UNIVERSITY OF EDINBURGH

COLLEGE OF SCIENCE AND ENGINEERING

SCHOOL OF INFORMATICS

INFR11020 ALGORITHMIC GAME THEORY AND ITS APPLICATIONS

Wednesday 4 th May 2022

13:00 to 15:00

INSTRUCTIONS TO CANDIDATES

Answer any TWO of the three questions. If more than two questions are answered, only QUESTION 1 and QUESTION 2 will be marked.

All questions carry equal weight.

This is an OPEN BOOK examination.

Year 4 Courses

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External Examiners: J.Bowles, A.Pocklington, H.Vandierendonck

THIS EXAMINATION WILL BE MARKED ANONYMOUSLY

1. (a) Consider the following 2-player normal form (bimatrix) game:

$$\begin{bmatrix} (2,5) & (5,4) & (7,6) & (8,5) \\ (3,3) & (5,2) & (6,2) & (9,2) \\ (2,4) & (5,2) & (6,4) & (8,3) \\ (2,9) & (4,8) & (6,8) & (10,7) \end{bmatrix}$$

Identify all Nash equilibria in this game (pure or mixed). Argue why there are no other Nash equilibria, other than the ones you have identified. [*9 marks*]

(b) Recall that Rock-Paper-Scissors is the two player zero-sum game defined by the following payoff matrix for player 1 (the row player):

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Suppose you know for sure that your opponent in Rock-Paper-Scissors will never play Rock, or even place any positive probability on playing Rock. You, on the other hand, are free to place any probabilities you wish on Rock, Paper, or Scissors. This corresponds to a modified zero-sum game whose payoff matrix for player 1 (you, the row player) is obtained by erasing the "Rock" column of the payoff matrix for Rock-Paper-Scissors.

What is an optimal (minmaximizer) strategy for you (the row player), and what is the minimax value (i.e., the highest expected payoff you can guarantee for yourself, no matter what the other player does), in this modified game? [*6 marks*]

(c) Consider any finite 2-player game, G, with pure strategy sets S_1 and S_2 for the two players, such that the payoff functions, $u_1(s_1, s_2)$ and $u_2(s_1, s_2)$ for the two players are given by:

$$u_1(s_1, s_2) := f(s_1, s_2) + g(s_1)$$

$$u_2(s_1, s_2) := -f(s_1, s_2) + h(s_2)$$

Here $f(s_1, s_2)$ is any real-valued function of $s_1 \in S_1$ and $s_2 \in S_2$, whereas $g(s_1)$ (and $h(s_2)$, respectively) is a real-valued function of s_1 (respectively, s_2) only. Note that this is not in general a zero-sum game (not quite).

Prove that if (x_1^*, x_2^*) is a mixed Nash equilibrium of this game, and (x_1', x_2') is another mixed Nash equilibrium of this game, then (x_1', x_2^*) is also a mixed Nash equilibrium of this game. [*10 marks*]

2. (a)

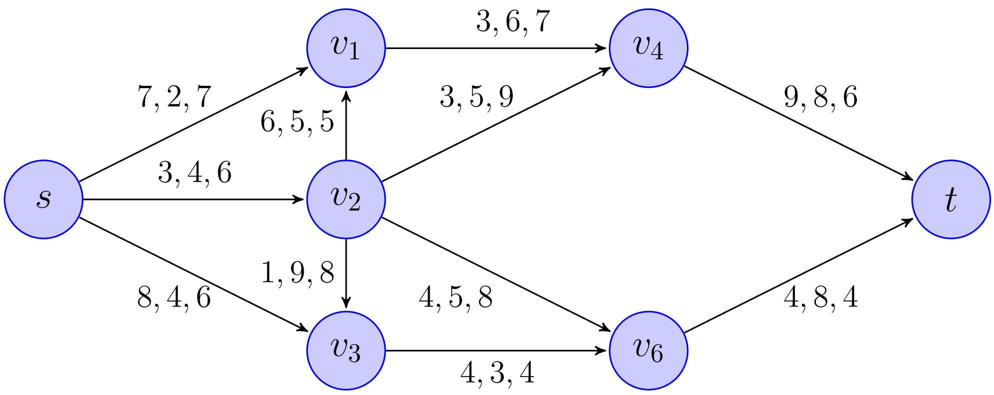

Figure 1: An atomic network congestion game

Consider the *atomic network congestion game*, with three players, described by the directed graph in Figure 1. In this game, every player i (for $i = 1, 2, 3$) needs to choose a directed path from the source s to the target t. Thus, every player i's set of possible actions (i.e., its set of pure strategies) is the set of all possible directed paths from s to t. Each edge e is labeled with a sequence of three numbers (c_1, c_2, c_3). Given a profile $\pi = (\pi_1, \pi_2, \pi_3)$ of pure strategies (i.e., s-t-paths) for all three players, the *cost* to player i of each directed edge, e, that is contained in player i's path π_i, is c_k, where k is the total number of players that have chosen edge e in their path. The total cost to player i, in the given profile π, is the sum of the costs of *all* the edges in its path π_i from s to t. Each player wants to minimize its own total cost.

Compute a pure strategy Nash Equilibrium in this atomic network congestion game. Compute the total cost to each player in that Nash Equilibrium. [*7 marks*]

(b) Consider the following linear programming problem, given as a "feasible dictionary", with "basis" $\{x_1, x_2\}$:

Maximize: $3 + 2x_3 + 5x_4$

Subject to:

$$x_1 = 4 - 4x_3 - 5x_4$$
$$x_2 = 8 - 6x_3 - 3x_4$$

$$x_1 \geq 0\ ,\ x_2 \geq 0\ ,\ x_3 \geq 0\ ,\ x_4 \geq 0$$

i. What is the "basic feasible solution" (BFS) corresponding to the basis $\{x_1, x_2\}$ of this feasible dictionary? [*2 marks*]

ii. Apply one "pivot" step (of the simplex algorithm) to this feasible dictionary, in order to move the variable x_3 into the basis and move the variable x_1 out of the basis.

Show the resulting dictionary after this pivot step, including the resulting new objective function. [*8 marks*]

iii. Is the new dictionary obtained via this pivot step a *feasible* dictionary? Explain your answer. [*2 marks*]

(c) Consider the following optimization problem. Suppose that you are given an $(m \times n)$ integer matrix, A, and an integer m-vector, b, and you wish to find a real-valued n-vector x, such that $Ax = b$ (if one exists), such that the objective $\sum_{i=1}^{n} |x_i|$ is minimized. Here $|x_i|$ denotes the absolute value of x_i. Show how to re-phrase this as a linear programming problem, and explain briefly why your formulation as a linear program is correct. [*6 marks*]

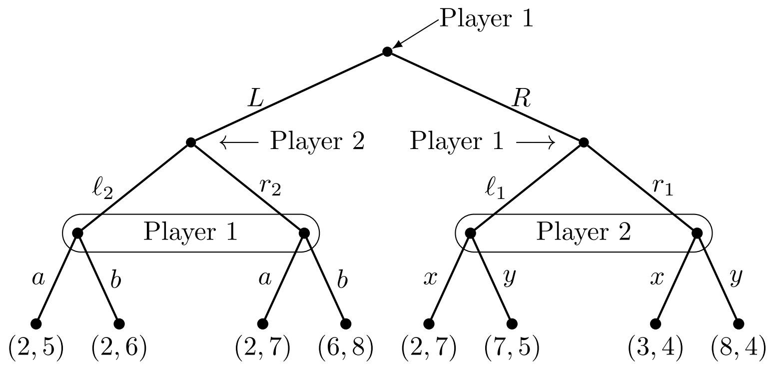

Figure 2: Question 3(a) and 3(b)

3. (a) Consider the 2-player extensive form game (of imperfect information) depicted in Figure 2. (At leaves, the left payoff is for player 1, and the right payoff is for player 2.) Describe all subgame-perfect Nash Equilibria in this game, in terms of behavior strategies. [*10 marks*]

(b) In a VCG-based auction, three distinct items are all being auctioned simultaneously. Let us denote the set of these three items as $E = \{1, 2, 3\}$. Suppose there are three potential buyers (bidders), A, B, and C, who provide their claimed valuation functions v_A, v_B, and v_C for this auction. For each subset $E' \subseteq E$ of items, $v_x(E')$ denotes the valuation in (millions of) British pounds that player x has for obtaining the subset E' of items:

	valuation							
bidder x	$v_x(\{\})$	$v_x(\{1\})$	$v_x(\{2\})$	$v_x(\{3\})$	$v_x(\{1,2\})$	$v_x(\{1,3\})$	$v_x(\{2,3\})$	$v_x(\{1,2,3\})$
$x := A$	0	5	1	6	12	15	13	16
$x := B$	0	3	4	6	18	17	17	22
$x := C$	0	9	7	5	17	19	18	20

A VCG allocation for this auction is specified by a partition of the items $\{1, 2, 3\}$ into three disjoint subsets $E_A, E_B, E_C \subseteq \{1, 2, 3\}$, such that E_A (E_B and E_C) is the set of items allocated to bidder A (respectively, to bidder B and to bidder C). Note that some of these sets may be empty. Each bidder will also be asked to pay a certain amount (in millions of British pounds), p_A, p_B, and p_C, respectively, for their allocation, by the VCG mechanism.

(You can assume the bidders will bid their true valuation functions, because the VCG mechanism is incentive compatible.)

Describe a VCG allocation, and corresponding VCG payments, for this auction. In other words, in that VCG allocation which items will each bidder get, and what price will each pay for the items they get, if the VCG mechanism is used? [*9 marks*]

(c) Consider a Bayesian game where player 1 (the row player) has 2 possible types, **I** and **II**, whereas player 2 (the column player) has only one possible type, **J**. There is a joint common prior probability distribution p over types given by $p(\mathbf{I}, \mathbf{J}) = p(\mathbf{II}, \mathbf{J}) = 1/2$. In other words, player 1 is equally likely to have each of its two types (and player 2 of course always has the same type **J** with probability 1), and both players know this common prior probability distribution on types.

Note that after the type of each player has been sampled from the distribution p, player 1 gets to find out its own type, and of course both players also know player 2's unique type **J**, but player 2 doesn't get to find out player 1's type (it only knows that both types for player 1 have equally probability 1/2).

Player 1 has two actions, A and B, and player 2 has two actions, C and D. The payoff bimatrix in the case where player 1 has type **I** is given by:

	C	D
A	$(2, 1)$	$(2, 0)$
B	$(0, 2)$	$(0, 0)$

On the other hand, the payoff bimatrix in the case where player 1 has type **II** is given by:

	C	D
A	$(1, 4)$	$(2, 1)$
B	$(2, 6)$	$(3, 2)$

Find a Bayesian Nash equilibrium (BNE) in this game, and say what the expected payoff to both players is under that BNE. [*6 marks*]