**Module Title: AGTA**
**Exam Diet (Dec/April/Aug): April 2019**
**Brief notes on answers:**

1. (a) Firstly, let's simplify the game by iterated elimination of strictly dominated strategies. This game can be simplified as follows:

   Player 2's second pure strategy is strictly dominated by $(1/2 + \epsilon, (1/2) - \epsilon)$ mixture of pure strategies 1 and 3, for a sufficiently small $\epsilon > 0$. (In particular, $\epsilon = 1/50$ works.)

   We thus get the residual game:

$$\begin{bmatrix} (6,7) & (3,2) \\ (4,9) & (4,7) \\ (2,3) & (7,6) \end{bmatrix}$$

   In this residual game, the 2nd pure strategy of player 1 is strictly dominated by a $(1/2 + \epsilon, (1/2) - \epsilon)$ mixture of pure strategies 1 and 3, for a sufficiently small $\epsilon > 0$. (In particular, $\epsilon = 1/50$ works again.)

   This leaves the following final residual game:

$$\begin{bmatrix} (6,7) & (3,2) \\ (2,3) & (7,6) \end{bmatrix}$$

   This game can not be reduced further. There are two pure Nash equilibria in this game. Namely, one pure NE is when both players play their first row/column, and the other is when both players play their second row/column. Those are the only pure NEs. Note that these correspond to the pure NEs in the original game where both players play their first pure strategy, or when both players play their <u>third</u> pure strategy.

   There is also a mixed NE in this game, namely when player 1 plays the mixed strategy $(3/8, 5/8)$ and player 2 plays the strategy $(1/2, 1/2)$. (Note that this corresponds to mixed NE $((3/8, 0, 5/8), (1/2, 0, 1/2))$, for the original game, before the elimination of pure strategies.)

   This can be calculated as follows: suppose that in an NE of the final residual game player 2 plays its first column with probability $q$ and its second column with probability $(1-q)$, such that $0 < q < 1$. Then (by the useful corollary to Nash's theorem), if player 1 is also playing a mixed strategy with positive probability on both of its pure strategies, then it must be the case that its expected payoff for switching to either pure strategy (against player 2's mixed strategy) is equal. In other words, it must be the case that $6q + 3(1-q) = 2q + 7(1-q)$. Hence $4q = 4(1-q)$. Hence $q = 1/2$. Thus, in this NE, player 2 plays its 1st and 3rd pure strategies (in the original game) with probability 1/2 each.

   Likewise, if in this NE of the final residual game player 1 plays its first row with probability $p$ and its second row with probability $(1-p)$, then if player 2's mixed strategy puts positive probability on both pure strategies, this implies that player $7p + 3(1-p) = 2p + 6(1-p)$ which implies $5p = 3(1-p)$ or $8p = 3$, i.e., $p = 3/8$. So, in this NE, player 1 puts 3/8 probability on its first pure strategy and 5/8 probability on its 3rd pure strategy in the original game.

(b) Using one iteration of Fourier-Motzkin elimination to eliminate the variable $x_2$, we obtain the following resulting LP:

**Minimize** $x_1$
**Subject to:**
$(1/4)x_1 - (1/2)x_3 \geq 4$
$(1/3)x_1 - (3/2)x_3 \geq 2$
$x_3 \geq 0$

It is easy to see that in an optimal feasible solution of this LP, the value of $x_3$ must be 0, because any higher value will force an increase in the value of $x_1$ via the constraints, and hence will force a higher objective value. Hence, letting $x_3^* = 0$, from the first constraint (which binds tighter than the second) we can obtain that an optimal (minimum) feasible solution for this LP has $x_1^* = 16$ and $x_3^* = 0$, and then plugging back in to the eliminated inequality $x_2 \geq 2 - x_3$, we can let $x_2^* = 2$ in an optimal feasible solution. Thus an optimal feasible solution is given by $x^* = (16, 2, 0)$.

(c) To solve this optimization problem, we can introduce two new variables, $z_1$ and $z_2$, four new constraints, and a new linear objective function, as follows:

Consider the following mathematical optimization problem, where $A$ is a $(m \times n)$ matrix of integers, $b$ is a $m$-vector of integers, and $x = (x_1, \ldots, x_n)^T$ is a $n$-vector of variables.

**Minimize** $z_1 + z_2$
**Subject to:**

$$\begin{aligned} Ax &\leq b \\ x_1 &\leq z_1 \\ -x_1 &\leq z_1 \\ x_2 &\leq z_2 \\ -x_2 &\leq z_2 \end{aligned}$$

We claim that in any optimal solution $(x^*, z^*)$ of the above LP, we have $z_i^* = |x_i^*|$, and since the objective minimizes $z_1 + z_2$, this solution also optimizes the original mathematical problem.

The reason why we must have $z_i^* = |x_i^*|$ is because the two constraints involving $z_i$ enforce the fact that $z_i \geq |x_i|$, and if we had $z_i^* > |x_i^*|$ then the solution would not be optimal, because we could lower the value of $z_i^*$ (keeping the values of other variables unchanged) and still satisfy all constraints.

2. (a) The price of anarchy in a normal form game with strictly positive payoffs is defined as follows. For a mixed strategy profile $x$, let welfare$(x) = \sum_{i=1}^{n} U_i(x)$ denote the sum total of all players' expected payoffs (the utilitarian social welfare), if they play according to profile $x$. For a game $G$, let $NE(G)$ denote the set of all Nash equilibria in the game $G$. Let $X$ denote the set of all profiles of mixed strategies in the game.

The price of anarchy is defined as the ratio of the best global welfare over all mixed strategies, over the worst welfare in Nash Equilibrium. In other words, it is:

$$\frac{\max_{x \in X} \text{welfare}(x)}{\min_{x \in NE(G)} \text{welfare}(x)}$$

(b) The price of anarchy in this game is $\frac{15}{5} = 3$.

To see this, note that this game is just a variant of Prisoner's dilemma. For both players playing their first pure strategy is a dominant strategy. So, in the unique NE, they get payoffs 2 and 3, respectively and their total welfare is 5.

On the other hand, if they both played pure strategy 2, they would get payoffs 8 and 7 respectively, yielding a total payoff of 15. And that's the maximum total payoff over all pure and mixed strategy profiles.

So, the price of anarchy is $\frac{15}{5} = 3$.

(c) A pure NE is given by the following paths for the three players:

$s \to v_1 \to v_4 \to t$

$s \to v_2 \to v_4 \to t$

$s \to v_2 \to t$

Under this combination of pure strategies (paths), the total cost of the first path is 9, the total cost of the second path is 10, and the cost of the last path is 11.

It is easy to check that no improvements can be made to any of these strategies by any player unilaterally deviating from their strategy. Hence this is a NE.

(d) One way to compute a pure NE in an atomic network congestion game is to start with any strategies (directed paths from source to target) for the respective players, and to then repeatedly do individual *best response dynamics*, until no player can unilaterally reduce its own cost by deviating from its current strategy. Here "best response dynamics", means that we check for each particular player whether there is a different directed path (pure strategy) that player could take from its source to its target, assuming everyone else's paths remain unchanged, such that the total cost to that player would get reduced as a result of this change. If such a better path exists, we switch that single player's strategy to this better path. We try this for all players (in some order). We repeat this, until no player can improve (lower) its cost by switching its own path unilaterally. At which point, we have found a pure Nash Equilibrium. By **Rosenthal's theorem** (which we proved in class, for all congestion games, not just atomic network congestion games), this procedure terminates: we never cycle back to the same pure strategy profile.

3. (a) (i). We can first convert this extensive form game into normal form. By computing the expected payoff to each player under each combination of actions (pure strategies) for both players in the (single) node they each control, we obtain the following bimatrix game:

$$\begin{bmatrix} & c & d \\ a & (3,9) & (3,9) \\ b & (5,4) & (2,2) \end{bmatrix}$$

Note that in this game, one pure Nash Equilibrium is when player 1 plays action $b$ and player 2 plays action $c$. In this case, their respective payoffs are 5 and 4.

Another pure NE is when player 1 plays $a$ and player 2 plays $d$. In this case, the expected payoff for player 1 is 3 and that of player 2 is 9.
Note that in fact, if player 2 plays $c$ with probability $q$ and $d$ with probability $(1-q)$, then the expected payoff to player 1 of playing strategy $a$ against this is 3 whereas its expected payoff from playing $b$ is $5q + 2(1-q)$.
So, as long as $3 \geq 5q + 2(1-q) = 3q + 2$, strategy $a$ is a best response for player 1, and of course for player 2 any mixed strategy is a best response against player 1's pure strategy $a$ (because it gets payoff 9 no matter what it plays against pure strategy $a$).
Note that $3 \geq 3q + 2$ precisely when $q \leq 1/3$.
Hence, the set of mixed strategy profiles where player 1 plays $a$ with probability 1 and player 2 plays $c$ with probability $q$ and $d$ with $(1-q)$ is a Nash Equilibrium for all values of $0 \leq q \leq 1/3$.
These and the Nash equilibrium where player 1 plays $b$ and player 2 plays $c$ constitute all the NEs in this game.

(ii). The unique subgame perfect equilibrium is when player 1 plays $b$ and player 2 plays $c$, both with probability 1. The expected payoff to player 1 is then 5 and the expected payoff to player 2 is 4.

(iii). All the equilibria other than the subgame perfect equilibrium involve non-credible threats. This is because in all of these equilibria, player 2 purportedly plays pure strategy $d$ with positive probability. But since the expected payoff to player 2 would be strictly higher in the subgame rooted at the chance node if it played pure strategy $c$, even if player 2 puts any positive probability on $d$, it could strictly increase its expected payoff in that subgame by shifting that probability over to action $c$. Hence the "threat" of playing $d$ with positive probability in that information set is not credible. When faced with the actual choice, player 2 would be strictly better off, in expectation, choosing $c$ with probability 1.

(b) The VCG allocation is $j_A = 2$ , $j_B = 1$, $j_C = 0$. This gives a total valuation of 18. No other allocation gives a valuation of more than 17.

The VCG payments are as follows:

$p_A = 17 - 7 = 10$, $p_B = 17 - 11 = 6$, and $p_C = 18 - 18 = 0$.

(c) The Gibbard-Satterthwaite Theorem states the following. Consider any social choice function $f(\prec_1, \ldots, \prec_n)$ that maps the preference total orderings $\prec_i$ specified by $n \geq 1$ voters, each giving their order of preference on all the candidates. The function $f(\prec_1, \ldots, \prec_n)$ maps such a list of preference orderings (one for each voter), to a "winning" candidate $c \in C$, where $C$ is the set of all candidates, and where $|C| \geq 3$, i.e., there are at least three candidates.

If the social choice function $f$ is *onto* $C$ (meaning each candidate is a possible winner), then either the function is a *dictatorship*, meaning the social choice produced by $f$ is always the top choice of a specific voter, regardless of how the other voters vote, or else the social choice function $f$ can be *strategically manipulated*, meaning that there exists a combination of voter preference orders by all voters, such that one of the voters, if it knew those preference orders of others, would be incentivized to lie about its own actual preference order, because lying about its preferences in a particular way (assuming others don't change their preference order) would cause the social choice function to output a winning candidate which is actually prefered by that voter to the winning candidate that would be output if that voter had told the truth about its own preferences.