Math tests

1. $var(\bar{y}_i) = var\left(\frac{1}{N_i}\sum_{j=1}^{N_i} y_{ij}\right) = \frac{1}{N_i^2}\sum_{j=1}^{N_i} var(y_{ij})$

$$= \frac{\sigma^2}{N_i}$$

So the variance of sample mean gets larger with smaller sample size

2. $y_{ij} \sim N(\theta_i, \sigma^2) \quad \theta_i \sim N(\mu, \tau^2\sigma^2) \qquad i=1,\dots n$

$j = 1, \dots N_i$

priors for $(\mu, \lambda = \frac{1}{\sigma^2}, \eta = \frac{1}{\tau^2\sigma^2}$

$\mu \sim N(m, v) \quad \lambda \sim Ga(a, b) \quad \eta \sim Ga(c, d)$

poterior

$p(\mu, \lambda, \eta, \theta \mid y)$

$\propto p(y \mid \theta, \lambda)\, p(\theta \mid \mu, \eta)\, p(\mu)\, p(\lambda)\, p(\eta)$

$= \prod_{i=1}^{n}\prod_{j=1}^{N_i} p(y_{ij} \mid \theta_i, \lambda) \prod_{i=1}^{n} p(\theta_i \mid \mu, \eta)\, p(\mu)\, p(\lambda)\, p(\eta)$

$\propto \prod_{i=1}^{n}\prod_{j=1}^{N_i} \lambda^{\frac{1}{2}} \exp\left[-\frac{\lambda}{2}(y_{ij}-\theta_i)^2\right] \prod_{i=1}^{n} \eta^{\frac{1}{2}} \exp\left[-\frac{\eta}{2}(\theta_i-\mu)^2\right]$

$\exp\left[-\frac{1}{2v^2}(\mu-m)^2\right] \lambda^{a-1} \exp(-b\lambda)\, \eta^{c-1} \exp(-c\eta)$

$$= \exp\left[-\frac{\lambda}{2}\sum_{i,j}(y_{ij}-\theta_i)^2 - \frac{\eta}{2}\sum_{i=1}^{n}(\theta_i-\mu)^2 - \frac{1}{2V^2}(\mu-m)^2\right]$$

$$\lambda^{a+\frac{1}{2}\sum_{i=1}^{n}N_i-1}\exp(-b\lambda)\ \eta^{c+\frac{n}{2}-1}\exp(-d\eta)$$

posterior full conditionals

$$p(\mu|-) \propto \exp\left[-\frac{\eta}{2}\sum_{i=1}^{n}(\theta_i-\mu)^2 - \frac{1}{2V}(\mu-m)^2\right]$$

$$\propto \exp\left[-\frac{\eta}{2}\left(n\mu^2 - 2\sum_{i=1}^{n}\theta_i\mu\right) - \frac{1}{2V}(\mu^2-2m\mu)\right]$$

$$= \exp\left\{-\frac{1}{2}\left[\left(n\eta+\frac{1}{V}\right)\mu^2 - 2\left(\eta\sum_{i=1}^{n}\theta_i+\frac{m}{V}\right)\mu\right]\right\}$$

$$\propto \exp\left\{-\frac{\left(n\eta+\frac{1}{V}\right)}{2}\left(\mu-\frac{\eta\sum_{i=1}^{n}\theta_i+\frac{m}{V}}{n\eta+\frac{1}{V}}\right)^2\right\}$$

$$\sim N\left(\left(n\eta+\frac{1}{V}\right)^{-1}\left(\eta\sum_{i=1}^{n}\theta_i+\frac{m}{V}\right),\ \left(n\eta+\frac{1}{V}\right)^{-1}\right)$$

$$p(\lambda|-) \sim Ga\left(a+\frac{\sum_{i=1}^{n}N_i}{2},\ b+\frac{1}{2}\sum_{ij}(y_{ij}-\theta_i)^2\right)$$

$$p(\eta|-) \sim Ga\left(c+\frac{n}{2},\ d+\frac{1}{2}\sum_{i=1}^{n}(\theta_i-\mu)^2\right)$$

$$p(\theta_i|-) \propto \exp\left[-\frac{\lambda}{2}\sum_{j=1}^{N_i}(y_{ij}-\theta_i)^2 - \frac{\eta}{2}(\theta_i-\mu)^2\right]$$

$$\sim N\left((\lambda N_i+\eta)^{-1}\left(\lambda\sum_{j=1}^{N_i}y_{ij}+\eta\mu\right),\ (\lambda N_i+\eta)^{-1}\right)$$

Cheese

$$\log Q = \beta_0 + \beta_1 \log P + \beta_2 D + \beta_3 D \log P$$

$$y_{ij} = x_{ij}^T \beta_i + \varepsilon_{ij} \qquad \varepsilon_{ij} \sim N(0, \sigma^2)$$

$$y_{ij} \sim N(x_{ij}^T \beta_i, \sigma^2)$$

$$i = 1, \dots, n \qquad j = 1, \dots, m_i \qquad N = \sum_{i=1}^{n} m_i$$

$$x_{ij} = (1, \log P_{ij}, D_{ij}, D_{ij} \log P_{ij}) \in \mathbb{R}^k \qquad k = 4$$

$$\beta_i \sim N(\mu, \Sigma)$$

Priors $\quad \lambda = \frac{1}{\sigma^2} \sim Ga(\frac{1}{2}, \frac{1}{2}) \qquad \mu \sim N(v, K)$

$\qquad\qquad \Sigma \sim IW(I, 1)$

Posterior $\quad p(\beta, \lambda, \mu, \Sigma \mid Y) \propto$

$$\prod_{i,j} p(y_{ij} \mid \beta_i, \lambda) \prod_i p(\beta_i \mid \mu, \Sigma) \; p(\lambda) \, p(\mu) \, p(\Sigma)$$

$$\propto \lambda^{\frac{N-1}{2}} \exp(-\tfrac{1}{2}\lambda) \, |\Sigma|^{-\frac{n}{2}} \; p(\Sigma)$$

$$\exp\left[-\frac{\lambda}{2} \sum_{i=1}^{n} \sum_{j=1}^{m_i} (y_{ij} - x_{ij}^T \beta_i)^2 - \frac{1}{2} \sum_{i=1}^{n} (\beta_i - \mu)^T \Sigma^{-1} (\beta_i - \mu) - \frac{1}{2} (\mu - v)^T K^{-1} (\mu - v)\right]$$

Full conditionals

$$p(\beta_i|-) \propto \exp\Big[-\frac{\lambda}{2}\sum_{j=1}^{m_i}(y_{ij}-x_{ij}^T\beta_i)^2 - \frac{1}{2}(\beta_i-\mu)^T\Sigma^{-1}(\beta_i-\mu)\Big]$$

$$= \exp\Big\{-\frac{1}{2}\Big[\beta_i^T\overset{A}{\Big(\lambda\sum_{j=1}^{m_i}x_{ij}x_{ij}^T+\Sigma^{-1}\Big)}\beta_i$$

$$-\beta_i^T\overset{B}{\Big(\lambda\sum_{j=1}^{m_i}y_{ij}x_{ij}+\Sigma^{-1}\mu\Big)} \quad - B^T\beta_i\Big]\Big\}$$

$$\propto \exp\Big[-\frac{1}{2}(\beta_i-A^{-1}B)^TA(\beta_i-A^{-1}B)\Big]$$

$$\sim N(A^{-1}B,\ A^{-1}) \qquad A^{-1} = \Big(\lambda\sum_{j=1}^{m_i}x_{ij}x_{ij}^T+\Sigma^{-1}\Big)^{-1}$$

$$B = \lambda\sum_{j=1}^{m_i}y_{ij}x_{ij}+\Sigma^{-1}\mu$$

$$p(\lambda|-) \propto Ga\Big(\frac{N+1}{2},\ \frac{1}{2}+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(y_{ij}-x_{ij}^T\beta_i)^2\Big)$$

$$p(\mu|-) \propto \exp\Big[-\frac{1}{2}\sum_{i=1}^{n}(\beta_i-\mu)^T\Sigma^{-1}(\beta_i-\mu)-\frac{1}{2}(\mu-\nu)^TK^{-1}(\mu-\nu)\Big]$$

$$\propto \exp\Big\{-\frac{1}{2}\Big[\mu^T\overset{A}{(n\Sigma^{-1}+K^{-1})}\mu + \mu^T\overset{B}{\Big(\Sigma^{-1}\sum_{i=1}^{n}\beta_i+K^{-1}\nu\Big)}$$

$$A^{-1} = (n\Sigma^{-1}+K^{-1})^{-1}$$

$$\sim N(A^{-1}B,\ A^{-1}) \qquad B = \Big(\Sigma^{-1}\sum_{i=1}^{n}\beta_i+K^{-1}\nu\Big)$$

$$p(\Sigma \mid -) \propto$$

$$|\Sigma|^{-\frac{n+k+2}{2}} \exp\left\{-\frac{1}{2} \operatorname{Tr}\left[\sum_{i=1}^{n} (\beta_i - \mu)(\beta_i - \mu)^T \Sigma^{-1} + \Sigma^{-1}\right]\right\}$$

$$\sim IW\left(\sum_{i=1}^{n} (\beta_i - \mu)(\beta_i - \mu)^T + I, \quad n+1\right)$$

Polls — Model 1

$y_{ij} \sim \text{Bernoulli}(p_{ij})$ $\qquad$ $p_{ij} = \Phi(z_{ij})$

Φ is the standard normal CDF

$i = 1, \ldots, n$ $\qquad$ $j = 1, \ldots, m_i$ $\qquad$ $N = \sum_{i=1}^{n} m_i$

$z_{ij} = \mu_i + x_{ij}^T \beta_i$

$\mu_i \sim N(v, \sigma^2)$ $\qquad$ $\lambda = \frac{1}{\sigma^2}$ $\qquad$ $\beta_i \sim N(\varphi, \Sigma)$

Priors : $f(v) \propto 1$ $\qquad$ $f(\lambda) = Ga(\frac{1}{2}, \frac{1}{2})$

$f(\varphi) \propto 1$ $\qquad$ $f(\Sigma) = IW(I, 1)$

Latent variable :

$d_{ij} \sim N(z_{ij}, 1)$ $\qquad$ $y_{ij} = 1$ if $d_{ij} > 0$

Posterior

$$\prod_{i,j} p(y_{ij} | d_{ij}) \, p(d_{ij} | \mu_i, \beta_i) \, p(\mu_i | v, \lambda) \, p(\beta_i | \varphi, \Sigma) \, p(v, \lambda, \varphi, \Sigma)$$

$$\prod_{i=1}^{n} \prod_{j=1}^{m_i} \left[I(y_{ij}=1) I(d_{ij} > 0) + I(y_{ij}=0) I(d_{ij} \leq 0) \right] \exp\left[-\frac{1}{2}(d_{ij} - z_{ij})^2\right]$$

$$\prod_{i=1}^{n} \lambda^{\frac{1}{2}} \exp\left[-\frac{\lambda}{2}(\mu_i - v)^2\right] |\Sigma|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\beta_i - \varphi)^T \Sigma^{-1} (\beta_i - \varphi)\right] \; p(\theta)$$

Gibbs sampler $(d_{ij}, \mu_i, \beta_i, v, \lambda, \varphi, \Sigma)$

$$p(d_{ij} \mid -) \propto I(y_{ij}=1) I(d_{ij}>0) \exp\left[-\tfrac{1}{2}(d_{ij}-z_{ij})^2\right]$$
$$+ I(y_{ij}=0) I(d_{ij}\le 0) \exp\left[-\tfrac{1}{2}(d_{ij}-z_{ij})^2\right]$$

$\sim$ positive $N(z_{ij}, 1)$ if $y_{ij}=1$

$\sim$ negative $N(z_{ij}, 1)$ if $y_{ij}=0$

$$p(\mu_i \mid -) \propto \prod_{j=1}^{m_i} \exp\left[-\tfrac{1}{2}(d_{ij}-x_{ij}^T\beta_i-\mu_i)^2\right] \exp\left[-\tfrac{\lambda}{2}(\mu_i-v)^2\right]$$

$$\propto \exp\left\{-\tfrac{1}{2}\sum_{j=1}^{m_i}\left[\mu_i^2-2(d_{ij}-x_{ij}^T\beta_i)\mu_i\right]-\tfrac{\lambda}{2}(\mu_i^2-2v\mu_i)\right\}$$

$$\sim N\left(\frac{1}{m_i+\lambda}\left[\sum_{j=1}^{m_i}(d_{ij}-x_{ij}^T\beta_i)+\lambda v\right], \quad \frac{1}{m_i+\lambda}\right)$$

$$p(\beta_i \mid -) \propto \exp\left[-\tfrac{1}{2}\sum_{j=1}^{m_i}(d_{ij}-\mu_i-x_{ij}^T\beta_i)^2-\tfrac{1}{2}(\beta_i-\varphi)^T\Sigma^{-1}(\beta_i-\varphi)\right]$$

$$\sim N\left(A^{-1}B, A^{-1}\right)$$

$$A^{-1} = \left(\sum_{j=1}^{m_i} x_{ij}x_{ij}^T + \Sigma^{-1}\right)^{-1} \qquad B = \sum_{j=1}^{m_i}(d_{ij}-\mu_i)\overset{1\times k}{x_{ij}} + \Sigma^{-1}\varphi$$

$$p(\nu|-) \propto \exp\left[-\frac{\lambda}{2}\sum_{i=1}^{n}(\mu_i-\nu)^2\right] \propto \exp\left[-\frac{\lambda}{2}\left(n\nu^2-2\sum_{i=1}^{n}\mu_i\nu\right)\right]$$

$$\sim N\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i, \quad \frac{1}{n\lambda}\right)$$

$$p(\lambda|-) \propto \lambda^{\frac{n-1}{2}} \exp\left[-\frac{\lambda}{2}\sum_{i=1}^{n}(\mu_i-\nu)^2 - \frac{1}{2}\lambda\right]$$

$$\sim Ga\left(\frac{n+1}{2}, \quad \frac{1}{2}+\frac{1}{2}\sum_{i=1}^{n}(\mu_i-\nu)^2\right)$$

$$p(\varphi|-) \propto \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(\beta_i-\varphi)^T\Sigma^{-1}(\beta_i-\varphi)\right]$$

$$\propto \exp\left\{-\frac{1}{2}\left[\varphi^T(n\Sigma^{-1})\varphi - \varphi^T\left(\Sigma^{-1}\sum_{i=1}^{n}\beta_i\right) - \left(\Sigma^{-1}\sum_{i=1}^{n}\beta_i\right)^T\varphi\right]\right\}$$

$$= \exp\left[-\frac{1}{2}\left(\varphi-\frac{1}{n}\sum_{i=1}^{n}\beta_i\right)^T n\Sigma^{-1}\left(\varphi-\frac{1}{n}\sum_{i=1}^{n}\beta_i\right)\right]$$

$$\sim N\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i, \quad \frac{1}{n}\Sigma\right)$$

$$p(\Sigma|-) \propto$$

$$|\Sigma|^{-\frac{n+k+2}{2}} \exp\left\{-\frac{1}{2}\mathrm{Tr}\left[\sum_{i=1}^{n}(\beta_i-\varphi)(\beta_i-\varphi)^T\Sigma^{-1}+\Sigma^{-1}\right]\right\}$$

$$\sim IW\left(\sum_{i=1}^{n}(\beta_i-\varphi)(\beta_i-\varphi)^T+I, \quad n+1\right)$$

Polls — Model 2

$y_{ij} \sim \text{Bernoulli}(p_{ij})$ $\qquad p_{ij} = \Phi(x_{ij}^T \beta_i)$

Φ is the standard normal CDF

$i = 1, \dots, n \qquad j = 1, \dots, m_i \qquad N = \sum_{i=1}^{n} m_i$

$x_{ij} \in \mathbb{R}^k \qquad x_{ij1} = 1$

Latent variable :

$z_{ij} \sim N(x_{ij}^T \beta_i, 1) \qquad y_{ij} = 1 \text{ if } z_{ij} > 0$

$\beta_i \sim N(\varphi, \sigma^2 I) \qquad \lambda = \frac{1}{\sigma^2}$

Priors : $f(\varphi) \propto 1 \qquad f(\lambda) = Ga(\frac{1}{2}, \frac{1}{2})$

Posterior: $p(z, \beta, \varphi, \lambda \mid Y) \propto$

$$\prod_{i=1}^{n} \prod_{j=1}^{m_i} p(y_{ij} \mid z_{ij})\, p(z_{ij} \mid \beta_i) \prod_{i=1}^{n} p(\beta_i \mid \varphi, \lambda)\, p(\varphi)\, p(\lambda)$$

$$\propto \prod_{i=1}^{n} \prod_{j=1}^{m_i} \left[I(y_{ij}=1) I(z_{ij} > 0) + I(y_{ij}=0) I(z_{ij} \leq 0) \right]$$

$$\exp\left[-\frac{1}{2}(z_{ij} - x_{ij}^T \beta_i)^2\right] \prod_{i=1}^{n} \lambda^{\frac{k}{2}} \exp\left[-\frac{\lambda}{2}(\beta_i - \varphi)^T(\beta_i - \varphi)\right] \lambda^{-\frac{1}{2}} e^{-\frac{1}{2}\lambda}$$

Gibbs sampler $(z_{ij}, \beta_i, \varphi, \lambda)$

$p(z_{ij}|-) \sim$ positive $N(x_{ij}^T\beta_i, 1)$ if $y_{ij}=1$

$\sim$ negative $N(x_{ij}^T\beta_i, 1)$ if $y_{ij}=0$

$$p(\beta_i|-) \propto \exp\left[-\frac{1}{2}\sum_{j=1}^{m_i}(d_{ij} - x_{ij}^T\beta_i)^2 - \frac{\lambda}{2}(\beta_i-\varphi)^T(\beta_i-\varphi)\right]$$

$$\sim N\left(A^{-1}B,\ A^{-1}\right)$$

$$A^{-1} = \left(\sum_{j=1}^{m_i} x_{ij}x_{ij}^T + \lambda I\right)^{-1} \qquad B = \sum_{j=1}^{m_i} z_{ij}x_{ij} + \lambda\varphi$$

$$p(\varphi|-) \propto \exp\left[-\frac{\lambda}{2}\sum_{i=1}^{n}(\beta_i-\varphi)^T(\beta_i-\varphi)\right]$$

$$\sim N\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i,\ \frac{I}{n\lambda}\right)$$

$$p(\lambda|-) \propto \lambda^{\frac{nk-1}{2}}\exp\left[-\frac{\lambda}{2}\sum_{i=1}^{n}(\beta_i-\varphi)^T(\beta_i-\varphi) - \frac{1}{2}\lambda\right]$$

$$\sim Ga\left(\frac{nk-1}{2},\ \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{n}(\beta_i-\varphi)^T(\beta_i-\varphi)\right)$$