Algorithmic Game theory and Applications
Solutions for Homework 1 (2022)

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1. (a)

$$\begin{bmatrix} (3,6) & (3,7) & (7,8) & (6,7) \\ (4,4) & (4,6) & (7,5) & (7,9) \\ (4,0) & (5,4) & (8,5) & (5,4) \\ (4,3) & (1,2) & (4,3) & (1,3) \end{bmatrix}$$

There are exactly four Nash Equilibria (NEs) in this game. Specifically, there are three pure NEs and one NE that is not pure. The four NEs, and their associated expected payoffs for each player, are as follows:

i. $(\pi_{1,2}, \pi_{2,4}) = ((0,1,0,0),(0,0,0,1))$.
The (expected) payoff for player 1 is $U_1(\pi_{1,2}, \pi_{2,4}) = u_1(2,4) = 7$, and for player 2 is $U_2(\pi_{1,2}, \pi_{2,4}) = u_2(2,4) = 9$.

ii. $(\pi_{1,4}, \pi_{2,1}) = ((0,0,0,1),(1,0,0,0))$.
The (expected) payoffs are: $U_1(\pi_{1,4}, \pi_{2,1}) = u_1(4,1) = 4$ for player 1 and $U_2(\pi_{1,4}, \pi_{2,1}) = u_2(4,1) = 3$ for player 2.

iii. $(\pi_{1,3}, \pi_{2,3}) = ((0,0,1,0),(0,0,1,0))$.
The (expected) payoffs are: $U_1(\pi_{1,3}, \pi_{2,3}) = u_1(3,3) = 8$, and $U_2(\pi_{1,3}, \pi_{2,3}) = u_2(3,3) = 5$.

iv. $x^* = (x_1^*, x_2^*) = ((0,1/5,4/5,0),(0,0,2/3,1/3))$.
The expected payoffs are $U_1(x_1^*, x_2^*) = 7$ for player 1, and $U_2(x_1^*, x_2^*) = 5$ for player 2.

Notice that it is easy to verify, by just looking at the payoff table, that each of the three pure strategy profiles listed above is indeed an NE. We will next justify in detail why the above four are all of the NEs in this game.

First, we claim that the 2nd pure strategy for player 2 (the column player), i.e, $\pi_{2,2}$, is strictly dominated by a $\frac{1}{2}$-$\frac{1}{2}$ average of its 3rd and 4th pure strategies $(1/2)\pi_{2,3} + (1/2)\pi_{2,4}$. To see this, note that the (1/2)-(1/2) averaging of the payoffs to player 2 in the 3rd and 4th columns gives us the following column vector of expected payoffs (against pure strategies of player 1):

$$\begin{bmatrix} 7.5 \\ 7 \\ 4.5 \\ 3 \end{bmatrix}$$

This column vector is clearly strictly greater, in every coordinate, than the payoffs to player 2 in the 2nd column, which are:

$$\begin{bmatrix} 7 \\ 6 \\ 4 \\ 2 \end{bmatrix}$$

Thus, the 2nd pure strategy for player 2, $\pi_{2,2}$, is strictly dominated, and hence can be eliminated from consideration, resulting in the following residual game:

$$\begin{bmatrix} (3,6) & (7,8) & (6,7) \\ (4,4) & (7,5) & (7,9) \\ (4,0) & (8,5) & (5,4) \\ (4,3) & (4,3) & (1,3) \end{bmatrix}$$

In this residual game, we claim the 1st row (1st pure strategy) for player 1, $\pi_{1,2}$, is strictly dominated by a 2/3-1/3 weighted average of the 2nd and 3rd pure strategies. Indeed, such a weighted average of the payoffs to player 1 in the 2nd and 3rd rows produces the following row vector:

$$\begin{bmatrix} 4 & 7.333333 & 6.333333 \end{bmatrix}$$

This is clearly a strictly greater row vector in every coordinate than the payoff row vector for player 1 in its first row of the residual game,which is:

$$\begin{bmatrix} 3 & 7 & 6 \end{bmatrix}$$

Thus, we can now eliminate the 1st pure strategy of player 1 from consideration, because it is strictly dominated, leaving us with the new residual game:

$$\begin{bmatrix} (4,4) & (7,5) & (7,9) \\ (4,0) & (8,5) & (5,4) \\ (4,3) & (4,3) & (1,3) \end{bmatrix}$$

In this residual game, no pure strategy is strictly dominated.

However, we can observe the following. There is a pure Nash equilibrium where the first player plays the 3rd row of this residual game (which corresponds to the 4th row of the original game) and where the second player plays the 1st column (corresponding to the 1st column of the original game). This corresponds to the pure NE $(\pi_{1,4}, \pi_{2,1}) = ((0,0,0,1),(1,0,0,0))$ we have already mentioned at the start, where payoff is 4 for player 1 and 3 for player 2. As already mentioned, it is clear by inspection that this is a pure NE, because neither player can strictly improve its own payoff by unilaterally deviating from this pure strategy combination.

Now we make the following claim regarding this residual game.

> **Claim:** In this residual game, if x_1 is ANY mixed strategy for player 1 in which the 4th pure strategy is played with probability strictly less than 1, i.e., if $x_1(4) < 1$, then for any mixed strategy x_2' for player 2 which is a best response to x_1 it must be that $x_2'(1) = 0$.
> Likewise, if x_2 is ANY mixed strategy for player 2 in which the 1st pure strategy is played with probability strictly less than 1, i.e., if $x_2(1) < 1$, then if x_1' is any best response mixed strategy by player 1 to the mixed strategy x_2, it must be the case that $x_1'(4) = 0$.

To see why this claim must hold, note, firstly, that if player 1 (respectively, player 2) plays its 4th (respectively, 1st) pure strategy with probability 1, then the payoff to player 2 (respectively, to player 1) is exactly 3 (respectively, exactly 4) regardless of which strategy it plays agains player 1 (respectively, against player 2).

On the other hand, if player 1 (respectively player 2), plays any pure strategy other than its 4th (respectively, 1st) pure strategy, then the payoff to player 2 (respectively, to player 1) will be strictly higher if it responds with ANY pure strategy other than its 1st (respectively, its 4th) pure strategy.

What this means is the following: unless player 1 is playing its 4th pure strategy with probability 1 (respectively, player 2 is playing its first pure strategy with probability 1), any best response of player 2 (respectively, any best response of player 1), must play its 1st pure strategy (respectively, is 4th pure strategy) with probability 0.

Hence, in any Nash equilibrium (x_1^*, x_2^*) of this residual game, we either have both ($x_1^*(4) = 1$ and $x_2^*(1) = 1$), or else we have both ($x_1^*(2) = 0$ and $x_2^*(1) = 0$).

Since we have already accounted for the pure NE of this residual game in which $x_1^*(4) = 1$ and $x_2^*(1) = 1$, we know that in the remaining NEs we must have that $x_1^*(2) = 0$ and $x_2^*(1) = 0$.

Hence, we can then "eliminate" those pure strategies and consider the remaining 2×2 residual game given by:

$$\begin{bmatrix} (7,5) & (7,9) \\ (8,5) & (5,4) \end{bmatrix}$$

In this residual game there are two pure NEs, which we have already identified, namely, the pure NE (of the original game) $(\pi_{1,3}, \pi_{2,3})$ which yields payoffs $(8, 5)$ for players 1 and 2, respectively; as well as the pure NE (of the original game) $(\pi_{1,2}, \pi_{2,4})$ which yields payoffs $(7, 9)$ for players 1 and 2, respectively.

There are no other pure NEs in this residual game, by inspection. We can also see, by inspection, that in this 2×2 game every pure strategy of either player has a unique pure best response by the other player. Hence it is not possible for there to be a Nash equilibrium in this residual game where one player plays a pure strategy while the other plays a non-pure strategy.

Hence, in any remaining NE in this residual game, BOTH players must play both of their pure strategies with positive probability. Hence (as in the solution to Tutorial 2, Question 2), using the useful corollary to Nash's theorem, we can see that if in such a NE player 1 is playing its two pure strategies with probabililites p and $(1 - p)$, where $0 < p < 1$, and player 2 is playing its two pure strategies with probabilities q and $(1 - q)$, with $0 < q < 1$, then in such an NE both of the pure strategies of both players must be best responses against the other player's mixed strategy.

Hence switching unilaterally to either of them must yield the same expected payoff for the respective player.

This yields the following two equations:

$$5p + 5(1-p) = 9p + 4(1-p)$$

and

$$7q + 7(1-q) = 8q + 5(1-q) \quad .$$

Solving the first equation yields $4p = 1 - p$, and hence $p = 1/5$, and $(1-p) = 4/5$.

Solving the second equation yields $q = 2(1-q)$, and hence $3q = 2$, or $q = 2/3$ and $(1-q) = 1/3$.

This yields the already mentioned mixed NE, x^* for the original game, given by $x^* = ((0, 1/5, 4/5, 0), (0, 0, 2/3, 1/3))$.

We can easily calculate the expected payoffs to the two players under this NE, which are $U_1(x^*) = 7$, and $U_2(x^*) = 5$.

We have systematically argued that there cannot be any other NEs in this game, because, firstly, we know that the pure strategies that were eliminated by eliminating strictly dominated strategies cannot be played with positive probability in any NE, and secondly, because after identifying the pure NE $(\pi_{1,4}, \pi_{2,1}) = ((0,0,0,1),(1,0,0,0))$, we argued that in the residual game if this pure NE is not played, then in any NE the pure strategies $\pi_{1,4}$ and $\pi_{2,1}$ must both be played with probability 0. Hence those strategies could rightly be "eliminated" from further consideration for the purposes of finding other NEs.

Finally, we identified the only three possible NEs in the remaining 2×2 residual game.

Thus, there are no other NEs.

(b) Actually this question was misphrased. Equation (1) should have said "$g_1(x_1) = 2$ if $x_{1,2} \geq 1/2$" instead of "$x_{1,1} \geq 1/2$".

As a result of this misphrasing, the question has become much easier to answer.

In particular, since there does not exist any NE (x_1, x_2) of the original game G in which $x_{1,1} \geq 1/2$, in the new game G' we can simply add an extra 3rd player, whose payoff $u'_3(s_1, s_2, s_3)$ is A if $s_3 = 1$ and is B if $s_3 = 2$, for any numbers A and B where

$A > B$. Moreover, we simply let $u'_i(s_1, s_2, s_3) = u_i(s_1, s_2)$, for $i \in \{1, 2\}$ and for (s_1, s_2, s_3) being any strategies whatsoever of the three players. This implies that in every NE of G' player 3 plays its first pure strategy with probability 1, and since player 3 doesn't influence the payoff of the other players at all, every NE $(x'_1, x'_2, \pi_{3,1})$ of G' corresponds to an NE (x'_1, x'_2) of G, and on the other hand every NE (x_1, x_2) of G, corresponds to an NE $(x_1, x_2, \pi_{3,1})$ of G.

If the question had not been misphrased, and instead of "$x_{1,1} \geq 1/2$" it had said "$x_{1,2} \geq 1/2$" as intended, then the solution below would have applied....

Consider the following 3-player game, G', obtained from the 2-player game G considered in the answer to part (a) of this question. The players' pure strategy sets are $S_1 = S_2 = \{1, 2, 3, 4\}$ and $S_3 = \{1, 2\}$. Let $S = S_1 \times S_2 \times S_3$.

The payoff functions $u_i : S \rightarrow, i \in \{1, 2, 3\}$ are defined as follows. Let $u_i(s_1, s_2)$ denote the payoff function for player i, $i \in \{1, 2\}$, in the original 2-player game G, when player 1 plays pure strategy $s_1 \in S_1$, and player 2 plays pure strategy $s_2 \in S_2$. For any pure strategies $(s_1, s_2, s_3) \in S$, For player $i \in \{1, 2\}$, we define:

$$u_i(s_1, s_2, s_3) := u_i(s_1, s_2)$$

In other words, for players 1 and 2, their payoff remains exactly the same as in the 2-player game G, irrespective of what player 3 does. In other words, their payoff DOES NOT DEPEND AT ALL on what the new 3rd player does. On the other hand, the payoffs for the new player 3 is defined as follows:

$$u_3(s_1, s_2, s_3) := \begin{cases} 1 & \text{if } (s_1 = 2 \text{ and } s_3 = 2) \text{ or } (s_1 \neq 2 \text{ and } s_3 = 1) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In other words, the payoff to player 3 is 1 if either player 1 plays pure strategy 2 and player 3 plays pure strategy 2, or player 1 doesn't play pure strategy 2 and player 3 plays pure strategy 1. Otherwise, in any other case, the payoff to player 3 is 0.

This completes the definition of the 3-player game G'.

Now, let us argue why it must be the case that

$$\mathrm{NE}(G') = \{(x_1, x_2, \pi_{3,g_1(x_1)}) \mid (x_1, x_2) \in \mathrm{NE}(G)\}.$$

Notice that any Nash Equilibrium $x^* = (x_1^*, x_2^*, x_3^*) \in \mathrm{NE}(G')$ must have the property that $(x_1^*, x_2^*) \in \mathrm{NE}(G)$. This is simply because in G' players 1 & 2 DON'T CARE what strategy player 3 is playing: their payoffs do not depend on player 3's strategy at all. So, irrespective of what strategy x_3^* is, x_1^* and x_2^* are best responses in G' if and only if they are best responses in the original game G. Hence (x_1^*, x_2^*) must be a Nash Equilibrium in G.

We know exactly what the four NEs of the game G are, because we found those NEs in our answer to Part (a) of this question. Specifically, there are 3 pure NEs, and one mixed NE. ** Importantly ** in the mixed NE, player 1 plays its 2nd pure strategy with probability 1/5.

Hence, in every NE of G, player 1 plays its 2nd pure strategy either with probability $< 1/2$ or with probability $> 1/2$ (and never with probability exactly equal to 1/2).

Now, consider the strategy x_3^* of player 3, in the NE (x_1^*, x_2^*, x_3^*). x_3^* must be a best response against (x_1^*, x_2^*). However, notice that if $x_{1,2}^* = x_1^*(2) > 1/2$ then the UNIQUE best response for player 3 is $\pi_{1,2}$. Likewise, if $x_1^*(2) < 1/2$, then the UNIQUE best response for player 3 is $\pi_{1,1}$. Since we already know that it cannot be the case that $x_1^*(2) = 1/2$, this implies that the best response x_3^* in an any NE x^* of G' is a pure strategy that is uniquely determined by x_1^* and is in fact $\pi_{3,g_1(x_1^*)}$.

2. (a) Let the matrix A be the matrix shown in the problem statement:

$$A = \begin{bmatrix} 3 & 3 & 9 & 6 & 2 \\ 7 & 8 & 4 & 5 & 3 \\ 1 & 2 & 5 & 6 & 4 \\ 1 & 4 & 4 & 5 & 9 \\ 4 & 7 & 7 & 8 & 3 \end{bmatrix}$$

Then, using x to denote a vector of variables denoting player 1's mixed strategy, and using an auxiliary variable v for the value of the game, and letting the vector $\mathbf{v}^T$ denote a row vector every coordinate of which is the variable v, we aim to solve the following LP for finding a minmaximizer strategy for player 1, and for finding the value v of the game:[1]

[1] Note that $\mathbf{1}$ ($\mathbf{0}$) denotes the all-one (respectively, all-zero) vector.

Maximize v
Subject to:
$x^T A \geq \mathbf{v}^T$
$x^T \mathbf{1} = 1$
$x \geq \mathbf{0}$

Plugging this LP into an LP solver, it turns out we can get more than one possible optimal solution vector. In particular,

$x^* = (2/15, 17/30, 0, 3/10, 0)$ and $v^* = 14/3$

is one possible solution vector. Another possible solution vector is:

$x' = (0, 1/2, 0, 5/18, 2/9)$ and $v' = 14/3$.

Note that, of course, in both cases, the optimal value v^* is the same. However, as we know, there can be more than one minmaximixer, andin the case both x^* and x' are minmaximzers in this game.

In fact, Note that any vector which is a convex combination (weighted average) of x^* and x', i.e., any vector of the form $z = \lambda x^* + (1 - \lambda)x'$, is also a minmaximizer for player 1, where $\lambda \in [0, 1]$ is any value between 0 and 1 (inclusive). This follows from the fact that the set of feasible solutions (and hence also feasible and optimal solutions) to a linear programming problem is a convex set, if it is non-empty.

Thus the minimax value of the game is $v^* = 14/3$, and a minmaximizer strategy for player 1 is given by either x^* or x' or any weighted average of these two vectors.

For player 2, we can solve the DUAL LP given by:

Miminize u
Subject to:
$Ay \leq \mathbf{u}$
$y\mathbf{1} = 1$
$y \geq \mathbf{0}$

The optimal solution (y^*, u^*) in this dual LP $y^* = (1/3, 0, 1/3, 0, 1/3)$, and, as we would expect $u^* = 14/3$, which gives us a maxminimizer for player 2, and (again) the value of game. (Of course the payoff to player 2 under this profile is: $-14/3$.)

It turns out that y^* is the unique maxminimizer for player 2 in this game.

Hence a minimax profile (NE) of this game is (x^*, y^*), or (x', y^*), or (z, y^*) for any vector z which is a convex combination of x^* and x'.

(b) (This question is taken from a paper by [Adler,2013].)

We are given that for an $(m \times n)$-matrix A, n-vector b, and m-vector c, there exist vectors $x' \in^n$ and $y' \in^m$, such that:

$$Ax' < b \ , \quad x' \geq 0, \qquad \text{and} \qquad A^T y' > c, \qquad , y' \geq 0 \tag{2}$$

As indicated in the hint, we let $w = (y^*, x^*, z)$ denote the COLUMN vector which is a maxminimizer strategy for player 2 (the minimizer) in the game symmetric 2-player zero-sum game with payoff matrix B.

$$B = \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix}$$

This means that $w \geq 0$, $\sum_k w_k = 1$ (because w is a probability distribution). And, since the minimax value of every symmetric 2-player zero-sum game is 0, it means that $Bw \leq 0$. If we expand the inequalities in $Bw \leq 0$, using the definition of block matrix B and vector w, we get the following inequalities (in block notation):

$$\begin{bmatrix} Ax^* - bz \\ -A^T y^* + cz \\ b^T y^* - c^T x^* \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, suppose contradiction that $z = 0$. Then the above inequalities yield:

$$Ax^* \leq 0, \quad x^* \geq 0 \tag{3}$$
$$A^T y^* \geq 0, \quad y^* \geq 0 \tag{4}$$
$$b^T y^* \leq c^T x^* \tag{5}$$

and

$$\sum_i x_i^* + \sum_j y_j^* = 1. \tag{6}$$

Now, notice that if $y^* \neq 0$, then by (2) and (4) $y^{*T}(Ax' - b) < 0$, which implies that $b^T y^* > y^{*T} A x' \geq 0$. But then using (5) this implies

$$c^T x^* \geq b^T y^* > 0 \tag{7}$$

which also implies that $x^* \neq 0$. But then this implies, again using (2) and (3) that $x^{*T}(A^T y' - c) > 0$, which implies that

$$c^T x^* \leq x^{*T} A^T y' = y' T A x^* \leq 0 \tag{8}$$

But by (7) and (8) we have $0 < c^T x^* < 0$, which is a contradiction. So, if $y^* \neq 0$, our assumption that $z = 0$ must have been false.

If, on the other hand $y^* = 0$, then by (6) we must have $x^* \neq 0$. In that case, again, by (2) and (3) we have $x^{*T}(A^T y' - c) > 0$, and hence $c^T x^* < x^{*T} A^T y' = y'^T A x^* \leq 0$. Hence by (5) $b^T y^* \leq c^T x^* < 0$, which means that $y^* \neq 0$. But this contradicts the assumption that $y^* = 0$.

Hence, under the assumption that $z = 0$, both the assumption that $y^* \neq 0$, and the assumption that $y^* = 0$ lead to a contradiction.

Which means that the assumption that $z = 0$ is a contradiction, meaning we must have $z > 0$ in any such maxminimizer strategy, which is what we wanted to prove. $\square$

3. Consider the "matching pennies" game given by the matrix A:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(a) It is easy to calculate that for the game given by matrix A, both players playing each of their strategies with probability $1/2$ gives the only minimax profile (i.e., only NE) of the Matching Pennies game. The minimax value of this game is 0.

Likewise, it is easy to calculate that for the game given by matrix B, the unique Nash Equilibrium is given by $x_1^* = (1/2, 1/2)$, and $x_2^* = (1/4, 3/4)$, meaning player 1 (row player) plays both its pure strategies with probability $1/2$, whereas player 2 plays its first pure strategy with probability $1/4$ and its second pure strategy with probability $3/4$. The minimax value of this game is $3/2$.

(b) For $i = 1, 2$, and $n \geq 1$, let H_n^i, T_n^i, denote the number of heads and tails played by player i after n iterations of the game.

Let $D_n^i = H_n^i - T_n^i$ denote the difference between the number of heads and tails played by player i after n iterations of the game.

We will show that $\lim_{n\to\infty} \frac{D_n^i}{n} = 0$. This then easily implies that $\lim_{n\to\infty} \frac{H_n^i}{H_n^i + T_n^i} = \frac{H_n^i}{n} = 1/2$, which is what we want to show.

First, we do something very simple: suppose we adopt the following tie break rule: if at some round n, $D_n^i = 0$, then in the next round $n+1$, player $j \neq i$ must play something different than what it did in round n.

If we adopt this rule, then it is easy to prove that if we start out in round 1 with $(D_1^1, D_1^2) = (A, B)$ for *any* $A, B \in \{1, -1\}$, then for all $n \geq 1$, and all $i \in \{1, 2\}$, we must have $|D_n^i| \leq |A| + |B|$. Indeed, starting for example from both players playing heads, the sequence of pairs (D_n^1, D_n^2), will always cycle as follows:
$(1, 1) \to (2, 0) \to (1, -1) \to (0, -2) \to (-1, -1) \to (-2, 0) \to (-1, 1) \to (0, 2) \to (1, 1)$.

(This crucially uses the tie break rule. It wouldn't be true without the tie break rule.)

Thus it is clear that, as $n \to \infty$ $\frac{H_n^i}{n} = 1/2$, using this tie-break rule.

It is harder to prove this without the assumption about the tie break rule.

Here is a *proof sketch only* for a general tie-break rule:

Let's first look at how the game must start. Suppose, e.g., that initially both players play heads.

Then the sequence of differences (D_n^1, D_n^2), starts off for $n = 0, 1, 2$, as:

$$(0, 0) \to (1, 1) \to (2, 0)$$

Consider what happens if, after some iteration n, we have $(D_n^1, D_n^2) = (A, 0)$, with $A > 0$.

There are two possibilities, based on how player 1 breaks the tie in the next round $(n + 1)$:

i. Player 1 plays tails. Then the sequence of differences thereafter looks like:
$(A - 1, -1) \to (A - 2, -2) \to \ldots \to (0, -A)$.

Once we reach $(0, -A)$ we can symmetrically consider how player 2 can break ties.

ii. Player 1 can play heads. Then the sequence of differences becomes:
$(A + 1, -1) \to (A, -2) \to \ldots \to (0, -(A + 2))$.

If at round n we have differences (D_n^1, D_n^2), let us define the SIZE of the "state" at round n, denoted $size_n = \max(|D_n^1|, |D_n^2|)$.

In both cases above, we see that it must take at least $A + 2$ iterations before the maximum size of the state is increased from A to $A + 2$ for the first time.

Using this, we claim that for all n, $size_n \leq c * \sqrt{n}$, for some fixed constants $c_1 \geq 0$.

This is clearly true at the beginning, say up to the step when the differences are $(2, 0)$. (We can pick, e.g., $c = 2$).

To see that it is always true, note that if the state at time n has differences $(N, 0)$ or $(0, N)$, then it will take $N + 2$ before we have $size_{n+N+2} = N + 2$ for the first time.

Thus, to add N to the size, we must do at least $(N + 2) * N/2$ iterations.

Thus in general, as n gets large, it must be the case that the statistical probability of heads at round n for player 1 is:

$$p_n^1 = \frac{H_n^1}{n} = \frac{1}{2} + \frac{D_n^1}{2n} \in [\frac{1}{2} - \frac{c}{2\sqrt{n}}, \frac{1}{2} + \frac{c}{2\sqrt{n}}]$$

Thus, we see that as $n \to \infty$, we have $p_n^1 \to \frac{1}{2}$.

Symmetrically, we can make the same arguments when we start at a state $(0, A)$, and to claim that $p_n^2 \to \frac{1}{2}$.

Discussion: (Not expected as part of the solution. This is just feedback for the students.) The approach of playing a best response against the statistical mixed strategy of the opponent in the repeated version of the game is called **ficticious play**. There is a large literature about ficticious play, and the following seminal result by Julia Robinson:

> Julia Robinson, "An iterative method of solving a game", *Annals of Mathematics*, 1951.

where she proved that for EVERY finite two-player zero-sum game (including of course the zero-sum game described by payoff matrix B), ficticious play converges to the set of statistical mixed strategies that correspond to minimax profiles (and thus to the set of NEs) of the zero-sum game. Her proof of this is quite amazing, but fairly short (only 7 pages). (She is one of my mathematical heros, not just for this theorem, but for numerous amazing theorems she proved. Her life story is also amazingly inspiring; if you are interested in biographies of great mathematicians, I would highly recommend reading about her.)

For the special case of matching pennies, the situation is of course much simpler. (In particular, there is a unique NE, so "converging to the set of NEs", means simply converging to that unique NE.)

In general, for 2-player non-zero sum games, ficticious play need not converge at all to any NE. Shapley (1964) gave examples of 3×3 non-zero sum 2-player games (bimatrix games) for which ficticious play does not converge to any equilibrium (regardless of what pure strategies we start with, and regardless of the tie-breaking rule that we use during ficticious play).

The following 3×3 bimatrix game (which can be viewed as a strange kind of non-zero-sum variant of rock-paper-scissors) is one of Shapley's examples:

$$\begin{bmatrix} (0,0) & (1,2) & (2,1) \\ (2,1) & (0,0) & (1,2) \\ (1,2) & (2,1) & (0,0) \end{bmatrix}$$

It has a unique Nash equilibrium, namely where both players play each pure strategy with probability $1/3$ in the unique NE. And yet ficticious play does not converge to this unique NE.

Over the years, researchers in game theory have tried to classify for which kinds of games, beyond zero-sum games, convergence of ficticious play to the set of NEs does/doesn't hold (and when it does hold, how "fast" or "slow" convergence happens). There are many such results.

4. (a) One variant of the Farkas Lemma says the following:

Lemma 1 Farkas Lemma: *A linear system of inequalities $Ax \leq b$ has a solution x if and only if there is no solution vector y satisfying $y \geq 0$, $y^T A = 0$, and $y^T b < 0$.*

Proof. One direction is easy: suppose there is a vector y such that $y \geq 0$, $y^T A = 0$, and $y^T b < 0$.

We show that this implies there is no solution to $Ax \leq b$. Suppose there is such a solution. Then $0 = (y^T A)x = y^T(Ax) \leq y^T b < 0$, which is a contradiction.

In the other (harder) direction, we wish to show that if there is no solution to $Ax \leq b$, then there is a solution $y \geq 0$ such that $y^T A = 0$, and $y^T b < 0$.

We will do this by using the fact that Fourier-Motzkin elimination works to solve linear systems of inequalities like $Ax \leq b$.

The key observation is the following: one round of Fourier-Motzkin elimination (i.e., the elimination of a variable) can be "implemented" by pre-multiplying a given linear system of inequalities $Ax \leq b$ by a *non-negative* matrix. In other words, FM-elimination achieves the following:

Claim 1 *For a linear system of inequalities $Ax \leq b$, and a variable x_i, there is a* non-negative *matrix C_i (corresponding to an FM-elimination step which eliminates variable x_i), such that $Ax \leq b$ is feasible if and only if $C_i Ax \leq C_i b$ is feasible, and furthermore such that the matrix $(C_i A)$ has only 0's in its i'th column. (In other words the variable x_i has been eliminated from the new system $C_i Ax \leq C_i b$ of inequalities.)*[2]

We can use this claim to easily establish the harder direction of the Farkas lemma as follows: proceed using FM-elimination to sequentially eliminate the variables $x_1, \ldots, x_n$.

This yields a new system of inequalities:

$$C_n C_{n-1} \ldots C_1 Ax \leq C_n C_{n-1} \ldots C_1 b$$

If we let $C := C_n C_{n-1} \ldots C_1$ then we can write this system as $CAx \leq Cb$.

Now notice that by Claim 1, all C_i matrices, and thus also the matrix C are non-negative. Furthermore notice that, again by Claim 1, $CA = 0$ (in other words, CA is the all zero matrix), because it must be all 0 in every column (note that once a column becomes all 0 it stays all 0 thereafter).

[2]Note that the number of rows in C may be much bigger then the number of columns, so the new system of inequalities $CAx \leq Cb$ may have many more constraints than $Ax \leq b$.

Thus, $CAx \leq Cb$ can be rewritten as $0 \leq Cb$.

But we have assumed that the original system, $Ax \leq b$, *does not* have a solution. Thus from the correctness of Fourier-Motzkin elimination we know that $0 \leq Cb$ does not have a solution.

But this is the case iff there is a row j' such that $(Cb)_{j'} < 0$.

Now, pick any such row j', and construct a (non-negative) vector z which is 1 in position j' and 0 everywhere else. Then we have $(z^T C)b = z^T(Cb) < 0$, and $(z^T C)A = z^T(CA) = 0$.

Thus, let $y^T := z^T C$. Note that $y \geq 0$, $y^T A = 0$, and $y^T b < 0$.

Thus, by assuming that $Ax \leq b$ has no solution, we have found a solution to $y \geq 0$, $y^T A = 0$, and $y^T b < 0$. We are done.

All that remains is to establish **Claim 1**. But this is fairly easy to do: consider one round of FM-elimination where, say, variable x_1 is being eliminated. Let us rephrase what happens in such a round of FM-elimination as follows:

Each inequality $a_{j,1}x_1 + a_{j,2}x_2 + \ldots + a_{j,n}x_n \leq b_j$ can be re-written (depending on the sign of $a_{j,1}$) in one of three forms (types) of inequalities:

i. if $a_{j,1} > 0$, then we can multiply the inequality by $1/a_{j,1}$ to rewrite it as an inequality of type (I):
$x_1 + (a_{j,2}/a_{j,1})x_2 + \ldots + (a_{j,n}/a_{j,1})x_n \leq b_j/a_{j,1}$

ii. if $a_{j,1} < 0$, then we can multiply by $1/-a_{j,1}$, to rewrite it as an inequality of type (II):
$-x_1 + (a_{j,2}/-a_{j,1})x_2 + \ldots + (a_{j,n}/-a_{j,1})x_n \leq b_j/-a_{j,1}$

iii. finally, if $a_{j,1} = 0$, we have an inequality of type (III) and we can leave it untouched (i.e., multiply the inequality by 1).

Note that all of these transformations can be carried out simultaneously by pre-multiplying $Ax \leq b$ by a non-negative matrix D which has the appropriate multiplication constant for each inequality sitting on the diagonal of D and with 0 everywhere else in D.

Now, we have a revised inequality system $DAx \leq Db$, and FM-elimination corresponds to the following action: find every pair of inequalities β and α of type I and II, respectively, and add them together to get a new inequality without x_1, and also retain all inequalities of type III.

But adding two such inequalities means pre-multiplying by a row vector with 0 everywhere except a 1 in the two columns corresponding to the two inequalities. Doing this for every such pair, and leaving the type III inequalities unchanged corresponds to pre-multiplying by a non-negative matrix C'. Thus we get a new system

$C'DAx \leq C'Db$, such that x_1 has been eliminated, i.e., such that $(C'DA)$ has only 0's in the first column. Let $C_1 := C'D$. Note that C_1 is non-negative.

We have shown the claim holds for eliminating the variable x_1. Of course x_1 was arbitrary, and we could have just as easily FM-eliminated any other variable by pre-multiplying by a non-negative matrix. We are done. ■

(b) There are many possible examples where both the primal and dual LP are infeasible.

Here's a simple 1-variable example. Consider the primal LP *maximize:* x , *Subject to:* $0 \cdot x \leq -1$, $x \leq 0$.

The dual LP is *minimize:* -y , *Subject to:* $0 \cdot y \geq 1$, $y \geq 0$.

Clearly, we cannot have $0x \leq -1$ and we cannot have $0y \geq 1$. So, both the primal and dual are infeasible.

If you are bothered by the 0 coefficients in this example (you shouldn't be), then here is another example with two variables in both the primal and dual LP:

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad b = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad , \quad c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The primal LP is *maximize:* $c^T x$, *Subject to:* $Ax \leq b$, $x \geq 0$.
The dual LP is *minimize:* $b^T y$, *Subject to:* $y^T A \geq c^T$, $y \geq 0$.

More explicitly, the primal LP is:

Maximize $x_1 + x_2$
Subject to:
$x_1 \leq -1$
$-x_2 \leq -1$
$x_1, x_2 \geq 0$

This is clearly infeasible, since we can't have both $x_1 \leq -1$ and $x_1 \geq 0$. The dual LP is:

Minimize $-y_1 - y_2$
Subject to:
$y_1 \geq 1$
$-y_2 \geq 1$
$y_1, y_2 \geq 0$

This is again clearly infeasible, since we can't have both $-y_2 \geq 1$, meaning $y_2 \leq -1$, and $y_2 \geq 0$.

So, both the primal and the dual in this example are infeasible.