1 Q1

(a) There are total 4 NE in this game G, one mixed NE and three pure NE. The results are showed in the Table 1.

player 1's strategy	player 2's strategy	player 1's payoff	player 2's payoff
(0, 1/5, 4/5, 0)	(0, 0, 2/3, 1/3)	7	5
(0, 0, 0, 1)	(1, 0, 0, 0)	4	3
(0, 1, 0, 0)	(0, 0, 0, 1)	7	9
(0, 0, 1, 0)	(0, 0, 1, 0)	8	5

Table 1: The NEs in game G.

For pure NEs, we first find player 1's best response given each player 2's pure strategy, which is marked in red in the following matrix. Likewise, we then find player 2's best response given each player 1's pure strategy, which is marked in cyan in the following matrix. Pure NEs exist when both elements in the same bracket of the bimatrix are marked, and we can see from the matrix given below that there are only three NEs in this bimatrix.

$$\begin{bmatrix} (3,6) & (3,7) & (7,8) & (6,7) \\ (4,4) & (4,6) & (7,5) & (7,9) \\ (4,0) & (5,4) & (8,5) & (5,4) \\ (4,3) & (1,2) & (4,3) & (1,3) \end{bmatrix}$$

For mixed NE, using the corollary of the Nash theorem, if player 2 is playing against player 1's mixed strategy, both of player 2's pure strategies must be the best response to player 1. The same argument applies to player 1. We assume the probabilities of player 1 plays its strategies are p_1, p_2, p_3, p_4 and the probabilities of player 1 plays its strategies are q_1, q_2, q_3, q_4, where $p_1 + p_2 + p_3 + p_4 = 1$ and $q_1 + q_2 + q_3 + q_4 = 1$. Based on that, we got a new mixed NE: ((0, 1/5, 4/5, 0), (0, 0, 2/3, 1/3)).

For player 1, the payoffs of its four pure strategies given fixed player 2's strategy are 20/3, 7, 7, 3, which are smaller or equal to player 1's expected payoff 7. Likewise, for player 2, the payoffs of its four pure strategies given fixed player 1's strategy are 4/5, 22/5, 5, 5, which are smaller or equal to player 2's expected payoff 5. Besides, there is only one mixed solution after solving the arithmetic, thus, ((0, 1/5, 4/5, 0), (0, 0, 2/3, 1/3)) is indeed an NE of G and there is no other mixed strategy in this game.

Similarly, the three pure NEs we got are also the best responses for both players separately, under other players' all other pure strategies. For example, for pure NE ((0, 0, 1, 0), (0, 0, 1, 0)), the payoffs of its four pure strategies given fixed player 2's strategy are 7,7,8,4, which are smaller or equal to player 1's expected payoff 8. Likewise, for player 2, the payoffs of its four pure strategies given fixed player 1's strategy are 0,4,5,4, which are smaller or equal to player 2's expected payoff 5. Besides, the overlaps indeed appear in the right corresponding positions in the bimatrix. Therefore, the three pure NEs are indeed NEs of G and there is no other pure strategy in this game.

(b) From the NEs we got from (a), we found that there are no strategies of player 1 where $x_{1,1}$ is bigger than 1/2, i.e., in the 3-player finite normal form game, player 3 will always play its strategy 1 to gain the best payoff, no matter what strategies player 1 and player 2 play. Below is one example of game G' that satisfies the conditions. Due to the difficulty of drawing a 3D matrix, we simply use two 2D matrices, where the left matrix is when player 3 plays its strategy 1, and the right matrix is when player 3 plays its strategy 2. After fixing the other two-player strategies, the best responses of player 3 are marked by brown.

$$\begin{bmatrix} (3,6,1) & (3,7,1) & (7,8,1) & (6,7,1) \\ (4,4,1) & (4,6,1) & (7,5,1) & (7,9,1) \\ (4,0,1) & (5,4,1) & (8,5,1) & (5,4,1) \\ (4,3,1) & (1,2,1) & (4,3,1) & (1,3,1) \end{bmatrix} \begin{bmatrix} (3,6,0) & (3,7,0) & (7,8,0) & (6,7,0) \\ (4,4,0) & (4,6,0) & (7,5,0) & (7,9,0) \\ (4,0,0) & (5,4,0) & (8,5,0) & (5,4,0) \\ (4,3,0) & (1,2,0) & (4,3,0) & (1,3,0) \end{bmatrix}$$

From this example, we can see that the new trimatrix keeps the all the pairs (a,b) in bimatrix that store the payoffs of player 1 and player 2. In this trimatrix, player 3's strategy 2 is strictly dominated by its strategy 1, which corresponds the definition of $\pi_{3,g1(x1)}$. Therefore, in this example game G', we can get the $NE(G')$ as Table 2. In summary, there exist a 3-player finite normal form game, G' that satisfies $NE(G')$.

player 1's strategy	player 2's strategy	player 3's strategy	player 1, 2, 3's payoffs
(0, 1/5, 4/5, 0)	(0, 0, 2/3, 1/3)	(1, 0)	7, 5, 1
(0, 0, 0, 1)	(1, 0, 0, 0)	(1, 0)	4, 3, 1
(0, 1, 0, 0)	(0, 0, 0, 1)	(1, 0)	7, 9, 1
(0, 0, 1, 0)	(0, 0, 1, 0)	(1, 0)	8, 5, 1

Table 2: The NEs in game G.