph.

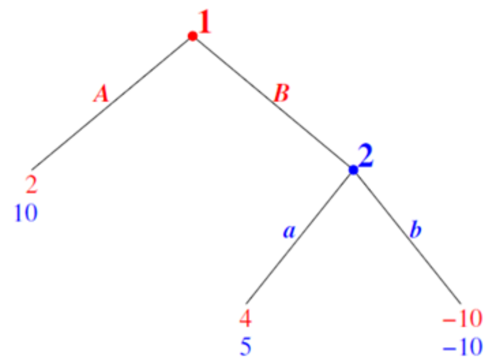

Figure 1: Example for Q1(a).

Using the 'Backward Induction' method, we can easily see that in subgame 1, player 2's must always choose strategy a to maximize its payoff in this subgame, i.e., the payoff of choosing a is 5 which is far greater than choosing b with payoff -10.

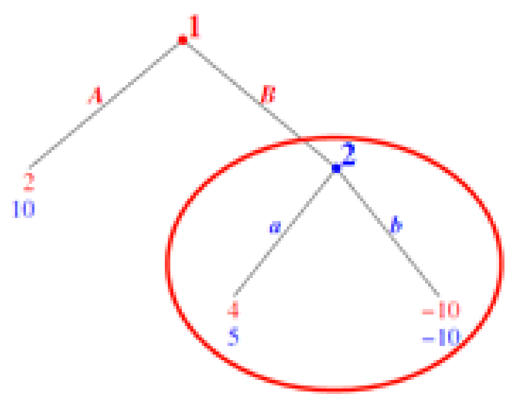

Figure 2: Subgame 1 of game G.

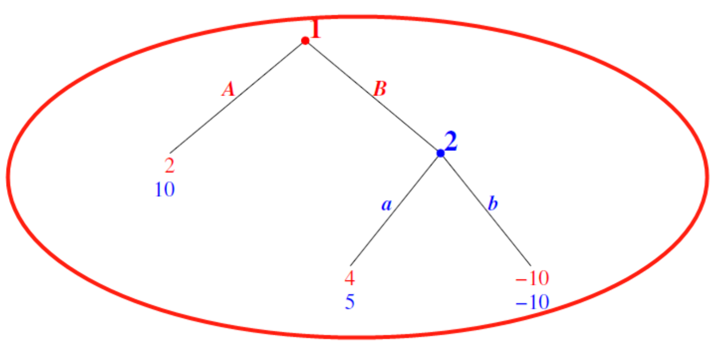

Figure 3: Subgame 2 of game G.

There are two subgames in game G, as shown in Figure 2 and Figure 3. Then, following the 'Backward Induction', we substitute player 2's node to a leaf node with payoff (4, 5). Then, we can find the SPNE of the game, that is player 1 plays B and player 2 plays a, i.e., (B, a) .

player 1/player 2	a	b
A	(2,10)	(2,10)
B	(4,5)	(-10,-10)

Table 1: The NEs in game G.

When we look back to the NE of this game, we can see that there are two NE in this game (A,b) and (B, a). Under both of these situations, neither player 1 nor player 2 can improve their payoff by unilaterally changing their strategy.

The first pure NE is (A,b). This is where player 1 plays A and player 2 plays b, we can see this as a pure Nash equilibrium since neither player can deviate from their strategy to increase their payoff.

The second pure NE is (B, a). This is where player 1 plays B and player 2 plays a, we can see this as a pure Nash equilibrium since neither player can deviate from their strategy to increase their payoff.

As a result, we can find out that (A, b) is NE but it is not SPNE, while (B, a) is NE and it is also SPNE.

(b) We could prove this by 'Backward Induction'.

First, we need to make some assumptions. For a game G with game tree T, and for $w \in T$, define a subtree $T_w \subseteq T$, by $T_w = \{w' = ww'' \; for \; w'' \in \Sigma^*\}$. Since the tree is finite, we can just associate payoff with leaf nodes. The depth of node w in T is its length $|w|$ as a string. The depth of tree T is the maximum depth of any node in T. The depth of a game G is the depth of its game tree. And there is a pure profile $s^* = (s_1^*, ..., s_n^*)$ is a SPNE.

To prove this statement, by using induction, we will show that every subgame G_w has a unique pure strategy SPNE. Because game G with perfect information, so the whole game can be identified as a subgame. Thus, until the final subgame (the whole game tree) is shown to have a unique pure strategy SPNE, the statement will be proved.

Depth 0: At leaf node w, each player i gets its payoff $u_i(w)$, and the strategies in the SPNE s* are empty.

Depth 1: Let's prove the case when depth=1, there must be a unique SPNE. As assumptions show in the statement, all the payoff at each leaf node is unique and there are no chance nodes in the game, therefore, there must be a leaf node at every subgame with depth=1 that has a strict better payoff than others and the subgame is controlled by only one player's strategy. As a result, the player to whom the node belongs, cannot change other strategies to maximize its payoff. In fact, there must be a unique edge or action that maximizes the payoff for the player who owns the node.

Inductive step: We assume the whole game tree's maximum depth is k+1. Let $Act(W) = a_1', ..., a_r'$ be the set of be the set of actions available at the root of game G_w. The subtrees $T_{wa_j'}$, for $j = 1, .., r$, each define a PI-subgame $G_{wa_j'}$, of depth $<= k$. Thus, by induction, each game $G_{wa_j'}$ has a pure strategy SPNE, $s^{wa_j'} = (s_1^{wa_j'}, ..., s_n^{wa_j'})$.

Next, we need to prove that this also works for game G_w when depth=k+1. For the root node $\epsilon \in PI_i$, which belongs to player i, where PI_i is the perfect information set for player i. Let $a' = argmax_{a \in Act(\epsilon)} h_i^{\epsilon a}(s^{\epsilon a}$ be the action that player i would take to get maximum payoff for itself, where $h_i^{\epsilon a}(s^{\epsilon a})$ is the expected payoff to player i in the subgame $G_{\epsilon a}$. As shown in Depth 0 and Depth 1, we know that each such node could be replaced by a leaf node with its maximum payoff based on the subgame's NE. Because all the nodes' payoffs are unique, so there must be unique action taken at each node, otherwise, the payoff wouldn't be the best one.

That is to say, each subgame with depth=k can be replaced by a leaf node that contains its subgame's expected payoff. And all the replaced node's payoffs are unique, therefore, the game with depth k+1 must have a unique choice among all the replaced nodes that maximize the root node's player i's payoff.

Therefore, we can define s_i^* be the union of player i's pure strategies in each subgame based on the subgame's NE. Composing all player's pure strategies together, we can get a unique SPNE $s^* = (s_1^*, ..., s_n^*)$ for game G. By using induction of showing the statement is true in Depth=0 and Depth=1, given the assumption in Depth=k, we successfully make our statement holds in Depth=k+1, thus, the statement 'every finite extensive game of perfect information where there are no chance nodes and where no player gets the same payoff at any two distinct leaves, must have a unique pure-strategy SPNE' is true.

(c) As shown in Figure.4, we can see the example for this sub-question. If we consider the normal form of this game, we will get a payoff matrix, shown in the following Table 3.

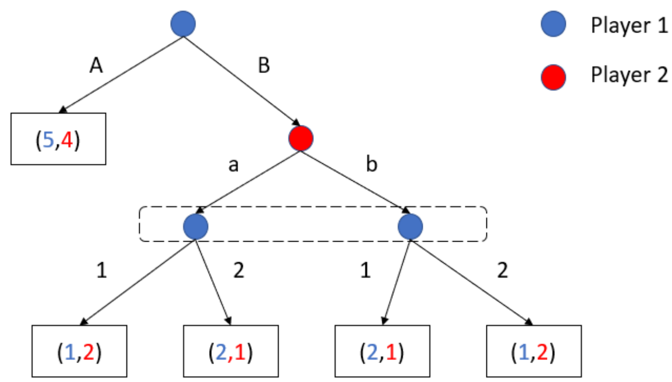

Figure 4: Example for Q1(c).

player 1/player 2	a	b
A1	**(5,4)**	**(5,4)**
A2	**(5,4)**	**(5,4)**
B1	(1,2)	(2,1)
B2	(2,1)	(1,2)

Table 2: The payoff matrix in game G.

We can see that we have four pure NEs in this game, namely $\{(A1, a), (A1, b), (A2, a), (A2, b)\}$, and no player can improve their payoff by unilaterally changing their strategy.

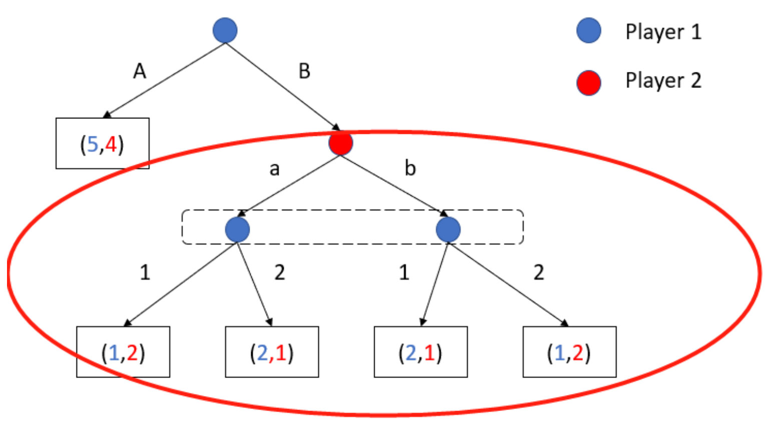

Figure 5: One subgame of game G..

For SPNE, we can use 'Backward Induction' again. One subgame is shown in Figure.5. Below is the payoff matrix of this subgame:

player 1/player 2	a	b
1	(1,2)	(2,1)
2	(2,1)	(1,2)

Table 3: The payoff matrix in one subgame of game G.

However, this subgame has no pure NE, since each pure strategy profile's payoff can be improved by one of the players unilaterally changing its strategy.

Because this subgame has no pure NE, the whole game wouldn't have a pure SPNE that covers all the subgames in the game. Thus, this example is the game that contains a pure NE but does not contain any subgame perfect pure Nash Equilibrium.