Module Title: ***Fill in***
Exam Diet (Dec/April/Aug): ***Fill in* 2019**
Brief notes on answers:

1. (a) The Nash equilibria in this game are the following three (the first two are pure NEs, the third is not pure):

   - $((1, 0, 0), (0, 0, 1))$
   - $((0, 1, 0), (0, 1, 0))$
   - $((7/11, 4/11, 0), (0, 3/5, 2/5))$

   The reason we know these are the only NEs in this game is as as follows: we can first eliminate the 1st pure strategy of player 2 from the game, by noticing that it is *strictly* dominated by a 2/3-(1/3) mixture of pure strategies 2 and 3, respectively.

   Next, in the remaining game, we can eliminate the 3rd pure strategy of player 1, by noticing that it is strictly dominated by a (3/4)-(1/4) mixture of pure strategies 1 and 2, respectively.

   That leaves us with the following residual game:

   $$\begin{bmatrix} (3,3) & (4,7) \\ (5,8) & (1,1) \end{bmatrix}$$

   This game has the two pure strategy NEs, as identified above. It does not have any other NE in which (at least) one of the two players plays a pure strategy (because there is a unique pure best response to each pure strategy for either player). The only remaining possibility is a fully mixed NE. Indeed, the game has one other fully mixed NE, which we can find using the useful corrollary to Nash's theorem, which tells us that if p_1 and p_2 are the positive probabilities player one plays strategies 1 and 2, then the payoff to player 2 of playing either of its pure strategies (both of which are in the support of its mixed strategy in the NE) against this must be equal, meaning: $3p_1 + 8p_2 = 7p_1 + p_2$, i.e., $7p_2 = 4p_1$. Hence, $p_1 = 7/11$ and $p_2 = 4/11$.

   In an exactly analogous way, we can compute the mixed strategy $(q_1, q_2) = (3/5, 2/5)$ for player 2 in the unique fully mixed NE of the above residual game.

   Since we obtained the residual game only by eliminating strictly dominated strategies, we did not eliminate any NEs. Moreover, as explained, we have found all (pure and mixed) NEs in the residual game. Hence we have found all NEs in the original game.

   (b) An example of a game without perfect recall is depicted in Figure 1.

   The reason this extensive form game does not satisfy perfect recall is because for the only non-trivial information set, which contains two nodes controlled by player 1, one of those nodes could only have been reached if in the root node (also controlled by player 1), player 1 had played action A, whereas the other node in the same information set could only have been reached if player 1 had played action B in the root node. Hence, the only way player 1 could not distinguish between the two nodes in its only non-trivial information set is if player 1 had "forgotten" which action it played in the root node. This is why this game does not satify perfect recall.

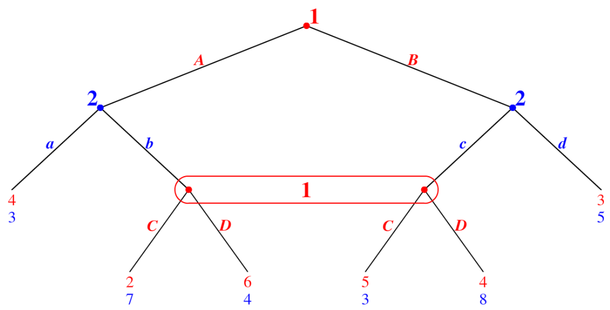

Figure 1: extensive form game

(c) (i). We prove that every finite 2-player symmetric game in which both players have exactly two pure strategies does have a pure NE. Note that, because the game is symmetric, the payoff bimatrix for such a game must look like the following:

$$\begin{bmatrix} (a,a) & (b,c) \\ (c,b) & (d,d) \end{bmatrix}$$

Now, suppose the pure strategy profile where both players play their first pure strategy, which yields payoffs (a, a) for the two players, is *not* a pure NE. Note that this can only be the case if $a < c$. So, suppose $a < c$. Next, suppose that the pure stategy combination that yields payoffs (c, b) for players 1 and 2, respectively, is *not* and NE. Since $a < c$, this can only be the case if $b < d$. But then, if $b < d$, then the pure profile where both players play their second pure strategy, and which yields payoffs (d, d) to the two players, is a pure NE.
Hence, in such a symmetric 2-player 2×2 game, the assumption that the pure profile yielding payoffs (a, a) is not a pure NE, and that the pure profile yielding payoffs (c, b) is not a pure NE, implies that the pure profile yielding (d, d) is a pure NE. Hence, there must exist a pure NE in any such game.

(ii). It is easy to disprove the claim that every finite 2-player symmetric game where both players have exactly 3 pure strategies has a pure NE. In particular, we saw in class that the game of Rock-Paper-Scissors is a symmetric 2-player zero-sum game in which both players have exactly 3 pure strategies, and in which the unique minimax profile (i.e., the unique NE), is the mixed strategy profile in which both players play each of rock, paper, and scissors, with probability $1/3$ each. Hence, Rock-Paper-Scissors has no pure NE, and this provides a counterexample which disproves the claim.

2. (a) (i). The dual LP (constructed using the formula given in class for computing the dual of a general LP) is:

Miminize $5y_1 + 8y_2$
Subject to:
$-y_1 = -1$

$$y_1 + 3y_2 = 2$$
$$y_1 + y_2 \geq 0$$
$$y_1 \geq 0$$

(ii). Using the dual LP, it is easy to see that the optimal feasible solution (in fact, the unique solution) for the dual is $(y_1, y_2) = (1, 1/3)$, and hence the optimal value of the dual is 23/3.
This therefore must also be the optimal value of the primal (by the strong duality theorem).
For the primal LP, the solution $(x_1, x_2, x_3) = (-7/3, 8/3, 0)$, yields the value $-x_1 + 2x_2 = 23/3$. Hence, this is an optimal feasible solution to the original LP.

(b) First, note that the unique (pure) strategy profile that yields the highest social welfare (i.e., highest sum total payoff to all players) is the one in which both players play their second pure stategy, and the payoffs there are $(7, 7)$, hence yielding a social welfare of 14.

Next, note that as far as NEs are concerned, we can actually eliminate the 3rd strategy of player 2, and then the 3rd strategy of player 1, via elimination of strictly dominated strategies. In the resulting (2×2) residual game, we see easily that the unique NE is the pure NE where both players play their first pure strategy, yielding payoffs of $(3, 4)$ and hence a total payoff of 7.

Hence, the *price of anarchy* is $14/7 = 2$.

(c) In this Bayesian game, it is not difficult to see that if the game type for both players is $(-1, -1)$ then the two players play a zero-sum 2×2 game, in which the payoff table for player 1 (the maximizer) is:

$$\begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$$

It is clear that in this zero sum game the unique (pure) Nash equilibrium is when player 1 plays its first pure strategy, and player 2 plays its second pure strategy, yielding payoff -1 for player 1 (and hence payoff 1 for player 2).

On the other hand, if the game type of both players is $(+1, +1)$, then the two players play a zero sum 2×2 game in which the payoff table for player 1 (the maximizer) is:

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

In this game, clearly in the unique (pure) NE player 1 plays its second pure strategy, and player 1 plays it first pure strategy, yielding payoff 1 for player 1 (and hence payoff -1 for player 2).

Now, since the common prior probability distribution p puts 1/2 probability on $(-1, -1)$, i.e., on both players being type -1, and 1/2 probability on $(+1, +1)$, i.e., on both players being type $+1$, it follows that each player knows exactly what the type of the other player is, after finding out its own type. Hence, after finding out its own type, each player knows precisely which of the above two zero-sum games it is playing in, and hence what its pure minimax stategy (in

the unique NE of that game) is. Hence, the unique pure Bayes Nash Equilibrium for this Bayesian Game is given by the following functions $s_1 : T_1 \rightarrow A_1$ and $s_2 : T_2 \rightarrow A_2$, for players 1 and 2 respectively:
$s_1(-1) := 0$, $s_2(+1) := 1$
$s_2(-1) := 1$, $s_2(+1) := 0$.
Note that the payoff to player 1 in the unique NE of the first zero-sum game above is -1, and the payoff to player 1 in the unique NE of the second zero-sum game above is +1. Since the common prior puts 1/2 probability on each of these games, the expected payoff to player 1 in the unique pure Bayesian NE of the Bayesian Game is $-1 * (1/2) + 1 * (1/2) = 0$.

3. (a) There is in fact only one Nash Equilibrium (which is hence also a subgame perfect equilibrium), in this extensive form game. We can compute it by first converting the game to normal form. (We can also do so by analyzing the extensive form game more carefully.)
   The unique NE is given by the following behavior strategies for the two players: In the root, Player 1 plays A with probability 3/5 and plays B with probability 2/5. In player 1's other node, it plays C with probability 1. Player 2 plays a with probability 3/5 and it plays b with probability 2/5.
   The Normal form for this game can be written as follows:

	a	b
AC	$(2,0)$	$(0,2)$
AD	$(2,0)$	$(0,2)$
BC	$(0,3)$	$(3,0)$
BD	$(1,1)$	$(1,1)$

   Notice that player 1's pure strategy BD is strictly dominated by a 4/7-3/7 mixture of the strategies AC (or equivalently AD), and BC, respectively. Hence the pure strategy BD can be eliminated. Moreover, since the strategies AC and AD are "equivalent" in the sense that they yield the same payoff for both players irrespective of the strategy of the other player, we can just focus on one of the two strategies AC and BC for player 2. We can solve the resulting 2×2 game to obtain the unique equilibrium above.
   The expected payoff for the two players in this NE is 6/5 for both of them.

   (b) The VCG outcome is that A gets item 1, B gets item 3, and C gets item 2. The payments are $p_A = 18 - 15 = 3$, and $p_B = 17 - 12 = 5$, and $p_C = 14 - 11 = 3$.

   (c) Consider any pure strategy profile s^* which maximizes the function $f(s)$, in other words, such that $f(s^*) = \max_{s \in S} f(s)$. (Since this is a finite game, $S = S_1 \times \ldots \times S_n$ is finite, i.e., are only finitely many pure strategy profiles s, and hence one of them does maximize $f(s)$.) We claim s^* is a pure Nash Equilbrium of the game. To see this, simply observe that for each player i, its payoff function is $u_i(s^*) = f(s^*) + g_i(s^*)$. Now, observe that changing s_i^* unilaterally can only decrease the value of $f(s^*)$ (or leave it unchanged), and by assumption changing s_i^* does not change the value of $g_i(s^*)$. Hence, if player i unilaterally changes s_i^* in s^* it can not increase its own payoff. This is true for all players i, and hence s^* is a pure Nash equilibrium.