Algorithmic Game Theory and Applications: Sample Solutions for Coursework 2

1. Recall that a Nash equilibrium in an extensive game is *subgame perfect nash equilibrium* (SPNE) if it is also a Nash equilibrium in every *subgame* of the original game. Formally, a *subgame*, is a game defined by a subtree, T_v of the original game tree, T, such that the subtree T_v, rooted at a node v, has the property that for every decendent u of v in the game tree (including v itself), every node in the same information set as u is also in the subtree T_v.

   (a) [14 points] Give an example of a pure NE which is *not* a SPNE, for a finite extensive form game of perfect information.

   Solution: In the 2-player finite extensive form game of perfect information depicted in Figure 1, the pure strategy profile (B, a), where player 1 plays action B, and player 2 plays action a, both with probability 1, is a pure Nash Equilibrium, but is not a subgame-perfect pure Nash equilibrium, because in the subgame rooted at the only node controlled by player 2, the only Nash equilibrium is for player 2 to choose b. ■

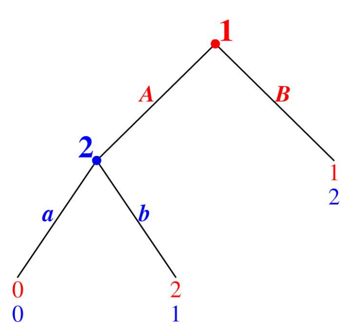

Figure 1: Exercise 1(a): (B, a) is an NE of this game, but it is not a SPNE

(b) [20 points] Show that every finite extensive game of perfect information where *there are no chance nodes* and where no player has the same payoffs at any two distinct terminal nodes has a <u>unique</u> pure-strategy SPNE.

Solution: Consider the proof of Kuhn's theorem, which shows the existence of a pure SPNE for any finite extensive form game of perfect information. Recall that the proof also yields an algorithm for computing a pure SPNE, by "backward induction".

Assuming that there are no chance nodes, and no player has the same payoffs at any two distinct terminal nodes, we wish to show that there must be a *unique* pure SPNE. Suppose $s = (s_1, \ldots, s_n)$ and $s' = (s'_1, \ldots, s'_n)$ are two different pure SPNEs for such a game. Let v be a node of the game tree belonging to some player i such that player i's choices of actions in s and s' differ. i.e., such that $s_i(v) \neq s'_i(v)$, and such that v is as close as possible to the leaves of the game tree, so that no descendants v' of v, belonging to some player j, satisfies $s_j(v') \neq s_j(v)$.

Thus, since there are no chance nodes in the game tree, the strategies s and s' in the subgame rooted at v yield distinct plays π_s and $\pi_{s'}$, ending at distinct leaves, and since the leaves all yield distinct payoffs to player i, without loss of generality $u_i(\pi_s) > u_i(\pi_{s'})$. But then s' is not an SPNE, because it would improve player i' payoff in the subgame rooted at node v to unilaterally switch its strategy in s', letting $s'_i(v) := s_i(v)$ (and keeping everything else the same). This contradicts the assumption that s and s' are two distinct SPNEs, so there must be a unique SPNE. ■

(c) [16 points] Give an example of a finite extensive form game that contains a pure Nash Equilibrium but does not contain any subgame-perfect pure Nash Equilibrium. Justify your answer.

Solution: In the example EFG given in Figure 2, (BC, a) is a pure NE, but it is not a SPNE. In fact, the subgame rooted at the unique node controlled by player 2 is simply a version of the "matching pennies" game, for which the unique equilibrium is the one where both players play a $(1/2, 1/2)$ mixed strategy on their actions. Thus the only SPNE in the game depicted in Figure 2 is described as follows: player 1 plays action B at the root (with probability 1), and also plays a $1/2 - 1/2$ mix of actions C and D in its other information set. Player 2 plays a $1/2 - 1/2$ mix of

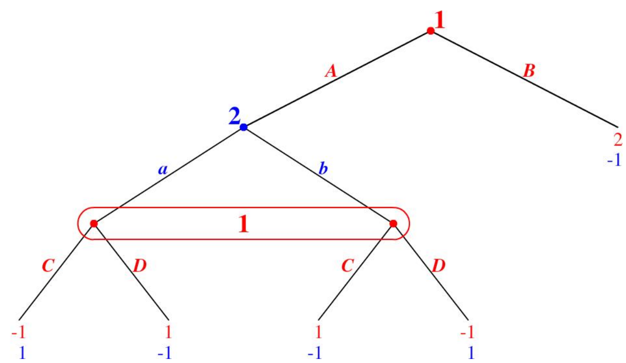

Figure 2: Exercise 1(c): a pure NE, but no pure SPNE; only a mixed SPNE

actions a and b in the only node (and the only information set) that it controls.

2. Consider the finite extensive form game of imperfect information depicted in Figure 3.

   (a) [6 points] Does this game satisfy "perfect recall"? Explain your answer.

   Solution:

   Yes. This game satisfies perfect recall. The reason is as follows: the only non-trivial information set (i.e., the only information set containing more than one node of the game tree), is that information set belonging to player 2 which contains 2 nodes. Note that, for both nodes in that information set, the prior "visible history" for player 2, is exactly the same: namely, in reaching either node, no prior information set belonging to the same player, player 2, has been visited (and also of course, no action has been taken by player 2). In other words, the "visible history" for player 2 at both of those nodes is EMPTY. Consequently, this game satisfies perfect recall.

   The following observations are not needed in the solution itself, but inform the subsequent questions:

   Hence, note that, as mentioned in lectures (due to a result of Kuhn [1953]), in such an extensive form game of perfect recall, it suffices to consider only "behavior strategies" (because any

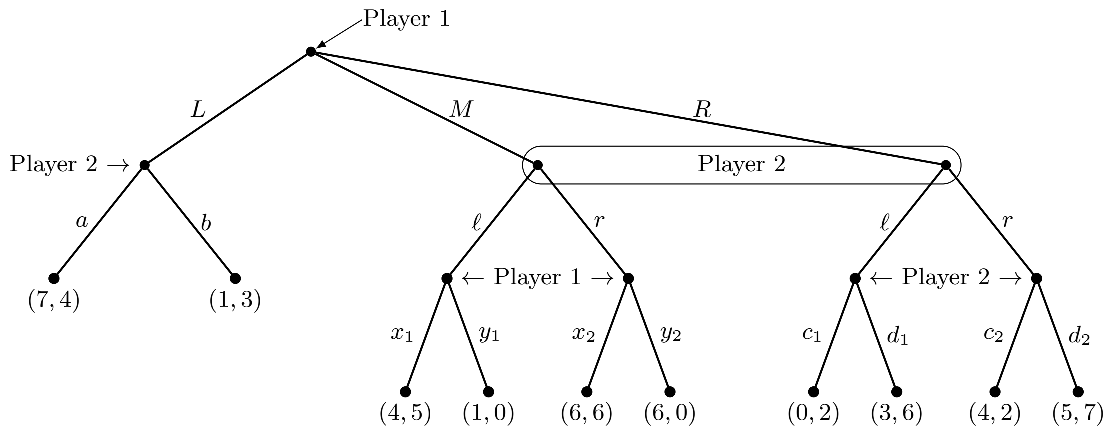

Figure 3: Question 2

possible distribution on the nodes, and hence on the "leaves", of the game tree that can be induced by any "mixed" strategies for any player can be induced by "behavior strategies").

Recall that a "behavior strategy" is a strategy in which each player chooses, for each information set that belongs to it, a probability distribution on the actions available at that information set. (Note that a pure strategy, which is a function that maps each information set (deterministically) to one action available at that information set is by definition a specific kind of behavior strategy.) ■

(b) [24 points] Identify all SPNEs in this game in terms of "behavior strategies". Explain why what you have identified constitutes all SPNEs.

Solution:

To identify all SPNEs, we work our way "bottom up" on the game tree, to compute NEs for the bottom-most subgames of this game tree.

In particular, let us first compute the unique NE for the subgame whose root is arrived at after the action L by player 1 at the root of the entire game tree.

- In that subgame, rooted at $\langle L \rangle$, which is a node controlled by player 2, it is clear that the unique NE is obtained by having player 2 play action a, which yields payoffs 7 and 4 to players 1 and 2, respectively.

Next, we similarly compute NEs for the subgames rooted at the node reached by each of the following action sequences:

- Subgame rooted at $\langle M, l\rangle$:
  For this subgame the root node (which is the only non-leaf node) is controlled by player 1, and it is clear that in the unique NE, player 1 plays action x_1 with probability 1, yielding payoffs 4 and 5 for players 1 and 2, respectively.
- Subgame rooted at $\langle M, r\rangle$:
  Here the root node (again, the only non-leaf node) is controlled by player 1, and it is clear that, since the (expected) payoff to player 1 is 6 regardless of whether it plays action x_2 or y_2, every probably distribution in which player 1 plays action x_2 with some probability $p \in [0, 1]$ and plays action y_2 with probability $(1 - p)$, yields a NE in this subgame, w with expected payoff 6 for player 1, and expected payoff $6 \cdot p + (1 - p) \cdot 0 = 6p$ for player 2.
- Subgame rooted at $\langle R, l\rangle$:
  Here the root node (again, the only non-leaf node) is controlled by player 2, and it is clear that in the unique NE, player 2 plays action d_1 with probability 1, yielding payoffs 3 and 6 for players 1 and 2, respectively.
- Subgame rooted at $\langle R, r\rangle$:
  Here the root node (again, the only non-leaf node) is controlled by player 2, and it is clear that in the unique NE, player 2 plays action d_2 with probability 1, yielding payoffs 5 and 7 for players 1 and 2, respectively.

Having computed all the NEs for each of these subgames (parametrized by the probability $p \in [0, 1]$ with which player 2 chooses to play action x_2 in the subgame rooted at M, r), and having computed the (expected) payoff to each player under these NEs, we can use the “bottom up” approach to “plug in” these payoffs at new “leaves” that replace these subgames. This yields a “reduced game” (for each such probability $p \in [0, 1]$), and reduces the problem of finding all SPNEs in the original game to finding all SPNEs in these reduced games (parametrized by the probability $p \in [0, 1]$), as shown in Figure 4.

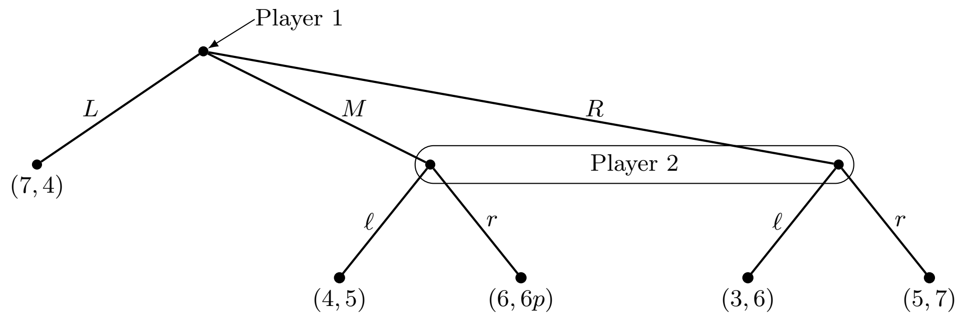

Figure 4: "Reduced game" obtained after "bottom up" computation of (unique) NEs for all the proper subgames of the original game. Note that this reduced game contains no proper subgames. In it, the expected payoff to player 2 at the leaf M, r is given by $6p$, parametrized by the probability $p \in [0, 1]$, with which player 1 plays action x_2 at the node after M, r.

Note that there are no further proper subgames in the original game (in other words, there are no other proper subtrees of the original game tree, rooted at any node of the game tree other than the original root or leaves, which fully contain all of their own information sets). Hence, as can be seen, there are no further proper subgames in the reduced game depicted in Figure 4.

Hence, any NE of this reduced game, when combined with the NE (parametrized by $p \in [0, 1]$) in the already eliminated subgames, would constitute an SPNE of the original game, and vice versa: every SPNE of the original game can be obtained by combining any NE of this reduced game with the respective NE for the eliminated subgames (which have already been computed above, parametrized by p).

Note that this reduced game is, when converted to normal form, a 3×2 game, where player 1 has three actions at the only node it controls (and hence 3 pure strategies), L, M, and R, and player 2 has two actions at the only information set it controls (and hence two pure strategies) l and r.

If we convert this subgame to normal form, it looks like this:

	l	r
L	$(7,4)$	$(7,4)$
M	$(4,5)$	$(6,6p)$
R	$(3,6)$	$(5,7)$

Notice that the pure action (strategy) L is a strictly dominant strategy for player 1: it yields payoff 7 for player 1, whereas any other action yields strictly lower payoff, irrespective of what player 2 plays.

Hence, regardless what strategy player 2 plays, player 1 is strictly better off playing L rather than any other pure or mixed strategy. Against the pure strategy L, player 2 can play any mixture of actions l and r, because irrespective what it plays it payoff will be 4.

Hence, the NEs in this reduced game are as follows: player 1 plays L with probability 1, and player 2 plays l with some probability $q \in [0, 1]$ and r with probability $(1 - q)$.

Finally, by combining all NEs in the reduced game, together with the NEs in the subgames that were removed to produce the reduced game, we obtain all SPNEs in the original extensive form game, and these can be described in terms of behavior strategies as follows.

A behavior strategy b_1 for player 1 consists of three probability distributions $b_1 = (b_{1,\varepsilon}, b_{1,Ml}, b_{1,Mr})$:

- Let $b_{1,\varepsilon}$ denote player 1's probability distribution on actions at the root of the game tree, namely its probability distribution on actions L, M, and R.
- $b_{1,Ml}$ denotes player 1's probability distribution on actions x_1 and y_1, at the node reached after the sequence of actions $\langle M, l\rangle$.
- $b_{1,Mr}$ deontes player 1's probability distribution on actions x_2 and y_2, at the node reached after the sequence of actions $\langle M, r\rangle$.

Analogously, a behavior strategy b_2 for player 2 is described by four probability distributions: $b_2 = (b_{2,L}, b_{2,\{M,R\}}, b_{1,Rl}, b_{1,Rr}))$. Here $b_{2,\{M,R\}}$ denotes player 2's probability distribution on actions l and r at the only non-trivial information set containing two nodes, reached from the root by the actions M and R, respectively. The other elements of b_2 should be clear from the analogous description for b_1.

Finally, based on the above computations, we can say that a behavior strategy profile $b' = (b'_1, b'_2)$ is a SPNE in this game if and only if, the following conditions hold:

- $b'_{1,\varepsilon}(L) = 1$ In other words, at the root, player 1 plays action L with probability 1.
- $b'_{1,Ml}(x_1) = 1$ and $b'_{1,Ml}(y_1) = 0$.
- $b'_{1,Mr}(x_2) = p$ and $b'_{1,Mr}(y_2) = (1 - p)$, for any probability $p \in [0, 1]$.
- $b'_{2,L}(a) = 1$ and $b'_{2,L}(b) = 0$.
- $b'_{2,Rl}(d_1) = 1$ and $b'_{2,Rl}(c_1) = 0$.
- $b'_{2,Rr}(d_2) = 1$ and $b'_{2,Rr}(c_2) = 0$.
- $b'_{2,\{M,R\}}(l) = q$ and $b'_{2,\{M,R\}}(l) = (1 - q)$, for any probability $q \in [0, 1]$.

As we have argued, these are all the SPNEs in this game, described in terms of behavior strategy profiles. ■

(c) [10 points] Are there any other Nash equilibria other than the SPNEs you have identified? Explain your answer.

Solution:

Yes, there are other NEs in this game, besides the SPNEs we have identified.

For example, consider the following pure NE:

- Player 1 plays actions M with probability 1, and plays action x_1 and x_2 with probability 1, at the respective information sets.
- Player 2 plays action b, action r, action d_1, and action d_2, all with probability 1, at the respective information sets.

Note that with this strategy profile the payoff to player 1 is 6 and the payoff to player 2 is 6.

We claim that this is an NE.

To see this, firstly, note that player 1 cannot unilaterally change its strategy and increase its own (expected) payoff of 6, because player 1 is already playing "optimally" from the two subgames where it took action x_1 and x_2, respectively, and moreover if it were to unilaterally switch its action at the root to L or R, it would change its payoff to 1 and 5, respectively, both of which are less that 6.

Secondly, note that player 2 cannot increase its own payoff by changing its own strategy unilaterally, because changing is action from b to a wouldn't impact its own payoff (because L is played with probability 0 by player 1), a moreover, its actions d_1 and d_2 are optimal for their respective subgames, and finally, given the stategy by player 1, action r gives player 2 strictly higher payoff than action l.

Hence, this is indeed a Nash Equilbrium, but it is not a SPNE, because player 2 is playing suboptimally in the subgame rooted at the node reached after action L.

Hence, there are certainly NEs in this game which are not SPNEs.

■

(d) [10 points] Are all of the equilibria in this game "credible"? Explain.

Solution:

This question is not phrased formally, because the notion of "credible" was not formally defined. So, the question should be interpreted informally, as asking whether there is any NE in which a player, in an information set that is reached with probability 0 under the given NE, is playing an action which constitutes a "non-credible threat" meaning that if any node in that information set were instead somehow reached with positive probability, the player would be strictly better off playing a different action at that information set.

A reasonable answer should say "No". In particular, the NE described in our solution to part (c) is not "credible" because it requires that player 2 plays action b with probability 1 in the information set rooted below action L. So, "if" the game were to reach the subgame after action L (which it doesn't under the given behaviour strategy profile, because player 1 plays action L with probability 0), then in this NE player 2 claims it would play action b. But this is a "non-credible threat" on the part of player 2, because if that subgame were actually reached (with positive probability), then player 2 would be strictly better off playing action a.

3. (a) [20 points] Recall *Rosenthal's Theorem*, namely that every finite *congestion game* has a *pure* Nash Equilibrium. In the proof we gave in the lecture slides for Rosenthal's theorem, we defined the potential function $\varphi(s)$, which for any pure strategy profile s is defined as:

$$\varphi(s) := \sum_{r \in R} \sum_{i=1}^{n_r(s)} d_r(i) \tag{1}$$

Later in the proof we claimed that $\varphi(s)$ can also be expressed as a different nested sum, but we didn't prove that fact, and instead said "check this yourself". This question asks you to prove that fact: Prove that for any pure strategy profile s the following equality holds:

$$\varphi(s) = \sum_{i=1}^{n} \sum_{r \in s_i} d_r(n_r^{(i)}(s)) \tag{2}$$

Solution:
Suppose that the set of resources R is $R = \{r_1, \ldots, r_m\}$. Note that there are n players. Consider a pure strategy profile $s = (s_1, s_2, \ldots, s_n)$.
Let us define the following $(m \times n)$ matrix, A, whose (j, i)'th entry $A_{j,i}$ is defined as follows:

$$A_{j,i} := \begin{cases} d_{r_j}(n_{r_j}^{(i)}(s)) & , \quad \text{if } r_j \in s_i \\ \\ 0 & , \quad \text{otherwise} \end{cases}$$

Next, consider two different ways of summing up all the entries of the matrix A: row-wise or column-wise. Obviously, we have:

$$\sum_{j=1}^{m} \sum_{i=1}^{n} A_{j,i} = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{j,i} \tag{3}$$

Now, first notice that by the formula defining $A_{j,i}$, it follows that, for any column $i \in \{1, \ldots, n\}$, we have:

$$\sum_{j=1}^{m} A_{j,i} = \sum_{r \in s_i} d_r(n_r^{(i)}(s))$$

Hence, the right hand side of equation (3) is equal to the formula for $\varphi(s)$ given in (2).

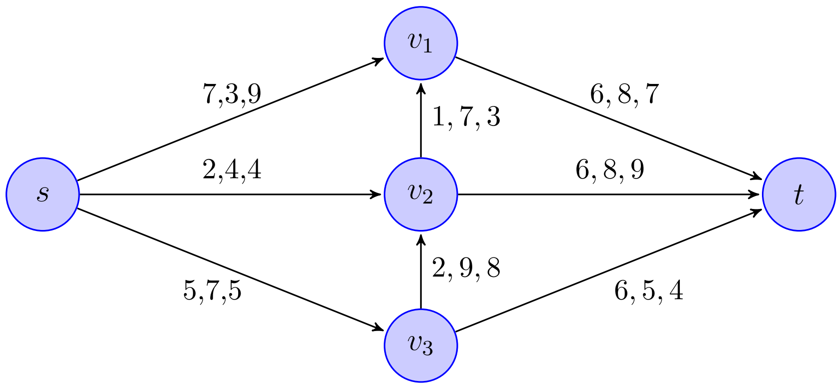

Figure 5: Question 3

So, the only thing left to do is to show that the left hand side of the equation (3) is equal to the formula defining $\varphi(s)$ in (1), in other words, to show that for any row $j \in \{1, \ldots, m\}$,

$$\sum_{i=1}^{n} A_{j,i} = \sum_{i=1}^{n_{r_j}(s)} d_{r_j}(i).$$

But this follows by considering the entries in row j of the matrix A. Consider the sequence of player indices $0 \leq i_1 < i_2 < \ldots < i_{n_{r_j}(s)} \leq n$, such that $r_j \in s_{i_k}$ for each $k \in \{1, \ldots, n_{r_j}(s)\}$. Note that for each $k \in \{1, \ldots, n_{r_j}(s)\}$, we have $A_{j,i_k} = d_{r_j}(k)$, whereas, for each $i' \in \{1, \ldots, n\}$ such that $i' \notin \{i_1, i_2, \ldots, i_{n_{r_j}(s)}\}$, we have $A_{j,i'} = 0$. Hence, the sum $\sum_{i=1}^{n} A_{j,i}$ of the entries in row j of matrix A is $\sum_{i=1}^{n_{r_j}(s)} d_{r_j}(i)$. This completes the proof.$\blacksquare$

(b) Consider the *atomic network congestion game*, with three players, described by the directed graph in Figure 5.

In this game, every player i (for $i = 1, 2, 3$) needs to choose a directed path from the source s to the target t. Thus, every player i's set of possible actions (i.e., its set of pure strategies) is the set of all possible directed paths from s to t.

Each edge e is labeled with a sequence of three numbers (c_1, c_2, c_3). Given a profile $\pi = (\pi_1, \pi_2, \pi_3)$ of pure strategies (i.e., s-t-paths) for all three players, the *cost* to player i of each directed edge, e, that is contained in player i's path π_i, is c_k, where k is the total number of players that have chosen edge e in their path. The total cost to player i, in the given profile π, is the sum of the costs of *all* the edges in its path π_i from s to t. Each player of course wants to minimize its own total cost.

i. [20 points] Compute a pure strategy NE in this atomic network congestion game. Explain why what you have computed is a pure NE.

Solution:
In this game, one pure NE consists of the three players playing the following three paths for the three players (the first two of which are the same):

Player 1: $s \rightarrow v_2 \rightarrow t$
Player 2: $s \rightarrow v_2 \rightarrow v_1 \rightarrow t$
Player 3: $s \rightarrow v_3 \rightarrow t$

The costs to the three players in this NE are:
Player 1: $4 + 6 = 10$
Player 2: $4 + 1 + 6 = 11$
Player 3: $5 + 6 = 11$

It is not difficult to check that this is an NE, by exhaustively examining all possible unilateral deviations by each of the three players, and seeing that these would not decrease the cost to that player.

Another pure NE consists of the three players playing the following three paths for the three players (the first two of which are the same):

Player 1: $s \rightarrow v_1 \rightarrow t$
Player 2: $s \rightarrow v_1 \rightarrow t$
Player 3: $s \rightarrow v_2 \rightarrow t$

The costs to the three players in this NE are:
Player 1: $3 + 8 = 11$
Player 2: $3 + 8 = 11$
Player 3: $2 + 6 = 8$

It is not difficult to check again that this is an NE, by exhaustively examining all possible unilateral deviations by each of the three players, and seeing that these would not decrease the cost to that player.

As discussed in class, one way of computing a pure NE in such an atomic network congestion game is to use an "iterated better response dynamics", where we start with some arbitrary pure strategy profile, and repeatedly look to see if there is any player who can switch unilaterally to a different pure strategy to lower its own cost. If there is such a player, then we make one such switch, and we then repeat. We do this until no further improvement for any player is possible. At that point (by Rosenthal's theorem) we have reached a pure Nash equilibrium. ■

ii. [10 points] Is the pure NE you have computed in part (i.) pareto optimal in terms of costs, amongst the set of all pure strategy combinations for the players? Explain.

Solution:
The first pure NE given above is not pareto optimal in terms of costs. In particular, consider the following pure strategy profile:

Player 1: $s \rightarrow v_3 \rightarrow t$
Player 2: $s \rightarrow v_3 \rightarrow t$
Player 3: $s \rightarrow v_3 \rightarrow t$

In this pure strategy profile the costs to all three players is $5 + 4 = 9$, which is strictly less than their cost in the first pure NE specified in the answer to part (i.). However, note that this profile is itself not an NE. In particular, any of the three players would be strictly better off by changing its pure strategy unilaterally from $s \rightarrow v_3 \rightarrow t$ to $s \rightarrow v_2 \rightarrow t$, because this unilateral change would lower its cost from 9 to $2 + 6 = 8$.
On the other hand, the second pure NE given above is pareto optimal. This is because the only combination of strategies where one player has cost 8 involves there being a unique player playing both the edge $s \rightarrow v_2$ and $v_2 \rightarrow t$, and in the remaining directed graph all paths from s to t for the remaining two players have cost at least 11.
■

4. The auction house Christie's of London is auctioning a triptych (a series of three related painting) by the famous artist Fracis Bacon, entitled "Three Studies of Isabel Rawsthorne". We will refer to the three paintings in the triptych series as T1, T2, and T3, respectively.

Suppose that Christie's hires you as an auction designer, and suppose that you decide to use the Vickery-Clarke-Groves mechanism as an auction, in order to determine which bidder should get which part(s) of the triptych, and at what price. Suppose that there are only three bidders. The three bidders' names are: Susanne (S), Lakshmi (L), and Bill (B).

Since you are running a VCG-based auction, you ask each bidder to give you their valuation for every subset of the paintings in the triptych, as part of the bidding process. Suppose that the valuation functions v_S , v_L, and v_B that you receive from the three bidders, S, L, and B, respectively, are as follows (the numbers denote 10^5 pounds):

	valuation							
bidder i	$v_i(\emptyset)$	$v_i(T_1)$	$v_i(T_2)$	$v_i(T_3)$	$v_i(T_1, T_2)$	$v_i(T_1, T_3)$	$v_i(T_2, T_3)$	$v_i(T_1, T_2, T_3)$
$i := S$	0	16	16	13	29	36	29	54
$i := L$	0	4	7	12	38	37	37	53
$i := B$	0	10	18	4	26	28	39	54

(a) [28 points] Give a VCG outcome for this auction. In other words, specify, in that VCG outcome, which bidders will get which of the painting(s), and what price they will each pay for the painting(s) they get. Justify your answer, and show your calculations.

Solution: It can be checked (by looking at all possible combinations) that one outcome that yields the maximum sum total value for the three bidders is if S gets nothing, L gets both T_1 and T_3, and B gets T_2. In this case, we get sum total value $v_S(\emptyset) + v_L(T_1, T_3) + v_B(T_2) = 0 + 37 + 18 = 55$.

Let this VCG outcome be denoted o^1. We can then calculate the payments for the three players under this outcome as follows. Firstly, it is clear that the price paid by player S under this outcome o^1 should be 0, because indeed the valuation of player S for this outcome is 0. Let's check this. The "best" total valuation for players L and B together, without player S, is again 55 (this is again given by letting L get T_1 and T_3, and letting B get T_2, yielding total valuation $v_L(T_1, T_3) + v_B(T_2) = 37 + 18 = 55$), and their total valuation under outcome o^1 is of course also the same, $37 + 18 = 55$. Thus the VCG payment of player S under VCG

outcome o^1 is:
$p^1_S = 55 - 55 = 0$.
This is no surprise. We showed in lectures that VCG is "individually rational", so the payment for a player, in this case S, should be at most its (declared) valuation for that outcome, and the valuation for S of the outcome of getting nothing is 0.
Next, consider the payment p^1_L of player L for this outcome VCG o^1. The best total valuation for players S and B together, without player L, is also 55 (namely, when player S gets T_1 and player B gets T_2 and T_3, their valuations are $v_S(T_1) + v_B(T_2, T_3) = 16 + 39 = 55$), whereas under outcome o^1 their total valuation is is $v_S(\emptyset) + v_B(T_2) = 0 + 18 = 18$, so player L's VCG payment is:
$p^1_L = 55 - 18 = 37$.
Finally, to determine p^1_B, note that the best total valuation for players S and L together, without player B, is 54 (which is obtained by allocating all three paintings to S), and their total valuation under allocation o^1 is of course 37, so player B's VCG payment is:
$p^1_B = 54 - 37 = 17$.
Under the outcome o^1, and payments p^1, it is worth noting (but this is not required as part of your answer) that the utility (valuation minus payment) for the players is as follows: for S it is $0 - 0 = 0$, for L it is $37 - 37 = 0$, and for B it is $18 - 17 = 1$.
There is also a completely different VCG outcome that maximizes the total valuation. Namely, consider the outcome o^2 where S gets T_1, L gets nothing, and B gets T_2 and T_3. This, as indicated before, also yields maximum total valuation $v_S(T_1) + v_L(\emptyset) + v_B(T_2, T_3) = 16 + 0 + 39 = 55$. Thus o^2 is also a VCG outcome.
We now have to also calculate the VCG payements for each bidder in this other VCG outcome, o^2. We already know that the "best" total valuation for players L and B together, without player S, is 55, and their total valuation under outcome o^2 is 39. Thus the VCG payment of player S under VCG outcome o^2 is:
$p^2_S = 55 - 39 = 16$.
Likewise, we already know the best total valuation for players S and B together, without L, is 55, which is the same as under outcome o^2, so player L's VCG payment in VCG outcome o^2 is:
$p^2_L = 55 - 55 = 0$, as expected (because player L gets nothing in this outocme).

Finally, as already calculated, the best total valuation for players S and L together, without B, is 54, and their total valuation under allocation o^2 is $16 + 0$, so player B's VCG payment under VCG outcome o^2 is:

$p^2_B = 54 - 16 = 38$.

Under the outcome o^2, and corresponding payments p^2, it is worth noting (but again this is not required as part of your answer) that the utility (valuation minus payment) for the players is as follows: for S it is $16 - 16 = 0$, for L it is $0 - 0 = 0$, and for B it is $39 - 38 = 1$.

Note that the sum total of payments (i.e., the *revenue* of the auction) in both these VCG outcomes are the same. Namely, $p^1_S + p^1_L + p^1_B = 54 = p^2_S + p^2_L + p^2_B$. Indeed, as already discussed in the solutions to Tutorial Sheet 8 the total revenue is always the same for any two different VCG outcomes. Note that this also implies that the sum total utility of all the players for any two different VCG outcomes is always the same.

This completes the answer. ■

(b) [16 points] Is the VCG outcome you have calculated in part (a) unique? Justify your answer, and show your calculations.

Solutions: No, clearly not, the VCG outcome is not unique, because in the above answer we have actually given two completely different VCG allocations with different corresponding payments. ■

(c) [6 points] Comment on the wisdom of choosing the VCG mechanism for this or any auction. Do you think it is a good idea to do so? What if instead of this triptych, Christie's wanted to do a simultaneous auction of 20 Andy Warhol paintings, and they knew that at least 30 viable bidders want to bid for (subsets of) those paintings. Would you suggest using the VCG mechanism for such an auction? What alternative auction would you use, and why? Explain, briefly.

Solutions: It is not realistic to use the VCG mechanism for a large multi-item auction for several reasons. Firstly, if there are 20 items and 30 bidders, under VCG each bidder would have to submit, as its "bid" a valuation function which puts a value on *every subset* of the 20 items. there are 2^{20} such subsets, so simply expressing its bid function would be too much for the bidders. (Of course, one can think of many ways to express bid valuation functions in a much more short-hand way, particularly if the bidders have specific types of valuation functions, such as "single-minded", etc. But in general, we have no idea what the valuation functions of the bidders might be.)

A further problem arises because, even assuming we have everybody's bid, given as shorthand single-minded valuation functions (which simply describe the subset each bidder wants and how much it values it), the problem of computing the VCG outcome in such a setting is NP-hard (as shown in lectures), so it is not easy to compute.

There are a variety of possible alternatives for such a "combinatorial" auctions (and this is a topic of much research), but there is no entirely ideal solution. In particular, auctioning each painting separately as a second-price or English auction is a possibility, but this does not take advantage of the fact that some bidders may *only* wish to have one painting if they can also have some other painting. Thus, a lot of potential for fulfilling bidder valuations can be unrealized by auctioning the paintings completely separately.

Anyway, good ways of designing combinatorial auctions in various settings is a large and ongoing research topic, at the intersection of Economics and CS. ■