UNIVERSITY OF EDINBURGH

COLLEGE OF SCIENCE AND ENGINEERING

SCHOOL OF INFORMATICS

INFR11020 ALGORITHMIC GAME THEORY AND ITS APPLICATIONS

Friday 21st May 2021

13:00 to 15:00

INSTRUCTIONS TO CANDIDATES

Answer any TWO of the three questions. If more than two questions are answered, only QUESTION 1 and QUESTION 2 will be marked.

All questions carry equal weight.

This is an OPEN BOOK examination.

Year 4 Courses

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External Examiners: J.Bowles, S.Rogers, H.Vandierendonck

THIS EXAMINATION WILL BE MARKED ANONYMOUSLY

1. (a) Consider the following 2-player normal form (bimatrix) game:

$$\begin{bmatrix} (7,1) & (4,3) & (8,7) \\ (3,5) & (9,4) & (5,4) \\ (2,5) & (3,4) & (8,3) \end{bmatrix}$$

Identify all Nash equilibria in this game. Explain why there are no other Nash equilibria, other than the ones you have identified. [*9 marks*]

(b) In the game considered in part (a), what is the greatest value r for which the following statement is true: "Player 2 is guaranteed to get expected payoff at least r in any Nash equilibrium of this game". Explain your answer. [*4 marks*]

(c) Consider a finite game, G, with pure strategy sets $S_1, \ldots, S_n$ for the n players, and with a payoff function $u_i(s)$ for each player $i \in \{1, \ldots, n\}$ that assigns a payoff to each pure strategy profile
$s = (s_1, \ldots, s_n) \in S = S_1 \times S_2 \times \ldots \times S_n$.

Now consider a different n-player game, G', which has exactly the same strategy sets $S_1, \ldots, S_n$, as G, but where the payoff function $u'_i(s)$ for each player i differs from $u_i(s)$ as follows:

$$u'_i(s) := u_i(s) + g_i(s_{-i})$$

where, for each player i, $g_i : S_{-i} \to \mathbb{R}$ is a function (any function) that depends *only* on the other players' pure strategies and not on player i's own pure strategy.

Prove that G and G' have exactly the same Nash equilibria. In other words, prove that a mixed strategy profile $x = (x_1, \ldots, x_n)$ is a NE for G if and only if it is a NE for G'.

[*12 marks*]

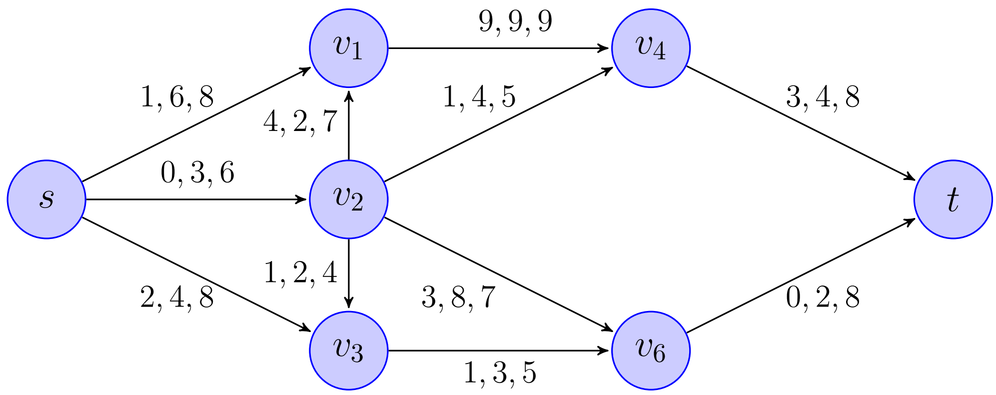

Figure 1: Question 2(b): An atomic network congestion game

2. (a) Consider the following Linear Program (LP).

Maximize $2x_1 + x_2$
Subject to:
$x_1 + x_2 + x_3 = 4$
$x_1 + 4x_2 + x_4 = 8$
$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.

i. Find the dual LP for this LP. [*5 marks*]

ii. Compute an optimal feasible solution & optimal value for the dual LP. [*5 marks*]

(b) Consider the *atomic network congestion game*, with three players, described by the directed graph in Figure 1. In this game, every player i (for $i = 1, 2, 3$) needs to choose a directed path from the source s to the target t. Thus, every player i's set of possible actions (i.e., its set of pure strategies) is the set of all possible directed paths from s to t. Each edge e is labeled with a sequence of three numbers (c_1, c_2, c_3). Given a profile $\pi = (\pi_1, \pi_2, \pi_3)$ of pure strategies (i.e., s-t-paths) for all three players, the *cost* to player i of each directed edge, e, that is contained in player i's path π_i, is c_k, where k is the total number of players that have chosen edge e in their path. The total cost to player i, in the given profile π, is the sum of the costs of *all* the edges in its path π_i from s to t. Each player wants to minimize its own total cost.

Compute a pure strategy Nash Equilibrium in this atomic network congestion game. Compute the total cost to each player in that NE. [*9 marks*]

(c) In lectures, we defined *best response dynamics* for *Congestion Games*, where in each "improvement step" a *single* player who can strictly reduce its own cost by switching its own pure strategy unilaterally, makes such a switch. We showed (in the proof of Rosenthal's theorem), that such a best response dynamics converges to a pure Nash equilibrium, starting from any pure strategy profile in any finite congestion game.

Now imagine that we instead define a *simultaneous best response dynamics* in which, at each "improvement step", *every* player who can strictly improve (i.e., reduce) its own cost by switching its own pure strategy unilaterally, makes such a switch. Does this "simultaneous best response dynamics" also converge to a pure Nash equilibrium starting with any pure strategy profile in any finite congestion game? Explain your answer. [*6 marks*]

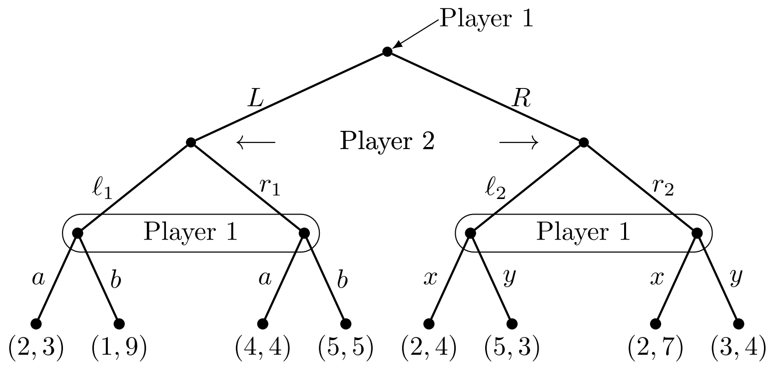

Figure 2: Question 3(a)

3. (a) Consider the 2-player extensive form game (of imperfect information) depicted in Figure 2. (At leaves, the left payoff is for player 1, and the right payoff is for player 2.) Describe all subgame-perfect Nash Equilibria in this game, in terms of behavior strategies. [*10 marks*]

   (b) In a VCG-based auction, three *identical* items are being auctioned simultaneously. Suppose there are three buyers (bidders), A, B, and C who provide their claimed valuation functions v_A, v_B, and v_C as follows; $v_x(j)$ denotes the value, in British pounds, that player x has for receiving j items:

	valuation			
bidder x	$v_x(0)$	$v_x(1)$	$v_x(2)$	$v_x(3)$
$x := A$	0	10	17	23
$x := B$	0	7	16	22
$x := C$	0	8	15	21

   An allocation outcome for this auction is specified by giving three numbers $j_A, j_B, j_C \in \{0,1,2,3\}$, such that for each $x \in \{A,B,C\}$, j_x is the number of (identical) items allocated to bidder x, and such that $j_A + j_B + j_C \leq 3$. Each bidder will also be asked to pay a certain amount (in British pounds), p_A, p_B, and p_C, respectively, for their allocation.

   What are VCG allocations, and VCG payments, for this auction? In other words, how many items will each bidder get, and what price will each pay for the items they get, if the VCG mechanism is used. [*10 marks*]

   (c) Recall that the Gibbard-Satterthwaite theorem says that if there are at least 3 choices, i.e., if $|C| \geq 3$, then any *incentive compatible* social choice function $f : L^n \to C$, $n \geq 1$, for which all outcomes are possible (i.e., f is a surjective (onto) function), is a *dictatorship.*

   Consider an election with 3 candidates, $C = \{X, Y, Z\}$. Consider the *plurality* voting rule, $h : L^n \to C$, which chooses, as the society's choice, the candidate who was ranked as the "top" candidate (the highest in the total order provided by a voter) by the most number of voters, and if there are ties for which candidate got the most top votes, the alphabetically first candidate among those with the most top votes is chosen.

Clearly, this voting rule, h, is not a dictatorship when there are $n > 1$ voters. Prove that it is not incentive compatible, by giving an example of an election with three voters, together with their actual preference orders, such that one of the three voters, knowing the preference orders provided by the other voters, would prefer to lie about its own actual preference order, under the above plurality voting rule. Explain your answer. [*5 marks*]