

## 2 Q2

(a) The answer is yes.

An imperfect information EFG has perfect recall if each player  $i$  never "forgets" its sequence of prior actions and information sets, i.e., an EFG has perfect recall if whenever  $w, w' \in Info_{i,j}$  belong to the same information set, then the "visible history" for player  $i$  (sequence of information sets and actions of player  $i$  during the play) before hitting node  $w$  and  $w'$  must be the same.

We can easily see from the game that it satisfies perfect recall. For each player  $i$  at node  $w$  after action took in the previous node  $x$ , all the nodes at the same information set which contains  $w$  also have the same "visible history". Those nodes that are in the information set with only one node do satisfy this requirement because the "visible histories" of one node and itself are the same. For the nodes of player 2 in the only information set with more than one node, we can see that the "visible history" of both nodes are  $\{\emptyset\}$ , so the nodes in this information set also satisfy the requirement.

After checking all nodes in the game tree, we can say that this game satisfies "perfect recall".

(b) The trivial subgames are shown in Figure 6. Apart from the game itself, there are 5 trivial subgames. Because the nodes for player 2 after player 1 plays M or R belong to the same information set, we cannot get the subgame rooted at these two nodes. Thus, the whole game contains 6 subgames. To find all SPNEs in this game, we plan to use the 'Backward Induction' algorithm.

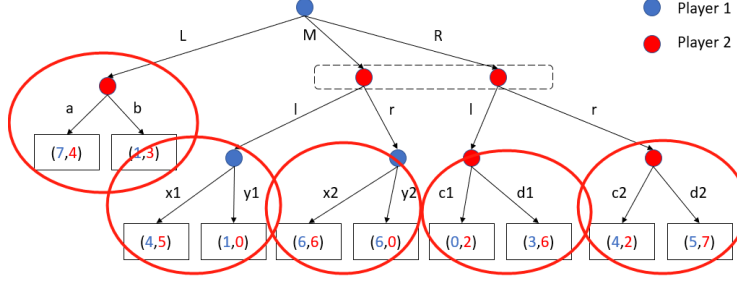


Figure 6: The trivial subgames in this extensive form game are highlighted in red circles.

First, we need to find all NEs in the subgames. We simply name the trivial subgames from left to right to subgame1, subgame2,..., subgame5.

- Subgame1: In this subgame, player 2 has two actions:  $a$  and  $b$ . Player 2's aim is to choose one action that maximizes its payoff in this subgame. Choosing action  $a$  gets payoff 4, while choosing action  $b$  gets payoff 3. Because  $3 < 4$ , player 2 will always choose to play action  $a$  with expected payoff (7,4).
- Subgame2: In this subgame, player 1 has two actions:  $x_1$  and  $y_1$ . Player 1's aim is to choose one action that maximizes its payoff in this subgame. Choosing action  $x_1$  gets payoff 4, while choosing action  $y_1$  gets payoff 1. Because  $4 > 1$ , player 1 will always choose to play action  $x_1$  with expected payoff (4,5).
- Subgame3: In this subgame, player 1 has two actions:  $x_2$  and  $y_2$ . Player 1's aim is choosing one action that maximizes its payoff in this subgame. Choosing action  $x_2$  gets payoff 6, while choosing action  $y_2$  gets payoff 6. Because  $6 = 6$ , so player 1 can either choose action  $x_2$  or  $y_2$  with expected payoffs (6,6) or (6,0) respectively.
- Subgame4: In this subgame, player 2 has two actions:  $c_1$  and  $d_1$ . Player 2's aim is choosing one action that maximizes its payoff in this subgame. Choosing action  $c_1$  gets payoff 2, while choosing action  $d_1$  gets payoff 6. Because  $2 < 6$ , so player 2 will always chooses to play action  $d_1$  with expected payoff (3,6).
- Subgame5: In this subgame, player 2 has two actions:  $c_2$  and  $d_2$ . Player 2's aim is choosing one action that maximizes its payoff in this subgame. Choosing action  $c_2$  gets payoff 2, while choosing action  $d_2$  gets payoff 7. Because  $2 < 7$ , so player 2 will always chooses to play action  $d_2$  with expected payoff (5,7).

After getting all NEs in these 5 trivial subgames, we can replace the subtrees of these subgames with leaf nodes that contain the expected payoffs. Since there are two possible actions in subgame3, we then have two versions of the new extensive form game, which are shown in Figure 7 and Figure 8.

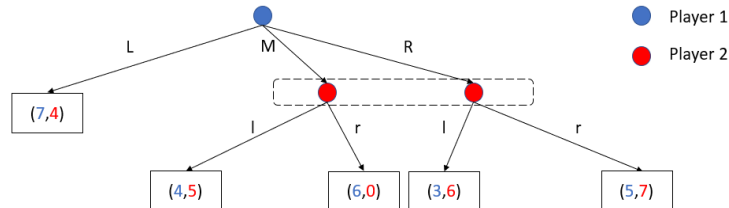


Figure 7: The reduced game  $G'$  when player 1 in subgame3 choose  $y_2$ .

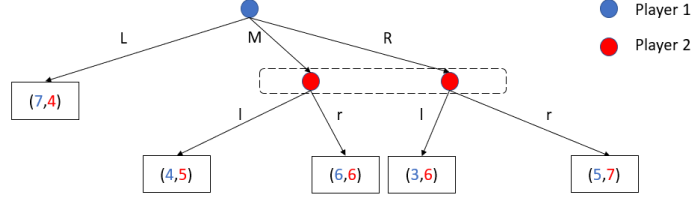


Figure 8: The reduced game  $G''$  when player 1 in subgame3 choose  $x_2$ .

In both two new games  $G'$  and  $G''$ , the only subgame is the game itself, which means that the NE of this new game is also the SPNE of the game. We plan to analyze these two situation one by one.

For situation shows in Figure 7, the expected payoff matrix can be described as Table 4. We can easily find the pure NE in this game  $G'$ , namely  $\{(L,l), (L,r)\}$ , since each player cannot increase their payoffs by unilaterally changing their strategies.

player 1/player 2	l	r
L	(7,4)	(7,4)
M	(4,5)	(6,0)
R	(3,6)	(5,7)

Table 4: The payoff matrix in reduced game  $G'$ .

For situation shows in Figure 8, the expected payoff matrix can be described as Table 5. We can easily find the pure NE in this game  $G'$ , namely  $\{(L,l), (L,r)\}$ , since each player cannot increase their payoffs by unilaterally changing their strategies.

player 1/player 2	l	r
L	(7,4)	(7,4)
M	(4,5)	(6,6)
R	(3,6)	(5,7)

Table 5: The payoff matrix in reduced game  $G''$ .

The 4 pure SPNEs we have found so far are  $\{(Lx_1x_2, ald_1d_2), (Lx_1x_2, ar d_1d_2), (Lx_1y_2, ald_1d_2), (Lx_1y_2, ar d_1d_2)\}$

Next, we need to be concerned about whether there are mixed SPNEs in the game. For subgame1, subgame2, subgame4, and subgame5, it is clear that there are no mixed NEs because there is only one player's choice and the payoff of each choice is unique. But for subgame3, the leaf nodes' payoffs are both 6 with player 1. We define a  $p \in [0, 1]$  that represents the probability of player 1 choosing action  $x_2$ , so the probability of player 1 choosing action  $y_2$  is  $(1 - p)$ . Based on the 'Backward Induction' algorithm and useful corollary for Nash Equilibria, the expected payoff of the leaf node that replaces the subgame3 would be  $(6, 6p)$ . And the new updated game  $G'''$  is shown in Figure 9. The corresponding payoff matrix in new game  $G'''$  is shown in Table 6.

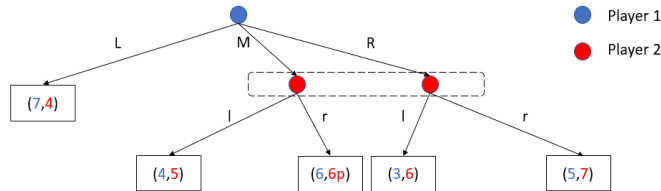


Figure 9: The reduced game  $G'''$  when player 1 in subgame3 choose  $x_2$  with  $p$  and  $y_2$  with  $(1 - p)$ .

player 1/player 2	l	r
L	<b>(7,4)</b>	<b>(7,4)</b>
M	(4,5)	(6,6p)
R	(3,6)	(5,7)

Table 6: The payoff matrix in reduced game  $G''$ .

We can see that in this payoff matrix, the small changes in subgame3's best payoff don't make any difference in the new game  $G''$ . Player 1's pure strategy L strictly dominates other pure strategies. Based on the assumption that player 1 will always choose L, both l and r are player 2's best response to it. We can use the corollary of the Nash Equilibrium theorem, i.e., if player 2 is playing against player 1's mixed strategy, both of player 2's pure strategies must be the best response to player 1. We define  $q \in [0, 1]$  as the probability of player 2 choosing l, and the probability of player 2 choosing r is  $(1 - q)$ . The mixed NE of game  $G''$  is  $\{(Lx_1(x_2^p, y_2^{(1-p)}), a(l^q, r^{(1-q)})d_1d_2)\}$ , where  $(x_2^p, y_2^{(1-p)})$  means the mixed strategy that player 1 has probability  $p$  to choose strategy  $x_2$  and probability  $(1 - p)$  to choose strategy  $y_2$ , the same to  $(l^q, r^{(1-q)})$ . If we set the range of  $p$  and  $q$  to  $[0, 1]$ , our expression will also contain the pure SPNEs.

As a result, all SPNEs are given by:

- $(Lx_1x_2, ald_1d_2)$ , in behavior strategy: Player 1:  $((1, 0, 0), (1, 0), (1, 0))$ ; Player 2:  $((1,0), (1,0), (0,1), (0,1))$
- $(Lx_1x_2, ard_1d_2)$ , in behavior strategy: Player 1:  $((1, 0, 0), (1, 0), (1, 0))$ ; Player 2:  $((1,0), (0,1), (0,1), (0,1))$
- $((Lx_1y_2), (ald_1d_2))$ , in behavior strategy: Player 1:  $((1, 0, 0), (1, 0), (0, 1))$ ; Player 2:  $((1,0), (1,0), (0,1), (0,1))$
- $(Lx_1y_2, ard_1d_2)$ , in behavior strategy: Player 1:  $((1, 0, 0), (1, 0), (0, 1))$ ; Player 2:  $((1,0), (0,1), (0,1), (0,1))$
- $(Lx_1(x_2^p, y_2^{(1-p)}), a(l^q, r^{(1-q)})d_1d_2)$ , in behavior strategy: Player 1:  $((1, 0, 0), (1, 0), (p, 1-p))$ ; Player 2:  $((1,0), (q,1-q), (0,1), (0,1))$ , where  $p \in [0, 1]$  and  $q \in [0, 1]$ .

Why these are all SPNEs in this game?

Following the 'Backward Induction' algorithm, we can get the NEs in those trivial subgames, four of which are fixed. Only subgame3 has two possible actions to choose from. Then, the reduced game doesn't have trivial subgames, thus the NEs in the reduced game is also fixed. A profile of behavior strategies can be named SPNE if defines a Nash equilibrium for every subgame in this game. And the subgames in this game contain those 5 trivial subgames and the whole game (or the reduced game). By composing the NEs of those subgames, we can easily prove that SPNEs we calculated are all SPNEs in this game.

(c) The answer is yes.

As shown in Figure 10, we can see that the pure SPNEs we find at the Q2(b) are in purple and highlighted by the blue box. But the NEs of this game are highlighted in the red box, especially, those profiles are NEs but no SPNEs are in red, any of which could be shown as examples to prove that this game contains NEs, other than the SPNEs identified above.

player1\player2	gdc1c2	gdc1d2	ghd1c2	ghd1d2	gpc1c2	gpc1d2	gpd1c2	gpd1d2	hdc1c2	hdc1d2	hdd1c2	hdd1d2	hpc1c2	hpc1d2	hpd1c2	hpd1d2
Lx1x2	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)
Lx1y2	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)
Ly1x2	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)
Ly1y2	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(7,4)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)
Mx1x2	(4,5)	(4,5)	(4,5)	(4,5)	(6,6)	(6,6)	(6,6)	(6,6)	(4,5)	(4,5)	(4,5)	(4,5)	(6,6)	(6,6)	(6,6)	(6,6)
Mx1y2	(4,5)	(4,5)	(4,5)	(4,5)	(6,6)	(6,6)	(6,6)	(6,6)	(4,5)	(4,5)	(4,5)	(4,5)	(6,6)	(6,6)	(6,6)	(6,6)
My1x2	(1,0)	(1,0)	(1,0)	(1,0)	(6,6)	(6,6)	(6,6)	(6,6)	(1,0)	(1,0)	(1,0)	(1,0)	(6,6)	(6,6)	(6,6)	(6,6)
My1y2	(1,0)	(1,0)	(1,0)	(1,0)	(6,6)	(6,6)	(6,6)	(6,6)	(1,0)	(1,0)	(1,0)	(1,0)	(6,6)	(6,6)	(6,6)	(6,6)
Rx1x2	(0,2)	(0,2)	(3,6)	(3,6)	(4,2)	(5,7)	(4,2)	(5,7)	(0,2)	(0,2)	(3,6)	(3,6)	(4,2)	(5,7)	(4,2)	(5,7)
Rx1y2	(0,2)	(0,2)	(3,6)	(3,6)	(4,2)	(5,7)	(4,2)	(5,7)	(0,2)	(0,2)	(3,6)	(3,6)	(4,2)	(5,7)	(4,2)	(5,7)
Ry1x2	(0,2)	(0,2)	(3,6)	(3,6)	(4,2)	(5,7)	(4,2)	(5,7)	(0,2)	(0,2)	(3,6)	(3,6)	(4,2)	(5,7)	(4,2)	(5,7)
Ry1y2	(0,2)	(0,2)	(3,6)	(3,6)	(4,2)	(5,7)	(4,2)	(5,7)	(0,2)	(0,2)	(3,6)	(3,6)	(4,2)	(5,7)	(4,2)	(5,7)

Figure 10: The payoff matrix of whole extensive form game.

For example, in the profile  $(Mx_1x_2, brd_1d_2)$ , player 1 cannot improve its current payoff 6 by unilaterally changing its strategy, neither player 2 can improve its current payoff 6 by unilaterally changing its strategy.

(d) The answer is no.

As we illustrated above, there are 4 pure SPNEs in this game. We take SPNE  $(Lx_1x_2, ard_1d_2)$  and NE  $(Mx_1x_2, brd_1d_2)$  as examples. Because NE  $(Mx_1x_2, brd_1d_2)$ , which is not SPNE, player 2 could make a non-credible threat at node we highlighted in Figure 11 to choose b. Player 2 threat player 1 that if player 1 choose L then player 2 will choose b to make player get payoff 1 instead of the maximum payoff 7. If player 1 believe this threat, then player 1 would choose M instead to get payoff 6, which at least is better than 1 by taking action L. Player 2 has the possibility to increase its own payoff from 4 to 6 by threatening if player 1 also takes action x2, so player 2 has a reasonable incentive to make non-credible threats.

Thus, this game exists some NEs that have a non-credible threat and the statement that all the equilibria in this game are “credible” is wrong.

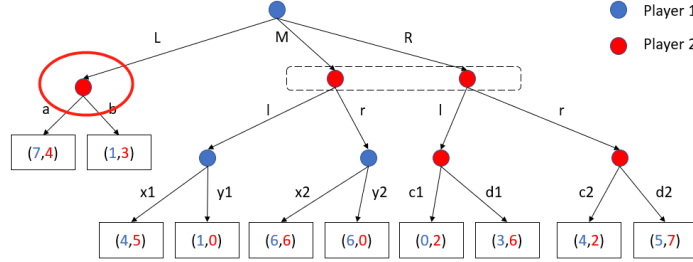


Figure 11: The example of Q2(d).