Natural Language Understanding, Generation, and Machine Translation

Lecture 4: Perceptrons

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Overview

Reading: Jurafsky and Martin (2021), Chapter 7.

Agenda for Today

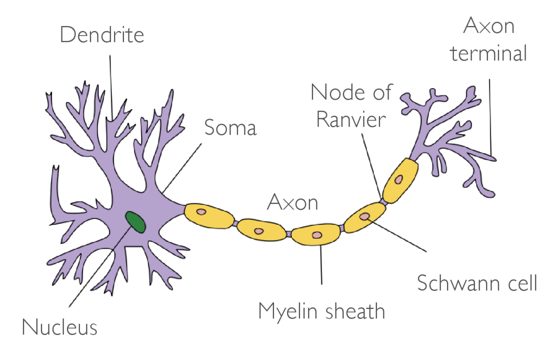
Last lecture: we reviewed some simple probabilistic models.

This lecture: we will review some basic nonlinear models: neural networks.

Next lecture: we'll combine these ideas by parameterizing a simple probabilistic model with a neural network!

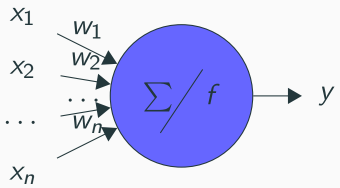
Simple Perceptrons

A (very) loose inspiration: biological neural networks



- Neuron receives inputs and combines these in the cell body.
- If the combined input reaches a threshold, then the neuron may fire (produce an output).
- Some inputs are excitatory, while others are inhibitory.

The perceptron: a very simple artificial neuron



- Input is a vector $\mathbf{x}$, whose ith element is x_i. To model a logical function, let each x_i be in the range [0, 1], where 0 is "off" and 1 is "on"—though this isn't necessary in general.
- Weight vector $\mathbf{w}$ is of same length as $\mathbf{x}$. Each w_i can be any real number (positive or negative).

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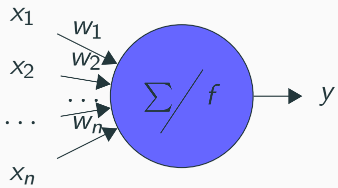

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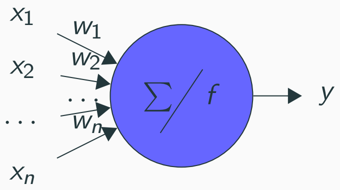
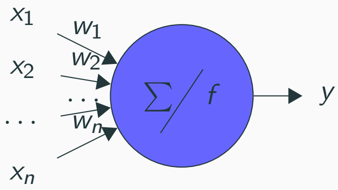


Input function:

$u(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i$

The perceptron: a very simple artificial neuron



Input function:

$u(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x}$

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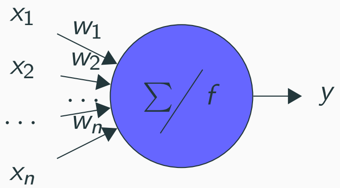

Activation function: threshold

Input function:
$u(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x}$

$$y = f(u(\mathbf{x})) = \begin{cases} 1, & \text{if } u(\mathbf{x}) > \theta \\ 0, & \text{otherwise} \end{cases}$$

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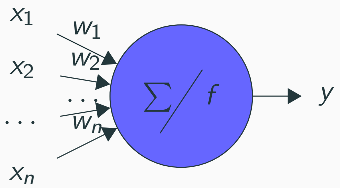

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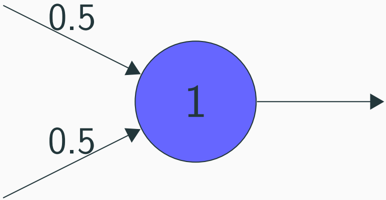
Activation state:
0 or 1

Perceptrons can represent logic functions

Perceptron for AND



f: if $u(\mathbf{x}) \geq 1$ then 1 else 0

x_1	x_2	x_1 AND x_2
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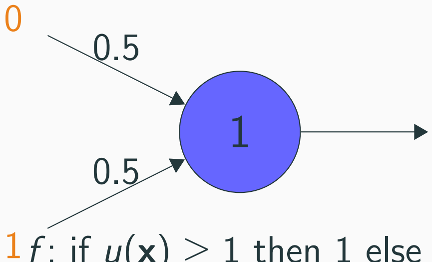

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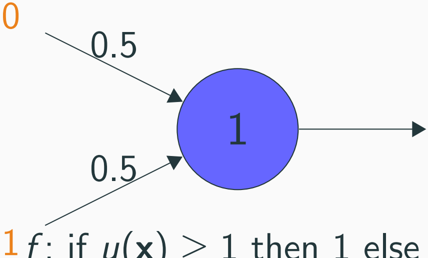

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$$0 \cdot 0.5 + 1 \cdot 0.5 = 0.5 < 1$$

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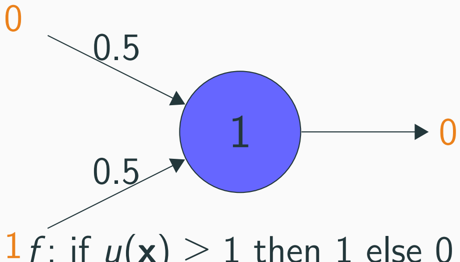

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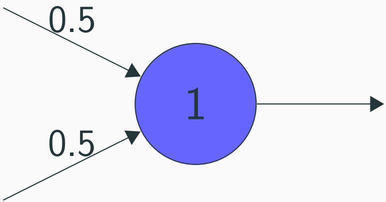

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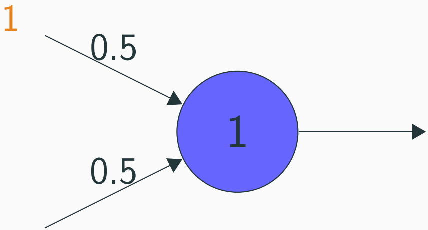

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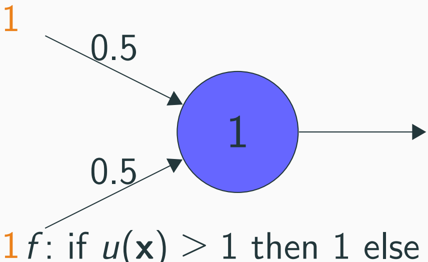

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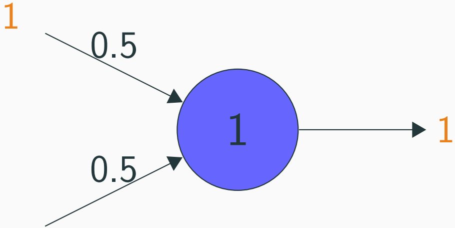

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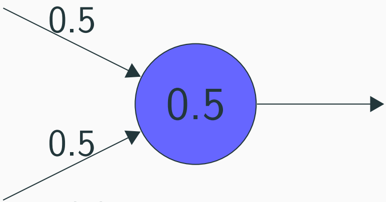
Perceptron for OR



f: if $u(\mathbf{x}) \geq 0.5$ then 1 else 0

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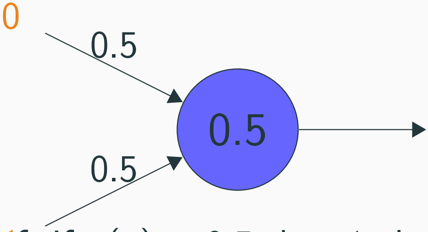

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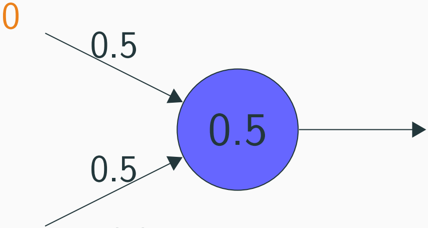

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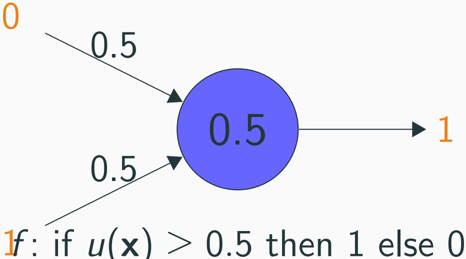

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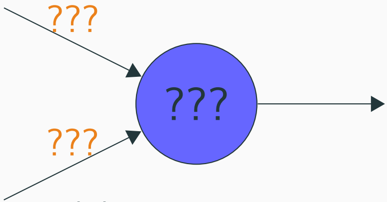
How would you represent NOT(OR)?

Perceptron for NOT(OR)



f: if $u(\mathbf{x}) \geq$??? then 1 else 0

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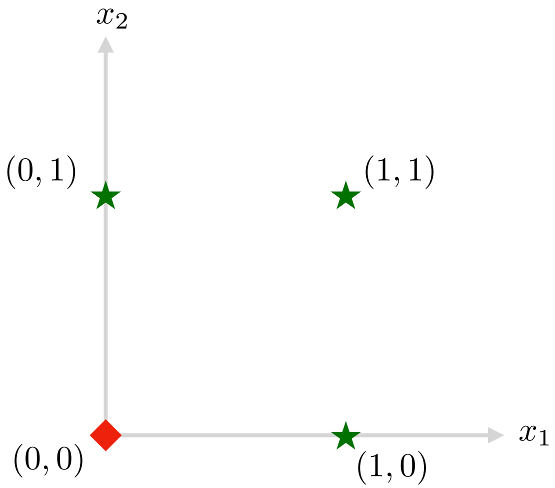
Perceptrons are linear classifiers (OR example)



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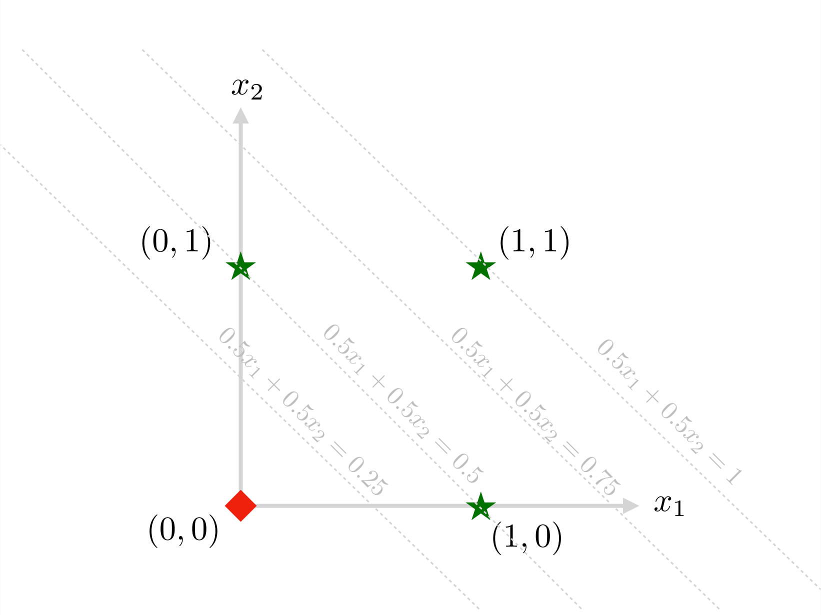

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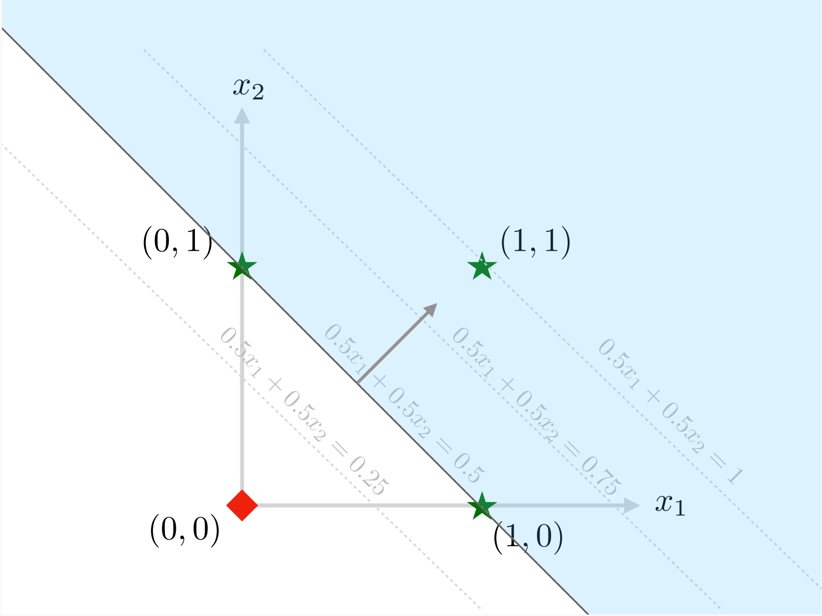

Perceptron can learn (some) logic functions from examples

Give some examples to the Perceptron:

N	**input** x	**target** t
1	(0,1,0,0)	1
2	(1,0,0,0)	0
3	(0,1,1,1)	0
4	(1,0,1,0)	0
5	(1,1,1,1)	1
6	(0,1,0,0)	1
...	...	...

- Input: a vector of 1's and 0's—a feature vector.
- Output: a 1 or 0, given as the target.

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- Input: a vector of 1's and 0's—a feature vector.
- Output: a 1 or 0, given as the target.
- How do we efficiently find the weights and threshold?

Learning

Q_1: Choosing weights and threshold θ for the perceptron is not easy! What's an effective way to learn the weights and threshold from examples?

A_1: We use a learning algorithm that adjusts the weights and threshold based on examples.

Simplify by converting θ into a weight

Original: $\mathbf{w} \cdot \mathbf{x} > \theta$

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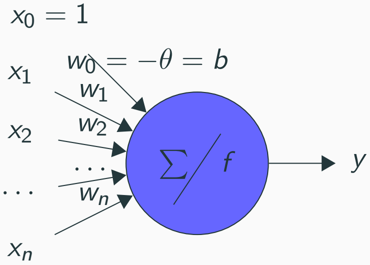

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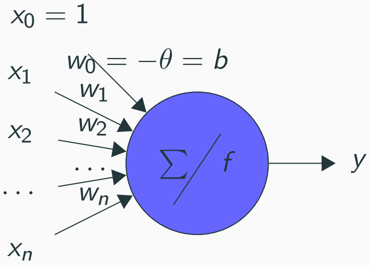

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Activation function: threshold

New input function:
$u(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x} + b$
Linear!

$$y = f(u(\mathbf{x})) = \begin{cases} 1, & \text{if } u(\mathbf{x}) > 0 \\ 0, & \text{otherwise} \end{cases}$$

Activation state:
0 or 1

Nonlinear!

Learn by adjusting weights whenever output $\neq$ target

Intuition: classification depends on the sign ($+$ or $-$) of the output. If output has a different sign than the target, adjust weights to move output in the direction of 0 (that is, towards the classification boundary).

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Notice: the sign of $t - o$ is the direction we want to move in.

Learn by adjusting weights whenever output $\neq$ target

Perceptron Learning Rule

$$w_i \leftarrow w_i + \Delta w_i$$

(t-o) gives the direction
x_i gives how much to update

$$\Delta w_i \ = \eta(t - o)x_i$$

- η, $0 < \eta \leq 1$ is a constant called the learning rate.
- t is the target output of the current example.
- o is the output of the Perceptron with the current weights.

Learning Rule

Perceptron Learning Rule

$$w_i \leftarrow w_i + \Delta w_i$$

$$\Delta w_i \; = \eta(t - o)x_i$$

$o = 1$ and $t = 1$

$o = 0$ and $t = 1$

- Learning rate η is positive; controls how big changes Δw_i are.
- If $x_i > 0$, $\Delta w_i > 0$. Then w_i increases in an so that $w_i x_i$ becomes larger, increasing $u(\mathbf{x})$.
- If $x_i < 0$, $\Delta w_i < 0$. Then w_i reduces so that the absolute value of $w_i x_i$ becomes smaller, increasing $u(\mathbf{x})$.

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Learning Algorithm

```
Initialize all weights randomly.
repeat
    for each training example do
        Apply the learning rule.
    end for
until the error is acceptable or a certain number of iterations is reached
```

update within batches could affect the weight changes

Learning Algorithm

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This algorithm is guaranteed to find a solution with zero error in a limited number of iterations **if** the examples are linearly separable.

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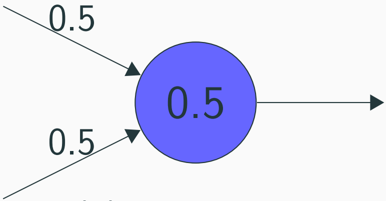
http://www.youtube.com/watch?v=vGwemZhPlsA&feature=youtu.be

Perceptrons can represent some logic functions... but not all!

Perceptron for XOR



f if $u(\mathbf{x}) \geq \theta$ then 1 else 0

x_1	x_2	x_1 XOR x_2
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XOR is an exclusive OR because it only returns a true value of 1 if the two values are exclusive, i.e., they are both different.

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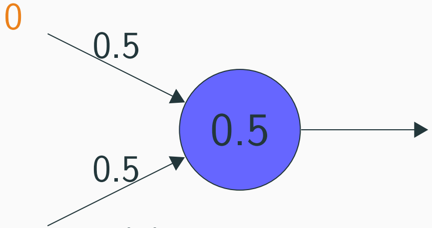

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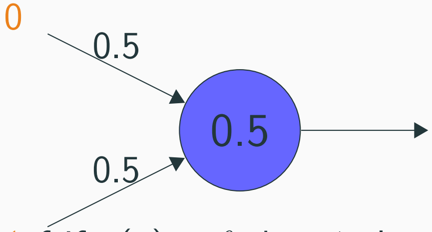

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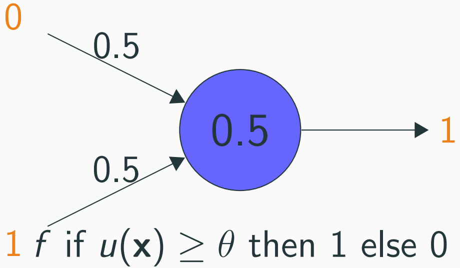
Perceptron for XOR



f if $u(\mathbf{x}) \geq \theta$ then 1 else 0

$$0 \cdot 0.5 + 1 \cdot 0.5 = 0.5$$

x_1	x_2	x_1 XOR x_2
0	0	0
0	1	1
1	0	1
1	1	0

XOR is an exclusive OR because it only returns a true value of 1 if the two values are exclusive, i.e., they are both different.

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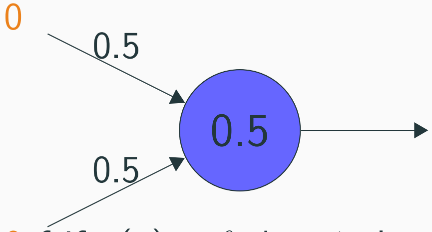

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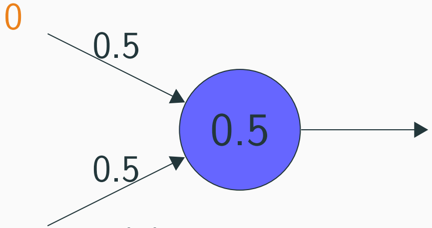

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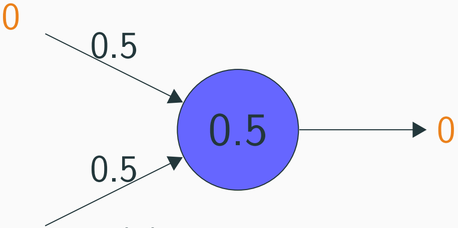

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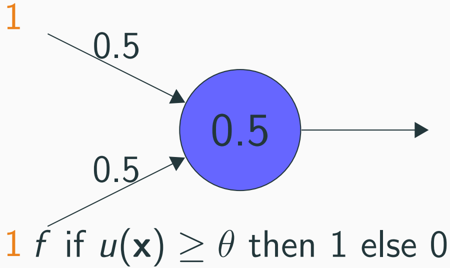

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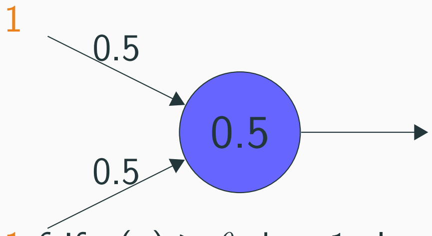

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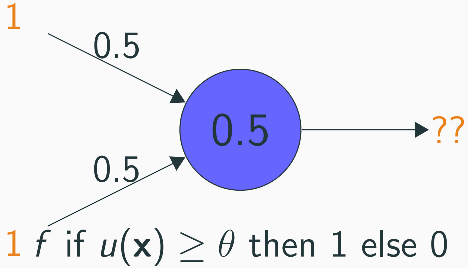

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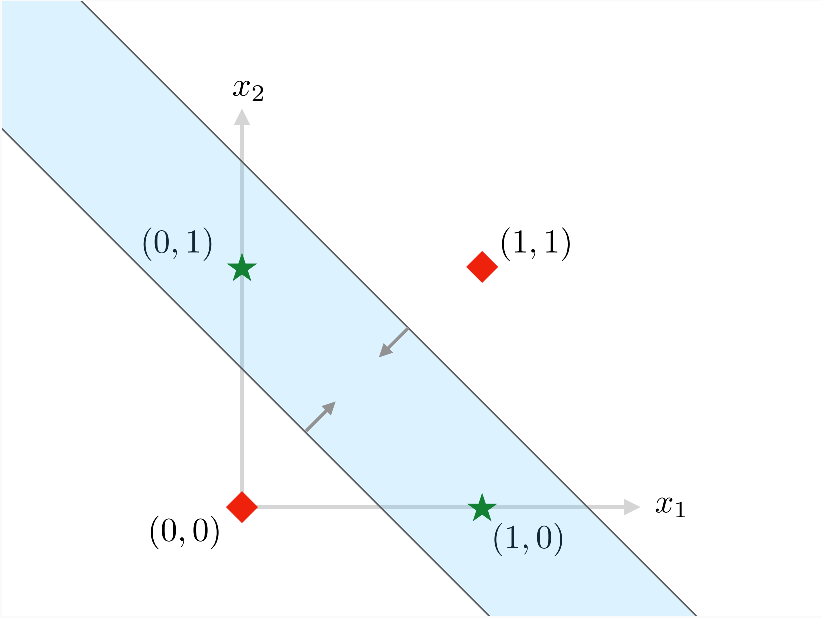
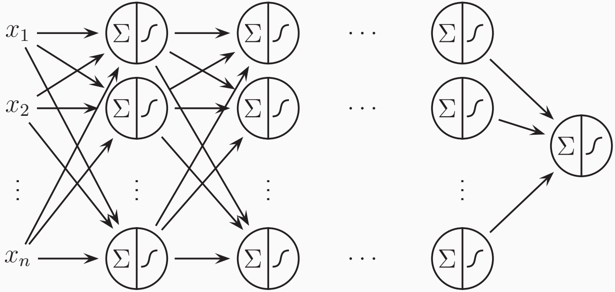
XOR is an exclusive OR because it only returns a true value of 1 if the two values are exclusive, i.e., they are both different.

Problem: XOR is not linearly separable



Multilayer perceptrons

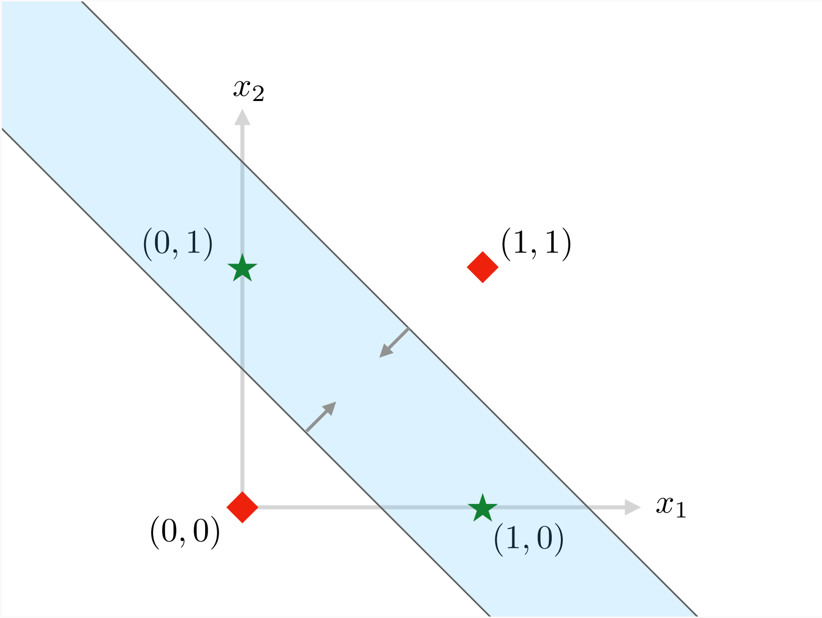
Multilayer Perceptrons (MLPs) are more expressive



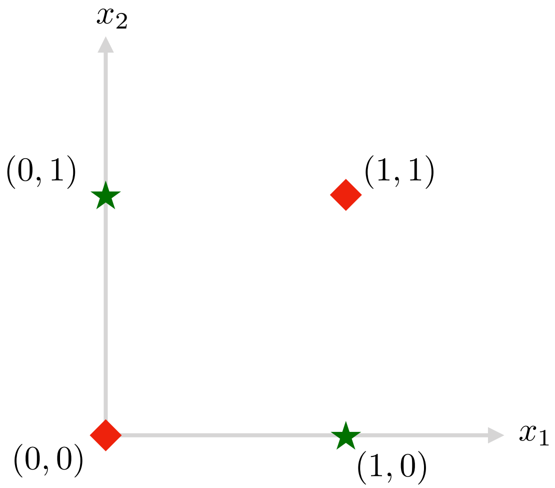
- MLPs are feed-forward neural networks, organized in layers.
- One input layer, one or more hidden layers, one output layer.
- Each node in a layer connected to all other nodes in next layer.
- Each connection has a weight (can be zero).

Q: How would you represent XOR?

MLP combines multiple separators



Hidden layer of MLP computes new representation of data

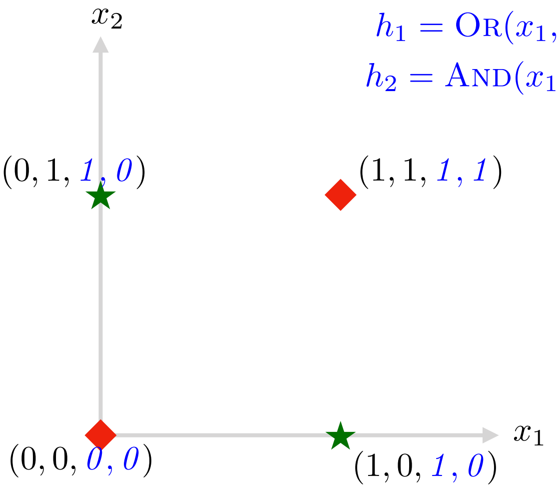


Hidden layer of MLP computes new representation of data

(x_1, x_2, h_1, h_2)

$h_1 = \textsc{Or}(x_1, x_2)$

$h_2 = \textsc{And}(x_1, x_2)$



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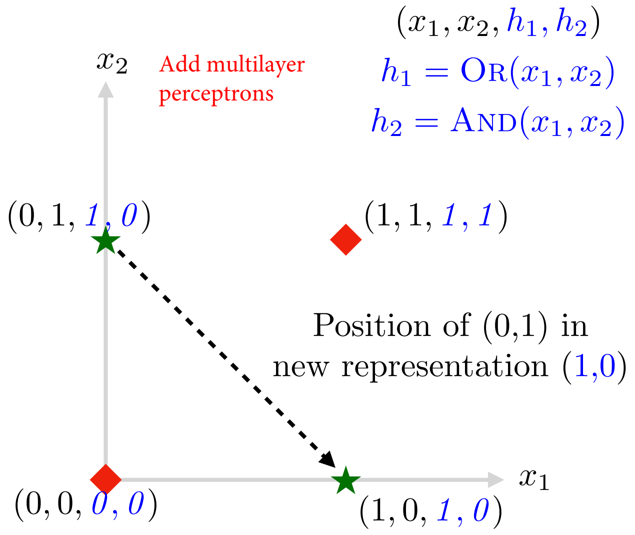

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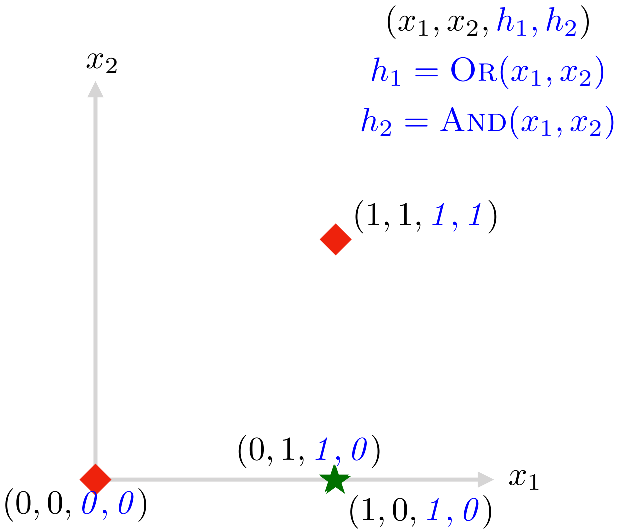

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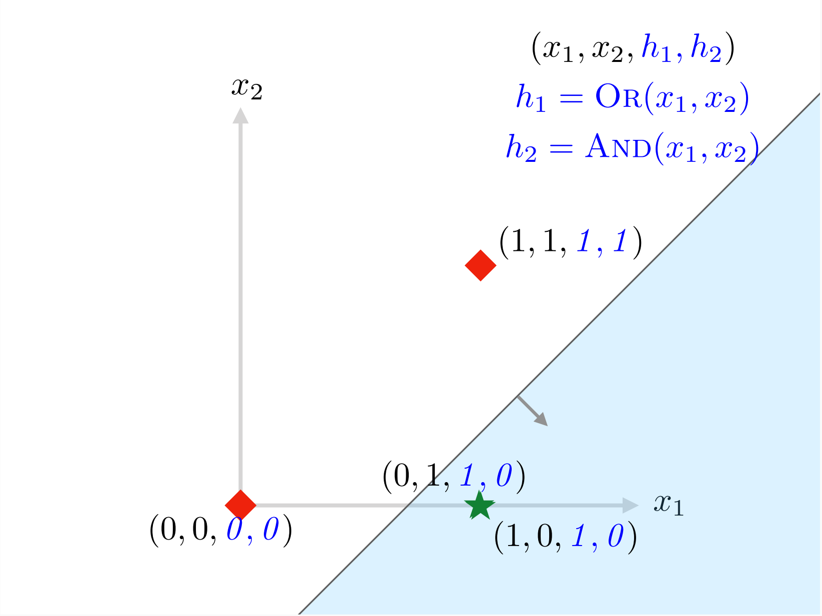
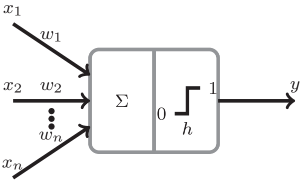
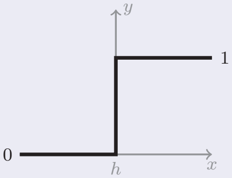


We can use activation functions other than thresholds



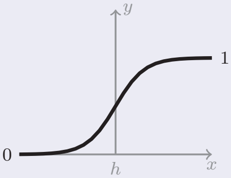
Q: Why many hidden layers rather than a big layer?
A: This is hard to train, overfitting, seperate the parameter into each samll hidden layers.

Step function



Outputs 0 or 1.

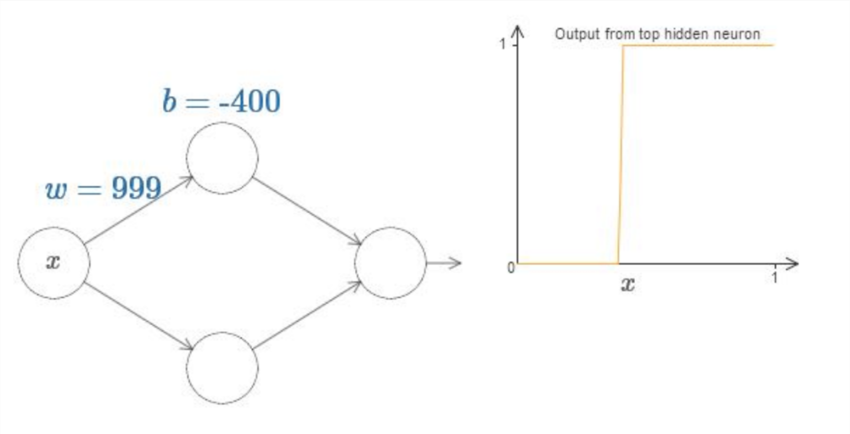
Sigmoid function



Outputs a real value between 0 and 1.

With sigmoid function we get **smooth transitions** instead of hard decision boundaries.

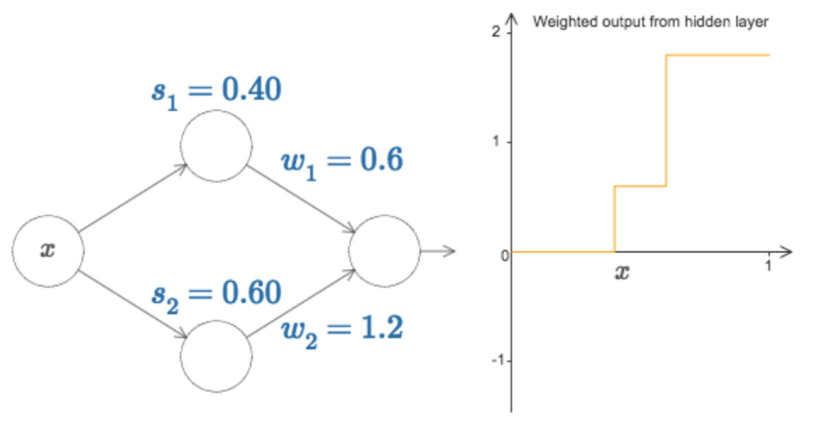
MLPs are universal function approximators



Each hidden unit can approximate a step function of the input.

Source: http://neuralnetworksanddeeplearning.com

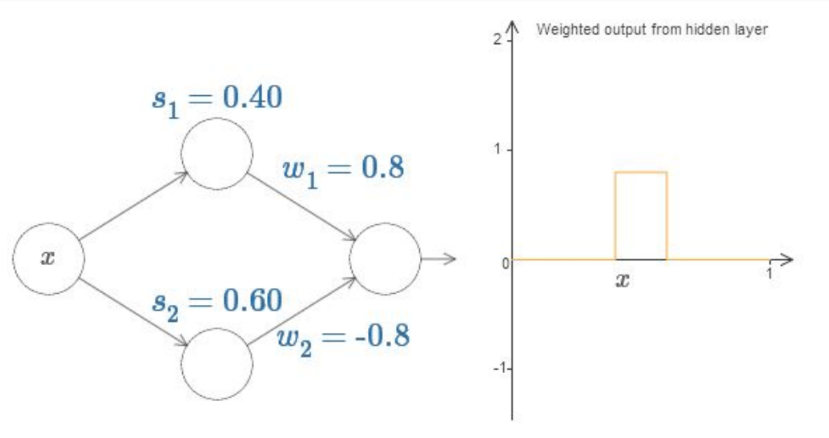
MLPs are universal function approximators



With multiple hidden units, you can get multiple steps.

Source: http://neuralnetworksanddeeplearning.com

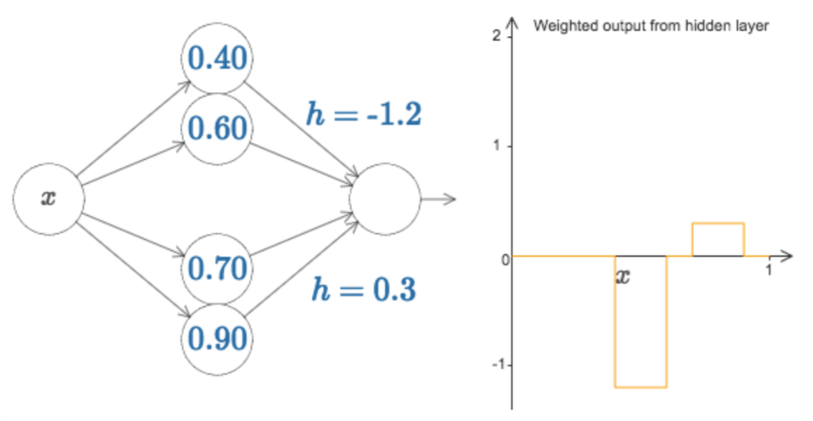
MLPs are universal function approximators



The steps can be positive or negative.

Source: http://neuralnetworksanddeeplearning.com

MLPs are universal function approximators



With more hidden units, you can represent more steps.

Source: http://neuralnetworksanddeeplearning.com

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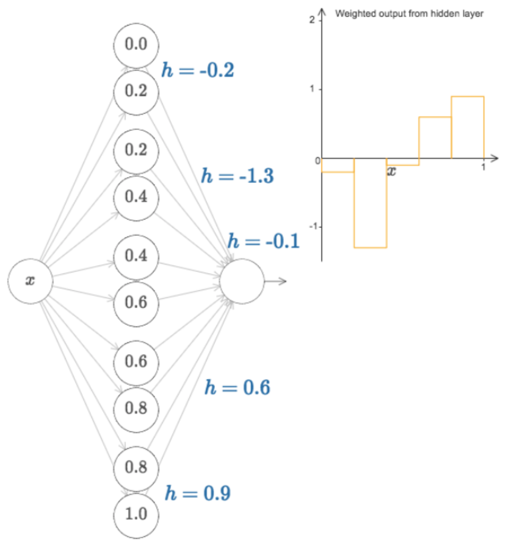
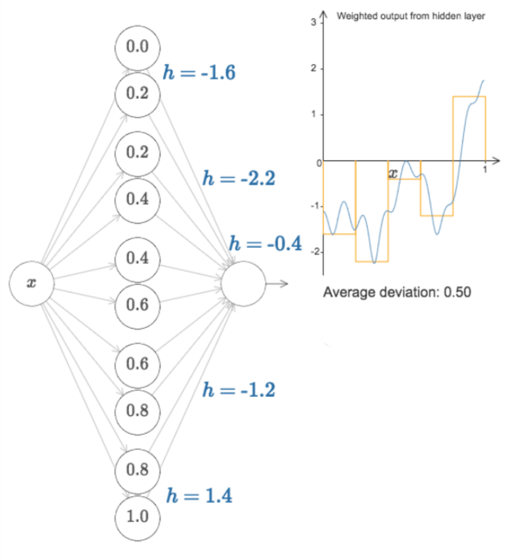


Many hidden units produce more complex functions.

MLPs are universal function approximators



Add hidden units to approximate very complex functions.

Summary

- A perceptron is a simple learnable function.
- A simple perceptron can express linearly separable functions.
- The perceptron itself is a non-linear function.
- There is a simple learning rule for the perceptron.
- A multilayer perceptron is more expressive than a perceptron, and in fact is a universal function approximator.
- The hidden layers of a multilayer perceptron compute a new representation of the input data.

Next lecture: We will use multilayer perceptrons to represent n-gram probabilities.

References

Jurafsky, Daniel, and James H. Martin. 2021. *Speech and Language Processing: An Introduction to Natural Language Processing, Computational Linguistics and Speech Recognition*. Upper Saddle River, NJ: Pearson Education, draft of 3rd edn. https://web.stanford.edu/~jurafsky/slp3/.